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CONVECTIVE INSTABILITY OF A POLYTROPIC LAYER IN A MAGNETIC FIELD AND A CORIOLIS FORCE

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16. Abstract

An infinite horizontal thin polytropic layer, heated from below and subjected to uniform magnetic field and rotation, is investigated. Rayleigh numbers defining the onset of instability are computed. For a small rotation rate, the presence of a magnetic field increases the stability. For a relatively large rotation rate, increasing magnetic field decreases the stability until a minimum value of Rayleigh number is reached. From that point, the presence of a magnetic field increases the stability.

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CONVECTIVE INSTABILITY OF A POLYTROPIC LAYER IN A MAGNETIC FIELD AND A CORIOLIS FORCE

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SUMMARY

The onset of convection induced by gravity is considered in a horizontal, infinite, polytropic fluid layer, heated from below, under a uniform vertical magnetic field and a coriolis acceleration resulting from a uniform rotation. The viscosity and thermal and electrical conductivities are included in the analysis. The principle of the exchange of stabilities is assumed valid. The numerical method is used to compute the critical Rayleigh number that defines the onset of the instability. It is found that the stability of the system depends on the layer thickness.

When the rotation rate is small, stability increases as the magnetic field increases. However, when the rotation rate is relatively large, complex interactions occur between the rotation and the magnetic field. The critical Rayleigh number decreases with increasing magnetic field until a minimum value is attained. From that point, the critical Rayleigh number increases as the magnetic field increases.

For a small magnetic field, rotation is effective in stabilizing the system. However, if the magnetic field is large, the stabilizing effect achieved by increasing the rotation rate becomes less prominent.

INTRODUCTION

The analysis of the thermal instability induced by gravity in an infinite layer of fluid medium is the subject of numerous examinations, because of the potential applications to both geophysical and astrophysical phenomena. Analyses using models without rotation have been studied extensively (for example, refs. 1 to 4). However, in the application to a stellar atmosphere, a coriolis force should be included in the study. Therefore, many investigations that include the rotation have been made. An incompressible medium, subjected to a coriolis force and a magnetic field simultaneously, has been examined by Chandrasekhar (ref. 5) and Talwar (ref. 6), who also

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studied the compressible model but considered the medium as an ideal fluid having infinite electrical conductivity (ref. 7). A similar model was analyzed in other studies (refs. 8 and 9) that included compressibility but ignored the electrical nature of the medium.

To provide a more realistic picture of thermal instability from the astrophysical standpoint, the finite transport coefficients, as well as the compressibility of the medium, have been included in the investigation discussed in this paper.

SYMBOLS

A_i, B_i	coefficients used in equation (45)
a	horizontal wave number defined by equation (37)
b	layer thickness
C	coefficient defined by equation (15)
c_p	specific heat capacity, constant pressure
c_v	specific heat capacity, constant volume
D	operator defined as $\frac{d}{d\zeta}$
E	electric field
e	exponential
g	gravitational field
H	magnetic field
H_x	x-component of the perturbation magnetic field
H_y	y-component of the perturbation magnetic field
H_z	z-component of the perturbation magnetic field
$i = \sqrt{-1}$	
J	current density

K	thermal conductivity
k	horizontal wave number, defined by equation (33)
k_x	wave number in the x-direction
k_y	wave number in the y-direction
m	parameter defined by equation (14)
P	integration constant
p	pressure
Q	Hartmann number
Q_1	parameter defined by equation (63)
R	Rayleigh number
R_c	critical Rayleigh number
R^*	universal gas constant
T	dimensionless thickness
τ	Taylor number defined by equation (62)
τ_1	parameter defined by equation (64)
τ'	parameter defined by equation (40)
t	time
u, v, W	x-, y-, and z-components of the perturbation velocity
$\overline{W}$	component defined in equation (27)
x,y,z	rotating coordinates
β	thermal gradient
Γ	polytropic index
ϵ	variable defined in equation (42)

ζ dimensionless variable defined by equation (35)

η electrical resistance, defined by equation (23)

θ dimensionless temperature

μ material viscosity

μ_e magnetic permeability

ρ density

σ electrical conductivity

$\tilde{\tau}$ stress tensor

Ω angular velocity

$\bar{\omega}$ amplitude of ω_z, defined in equation (29)

ω_z z-component of $\nabla \times \vec{u}$

$\vec{(\ )}$ vector

$(\ )'$ perturbation quantity

$\tilde{(\ )}$ tensor

$\hat{(\ )}$ unit vector

$\bar{(\ )}$ amplitude of the perturbation quantity

Subscripts:

0 equilibrium condition

i index, $i = 1, 2, 3, \ldots$

Operator:

∇ del-operator

GOVERNING EQUATIONS OF THE PROBLEM

The system considered in this paper is a free infinite layer of thickness b, rotating with a constant angular velocity $\vec{\Omega}$ about the vertical axis. The configuration is shown in figure 1. Both the applied magnetic field $\vec{H}_0$ and the gravitational field $\vec{g}$ are parallel to the z-axis and are considered to be constant.

In rotating coordinates x, y, and z, the governing hydrodynamic equations are

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho(\vec{u} \cdot \nabla)\vec{u} - \mu_e \vec{J} \times H = -\nabla p + \mu \nabla^2 \vec{u}$$

$$+ \rho\vec{g} + \frac{1}{3}\mu\nabla(\nabla \cdot \vec{u}) + 2\rho\vec{u} \times \vec{\Omega} \qquad (1)$$

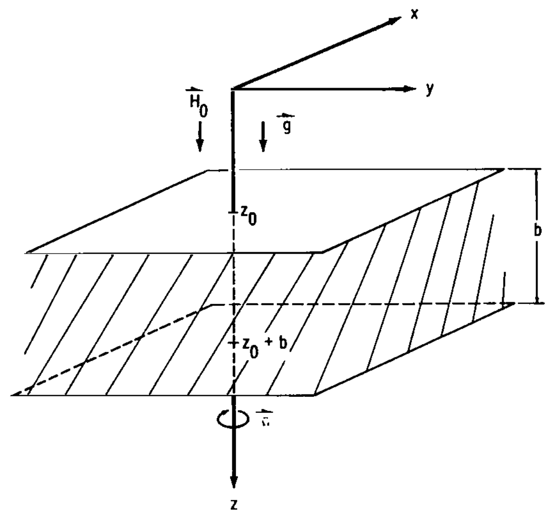

Figure 1. - Configuration of the system with a free infinite layer of thickness b, rotating with angular velocity $\vec{\Omega}$ about the z-axis.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{u}) = 0 \qquad (2)$$

$$\rho c_v \left[\frac{\partial T}{\partial t} + (\vec{u} \cdot \nabla)T\right] = K\nabla^2 T - p\nabla \cdot \vec{u} + \frac{|\vec{J}|^2}{\sigma} + (\tilde{\tau} \cdot \nabla) \cdot \vec{u} + \mu\nabla \cdot (\vec{u} \cdot \nabla\vec{u}) - \frac{2}{3}\mu(\nabla \cdot \vec{u})^2 \qquad (3)$$

and the equation of state is

$$p = R^*\rho T \qquad (4)$$

The equations governing the electromagnetic conditions are

$$\nabla \times \vec{H} = 4\pi\vec{J} \qquad (5)$$

$$\nabla \times \vec{E} = -\mu_e \frac{\partial \vec{H}}{\partial t} \tag{6}$$

$$\nabla \cdot \vec{H} = \vec{0} \tag{7}$$

and

$$\vec{J} = \sigma\left(\vec{E} + \mu_e \vec{u} \times \vec{H}\right) \tag{8}$$

In equations (1) to (8), μ is the material viscosity, μ_e is the magnetic permeability, and $\tilde{\tau}$ is the stress tensor, which is a function of derivatives of $\vec{u}$. The electromagnetic units are used in this paper.

From the static equilibrium

$$p_0 = Pz^{m+1} \tag{9}$$

$$\rho_0 = \frac{P}{R^*\beta_0} z^m \tag{10}$$

and

$$T_0 = \beta_0 z \tag{11}$$

where P is an integration constant, β_0 is the applied thermal gradient, and the origin of the coordinates is chosen at the level where $T_0 = 0$. Variables with a subscript zero represent the equilibrium conditions. Equations (9) to (11) are indicative that

$$p_0 = C\rho_0^{\Gamma} \tag{12}$$

with

$$\Gamma = \frac{m + 1}{m} = \text{polytropic index} \tag{13}$$

$$m = \frac{g}{\beta_0 R^*} - 1 \tag{14}$$

and

$$C = P\left(\frac{R^*\beta_0}{P}\right)^{1+(1/m)} \tag{15}$$

Hence, the equilibrium state follows the polytropic relation.

THE PERTURBATION STATE

Consider a slightly perturbed state in which the variables can be expressed as

$$\left.\begin{aligned}
\rho &= \rho_0 + \rho' \\
p &= p_0 + p' \\
T &= T_0 + T' \\
\vec{u} &= \vec{u}' = (u, v, W) \\
\vec{E} &= \vec{E}' \\
\vec{J} &= \vec{J}' \\
\vec{H} &= \vec{H}_0 + \vec{H}' = \left(H_x', H_y', H_0 + H_z'\right)
\end{aligned}\right\} \tag{16}$$

The primed terms are the perturbation quantities that have negligible products. Assuming that the marginal stability is valid (that is, the $\partial/\partial t$ terms vanish), the governing equations can be combined to form the following equations.

$$-\frac{\mu_e}{4\pi}\left(H_0 \frac{\partial}{\partial z}\right)\vec{H}' = -\nabla\left(p' + \frac{\mu_e}{4\pi} H_0 H_z'\right) + \mu\nabla^2\vec{u}' + \rho'\vec{g}$$

$$+ \mu\nabla(\nabla \cdot \vec{u}') + 2\rho_0\Omega\left(v\hat{e}_x - u\hat{e}_y\right) \tag{17}$$

$$\nabla \cdot \vec{u}' = -\frac{m}{z} W \tag{18}$$

$$\beta\rho_0 c_p W = K\nabla^2 T' \tag{19}$$

$$\vec{H}_0\nabla \cdot \vec{u}' - \left(H_0 \frac{\partial}{\partial z}\right)\vec{u}' = \eta\nabla^2\vec{H}' \tag{20}$$

$$\rho' = \frac{1}{R^*T_0}\left(p' - \frac{Pz^m}{\beta_0} T'\right) \tag{21}$$

$$\beta = \beta_0 - \frac{g}{c_p} \tag{22}$$

and

$$\eta = \frac{1}{4\pi\mu_e\sigma} \tag{23}$$

When standard eliminations by means of curl and divergence operations are used on equation (17), the results can be combined with equations (18) to (21) to give

$$\frac{\mu_e H_0^2}{4\pi\eta}\left(1 - \frac{z}{m+1}\frac{\partial}{\partial z}\right)\left(\frac{m}{z} + \frac{\partial}{\partial z}\right) W - \mu\left[\left(\nabla^2 + \frac{1}{3}\nabla_1^2 - \frac{z}{m+1}\nabla\frac{\partial}{\partial z}\right)\left(\frac{m}{z}\right)\right.$$

$$\left. + \nabla^2\frac{\partial}{\partial z} - \frac{z}{m+1}\nabla^4\right] W - \frac{Pz^m}{\beta_0}\nabla_1^2 T' + 2\Omega\left(1 - \frac{z}{m+1}\frac{\partial}{\partial z}\right)\rho_0\omega_z = 0 \tag{24}$$

$$\nabla^4 \omega_z - \frac{\mu_e H_0^2}{4\pi\mu\eta} \frac{\partial^2}{\partial z^2} \omega_z + 2 \frac{\Omega}{\mu} \frac{\partial}{\partial z} \nabla^2 \left(\rho_0 W\right) = 0 \tag{25}$$

and

$$\beta \rho_0 c_p W = K \nabla^2 T' \tag{26}$$

where ω_z is the z-component of $\nabla \times \vec{u}'$, and $\nabla_1^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Because the coefficients in equations (24) to (26) are not functions of x and y, the solutions are assumed to be in the form

$$W = \overline{W}(z) e^{i\left(k_x x + k_y y\right)} \tag{27}$$

$$T' = \theta(z) e^{i\left(k_x x + k_y y\right)} \tag{28}$$

and

$$\omega_z = \overline{\omega}(z) e^{i\left(k_x x + k_y y\right)} \tag{29}$$

where k_x and k_y are the wave numbers in the x- and y-directions, respectively. By substituting equations (27) to (29) into equations (24) to (26), the following equations are obtained.

$$\begin{aligned} &\left(\frac{d^2}{dz^2} - k^2\right)^2 \overline{W} + \left(\frac{d^2}{dz^2} - k^2\right) \frac{d}{dz}\left(\frac{m\overline{W}}{z}\right) + \frac{1}{3} \frac{(m+1)m}{z^2} k^2 \overline{W} \\ &- \frac{(m+1)}{z} \left(\frac{d^2}{dz^2} - k^2\right)\left(\frac{d}{dz} + \frac{m}{z}\right) \overline{W} - \frac{\mu_e H_0^2}{4\pi\mu\eta} \left(\frac{d}{dz} - \frac{m+1}{z}\right)\left(\frac{m}{z} + \frac{d}{dz}\right) \overline{W} \\ &+ \frac{PR^*}{\mu g} (m+1)^2 k^2 z^{m-1} \theta + \frac{2\Omega}{\mu} \left(\frac{m+1}{z} - \frac{d}{dz}\right) \rho_0 \overline{\omega} = 0 \end{aligned} \tag{30}$$

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 \bar{\omega} - \frac{\mu_e H_0^{\;2}}{4\pi\mu\eta} \frac{d^2}{dz^2} \bar{\omega} + 2\frac{\Omega}{\mu} \frac{d}{dz}\left(\frac{d^2}{dz^2} - k^2\right)\left(\rho_0 \overline{W}\right) = 0 \tag{31}$$

and

$$\beta c_p \frac{Pz^m}{R^*\beta_0} \overline{W} = K\left(\frac{d^2}{dz^2} - k^2\right)\theta \tag{32}$$

with

$$k^2 = k_x^{\;2} + k_y^{\;2} = \text{horizontal wave number} \tag{33}$$

By use of equations (30) to (32), the following equation governing θ can be formed.

$$\begin{aligned}
&\Bigg\{\left[D\left(\frac{1}{\zeta}\left\{D^2\left[\frac{1}{\zeta} D(\zeta)\right]\right\}\right) - \left(2a^2 + Q\right)D\left\{\frac{1}{\zeta}\left[\frac{1}{\zeta} D(\zeta)\right]\right\} + \frac{a^4}{\zeta}\right] \\
&\cdot \frac{1}{\zeta^m}\left[\left(D^2 - a^2\right)^2 + \left(D^2 - a^2\right)D\left(\frac{m}{\zeta}\right) + \frac{1}{3}\frac{(m+1)m}{\zeta^2} a^2\right. \\
&\left. - \frac{(m+1)}{\zeta}\left(D^2 - a^2\right)\left(D + \frac{m}{\zeta}\right) - Q\left(D - \frac{m+1}{\zeta}\right)\left(\frac{m}{\zeta} + D\right)\right] \\
&+ \tau' D\left[\frac{1}{\zeta} D\left(D^2 - a^2\right)\left(\zeta^m\right)\right]\Bigg\}\left[\zeta^{-m}\left(D^2 - a^2\right)\right]\theta \\
&= -\zeta_0^{\;-(2m-1)} Ra^2\left\{D\left[\frac{1}{\zeta} D^2\left(\frac{1}{\zeta} D\right)\right] - \left(2a^2 + Q\right)D\left(\frac{1}{\zeta^2} D\right) + \frac{a^4}{\zeta^2}\right\}\theta
\end{aligned} \tag{34}$$

This equation has been normalized by the layer thickness b, and the dimensionless quantities are defined as

$$\zeta = \frac{z}{b} \tag{35}$$

$$D = \frac{d}{d\left(\frac{z}{b}\right)} = \frac{d}{d\zeta} \qquad (36)$$

$$a^2 = k^2b^2 = \text{horizontal wave number} \qquad (37)$$

$$R = \zeta_0^{2m-1} \frac{P^2R^*\beta c_p(m + 1)^3 b^{2m+3}}{K\mu g^2} = \text{Rayleigh number} \qquad (38)$$

$$Q = \frac{\mu_e H_0^2 b^2}{4\pi\mu\eta} = \text{Hartmann number} \qquad (39)$$

and

$$\tau' = b^4 \left(\frac{2\Omega}{\mu}\right)^2 \left(\frac{Pb^m}{\beta_0 R^*}\right)^2 \qquad (40)$$

THIN-LAYER ANALYSIS

Because equation (34) is a 10th-order ordinary differential equation with variable coefficients, an analytical solution is difficult to obtain. For simplification, the analysis will be restricted to the thin-layer case.

Consider a layer that has a small thickness b and is located at a large distance z_0 from the origin such that

$$T \equiv \frac{1}{\zeta_0} = \frac{b}{z_0} << 1 \qquad (41)$$

The parameter T is defined as the dimensionless thickness. The variable ζ can now be written as

$$\zeta = \zeta_0 + \epsilon \qquad (42)$$

where ϵ varies from zero to one. Consequently, quantities in the form of $1/\zeta^n$ can be expressed as

$$\frac{1}{\zeta^n} \simeq \frac{1}{\zeta_0{}^n}\left(1 - n\frac{\epsilon}{\zeta_0}\right) \tag{43}$$

and

$$\frac{d}{d\zeta} = \frac{d}{d\epsilon} \tag{44}$$

With this approximation, equation (34) can be written as

$$\begin{aligned} & A_{10}\frac{d^{10}\theta}{d\epsilon^{10}} + A_9\frac{d^9\theta}{d\epsilon^9} + A_8\frac{d^8\theta}{d\epsilon^8} + A_7\frac{d^7\theta}{d\epsilon^7} + A_6\frac{d^6\theta}{d\epsilon^6} + A_5\frac{d^5\theta}{d\epsilon^5} \\ & + \left(A_4 + Ra^2B_4\right)\frac{d^4\theta}{d\epsilon^4} + \left(A_3 + Ra^2B_3\right)\frac{d^3\theta}{d\epsilon^3} \\ & + \left(A_2 + Ra^2B_2\right)\frac{d^2\theta}{d\epsilon^2} + \left(A_1 + Ra^2B_1\right)\frac{d\theta}{d\epsilon} \\ & + \left(A_0 + Ra^2B_0\right)\theta = 0 \end{aligned} \tag{45}$$

where

$$\begin{aligned} A_0 = & -a^{10} + 5a^{10}\frac{\epsilon}{\zeta_0} + \frac{1}{\zeta_0{}^2}\left(12a^8 + 4Qa^6\right) \\ & + \frac{\epsilon}{\zeta_0{}^3}\left(34a^8 + 28Qa^6\right) + \frac{1}{\zeta_0{}^4}\left(236a^6 + 288Qa^4 + 84Q^2a^2\right) \\ & - \frac{\epsilon}{\zeta_0{}^5}\left(208a^6 + 248Qa^4 + 72Q^2a^2\right) - \frac{1}{\zeta_0{}^6}\left(52a^4 + 36a^2Q\right) \end{aligned} \tag{46}$$

$$A_1 = -\frac{1}{\zeta_0}\left(25a^8 + 13Qa^6 + a^4\tau\right) + \frac{\epsilon}{\zeta_0^{\,2}}\left(74a^8 + 40Qa^6\right)$$

$$-\frac{1}{\zeta_0^{\,3}}\left(112a^6 + 130Qa^4 + 38Q^2a^2\right)$$

$$+\frac{\epsilon}{\zeta_0^{\,4}}\left(660a^6 + 624Qa^4 + 180Q^2a^2\right)$$

$$+\frac{1}{\zeta_0^{\,5}}\left(858a^4 + 510Qa^2\right) - \frac{\epsilon}{\zeta_0^{\,6}}\left(444a^4 + 252Qa^2\right) - \frac{72a^2}{\zeta_0^{\,7}} \tag{47}$$

$$A_2 = \left(5a^8 + 2Qa^6 + a^4\tau\right) - \frac{\epsilon}{\zeta_0}\left(25a^8 + 10Qa^6 + a^4\tau\right)$$

$$-\frac{1}{\zeta_0^{\,2}}\left(170a^6 + 118Qa^4 + 30Q^2a^2\right) + \frac{\epsilon}{\zeta_0^{\,3}}\left(24a^6 + 10Qa^4 + 12Q^2a^2\right)$$

$$-\frac{1}{\zeta_0^{\,4}}\left(1354a^4 + 958Qa^2 + 84Q^2\right) + \frac{\epsilon}{\zeta_0^{\,5}}\left(1794a^4 + 1166Qa^2 + 72Q^2\right)$$

$$+\frac{1}{\zeta_0^{\,6}}\left(712a^2 + 36Q\right) - 216\,\frac{a^2\epsilon}{\zeta_0^{\,7}} \tag{48}$$

$$A_3 = \frac{1}{\zeta_0}\left(100a^6 + 64Qa^4 + 13Q^2a^2 + 2a^2\tau\right)$$

$$-\frac{\epsilon}{\zeta_0^{\,2}}\left(298a^6 + 194Qa^4 + 40Q^2a^2\right)$$

$$+\frac{1}{\zeta_0^{\,3}}\left(150a^4 + 146Qa^2 + 38Q^2\right) - \frac{\epsilon}{\zeta_0^{\,4}}\left(1516a^4 + 1124Qa^2 + 180Q^2\right)$$

$$-\frac{1}{\zeta_0^{\,5}}\left(2112a^2 + 510Q\right) + \frac{\epsilon}{\zeta_0^{\,6}}\left(1488a^2 + 252Q\right) + \frac{72}{\zeta_0^{\,7}} \tag{49}$$

$$A_4 = -\left(10a^6 + 6a^4 Q + Q^2 a^2 + 2a^2 \tau\right)$$

$$+ \frac{\epsilon}{\zeta_0}\left(50a^6 + 30Qa^4 + 5Q^2 a^2 + 2a^2 \tau\right)$$

$$+ \frac{1}{\zeta_0^{\,2}}\left(438a^4 + 262Qa^2 + 30Q^2\right)$$

$$- \frac{\epsilon}{\zeta_0^{\,3}}\left(294a^4 + 178Qa^2 + 12Q^2\right)$$

$$+ \frac{1}{\zeta_0^{\,4}}\left(1672a^2 + 670Q\right) - \frac{\epsilon}{\zeta_0^{\,5}}\left(2672a^2 + 918Q\right) - \frac{660}{\zeta_0^{\,6}} + 216\,\frac{\epsilon}{\zeta_0^{\,7}} \tag{50}$$

$$A_5 = -\frac{1}{\zeta_0}\left(150a^4 + 89Qa^2 + 13Q^2 + \tau\right)$$

$$+ \frac{\epsilon}{\zeta_0^{\,2}}\left(450a^4 + 270Qa^2 + 40Q^2\right) + \frac{1}{\zeta_0^{\,3}}\left(30a^2 - 16Q\right)$$

$$+ \frac{\epsilon}{\zeta_0^{\,4}}\left(1048a^2 + 500Q\right) + \frac{1254}{\zeta_0^{\,5}} - 1044\,\frac{\epsilon}{\zeta_0^{\,6}} \tag{51}$$

$$A_6 = \left(10a^4 + 6Qa^2 + Q^2 + \tau\right) - \frac{\epsilon}{\zeta_0}\left(50a^4 + 30Qa^2 + 5Q^2 + \tau\right) \tag{52}$$

$$A_7 = \left(100a^2 + 38Q\right)\frac{1}{\zeta_0} - \left(302a^2 + 116Q\right)\frac{\epsilon}{\zeta_0^{\,2}} - \frac{70}{\zeta_0^{\,3}} - \frac{192\epsilon}{\zeta_0^{\,4}} \tag{53}$$

$$A_8 = -\left(5a^2 + 2Q\right) + 5\left(5a^2 + 2Q\right)\frac{\epsilon}{\zeta_0} + \frac{134}{\zeta_0^{\,2}} - \frac{144\epsilon}{\zeta_0^{\,3}} \tag{54}$$

$$A_9 = -\frac{25}{\zeta_0} + 76\,\frac{\epsilon}{\zeta_0^{\,2}} \tag{55}$$

$$A_{10} = 1 - \frac{5\epsilon}{\zeta_0} \tag{56}$$

$$B_0 = a^4 - 2a^4 \frac{\epsilon}{\zeta_0} \tag{57}$$

$$B_1 = 2\left(2a^2 + Q\right)\frac{1}{\zeta_0} \tag{58}$$

$$B_2 = -\left(2a^2 + Q\right) + \frac{2\epsilon}{\zeta_0}\left(2a^2 + Q\right) + \frac{2}{\zeta_0^{\,2}} \tag{59}$$

$$B_3 = -\frac{4}{\zeta_0} + \frac{2\epsilon}{\zeta_0^{\,2}} \tag{60}$$

and

$$B_4 = 1 - \frac{2\epsilon}{\zeta_0} \tag{61}$$

Note that $\mathcal{T}'$ in equation (34) is now replaced by $\mathcal{T}$ which is defined as

$$\mathcal{T} = b^4\left(\frac{2\Omega}{u}\right)^2\left(\frac{Pz_0^{\,m}}{\beta_0 R^*}\right)^2 = \text{Taylor number} \tag{62}$$

For convenience in comparing the results with those obtained by Chandrasekhar (ref. 5) for the incompressible model

$$Q_1 = \frac{Q}{\pi^2} \tag{63}$$

and

$$\tau_1 = \frac{\tau}{\pi^4} \tag{64}$$

will be used in the computations, for which m is set equal to 2.

BOUNDARY CONDITIONS

Equation (45) must be solved with appropriate boundary conditions. Assuming that the perturbations vanish at the boundaries, the following equation is derived.

$$\overline{W} = \theta = 0 \quad \text{at} \quad \epsilon = 0,\ 1 \tag{65}$$

Because only the free layer is considered, no boundary stresses exist at the surfaces. This condition leads to

$$D^2\overline{W} + \frac{m}{\zeta}\,\overline{W} = 0 \tag{66}$$

$$DH = 0 \tag{67}$$

$$D\overline{\varpi} = 0 \tag{68}$$

and

$$D^3\overline{\varpi} = 0 \quad \text{at} \quad \epsilon = 0,\ 1 \tag{69}$$

The adjoining vacuum acts as a perfect insulator, thus requiring that

$$E_z = 0 \quad \text{at} \quad \epsilon = 0,\ 1 \tag{70}$$

Because of the variable coefficients of the equations, the other boundary conditions for θ are difficult to obtain. To simplify the boundary conditions, all terms that contain $1/\zeta_0{}^n$, where $n \geq 1$, will be neglected. Only the thin-layer analysis is considered; therefore, such an approximation is justified, because $1/\zeta_0^{\,n}$ is much smaller than unity. The appendix contains a discussion of the use of approximate boundary conditions for the thin-layer analysis.

The following two points must be clarified: (1) In deriving the governing equation (eq. (45)), all $1/\zeta^n$ terms are approximated with their linear forms obtained from the binomial expansion, as shown by equation (43). By using this method, the compressibility of the medium is retained. If all $1/\zeta^n$ terms are allowed to vanish, the problem is reduced simply to that of the incompressible model used by Chandrasekhar (ref. 5). (2) For the boundary conditions, the $1/\zeta^n$ terms for the incompressible model are used. The approximation greatly simplifies the problem and is justifiable for the thin-layer case. Because $\zeta_0 >> 1$, as defined by equation (41), $\zeta \simeq \zeta_0$ = constant, and ρ_0 will be treated as constant in deriving the boundary conditions. Then, equation (30) can be normalized by b and can be written in the form

$$\left[\left(D^2 - a^2\right)^2 - QD^2\right]\overline{W} + \frac{PR^*}{\mu g}(m + 1)^2 a^2 b^{-m+3} \zeta_0^{\,m+1} \theta$$

$$- \frac{2\Omega}{\mu} \frac{Pb^{-m+3}}{R^*\beta_0} \zeta_0^{\,m} D\overline{\omega} = 0 \tag{71}$$

Equation (66) is then reduced to

$$D^2\overline{W} = 0 \text{ at } \epsilon = 0,\ 1 \tag{72}$$

By use of equation (65), equation (32) can be shown as

$$D^2\theta = 0 \text{ at } \epsilon = 0,\ 1 \tag{73}$$

By differentiating equation (32) twice and combining the result with equations (72) and (73)

$$D^4\theta = 0 \text{ at } \epsilon = 0,\ 1 \tag{74}$$

When equations (65), (68), and (72) are substituted into equation (71), the result is

$$D^4\overline{W} = 0 \text{ at } \epsilon = 0, 1 \tag{75}$$

When the fourth differentiation of equation (32) is combined with equations (74) and (75)

$$D^6\theta = 0 \text{ at } \epsilon = 0, 1 \tag{76}$$

If equation (71) is differentiated twice and equations (69), (72), and (75) are used, the result is

$$D^6\overline{W} = 0 \text{ at } \epsilon = 0, 1 \tag{77}$$

The substitution of equations (76) and (77) into the sixth differentiation of equation (32) gives

$$D^8\theta = 0 \text{ at } \epsilon = 0, 1 \tag{78}$$

Hence, the 10 approximate boundary conditions for θ are

$$\left.\begin{aligned} \theta &= 0 \\ D^2\theta &= 0 \\ D^4\theta &= 0 \\ D^6\theta &= 0 \\ &\text{and} \\ D^8\theta &= 0 \end{aligned}\right\} \text{ at } \epsilon = 0, 1 \tag{79}$$

These boundary conditions, together with equation (45), form an eigenvalue problem, having Q_1 and τ_1 as the parameters and the Rayleigh number R as the characteristic value. Because R indicates the stability of the system, the minimum value R_c, with respect to the horizontal wave number, defines the onset of the thermal convection.

RESULTS

Equation (45) can be reduced to the equation obtained by Chandrasekhar (ref. 5) by setting $T = 1/\zeta_0 = 0$. Hence, the results obtained for the incompressible case correspond to the thin-layer case with infinitesimal thickness.

As shown in figure 2, the critical Rayleigh number R_c decreases as the dimensionless thickness T increases for a given value of Q_1 and τ_1. Therefore, in a compressible model, stability depends on the layer thickness. Because it is assumed that the perturbations vanish at the boundary, a thicker layer therefore allows more freedom for the disturbance than the thinner layer.

As shown in table I, for a given thickness and a given small magnetic field Q_1, the rotation rate τ_1 increases

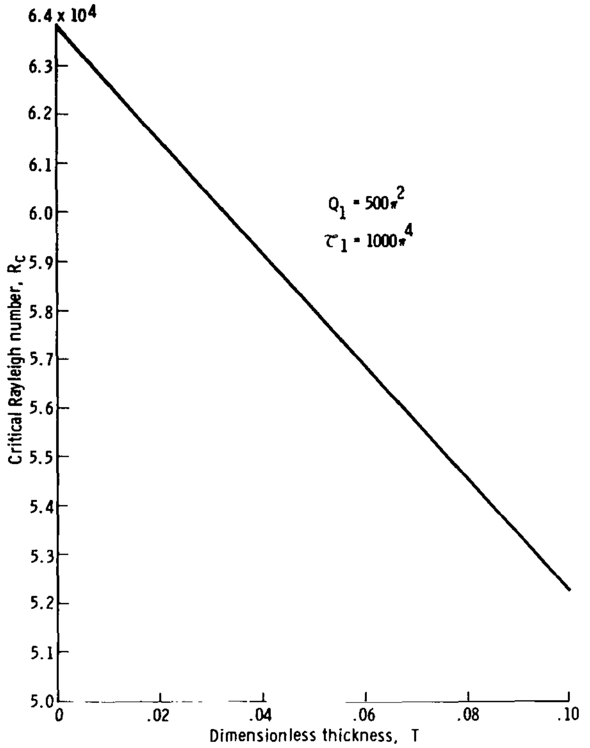

Figure 2. - Critical Rayleigh number as a function of dimensionless thickness T.

TABLE I. - CRITICAL RAYLEIGH NUMBER AS A FUNCTION OF τ_1 FOR $Q_1 = 10$, $Q_1 = 500$, $Q_1 = 10\ 000$, AND $T = 0.06$

Q_1	τ_1	R_c	a
10	1	2 351	3.60
	50	3 611	4.00
	100	4 848	4.30
	500	12 802	6.76
	1000	20 009	8.10
500	1	55 350	7.50
	50	55 421	7.50
	100	55 493	7.50
	500	56 337	7.50
	1000	57 054	7.50
10 000	1	944 576	12.6
	50	949 756	12.6
	100	949 765	12.6
	500	949 837	12.6
	1000	949 926	12.6

the stability, because R_c increases rapidly as $\mathcal{T}_1$ increases. The size of the cell also increases accordingly, as indicated by an increasing a. However, for a large value of Q_1, a large increase of $\mathcal{T}_1$ only increases the stability to a slight extent, and the size of the cell remains the same. An increased rotation rate can promote stability because the vortexes generated by the rotation increase the tendency of the fluid particles to remain at the same altitude and to resist being moved to a higher level by convection.

When the magnetic field is very large, the fluid particles are strongly bound to gyrate about the magnetic lines and have less drift tendency. The circulating motion of the fluid element is caused primarily by the Larmor gyration rather than by the applied rotation. Consequently, the effect of the applied rotation upon the stability becomes less dominant.

The combination of the magnetic field and the rotation rate helps to delay the onset of the thermal convection (table I). However, this statement is not exactly true when $\mathcal{T}_1$ is very large, as indicated in table II. For $\mathcal{T}_1$ = 100 000 and for Q_1

TABLE II. - CRITICAL RAYLEIGH NUMBER AS A FUNCTION OF Q_1

FOR $\mathcal{T}_1$ = 100 000 AND T = 0.06

Q_1	a	R_c
20	19.2	399 111
30	19.1	398 336
40	19.0, 3.6	397 460, 876 700
50	18.8, 3.5	396 589, 719 920
60	18.8, 3.4	395 710, 611 410
80	18.8, 3.3	393 982, 472 281
100	18.6, 3.3	392 156, 387 757
200	3.4	222 611
500	4.2	154 566
1 000	5.5	170 008
2 000	7.3	247 031
4 000	9.4	422 003
10 000	12.0	965 501
50 000	16.6	4 482 024
100 000	18.7	8 861 127

ranging from 20 to 100 000, two minimums occur, just as they occur for the previously mentioned incompressible case (ref. 5). As Q_1 increases, R_c first decreases and then increases monotonically.

The vertical velocity distribution $\overline{W}$ across the thickness of the layer is illustrated in figure 3. When T is increased, the profile is distorted to the left, because a thicker layer is less stable, and $\overline{W}$ reaches maximum value sooner and takes a longer time to decrease to the vanishing value at the upper boundary.

CONCLUSIONS

It is found that the compressible medium is less stable than the incompressible medium when heated from below. When the rotation rate is small, the inhibition caused by a magnetic field on the onset of thermal convection is very effective. When the rotation rate is large, increasing the magnetic field offers only decreasing inhibition until a minimum value of the critical Rayleigh number is attained. After this point, the inhibition increases as the magnetic field increases.

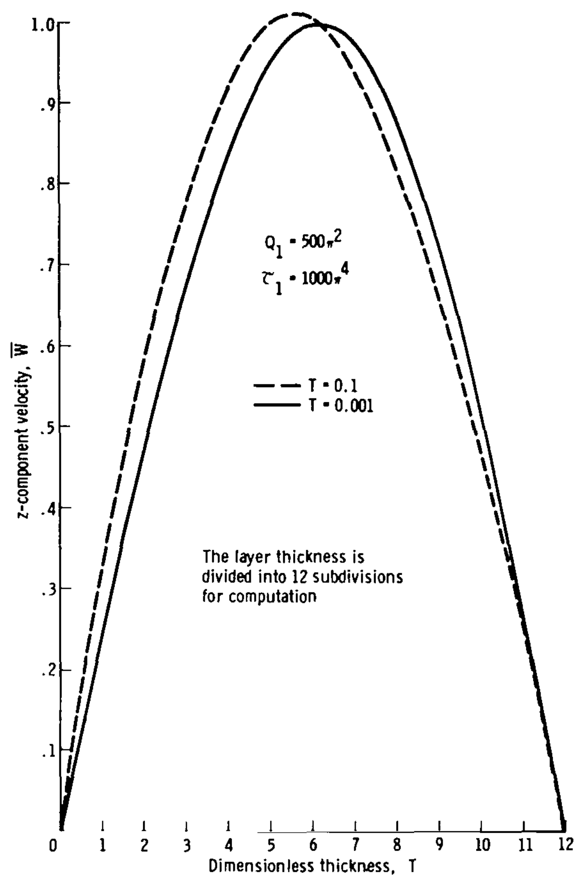

Figure 3. - The z-component velocity $\overline{W}$ profiles for dimensionless thickness T = 0.001 and T = 0.1.

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National Aeronautics and Space Administration
Houston, Texas, November 19, 1970
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APPENDIX

USE OF APPROXIMATE BOUNDARY CONDITIONS FOR THE THIN-LAYER CASE

For the thin-layer case, the use of approximate boundary conditions can be justified from the following comparison in the case without rotation (ref. 3). The exact boundary conditions are

$$\theta = 0 \tag{A1}$$

$$\overline{W} = 0 \tag{A2}$$

and

$$D^2\overline{W} + \frac{m}{\zeta} D\overline{W} = 0 \quad \text{at} \quad \zeta = \zeta_0, \zeta_0 + 1 \tag{A3}$$

The critical Rayleigh numbers R_c are obtained analytically by use of the variational method. The critical Rayleigh numbers are then recalculated with the approximate boundary conditions

$$\theta = 0 \tag{A4}$$

$$\overline{W} = 0 \tag{A5}$$

and

$$D^2\overline{W} = 0 \quad \text{at} \quad \zeta = \zeta_0, \zeta_0 + 1 \tag{A6}$$

The results are shown in table A-I for $Q = 10$ and 50. The approximate results differ from the exact results by only approximately 2 percent. Hence, the use of approximate boundary conditions for the present problem is acceptable.

TABLE A-I. - COMPARISON BETWEEN R_c FROM THE EXACT AND FROM THE APPROXIMATE BOUNDARY CONDITIONS

Q	T	R_c		Error, percent
		Exact boundary conditions	Approximate boundary conditions	
10	0.001	921.7	921.7	0
	.01	909.4	909.3	-0.01
	.1	803.8	787.5	-2.00
50	0.001	1759	1759	0
	.01	1736	1736	0
	.1	1535	1508	-1.76

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