X-552-71-60
PREPRINT

NASA TM X-65476

LUNISOLAR PERTURBATIONS OF THE MOTION OF ARTIFICIAL SATELLITES

DAVID FISHER

JANUARY 1971

GSFC

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

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CONTENTS

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SUMMARY

An expression for the lunisolar disturbing function is given which provides the basis for the calculation of perturbations in the motion of artificial satellites. Previous work in this field is described in references 1, 2, 3, and 4. The Lagrange planetary equations are derived in a form suitable for satellites of arbitrary eccentricity in contrast to the standard form which depends upon first developing the disturbing function in powers of eccentricity.

The spectrum of the disturbing function is partitioned into secular, long period, intermediate, and short period terms. The motion of the short period terms is dominated by the mean anomaly of the satellite. The remaining terms are derived by averaging the disturbing function over the satellite's mean anomaly. Those terms which contain the mean longitude of the sun or the moon are called terms of intermediate period. For the sun these terms have periods of multiples of 180 days, for the moon the corresponding periods are multiples of 14 days. Additional averaging over the mean anomaly of the disturbing body results in the secular and long period terms.

The perturbations arising from the secular and long period terms have been discussed in references 5 and 6. In this paper analytic formulas for lunisolar perturbations of intermediate periods are derived. The disturbing function containing the lunar inclination factors are developed analytically in terms of N, the longitude of the lunar node along the ecliptic, rather than numerically as in reference 1.

The results are given in tabular form making it easier to prepare computer programs and to select terms of given amplitude or period when hand calculations are desired.

Since the development is literal, the results are applicable to three body systems having configurations similar to the sun, Earth, moon system.

SYMBOLS

$A_{\nu q}$	function of i
$A'_{\nu q'}$	function of i'
$A''_{\nu q' p}$	function of ϵ and J
B_{iq}	function of e
$B'_{i' q'}$	function of e'
G	Delaunay variable $= \sqrt{\mu a (1 - e^2)}$
H	Delaunay variable $= \sqrt{\mu a (1 - e^2)} \cos i$
J	Inclination of moon's orbit to ecliptic
J_2	Earth's second order zonal harmonic
L	Delaunay variable $= \sqrt{\mu a}$
N	Longitude of lunar node along ecliptic
R'	disturbing function of the moon
R'_{sec}	secular part of R'
R'_{LP}	long period part of R'
R'_{INT}	intermediate part of R'
S'_{INT}	determining function of R'_{INT}
$S_{\nu q'}$	function of ϵ and J
a	semi-major axis of satellite
a_e	mean equatorial radius of earth
$c_{\nu q' q}$	constants of the disturbing function

e	eccentricity of the satellite
e'	eccentricity of the moon
f	true anomaly of satellite
i	inclination of satellite orbit
i'	inclination of lunar orbit
ℓ	mean anomaly of satellite
ℓ'	mean anomaly of moon
m'	ratio of mass of moon to mass of earth plus moon
n	mean motion of satellite
p	summation index
n'	mean motion of moon
q	summation index
q'	summation index
r	geocentric distance of satellite
r'	geocentric distance of moon
Γ'	mean longitude of lunar pericenter
α	argument of disturbing function
$\dot{\alpha}$	mean motion of α
ϵ	obliquity of ecliptic
ϕ	argument of latitude of satellite
ϕ'	argument of latitude of moon

LUNISOLAR PERTURBATIONS OF THE MOTION OF ARTIFICIAL SATELLITES

INTRODUCTION

For analytic purposes it is convenient to start the development of the disturbing function resulting from the expansion of the second order Legendre polynomial as given by Kozai Reference 1. The lunar disturbing function R', is written in the form

$$R' = n'^2 m' a^2 \left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \sum_{q,q',\nu} A_{\nu q} A'_{\nu q'} c_{\nu q' q} \cos (q\phi + q'\phi' + \nu\theta) \tag{1}$$

where

n'	mean motion of the moon
m'	ratio of mass of the moon to combined mass of moon and earth
r	radius from center of the earth to the satellite
r'	radius from center of the earth to the moon
a, a'	semi-major axis of satellite and moon respectively
q, q', ν	indices of summation
$c_{\nu q' q}$	constants
ϕ	argument of latitude of the earth
ϕ'	argument of latitude of the moon
θ	difference between the longitude of the ascending node of the earth and that of the moon
i	inclination of the satellite orbit to the earth's equator

i' inclination of the lunar orbit to the earth's equator

$A_{\nu q}$ function of i

$A'_{\nu q'}$ function of i'

The corresponding disturbing function due to the sun is obtained by replacing the prime quantities by similar quantities with the subscript $\odot$. Thus we set

$$n' = n_{\odot},\ r' = r_{\odot},\ a' = a_{\odot},\ m' = 1$$

$$i' = \epsilon,\ h' = 0,\ \phi' = \phi_{\odot}$$

The symbol ϵ represents the obliquity of the ecliptic.

The nomenclature and symbols appearing in Equation (1) are described in the section on symbols.

The functions $A_{\nu q}, A'_{\nu q'}$, and the constants $c_{\nu q' q}$ are given in Table 1. The index q takes on the two values 0 and 2, the index q' takes the value -2, 0, 2, while the index ν varies over the range -2, -1, 0, 1, 2.

In certain applications, particularly when precise long range trajectories are required, numerical integration of the planetary equations may be useful. The planetary equations are therefore derived. In order to extend the accuracy of the computations to satellites with highly eccentric orbits, the Lagrange planetary equations are derived directly from Equation (1) without first expanding in powers of eccentricity.

In order to obtain analytic formulas the disturbing function, Equation (1), is further expanded. After the expansion is made, the various harmonics of the disturbing function are shown to consist of short, intermediate, and long period terms. The short period terms contain the mean anomaly of the satellite. In the intermediate period terms the mean anomaly of the satellite is absent in the argument, but the mean anomaly of the perturbing body is present. For example, in the intermediate case for the sun, we find approximately semi-annual period terms, as well as multiples of this period. The corresponding terms of the lunar function consist of semi-monthly terms and multiples thereof. In the long period terms and the secular terms neither the mean anomalies of the satellite or the perturbing body appear. These terms have been treated in References 4, 5 and 6.

TABLE 1

q	q'	ν	$A_{\nu q}$	$c_{\nu q' q}$	$A'_{\nu q'}$	$\frac{\partial A_{\nu q}}{\partial(\cos i)}$	B_{eq}	$-(\nu + q')$
0	0	0	$1 - 3/2 \sin^2 i$	1/4	$1 - 3/2 \sin^2 i'$	$3 \cos i$	$1 + 3/2\ e^2$	0
0	2	0		3/8	$\sin^2 i'$			-2
0	0	1	$\sin 2i$	3/16	$\sin 2i'$	$-2 \cos 2i/\sin i$		-1
0	2	-1		-3/8	$\sin i' \cos^2 i'/2$			-1
0	2	1		3/8	$\sin i' \sin^2 i'/2$			-3
0	2	-2	$\sin^2 i$	3/8	$\cos^4 i'/2$	$-2 \cos i$		0
0	0	2		3/16	$\sin^2 i'$			-2
0	2	2		3/8	$\sin^4 i'/2$			-4
2	-2	0	$\sin^2 i$	9/32	$\sin^2 i'$	$-2 \cos i$	$5/2\ e^2$	2
2	0	0		3/8	$1 - 3/2 \sin^2 i'$			0
2	2	0		9/32	$\sin^2 i'$			-2
2	-2	2	$\cos^4 i/2$	3/4	$\cos^4 i'/2$	$\cos^2 i/2$		0
2	0	2		3/8	$\sin^2 i'$			-2
2	2	2		3/4	$\sin^4 i'/2$			-4
2	2	-2	$\sin^4 i/2$	3/4	$\cos^4 i'/2$	$-\sin^2 i/2$		0
2	0	-2		3/8	$\sin^2 i'$			2
2	-2	-2		3/4	$\sin^4 i'/2$			4
2	-2	1	$\sin i \cos^2 i/2$	3/4	$\sin i' \cos^2 i'/2$	$\frac{1 - \cos i - 2 \cos^2 i}{2 \sin i}$		1
2	0	1		-3/8	$\sin 2i'$			-1
2	2	1		-3/4	$\sin i' \sin^2 i'/2$			-3
2	2	-1	$\sin i \sin^2 i/2$	-3/4	$\sin i' \cos^2 i'/2$	$-\frac{(1 + \cos i - 2 \cos^2 i)}{2 \sin i}$		-1
2	0	-1		3/8	$\sin 2i'$			1
2	-2	-1		3/4	$\sin i' \sin^2 i'/2$			3

Since terms of intermediate period have been observed in satellite data, these terms are treated in this report. Analytic formulas for the perturbations in the motion of artificial satellites due to these terms are derived.

THE DISTURBING FUNCTION FOR THE LAGRANGE PLANETARY EQUATIONS

In order to obtain the differential coefficients needed for the planetary equations, it is convenient to expand Equation (1). Since q takes the two values 0 and 2, as seen from Table 1, we obtain

$$R' = n'^2 m' a^2 \left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \sum_{q',\nu} \{A_{\nu 0} A'_{\nu q'} c_{\nu q' 0} \cos (q'\phi' + \nu \theta)$$

$$+ A_{\nu 2} A'_{\nu q'} c_{\nu q' 2} \cos (2\phi + q' \phi' + \nu \theta)\} \tag{2}$$

For the convenience of the reader, the ephemeris quantities needed for Equation (2) are given in Appendix 1.

The angles ϕ and θ are defined by

$$\phi = \ell + g$$

$$\theta = h - h'$$

where

ℓ = mean anomaly of the satellite

g = argument of perigee of the satellite

h = longitude of the ascending node of the satellite

h′ = longitude of the ascending node of the moon

Some useful geometric relations, which may be derived from Figure 1, are

$$\cos i' = \cos \epsilon \cos J - \sin \epsilon \sin J \sin N$$

$$\sin h' \sin i' = \sin J \sin N$$

$$
\begin{aligned}
\sin i' \cos h' &= \sin \epsilon \cos J + \cos \epsilon \sin J \cos N \\
\sin i' \sin [g' - (\Gamma' - N)] &= \sin \epsilon \sin N \\
\sin i' \sin J \cos [g' - (\Gamma' - N)] &= \cos \epsilon - \cos J \cos i' \\
\ell' &= \mathbf{C} - \Gamma' \\
\phi' &= \ell' + g' \\
\lambda' &= \phi' + h'
\end{aligned}
\tag{3}
$$

Equation (2) is a form of the disturbing function useful in obtaining the Lagrange planetary equations.

FURTHER EXPANSIONS OF THE DISTURBING FUNCTION

Equation (2) while useful for forming the planetary equations is further expanded. Various characteristics of the disturbing function are then displayed preparing the way for analytic treatment. First, we give q' the values -2, 0, and and 2 as indicated in Table 1. Then by means of Cayley's tables, Reference 7, the disturbing function R' of Equation (2) is expanded in terms of ℓ', the mean anomaly of the disturbing body to obtain

$$
\begin{aligned}
R' = n'^2 m' a^2 \left(\frac{r}{a}\right)^2 \sum \{ & A_{\nu 0} A'_{\nu 0} c_{\nu 00} B'_{00} \cos \nu\theta \\
& + A_{\nu 0} A'_{\nu 0} c_{\nu 00} B'_{i'0} \cos (i'\ell' + \nu\theta) + A_{\nu 0} A'_{\nu 2} c_{\nu 20} B'_{i'2} \cos (i'\ell' + 2g' + \nu\theta) \\
& + A_{\nu 2} A'_{\nu 0} c_{\nu 02} B'_{00} \cos (2\phi + \nu\theta) + A_{\nu 2} A'_{\nu 0} c_{\nu 02} B'_{i'0} \cos (i'\ell' + 2\phi + \nu\theta) \\
& + A_{\nu 2} A'_{\nu -2} c_{\nu -22} B'_{i'-2} \cos (i'\ell' - 2g' + 2\phi + \nu\theta) \\
& + A_{\nu 2} A'_{\nu 2} c_{\nu 22} B'_{i'2} \cos (i'\ell' + 2g' + 2\phi + \nu\theta) \}
\end{aligned}
\tag{4}
$$

The coefficients $B'_{i'q'}$ arise from the expansion by means of Cayley's tables, Reference 7, and are listed in Table 2.

TABLE 2

$B'_{i',0}$	i'	$B'_{i',-2}$	i'	$B'_{i',2}$	i'
$53/16\ e'^3$	-3	$845/48\ e'^3$	-5	$1/48\ e'^3$	-1
$9/4\ e'^2$	-2	$17/2\ e'^2$	-4	0	0
$3/2\ e' + 27/16\ e'^3$	-1	$7/2\ e' - 123/16\ e'^3$	-3	$-1/2\ e' + 1/16\ e'^3$	1
$(1-e'^2)^{-3/2}$	0	$1 - 5/2\ e'^2$	-2	$1 - 5/2\ e'^2$	2
$3/2\ e' + 27/16\ e'^3$	1	$-1/2\ e' + 1/16\ e'^3$	-1	$7/2\ e' - 123/16\ e'^3$	3
$9/4\ e'^2$	2	0	0	$17/2\ e'^2$	4
$53/16\ e'^3$	3	$1/48\ e'^3$	1	$845/48\ e'^3$	5

Of particular interest is the quantity $\langle R' \rangle$, representing the mean value of R'. The definition of $\langle R' \rangle$ is given by

$$\langle R' \rangle = \frac{1}{2\pi} \int_0^{2\pi} R'\, d\ell \tag{5}$$

Since

$$\left\langle \left(\frac{r}{a}\right)^2 \right\rangle = 1 + \frac{3}{2} e^2$$

$$\left\langle \left(\frac{r}{a}\right)^2 \cos\,(2f + \beta) \right\rangle = \frac{5}{2} e^2 \cos \beta$$

where β is an arbitrary angle, $\langle R' \rangle$ is readily obtained.

We find

$$\langle R' \rangle = \sum R'_{SEC} + \sum R'_{LP} + \sum R'_{INT}. \tag{6}$$

where R_{SEC} represents the secular R'_{LP} the long period and R'_{INT} the intermediate parts of R'.

The components of Equation (6) are,

$$\sum R'_{SEC} = n'^2 m' a^2 \left(1 + \frac{3}{2} e^2\right) \sum A_{00} A'_{00} c_{000} B'_{00} \tag{7}$$

$$\begin{aligned} \sum R'_{LP} = {} & n'^2 m' a^2 \left(1 + \frac{3}{2} e^2\right) \sum A_{\nu 0} A'_{\nu 0} c_{\nu 00} B'_{00} \cos (\nu h - \nu h') \\ & + n'^2 m' a^2 \frac{5}{2} e^2 \sum A_{\nu 2} A'_{\nu 0} c_{\nu 02} B'_{00} \cos (2g + \nu h - \nu h') \end{aligned} \tag{8}$$

$$\begin{aligned} \sum R'_{INT} = {} & n'^2 m' a^2 \left(1 + \frac{3}{2} e^2\right) \sum \{A_{\nu 0} A'_{\nu 0} c_{\nu 00} B'_{i'0} \cos (\nu h + i'\ell' - \nu h') \\ & + A_{\nu 0} A'_{\nu 2} c_{\nu 20} B'_{i'0} \cos [\nu h + 2\lambda' + (i' - 2)\ell' - (\nu + 2) h']\} + \\ & \frac{5}{2} n'^2 m' a^2 e^2 \sum \{A_{\nu 2} A'_{\nu 0} c_{\nu 02} B'_{i'0} \cos (2g + \nu h + i'\ell' - \nu h') + \\ & A_{\nu 2} A'_{\nu -2} c_{\nu -22} B'_{i'-2} \cos [2g + \nu h - 2\lambda' + (i' + 2)\ell' - (\nu - 2) h'] \\ & + A_{\nu 2} A'_{\nu 2} c_{\nu 22} B'_{i'2} \cos [2g + \nu h + 2\lambda' + (i' - 2)\ell' - (\nu + 2) h']\} \end{aligned} \tag{9}$$

Equations (7), (8), and (9) can be used directly to find analytic perturbations due to the sun.

However, as shown in Reference 1, the angle i' appearing in the functions $A'_{\nu q'}$, should be eliminated when obtaining perturbations due to the moon analytically. This is done in combination with angle h' by using the first three of Equations (3). The quantities $A'_{\nu q'}$ are replaced by quantities $A''_{\nu q' p}$ which are functions of the obliquity ϵ and the inclination of the lunar plane to the ecliptic J. The arguments of the trigonometric terms will now contain N, the longitude of the lunar node along the ecliptic, replacing h' the longitude of the node along the earth's equator.

To derive the functions $A''_{\nu q' p}$ we set

$$A'_{\nu q'} \cos n h' = C_{\nu q' 0} + C_{\nu q 1'} \sin J \cos N + C_{\nu q' 2} \sin^2 J \cos 2N$$

$$A'_{\nu q'} \sin nh' = S_{\nu q' 1} \sin J \sin N + S_{\nu q' 2} \sin^2 J \sin 2N \quad (10)$$

From trigonometry we find

$$A'_{\nu q'} \cos (\beta + n h') = \sum_{p=-2}^{2} (C'_{\nu q' p} + s' S'_{\nu q' p}) (\sin J)^{|p|} \cos (\beta + p N) \quad (11)$$

In Equation (11), β is an arbitrary angle, and

$$n = -(q + \nu).$$

The index p takes on the values -2, -1, 0, 1, 2. We also have the following relations

$$C'_{\nu q' p} = C'_{\nu q' -p}$$

$$S'_{\nu q' p} = S'_{\nu q' -p}$$

for $p = 0$, $\quad c'_{\nu q' 0} = c_{\nu q' 0}, \; S'_{\nu q' 0} = S_{\nu q' 0} = 0.$

for $p \neq 0$

$$C'_{\nu q' p} = \frac{1}{2} C_{\nu q' p}$$

$$S'_{\nu q' p} = \frac{1}{2} S_{\nu q' p}$$

also

$$s' = 1 \quad \text{for } np > 0$$

$$s' = 0 \quad \text{for } np = 0$$

$$s' = -1 \quad \text{for } np < 0$$

We therefore find

$$A''_{\nu q' p} = C'_{\nu q' p} + s' \, S'_{\nu q' p} ,$$

$C_{\nu q' p}$ and $S_{\nu q' p}$ are given in Table 3.

The quantities d_0, d_1 and d_2 arise from the expansion of $\cos^{-2} i'/2$ to order sin J. We have

$$\frac{1}{4} \cos^{-2} \frac{i'}{2} = d_0 + d_1 \sin J \cos N + d_2 \sin^2 J \cos 2N$$

where

$$d_0 = \frac{1}{2(1 + \cos \epsilon \cos J)} \left[1 + \frac{1}{2} \frac{\sin^2 \epsilon \sin^2 J}{(1 + \cos \epsilon \cos J)^2} \right]$$

$$d_1 = \frac{\sin \epsilon}{2(1 + \cos \epsilon \cos J)^2}$$

$$d_2 = \frac{\sin^3 \epsilon}{4(1 + \cos \epsilon \cos J)^3}$$

By substituting Equation (11) into Equations (7), (8), and (9) we obtain

$$\sum R'_{SEC} = n'^2 m' a^2 \left(1 + \frac{3}{2} e^2\right) \sum_{p \neq 0}^{2} A_{00} A''_{00p} \, c_{000} B'_{00} (\sin J)^{|p|} \cos p N \quad (12)$$

$$\sum R'_{LP} = n'^2 m' a^2 \left(1 + \frac{3}{2} e^2\right) \sum A_{\nu 0} A''_{\nu 0 p} \, c_{\nu 00} B'_{00} (\sin J)^{|p|} \cos (\nu h + p N)$$

$$+ \frac{5}{2} n'^2 m' a^2 e^2 \sum A_{\nu 2} A''_{\nu 0 p} \, c_{\nu 02} B'_{00} (\sin J)^{|p|} \cos (2g + \nu h + p N) \quad (13)$$

TABLE 3*

	$1 - 3/2 \sin^2 i'$		
$C_{\nu q' 0}$	$(1 - 3/2 \sin^2 \epsilon)(1 - 3/2 \sin^2 J)$		
$C_{\nu q' 1}$	$-3/2 \sin 2\epsilon \cos J$		
$C_{\nu q' 2}$	$3/4 \sin^2 \epsilon$		
	$\sin^2 i' \cos 2h'$		$\sin^2 i' \sin 2h'$
$C_{\nu q' 0}$	$\sin^2 \epsilon (1 - 3/2 \sin^2 J)$		
$C_{\nu q' 1}$	$\sin 2\epsilon \cos J$	$S_{\nu q' 1}$	$2 \sin \epsilon \cos J$
$C_{\nu q' 2}$	$1/2 (1 + \cos^2 \epsilon)$	$S_{\nu q' 2}$	$\cos \epsilon$
	$\sin 2i' \cos h'$		$\sin 2i' \sin h'$
$C_{\nu q' 0}$	$\sin 2\epsilon (1 - 3/2 \sin^2 J)$		
$C_{\nu q' 1}$	$2 \cos 2\epsilon \cos J$	$S_{\nu q' 1}$	$2 \cos \epsilon \cos J$
$C_{\nu q' 2}$	$- 1/2 \sin 2\epsilon$	$S_{\nu q' 2}$	$- \sin \epsilon$
	$\sin i' \cos^2 i'/2 \cos h'$		$\sin i' \cos^2 i'/2 \sin h'$
$C_{\nu q' 0}$	$1/2 \sin \epsilon \, [\cos J + \cos \epsilon \, (1 - 3/2 \sin^2 J)]$		
$C_{\nu q' 1}$	$1/2 (\cos \epsilon + \cos 2\epsilon \cos J)$	$S_{\nu q' 1}$	$1/2 (1 + \cos \epsilon \cos J)$
$C_{\nu q' 2}$	$- 1/8 \sin 2\epsilon$	$S_{\nu q' 2}$	$- 1/4 \sin \epsilon$
	$\sin i' \sin^2 i'/2 \cos 3h'$		$\sin i' \sin^2 i'/2 \sin 3h'$
$C_{\nu q' 0}$	$\sin^3 \epsilon \cos J (1 - 5/2 \sin^2 J) d_0 + 3/2 \sin^2 \epsilon \cos \epsilon \sin^2 J d_1$		
$C_{\nu q' 1}$	$\sin^3 \epsilon \cos J d_1 + \sin^2 \epsilon \cos \epsilon \, (1 + 2 \cos^2 J) d_0$	$S_{\nu q' 1}$	$3 \sin^2 \epsilon \, d_0$
$C_{\nu q' 2}$	$3/2 \sin \epsilon \cos J (1 + \cos^2 \epsilon) d_0$	$S_{\nu q' 2}$	$3/2 \sin 2\epsilon \cos J d_0 + 3/2 \sin^2 \epsilon \, d_1$
C	$+ \sin^2 \epsilon \cos \epsilon \, (1/2 + \cos^2 J) d_1 + \sin^3 \epsilon \cos J d_2$		
	$\cos^4 i'/2$		
$C_{\nu q' 0}$	$1/4 (1 + \cos \epsilon \cos J)^2 + 1/8 \sin^2 \epsilon \sin^2 J$		
$C_{\nu q' 1}$	$- 1/2 \sin \epsilon \, (1 + \cos \epsilon \cos J)$		
$C_{\nu q' 2}$	$1/8 \sin^2 \epsilon$		
	$\sin^4 i'/2 \cos 4h'$		$\sin^4 i'/2 \sin 4h'$
$C_{\nu q' 0}$	$1/4 (1 - \cos \epsilon \cos J)^2 + 1/8 \sin^2 \epsilon \, (1 - 32 \cos^2 J d_0^2) \sin^2 J$		
$C_{\nu q' 1}$	$1/2 \sin \epsilon \, (1 - \cos \epsilon \cos J)$	$S_{\nu q' 1}$	$\sin^3 \epsilon \cos J d_0^2$
$C_{\nu q' 2}$	$1/8 \sin^2 \epsilon \, (1 + 32 \cos^2 J A_0^2)$	$S_{\nu q' 2}$	$\sin^3 \epsilon \cos J d_0 d_1 + 1/2 \sin^2 \epsilon \cos \epsilon \, (1 + 2 \cos^2 J) d_0^2$

*The subscripts ν and q' are chosen to conform with Table 1.

$$
\begin{aligned}
\sum R'_{INT} = {} & n'^2 m' a^2 \left(1 + \frac{3}{2} e^2\right) \sum \{ A_{\nu 0} A''_{\nu 0 p} \; c_{\nu 00} B'_{i' 0} (\sin J)^{|p|} \cos(\nu h + i' \ell' + p N) \\
& + A_{\nu 0} A''_{\nu 2 p} \; c_{\nu 20} B'_{i' 2} (\sin J)^{|p|} \cos[\nu h + 2\lambda' + (i' - 2)\ell' + p N] \} + \\
& \frac{5}{2} n'^2 m' a^2 e^2 \sum \{ A_{\nu 2} A''_{\nu 0 p} \; c_{\nu 02} B'_{i' 0} (\sin J)^{|p|} \cos(2g + \nu h + i \; \ell' + p N) + \\
& A_{\nu 2} A''_{\nu - 2 p} \; c_{\nu - 22} B'_{i - 2} (\sin J)^{|p|} \cos[2g + \nu h - 2\lambda' + (i' + 2)\ell' + p N] \\
& + A_{\nu 2} A''_{\nu 2 p} \; c_{\nu 22} B'_{i' 2} (\sin J)^{|p|} \cos[2g + \nu h + 2\lambda' + (i' - 2)\ell' + p N] \}
\end{aligned} \tag{14}
$$

Equations 12, 13, and 14 represent the division of the lunar disturbing function into secular, long period, and intermediate portions. It is interesting to note that terms with the period of N of 18.6 years, arise from Equation (12) for p equal to 1 and 2. In Equations (14), i' does not take the value zero.

Equation (14) is used to find the analytic perturbations in the elements due to terms of 14 day periods or multiples of this period.

THE LAGRANGE PLANETARY EQUATIONS

The Lagrange planetary equations in canonical form are given by

$$
\frac{dL}{dt} = \frac{\partial R}{\partial \ell}, \quad \frac{dG}{dt} = \frac{\partial R}{\partial g}, \quad \frac{dH}{dt} = \frac{\partial R}{\partial h} \tag{15}
$$

$$
\frac{d\ell}{dt} = -\frac{\partial R}{\partial L}, \quad \frac{dg}{dt} = -\frac{\partial R}{\partial G}, \quad \frac{dh}{dt} = -\frac{\partial R}{\partial H} \tag{16}
$$

The parameters a, e, and i are the semi-major axis, the eccentricity of the satellite and the inclination of the satellite orbit to the earth's equator, respectively. The Delaunay variables appearing in Equations (15) and (16) are defined by

$$
L = \sqrt{\mu a}, \quad G = L\sqrt{1 - e^2}, \quad H = G \cos i \tag{17}
$$

ℓ = mean anomaly

g = argument of the pericenter

h = longitude of the ascending node

We now substitute Equation (2) into Equations (15) and (16) to obtain the planetary equations.

From Equation (2) let us introduce the abbreviations

$$A_1 = A_{\nu 0}\, A'_{\nu q'}\, c_{\nu q' 0}$$

$$A_2 = A_{\nu 2}\, A'_{\nu q'}\, c_{\nu q' 2}$$

$$\alpha_{\nu q'} = q'\,\phi' + \nu\,\theta$$

We also have from Reference 8 that

$$\frac{\partial \left(\frac{r}{a}\right)^2}{\partial \ell} = 2\,\frac{r}{a}\, e\, \frac{\sin f}{\sqrt{1 - e^2}}$$

$$\frac{\partial \phi}{\partial \ell} = \frac{a^2}{r^2}\sqrt{1 - e^2}$$

Consequently, we find

$$\frac{dL}{dt} = 2 n'^2 m' a^2 \left(\frac{a'}{r'}\right)^3 \left\{ -\frac{e}{\sqrt{1-e^2}} \left(\frac{r}{a}\right)\left[\sum A_1 \cos \alpha_{\nu q'} + \sum A_2 \cos(2\phi + \alpha_{\nu q'})\right] \sin f \right.$$

$$\left. - \sqrt{1 - e^2} \sum A_2 \sin(2\phi + \alpha_{\nu q'}) \right\} \tag{18}$$

$$\frac{dG}{dt} = -2 n'^2 m' a^2 \left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \sum A_2 \sin(2\phi + \alpha_{\nu q'}) \tag{19}$$

$$\frac{dH}{dt} = -n'^2 m' a^2 \left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \sum \{\nu A_1 \sin \alpha\nu q' + \nu A_2 \sin (2\phi + \alpha_{\nu q'})\} \tag{20}$$

Noting that the disturbing function is written in terms of the Keplerian elements a, e, and i it is convenient to write the planetary equations (16) in the form

$$\frac{d\ell}{dt} = -\frac{2}{na}\left(\frac{\partial R}{\partial a}\right) - \frac{1-e^2}{na^2e}\left(\frac{\partial R}{\partial e}\right) \tag{21}$$

$$\frac{dg}{dt} = \frac{\cos i}{na^2 e\sqrt{1-e^2}}\left[\frac{\partial R}{\partial(\cos i)}\right] + \frac{\sqrt{1-e^2}}{na^2 e}\left(\frac{\partial R}{\partial e}\right) \tag{22}$$

$$\frac{dh}{dt} = -\frac{1}{na^2 e\sqrt{1-e^2}}\left[\frac{\partial R}{\partial(\cos i)}\right] \tag{23}$$

Equations (18), (19), and (20) have been derived from Equations (16) by applying the relation

$$n^2 a^3 = \mu$$

$$\frac{da}{dL} = \frac{2}{na}$$

$$\frac{\partial e}{\partial L} = \frac{1-e^2}{na^2e}$$

$$\frac{\partial e}{\partial G} = -\frac{\sqrt{1-e^2}}{na^2e} \tag{24}$$

We now evaluate the differential coefficients appearing in Equations (18), (19), and (20). From Reference (9) we find

$$\frac{\partial\left(\frac{r}{a}\right)}{\partial e} = -\cos f \qquad (25)$$

$$\frac{\partial f}{\partial e} = \left(\frac{a}{r} + \frac{1}{1 - e^2}\right) \sin f .$$

If we make the substitutions

$$B_1 = n'^2\, m' \left(\frac{a'}{r'}\right)^3 A'_{\nu q'}\, c_{\nu q' 0} \cos\left(q'\, \phi' + \nu\, \theta\right)$$

$$B_2 = n'^2\, m' \left(\frac{a'}{r'}\right)^3 A'_{\nu q'}\, c_{\nu q' 2}$$

$$\alpha_{\nu q'} = q'\, \phi' + \nu\, \theta$$

we find

$$\frac{\partial R'}{\partial a} = 2a \left(\frac{r^2}{a^2}\right) \sum \left\{A_{\nu 0}\, B_1 + A_{\nu 2}\, B_2 \cos\left(2\phi + \alpha_{\nu q'}\right)\right\} \qquad (26)$$

$$\frac{\partial R'}{\partial e} = -2a^2 \left(\frac{r}{a}\right) \cos f \sum A_{\nu 0}\, B_1 + A_{\nu 2}\, B_2 \cos\left(2\phi + \alpha_{\nu q'}\right)$$

$$-2a^2 \left(\frac{r}{a}\right)^2 \left(\frac{a}{r} + \frac{1}{1 - e^2}\right) \sin f \sum A_{\nu 2}\, B_2 \sin\left(2\phi + \alpha_{\nu q'}\right) \qquad (27)$$

$$\frac{\partial R'}{\partial (\cos i)} = a^2 \left(\frac{r}{a}\right)^2 \sum \left\{\frac{\partial A_{\nu 0}}{\partial (\cos i)}\, B_1 + \frac{\partial A_{\nu 2}}{\partial (\cos i)}\, B_2 \cos\left(2\phi + \alpha_{\nu q'}\right)\right\} . \qquad (28)$$

The coefficients $\partial A_{\nu q} / \partial (\cos i)$ are given in Table 1.

ANALYTIC PERTURBATIONS OF INTERMEDIATE PERIODS

The method of canonical variables with the aid of a determining function, References 9 and 10, is used for computing perturbations of intermediate periods.

We first write the lunar function, R'_{INT}, Equation (14) in the condensed form

$$R'_{INT} = n'^2 m' a^2 A_{\nu q} A''_{\nu q' p} B_{0q} B'_{i' q'} c_{\nu q' q} (\sin J)^{|p|} \cos \alpha \tag{29}$$

where

$$\alpha = q g + \nu h + \nu \lambda' + (i' - \nu) \ell' + p N$$

The complementary function $\bar{R}'_{INT}$ is formed by simply replacing the cosine terms by sine terms, to obtain

$$\bar{R}'_{INT} = n'^2 m' a^2 A_{\nu q} A''_{\nu q' p} B_{0q} B'_{i' q'} c_{\nu q' q} (\sin J)^{|p|} \sin \alpha \tag{30}$$

The lunar determining function S_{INT} is defined by

$$S'_{INT} = \sum \frac{\bar{R}'_{INT}}{\dot{\alpha}} \tag{31}$$

where

$$\dot{\alpha} = q \dot{g} + \nu \dot{h} + i' n' + p \dot{N}$$

The functions appearing in Equation (31) are given in Tables 1, 2, and 3.

To obtain the solar determining function, we set p equal to zero in Equation (27), put $A''_{\nu q' 0}$ equal to $A'_{\nu q'}$, and make the substitution indicated below Equation (1).

From Reference 6, we have

$$\dot{g} = \frac{3 n J_2 \; a_e^2 \; (-1 + 5 \cos^2 i)}{4 a^2 (1 - e^2)^2} \tag{32}$$

$$\dot{h} = \frac{-3 n J_2 \; a_e^2 \; \cos i}{2 a^2 (1 - e^2)^2}$$

where J_2 is the second order zonal harmonic of the earth's gravitational field, and a_e is the mean earth radius.

By the theory of canonical variables we have

$$\delta L = \frac{\partial S}{\partial \ell} \qquad \delta \ell = -\frac{\partial S}{\partial L}$$

$$\delta G = \frac{\partial S}{\partial g} \qquad \delta g = -\frac{\partial S}{\partial G}$$

$$\delta H = \frac{\partial S}{\partial h} \qquad \delta h = -\frac{\partial S}{\partial H} \tag{33}$$

By substituting the expression for S given by Equation (31) into Equation (33), we find that the perturbations in the elements L, G, and H are

$$\delta L = 0$$

$$\delta G = \sum \frac{q\, R'_{INT}}{\dot{\alpha}} \tag{34}$$

$$\delta H = \sum \frac{\nu\, R'_{INT}}{\dot{\alpha}}$$

The Delaunay elements L, G, and H are related to the Keplerian elements a, e, and i by Equation (17).

To find the perturbations in the angular elements ℓ, g, and h, we let

$$D = n'^2\, m'\, A''_{\nu q' p}\, B'_{i' q'}\, c_{\nu q' q}\, (\sin J)^{|p|} \sin \alpha . \tag{35}$$

Then

$$\delta \ell = -\frac{\partial S}{\partial L} = -\sum \left\{ \frac{\frac{1}{n}\left(4 B_{0q} + L \frac{\partial B_{0q}}{\partial L}\right)}{\dot{\alpha}} A_{\nu q} D - \frac{a^2 B_{0q} A_{\nu q} D}{\dot{\alpha}^2} \left(\frac{\partial \dot{\alpha}}{\partial L}\right) \right\} \quad (36)$$

where

$$L \frac{\partial B_{0\upsilon}}{\partial L} = 3(1 - e^2)$$

$$L \frac{\partial B_{02}}{\partial L} = 5(1 - e^2)$$

$$\frac{\partial \dot{\alpha}}{\partial L} = q \frac{\partial \dot{g}}{\partial L} + \nu \frac{\partial \dot{h}}{\partial L}$$

By Reference 6

$$\frac{\partial \dot{g}}{\partial L} = \frac{9 J_2 a_e^2 (1 - 5 \cos^2 i)}{4 a^6 (1 - e^2)^2}$$

$$\frac{\partial \dot{h}}{\partial L} = -\frac{9 J_2 a_e^2 \cos i}{2 a^6 (1 - e^2)^2}$$

Similarly the perturbation in the agreement of perigee is

$$\delta g = -\frac{\partial S}{\partial G} = -\sum \left\{ \frac{\left[-\frac{\partial A_{\nu q}}{\partial (\cos i)} B_{0q} \cos i + A_{\nu q} G \frac{\partial B_{0q}}{\partial G}\right]}{(n\sqrt{1 - e^2})\,\dot{\alpha}} - \frac{B_{0q} A_{\nu q} G \frac{\partial \dot{\alpha}}{\partial G}}{(n\sqrt{1 - e^2})\,\dot{\alpha}^2} \right\} D$$

where

$$G \frac{\partial B_{00}}{\partial G} = -3 (1 - e^2)$$

$$G \frac{\partial B_{02}}{\partial G} = -5 (1 - e^2)$$

$$G \frac{\partial \dot{g}}{\partial G} = \frac{3 J_2 a_e^2 n (2 - 15 \cos^2 i)}{2 a^2 (1 - e^2)^2}$$

$$G \frac{\partial \dot{h}}{\partial G} = \frac{15 J_2 a_e^2 n \cos i}{2 a^2 (1 - e^2)^2}$$

Finally the perturbation in the node of the satellite is

$$\delta h = - \frac{\partial S}{\partial H} = - \sum \left\{ \frac{\dfrac{\partial A_{\nu q}}{\partial \cos i}}{(n \sqrt{1 - e^2}) \dot{\alpha}} - \frac{G \dfrac{\partial \dot{\alpha}}{\partial H} A_{\nu q}}{n \sqrt{1 - e^2} \dot{\alpha}^2} \right\} B_{0q} D \tag{38}$$

where

$$G \frac{\partial \dot{g}}{\partial H} = \frac{15 J_2 a_e^2 n \cos i}{2 a^2 (1 - e^2)^2}$$

$$G \frac{\partial \dot{h}}{\partial H} = \frac{-3 J_2 a_e^2 n}{2 a^2 (1 - e^2)^2}$$

CONCLUSION

In this report formulas have been derived for calculating the perturbations on the orbits of artificial satellites due to the gravitational influences of the sun and the moon. Formulas 18-23 are the Lagrange planetary equations for the Delaunay elements which are required for predicting orbits of arbitrary eccentricity.

Equations 34-38 have been derived in order to be able to calculate analytically solar and lunar perturbations to the orbits of satellites to within an estimated 0.5 meter. These equations take into account the perturbations of approximately 180 day periods due to the sun and the corresponding 14 day periods due to the moon.

REFERENCES

1. Kozai, Y., "Lunisolar Perturbations With Short Periods," SAO Special Report 235, Smithsonian Institution Astrophysical Observatory, Dec. 20, 1966.

2. Kaula, Wm., M., "Theory of Satellite Geodesy," Blaisdell Publishing Co., 1966, Chap. 3.

3. Fisher, D. and Murphy, J. P., "Short Period Lunar and Solar Perturbations for Artificial Satellites," Goddard Space Flight Center – X-547-66-493, Oct. 1966.

4. Murphy, J. P., and Felsentreger, T. L., "Analysis of Lunar and Solar Effects on the Motion of Close Earth Satellites," National Aeronautics and Space Administration, NASA TN-D-3559, Aug. 1966.

5. Kozai, Y., "On the Effects of the Sun and the Moon Upon the Motions of a Close Earth Satellite," SAO Special Report 22, Smithsonian Institution Astrophysical Observatory, March 20, 1959.

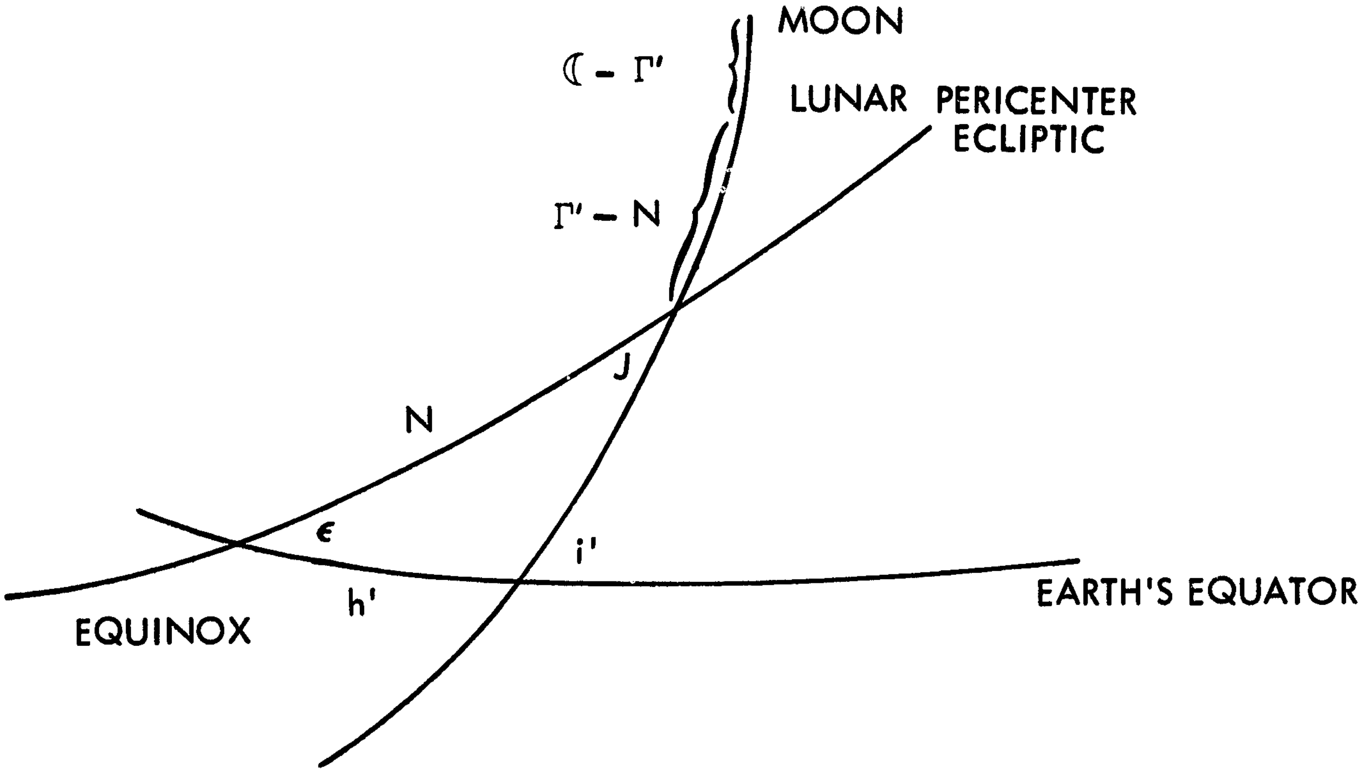
6. Fisher, D., "Formulas for Long Period Motion of Artificial Satellites," Goddard Space Flight Center – X-550-69-36, Feb, 1969.

7. Cayley, A., "Tables of the Development of Functions in the Theory of Elliptic Motion," Memoires Royal Astronomical Society, 29-1861, pp. 191-306.

8. Smart, Wm. M., "Celestial Mechanics," 1953, p. 38.

9. Brouwer, D., "Solutionof the Problem of Artificial Satellite Theory Without Drag," Astronomical Journal, Vol. 64, No. 9, Nov. 1959, p. 379.

1 . Brown, E. W. and Shook, C. A., "Planetary Theory," Dover Edition, 1964, p. 118.



MOON
☾ – Γ′
LUNAR PERICENTER
ECLIPTIC
Γ′ – N
J
N
ε
i′
h′
EQUINOX
EARTH'S EQUATOR

Figure 1.

APPENDIX I

Ephemeris Quantities

For the sun

$$m_\odot = .999997$$

$$e_\odot = 0.01675104 - 0.00004180T$$

$$n_\odot = 0.98560027/\text{day}$$

$$\lambda_\odot = 279^\circ\!.69668 + 36000^\circ\!.76892T + 0.00030T^2$$

$$g_\odot = 281^\circ\!.22083 + 1^\circ\!.71918T + 0^\circ\!.00045T^2$$

$$h_\odot = 0$$

$$i_\odot = \epsilon = 23^\circ\!.452294 - 0^\circ\!.013013T - 0^\circ\!.000002T^2$$

For the moon

$$m' = 0.012150668$$

$$e' = 0.054900489$$

$$☾ = 270^\circ\!.43416 + 481267^\circ\!.88314T - 0^\circ\!.00113T^2$$

$$\Gamma' = 334^\circ\!.32956 + 4069^\circ\!.03403T - 0^\circ\!.01033T^2$$

$$N = 259^\circ\!.18328 - 1934^\circ\!.14201T$$

$$n' = 13^\circ\!.064993/\text{day} = .036291647 \text{ rev/day}$$

$$J = 5^\circ\!.1453964$$

where T = number of days since January 0.5, 1900, divided by 36525.