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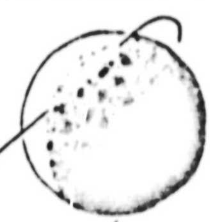

AUTOMATED, CLOSED FORM INTEGRATION OF FORMULAS IN ELLIPTIC MOTION

BY
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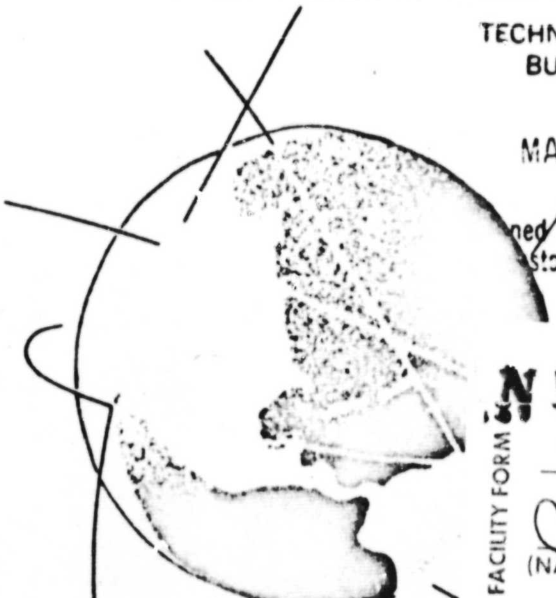

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Automated, Closed Form Integration of
Formulas in Elliptic Motion

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Abstract: In some perturbation theories it is possible to avoid expansion of the perturbations in powers of the eccentricity, obtaining results in closed form by using the true or eccentric anomaly instead of the mean anomaly. This paper describes an algorithm (which has been programmed for the 6600 computer using the formula manipulation system TRIGMAN) for automatically performing the integrals which arise in these theories.

Some perturbation theories (e.g. Brouwer 1959; Hori 1963, Harrington 1969; Aksnes 1969) avoid expansion of the perturbations in powers of the eccentricity by expressing them in terms of the true or eccentric anomaly instead of the mean anomaly. To carry these or other theories to high order, it is necessary to perform the calculations on computers, using a formula manipulation language such as FORMAC (Tobey, et.al. 1967), or one of the special purpose systems developed for use in celestial mechanics, such as that of Barton (1967), Deprit and Rom (1968), Hall and Cherniack (1969) or Jefferys (1970).

In the computation of the short-period perturbations, one has to integrate functions of the form

$$F\ (r,\dot{r},u,f) \tag{1}$$

with respect to the mean anomaly ℓ; where r is the distance to the primary, $\dot{r} = dr/d\ell$, u is the eccentric anomaly and f the true anomaly. For convenience we will also assume that the semimajor axis has been normalized to unity.

In hand computations the integrals are computed case-by-case as they arise, using any available method and some well-known tricks. For certain cases, general formulae are available (Kozai 1962).

On the other hand, when we attempt to automate the calculation, it is necessary to have an algorithm which will invariably work on some well-defined but rather general class of functions. Of course, it is not possible to guarantee that the integrated function will remain in that class of functions, or even that it will itself be integrable in closed form (as will be seen below). Futhermore, it may be necessary to sacrifice elegance and speed in the algorithm in order to obtain reliability and the desired degree of generality.

We will assume that the function to be integrated is a sum of the form

$$F = \Sigma\, P(r,\dot{r}) {\cos \atop \sin}(jf + ku + \varphi) \quad , \tag{2}$$

where P is a polynomial in r and $\dot{r}$, j and k are integers, and φ is independent of the three anomalies. In general, negative powers of $\dot{r}$ cannot occur, owing to the resulting singularity that appears at $\ell = 0, \pi$; but they may occur in Eq. (2), provided that they do not occur in Eq. (6), below. (For example, the algorithm will successfully handle the function $(\sin u)/\dot{r} \equiv r/e$).

Also, the fact that we do not normally have both $j \neq 0$ and $k \neq 0$ in a given term does not affect the algorithm.

The algorithm proceeds as follows:

Step 1. Using standard trigonometric formulas and the identities of elliptic motion

$$\begin{aligned} \sin f &= \dot{r}\eta/e, \\ \cos f &= (\eta^2/r - 1)/e, \\ \sin u &= r\dot{r}/e, \\ \cos u &= (1 - r)/e, \end{aligned} \tag{3}$$

where e is the eccentricity and $\eta = (1 - e^2)^{1/2}$, we may express sin nf, cos nf, sin nu, and cos nu as power series in r and $\dot{r}$ for those multiples of f and u appearing in the series to be integrated.

Step 2. Using the results from Step 1 and the identities

$$\cos(nf + \Psi) = \cos nf \cos \Psi - \sin nf \sin \Psi ,$$
$$\sin(nf + \Psi) = \sin nf \cos \Psi + \cos nf \sin \Psi , \qquad (4)$$

and similar ones in the eccentric anomaly, we express the function to be integrated as a polynomial in r and $\dot{r}$, in which u and f do not appear explicitly.

Step 3. The identity

$$(\dot{r})^2 = -1 + 2/r - \eta^2/r^2 \qquad (5)$$

is used to eliminate all powers of $\dot{r}$ higher than the first. Thus (assuming F does not have any singularities, as mentioned above) we obtain

$$F = A(r) + B(r)\dot{r} , \qquad (6)$$

where A and B are polynomials.

Step 4. The second term of Eq. (6) is separated off and immediately integrated:

$$\int B(r)\dot{r}\, d\ell = \int B(r)dr. \qquad (7)$$

In this integration it is possible for terms in log(r) to appear. Such a term arises, for example, in the third-order Brouwer theory of an artificial satellite, from the term

$$\frac{\partial F_2^*}{\partial g} \cdot \frac{\partial S_1}{\partial G} , \qquad (8)$$

owing to the fact that S_1 contains a term in the equation of the center $f - \ell$, and

$$\frac{\partial f}{\partial e} = \left\{ \frac{1}{r} + \frac{1}{\eta^2} \right\} \sin f$$

$$= \frac{\dot{r}}{\eta e} + \frac{\eta}{e}\frac{\dot{r}}{r} \qquad (9)$$

In addition, the third-order Brouwer theory requires the integration of $f - \ell$ itself, which cannot be obtained in closed form.

Step 5. The first term of Eq. (6) is rewritten

$$A(r) = \frac{1}{r^2} A_1(r) + \frac{1}{r} A_2(r) \quad , \tag{10}$$

where A_1 contains no positive powers of r, and A_2 no negative powers of r.

Step 6. The identities

$$1/r = (1 + e \cos f)/\eta^2,$$

$$r = 1 - e \cos u \tag{11}$$

are substituted into A_1 and A_2, respectively, to obtain

$$A(r) = \frac{1}{r^2} B_1(f) + \frac{1}{r} B_2(u), \tag{12}$$

where B_1 and B_2 are Fourier series in the indicated variables.

Step 7. The terms in B_1 and B_2 of Eq. (12) are integrated with the aid of the formulae

$$d\ell = \frac{r^2}{\eta} df \ , \text{ and}$$

$$d\ell = r \, du \quad , \tag{13}$$

respectively. Thus, we get the final result as

$$\int F \, d\ell = \int B(r) \, dr + \frac{1}{\eta} \int B_1(f) \, df + \int B_2(u) du \tag{14}$$

Because of the requirements of perturbation theory, it is usual to separate out the secular terms in B_1 and B_2 before integration. These are added to the new Hamiltonian, and the result is to produce in the determining function terms of the form

$$(f-\ell)C_1 \text{ and}$$

$$(u-\ell)C_2 \quad , \tag{15}$$

from B_1 and B_2 respectively. The terms from Eq. (15.2) are easily converted into the form of the original function F by using Kepler's equation $\ell = u - e \sin u$; but (as is well known) there is no such simplification for the term in the equation of the center, $f-\ell$. Careful choice of the imtermediary orbit (cf. Aksnes 1969) may postpone to a higher order the introduction of such terms; but it is probable that they will eventually appear in any case.

Our algorithm does not stop here, however. We have found that the resulting series is more complicated than it has to be, owing to the existence of the identity $e^2 + \eta^2 = 1$, which has not been made use of because e and η are carried as independent variables. The problem of simplifying the final result is far from solved, but the following two steps appear to give good results in some cases (an approximate 40% reduction in the length of the series).

Step 8. Suppose that the largest negative power of e in the expression Φ to be simplified is e^{-k}. We multiply Φ by e^k and then make the substitution $e^2 \rightarrow 1 - \eta^2$ so as to obtain an expression at most linear in e. We then multiply by e^{-k} again. (To see the reason for this, consider the effect on the series $\eta^{-19}e^{-1} - \eta^{-19}e - \eta^{-17}e - \eta^{-15}e$).

Step 9. Repeat Step 8, but reverse the roles of e and η. This expresses all positive powers in terms of e instead of η, which appears to be the more compact form.

The algorithm as presented may result in a determining function which differs from the usual one by a function independent of ℓ. This, of course, does not affect the validity of the results, although it does change the meaning of the constants in the theory. It is up to the judgement of the investigator as to what constant of integration in the determining function will produce the most convenient theory.

It should also be noted that this algorithm works just as well when Hill's variables r, f+g ,h; $\dot{r}$,G,H are used instead of the more usual Delaunay variables, since both r and $\dot{r}$ appear in the algorithm. In fact, Aksnes has shown (1969) that the resulting theory may be more compact; and in addition, if the determining function is expressed in terms of r and $\dot{r}$ instead of f or u , the evaluation of the perturbations may be more convenient, because instead of evaluating several trigonometric functions in multiples of an angle, we evaluate shorter polynomials in r and $\dot{r}$.

Finally, Professor Hori has noted in a private conversation that this algorithm may be able to handle certain functions of the form

$$G = \sum_k (f-\ell)^k F_k ,$$

where F is as before, if one integrates by parts and uses $\frac{d}{d\ell}(f-\ell)=(\frac{\eta}{r^2}-1)$, so long as F_k has no secular part, or if its secular part cancels that of another term (as happens in the second order theory of the Artificial Satellite by the Hori-Lie method in Hill's variables, using an elliptic reference orbit). When the integration cannot be obtained in closed form, it may be necessary to apply the identity

$$f - \ell = (f - u) + (u - \ell)$$

$$= 2\sum_{n=1}^{\infty}\left\{\frac{\beta^n}{n} \sin nu\right\} + e \sin u ,$$

where $\beta = \beta(e) = \frac{1}{e}(1 - \eta)$. An alternative is to define new functions such as

$$\int_0^{\ell} (f-\ell)^k d\ell , \text{ as they are required.}$$

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