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EFFECT OF STATIC STRESS AND EDGE RESTRAINT ON THE VIBRATION OF NEARLY CYLINDRICAL SHELLS WITH CIRCULAR CROSS SECTION

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16. Abstract

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SUMMARY

The effect of stress due to statically applied axial compressive and tensile loads and external or internal pressure on the minimum natural frequencies of freely supported nearly cylindrical shells with circular cross section and slight meridional curvature is investigated. Also the effect of various edge restraints on the natural vibration of unstressed nearly cylindrical shells is investigated. Results show that axial tension loads and external hydrostatic pressure can cause sizable reductions in minimum frequencies for positive Gaussian curvature shells. While axial compression loads and internal pressure increase the frequencies for these shells, the compression loading increases frequencies only slightly before onset of static instability. In contrast, negative-curvature shells are highly sensitive to axial compression and external hydrostatic pressure and evidence large reductions in frequencies at low load levels. Both axial tension and internal hydrostatic pressure cause large increases in frequencies in negative-curvature shells, but generally not higher than similarly loaded cylinders.

The minimum frequencies of negative-curvature shells are sensitive to the degree of meridional in-plane restraint at the shell edges, with large increases occurring when the edges are completely restrained, whereas they are relatively insensitive to slope restraint at the edges. The minimum frequencies of positive-curvature shells, on the other hand, are relatively insensitive to both in-plane and slope restraints.

INTRODUCTION

Doubly curved shell elements are commonly used in aerospace vehicle structures. With the exception of the special case of spherical shells, very little general information on the dynamic behavior of doubly curved shells exists in the literature. One class of shells which is useful for characterizing the behavior of doubly curved shells is the nearly cylindrical shell with circular cross section which contains slight meridional curvature. The vibration behavior of such shells has been studied analytically in reference 1, for example, in which unstressed shells with "freely supported" edges are treated. The term freely supported denotes edges which have zero displacement in the direction normal to the shell middle surface but which are free to displace in the direction along the middle surface meridian and free to rotate. It was found in reference 1 that for nearly cylindrical freely supported shells with negative Gaussian curvature, the minimum natural frequencies are generally below those of the corresponding cylinder and evidence wide variations in value as the meridional curvature is varied. At certain critical values of negative curvature, exceptionally large reductions in frequency occur. These large reductions result from a condition of essentially inextensional deformation during the vibration. Positive curvature shells, on the other hand, exhibit higher minimum frequencies than those of the corresponding cylinder.

The purpose of the present paper is to extend the investigation of reference 1 in two ways. First, the effects of several constant biaxial membrane stress fields on the natural vibration of freely supported shells are presented. Second, the effects of various types of edge support as means of preventing inextensional vibration modes are investigated for unstressed shells. As in reference 1, the minimum natural frequency is used as an indicator of the vibration behavior of the doubly curved shells.

The governing equations used for this analysis were developed in reference 1 and are restricted to nearly cylindrical thin shells having shallow meridional curvature. The equations are solved for a general linear elastic edge restraint by methods similar to those reported in references 2 and 3, where trial values of frequency are introduced to force the vanishing of a characteristic determinant. The method of solution is "exact" in the sense that the natural frequencies can be found to any degree of accuracy desired.

SYMBOLS

A_j	displacement coefficients defined by equations (5)
B	extensional stiffness
c	central rise of the shell meridian
D	bending stiffness
E	Young's modulus of elasticity
h_j,g_j	displacement coefficients (eqs. (7))
h	shell thickness
m	number of axial half-waves

$M_\xi, M_\theta, M_{\xi\theta}$	moment resultants associated with vibration state
n	number of circumferential waves
$N_\xi, N_\theta, N_{\xi\theta}$	stress resultants associated with vibration state
$\overline{N}_\xi, \overline{N}_\theta$	static stress resultants
$\overline{p}, \overline{N}$	surface loading, edge loading
r	circumferential radius of doubly curved shell
R	radius of cylinder; circumferential radius at midlength of doubly curved shell
R_ξ	meridional radius of curvature
s	total meridional arc length
t	time
u,v,w	displacement variables in meridional (ξ), circumferential (θ), and normal directions, respectively, defining the vibration state
U,V,W	coefficients in equations (8)
$\alpha, \beta, \gamma, \delta$	coefficients of characteristic roots (eqs. (10))
θ	circumferential coordinate
λ_j	characteristic roots
λ	thickness parameter, h/R
Λ_j	coefficients of equation (20)
μ	Poisson's ratio (taken as $\mu = 0.3$ for all calculations)
ν	mass density

$\bar{\sigma}$ nondimensional meridional stress, $\frac{\overline{N}_\xi}{B} \times 10^4$

ξ meridional coordinate

τ percent ratio of meridional rise to length, $\frac{c}{s}(100) \approx \frac{s}{8R_\xi}(100)$

ω natural circular frequency

Ω frequency parameter, $R\omega\sqrt{\frac{\nu\left(1 - \mu^2\right)}{E}}$

∇^4 biharmonic differential operator $R^4 \frac{\partial^4}{\partial \xi^4} + 2R^2 \frac{\partial^4}{\partial \xi^2 \partial \theta^2} + \frac{\partial^4}{\partial \theta^4}$

Matrices:

$\{B\}$ 8×1 column matrix

$[\Phi], [N], [K], [Y]$ 8×8 square matrices

Subscripts following commas represent derivatives with respect to variables indicated by the subscript.

DESCRIPTION OF ANALYSIS

Governing Equations

The equations governing the motion of symmetrically stressed nearly cylindrical thin elastic shells of revolution with shallow meridional constant curvature of the type shown in figure 1 are given in terms of displacements in reference 1 (with origin of ξ taken at midlength) as

$$u_{,\xi\xi} + \frac{1 - \mu}{2} \frac{u_{,\theta\theta}}{R^2} + \frac{1 + \mu}{2} \frac{v_{,\theta\xi}}{R} + \left(\frac{1}{R_\xi} + \frac{\mu}{R}\right) w_{,\xi} - \frac{\nu h}{B} u_{,tt} = 0 \tag{1a}$$

$$\frac{1 + \mu}{2} \frac{u_{,\xi\theta}}{R} + \frac{1}{R^2}\left(1 + \frac{\lambda^2}{12}\right) v_{,\theta\theta} + \frac{1 - \mu}{2}\left(1 + \frac{\lambda^2}{3}\right) v_{,\xi\xi} + \left(\frac{1}{R} + \frac{\mu}{R_\xi}\right)\frac{w_{,\theta}}{R} - \frac{\lambda^2}{12} \frac{w_{,\theta\theta\theta}}{R^2}$$

$$- \frac{(2 - \mu)\lambda^2}{12} w_{,\xi\xi\theta} - \frac{\nu h}{B} v_{,tt} = 0 \tag{1b}$$

$$\left(\frac{1}{R_\xi} + \frac{\mu}{R}\right)u_{,\xi} + \left(\frac{1}{R} + \frac{\mu}{R_\xi}\right)\frac{v_{,\theta}}{R} - \frac{\lambda^2}{12R^2} v_{,\theta\theta\theta} - \frac{\lambda^2(2 - \mu)}{12} v_{,\xi\xi\theta} + \frac{\lambda^2}{12R^2} \nabla^4 w + \left(\frac{1}{R_\xi^2} + \frac{2\mu}{R_\xi R} + \frac{1}{R^2}\right)w$$

$$- \frac{\overline{N}_\xi}{B} w_{,\xi\xi} - \frac{\overline{N}_\theta}{B} \frac{w_{,\theta\theta}}{R^2} + \frac{\nu h}{B} w_{,tt} = 0 \tag{1c}$$

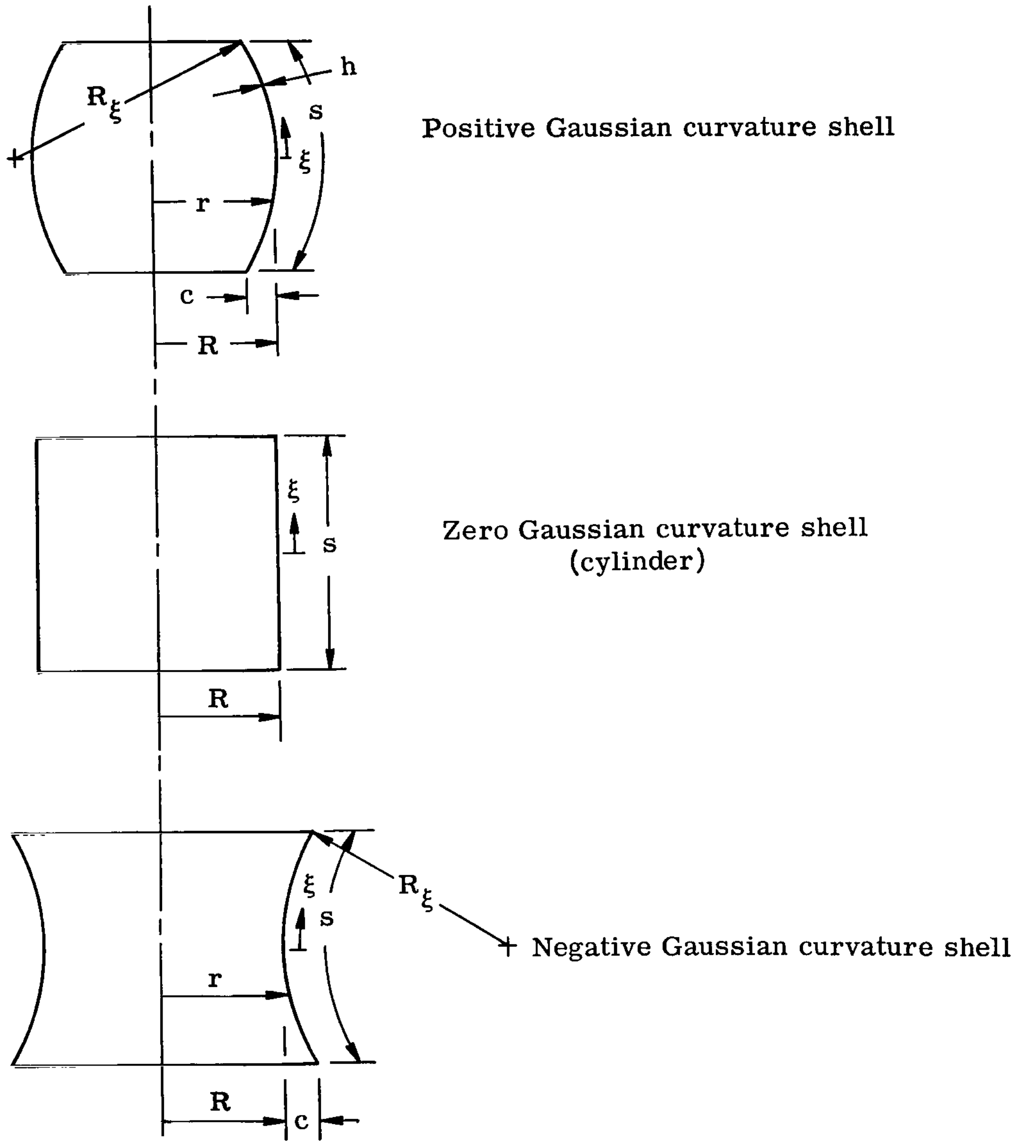

Figure 1.- Geometry of class of shell investigated.

where

$$\lambda = \frac{h}{R}$$

and the circumferential radius is defined by $r = R - \frac{\xi^2}{2R_\xi}$. The quantities u, v, and w are the displacements in the meridional, circumferential, and normal directions of the shell surface, respectively; R is the circumferential radius of the shell at midlength; R_ξ is the constant meridional radius of curvature; h is the shell thickness; and μ is Poisson's ratio. The variable ξ is the measure of length along the meridian with its origin taken at midlength (fig. 1), θ is the angular circumferential measure, and t is time. The quantity ν is the mass density of the material and B is the extensional stiffness of the shell surface given by

$$B = \frac{Eh}{1 - \mu^2} \tag{2}$$

where E is Young's modulus of elasticity. Equations (1) have been derived on the assumption that the shell material is homogeneous, isotropic, and linearly elastic. Transverse shear and rotary inertia terms have been neglected.

Deformations due to initial static loading have been assumed to be uniform, and variation in stress along the meridian due to double curvature is zero in the approximate theory, so that the static stress quantities $\overline{N}_\xi$ and $\overline{N}_\theta$ are constant membrane stress resultants given by

$$\left.\begin{aligned} \overline{N}_\xi &= \overline{N} \\ \overline{N}_\theta &= R\left(\overline{p} - \frac{\overline{N}}{R_\xi}\right) \end{aligned}\right\} \tag{3}$$

where $\overline{N}$ is an applied tensile edge meridional force resultant and $\overline{p}$ is an applied uniform internal constant directional pressure.

The corresponding boundary conditions at $\xi = \pm\frac{s}{2}$ are given by

$$\left.\begin{aligned} & u_{,\xi} + \frac{w}{R_\xi} + \mu\left(\frac{v_{,\theta}}{R} + \frac{w}{R}\right) = 0 && \text{or} \quad u = 0 \\ & \frac{u_{,\theta}}{R} + \left(1 + \frac{\lambda^2}{3}\right)v_{,\xi} - \frac{\lambda^2}{3} w_{,\theta\xi} = 0 && \text{or} \quad v = 0 \\ & \frac{\lambda^2}{12}\left[-(2 - \mu)v_{,\xi\theta} + R^2 w_{,\xi\xi\xi} + (2 - \mu)w_{,\theta\theta\xi}\right] - \frac{\overline{N}_\xi}{B} w_{,\xi} = 0 && \text{or} \quad w = 0 \\ & w_{,\xi\xi} + \frac{\mu}{R^2}\left(w_{,\theta} - v\right)_{,\theta} = 0 && \text{or} \quad w_{,\xi} = 0 \end{aligned}\right\} \tag{4}$$

Stressed Shells With General Boundary Conditions

Equations (1) are constant-coefficient partial-differential equations which are satisfied by a solution of the form

$$\left.\begin{aligned} u &= A_j h_j e^{\frac{\lambda_j \xi}{R}} \sin n\theta \cos \omega t \\ v &= A_j g_j e^{\frac{\lambda_j \xi}{R}} \cos n\theta \cos \omega t \\ w &= A_j e^{\frac{\lambda_j \xi}{R}} \sin n\theta \cos \omega t \end{aligned}\right\} \tag{5}$$

where n is the number of circumferential waves, ω is the natural frequency, and the λ_j's are the characteristic roots of equations (1) and must satisfy the biquartic characteristic equation

$$\lambda_j^{\,8} + a_6\lambda_j^{\,6} + a_4\lambda_j^{\,4} + a_2\lambda_j^{\,2} + a_0 = 0 \tag{6}$$

The coefficients a_0, a_2, a_4, and a_6 are given in appendix A and are functions of μ, n, R, R_ξ, and Ω.

The amplitude coefficients h_j and g_j of equations (5) may be found from any two of equations (1). From equations (1b) and (1c), they become

$$\left.\begin{aligned} h_j &= \frac{-\left(f_6\lambda_j^{\,2} + f_7\right)^2 + \left(f_8\lambda_j^{\,4} + f_9\lambda_j^{\,2} + f_{10}\right)\left(f_4\lambda_j^{\,2} + f_5\right)}{\lambda_j\left[f_2\left(f_6\lambda_j^{\,2} + f_7\right) - f_3\left(f_4\lambda_j^{\,2} + f_5\right)\right]} \\ g_j &= \frac{-f_2\left(f_8\lambda_j^{\,4} + f_9\lambda_j^{\,2} + f_{10}\right) + f_3\left(f_6\lambda_j^{\,2} + f_7\right)}{f_2\left(f_6\lambda_j^{\,2} + f_7\right) - f_3\left(f_4\lambda_j^{\,2} + f_5\right)} \end{aligned}\right\} \tag{7}$$

where the functions $f_1, \ldots, f_{10}$ are given in appendix A.

For the special case of axisymmetric vibrations ($n = 0$), the terms g_j vanish and equation (1b), the circumferential equilibrium equation, uncouples from the remaining equations (1) so that the procedure must be modified. The axisymmetric case is handled separately in appendix B.

The complete solution for u, v, and w is found by summing the contributions from each of the eight characteristic roots of equation (6) so that

$$\left.\begin{aligned} u &= U(\xi)\sin n\theta \cos \omega t \\ v &= V(\xi)\cos n\theta \cos \omega t \\ w &= W(\xi)\sin n\theta \cos \omega t \end{aligned}\right\} \tag{8}$$

where

$$\left.\begin{aligned} U(\xi) &= \sum_{j=1}^{8} A_j h_j e^{\lambda_j \xi} \\ V(\xi) &= \sum_{j=1}^{8} A_j g_j e^{\lambda_j \xi} \\ W(\xi) &= \sum_{j=1}^{8} A_j e^{\lambda_j \xi} \end{aligned}\right\} \tag{9}$$

For the problems treated in this analysis, the natural frequencies of interest are always associated with characteristic roots of equation (6) of the following form:

Case (1):

$$\lambda_j = \pm(\alpha \pm i\beta),\ \pm\gamma,\ \pm i\delta \tag{10a}$$

Case (2):

$$\lambda_j = \pm(\alpha \pm i\beta),\ \pm i\gamma,\ \pm i\delta \tag{10b}$$

where α, β, γ, and δ are real and in general distinct. The natural frequencies of a cylinder were generally found to occur for characteristic roots of the form of case (1) in reference 2. In the present analysis, case (2) can also occur.

The displacement coefficients given in equations (9) can be written after some manipulation in terms of real functions as

$$
\left.
\begin{aligned}
U(\xi) &= \sum_{j=1}^{8} B_j U_j(\xi) \\
V(\xi) &= \sum_{j=1}^{8} B_j V_j(\xi) \\
W(\xi) &= \sum_{j=1}^{8} B_j W_j(\xi)
\end{aligned}
\right\} \tag{11}
$$

where U_j, V_j, and W_j are real functions of ξ given in appendix A. The amplitude coefficients B_j are undetermined real constants which take the role of the eigenvector elements in the vibration solution.

With the origin of ξ taken at the midpoint of the shell, the boundary conditions of equations (4) may be written in general at two edges $\xi = \frac{s}{2}$ and $\xi = -\frac{s}{2}$ as

$$
\Big[[\Phi][N] + [K][Y]\Big]\{B\} = \{0\} \tag{12}
$$

where $\{B\}$ is an eight-element column matrix for which the elements are B_j and where $[N]$ and $[Y]$ are 8×8 square matrices for which the elements are given in appendix A and where $[\Phi]$ and $[K]$ are 8×8 square boundary selection matrices for which the elements are defined as

$$
\Phi_{jk} = K_{jk} = 0 \qquad (\text{for} \quad j \neq k) \tag{13a}
$$

and

$$
K_{jk} = 1 - \Phi_{jk} \qquad (\text{for} \quad j = k) \tag{13b}
$$

The elements Φ_{jj} take on the value one or zero according to the prescribed edge conditions. If a stress-free condition is prescribed, the corresponding Φ_{jj} term is set equal to one. If a displacement variable or meridional slope is fully constrained, the corresponding Φ_{jj} term is set equal to zero. The $[\Phi]$ and $[K]$ matrices may be generalized to enforce coupled linear elastic or directional constraints.

For a given set of homogeneous boundary conditions and a particular circumferential mode number n, the natural frequencies of the system are those which cause the determinant of the coefficient matrix in equation (12) to vanish; that is,

$$\left| [\Phi][N] + [K][Y] \right| = 0 \tag{14}$$

There are an infinite number of natural frequencies associated with each circumferential mode number, with each frequency in general associated with a unique natural mode. Because of the complicated transcendental character of the determinantal equation (14), a trial procedure is used to find the natural frequencies. The procedure is exact, however, in the sense that the frequency parameter can be found to within any desired degree of accuracy.

With a particular circumferential mode number n given and a trial value for the nondimensional frequency Ω selected, the characteristic roots are determined from equation (6), h_j and g_j are found from equations (7) and U_j, V_j, and W_j are found from equations (A2) and (A3) in appendix A. With the proper case number determined, the elements of $[N]$ and $[Y]$ needed to form the boundary conditions are calculated. The determinant of equation (14) is then evaluated and the result compared with zero. A new trial value is selected for Ω and the procedure is repeated until a change in sign of the determinant is observed. A Newton-Raphson convergence routine is then employed to select a new Ω and the same procedure is repeated until equation (14) is satisfied to within the desired accuracy.

The success of this method as a rapid means of obtaining solutions depends on the selection of a good initial estimate of the frequency and a proper increment in frequency such that two roots are not traversed in one step. The closed-form solution for the natural vibration of nearly cylindrical shells with freely supported edges (see next section) can be used to good advantage as an aid in the selection of initial trial frequencies for systems with other boundary conditions.

When a natural frequency has been determined, its corresponding mode may be found from equations (11) and (12). The 7×8 augmented matrix C_{jj} is formed by deleting one of the dependent equations from equation (12). With Δ_j defined to be the determinant formed from the square matrix found by deleting the jth column of C_{jj}, the modal amplitude coefficients B_j are proportional to $(-1)^{j+1}\Delta_j$ and can thus be defined as

$$B_j = \frac{(-1)^{j+1}\Delta_j}{\epsilon} \tag{15}$$

where ϵ is some arbitrary scale factor. With B_j known, the meridional mode corresponding to Ω may be found from equations (11). Similarly, the vibratory stress and moment resultant distributions become, from the definitions of reference 1 and equations (8) and (11):

$$\left.\begin{aligned}
N_\xi &= \frac{B}{R} n_\xi \sin n\theta \cos \omega t \\
N_\theta &= \frac{B}{R} n_\theta \sin n\theta \cos \omega t \\
N_{\xi\theta} &= \frac{B}{R} n_{\xi\theta} \cos n\theta \cos \omega t \\
M_\xi &= -\frac{D}{R^2} m_\xi \sin n\theta \cos \omega t \\
M_\theta &= -\frac{D}{R^2} m_\theta \sin n\theta \cos \omega t \\
M_{\xi\theta} &= -\frac{D}{R^2} m_{\xi\theta} \cos n\theta \cos \omega t
\end{aligned}\right\} \tag{16}$$

where

$$D = \frac{Eh^3}{12\left(1 - \mu^2\right)} \tag{17}$$

and

$$\left.\begin{aligned}
n_\xi &= \sum_{j=1}^{8} B_j\left[U_j' - \mu n V_j + \left(\frac{R}{R_\xi} + \mu\right)W_j\right] \\
n_\theta &= \sum_{j=1}^{8} B_j\left[\mu U_j' - nV_j + \left(1 + \mu \frac{R}{R_\xi}\right)W_j\right] \\
n_{\xi\theta} &= \sum_{j=1}^{8} B_j\left(\frac{1 - \mu}{2}\right)\left(nU_j + V_j'\right) \\
m_\xi &= \sum_{j=1}^{8} B_j\left(\mu n V_j + W_j'' - \mu n^2 W_j\right)
\end{aligned}\right\} \tag{18}$$

Equations continued on next page

$$m_\theta = \sum_{j=1}^{8} B_j\left(-nV_j + \mu W_j'' - n^2 W_j\right)$$

$$m_{\xi\theta} = \sum_{j=1}^{8} B_j(1 - \mu)\left(nW_j' - V_j'\right)$$

The primes on symbols in these equations indicate differentiation with respect to ξ.

Stressed Shells With Freely Supported Edges

A simplified procedure of solution is available for one set of boundary conditions. The exact solution of equations (1) for freely supported boundary conditions $\left(N_\xi = v = w = M_\xi = 0\right)$ is given by

$$\left.\begin{aligned} u &= U_{mn} \cos \frac{m\pi\left(\xi + \frac{s}{2}\right)}{s} \sin n\theta \cos \omega t \\ v &= V_{mn} \sin \frac{m\pi\left(\xi + \frac{s}{2}\right)}{s} \cos n\theta \cos \omega t \\ w &= W_{mn} \sin \frac{m\pi\left(\xi + \frac{s}{2}\right)}{s} \sin n\theta \cos \omega t \end{aligned}\right\} \tag{19}$$

where m is the number of meridional half waves, n is the number of circumferential waves, and ω is the natural circular frequency. Substitution of equations (19) into equations (1) leads to the following characteristic equation for natural frequencies:

$$-\Omega^6 + \Lambda_2\Omega^4 - \Lambda_1\Omega^2 + \Lambda_0 = 0 \tag{20}$$

where Ω is a nondimensional frequency given by

$$\Omega = R\omega\sqrt{\frac{\nu\left(1 - \mu^2\right)}{E}} \tag{21}$$

The coefficients Λ_j depend upon the shell geometry, static stresses, and mode shape and are given in appendix A.

RESULTS AND DISCUSSION

The analysis just described has been used to investigate effects of various stress states and edge restraints on the vibration behavior of doubly curved shells. Stress states treated include those caused by constant directional axial loads and internal and external hydrostatic pressure on vibrating freely supported shells. Edge restraints treated include zero in-plane displacement and zero slope on unstressed vibrating shells. All results presented are for nearly cylindrical shells having both positive and negative Gaussian curvature with a ratio of circumferential radius to thickness of 500, ratio of length to circumferential radius of 2, and Poisson's ratio of 0.3.

Results are presented in terms of an auxiliary parameter τ, the ratio of the meridional rise to shell length expressed in percent:

$$\tau = \frac{c}{s}(100)$$

where c is the central rise of the shell meridian. (See fig. 1.) For small percentages of central rise with respect to length, τ can be closely approximated by

$$\tau = \frac{s}{8R_\xi}(100) \tag{22}$$

It is shown in reference 1 that for unstressed shells the approximate theory from which the basic equations (1) are derived gives reasonable estimates of frequencies and predicts trends adequately for values of τ between ± 5 percent.

Freely Supported Stressed Shells

Axial tension.- Effects of axial tension on the vibration of freely supported shells of double curvature are shown in figures 2 and 3. In figure 2, the envelope of frequencies are plotted as a function of the ratio of meridional rise to shell length τ, for various values of an in-plane stress parameter $\overline{\sigma}$, which is essentially the meridional stress resultant normalized with respect to the in-plane axial stiffness of the shell wall $\left(\overline{\sigma} = \frac{\overline{N}}{B} \times 10^4\right)$.

As the positive curvature increases from zero (cylinder) on the frequency envelope for $\overline{\sigma} = 0$, the lowest natural frequency and its associated circumferential wave number increase. In reference 1 it was shown that the membrane frequency (where the shell is regarded as a membrane) can vanish for unstressed freely supported shells when

$$\frac{R}{R_\xi} = -\left(\frac{m\pi R}{ns}\right)^2 \tag{23}$$

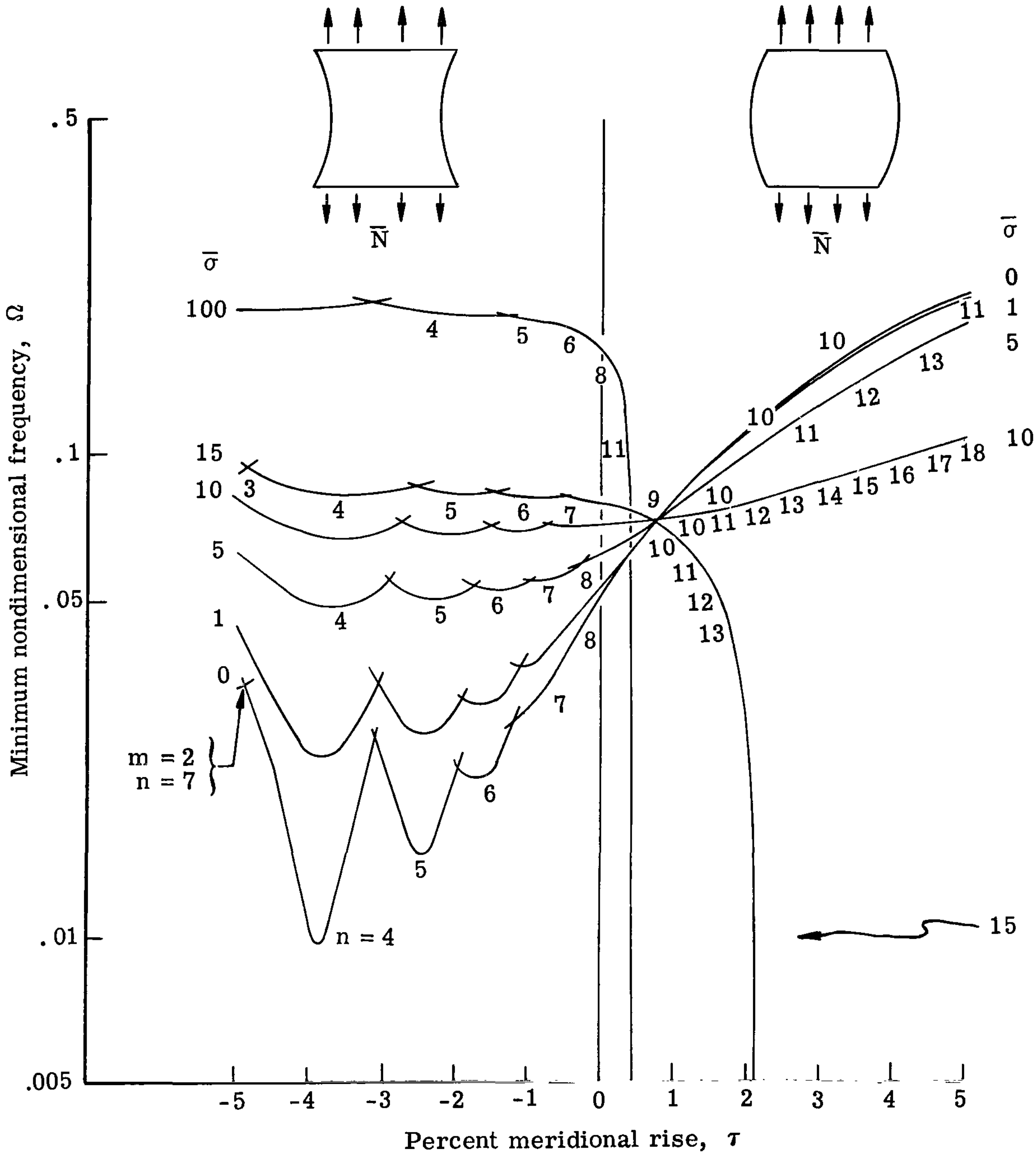

Figure 2.- Effect of meridional curvature on the minimum frequencies of freely supported, nearly cylindrical shells for several tensile axial loads. The meridional wave number m is 1 unless otherwise noted; the circumferential wave number n is given on curves.

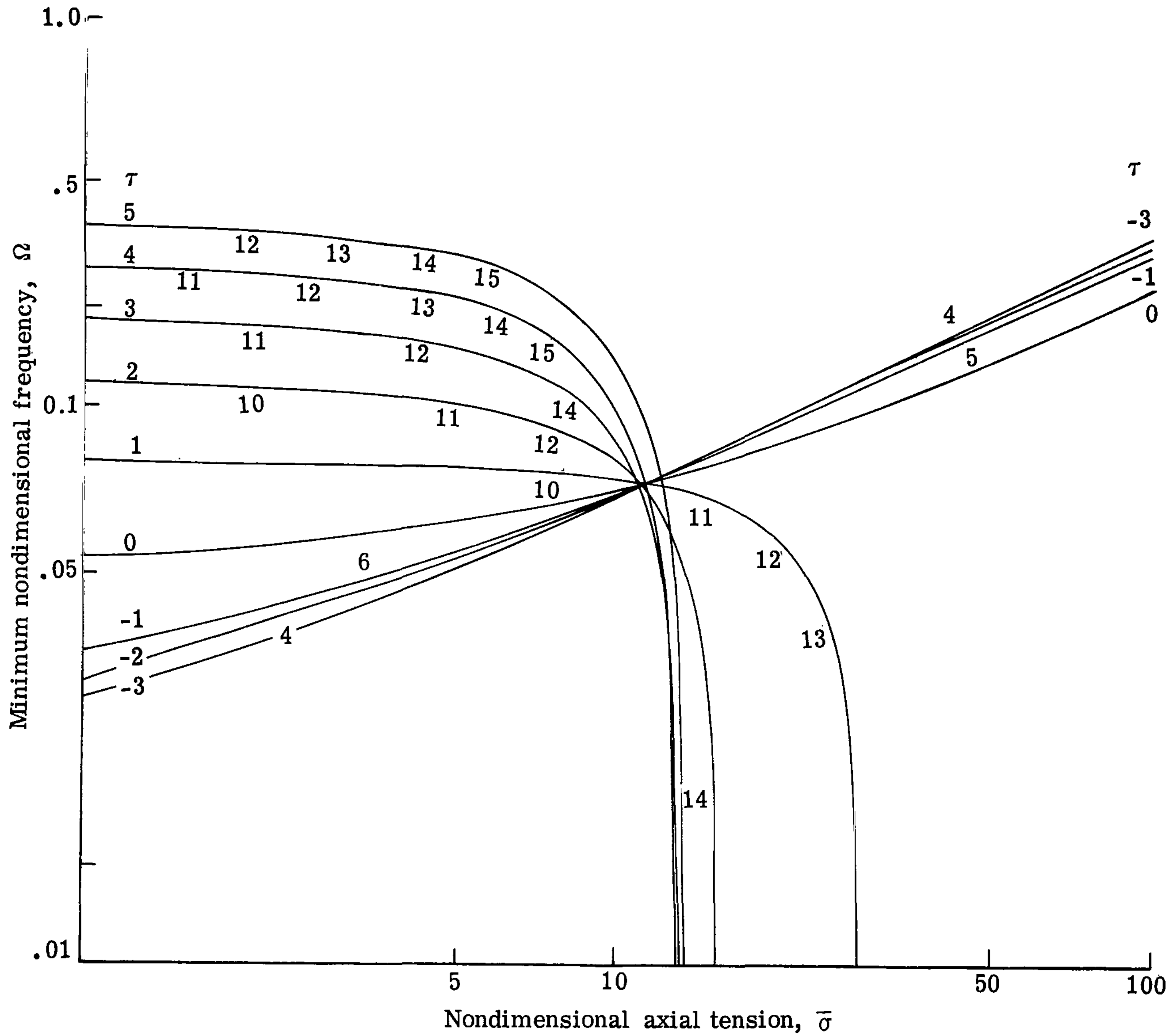

Figure 3.- Effect of axial tension on the minimum frequencies of several freely supported nearly cylindrical shells. The meridional wave number is 1; the circumferential wave number n is given on curves.

Thus, unlike positive-curvature shells and finite-length cylinders, negative-curvature shells can develop an inextensional vibrational mode. As the negative curvature increases from zero, the minimum frequency decreases and occurs in a progressively lower circumferential mode. For each circumferential mode, there is a distinct minimum in the variation of minimum frequency with curvature. These minimums occur at specific values of curvature for which membrane theory predicts inextensional behavior in the associated mode. (See ref. 1.) The frequencies at these minimums are governed almost entirely by bending stiffness.

Thus, for specific combinations of modal wavelength ratio and meridional curvature, negative-curvature shells vibrate in a nearly inextensional mode and have effective vibration stiffnesses similar to those of simply supported plates vibrating with the same wavelength ratios. Reference 1 discusses in more detail the effect of curvature on the minimum natural frequencies for unstressed freely supported shells. The natural frequency envelope for $\bar{\sigma} = 0$ is repeated in all subsequent figures of this type for purposes of comparison.

For an applied tensile stress resultant $\overline{N}$ at the shell edges, a constant biaxial membrane stress field is developed. From equations (3), the circumferential and meridional stress resultants are

$$\left.\begin{aligned} \overline{N}_\xi &= \overline{N} \\ \overline{N}_\theta &= -\frac{R\overline{N}}{R_\xi} \end{aligned}\right\} \tag{24}$$

The axial-tension load causes tensile stresses in both the meridional and circumferential directions in the negative-curvature shells which introduce a stiffening effect. As the loading increases, this biaxial tensile state eventually drives the minimum frequencies up above that of a similarly loaded cylinder ($\bar{\sigma} > 11$ in fig. 3). For the higher loadings, there is only a very slight variation in frequency with the degree of curvature, contrasted with the large variation that occurred with no load.

For positive-curvature shells, on the other hand, the axial tensile edge load introduces compressive circumferential stresses which cause reductions in the minimum frequencies. Increased loading on these shells may lead to large enough circumferential compression to cause buckling. This condition is evidenced in figure 2 where the curves for $\bar{\sigma} = 15$ and 100 become almost vertical (i.e., Ω approaches zero) in the positive-curvature-shell regime.

From the third of equations (1) and equations (24) it can be shown that the natural frequencies of a nearly cylindrical shell are independent of applied axial tensile or compressive edge loading when the associated mode has the relationship

$$w_{,\xi\xi} = \frac{w_{,\theta\theta}}{RR_\xi} \tag{25}$$

if $w = 0$ at the edges. For a freely supported shell, this condition is satisfied when

$$\frac{R}{R_\xi} = \left(\frac{m\pi R}{ns}\right)^2$$

Every positive-curvature shell, therefore, has modes of vibration with a specific wavelength ratio for which the corresponding natural frequency is independent of the static stress. These modes of vibration are usually not associated with the minimum frequency of the shell. For the particular shell configuration treated in this report, however, this condition does occur on the minimum frequency envelope in figure 2 where the $\bar{\sigma}$ = Constant curves intersect with $m = 1$ and $n = 9$ for $\tau = 0.76$ percent when $\bar{\sigma}$ is approximately equal to or less than 10.

Figure 3 is a plot of the lowest natural frequency as a function of meridional nondimensional stress for shells with various curvatures under axial tension. The minimum frequencies of positive curvature shells are very nearly insensitive to stress changes at low stress levels but become highly sensitive to changes in stress as the instability stress level is approached.

The negative-curvature-shell frequencies are almost linear with stress on the log-log plot of figure 3 so that a simple power relationship ($\Omega \propto \sigma^a$) would be a good indicator of the effect of axial tension. Figure 3 clearly displays curvature independence at a nondimensional meridional stress level of approximately $\bar{\sigma} = 11$. At this stress level the natural frequency is nearly independent of the degree of curvature; hence, this level can be interpreted as the level where the stiffening effect of positive curvature is offset by the weakening effect of circumferential compressive stress. This effect can be observed in figure 2 where for $\bar{\sigma} = 10$ the minimum frequency envelope is almost horizontal for most of the range of curvature, the influence of curvature being felt only in the upper range of positive curvature.

Axial compression.- Figure 4 is a set of envelopes of minimum natural frequencies plotted as a function of the central rise-length ratio for a range of nondimensional statically applied edge compressive stress resultants $\bar{\sigma}$. The axial compressive load causes compressive meridional and circumferential stresses in the negative-curvature shells which severely weaken the shell and cause large reductions in frequency in local ranges of negative curvature and loss of stability at levels ($\bar{\sigma} = -0.5$) an order of magnitude below the instability level for the corresponding cylinder ($\bar{\sigma} = -10.9$).

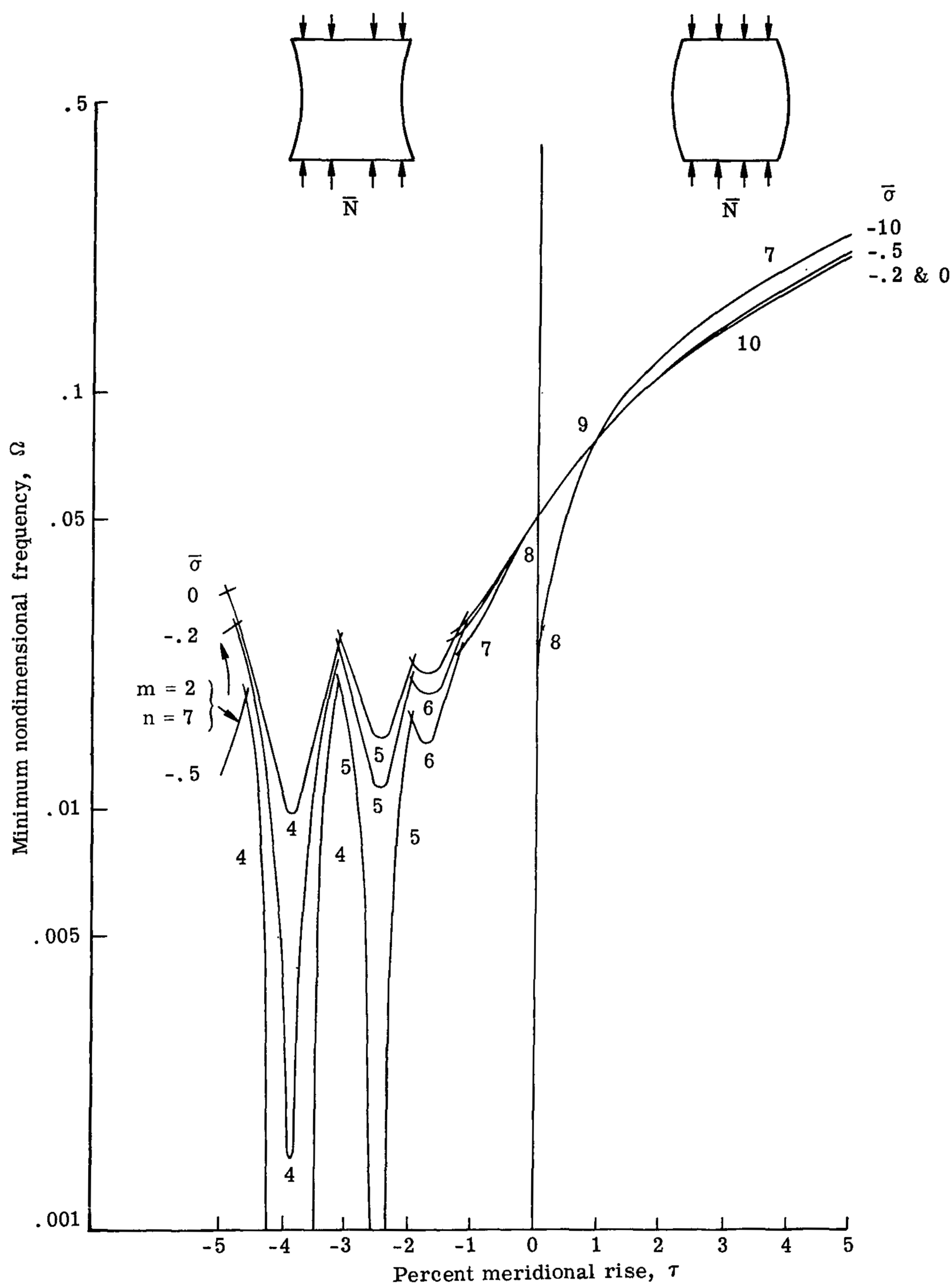

Figure 4.- Effect of meridional curvature on the minimum frequencies of freely supported nearly cylindrical shells for several compressive axial loads. The meridional wave number m is 1 unless otherwise noted; the circumferential wave number n is given on curves.

In figure 5, the minimum natural frequency is plotted as a function of meridional compressive nondimensional stress for shells with various degrees of curvature. The axial compression introduces a static positive circumferential stress which tends to stiffen slightly the positive-curvature shells, but in general the effect is small. The positive-curvature shell becomes unstable at the same stress level as the cylinder

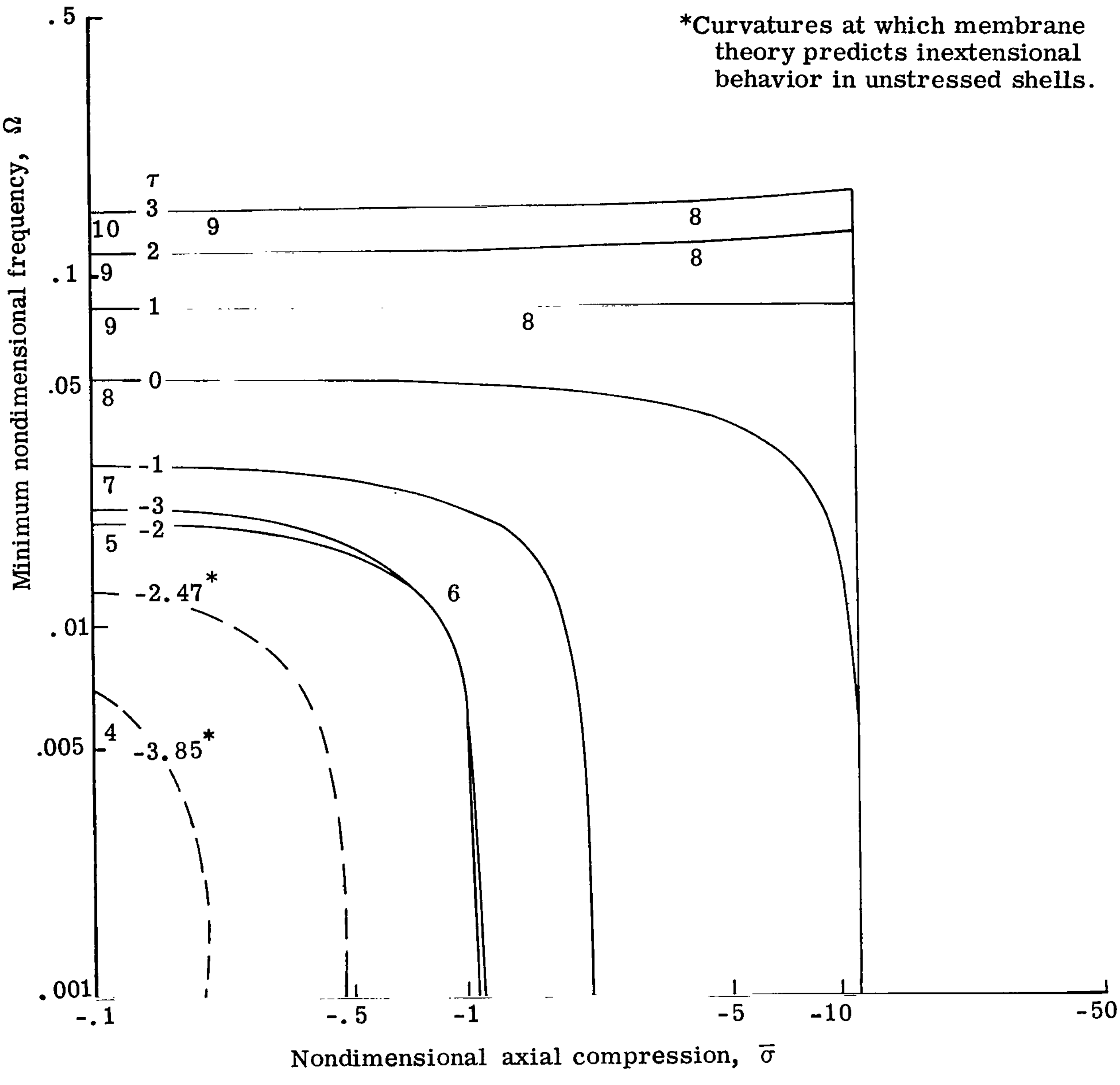

Figure 5.- Effect of axial compression on the minimum frequencies of several freely supported nearly cylindrical shells. The meridional wave number is 1; the circumferential wave number n is given on curves.

(ref. 4). The curves end abruptly at the buckling stress of the cylinder and do not exhibit a continuous decrease in frequency when the buckling stress is approached as does the cylinder. The minimum frequencies of the negative-curvature shells show more dependence on stress at low stress levels than do those of the positive-curvature shells and are highly dependent on stress only at stress levels close to instability. The particular curvatures at which membrane theory predicts inextensional behavior are shown as dashed lines and indicate the extreme sensitivity of shells to stress in this region of curvature.

Internal hydrostatic pressure.- Figure 6 shows the effect of curvature on the minimum natural frequencies of internally pressurized shells. The internal pressure introduces a tensile meridional stress as well as circumferential stress in both positive- and negative-curvature shells and increases the stiffnesses of both shells above that of unstressed shells. For pressurized shells $\left(\overline{\sigma} = \frac{\overline{N}_{\xi}}{B} \times 10^4\right)$ the envelope of natural frequencies in the negative-curvature range is consistently lower than that of the cylinder, although the difference in minimum frequency is no longer as large as for the unstressed shell. The positive-curvature range does not show large increases in stiffness. Thus the internal pressure acts as a stabilizing influence and generally moderates any large effects due to meridional curvature.

External hydrostatic pressure.- Figure 7 shows the effect of curvature on the lowest natural frequency of externally pressurized nearly cylindrical shells. External pressure introduces compressive meridional and circumferential stresses. The effect of static external pressure on the lowest frequencies of negative-curvature shells is very similar to the effect of axial compression. Large reductions of frequency occur at curvatures corresponding to nearly inextensional vibration.

Figure 8 shows the buckling pressure for negative-curvature shells to be well below that of the corresponding cylinder (on the order of 50 percent lower for $\tau = -1$). Unlike the results under axial compression, the positive-curvature shells are stable at pressures higher than the buckling pressure of the corresponding cylinder. The frequencies for both the positive- and negative-curvature shells are essentially insensitive to the pressure at low stress levels but are highly sensitive to stress as their respective buckling pressures are approached.

Effects of Edge Restraint

The nearly inextensional vibrations that can occur in negative-curvature shells and the attendant reduced effective stiffness associated with this type of vibration can be avoided with proper edge restraints.

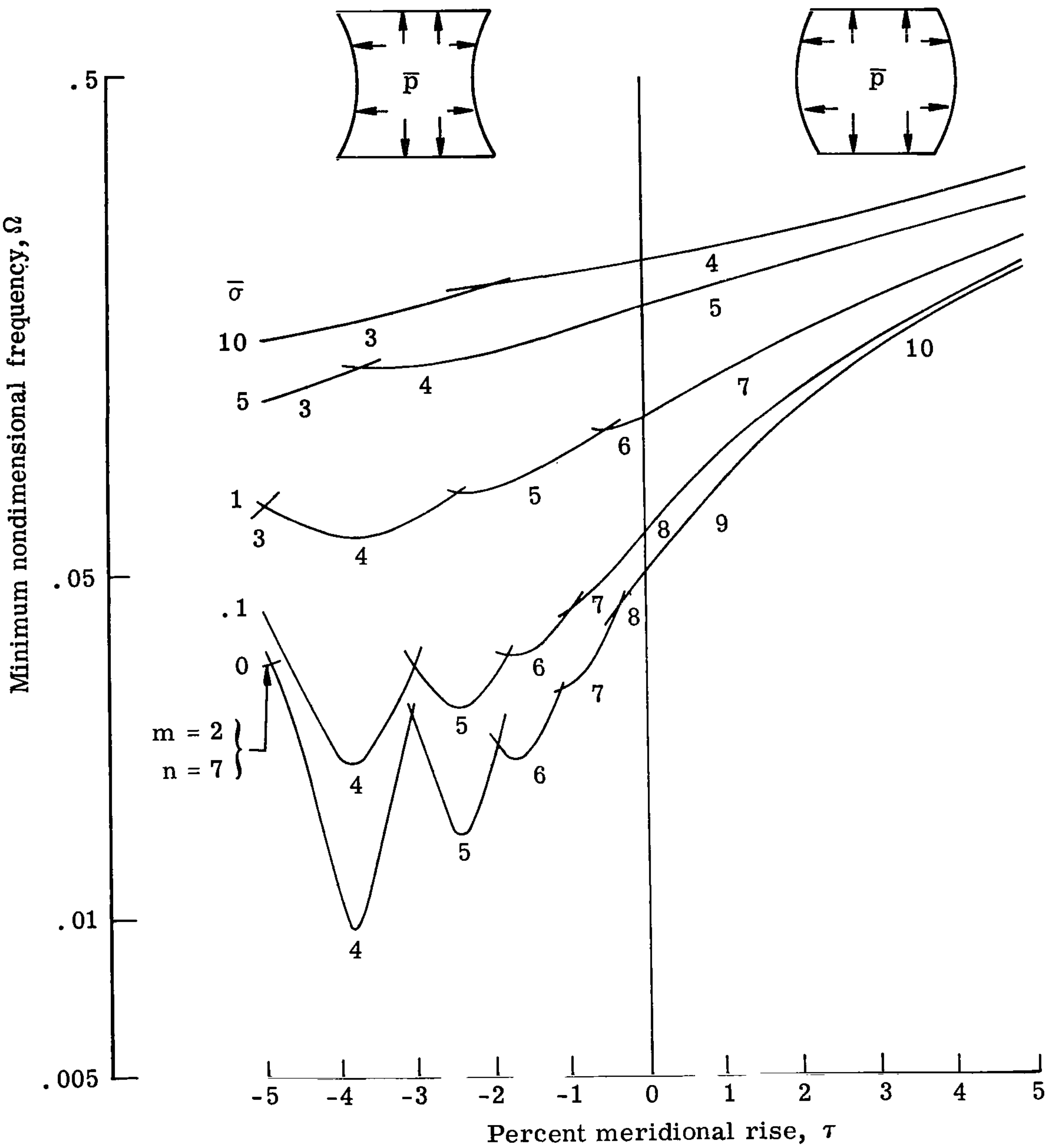

Figure 6.- Effect of meridional curvature on the minimum frequencies of freely supported nearly cylindrical shells for several levels of internal hydrostatic pressure. The meridional wave number m is 1 unless otherwise noted; the circumferential wave number n is given on curves.

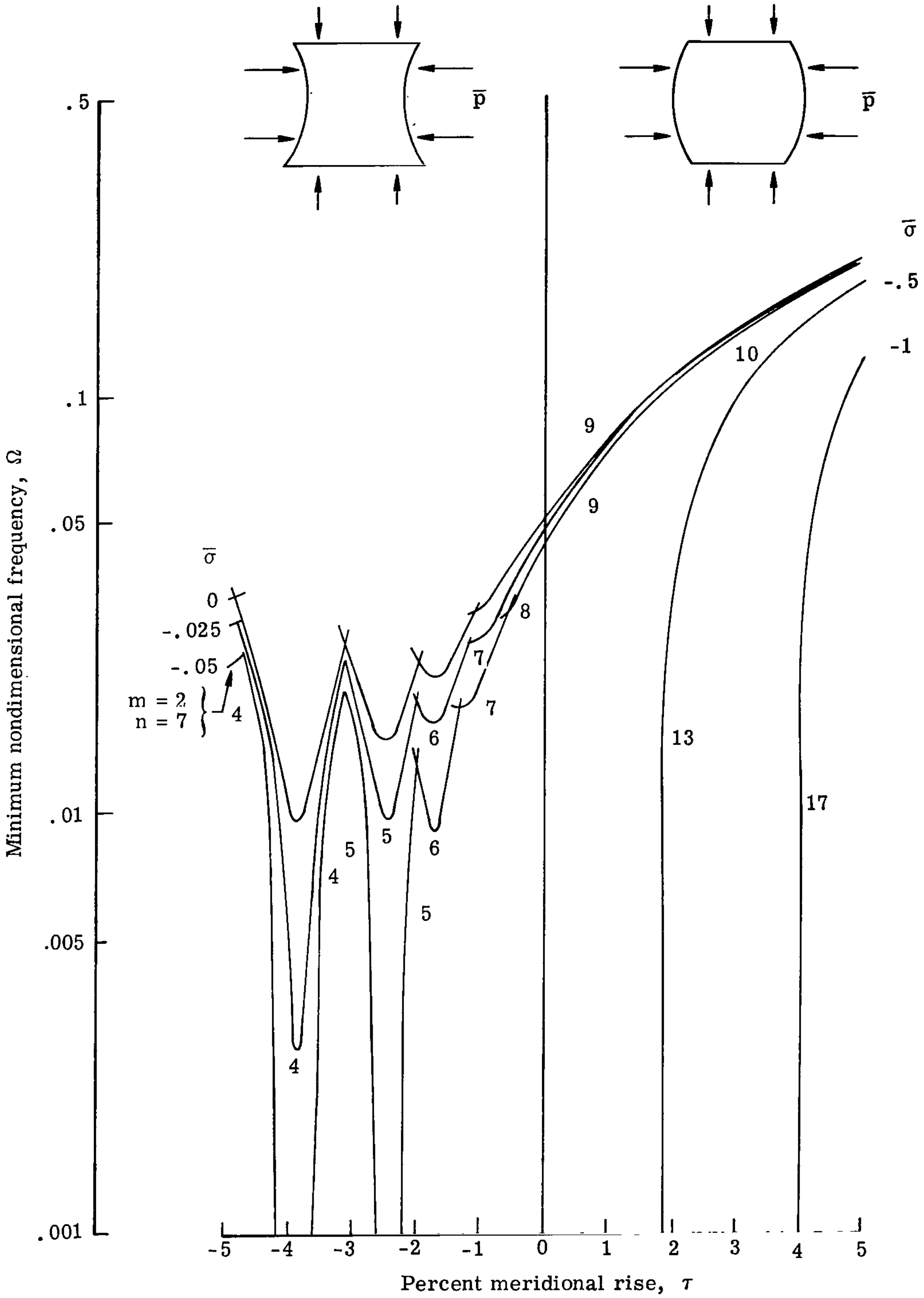

Figure 7.- Effect of meridional curvature on the minimum frequencies of freely supported nearly cylindrical shells for several levels of external hydrostatic pressure. The meridional wave number m is 1 unless otherwise noted; the circumferential wave number n is given on curves.

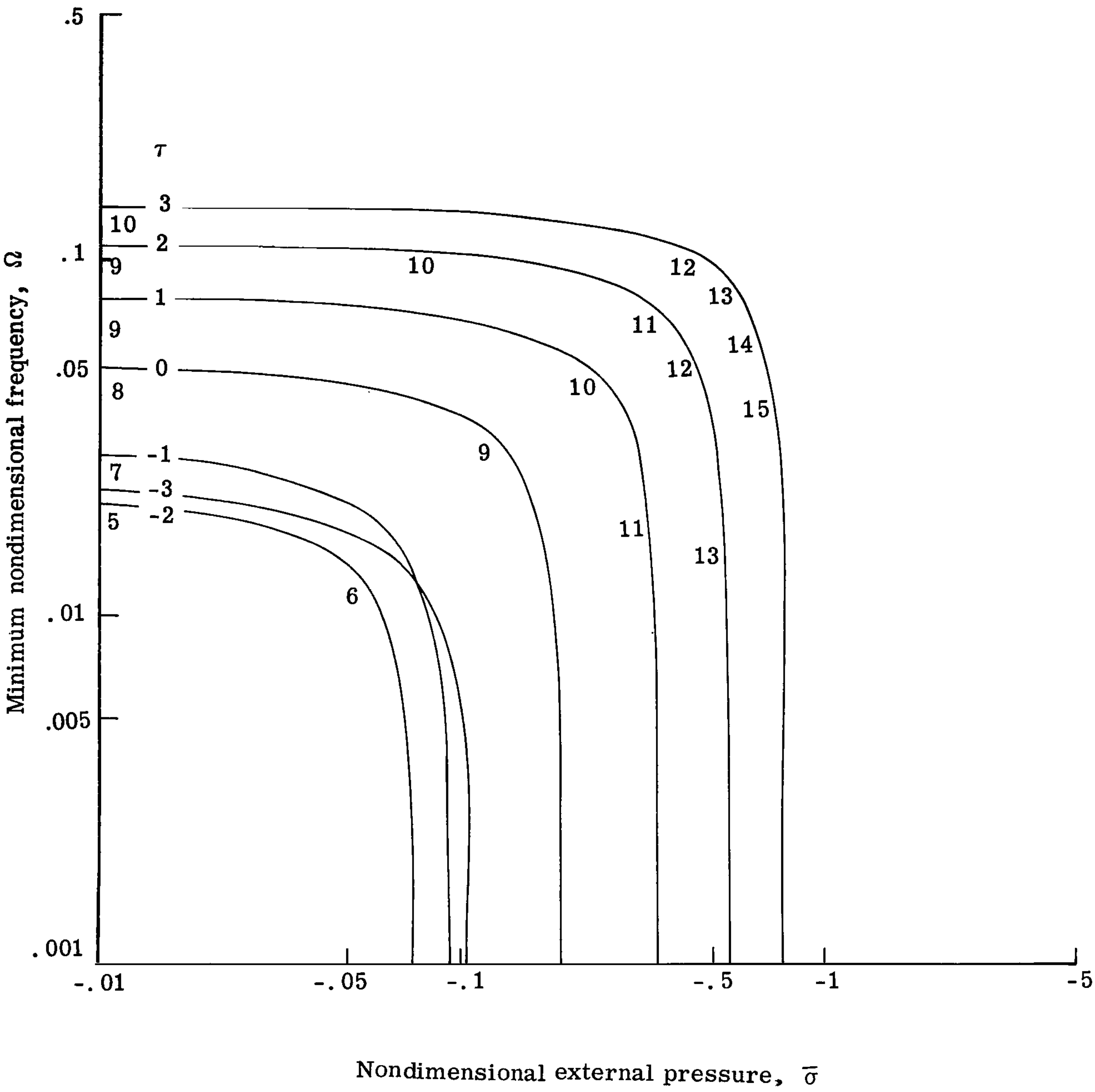

Figure 8.- Effect of external hydrostatic pressure on the minimum frequencies of several freely supported nearly cylindrical shells. The meridional wave number m is 1; the circumferential wave number n is given on curves.

The lowest natural frequency for the class of unstressed freely supported shells investigated occurred for $\tau = -3.85$ percent with $m = 1$ and $n = 4$. (See fig. 2.) Figure 9 has a comparison between the minimum frequencies of this shell for each n from 1 to 15 with freely supported edges and the minimum frequencies of the same shell with meridional restraint at the edges ($u = 0$). From equation (23), inextensional behavior occurs for the freely supported shell when $m/n = 1/4$. The open symbols in figure 9 at $m = 1$, $n = 4$; $m = 2$, $n = 8$; and $m = 3$, $n = 12$ locate the inextensional frequencies for these particular modes. The inextensional frequencies closely approximate the minimum for each m-branch of the envelope for the shell with freely supported edges in the lower n-range. The in-plane edge restraint had a sizable influence in preventing the inextensional vibration and in increasing the frequency for moderate values of n but diminished in influence for high and low values of n. The lowest natural frequency shifted from $m = 1$, $n = 4$ to a more complex mode $m = 2$, $n = 10$ and increased by an order of magnitude.

The results for different combinations of edge restraint of the in-plane displacement u and the slope at the edge $w_{,\xi}$ given in table I for $\tau = -3.85$ percent show that the natural frequencies are highly dependent on the displacement constraint. The slope restraint $w_{,\xi} = 0$ has essentially no influence on the natural frequencies.

Figure 10 is a plot of the minimum frequency envelope as the curvature varies for shells with unrestrained in-plane displacements (freely supported) on each edge and with restrained in-plane displacements (simply supported) on each edge. The positive-curvature shells evidence only a small increase in minimum frequency due to the edge restraint, and the effect decreases as curvature increases. The negative-curvature shells on the other hand show a large increase in frequency with in-plane edge restraint and the dramatic drops in frequency in local regions of negative curvature are no longer evident since the in-plane restraint has prevented the development of inextensional vibrations.

For small negative curvature and in-plane restraint there is an initial reduction in frequency from that of the corresponding cylinder with a maximum reduction of approximately 8 percent occurring at $\tau = -0.7$ percent. As the negative curvature increases beyond this point, frequency increases and for $\tau = -1.6$ percent the shell becomes stiffer than the corresponding cylinder.

Although the analysis and conclusions are restricted to simple shell configurations, the behavior of these shells can be used to predict general phenomena in shell structures of more complex configurations. For example, it can be expected that local negative-curvature areas of more complex shells can develop nearly inextensional modes of vibration if the adjoining structure does not provide sufficient in-plane support. Further, in-plane compressive meridional stresses can cause severe reductions in the local

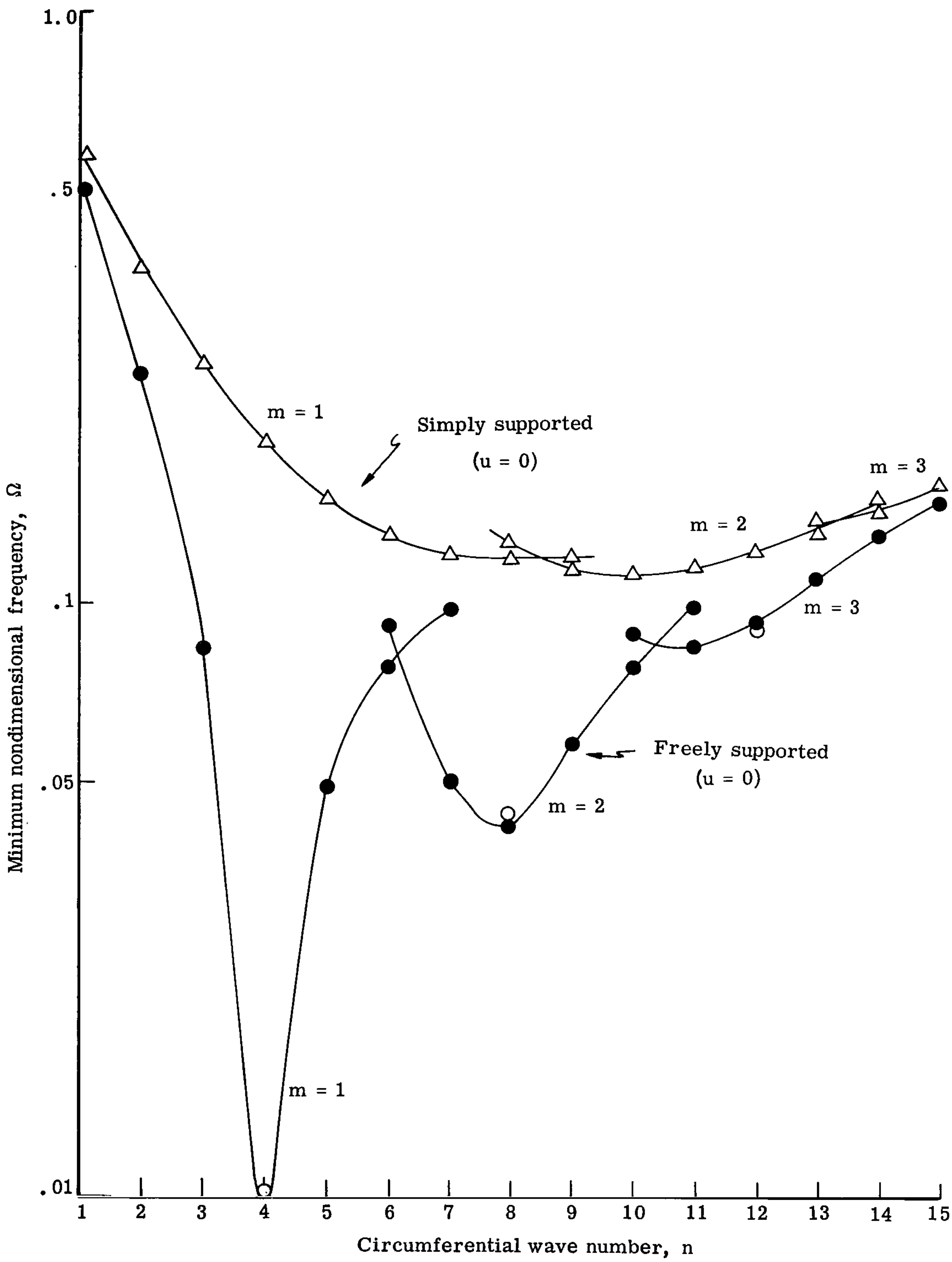

Figure 9.- Effect of in-plane edge restraint on the natural frequencies of a negative-curvature shell with $\tau = -3.85$ percent.

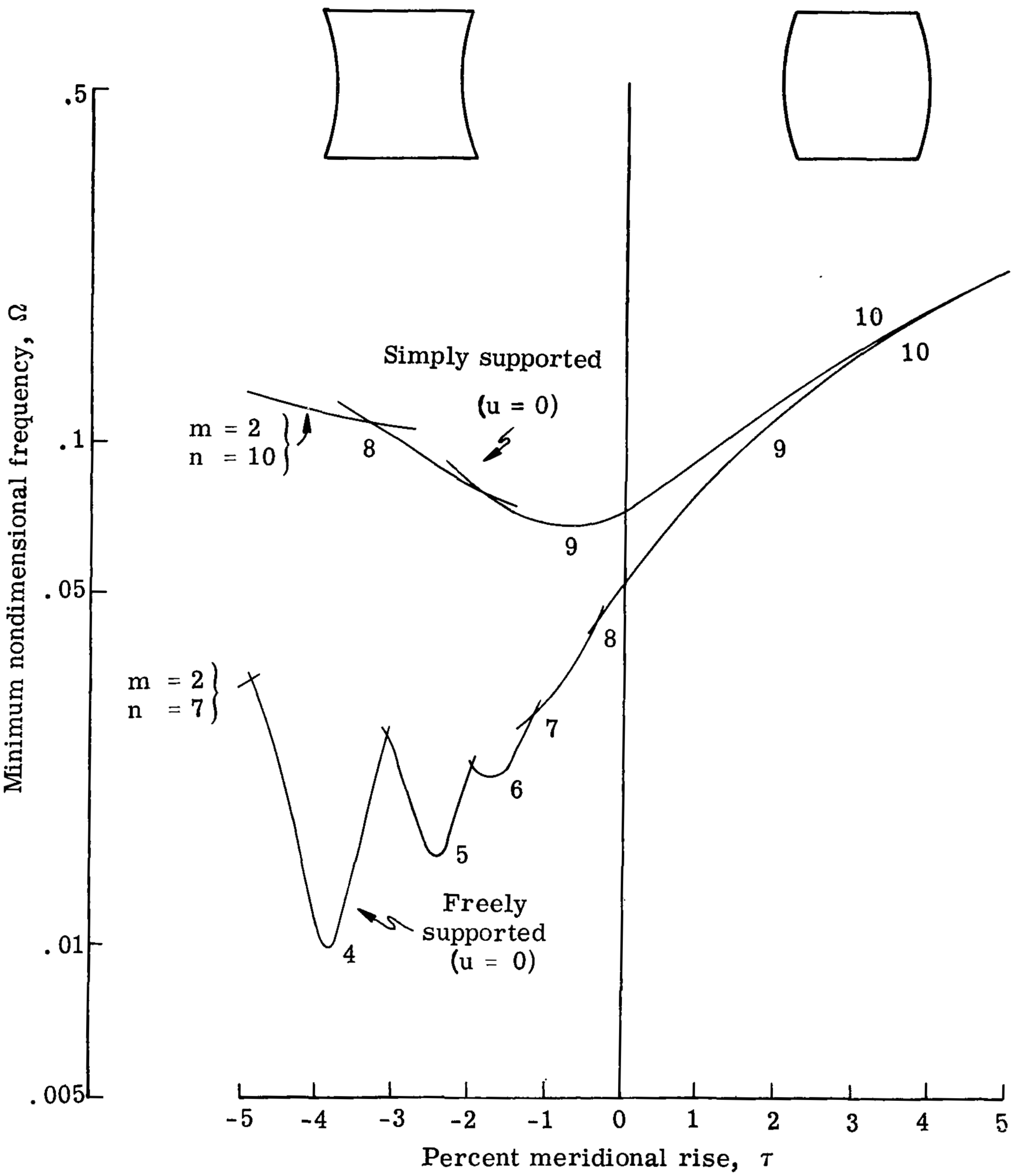

Figure 10.- Comparison of the effects of meridional curvature on the minimum frequencies of freely supported and simply supported nearly cylindrical unstressed shells. The number of meridional half waves m is 1 unless otherwise noted; the circumferential wave number n is given on curves.

TABLE I.- EFFECT OF IN-PLANE AND ROTATIONAL EDGE RESTRAINT ON THE NATURAL FREQUENCIES OF A NEGATIVE CURVATURE SHELL

[τ = -3.85 percent; s/R = 2; R/h = 500]

n	Ω			
	Freely supported $N_\xi = v = w = M_\xi = 0$ (a)	Clamped with no meridional restraint $N_\xi = v = w = w_{,\xi} = 0$ (a)	Simply supported $u = v = w = M_\xi = 0$ (a)	Fixed $u = v = w = w_{,\xi} = 0$ (a)
1	0.5278(1)	0.5278(1)	0.5709(1)	0.5711(1)
2	.2429(1)	.2430(1)	.3663(1)	.3665(1)
3	.0847(1)	.0852(1)	.2532(1)	.2534(1)
4	.00969(1)	.0134(1)	.1879(1)	.1881(1)
5	.0493(1)	.0502(1)	.1506(1)	.1507(1)
6	.0782(1)	.0787(1)	.1307(1)	.1308(1)
7	.0497(2)	.0532(2)	.1218(1)	.1218(1)
8	.0416(2)	.0457(2)	.1190(1)	.1190(1)
9	.0582(2)	.0611(2)	.1153(2)	.1154(2)
10	.0786(2)	.0807(2)	.1118(2)	.1118(2)
11	.0850(3)	.0893(3)	.1150(2)	.1151(2)
12	.0949(3)	.0987(3)	.1229(2)	.1229(2)

[a]Integer enclosed by parentheses represents meridional wave number.

stiffness of negative-curvature regions of shells which have only slight in-plane restraint. These results suggest that a proper design for a shell structure with negative-curvature regions should contain either sufficient in-plane stiffness, such as intermediate rings or doubler strips at the boundaries of these regions, or appropriate loadings which provide a sufficient positive in-plane stress field to stabilize such regions.

CONCLUDING REMARKS

The effect of the interaction between slight meridional curvature and static stress due to simple loading conditions on the minimum natural frequencies of nearly cylindrical shells is investigated. The effect of various edge restraints is also investigated.

Axial tension increases the minimum frequencies of negative-curvature shells but because of compressive circumferential stresses introduced by this loading in positive-curvature shells, the minimum frequencies of these shells are reduced and for sufficiently high tensile loading these shells can become unstable. Axial compression decreases the minimum frequencies of negative-curvature shells but increases only slightly the minimum frequencies of positive-curvature shells. The negative-curvature shells become unstable at stress levels well below that of the corresponding cylinder, whereas positive-curvature shells offer no increase in stability and become unstable at the same stress level as the corresponding cylinder.

Internal pressure increases the minimum frequencies of both positive- and negative-curvature shells and decreases the effect of curvature on the frequency. External pressure accentuates the destabilizing effect of negative curvature by reducing the minimum frequencies, whereas it only slightly reduces the minimum frequencies of positive-curvature shells except near instability stress levels.

In-plane meridional restraint on the boundaries has only a slight effect on the minimum natural frequencies of positive-curvature shells with moderately small positive meridional curvature with the effect increasing as the degree of curvature diminishes. The in-plane meridional restraint prevents inextensional vibration from developing in negative-curvature shells, and as a consequence negative-curvature shells are highly sensitive to in-plane restraint, evidencing large increases in frequency above that obtained with unrestrained edges. In fact, for a sizable range of curvature the restrained negative-curvature shells have higher minimum natural frequencies than the corresponding cylinder. Rotational restraint has a negligible effect on the minimum frequencies of both positive- and negative-curvature shells.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., February 2, 1971.

APPENDIX A

COEFFICIENTS IN EQUATIONS (6), (11), (12), AND (20)

The coefficients of equation (6) are defined as follows:

$$\left.\begin{aligned}
a_0 &= \frac{-f_1\left(f_7\right)^2 + f_1 f_5 f_{10}}{f_4 f_8} \\
a_2 &= \frac{f_3\left(2f_2 f_7 - f_3 f_5\right) - 2f_1 f_6 f_7 - \left(f_7\right)^2 + f_1 f_5 f_9 + f_{10}\left[f_5 + f_1 f_4 - \left(f_2\right)^2\right]}{f_4 f_8} \\
a_4 &= \frac{f_3\left(2f_2 f_6 - f_3 f_4\right) - f_6\left(2f_7 + f_1 f_6\right) + f_1 f_5 f_8 + f_9\left[f_5 + f_1 f_4 - \left(f_2\right)^2\right] + f_4 f_{10}}{f_4 f_8} \\
a_6 &= \frac{-\left(f_6\right)^2 + f_8\left[f_5 + f_1 f_4 - \left(f_2\right)^2\right] + f_4 f_9}{f_4 f_8}
\end{aligned}\right\} \qquad \text{(A1)}$$

where

$$f_1 = -\left(\frac{1 - \mu}{2}\right)n^2 + \Omega^2$$

$$f_2 = -\left(\frac{1 + \mu}{2}\right)n$$

$$f_3 = \frac{R}{R_\xi} + \mu$$

$$f_4 = -\left(\frac{1 - \mu}{2}\right)\left(1 + \frac{\lambda^2}{3}\right)$$

$$f_5 = \left(1 + \frac{\lambda^2}{12}\right)n^2 - \Omega^2$$

$$f_6 = \frac{\lambda^2}{12}(2 - \mu)n$$

$$f_7 = -\left(1 + \mu\,\frac{R}{R_\xi}\right)n - \frac{\lambda^2 n^3}{12}$$

$$f_8 = \frac{\lambda^2}{12}$$

$$f_9 = -\frac{\lambda^2 n^2}{6} - \frac{\overline{N}}{B}$$

$$f_{10} = \frac{\lambda^2 n^4}{12} + \left[\left(\frac{R}{R_\xi}\right)^2 + 2\mu\left(\frac{R}{R_\xi}\right) + 1\right] + \frac{\overline{N}_\theta}{B} n^2 - \Omega^2$$

The coefficients U_j, V_j, and W_j of equations (11) are as follows:

For cases 1 and 2, U_j coefficients U_1 to U_4 are defined as

$$U_1 = \frac{1}{2}\left[\mathrm{Re}(h_1) + \mathrm{Re}(h_3)\right]\sinh\frac{\alpha\xi}{R^2}\sin\frac{\beta\xi}{R^2} + \frac{1}{2}\left[\mathrm{Im}(h_1) + \mathrm{Im}(h_3)\right]\sinh\frac{\alpha\xi}{R^2}\cos\frac{\beta\xi}{R^2}$$
$$+ \frac{1}{2}\left[\mathrm{Re}(h_1) - \mathrm{Re}(h_3)\right]\cosh\frac{\alpha\xi}{R^2}\sin\frac{\beta\xi}{R^2} + \frac{1}{2}\left[\mathrm{Im}(h_1) - \mathrm{Im}(h_3)\right]\cosh\frac{\alpha\xi}{R^2}\cos\frac{\beta\xi}{R^2} \qquad \text{(A2a)}$$

$$U_2 = -\frac{1}{2}\left[\mathrm{Im}(h_1) + \mathrm{Im}(h_3)\right]\sinh\frac{\alpha\xi}{R^2}\sin\frac{\beta\xi}{R^2} + \frac{1}{2}\left[\mathrm{Re}(h_1) + \mathrm{Re}(h_3)\right]\sinh\frac{\alpha\xi}{R^2}\cos\frac{\beta\xi}{R^2}$$
$$- \frac{1}{2}\left[\mathrm{Im}(h_1) - \mathrm{Im}(h_3)\right]\cosh\frac{\alpha\xi}{R^2}\sin\frac{\beta\xi}{R^2} + \frac{1}{2}\left[\mathrm{Re}(h_1) - \mathrm{Re}(h_3)\right]\cosh\frac{\alpha\xi}{R^2}\cos\frac{\beta\xi}{R^2} \qquad \text{(A2b)}$$

$$U_3 = \frac{1}{2}\left[\mathrm{Re}(h_1) - \mathrm{Re}(h_3)\right]\sinh\frac{\alpha\xi}{R^2}\sin\frac{\beta\xi}{R^2} + \frac{1}{2}\left[\mathrm{Im}(h_1) - \mathrm{Im}(h_3)\right]\sinh\frac{\alpha\xi}{R^2}\cos\frac{\beta\xi}{R^2}$$
$$+ \frac{1}{2}\left[\mathrm{Re}(h_1) + \mathrm{Re}(h_3)\right]\cosh\frac{\alpha\xi}{R^2}\sin\frac{\beta\xi}{R^2} + \frac{1}{2}\left[\mathrm{Im}(h_1) + \mathrm{Im}(h_3)\right]\cosh\frac{\alpha\xi}{R^2}\cos\frac{\beta\xi}{R^2} \qquad \text{(A2c)}$$

$$U_4 = -\frac{1}{2}\left[\mathrm{Im}(h_1) - \mathrm{Im}(h_3)\right]\sinh\frac{\alpha\xi}{R^2}\sin\frac{\beta\xi}{R^2} + \frac{1}{2}\left[\mathrm{Re}(h_1) - \mathrm{Re}(h_3)\right]\sinh\frac{\alpha\xi}{R^2}\cos\frac{\beta\xi}{R^2}$$
$$- \frac{1}{2}\left[\mathrm{Im}(h_1) + \mathrm{Im}(h_3)\right]\cosh\frac{\alpha\xi}{R^2}\sin\frac{\beta\xi}{R^2} + \frac{1}{2}\left[\mathrm{Re}(h_1) + \mathrm{Re}(h_3)\right]\cosh\frac{\alpha\xi}{R^2}\cos\frac{\beta\xi}{R^2} \qquad \text{(A2d)}$$

For case 1, the coefficients U_5 and U_6 are

$$U_5 = \frac{1}{2}\left(h_5 + h_6\right)\sinh\frac{\gamma\xi}{R^2} + \frac{1}{2}\left(h_5 - h_6\right)\cosh\frac{\gamma\xi}{R^2} \qquad \text{(A2e)}$$

$$U_6 = \frac{1}{2}(h_5 - h_6)\sinh\frac{\gamma\xi}{R^2} + \frac{1}{2}(h_5 + h_6)\cosh\frac{\gamma\xi}{R^2} \tag{A2f}$$

and for case 2, these coefficients are

$$U_5 = \mathrm{Re}(h_5)\sin\frac{\gamma\xi}{R^2} + \mathrm{Im}(h_5)\cos\frac{\gamma\xi}{R^2} \tag{A2g}$$

$$U_6 = -\mathrm{Im}(h_5)\sin\frac{\gamma\xi}{R^2} + \mathrm{Re}(h_5)\cos\frac{\gamma\xi}{R^2} \tag{A2h}$$

For both cases 1 and 2, the coefficients U_7 and U_8 are

$$U_7 = \mathrm{Re}(h_7)\sin\frac{\delta\xi}{R^2} + \mathrm{Im}(h_7)\cos\frac{\delta\xi}{R^2} \tag{A2i}$$

$$U_8 = -\mathrm{Im}(h_7)\sin\frac{\delta\xi}{R^2} + \mathrm{Re}(h_7)\cos\frac{\delta\xi}{R^2} \tag{A2j}$$

For the V_j coefficients, replace h_j with g_j in equations (A2) for U_j.

For coefficients W_j, the following apply:

For both cases 1 and 2,

$$W_1 = \sinh\frac{\alpha\xi}{R^2}\sin\frac{\beta\xi}{R^2} \tag{A3a}$$

$$W_2 = \sinh\frac{\alpha\xi}{R^2}\cos\frac{\beta\xi}{R^2} \tag{A3b}$$

$$W_3 = \cosh\frac{\alpha\xi}{R^2}\sin\frac{\beta\xi}{R^2} \tag{A3c}$$

$$W_4 = \cosh\frac{\alpha\xi}{R^2}\cos\frac{\beta\xi}{R^2} \tag{A3d}$$

For case 1,

$$W_5 = \sinh\frac{\gamma\xi}{R^2} \tag{A3e}$$

$$W_6 = \cosh\frac{\gamma\xi}{R^2} \tag{A3f}$$

For case 2,

$$W_5 = \sin\frac{\gamma\xi}{R^2} \tag{A3g}$$

$$W_6 = \cos\frac{\gamma\xi}{R^2} \tag{A3h}$$

For both cases 1 and 2,

$$W_7 = \sin\frac{\delta\xi}{R^2} \tag{A3i}$$

$$W_8 = \cos\frac{\delta\xi}{R^2} \tag{A3j}$$

Elements of boundary matrices of equation (12) are as follows:

$\lceil N \rfloor$ matrices:

$$N_{1j} = \left[U_j' - \mu n V_j + \left(\frac{R}{R_\xi} + \mu\right)W_j\right]_{\xi=\frac{s}{2}} \tag{A4a}$$

$$N_{2j} = \left[nU_j + \left(1 + \frac{\lambda^2}{12}\right)V_j' - \frac{\lambda^2 n}{3} W_j'\right]_{\xi=\frac{s}{2}} \tag{A4b}$$

$$N_{3j} = \left\{\frac{\lambda^2 n}{12}(2 - \mu)V_j' + \frac{\lambda^2}{12} W_j''' + \left[\frac{\lambda^2 n^2}{12}(2 - \mu) - \frac{\overline{N}_\xi}{B}\right]W_j'\right\}_{\xi=\frac{s}{2}} \tag{A4c}$$

$$N_{4j} = \left(\mu n V_j + W_j'' - \mu n^2 W_j\right)_{\xi=\frac{s}{2}} \tag{A4d}$$

$$N_{5j} = \left[U_j' - \mu n V_j + \left(\frac{R}{R_\xi} + \mu\right)W_j\right]_{\xi=-\frac{s}{2}} \tag{A4e}$$

$$N_{6j} = \left[nU_j + \left(1 + \frac{\lambda^2}{12}\right)V_j' - \frac{\lambda^2 n}{3} W_j'\right]_{\xi=-\frac{s}{2}} \tag{A4f}$$

$$N_{7j} = \left\{\frac{\lambda^2 n}{12}(2 - \mu)V_j' + \frac{\lambda^2}{12} W_j''' + \left[\frac{\lambda^2 n^2}{12}(2 - \mu) - \frac{\overline{N}_\xi}{B}\right] W_j'\right\}_{\xi=-\frac{s}{2}} \tag{A4g}$$

$$N_{8j} = \left(\mu n V_j + W_j'' - \mu n^2 W_j\right)_{\xi=-\frac{s}{2}} \tag{A4h}$$

where the indices $j = 1, 2, \ldots, 8$.

$\left[Y\right]$ matrices:

$$Y_{1j} = \left(U_j\right)_{\xi=\frac{s}{2}} \tag{A5a}$$

$$Y_{2j} = \left(V_j\right)_{\xi=\frac{s}{2}} \tag{A5b}$$

$$Y_{3j} = \left(W_j\right)_{\xi=\frac{s}{2}} \tag{A5c}$$

$$Y_{4j} = \left(W_j'\right)_{\xi=\frac{s}{2}} \tag{A5d}$$

$$Y_{5j} = \left(U_j\right)_{\xi=-\frac{s}{2}} \tag{A5e}$$

$$Y_{6j} = \left(V_j\right)_{\xi=-\frac{s}{2}} \tag{A5f}$$

$$Y_{7j} = \left(W_j\right)_{\xi=-\frac{s}{2}} \tag{A5g}$$

$$Y_{8j} = \left(W_j'\right)_{\xi=-\frac{s}{2}} \tag{A5h}$$

where the indices $j = 1, 2, \ldots, 8$.

The coefficients of equation (20) are defined as

$$\left.\begin{aligned}
\Lambda_0 &= b_{11}\left[b_{22}b_{33} - \left(b_{23}\right)^2\right] - b_{12}\left(b_{12}b_{33} - b_{13}b_{23}\right) + b_{13}\left(b_{12}b_{23} - b_{13}b_{22}\right) \\
\Lambda_1 &= b_{22}b_{33} - \left(b_{23}\right)^2 + b_{33}b_{11} - \left(b_{13}\right)^2 + b_{11}b_{22} - \left(b_{12}\right)^2 \\
\Lambda_2 &= b_{11} + b_{22} + b_{33}
\end{aligned}\right\} \tag{A6}$$

where

$$b_{11} = \left(\frac{m\pi R}{s}\right)^2 + \frac{1 - \mu}{2} n^2$$

$$b_{12} = \frac{1 + \mu}{2} \frac{m\pi R}{s} n$$

$$b_{13} = -\frac{m\pi R}{s}\left(\frac{R}{R_\xi} + \mu\right)$$

$$b_{22} = \left(1 + \frac{\lambda^2}{12}\right)n^2 + \frac{1 - \mu}{2}\left(1 + \frac{\lambda^2}{3}\right)\left(\frac{m\pi R}{s}\right)^2$$

$$b_{23} = -n\left(1 + \mu \frac{R}{R_\xi}\right) - \frac{\lambda^2}{12} n\left[n^2 + (2 - \mu)\left(\frac{m\pi R}{s}\right)^2\right]$$

$$b_{33} = \frac{\lambda^2}{12}\left[\left(\frac{m\pi R}{s}\right)^2 + n^2\right]^2 + \left(\frac{R}{R_\xi}\right)^2 + 2\mu \frac{R}{R_\xi} + 1 + \frac{\bar{N}_\xi}{B}\left(\frac{m\pi R}{s}\right)^2 + \frac{\bar{N}_\theta}{B} n^2$$

APPENDIX B

AXISYMMETRIC VIBRATIONS

For the particular case of axisymmetric vibrations ($n = 0$), the circumferential equilibrium equation, equation (1b), uncouples from the remaining of equations (1); hence, the torsional frequency is independent of u and w. This equation, written in terms of the nondimensional meridional length ξ, becomes

$$\left(\frac{1-\mu}{2}\right)\left(1+\frac{\lambda^2}{3}\right)v_{,\xi\xi} - \frac{\nu h}{B} v_{,tt} = 0 \qquad \text{(B1)}$$

Since R_ξ does not occur in this equation, the torsional frequency will be independent of the meridional curvature.

The general solution of equation (B1) is

$$v = \left\{v_1 \sin\left[\frac{\Omega\xi}{R\sqrt{\left(\frac{1-\mu}{2}\right)\left(1+\frac{\lambda^2}{3}\right)}}\right] + v_2 \cos\left[\frac{\Omega\xi}{R\sqrt{\left(\frac{1-\mu}{2}\right)\left(1+\frac{\lambda^2}{3}\right)}}\right]\right\} \cos \omega t \qquad \text{(B2)}$$

so that for the boundary conditions $v = 0$ the torsional vibration frequencies are given by

$$\frac{\Omega s}{R\sqrt{\left(\frac{1-\mu}{2}\right)\left(1+\frac{\lambda^2}{3}\right)}} = k\pi \qquad (k = 1, 3, 5, \ldots)$$

The minimum torsional axisymmetric vibration frequency is thus given by

$$\Omega_{torsion} = \frac{\pi R\sqrt{\left(\frac{1-\mu}{2}\right)\left(1+\frac{\lambda^2}{3}\right)}}{s} \qquad \text{(B3)}$$

The two equilibrium equations (1a) and (1c) of equations (1) reduce to

$$\left.\begin{aligned} & u_{,\xi\xi} + \left(\frac{1}{R_\xi} + \frac{\mu}{R}\right)w_{,\xi} - \frac{\nu h}{B} u_{,tt} = 0 \\ & \left(\frac{1}{R_\xi} + \frac{\mu}{R}\right)u_{,\xi} + \frac{\lambda^2}{12} w_{,\xi\xi\xi\xi} + \left(\frac{1}{R_\xi^{\,2}} + \frac{2\mu}{R_\xi R} + \frac{1}{R^2}\right)w - \frac{\overline{N}_\xi}{B} w_{,\xi\xi} + \frac{\nu h}{B} w_{,tt} = 0 \end{aligned}\right\} \qquad \text{(B4)}$$

These equations are handled in the same manner as were equations (1). The characteristic roots of equations (B4) are determined from a sixth degree equation formed by equating the determinant of the matrix which results after deleting the second row and column of the coefficient matrix of equation (12) to zero.

With these modifications, the terms g_j vanish as they should, since there is no longer any interdependency between w and v. Since only six roots are present, the sum of linear solutions in equations (9) range over six rather than eight terms. With the dependence on v removed, the boundary conditions in equations (A4) and (A5) associated with v must be deleted, that is, N_{2j}, N_{6j}, Y_{2j}, and Y_{6j}, leaving three boundary conditions on each edge. The solution of the axisymmetric frequencies is determined in the same manner as is indicated in the body of the text. Since the shell of interest in this report has its torsional frequency as the minimum frequency for $n = 0$ and is thus independent of τ, no calculations for $n = 0$ have been presented.

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