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LONGITUDE REPOSITIONING OF HIGH POWER COMMUNICATION SATELLITES

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16. Abstract

An analysis of the methods and requirements for longitude repositioning of a satellite in synchronous equatorial orbit is presented. The fuel requirement for the repositioning maneuver is a function of the change in longitude, the time of transit from the initial to the final longitude, and the acceleration imparted to the satellite by the propulsion system. For low-thrust electric propulsion, continuous thrust over many orbits may be necessary to effect the required change in longitude.

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SUMMARY

Communication satellites in synchronous equatorial orbit may have to be repositioned from one stationary longitude to another. To minimize fuel consumption, the longitude repositioning maneuver is best effected by changing the orbit period with a thrust impulse collinear with the orbit velocity vector. The change in orbit period produces a steady longitudinal drift of the satellite. When the satellite reaches the desired longitude, a second thrust impulse is applied to restore the synchronous condition. Impulsive thrusting, however, may not be possible if an electric propulsion system is used. Due to very low thrust levels, continuous thrust over many orbits may be necessary to effect a longitude change in a reasonable length of time. The velocity increment ΔV of the repositioning maneuver is a function of the change in longitude, the time of transit from the initial to the final longitude, and the acceleration imparted to the satellite by the propulsion system.

For a fixed change in longitude and time of transit, there exists a minimum acceleration for which longitude repositioning can be accomplished. As the acceleration approaches infinity, the ΔV approaches a value equal to one-half the value corresponding to the minimum acceleration. The nature of this variation in ΔV is such that a major portion of this possible ΔV reduction is obtained for an acceleration of approximately twice the minimum.

For a fixed change in longitude and fixed acceleration, there exists a minimum time of transit for which longitude repositioning can be accomplished. The ΔV for a transit time slightly larger than the minimum is significantly less than the ΔV for the minimum transit time.

The ΔV required to remove any residual eccentricity at the end of the repositioning maneuver is negligible. The Earth's triaxiality has a negligible effect on the ΔV requirement.

INTRODUCTION

Communication satellites in synchronous equatorial orbit may have to be repositioned from one stationary longitude to another. For example the next generation of high power communication satellites may be used for communications over sections of Africa, Europe, and Asia. After some time, the satellite may be shifted in longitude by as much as 180° for communications over North and South America. The systems engineer must allow for the additional fuel required for the repositioning maneuver.

The term "stationwalking" is used in referring to the changing of the stationary position of a synchronous equatorial satellite from one longitude to another. For minimum propellant consumption the stationwalking maneuver is most effectively accomplished by changing the orbit period with a thrust impulse collinear with the orbit velocity vector. The change in orbit period produces a steady longitudinal drift of the satellite. When the satellite reaches the desired longitude, a second thrust impulse is applied to restore the synchronous condition. The second impulse is equal in magnitude but opposite in direction to the first.

Impulsive thrusting, however, may not always be permissible. Communication satellites of the future may use electric propulsion for attitude control and stationkeeping (see ref. 1) in order to keep the total propellant weight to a practical percentage of spacecraft weight. To avoid duplication of hardware, the same propulsion system may also be used for longitude repositioning. Due to very low thrust levels, continuous thrust over many orbits may be necessary to effect a station change.

This report analyzes the stationwalking maneuver including the effects of nonimpulsive thrusting. Design curves are included in the report for calculating the ΔV requirement for the stationwalking maneuver.

SYMBOLS

A	acceleration imparted to satellite by propulsion system
$A_{min}\|_{\Delta L, T}$	minimum acceleration for a given change in longitude and time of transit
a	semimajor axis of orbit
Δa	variation of semimajor axis from semimajor axis of synchronous orbit
e	eccentricity of orbit
e_{max}	maximum eccentricity of orbit during nonimpulsive thrusting
f	true anomaly

L	satellite longitude
ΔL	change in longitude effected by repositioning maneuver
L_1	satellite longitude before repositioning maneuver
L_2	satellite longitude after repositioning maneuver
Δr	variation of orbit geocentric radius from geocentric radius of synchronous orbit
T	time of transit from initial longitude to final longitude
$T_{min}\vert_{\Delta L, A}$	minimum time of transit for a given change in longitude and acceleration
t	time
t_1	time duration of one thrusting period
t_2	time duration of coast period
V_0	velocity of satellite in circular synchronous orbit
ΔV	velocity increment required for repositioning maneuver
$\Delta V'$	velocity increment required for repositioning maneuver with triaxiality effects accounted for
ΔV_e	velocity increment required for removing residual eccentricity
$\Delta V_{max}\vert_{\Delta L, A}$	maximum velocity increment for a given change in longitude and acceleration
$\Delta V_{max}\vert_{\Delta L, T}$	maximum velocity increment for a given change in longitude and time of transit
$\Delta V_{min}\vert_{\Delta L, A}$	minimum velocity increment for a given change in longitude and acceleration
$\Delta V_{min}\vert_{\Delta L, T}$	minimum velocity increment for a given change in longitude and time of transit
γ	satellite longitude relative to nearest minor axis of Earth's equatorial section
$\dot{\theta}$	average angular velocity of osculating orbit
$\Delta\dot{\theta}$	variation of average angular velocity from angular velocity of circular synchronous orbit
$\dot{\theta}_e$	angular velocity of circular synchronous orbit
$\Delta\lambda$	variation of satellite longitude from reference longitude

μ Earth's gravitational constant

φ angular distance of satellite from x-axis

ω longitude of perigee measured from x-axis

Subscript:

0 initial condition

STATIONWALKING MANEUVERS

Impulsive Thrusting

A synchronous satellite has an average angular velocity of 360° per sidereal day. To change the satellite position from longitude L_1 to longitude L_2, the angular velocity, and hence the orbital period, must be modified. For example, suppose that the change in longitude ΔL is 30° and the time of transit T from L_1 to L_2 is 10 days. If the adjustment in orbital period is done impulsively, then the adjusted period must be such as to cause the satellite to drift an average of 3° per day. So the adjusted average angular velocity is either 363° per day or 357° per day, depending on whether the satellite drifts eastward or westward. After 10 days, when the satellite is at longitude L_2, the orbital period must be restored to the synchronous condition.

The orbital period P is given by

$$P = 2\pi\sqrt{\frac{a^3}{\mu}}$$

where a is the semimajor axis and μ is the Earth's gravitational constant. Thus the only way to change the orbit period is to change the semimajor axis. To produce an eastward drift, P must be made smaller and a must be decreased. For westward drift, P must be made larger and a must be increased.

The most effective method of changing the semimajor axis is to thrust eastward or westward. Assume that L_2 is to the east of L_1. For impulsive thrusting, the first impulse must be directed westward in order to decrease the semimajor axis (see fig. 1(a)). An apogee is created at the point of thrust application. After the satellite has drifted to the desired longitude, a second thrust impulse is applied as shown in figure 1(b). The second impulse, equal in magnitude but opposite in direction to the first, restores the synchronous condition. The second impulse must be applied at the apogee of the stationwalking orbit in order to remove the eccentricity. For the case of

L_2 to the west of L_1, the direction of the thrust vectors in figure 1 would be reversed.

Figure 2 shows orbital velocity and satellite longitude as functions of time for the case of L_2 to the east of L_1. The solid portions of the curves represent the average velocity and average satellite longitude. The dashed portions represent the sinusoidal oscillations due to the eccentricity of the stationwalking orbit. Although the first impulse reduces the orbital velocity, the average velocity of the stationwalking orbit is greater than the velocity of the synchronous orbit.

Nonimpulsive Thrusting

If low-thrust propulsion is used for stationwalking, impulsive thrusting is not possible. For nonimpulsive thrusting, the stationwalking maneuver is carried out in three phases. Let us assume that L_2 is to the east of L_1. Let A be the acceleration imparted to the satellite by the thruster. In the first phase, westward thrust is applied for a time t_1 determined by ΔL, T, and A. The second phase is a coast phase during which the satellite drifts steadily eastward for a time t_2. In the third phase, eastward thrust is applied in order to increase the semimajor axis to its synchronous value. If it is assumed that A is the same for both thrust periods, the third-phase thrusting time is again t_1. Figure 3 shows orbital velocity and satellite longitude for the case of $L_1 = 0^\circ$, $L_2 = 170^\circ$, $T = 28$ days, $A = 3.4\times10^{-6}$ g's. The oscillatory components of the velocity and longitude due to orbit eccentricity are not shown in figure 3.

In order to minimize fuel consumption, the thrusting time $2t_1$ should be much smaller than the coast time t_2. For very low thrust levels, however, $2t_1$ may be of the same order of magnitude as t_2 (see fig. 3). In the general case, for a given L_1, L_2, A, and T, longitude as a function of time (neglecting oscillatory components due to eccentricity) is given by

$$L = \mp\left(\frac{3|A|}{2a}\right)t^2 + L_1, \qquad 0 \le t \le t_1 \tag{1a}$$

$$L = \mp\left(\frac{3|A|t_1}{a}\right)t \pm \frac{3|A|t_1^2}{2a} + L_1, \qquad t_1 \le t \le t_1 + t_2 \tag{1b}$$

$$L = \pm\left(\frac{3|A|}{2a}\right)t^2 \mp \left(\frac{3|A|T}{a}\right)t \pm \frac{3|A|}{2a}\left(T^2 - 2t_1t_2 - 2t_1^2\right) + L_1, \qquad t_1 + t_2 \le t \le T \tag{1c}$$

When the double sign, $\pm$ or $\mp$, is used, the upper sign is for the case of L_2 to the east of L_1. The lower sign is for the case of L_2 to the west of L_1. Notice that $L(t)$ is quadratic during the thrust periods and linear during the coast period.

Equations (1) along with the equations of the following sections are derived in appendix A. The following assumptions were made in deriving these equations:

(1) The acceleration level A is constant through both thrusting periods.

(2) Errors in the magnitude and direction of thrust and in the time-off and time-on of the thrusters are neglected.

(3) The effect that the Earth's triaxiality has on satellite longitude is neglected.

Appendix B presents a discussion of how the Earth's triaxiality affects satellite longitude. The analysis shows that for T less than 100 days, triaxiality has a negligible effect on the repositioning maneuver requirements.

STATIONWALKING REQUIREMENTS

ΔV Requirement

If ΔL, T, and A are known parameters, then one can calculate the thrusting time required for the stationwalking maneuver. From appendix A, the thrusting time is given by

$$2t_1 = T - \sqrt{T^2 - \frac{4a}{3}\left|\frac{\Delta L}{A}\right|} \tag{2}$$

The coast time, by the definitions of T and t_1, is

$$t_2 = T - 2t_1$$

The velocity increment ΔV needed to carry out the stationwalking maneuver is

$$\Delta V = 2t_1 A = \left(T - \sqrt{T^2 - \frac{4a}{3}\left|\frac{\Delta L}{A}\right|}\right)A \tag{3}$$

In system design, two cases often occur. In one case ΔL and T are fixed parameters, and the system designer is interested in the functional relation of ΔV and A. In the second case ΔL and A are the fixed parameters, and the functional relation of ΔV and T is desired. These two cases are analyzed in the following sections.

Fixed ΔL and T

If ΔL and T are fixed parameters, the acceleration level A must be greater than a minimum value $A_{min}|_{\Delta L, T}$ which corresponds to the case of continuously thrusting from L_1 to L_2; that is, $2t_1 = T$. The minimum acceleration for fixed ΔL and T is given by (see appendix A)

$$A_{min}\Big|_{\Delta L, T} = \frac{4a|\Delta L|}{3T^2} \tag{4}$$

Figure 4 is a plot of $A_{min}|_{\Delta L, T}$ as a function of $\Delta L/T^2$. By using equation (3), it can be shown that the partial derivative of ΔV with respect to A is less than zero, implying that the ΔV decreases as A increases when ΔL and T are fixed. Equivalently, the ΔV has a maximum value $\Delta V_{max}|_{\Delta L, T}$ when $A = A_{min}|_{\Delta L, T}$ and a minimum value $\Delta V_{min}|_{\Delta L, T}$ when $A = \infty$. From appendix A, $\Delta V_{min}|_{\Delta L, T}$ is one-half of $\Delta V_{max}|_{\Delta L, T}$ or

$$\Delta V_{min}\Big|_{\Delta L, T} = \frac{1}{2}\,\Delta V_{max}\Big|_{\Delta L, T} = \frac{2a|\Delta L|}{3T} \tag{5}$$

Figure 5 is a plot of $\Delta V_{max}|_{\Delta L, T}$ as a function of $\Delta L/T$.

The functional relation of ΔV and A for a fixed ΔL and T is given in normalized form by (see appendix A)

$$\frac{\Delta V}{\Delta V_{max}\Big|_{\Delta L, T}} = \frac{A}{A_{min}\Big|_{\Delta L, T}}\left[1 - \sqrt{1 - \frac{A_{min}\Big|_{\Delta L, T}}{A}}\right] \tag{6}$$

Figure 6 is a plot of $\Delta V/\Delta V_{max}|_{\Delta L, T}$ as a function of $A/A_{min}|_{\Delta L, T}$. By differentiating equation (6), it can be shown that the derivative of $\Delta V/\Delta V_{max}|_{\Delta L, T}$ evaluated at $A/A_{min}|_{\Delta L, T} = 1$ is infinite, indicating that a significant saving in ΔV is achieved when A is only slightly larger than $A_{min}|_{\Delta L, T}$. A major portion of the possible reduction in ΔV is achieved when A is approximately twice $A_{min}|_{\Delta L, T}$. Figures 4, 5, and 6 can be used to calculate ΔV for given values of ΔL, T, and A. Figures 4 and 5 are used to find $A_{min}|_{\Delta L, T}$ and $\Delta V_{max}|_{\Delta L, T}$. Knowing $\Delta V_{max}|_{\Delta L, T}$, $A_{min}|_{\Delta L, T}$, and A, the ΔV can be found from figure 6.

Fixed ΔL and A

If ΔL and A are fixed parameters, the time of transit T must be greater than a minimum value $T_{min}|_{\Delta L,A}$ which corresponds to the case of continuously thrusting from L_1 to L_2. From appendix A, the minimum transit time for fixed ΔL and A is given by

$$T_{min}|_{\Delta L,A} = \sqrt{\frac{4a|\Delta L|}{3A}} \tag{7}$$

Figure 7 is a plot of $T_{min}|_{\Delta L,A}$ as a function of $\Delta L/A$. By using equation (3), it can be shown that the partial derivative of ΔV with respect to T is less than zero, implying that the ΔV decreases as T increases when ΔL and A are fixed. Equivalently, the ΔV has a maximum value $\Delta V_{max}|_{\Delta L,A}$ when $T = T_{min}|_{\Delta L,A}$ and a minimum value $\Delta V_{min}|_{\Delta L,A}$ when $T = \infty$. From appendix A, $\Delta V_{min}|_{\Delta L,A}$ and $\Delta V_{max}|_{\Delta L,A}$ are given by

$$\Delta V_{min}|_{\Delta L,A} = 0 \tag{8a}$$

$$\Delta V_{max}|_{\Delta L,A} \doteq \sqrt{\frac{4aA|\Delta L|}{3}} \tag{8b}$$

Figure 8 is a plot of $\Delta V_{max}|_{\Delta L,A}$ as a function of $A\ \Delta L$.

The functional relation of ΔV and T for a fixed ΔL and A is given in normalized form by (see appendix A)

$$\frac{\Delta V}{\Delta V_{max}|_{\Delta L,A}} = \frac{T}{T_{min}|_{\Delta L,A}}\left[1 - \sqrt{1 - \left(\frac{T_{min}|_{\Delta L,A}}{T}\right)^2}\right] \tag{9}$$

Figure 9 is a plot of $\Delta V/\Delta V_{max}|_{\Delta L,A}$ as a function of $T/T_{min}|_{\Delta L,A}$. The functional relation of ΔV and T for a fixed ΔL and A is similar to the functional relation of ΔV and A for a fixed ΔL and T (compare eq. (9) with eq. (6) and fig. 9 with fig. 6). One significant difference is that $\Delta V/\Delta V_{max}|_{\Delta L,A} \to 0$ as $T/T_{min}|_{\Delta L,A} \to \infty$, whereas $\Delta V/\Delta V_{max}|_{\Delta L,T} \to \frac{1}{2}$ as $A/A_{min}|_{\Delta L,T} \to \infty$.

By differentiating equation (9), it can be shown that the derivative of $\Delta V/\Delta V_{max}|_{\Delta L,A}$ evaluated at $T/T_{min}|_{\Delta L,A} = 1$ is infinite, indicating that a significant saving in ΔV is achieved when T is slightly larger than $T_{min}|_{\Delta L,A}$. Figures

7, 8, and 9 can be used to calculate ΔV for given values of ΔL, T, and A. Figures 7 and 8 are used to find $T_{min}|_{\Delta L,A}$ and $\Delta V_{max}|_{\Delta L,A}$. Knowing $\Delta V_{max}|_{\Delta L,A}$, $T_{min}|_{\Delta L,A}$, and T, the ΔV can be found from figure 9.

If the ratio of propellant weight to spacecraft weight is assumed small, the propellant weight as a function of ΔV is given by

$$W_p = \frac{W\ \Delta V}{gI_{sp}} \tag{10}$$

where W_p is the propellant weight, W is the spacecraft weight, g is the acceleration of gravity, and I_{sp} is the specific impulse. Equation (10) can be used to calculate the propellant weight once the ΔV is known.

Orbit Eccentricity During Thrusting

Any residual eccentricity remaining at the end of the stationwalking maneuver must be removed. With the assumption that the orbit is circular when thrusting begins, eccentricity during the thrusting phase is given by

$$e(t) = e_{max}\left|\sin\frac{\dot{\theta}_e t}{2}\right| \tag{11}$$

where $\dot{\theta}_e$ is the angular velocity of the synchronous orbit and e_{max} is

$$e_{max} = \frac{4|A|}{a\dot{\theta}_e^2} \tag{12}$$

Equation (11), which is derived in appendix A, is plotted in figure 10. If nonimpulsive thrusting is used, both thrusting periods should be an integral number of days so that the stationwalking orbit and final synchronous orbit will be circular. If the thrusting periods are not an integral number of days, the residual eccentricity will not be greater than e_{max}. Let ΔV_e be the velocity increment needed to remove the residual eccentricity. From reference 1, ΔV_e is bounded by

$$\Delta V_e \leq \Delta V_{e_{max}} = \frac{e_{max} a\dot{\theta}_e}{2} \tag{13}$$

Substituting equation (12) into equation (13) results in

$$\Delta V_e \leq \frac{2|A|}{\dot{\theta}_e} \tag{14}$$

For A expressed in g's and ΔV_e in meters per second, equation (14) becomes

$$\Delta V_e\left(\frac{m}{sec}\right) \leq 2.9\times10^6 |A| (g's) \tag{15}$$

From equation (15), ΔV_e is insignificant for very low accelerations (less than 10^{-5} g's). For larger accelerations, the upper bound for ΔV_e may become significant. It should be noted, however, that by a judicious choice of thrusting times, the eccentricity at thrust cutoff can be made much closer to zero than to e_{max}.

CONCLUDING REMARKS

The purpose of this report is to analyze the methods and requirements for longitude repositioning of a satellite in synchronous equatorial orbit. The method of longitude repositioning consists of an initial thrusting phase, a coast phase, and a final thrusting phase. The thrust vector is directed eastward in one thrusting phase and westward in the other.

The requirements for longitude repositioning are given in terms of the velocity increment ΔV of the maneuver. The ΔV is a function of the change in longitude, the time of transit from the initial to the final longitude, and the acceleration imparted to the satellite by the propulsion system.

For a fixed change in longitude and time of transit, there exists a minimum acceleration for which longitude repositioning can be accomplished. As the acceleration approaches infinity the ΔV approaches a value equal to one-half the value corresponding to the minimum acceleration. The nature of this variation in ΔV is such that a major portion of this possible ΔV reduction is obtained for an acceleration of approximately twice the minimum.

For a fixed change in longitude and fixed acceleration, there exists a minimum time of transit for which longitude repositioning can be accomplished. As the time of transit approaches infinity, the ΔV approaches zero. The nature of this variation in ΔV is such that a major portion of the possible ΔV reduction is obtained for a transit time of approximately twice the minimum.

The ΔV required to remove any residual eccentricity at the end of the repositioning maneuver is negligible. The Earth's triaxiality has a negligible effect on the ΔV requirement.

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APPENDIX A

DERIVATION OF EQUATIONS

Assume that ΔL, T, and A are known quantities and that t_1, t_2, and ΔV are to be calculated. First we must find the change in semimajor axis due to tangential thrusting. From reference 2, the time rate of change of the semimajor axis for a nearly circular orbit is

$$\frac{da}{dt} = \frac{2A}{\dot{\theta}} \tag{A1}$$

where $\dot{\theta}$ is the average angular velocity of the osculating orbit. The acceleration A is assumed positive for eastward thrust and negative for westward thrust. If changes in $\dot{\theta}$ are assumed to be small, Δa is given by

$$\Delta a = \frac{2At}{\dot{\theta}} \tag{A2}$$

where t is the time of thrusting.

We now wish to find a relation between Δa and the change in average angular velocity $\Delta\dot{\theta}$. The average angular velocity is given by

$$\dot{\theta} = \sqrt{\frac{\mu}{a^3}} \tag{A3}$$

where μ is the Earth's gravitational constant. If the variations Δa and $\Delta\dot{\theta}$ are assumed small, we obtain

$$\Delta\dot{\theta} = \frac{-3}{2a}\sqrt{\frac{\mu}{a^3}}\,\Delta a = \frac{-3\dot{\theta}}{2a}\,\Delta a \tag{A4}$$

Substituting equation (A2) into equation (A4) gives

$$\Delta\dot{\theta} = \frac{-3At}{a} \tag{A5}$$

The stationwalking maneuver is carried out in three phases. The first and third phases are thrusting phases, and the second phase is a coast. Let ΔL_a be the change in longitude during the first phase, ΔL_b the change in longitude during the second phase, and ΔL_c the change in longitude during the third phase. We now proceed to find ΔL_a, ΔL_b, ΔL_c as functions of t_1 and t_2. The change in longitude ΔL_a is given by

$$\Delta L_a = \int_0^{t_1} \Delta\dot{\theta}\, dt \tag{A6}$$

Substituting equation (A5) into equation (A6) and carrying out the integration gives

$$\Delta L_a = \frac{-3At_1^2}{2a} \tag{A7}$$

During the coasting phase, $\Delta\dot{\theta}$ is constant and is equal to

$$\Delta\dot{\theta} = \frac{-3At_1}{a} \tag{A8}$$

The change in longitude ΔL_b is given by

$$\Delta L_b = \int_0^{t_2} \Delta\dot{\theta}\, dt \tag{A9}$$

Substituting equation (A8) into equation (A9) and carrying out the integration results in

$$\Delta L_b = \frac{-3At_1t_2}{a} \tag{A10}$$

We make the convention that the sign of A is determined by the direction of thrust during the first phase. During the third phase, the direction of thrust is reversed. So $\Delta\dot{\theta}$ during the third phase, from equations (A5) and (A8), is given by

$$\Delta\dot{\theta} = \frac{-3At_1}{a} + \frac{3At}{a} \tag{A11}$$

where t is the time elapsed in the third phase. If it is recalled that the first and third stages are of the same duration, then

$$\Delta L_c = \int_0^{t_1} \Delta\mathring{\theta}\, dt \tag{A12}$$

Substituting equation (A11) into equation (A12) and carrying out the integration gives

$$\Delta L_c = \frac{-3At_1^2}{a} + \frac{3At_1^2}{2a} \tag{A13}$$

By adding equations (A7), (A10), and (A13), we obtain

$$\Delta L = \Delta L_a + \Delta L_b + \Delta L_c = \frac{-3At_1t_2}{a} - \frac{3At_1^2}{a} \tag{A14}$$

Now t_2 is given by

$$t_2 = T - 2t_1 \tag{A15}$$

Substituting equation (A15) into equation (A14) yields

$$\Delta L = \frac{3A}{a}\left(t_1^2 - Tt_1\right) \tag{A16}$$

Rearranging terms results in

$$t_1^2 - Tt_1 - \frac{a\ \Delta L}{3A} = 0 \tag{A17}$$

Using the quadratic formula, we solve for the thrusting time $2t_1$ as follows:

$$2t_1 = T - \sqrt{T^2 + \frac{4a\ \Delta L}{3A}} \tag{A18}$$

From equation (A14), it is clear that ΔL and A are of opposite sign. So equation (A18) can be written as

$$2t_1 = T - \sqrt{T^2 - \frac{4a}{3}\left|\frac{\Delta L}{A}\right|} \tag{A19}$$

The ΔV is then given by

$$\Delta V = 2t_1 |A| = \left(T - \sqrt{T^2 - \frac{4a|\Delta L|}{3|A|}}\right)|A| \tag{A20}$$

We now wish to find $\Delta V_{min}|_{\Delta L, T}$, the minimum ΔV for a fixed ΔL and T. By taking the partial derivative of equation (A20) with respect to $|A|$, it can be shown that

$$\frac{\partial(\Delta V)}{\partial |A|} < 0 \tag{A21}$$

for all permissible values of ΔL, T, and A. Thus equation (A20) is a monotonically decreasing function of $|A|$, implying that the minimum ΔV occurs when $|A|$ is infinite. By taking the limit of equation (A20) as $|A|$ approaches infinity, $\Delta V_{min}|_{\Delta L, T}$ is calculated to be

$$\Delta V_{min}|_{\Delta L, T} = \frac{2a}{3}\left|\frac{\Delta L}{T}\right| \tag{A22}$$

Equation (A21) implies that $\Delta V_{max}|_{\Delta L, T}$, the maximum ΔV for a fixed ΔL and T, occurs when A is equal to its minimum permissible value $A_{min}|_{\Delta L, T}$. The minimum acceleration occurs when $2t_1 = T$, or when the expression under the square root sign in equation (A19) is zero. Solving for $A_{min}|_{\Delta L, T}$ gives

$$A_{min}|_{\Delta L, T} = \frac{4a|\Delta L|}{3T^2} \tag{A23}$$

By evaluating equation (A20) when $A = A_{min}|_{\Delta L, T}$, we obtain an expression for $\Delta V_{max}|_{\Delta L, T}$

$$\Delta V_{max}|_{\Delta L, T} = \frac{4a}{3}\left|\frac{\Delta L}{T}\right| \tag{A24}$$

Equations (A22) and (A24) imply that

$$\Delta V_{max}\Big|_{\Delta L, T} = 2\Delta V_{min}\Big|_{\Delta L, T} \tag{A25}$$

Combining equations (A20), (A23), and (A24) gives

$$\frac{\Delta V}{\Delta V_{max}\Big|_{\Delta L, T}} = \frac{A}{A_{min}\Big|_{\Delta L, T}}\left[1 - \sqrt{1 - \frac{A_{min}\Big|_{\Delta L, T}}{A}}\right] \tag{A26}$$

We now wish to find $\Delta V_{max}\Big|_{\Delta L, A}$, the maximum ΔV for a fixed ΔL and A. By taking the partial derivative of equation (A20) with respect to T, it can be shown that

$$\frac{\partial(\Delta V)}{\partial T} < 0 \tag{A27}$$

for all permissible values of ΔL, T, and A. Thus equation (A20) is a monotonically decreasing function of T, implying that the maximum ΔV occurs when T is equal to its minimum permissible value, $T_{min}\Big|_{\Delta L, A}$. The minimum time occurs when the expression under the square root sign in equation (A19) is zero. Solving for $T_{min}\Big|_{\Delta L, A}$ results in

$$T_{min}\Big|_{\Delta L, A} = \sqrt{\frac{4a}{3}\left|\frac{\Delta L}{A}\right|} \tag{A28}$$

Evaluating equation (A20) when $T = T_{min}\Big|_{\Delta L, A}$, we obtain the following expression for $\Delta V_{max}\Big|_{\Delta L, A}$:

$$\Delta V_{max}\Big|_{\Delta L, A} = \sqrt{\frac{4a|A\ \Delta L|}{3}} \tag{A29}$$

The minimum ΔV occurs when T is infinite. Taking the limit of equation (A20) as T approaches infinity gives

$$\Delta V_{min}\Big|_{\Delta L, A} = 0 \tag{A30}$$

Combining equations (A20), (A28), and (A29) yields

$$\frac{\Delta V}{\Delta V_{max}\Big|_{\Delta L, A}} = \frac{T}{T_{min}\Big|_{\Delta L, A}}\left[1 - \sqrt{1 - \left(\frac{T_{min}\Big|_{\Delta L, A}}{T}\right)^2}\right] \tag{A31}$$

Equations will now be derived for finding orbit eccentricity during the time of thrusting. The coordinate system adopted is shown in figure 11, where the x-y system is an inertial reference with origin at the Earth's center. The x-y plane is the equatorial plane. The angle ω is the longitude of perigee measured from the x-axis, f is the true anomaly, and φ is the angular distance of the satellite from the x-axis. From figure 11 the angles ω, f, and φ are related by the equation

$$f = \varphi - \omega \tag{A32}$$

The point B on the Earth's equator is assumed to lie on the positive x-axis at $t = 0$. Thus, at time t, the angle between the line OB and the positive x-axis is $\dot{\theta}_e t$, where $\dot{\theta}_e$ is the angular velocity of the Earth's rotation (360^o per sidereal day). Let $\Delta\lambda(t)$ be the variation at time t of the satellite longitude from the longitude of point B. Then $\Delta\lambda(t)$ is given by

$$\Delta\lambda(t) = \varphi(t) - \dot{\theta}_e t \tag{A33}$$

We now proceed to find eccentricity as a function of time with thrust beginning at $t = 0$. We first derive equations for $d(\Delta\lambda)/dt$, de/dt, and $d\omega/dt$. For an orbit with small eccentricity, $d\varphi/dt$ is given to first order in e by (ref. 2)

$$\frac{d\varphi}{dt} = \dot{\theta} + 2e\dot{\theta} \cos f \tag{A34}$$

where $\dot{\theta}$ is the average angular velocity of the osculating orbit. The variation in $\dot{\theta}$ is given by

$$\Delta\dot{\theta} = \dot{\theta} - \dot{\theta}_e \tag{A35}$$

Using equations (A32) to (A35) and recalling that $\dot{\theta} \approx \dot{\theta}_e$ result in

$$\frac{d(\Delta\lambda)}{dt} = \Delta\dot{\theta} + 2e\dot{\theta}_e \cos(\Delta\lambda + \dot{\theta}_e t - \omega) \tag{A36}$$

From equation (A5), $\Delta\dot{\theta}$ is given by

$$\Delta\dot{\theta} = \frac{-3At}{a}$$

where A is the acceleration and a is the semimajor axis. The acceleration A is assumed positive for eastward thrust and negative for westward thrust. Equation (A36) can now be written as

$$\frac{d(\Delta\lambda)}{dt} = \frac{-3At}{a} + 2e\dot{\theta}_e \cos(\Delta\lambda + \dot{\theta}_e t - \omega) \tag{A37}$$

From reference 2, de/dt and $d\omega/dt$ are given to first order in e by

$$\frac{de}{dt} = \frac{2A \cos f}{a\dot{\theta}_e} = \frac{2A \cos(\Delta\lambda + \dot{\theta}_e t - \omega)}{a\dot{\theta}_e} \tag{A38}$$

$$\frac{d\omega}{dt} = \frac{2A \sin f}{ea\dot{\theta}_e} = \frac{2A \sin(\Delta\lambda + \dot{\theta}_e t - \omega)}{ea\dot{\theta}_e} \tag{A39}$$

Equations (A37), (A38), (A39) from a coupled system of three first-order differential equations. If we assume that the change in $\Delta\lambda$ is small over a one-orbit period, then equations (A38) and (A39) form a coupled system of two equations to be solved over a one-orbit period

$$\frac{de}{dt} = \frac{2A \cos(\Delta\lambda + \dot{\theta}_e t - \omega)}{a\dot{\theta}_e} \tag{A40}$$

$$\frac{d\omega}{dt} = \frac{2A \sin(\Delta\lambda + \dot{\theta}_e t - \omega)}{ea\dot{\theta}_e} \tag{A41}$$

In equations (A40) and (A41), $\Delta\lambda$ is assumed to remain approximately equal to its initial value $\Delta\lambda_0$. To this point, we have not specified initial conditions. We now impose the initial condition that $e_0 = 0$. To determine ω_0, consider the case where the thrust is in the eastward direction. At $t = 0$, the satellite's angular distance from the x-axis is found from equation (A33) to be $\Delta\lambda_0 = \varphi(0)$. An eastward tangential thrust applied to a circular orbit at the angular position $\Delta\lambda_0$ will create an instantaneous perigee at that position. Thus, $\omega_0 = \Delta\lambda_0$. For thrust in the westward direction, $\omega_0 = \Delta\lambda_0 + \pi$. The solution to equations (A40) and (A41) satisfying the initial conditions is found by assuming ω is a linear function of time

$$e(t) = \frac{4|A|}{a\dot{\theta}_e^2} \sin \frac{\dot{\theta}_e t}{2} \tag{A42}$$

$$\omega(t) = \frac{\dot{\theta}_e t}{2} + \omega_0 \tag{A43}$$

Substituting equation (A42) into equation (A37) and integrating, we obtain the following solution for $\Delta\lambda(t)$:

$$\Delta\lambda(t) = \Delta\lambda_0 - \frac{3At^2}{2a} + \frac{4A}{a\dot{\theta}_e^2} (1 - \cos \dot{\theta}_e t) \tag{A44}$$

As an application of equations (A42) to (A44), consider first the case where $t = 0$ corresponds to the beginning of thrusting. Equation (A42) implies that eccentricity is a half sine wave over a 24-hour period with a maximum value of

$$e_{max} = \frac{4|A|}{a\dot{\theta}_e^2} \tag{A45}$$

At the end of the 24-hour period, the eccentricity is again 0. We now re-initialize the problem, the only change being that $\Delta\lambda_0$ is no longer $\varphi(0)$. From equation (A44), the new $\Delta\lambda_0$ is given by

$$\Delta\lambda_0 = \varphi(0) - \frac{3A}{2a} (24 \text{ hr})^2 = \varphi(0) - \frac{6\pi^2 A}{a\dot{\theta}_e^2} \tag{A46}$$

Again the solution for e is a half sine wave over the second 24-hour period. By continuing the process, the eccentricity will be a succession of half sine waves with amplitude given by equation (A45).

APPENDIX B

EFFECT OF TRIAXIALITY ON STATIONWALKING MANEUVERS

An analysis of the effect of the Earth's triaxiality on a synchronous satellite is given in reference 3. A brief synopsis of that analysis is presented here. The Earth's equatorial cross section is approximately an ellipse whose minor axis passes through 74.6° east longitude and 105.4° west longitude (ref. 4). These two longitudes are stable points. Neglecting other perturbations, a synchronous satellite placed at either of these longitudes will tend to stay there. If a satellite is positioned at any other longitude, it will undergo a longitudinal oscillation about the nearest minor axis. A simultaneous oscillation in orbit radius also occurs. The period of the oscillations in longitude and radius is greater than 2.2 years.

In this appendix, the symbol γ will denote longitude relative to the nearest minor axis of the Earth's equatorial section. Figure 12 presents the oscillations in radius Δr and longitude γ for two cases. The initial longitude γ_0 is 45° in the first case and 25° in the second case. The period of oscillation is 2.7 years for $\gamma_0 = 45^{\circ}$ and 2.3 years for $\gamma_0 = 25^{\circ}$.

In figure 13, we present another $\Delta r - \gamma$ curve, except now we neglect triaxiality effects and consider what happens when stationwalking from longitude γ_1 to longitude γ_2. We assume that the thrusting time is much smaller than the coasting time. If $T = 30$ days is the given time of transit from γ_1 to γ_2, then the first thrusting period would produce a change Δr in the orbit radius. This maneuver corresponds to moving from point A to point B in figure 13. During the coast phase, we move from B to C. In the second thrusting period, the synchronous radius is restored and we move from C to D. Now assume that the time of transit is $T' = 15$ days, where $T' < T$. Then the change in radius $\Delta r'$ will have to be greater than Δr. The path followed is AB'C'D instead of ABCD.

Figure 14 presents the $\Delta r - \gamma$ curve when the triaxiality effect is taken into account. Again assume that the time of transit is T or T'. The triaxiality produces a change in radius during the coast phase. So the first change in radius, Δr_1 or $\Delta r_1'$, will not be the same as the second change, Δr_2 or $\Delta r_2'$. The dashed portion of the curve represents the path that would occur if there were no second thrusting period. Here it is assumed that Δr_1, $\Delta r_1'$, Δr_2, $\Delta r_2'$ are less than 30 kilometers. For changes in radius greater than 30 kilometers, the triaxiality effect is not great enough to "capture" the satellite and it would continue drifting westward in the absence of a second thrusting period.

Assume that γ_1, γ_2, and T are given parameters. Let Δr be the necessary change in radius, neglecting the triaxiality effect. Let Δr_1 and Δr_2 be the first and second changes in radius when triaxiality is taken into account. Let ΔV be the velocity increment for the stationwalking maneuver when triaxiality is neglected. Let $\Delta V'$ be the velocity increment when triaxiality is taken into account. Table I presents an array

TABLE I. - DATA FOR CASE OF $\gamma_1 = 20^o$ AND $\gamma_2 = 10^o$

Time, T, days	Velocity increment when triaxiality is neglected, ΔV, m/sec	Velocity increment when triaxiality is accounted for, $\Delta V'$, m/sec	Change in radius when triaxiality is neglected, Δr, m	First change in radius when triaxiality is accounted for, Δr_1, m	Second change in radius when triaxiality is accounted for, Δr_2, m
26.0	2.19	2.18	30 000	29 000	30 700
46.2	1.23	1.22	16 900	15 200	18 200
66.4	.86	.84	11 700	9 300	13 700
86.5	.66	.64	9 000	5 700	11 600
106.7	.53	.50	7 300	3 300	10 600
126.9	.45	.41	6 100	1 100	10 200

of values of T, ΔV, $\Delta V'$, Δr, Δr_1, and Δr_2 for the case $\gamma_1 = 20^o$ and $\gamma_2 = 10^o$. The thrusting time is assumed to be much smaller than T. As T becomes larger, the difference between Δr_1 and Δr_2 becomes larger. Notice, however, that ΔV is only slightly larger than $\Delta V'$ in all cases, the reason being that the sum of Δr_1 and Δr_2 is nearly equal to 2 Δr in all cases.

One would suspect that $\Delta V - \Delta V'$ is greatest when $\gamma_1 = -\gamma_2$. In this case the stable longitude is midway between γ_1 and γ_2, and $\Delta V' = 0$ if T is equal to one-half the period of oscillation. But T would have to be greater than 1 year in such a case. For T less than 100 days, ΔV and $\Delta V'$ are nearly the same, even for the case $\gamma_1 = -\gamma_2$. Figure 15 presents curves of ΔV and $\Delta V'$ when $\gamma_1 = -\gamma_2$.

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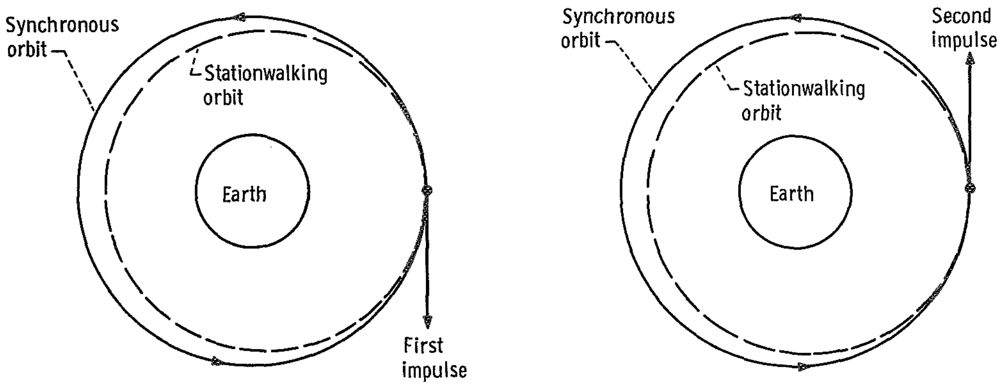
3. Frick, R. H.; and Garber, T. B.: Perturbations of a Synchronous Satellite Due to Triaxiality of the Earth. J. Aerospace Sci., vol. 29, no. 9, Sept. 1962, pp. 1105-1111, 1144.

4. Wagner, C. A.: Longitude Variations of the Earth's Gravity Field as Sensed by the Drift of Three Synchronous Satellites. J. Geophys. Res., vol. 71, no. 6, March 15, 1966, pp. 1703-1711.



(a) First impulse. (b) Second impulse.

Figure 1. - Stationwalking maneuver with impulsive thrusting.

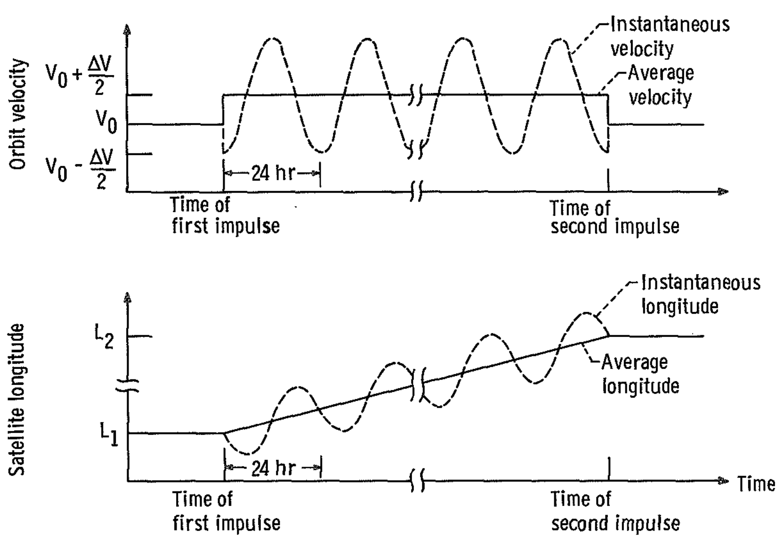


Figure 2. - Orbit velocity and satellite longitude as functions of time for impulsive thrusting maneuver. L_2 is east of L_1.

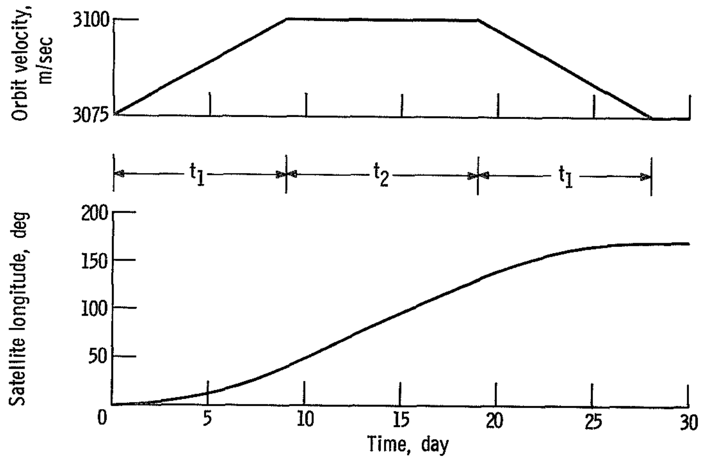


Figure 3. - Orbit velocity and satellite longitude as functions of time for nonimpulsive thrusting maneuver. L_2 is east of L_1. Oscillatory components due to eccentricity are not shown.

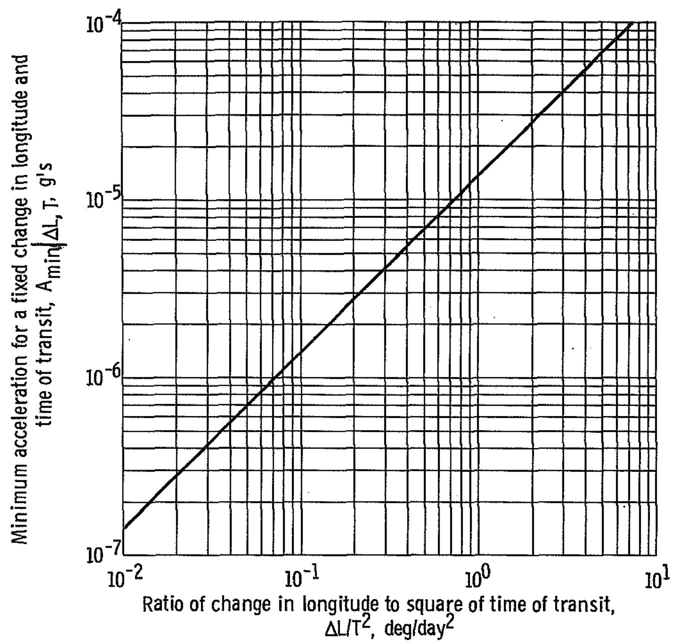


Figure 4. - Minimum acceleration for a fixed change in longitude and time of transit as function of ratio of change in longitude to square of time of transit.

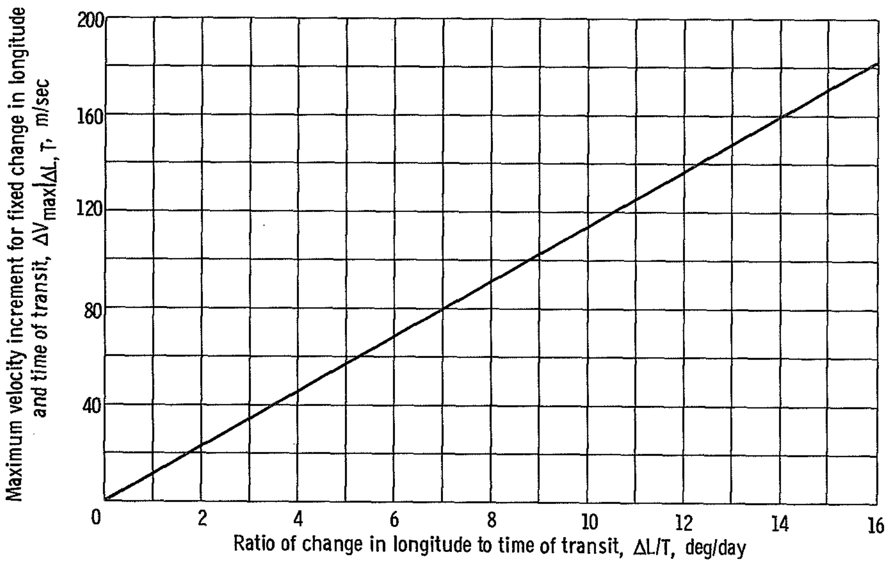


Figure 5. - Maximum velocity increment for fixed change in longitude and time of transit as function of ratio of change in longitude to time of transit.

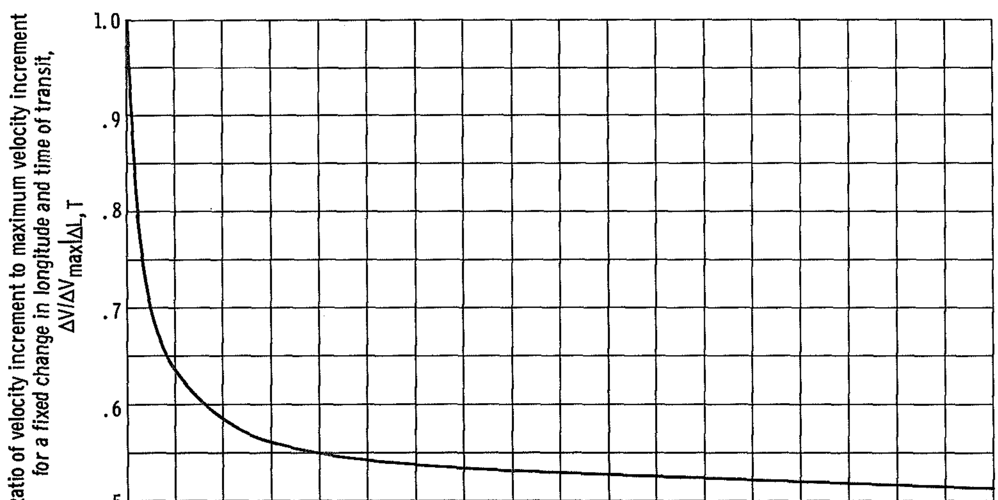


Figure 6. - Normalized velocity increment as function of normalized acceleration. Normalization is with respect to fixed change in longitude and time of transit.

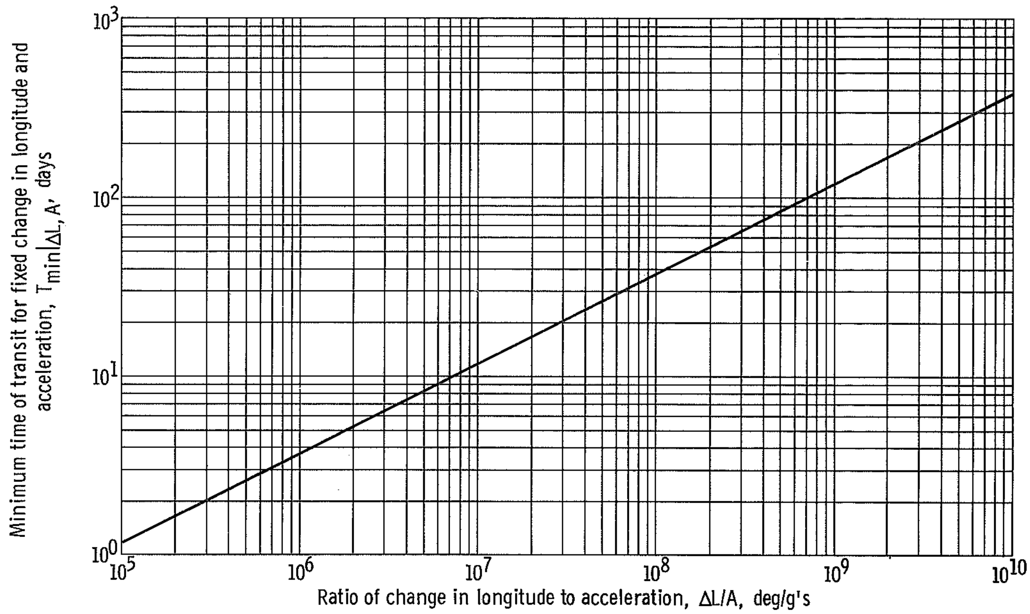


Figure 7. - Minimum time of transit for fixed change in longitude and acceleration as a function of the ratio of change in longitude to acceleration.

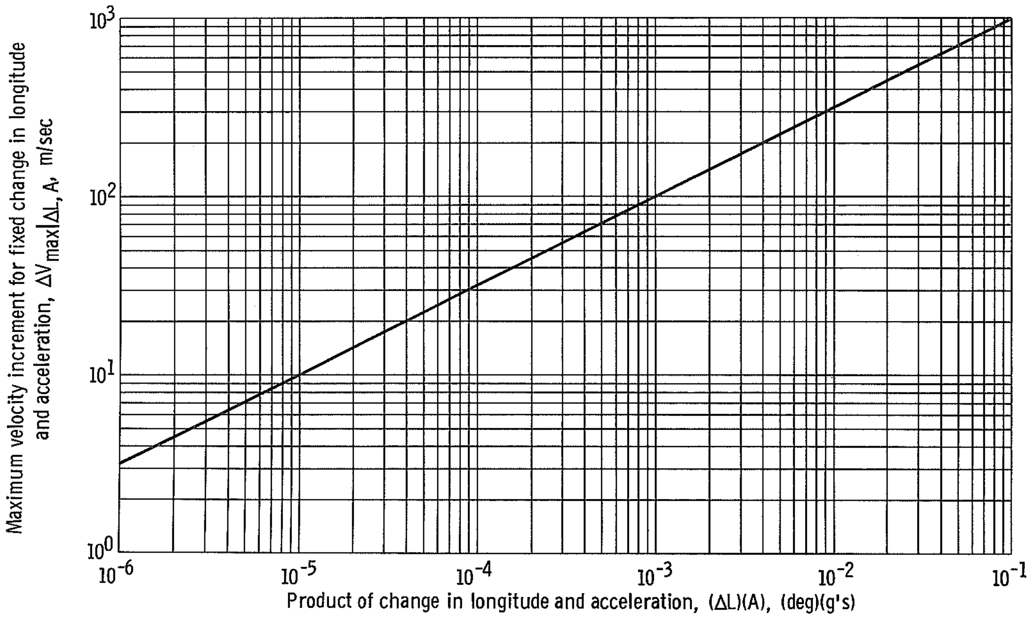


Figure 8. - Maximum velocity increment for fixed change in longitude and acceleration as a function of the product of change in longitude and acceleration.

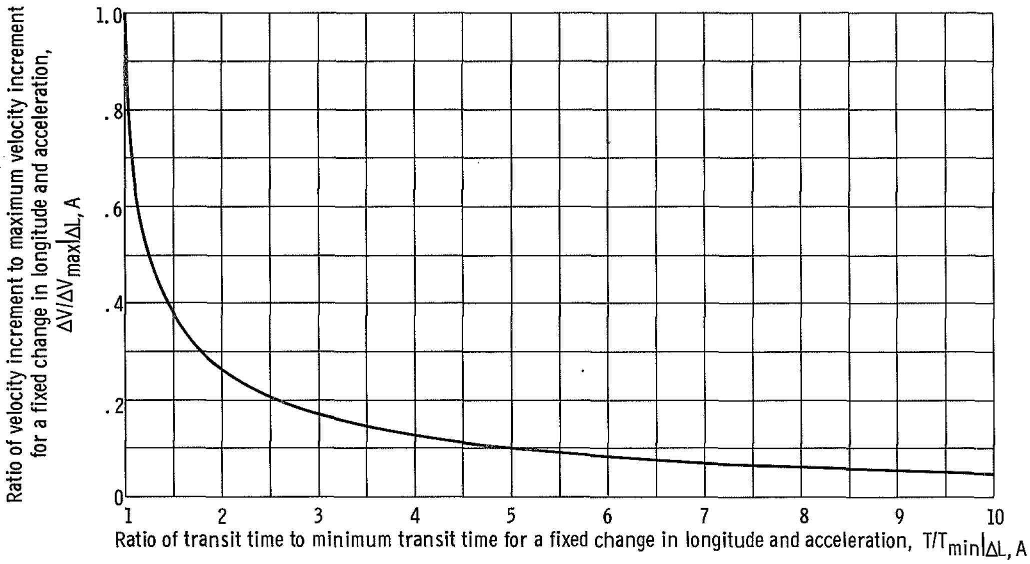


Figure 9. - Normalized velocity increment as function of normalized transit time. Normalization is with respect to fixed change in longitude and acceleration.

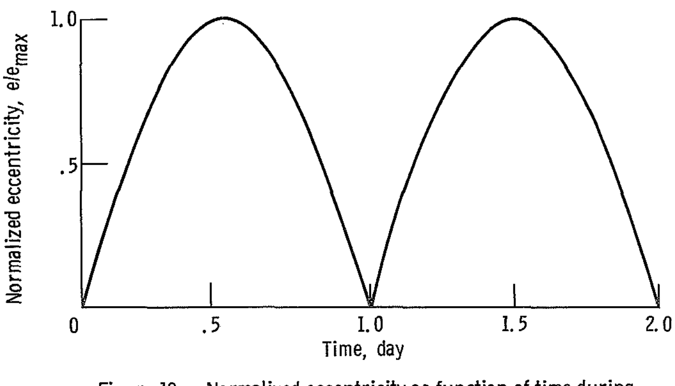


Figure 10. - Normalized eccentricity as function of time during thrusting period. $e/e_{max} = |\sin(\dot{\theta}_e t/2)|$; $e_{max} = 4|A|/a\dot{\theta}_e^2$.

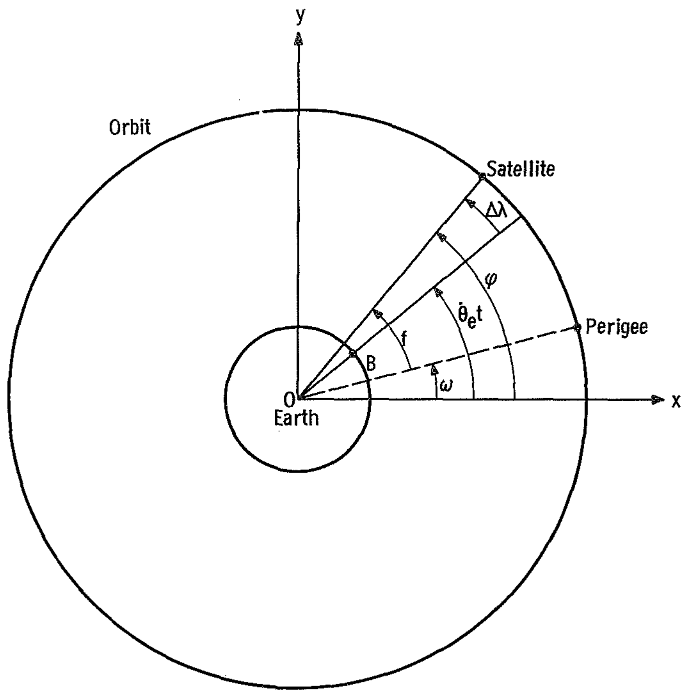


Figure 11. - Coordinate system.

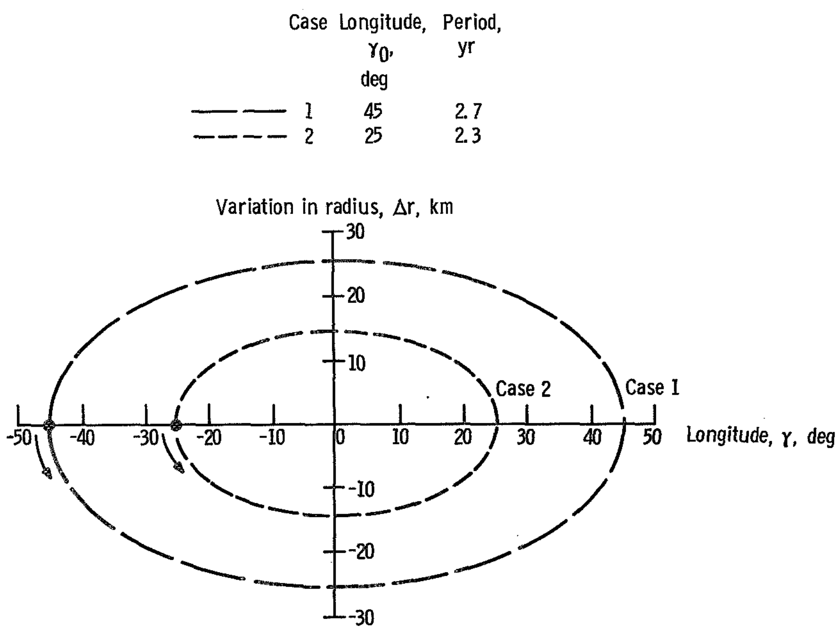


Figure 12. - Variation in radius as function of longitude; $\Delta r_0 = 0$.

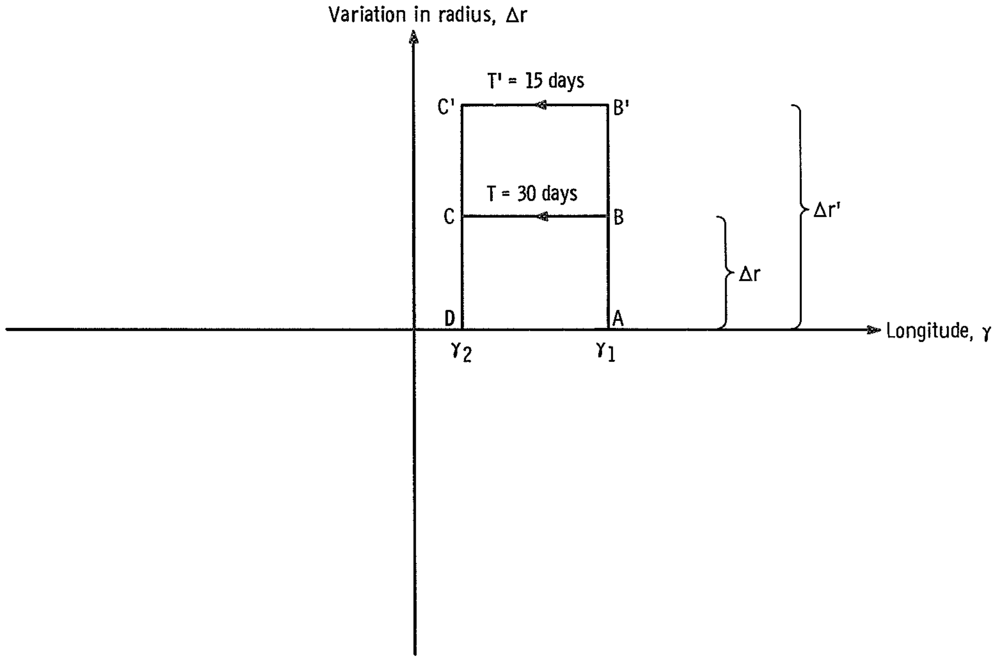


Figure 13. - Variation in radius as function of longitude for two stationwalking maneuvers. Triaxiality effects are neglected.

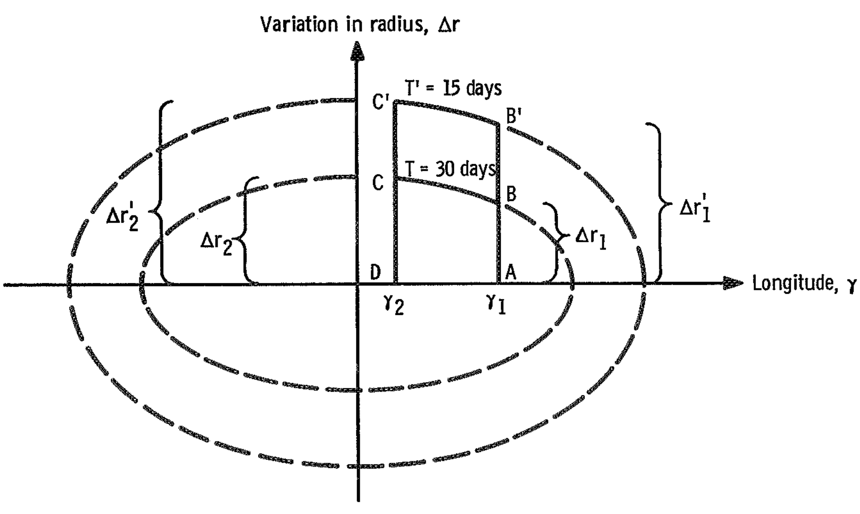


Figure 14. - Variation in radius as function of longitude for two stationwalking maneuvers. Triaxiality effects are accounted for.

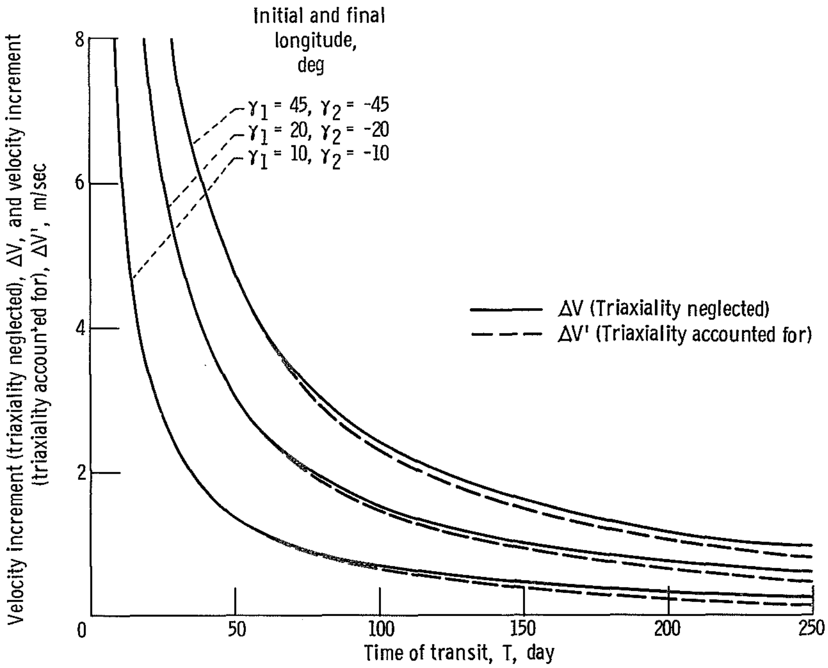


Figure 15. - Velocity increments with and without triaxiality effects as functions of time of transit.

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