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OPTIMAL CONTROL USING IMBEDDING OF THE TERMINAL CONDITIONS

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16. Abstract

Imbedding theory is used to construct solutions of autonomous optimal control problems with free termination time; this is accomplished by imbedding the trajectory optimization problem of interest in a continuous family of problems which are parameterized by terminal conditions. Necessary conditions are derived which determine the modifications in the control functions required when passing from the solution corresponding to one member of the family to that corresponding to another member when their terminal conditions differ infinitesimally. By continuously collapsing the terminal conditions of the family of problems onto those of the original problem while appropriately modifying the control function, the solution to the original optimization problem can be obtained.

A problem is solved for finding the time-optimal maneuvers for an airplane to maneuver in such a way that the target vehicle (assumed to be nonmaneuvering) is placed in a situation favorable to deployment of the airplane's armament. The solutions to this problem are composed of both singular and nonsingular subarcs of the trajectory. Thus, the utility of imbedding in solving singular optimal control problems is illustrated.

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SUMMARY

Imbedding theory is used to construct solutions of autonomous optimal control problems with free termination time; this is accomplished by imbedding the trajectory optimization problem of interest in a continuous family of problems which are parameterized by terminal conditions. The imbedding family used is quite arbitrary but it must contain the original problem as one member and a problem that possesses a known solution as another member.

Necessary conditions are derived which determine the modifications in the control functions required when passing from the solution corresponding to one member of the family to that corresponding to another member when their terminal conditions differ infinitesimally. By continuously collapsing the terminal conditions of the family of problems onto those of the original problem while appropriately modifying the control function, the solution to the original optimization problem can be obtained.

One important reason for using imbedding theory to solve optimization problems is its utility in solving singular control problems. This utility is illustrated with an example. A problem of finding the time-optimal maneuvers for an aerial attack where maneuvers are restricted to a horizontal plane is solved. The model of the airplane uses a square-law drag term. The controls used are airplane thrust and turning acceleration. Both controls are limited. The airplane maneuvers in such a way that the target vehicle (assumed to be nonmaneuvering) is placed in a situation favorable to deployment of the airplane armament in the shortest possible time. The solutions to this problem are composed of both singular and nonsingular subarcs of the trajectory. Thus, the utility of imbedding in solving singular optimal control problems is illustrated.

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INTRODUCTION

Applications of optimization techniques in aeronautics and astronautics have appeared as early as the 17th century. Indeed Sir Isaac Newton used a variational approach to study the problem of finding minimum drag shapes for a body of revolution which was submersed in an extremely high-speed fluid flow at zero angle of attack. (See ref. 1.) For this problem the body had a given length and base diameter. More recently the rising interest in problems related to atmospheric flight mechanics has acted as a catalyst in advancing the development and application of optimal control theory. One example of this is the defining of optimal guidance and control for the powered flight of a missile considered by Duersch (ref. 2). Others have studied the minimum time to climb problem for airplanes. (For example, see ref. 3.) This later research led to significant improvements in guidance for achieving a given altitude in the shortest possible time.

The optimization problem considered in this paper may be formulated as follows: "Given a deterministic system with a state $x(t)$, governed by a set of first-order differential equations; find the control function $u(t)$ which satisfies the given control constraints, such that at some final time the state satisfies prescribed terminal constraints and renders a given integral cost function a minimum."

The stated optimization problem has many aerospace applications. As formulated, there are four popular methods for extracting optimal controls: the calculus of variations, the maximum principle of Pontryagin, dynamic programing, and direct iterative methods. Each of these four methods has proved to be a strong tool for obtaining optimal controls although each possesses inherent difficulties. Convergence of direct iterative methods is often a problem. Computational implementation of dynamic programing is sometimes prohibitive due to excessive memory requirements. Both the calculus of variations and the maximum principle suffer from difficulties inherent in multipoint boundary value problems as well as singular control problems.

Ideally, the application of the maximum principle leads to a two-point boundary value problem (ref. 4) which may be solved by a method, introduced by Bellman (ref. 5), called invariant imbedding. However, certain problems may arise in which the conditions set down by the maximum principle fail to restrict the control functions sufficiently to enable a unique optimal control to be extracted. If only a finite number of control functions satisfy the conditions of the maximum principle, a direct comparison of the "cost" associated with each control serves to eliminate all but the optimal control. If, on the contrary, there is an infinite selection of controls which satisfy the maximum principle, then the optimization problem is not reducible to a simple two-point boundary-value problem and the concept of invariant imbedding will not define the optimal control.

Problems possessing this difficulty have been studied by Rozonoêr (ref. 6) who referred to them as singular control problems and by Johnson and Gibson (ref. 7). In reference 7 is outlined a systematic procedure for obtaining singular controls. However, he does not say whether the arc derived from the singular control is a portion of the optimal trajectory. An illustration of the approach of Johnson and Gibson applied to solving a singular control problem is presented in appendix A. Rozonoêr has shown, for restricted cases where the Hamiltonian used in the maximum principle as well as the canonical equations are all linear in the control, that the conditions of the maximum principle are both necessary and sufficient to define the optimal control. However, the general problem of singular control is unanswered and the problem of piecing together optimal trajectories that contain singular controls is somewhat more complex than has been resolved by present control theory.

The primary purpose of this report is to develop a field theory for optimal control problems that more effectively deals with the problem of piecing together singular and nonsingular subarcs to construct optimal trajectories. This field theory is called "terminal imbedding" since optimal trajectories are determined by imbedding the given optimization problem of interest in a family of optimization problems parameterized by their termination conditions. This terminal imbedding is in contrast with "general imbedding" (ref. 8) in which the imbedding is accomplished by considering a family of optimization problems parameterized by different cost functions and with dynamic programing in which surfaces of constant optimal cost can be considered as the imbedding parameterization. Note, however, that with dynamic programing, the surfaces of constant optimal cost are not known a priori, whereas in terminal imbedding the imbedding termination surfaces are selected a priori. The closest analogy to other methods of extracting optimal trajectories is invariant imbedding in which a terminal manifold parameterization is made holding a boundary condition constant for the purpose of solving a two-point boundary value problem. In the case of terminal imbedding, the terminal manifold parameterization is made obtaining a minimum cost for each member of the family of solutions.

The basic concept underlying terminal imbedding was derived from a synthesis procedure for determining the feedback gains for a linear feedback control system that result in arbitrarily specified closed-loop pole-zero properties. The basic tool used in that synthesis procedure, referred to herein as the method of conversion to differential form, was proposed for solving algebraic equations by Yakovlev (ref. 9) and was extended, without vigorous analytical justification, to the synthesis of linear feedback control systems by Montgomery and Hatch (ref. 10). Since the method of terminal imbedding relies heavily on the method of conversion to differential form, first a strong analytical base is provided for the method of conversion to differential form. Next, the general theory of terminal imbedding, as applied to trajectory optimization problems, is developed. Then, singularities that arise in the analysis which must be considered in the practical application of terminal imbedding are analyzed. Finally, several applications of terminal imbedding are presented, among which are linear and nonlinear optimization problems, and singular control problems.

SYMBOLS

$A(t,\sigma)$	matrix defined by equation (10)
A_k'	matrix defined by equation (30)
$a_1,\ldots,a_6$	constants defined by equation (66)
$B(t,\sigma)$	matrix defined by equation (11)
$C(x^f,u^f,\sigma)$	matrix function defined by equation (18)
$c(\sigma)$	vector function of σ
c^0	$c(\sigma)$ evaluated at $\sigma = 0$
c^1	$c(\sigma)$ evaluated at $\sigma = 1$
d	scalar variable
$d\lambda/d\sigma$	vector of Lagrange multipliers
$F(x)$	matrix function of x defined by equation (5)
$F(x^f,\sigma)$	matrix function of x^f and σ defined by equation (20)
$\tilde{F}(x,\psi)$	coefficient of the control $u(t,\sigma)$ in $H(x,\psi,u)$
$f(x)$	algebraic vector function
$f(x,u)$	the state derivative as a function of $x(t,\sigma)$ and $u(t,\sigma)$
$G(x) = \nabla_x f(x)\ \nabla_x^T f(x)$	
$G_1(x^f,u^f,\sigma)$	scalar function defined by equation (19)
$G_2(x^f,u^f,\sigma)$	total derivative of G_1 with respect to σ

$H(x,\psi,u)$	Hamiltonian function
$h(t,\sigma)$	vector function defined by equation (B1)
h_i	vector defined by equation (39)
I	identity matrix
$\tilde{I}(x,\psi)$	part of $H(x,\psi,u)$ independent of control
$J(x^f)$	scalar function describing terminal manifold
$J(x^f,\sigma)$	imbedded scalar function describing terminal manifolds
$J[\cdot]$	functional defined by equation (B3)
$K(t)$	number of discontinuities in control over interval $[t_o,t)$
K_1	maximum normal acceleration
K_2	constant depending on airplane drag
$M(t,\sigma)$	vector function of t,σ
m	dimension of control vector u
N	number of discontinuities in $\partial x/\partial t$ occurring in time interval $[t_o,t)$
n	dimension of state vector x
P	scalar cost function
R	distance from airplane to center of termination zone
r	radial distance
r^f	radius of termination zone
$S(x^o,r)$	hypersphere of radius r centered at x^o
$S_i(\sigma)$	time at which ith discontinuity occurs

t	time
t_o	initial time
$u(t,\sigma)$	control function
$u^f(\sigma)$	terminal control
u_c^f	airplane control of normal acceleration
V	airplane speed
V_T	target speed
V_c	airplane speed control
V_{max}	maximum airplane speed
V_{min}	minimum airplane speed
v	vector
$v(\sigma)$	vector of independent variables
$x(t,\sigma)$	state vector with x_{n+1} being cost function
x^o	$x(\sigma)$ evaluated at $\sigma = 0$ or initial state
x^1	$x(\sigma)$ evaluated at $\sigma = 1$
x_A, y_A	coordinates of position of airplane in horizontal plane
x_h	position of harmonic oscillator
$\alpha(x^f)$	function in equation (57)
α_i	lower control limit on u_i
$\beta(x^f)$	function in equation (57)
β_i	upper control limit on u_i

γ_1, γ_2	functions independent of speed control
$\Delta f_i(\sigma)$	change in $f(x,u)$ across ith discontinuity
Δu_i	change in u across ith discontinuity
$\delta\left(x^f\right)$	function defined by equation (59)
ϵ	is an element of
ϵ	positive number
θ	airplane orientation in horizontal plane
μ	(n+1)-dimensional unit vector with μ_{n+1} unity
ξ_i	region of t,σ space free of discontinuities
σ	imbedding parameter
$\Phi\left(t,t_o,\sigma\right)$	matrix function defined by equations (13) and (14)
ϕ_{ij}	element of ith row and jth column of Φ
$\chi(t,\sigma)$	vector function
ψ	costate variables
ψ^n	column vector whose components are first n components of vector ψ
Ω	control space
Ω_1	restricted control space
$\nabla_u f$	matrix whose element of ith row and jth column is $\partial f_i / \partial u_j$
$\nabla_x f$	matrix whose i,j element is $\partial f_i / \partial x_j$
$\nabla_x f(x)$	matrix whose element of ith row and jth column is $\partial f_i / \partial x_j$

$\nabla_x H$	column vector whose ith element is $\frac{\partial H}{\partial x_i}(x,\psi,u)$
$\nabla_x J$	column vector whose ith element is $\frac{\partial H}{\partial x_i}\left(x^f\right)$
$\nabla_x J\left(x^f,\sigma\right)$	column vector whose ith element is $\frac{\partial J}{\partial x_i}\left(x^f,\sigma\right)$
$\nabla_{xx} J$	matrix whose element of ith row and jth column is $\frac{\partial^2 J}{\partial x_i}\ \partial x_j$
$\nabla_\psi H$	column vector whose ith element is $\frac{\partial H}{\partial \psi_i}(x,\psi,u)$
$\{x \mid P\}$	set x satisfying property P
$\triangleq$	equality by definition
$\|\cdot\|$	norm
$[a,b)$	range of a real variable x such that $a \leqq x < b$
$\int_\xi [\ ]\,dt$	a Riemann integral taken over the interval ξ

A dot indicates differentiation with respect to time. A superscript T indicates a transpose whereas a subscript T indicates target. Superscript and subscript f indicate that the function is evaluated at termination time.

IMBEDDING FOR SYSTEMS OF ALGEBRAIC EQUATIONS

The recent development of high-speed digital computers has created a tool which was not previously available to aerospace scientists. Problems which heretofore have been intractable due to immense computational requirements can now be considered. One example of such is the analysis and design of high-order linear feedback control systems. Much of the early effort in this area was devoted to the development of simplified methods of analysis which could analyze the stability and control characteristics of single-input, single-output systems.

Generally the synthesis of multiple input feedback control systems leads to sets of nonlinear algebraic equations which must be solved for feedback gains. The modern high-speed computer enables one to apply iterative approaches to the solving of algebraic equations which were previously undesirable. Newton's procedure is a popular method for obtaining solutions of algebraic equations (ref. 11). The convergence of Newton's method, as with most iterative methods, is not usually known a priori. This difficulty with convergence can sometimes be avoided by using imbedding theory to obtain the solution desired. This is accomplished by considering the solution of the algebraic problem of interest as one member of a continuous family of solutions wherein one member of the family is known. Then conditions are derived that allow determination of one member of the family given a neighboring member. This concept of imbedding when applied to the solution of algebraic problems will be referred to as the method of conversion to differential form, for reasons which will be evident later.

As an aid in explaining the method of conversion to differential form for solving nonlinear algebraic equations consider the scalar equation

$$f(x) = x^2 - 3x + 2 = c \tag{1}$$

A graph of $f(x)$ against x satisfying equation (1) is presented in figure 1. To solve the equation $f(x) = 0$, one should first let both c and x in equation (1) be functions of a dummy variable σ which varies from 0 and 1. That is, let $c \triangleq c(\sigma)$ and $x \triangleq x(\sigma)$. The basic idea is to arbitrarily select an initial point, say $x^o = x(0)$ and a variation $c(\sigma)$ such that $c(\sigma)$ passes through the point $c(0) = f(x^o)$ and the desired point $c(1) = 0$.

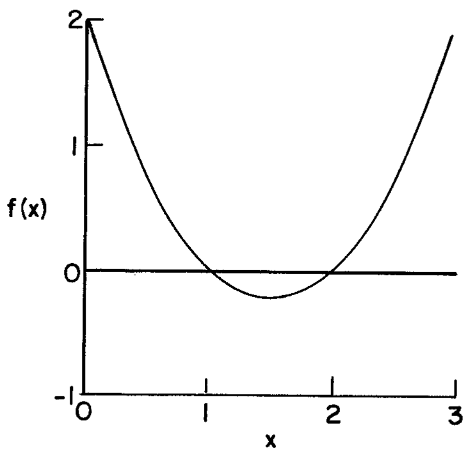


Figure 1.- Graph of $f(x)$ against x satisfying equation (1).

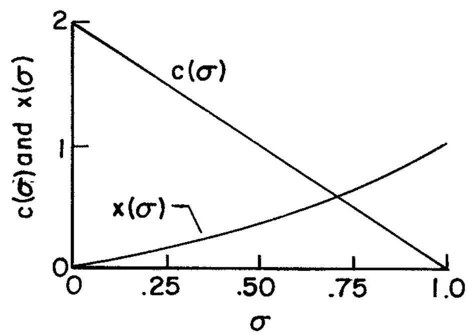


Figure 2.- Solution of equation (1) using imbedding theory.

A typical curve $c(\sigma)$ satisfying these requirements is the straight-line segment illustrated in figure 2. For that graph the value of $c(0)$ was calculated by substituting $x(0) = x^o = 0$ into equation (1). Then, the variation in $x(\sigma)$ required by the identity $f(x(\sigma)) \equiv c(\sigma)$ is determined by using a differential form of equation (1); namely,

$$\frac{\partial f}{\partial x}(x(\sigma))\,\frac{dx}{d\sigma}(\sigma) = \frac{dc}{d\sigma}(\sigma)$$

For this particular example, using the linear variation for $c(\sigma)$ as indicated in figure 2 where $dc/d\sigma$ has been made equal to -2, the derivative $dx/d\sigma$ must satisfy the equation

$$\frac{dx}{d\sigma}(\sigma) = \frac{-2}{2x - 3}$$

This equation can be integrated (numerically) over the interval $0 \leqq \sigma \leqq 1$ with an initial condition $x(0) = 0$. The value of x at $\sigma = 1$ satisfies the identity $f(x(1)) = c(1) = 0$ and a solution of the equation $f(x) = 0$ is obtained. Note that if an initial value of $x = 3$ was selected, the solution obtained would be $x = 2$ and not the one indicated in figure 2, which is $x = 1$. This example illustrates the nonuniqueness of solutions of algebraic equations and the fact that the particular solution obtained by using imbedding is dependent on the imbedding parameterization selected, for example, the choice of x^o and $c(\sigma)$.

For this simple example, it is obviously easier to solve the nonlinear algebraic equation using the quadratic formula than to use the method of conversion to differential form. However, for complex sets of nonlinear algebraic equations such as those arising in the synthesis of linear feedback control systems no general closed-form solutions are available; however, the method of conversion to differential form does remain applicable.

Assume that one wishes to solve the nonlinear set of algebraic equations $f(x) = c$ for a vector x, say x^1, which leads to a specific vector c, say c^1. Stipulate that n, the dimension of x, is greater than or equal to m, the dimension of c. Let c and x be functions of the imbedding parameter σ, and select $c(\sigma)$ such that it satisfies $c(0) = c^o$. Hence,

$$c^o \triangleq f(x^o) \tag{2}$$

and is calculated from an arbitrary x^o. Other requirements on $c(\sigma)$ are that $c(1) = c^1$ and $c(\sigma)$ is differentiable on the interval $0 \leqq \sigma \leqq 1$. The linear function

$$c(\sigma) = c^o + \sigma\left(c^1 - c^o\right)$$

is one such example. The variation in $x(\sigma)$ – required by the identity $f(x(\sigma)) \equiv c(\sigma)$ – must satisfy the relation

$$\nabla_x f(x(\sigma)) \frac{dx}{d\sigma}(\sigma) = \frac{dc}{d\sigma}(\sigma) \tag{3}$$

Equation (3) is a set of implicit differential equations that are linear in $dx/d\sigma$, and which usually can be integrated numerically once the function $\frac{dc}{d\sigma}(\sigma)$ over the interval $0 \leqq \sigma \leqq 1$ and the initial conditions $x(0) = x^o$ are given. When the dimension of x is greater than that of c the integration process is usually nonunique. The function $\frac{dc}{d\sigma}(\sigma)$

can be thought of as a forcing function which forces $x(\sigma)$ to follow some path which preserves the identity $c(\sigma) \equiv f(x(\sigma))$. Hence, the value of $x(\sigma)$, at $\sigma = 1$, should satisfy the equation $f(x(1)) = c^1$.

Since equation (3) is an implicit set of differential equations one does not know, a priori, whether the solution $x(\sigma)$ can be continued from $\sigma = 0$ to $\sigma = 1$. Therefore, it is of interest to determine the extent to which solutions of $x(\sigma)$ can be continued in terms of restrictions on both $\nabla_x f(x)$ and $c(\sigma)$. Let x and c satisfy

$$c = f(x) \tag{4}$$

and let $x(0) = x^o$ and $c(0) = c^o$ satisfy equation (2). Assume that the matrix $\nabla_x f(x)$ is of rank $m < n$ for vectors $x \in S(x^o,r)$ where $S(x^o,r)$ is the set of vectors x such that $\|x - x^o\| \leq r$. In $S(x^o,r)$ equation (3) has an $n - m$ parameter family of solutions for $dx/d\sigma$. The particular member of this family of solutions for $dx/d\sigma$ which minimizes the function $\frac{1}{2}\frac{dx^T}{d\sigma}\frac{dx}{d\sigma}$ tends to minimize the rate at which the solution $x(\sigma)$ leaves the sphere $S(x^o,r)$ for any given $c(\sigma)$ variation. Hence, consider the solution of equation (3) which minimizes $\frac{1}{2}\frac{dx^T}{d\sigma}\frac{dx}{d\sigma}$ subject to the constraint of equation (3). This problem is considered in reference 12. Only the essential results of the analysis will be presented here. To incorporate the constraint, construct an alternate function

$$P \triangleq \frac{1}{2}\frac{dx^T}{d\sigma}\frac{dx}{d\sigma} + \frac{d\lambda^T}{d\sigma}\left[\nabla_x f(x)\frac{dx}{d\sigma} - \frac{dc}{d\sigma}\right]$$

to be minimized where $d\lambda/d\sigma$ is an m dimensional set of Lagrange multipliers appending the constraints to the original cost function. The solution to the optimization problem posed must satisfy

$$F(x)\begin{bmatrix} \frac{dx}{d\sigma} \\ \hdashline \frac{d\lambda}{d\sigma} \end{bmatrix} = \begin{bmatrix} 0 \\ \hdashline \frac{dc}{d\sigma} \end{bmatrix}$$

where

$$F(x) \triangleq \left[\begin{array}{c:c} I & \nabla_x^T f(x) \\ \hdashline \nabla_x f(x) & 0 \end{array}\right] \tag{5}$$

Since $\nabla_x f(x)$ is of rank m for $x \in S(x^o,r)$, $F(x)$ is a nonsingular matrix for $x \in S(x^o,r)$ whose inverse is

$$F_{(x)}^{-1} = \left[\begin{array}{c|c} I - \nabla_x^T f G^{-1} \nabla_x f & \nabla_x^T f G^{-1} \\ \hline G^{-1} \nabla_x f & -G^{-1} \end{array}\right]$$

where

$$G = G(x) \triangleq \nabla_x f(x)\ \nabla_x^T f(x)$$

Note that $G(x)$ is nonsingular since $\nabla_x f(x)$ is of rank m for any $x \in S(x^o,r)$. Next, define a function $r(d)$ such that r is the largest radius of the hypersphere $S(x^o,r)$ with $\|F^{-1}(x)\| \leqq d$ for all $x \in S(x^o,r)$. Then one can see that

$$\left\|\frac{dx}{d\sigma}\right\| \leqq \left\|\begin{bmatrix} \frac{dx}{d\sigma} \\ --- \\ \frac{d\lambda}{d\sigma} \end{bmatrix}\right\| \leqq \left\|F_{(x)}^{-1}\right\| \cdot \left\|\frac{dc}{d\sigma}\right\| \leqq d\left\|\frac{dc}{d\sigma}\right\|$$

If $m = n$ then $\|dx/d\sigma\| \leqq \left\|\nabla_x^{-1} f(x)\right\| \cdot \|dc/d\sigma\|$. In this case, define $r(d)$ as the largest r such that $\left\|\nabla_x^{-1} f(x)\right\| \leqq d$, $x \in S(x^o,r)$.

Now, since

$$\|x(\sigma) - x^o\| = \left\|\int_0^\sigma \frac{dx}{d\sigma}(\beta)\, d\beta\right\| \leqq \int_0^\sigma \left\|\frac{dx}{d\sigma}(\beta)\right\| d\beta \leqq d \int_0^\sigma \left\|\frac{dc}{d\sigma}(\beta)\right\| d\beta$$

if one requires that

$$r(d) \geqq d \int_0^1 \left\|\frac{dc}{d\sigma}(\beta)\right\| d\beta \geqq d \int_0^\sigma \left\|\frac{dc}{d\sigma}(\beta)\right\| d\beta$$

then for some value of d, $\|x(\sigma) - x^o\| \leqq r$ if $0 \leqq \sigma \leqq 1$, and thus $x(\sigma)$ is contained in $S(x^o,r)$. The method of conversion to differential form can therefore be successfully applied, and the following sufficient condition has been demonstrated.

Theorem 1: Consider equation (4) with the condition $m \leqq n$ and given point x^o and function $c(\sigma)$ which is differentiable on the interval $0 \leqq \sigma \leqq 1$ and satisfies $c(0) = f(x^o)$. Then if $m < n$, let $\nabla_x f(x)$ be of rank m, $x \in S(x^o,r(d))$ where $r(d)$ is the largest r for which $\|F^{-1}(x)\| \leqq d$, $x \in S(x^o,r)$; if $m = n$, define $r(d)$ as the maximum r for which $\left\|\nabla_x^{-1} f(x)\right\| \leqq d$, $x \in S(x^o,r)$. If for some value of d,

$$r(d) \geqq d \int_0^1 \left\| \frac{dc}{d\sigma}(\beta) \right\| d\beta$$

then there exists a solution $x(\sigma)$ of equation (4) such that

$$f(x(\sigma)) = c(\sigma)$$

and

$$\nabla_x f(x) \frac{dx}{d\sigma} = \frac{dc}{d\sigma}$$

On the interval $0 \leqq \sigma \leqq 1$, the rank of $\nabla_x f(x)$ is m.

To illustrate the use of this theorem, consider the example noted as equation (1) except that c has been replaced with $c(0) = c^o$. The gradient matrix is $\nabla_x f = 2x - 3$, and taking $x(0) = 0$, the function $r(d)$ can be calculated from $\left\| \nabla_x^{-1} f \right\| \leqq d$ to give

$$r(d) = \frac{3d - 1}{2d}$$

where $d \geqq \frac{1}{3}$. If one designates $c(\sigma)$ to be linear and of the form

$$c(\sigma) = c^o(1 - \sigma)$$

then

$$\int_0^1 \left\| \frac{dc}{d\sigma}(\beta) \right\| d\beta = |c^o|$$

Consequently, the conditions of theorem 1 imply that a solution of the equation

$$x^2 - 3x + c^o = 0 \tag{6}$$

exists provided that $r(d) \geqq |c^o| d$ for some d. This condition reduces to

$$\frac{3d - 1}{2d} \geqq |c^o| d$$

This condition is graphically illustrated in figure 3 where the function $r(d)$ is plotted against d; note that if the straight line passing through the origin, with a slope of $|c^o|$, intersects the curve $r(d)$ a solution to equation (6) is assured and can be obtained by the method of conversion to differential form. Also, from figure 3 it is apparent that solutions cannot be assured for all $|c^o|$. Indeed, if $|c^o| > 9/8$, theorem 1 cannot be applied even though solutions of equation (6) are known to exist. This example emphasizes the fact that theorem 1 is only a sufficient condition and is not generally a necessary one.

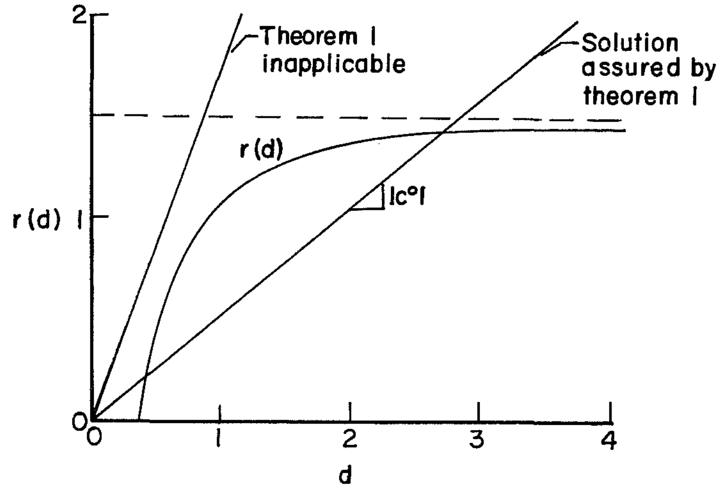


Figure 3.- Graph of r(d) against d which illustrates the applicability of theorem 1.

The method of conversion to differential form has been applied to the design of multiaxis stability augmentation systems for aerospace vehicles in reference 10.

TERMINAL IMBEDDING IN OPTIMAL CONTROL THEORY

In this section the method of terminal imbedding is applied to trajectory optimization problems. The problem to which the method of terminal imbedding is directed is precisely stated; the general concept, underlying the terminal imbedding, is outlined; geometric implications and assumptions of the theory are discussed; and the principle result of the analysis is presented.

Statement of the Problem

The method of terminal imbedding is directed to a slightly restricted form of the general optimization problem mentioned in the introduction. Let a control process be given which is governed by the autonomous differential equation

$$\dot{x} = f(x,u) \tag{7}$$

where x is an $(n+1)$-dimensional state vector and u is an m-dimensional control vector. The control vector u is assumed to be constrained to some manifold region of an m-dimensional Euclidean space Ω. A manifold region is defined as a region of Euclidean space such that $u \in \Omega$ implies that each component of u satisfies the inequalities $\alpha_i \leqq u_i \leqq \beta_i$ for $i = 1,2,\ldots,m$ and α_i,β_i either finite or infinite. The function $f(x,u)$ is assumed to have first partial derivatives with respect to the components of the x and u vectors.

Suppose that there is given a simply connected (n-1)-dimensional space called the terminal manifold. Let it be described by the set of vectors x satisfying the equality $J(x) = 0$. The function $J(x)$ is a scalar function of x, taken to be independent of x_{n+1}, having $\nabla_{xx}J(x)$ defined at each boundary point of the terminal manifold. An admissible trajectory is defined to be a continuous and piecewise differentiable vector function $x(t)$ defined over the interval $t_o \leqq t \leqq t_f$ of the independent variable t which satisfies

(1) $x(t_o) = x^o$ with $x^o_{n+1} = 0$

(2) $J(x(t)) > 0$ for $t_o \leqq t < t_f$

(3) $J\left(x(t_f)\right) = 0$

(4) the governing differential equation (7) on each differentiable subarc of $x(t)$ for some piecewise continuous control function $u(t) \in \Omega$ for $t_o \leqq t \leqq t_f$

The control functions $u(t)$ corresponding to admissible trajectories are called admissible controls. The problem considered in this section is to find an admissible control $u(t)$ which renders $x_{n+1}(t_f)$ a minimum.

The Concept of Terminal Imbedding

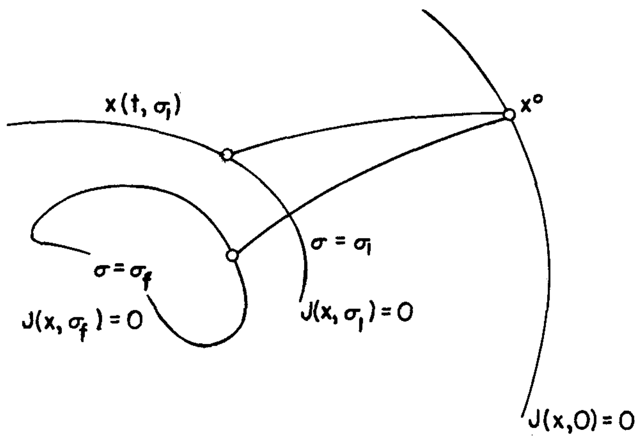
In attacking the stated optimization problem using the method of terminal imbedding the original problem is considered to be imbedded in a family of optimization problems, transferring the vector x from the initial point x^o to each of a family of terminal manifolds which are parameterized by a scalar σ. The basic concept underlying the method of terminal imbedding is "with a knowledge of the control function and corresponding motion trajectory, which will optimally transfer the state x, from the initial point x^o to a surface described by $J(x^f,\sigma) = 0$, find the modifications required to transfer the state from x^o to the surface described by $J\left(x^T,\sigma+\epsilon\right) = 0$ optimally for $\epsilon > 0$." Continued application of this idea will allow one to construct the control function and corresponding trajectory of motion required to transfer the state from x^o to the surface described by $J(x^f) = 0$ in an optimal manner. Requirements on the parameterized family of terminal manifolds are such that the stated optimization problem has a known solution or one easily obtained for one member of the family of imbedded manifolds corresponding to a value of σ, say, $\sigma = 0$, and that the family of terminal manifolds includes the terminal manifold of the original problem statement for some value of σ, say, $\sigma = \sigma_f$. The parameterized family of terminal manifolds is described by the scalar function $J(x,\sigma)$ which is identical to $J(x)$ for $\sigma = \sigma_f$.

The parameterization of the terminal manifolds is not generally unique. One possible choice is the family described by

$$J(x,\sigma) = J(x) + J(x^o)(\sigma - 1)$$

where $J(x)$ is the function describing the terminal manifold of the original problem. Here, x^o is the initial condition of the original problem, and σ_f is assumed to be 1. This family satisfies both of the stated conditions. At $\sigma = 0$ the initial condition satisfies $J(x^o,0) = 0$ and the optimal trajectory is known; at $\sigma = 1$ the function $J(x,1)$ reduces to $J(x)$ which describes the original terminal manifold.

The stated optimization problem is therefore imbedded in an initial value problem. The schematic diagram at the top of figure 4 illustrates the evolution of the optimal trajectories for various values of σ. Identify a particular optimal trajectory $x(t)$ and the associated optimal control $u(t)$, that correspond to a particular value of σ as $x(t,\sigma)$ and $u(t,\sigma)$, respectively. In the analysis which follows, the continuity of the vector $x(t,\sigma)$ in both t and σ is required. Because of the governing differential equations, continuity in t is assured; however, continuity in σ is not. Furthermore it is assumed



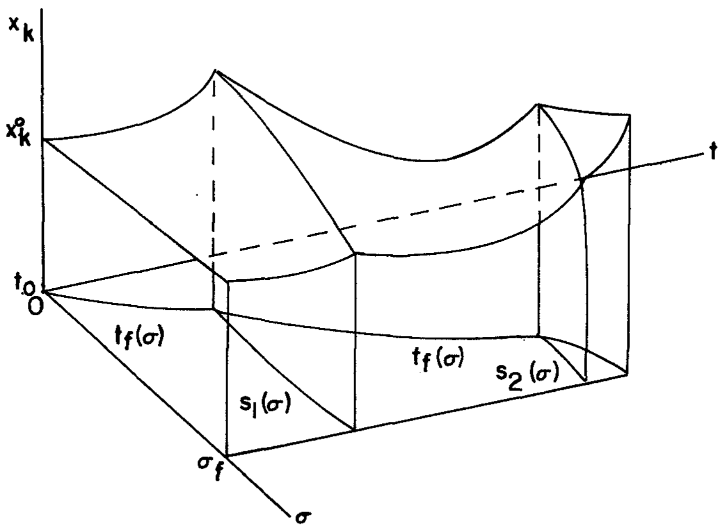


Figure 4.- Terminal imbedding applied to trajectory optimization.

that $x(t,\sigma)$ has continuous partial derivatives in t and σ on each interval, ξ_i for $i = 1,2,\ldots,N+1$, where

$$\xi_i : \left\{(t,\sigma) \middle| t_o < t < S_1(\sigma); \quad 0 \leqq \sigma \leqq \sigma_f\right\}$$

$$\xi_i : \left\{(t,\sigma) \middle| S_{i-1}(\sigma) < t < S_i(\sigma); \quad 0 \leqq \sigma \leqq \sigma_f\right\}$$

and

$$\xi_{N+1} : \left\{(t,\sigma) \middle| S_N(\sigma) < t < t_f(\sigma); \quad 0 \leqq \sigma \leqq \sigma_f\right\}$$

The values of $S_1(\sigma),\ldots,S_N(\sigma)$ represent values of t for which the function $x(t,\sigma)$ may have discontinuous partial derivatives in t. These times are not known a priori and must be determined. They normally represent points of discontinuity in the control function $u(t,\sigma)$.

The lower sketch of figure 4 graphically illustrates a typical variation of the kth component of x, say, x_k with both t and σ. Note that the locus of discontinuities in $\frac{\partial x_k}{\partial t}(t,\sigma)$ is indicated by the curves $S_1(\sigma)$ and $S_2(\sigma)$. The terminal time t_f in general depends on σ; this is indicated by $t_f(\sigma)$ in the sketch. At $\sigma = \sigma_f$ the function $x_k(t,\sigma_f)$ is the desired optimal trajectory. The functions $S_1(\sigma),\ldots,S_N(\sigma)$ and $u(t,\sigma)$ are assumed to be bounded and differentiable functions of σ over their region of definition.

Theoretical Results

The principal result.- From the concept of terminal imbedding, the state x is considered a function of two scalar variables t and σ. According to equation (7), the solution $x(t,\sigma)$ must satisfy the equation

$$\frac{\partial x}{\partial t}(t,\sigma) = f\big(x(t,\sigma),u(t,\sigma)\big) \tag{8}$$

on every interval ξ_i; hence,

$$x(t,\sigma) = x^o + \int_{t_o}^{t} f\big(x(\tau,\sigma),u(\tau,\sigma)\big)\, d\tau$$

Because of the requirement that $x(t,\sigma)$ and $u(t,\sigma)$ must be differentiable in both t and σ on intervals ξ_i and that the functions $S_i(\sigma)$ be differentiable, it follows that

$$\frac{\partial x}{\partial \sigma}(t,\sigma) = \int_{t_o}^{t}\left[A(\tau,\sigma)\,\frac{\partial x}{\partial \sigma}(\tau,\sigma) + B(\tau,\sigma)\,\frac{\partial u}{\partial \sigma}(\tau,\sigma)\right] d\tau + \sum_{i=1}^{K(t)} \Delta f_i(\sigma)\,\frac{dS_i}{d\sigma}(\sigma) \tag{9}$$

for t,σ in ξ_{K+1}. In equation (9), $K(t)$ represents the number of $S_i(\sigma)$ functions contained in the interval $[t_o,t)$ and the following definitions have been used

$$A(t,\sigma) \triangleq \nabla_x f\big(x(t,\sigma),u(t,\sigma)\big) \tag{10}$$

$$B(t,\sigma) \triangleq \nabla_u f\big(x(t,\sigma),u(t,\sigma)\big) \tag{11}$$

and

$$\Delta f_i(\sigma) \triangleq \lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\Big[f\big(x(S_i,\sigma),u(S_i-\epsilon,\sigma)\big) - f\big(x(S_i,\sigma),u(S_i+\epsilon,\sigma)\big)\Big]$$

Equation (9) is a linear Volterra integral equation for $\frac{\partial x}{\partial \sigma}(t,\sigma)$. Its solution for $(t,\sigma) \in \xi_{K+1}$ is

$$\frac{\partial x}{\partial \sigma}(t,\sigma) = \sum_{i=1}^{K(t)} \Phi(t,S_i,\sigma)\,\Delta f_i(\sigma)\,\frac{dS_i}{d\sigma}(\sigma) + \int_{t_o}^{t} \Phi(t,\tau,\sigma)\,B(\tau,\sigma)\,\frac{\partial u}{\partial \sigma}(\tau,\sigma)\,d\tau \tag{12}$$

where $K(t)$ is the total number of discontinuities in the slope $\partial x/\partial t$ or the control u in the time interval $[t_o,t)$. In equation (12) the matrix $\Phi(t,t_o,\sigma)$ satisfies the equations

$$\frac{\partial \Phi}{\partial t}(t,t_o,\sigma) = A(t,\sigma)\,\Phi(t,t_o,\sigma) \tag{13}$$

and

$$\Phi(t_o,t_o,\sigma) \equiv I \tag{14}$$

Here, I is the identity matrix and $\Phi(t,t_o,\sigma)$ is a (n+1)-square matrix function. Equations (8) and (12) may be used to evaluate the total derivative of $x(t,\sigma)$ with respect to σ along the line $t = t_f(\sigma)$. This result is

$$\frac{dx^f}{d\sigma}(\sigma) = \sum_{i=1}^{K(t_f)} \Phi(t_f,S_i,\sigma)\,\Delta f_i(\sigma)\,\frac{dS_i}{d\sigma}(\sigma) + \int_{t_o}^{t_f} \Phi(t_f,\tau,\sigma)\,B(\tau,\sigma)\,\frac{\partial u}{\partial \sigma}(\tau,\sigma)\,d\tau$$

$$+ f\big(x^f,u^f\big)\,\frac{dt_f}{d\sigma}(\sigma) \tag{15}$$

where the notations $x^f(\sigma) \triangleq x\big(t_f(\sigma),\sigma\big)$ and $u^f(\sigma) \triangleq u\big(t_f(\sigma),\sigma\big)$ have been used.

Each trajectory of the imbedding family must satisfy the terminal constraint

$$J\left(x^f,\sigma\right) \equiv 0$$

for all σ. Hence variations in x^f, given by equation (15), must satisfy

$$\frac{\partial J}{\partial \sigma}\left(x^f,\sigma\right) + \nabla_x^T J\left(x^f,\sigma\right) \frac{dx^f}{d\sigma}(\sigma) = 0 \tag{16}$$

identically in σ. An expression for the required variation in terminal time t_f can be obtained by substituting equation (15) into equation (16) and solving the resulting expression for $\frac{dt_f}{d\sigma}(\sigma)$. The result $dt_f/d\sigma$ can be inserted into equation (15) to obtain an expression for $dx^f/d\sigma$, which is required to satisfy the differential constraint equation (16). Thus, one obtains

$$\frac{dx^f}{d\sigma} = - \frac{\partial J}{\partial \sigma}\left(x^f,\sigma\right) \frac{f\left(x^f,u^f\right)}{\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right)} + C\left(x^f,u^f,\sigma\right)\left[\sum_{i=1}^{K(t_f)} \Phi\left(t_f,S_i,\sigma\right) \Delta f_i(\sigma) \frac{dS_i}{d\sigma}(\sigma) + \int_{t_o}^{t_f} \Phi\left(t_f,\tau,\sigma\right) B(\tau,\sigma) \frac{\partial u}{\partial \sigma}(\tau,\sigma)\, d\tau\right] \tag{17}$$

wherein

$$C\left(x^f,u^f,\sigma\right) \triangleq I - \frac{f\left(x^f,u^f\right) \Delta_x^T J\left(x^f,\sigma\right)}{\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right)} \tag{18}$$

With the definitions provided in the preceding paragraphs the field of solutions to the optimization problem may be shown to satisfy theorem 2. The proof of theorem 2 is contained in appendix B.

Theorem 2: If the stated optimal control problem possesses a field of solutions $x(t,\sigma)$ which are continuous and piecewise differentiable, along with the associated optimal controls $u(t,\sigma)$ which are piecewise continuous and piecewise differentiable, then $x(t,\sigma)$ and $u(t,\sigma)$ must satisfy the following three conditions:

(1) Switching point condition – The functions $S_i(\sigma)$ which describe the intervals ξ_i satisfy

$$\mu^T C\left(x^f,u^f,\sigma\right) \Phi\left(t_f,S_i,\sigma\right) \Delta f_i(\sigma) = 0$$

for $i = 1,2,\ldots,K(t_f)$.

(2) General point condition – Over any interval of finite duration in t, where a component u_j of the control function $u(t,\sigma)$ is not on the boundary of the control space Ω,

$$M_j(t,\sigma) = \left[\mu^T C\left(x^f,u^f,\sigma\right) \Phi\left(t_f,t,\sigma\right) B(t,\sigma)\right]_j = 0$$

for t,σ in ξ_i. Also if u_j is on the boundary of the control space Ω then

$$M_j(t,\sigma) \geqq 0 \quad \text{if} \quad u_j = \alpha_j$$

$$M_j(t,\sigma) \leqq 0 \quad \text{if} \quad u_j = \beta_j$$

and

$$\left[\frac{\partial u}{\partial \sigma}\right]_j = 0 \quad \text{if} \quad M_j(t,\sigma) \neq 0$$

(3) Terminal control condition – The terminal control $u^f(\sigma)$ must minimize $G_1\left(x^f,u^f,\sigma\right)$ and the index of performance must satisfy

$$\left(\frac{dx}{d\sigma}\right)_{n+1}^f (\sigma) = \min_{u^f \in \Omega} G_1\left(x^f,u^f,\sigma\right)$$

where

$$G_1\left(x^f,u^f,\sigma\right) \triangleq - \frac{\partial J}{\partial \sigma}\left(x^f,\sigma\right) \frac{f_{n+1}\left(x^f,u^f\right)}{\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right)} \tag{19}$$

These three conditions are subject to the constraint that

$$\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right) < 0$$

Any practical application of theorem 2 involves a consideration of the singular situations where the terminal control u^f cannot be defined by the terminal control condition of theorem 2. The next section presents extensions to theorem 2 which must be considered in order to cope with singularities which arise in problem solutions.

Extensions of the principal result.- As in algebraic problems there is no assurity that the imbedding process may be continued until the family of terminal manifolds collapses onto the one which is of interest. Indeed there are singular situations which arise during the application of terminal imbedding when the continuability of the process must be determined. The singularities mentioned above depend on the parameterization of the family of terminal manifolds. The singularities which can prevent continuation of the process can be divided into two categories: class A singularities which are due to the failure of the function $G_1\left(x^f,u^f,\sigma\right)$ of theorem 2 to define the terminal control u^f and class B singularities due to the nonexistence of the derivative $dx^f/d\sigma$ of equation (17). These two classes of singularities are discussed, in order, in the following paragraphs.

By definition, class A singularities arise because of the failure of the terminal control condition (theorem 2) to uniquely define the terminal control u^f. These singularities occur when $G_1\left(x^f,u^f,\sigma\right)$ becomes independent of some component of the vector u^f. The terminal control condition requires u^f to minimize the function $G_1\left(x^f,u^f,\sigma\right)$ subject to the constraints that u^f must be in Ω and that $\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right) < 0$.

Let Ω_1 be the set of controls u^f which satisfy this set of conditions. If this set does not define a unique u^f the series representation of $x_{n+1}^f(\sigma+\epsilon)$ is extended to include the second-order terms:

$$x_{n+1}^f(\sigma+\epsilon) = x_{n+1}^f(\sigma) + G_1\left(x^f,u^f,\sigma\right)\epsilon + G_2\left(x^f,u^f,\sigma\right)\frac{\epsilon^2}{2}$$

Then for u^f in Ω_1 the quantity u^f must minimize $G_2\left(x^f,u^f,\sigma\right)$. Let Ω_2 be the set of controls u^f which are in Ω_1 and minimize G_2. If this set does not uniquely define a terminal control u^f, then continue this procedure, considering higher-order terms to further restrict the set of terminal controls. An illustration of this procedure is presented in the section "Applications to Nonlinear Analysis."

The class B singularities are defined by the zeros of the function $\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right)$ since at such points the derivative $dx^f/d\sigma$ cannot be defined according to equation (17). Consider u^f to be determined as a function of the terminal state x^f and σ, that is, $u^f = u^f\left(x^f,\sigma\right)$ according to the terminal control condition of theorem 2. Define

$$F\left(x^f,\sigma\right) \triangleq \nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\left(x^f,\sigma\right)\right) \tag{20}$$

The function $F\left(x^f,\sigma\right)$ vanishes for some x at any given value of σ. The nature of this function is schematically illustrated in the upper sketch of figure 5. The shaded region represents a zone in x^f space where $F\left(x^f,\sigma\right)$ is not defined. This zone will generally appear since there may be terminal states x^f for which there is no control u^f directing the velocity vector $f\left(x^f,u^f\right)$ into the terminal manifold for a given value of σ. The point A in this sketch is a point for which there is no vector $f\left(x^f,u^f\right)$ directed into the terminal manifold with u^f in Ω. This is indicated by the vectorgram at point A representing the set of vectors $f\left(x^f,u^f\right)$ with u^f in Ω. A similar vectorgram at point B shows that there are vectors $f\left(x^f,u^f\right)$ with u^f in Ω directed into the terminal manifold.

Whether a class B singularity will result in termination of the process of evolution of the optimal trajectories depends on the behavior of $dx^f/d\sigma$ at the singular point and on the manner in which the terminal manifolds evolve with respect to σ. For example, if in a class B singularity the component of the vector $dx^f/d\sigma$ in the direction of $\nabla_x J\left(x^f,\sigma\right)$ can be made equal to $-\frac{\partial J}{\partial \sigma}\left(x^f,\sigma\right)$, then the evolution of the optimal trajectories may be continued. In order to do this, however, one may be required to add another discontinuity into the field of optimal trajectories. The procedure for accomplishing this is illustrated in the section entitled "Applications of Terminal Imbedding." A singularity

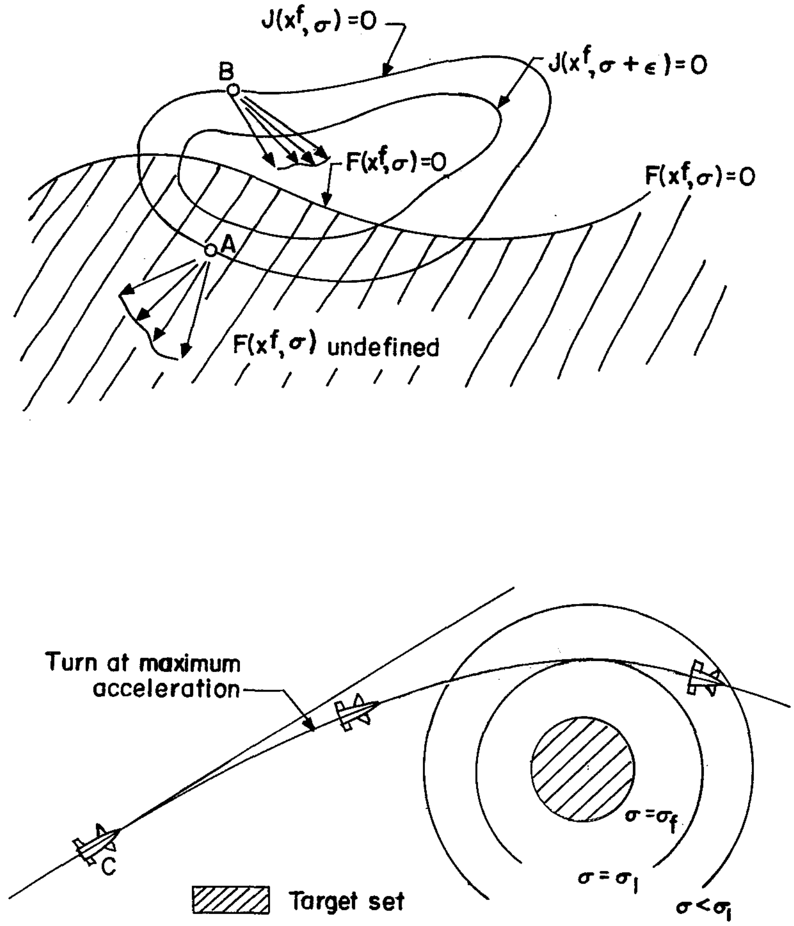


Figure 5.- Nature of class B singularities.

for which the addition of another discontinuity to the imbedded family of control functions will allow continuation of the solutions is said to be a simple singularity. Sometimes it is not possible by the addition of discontinuities at the end of the trajectory to make the component of $dx^f/d\sigma$ in the $\nabla_x J\left(x^f,\sigma\right)$ direction equal to $-\frac{\partial J}{\partial \sigma}\left(x^f,\sigma\right)$. Then the singularity is said to be terminal since the evolution of optimal trajectories, with the parameterization of terminal manifolds attempted, cannot be continued. The lower sketch of figure 5 illustrates a class B singularity for which a continuation of a continuous field of optimal trajectories is not possible. The problem considered is for the missile initially at point C to maneuver in such a way as to strike the target set indicated minimum time. For $\sigma < \sigma_1$ it is possible for the missile to strike the set of terminal conditions indicated in the sketch. However, at $\sigma = \sigma_1$ the trajectory of the missile is just tangent to the set of terminal conditions indicated. For $\sigma > \sigma_1$ the missile must fly by the target set and return to strike it. For the imbedding parameterization selected, the field of optimal trajectories is not continuous and by continuous imbedding it is not possible to continue the imbedding past $\sigma = \sigma_1$. This does not mean that no solution exists; only that, for the imbedding parameterization selected, one cannot move continuously from $\sigma < \sigma_1$ to $\sigma = \sigma_f > \sigma_1$ in order to obtain the solution.

For many problems the condition that the function $\mu^T C(x^f, u^f, \sigma)\, \Phi(t_f, t, \sigma)\, B(t, \sigma)$ must vanish cannot be realized over an interval of finite measure in t. In these cases the general point condition of theorem 2 implies that $\partial u/\partial\sigma \equiv 0$ (almost everywhere) and the main points of concern are the switching point condition and the terminal control condition. The switching point condition is

$$\mu^T C(x^f, u^f, \sigma)\, \Phi(t_f, S_i, \sigma)\, \Delta f_i = 0 \tag{21}$$

The terminal state variation can be obtained from equation (15), for the special case where $\partial u/\partial\sigma \equiv 0$, as

$$\frac{dx^f}{d\sigma} - f(x^f, u^f)\frac{dt_f}{d\sigma} - \sum_{i=1}^{K(t_f)} \Phi(t_f, S_i, \sigma)\, \Delta f_i \frac{dS_i}{d\sigma} = 0 \tag{22}$$

and the terminal constraints are

$$J(x^f, \sigma) = 0 \tag{23}$$

Equations (21) and (23) are algebraic relations between the variables t_f, x^f, $S_1, S_2, \ldots, S_{K(t_f)}$ which constrain the integration of equation (22). As such they are difficult to use in their algebraic form with equation (22). Equations (21) and (23) are most easily used by applying a conversion to differential form similar to that previously developed herein for algebraic systems.

The differential form of the constraint equation (23) is

$$\nabla_x^T J(x^f, \sigma)\frac{dx^f}{d\sigma} = -\frac{\partial J}{\partial\sigma}(x^f, \sigma) \tag{24}$$

Now consider equation (21). Differentiation of this equation with respect to σ results in the expression:

$$\left[\frac{dx^f}{d\sigma}\right]^T \nabla_x\left(\mu^T C\right)\left(x^f, u^f, \sigma\right) \Phi\left(t_f, S_i, \sigma\right) \Delta f_i + \left[\frac{du^f}{d\sigma}\right] \nabla_u\left(\mu^T C\right)\left(x^f, u^f, \sigma\right) \Phi\left(t_f, S_i, \sigma\right) \Delta f_i$$

$$+ \mu^T \frac{\partial C}{\partial \sigma}\left(x^f, u^f, \sigma\right) \Phi\left(t_f, S_i, \sigma\right) \Delta f_i + \mu^T C\left(x^f, u^f, \sigma\right) \left\{\frac{\partial \Phi}{\partial t}\left(t_f, S_i, \sigma\right) \frac{dt_f}{d\sigma}\right.$$

$$\left. + \frac{\partial \Phi}{\partial t_o}\left(t_f, S_i, \sigma\right) \frac{dS_i}{d\sigma} + \frac{\partial \Phi}{\partial \sigma}\left(t_f, S_i, \sigma\right)\right\} \Delta f_i = 0 \tag{25}$$

for $i = 1,2,\ldots,K$. In equation (25) the additional assumption that Δf_i is constant has been used. From the properties of the transition matrix $\Phi(t,t_o,\sigma)$, the following expressions are obtained:

$$\frac{\partial \Phi}{\partial t}\left(t_f, S_i, \sigma\right) = A\left(t_f, \sigma\right) \Phi\left(t_f, S_i, \sigma\right) \tag{26}$$

$$\frac{\partial \Phi}{\partial t_o}\left(t_f, S_i, \sigma\right) = -\Phi\left(t_f, S_i, \sigma\right) A\left(S_i, \sigma\right) \tag{27}$$

Evaluation of the matrix $\frac{\partial \Phi}{\partial \sigma}\left(t_f, S_i, \sigma\right)$ requires further consideration. First one notes that

$$\frac{\partial \Phi}{\partial \sigma}\left(t, t_o, \sigma\right) = \frac{\partial}{\partial \sigma}\left[I + \int_{t_o}^{t} A(\tau, \sigma) \Phi\left(\tau, t_o, \sigma\right) d\tau\right]$$

thus,

$$\frac{\partial \Phi}{\partial \sigma}\left(t, t_o, \sigma\right) = \int_{t_o}^{t} \frac{\partial A}{\partial \sigma}(\tau, \sigma) \Phi\left(\tau, t_o, \sigma\right) d\tau + \int_{t_o}^{t} A(\tau, \sigma) \frac{\partial \Phi}{\partial \sigma}\left(\tau, t_o, \sigma\right) d\tau$$

This last equation is a linear integral equation for $\frac{\partial \Phi}{\partial \sigma}\left(t, t_o, \sigma\right)$ of the same form as equation (9). Its solution is

$$\frac{\partial \Phi}{\partial \sigma}\left(t, t_o, \sigma\right) = \int_{t_o}^{t} \Phi(t, \tau, \sigma) \frac{\partial A}{\partial \sigma}(\tau, \sigma) \Phi\left(\tau, t_o, \sigma\right) d\tau \tag{28}$$

which can be verified by direct substitution. Equations (26), (27), and (28) may be used in equation (25) to obtain

$$\Delta f_i^T\, \Phi^T\left(t_f,S_i,\sigma\right) \nabla_x^T\left(\mu^T C\right)\left(x^f,u^f,\sigma\right) \frac{dx^f}{d\sigma} + \Delta f_i^T\, \Phi\left(t_f,S_i,\sigma\right) \nabla_u^T\left(\mu^T C\right)\left(x^f,u^f,\sigma\right) \frac{du^f}{d\sigma}$$

$$+ \mu^T\, C\left(x^f,u^f,\sigma\right)\left[A\left(t_f,\sigma\right) \Phi\left(t_f,S_i,\sigma\right) \frac{dt_f}{d\sigma} - \Phi\left(t_f,S_i,\sigma\right) A\left(S_i,\sigma\right) \frac{dS_i}{d\sigma}\right] \Delta f_i$$

$$+ \mu^T\, C\left(x^f,u^f,\sigma\right) \int_{S_i}^{t_f} \Phi\left(t_f,\tau,\sigma\right) \sum_{j=1}^{K(t)} A_j'(\tau,\sigma)\, \Phi\left(\tau,S_i,\sigma\right) d\tau\, \Delta f_i \frac{dS_j}{d\sigma}$$

$$= -\mu^T \frac{\partial C}{\partial\sigma}\left(x^f,u^f,\sigma\right) \Phi\left(t_f,S_i,\sigma\right) \Delta f_i \tag{29}$$

for $i = 1,2,\ldots,K$ wherein the following relation has been used:

$$\frac{\partial A}{\partial\sigma}(t,\sigma) = \sum_{k=1}^{K(t)} A_k'(t,\sigma) \frac{dS_k}{d\sigma}$$

with

$$A_k'(t,\sigma) \triangleq \left[\frac{\partial A}{\partial x_i}(t,\sigma)\, \Phi\left(t,S_k,\sigma\right) \Delta f_k, \ldots, \frac{\partial A}{\partial x_{n+1}}(t,\sigma)\, \Phi\left(t,S_k,\sigma\right) \Delta f_k\right] \tag{30}$$

Since the control space is such that $\alpha_j \leqq u_j \leqq \beta_j$ for each component of the control vector with α_j,β_j either finite or infinite then the control u_j^f is either against a limit where $du_j^f/d\sigma = 0$, or the function $G_1\left(x^f,u^f,\sigma\right)$ has an internal minimum where $\frac{\partial G_1}{\partial u_j^f}\left(x^f,u^f,\sigma\right) = 0$. In either event

$$\nabla_u^T\left(\mu^T C\right)\left(x^f,u^f,\sigma\right) \frac{du^f}{d\sigma} = 0$$

since $\nabla_u\left(\mu^T C\right) = \left[\frac{\partial J}{\partial\sigma}\left(x^f,\sigma\right)\right]^{-1} \nabla_u G_1$. Hence, the final differential form of the switching point constraint equation (21) is:

$$\Delta f_i^T\ \Phi^T(t_f,S_i,\sigma)\ \nabla_x^T\left(\mu^T C\right)\left(x^f,u^f,\sigma\right)\frac{dx^f}{d\sigma} + \mu^T\ C\left(x^f,u^f,\sigma\right)A\left(t_f,\sigma\right)\Phi\left(t_f,S_i,\sigma\right)\Delta f_i\ \frac{dt_f}{d\sigma}$$

$$- \mu^T\ C\left(x^f,u^f,\sigma\right)\Phi\left(t_f,S_i,\sigma\right)A\left(S_i,\sigma\right)\Delta f_i\ \frac{dS_i}{d\sigma}$$

$$+ \mu^T\ C\left(x^f,u^f,\sigma\right)\int_{S_i}^{t_f}\Phi\left(t_f,\tau,\sigma\right)\sum_{k=1}^{K(t)} A_k'(\tau,\sigma)\ \Phi\left(\tau,S_i,\sigma\right)d\tau\ \Delta f_i\ \frac{dS_k}{d\sigma}$$

$$= - \frac{\partial}{\partial\sigma}\left(\mu^T C\right)\left(x^f,u^f,\sigma\right)\Phi\left(t_f,S_i,\sigma\right)\Delta f_i \tag{31}$$

which must hold for $i = 1,2,\ldots,K\left(t_f\right)$.

Equations (22), (24), and (31) along with the terminal control condition of theorem 2 form an implicit set of ordinary differential equations which are linear in the variables $\frac{dx^f}{d\sigma}, \frac{dt_f}{d\sigma}, \frac{dS_1}{d\sigma},\ldots,\frac{dS_K}{d\sigma}$. At simple singularities the set of differential equations (eq. (24)) which was the condition used to define $dt_f/d\sigma$ in the derivation of equation (17) for theorem 2 cannot be satisfied. By elimination of this equation and converting the remaining ordinary differential equations to equations with t_f as the independent variable, instead of σ, the behavior of the system of equations obtained by admitting another singularity to the field of solutions may be determined. Whether the singularity is simple or terminal may be verified by using this procedure which is illustrated by means of examples in the next section.

APPLICATIONS OF TERMINAL IMBEDDING

In this section, the concepts which were previously outlined are applied to problems in the aerospace sciences. An example of a standard optimization problem, illustrating the concepts of terminal imbedding for linear time invariant systems, is first presented. Then, a nonlinear singular trajectory optimization problem, using a simplified model of an airplane attacking a target vehicle, is solved. The purpose of these examples is to clarify the application of the various concepts which have been presented previously.

Time-Optimal Control for Time-Invariant Linear Systems

Consider the time optimal control of a linear system, described by

$$\dot{x} = Ax + Bu \tag{32}$$

to the manifold described by $J(x^f) = 0$ from an initial point x^o with the control space Ω taken such that $|u_j| \leqq 1$ for $j = 1,2,\ldots,m$. Applying the parameterization of the terminal manifolds, one obtains

$$J(x^f,\sigma) = J(x^f) - J(x^o) + \sigma \tag{33}$$

Here $\sigma_f = J(x^o)$, and σ is varied between 0 and σ_f. The terminal control condition of theorem 2 for this example is

$$\frac{dt_f}{d\sigma} = \min_{u^f \in \Omega} \frac{-1}{\nabla_x^T J(x^f)\left[Ax^f + Bu^f\right]} \tag{34}$$

The minimization here requires that

$$u_j^f = -\text{sgn}\left[B^T \nabla_x J(x^f)\right]_j \tag{35}$$

for $j = 1,2,\ldots,m$ where

$$\text{sgn } x = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{Undefined} & x = 0 \end{cases}$$

Under the parameterization of the terminal manifolds used for this example, equations (22) and (31) may be rewritten as

$$\frac{dx^f}{dt_f} - \sum_{k=1}^{K(t_f)} h_k \frac{dS_k}{dt_f} = Ax^f + Bu^f \tag{36}$$

$$h_i^T \nabla_{xx} J(x^f) \frac{dx^f}{dt_f} - \nabla_x^T J(x^f) Ah_i \frac{dS_i}{dt_f} = \nabla_x^T J(x^f) Ah_i \tag{37}$$

for $i = 1,2,\ldots,K$, and equation (34) may be written as

$$\frac{d\sigma}{dt_f} = \left\|B^T \nabla_x J(x^f)\right\| - \nabla_x^T J(x^f) Ax^f \tag{38}$$

where the norm definition $\|v\| = \sum_{i=1}^{n} |v_i|$ and the function definition

$$h_i \triangleq e^{A(t_f - S_i)} B \Delta u_i \tag{39}$$

for $i = 1,2,\ldots,K$ have been used in equations (36), (37), and (38). Note that the class B singularities arising have been treated by a change of independent variable from σ to t_f. However, if the coefficient preceding dS_i/dt_f in equation (37) vanishes, then the equation will fail to define dS_i/dt_f and another type of singularity may be encountered. These have not been investigated since this problem has not yet occurred in application.

To illustrate the application of this analysis, consider the problem of the undamped harmonic oscillator with a natural frequency of unity. The equation of motion of the oscillator is presumed to be

$$\ddot{x}_h + x_h = u$$

Now, let the control force u be bounded so that $|u| \leq 1$. If the oscillator is disturbed from its equilibrium position, then consider the problem of applying the control force in such a way that $\dot{x}_h^2 + x_h^2$ becomes less than some arbitrary r^2 in the shortest possible time.

Formulating this problem in state vector notation, the equation of motion is

$$\dot{x} = Ax + Bu \tag{40}$$

where

$$x^T = (x_1, x_2) = (x_h, \dot{x}_h)$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{41}$$

and

$$B^T = (0,1) \tag{42}$$

The equation for the terminal manifold can be written as

$$J(x^f) = \frac{\left(x_1^f\right)^2 + \left(x_2^f\right)^2 - r^2}{2} = 0 \tag{43}$$

Now for this problem the fundamental matrix $\Phi(t,0) = e^{At}$ is

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \tag{44}$$

and the gradient matrices to be used in equations (36), (37), and (38) are

$$\nabla_x J(x^f) = x^f \tag{45}$$

and

$$\nabla_{xx} J\left(x^f\right) = I \tag{46}$$

From equation (35) the terminal control is

$$u^f = -\mathrm{sgn}\, x_2^f \tag{47}$$

Equations (41) to (47) may be substituted into equations (36), (37), and (38), along with equation (39) to construct a set of $3 + K(t_f)$ ordinary differential equations in the $3 + K(t_f)$ dependent variables $x_1^f, x_2^f, S_1, \ldots, S_K$ and σ. For the specific initial conditions $(x^o)^t = (2,2)$, figure 6 illustrates the evolution of the terminal time t_f and switching points $S_1(\sigma)$ and $S_2(\sigma)$ obtained by integration of the differential equations for this problem. In the range of values from $\sigma = 0$ to $\sigma \approx 0.6$, $K(t_f) = 0$; but at $\sigma \approx 0.6$, the final state crosses the curve $x_2^f = 0$ in the x_1^f, x_2^f plane (fig. 7), an

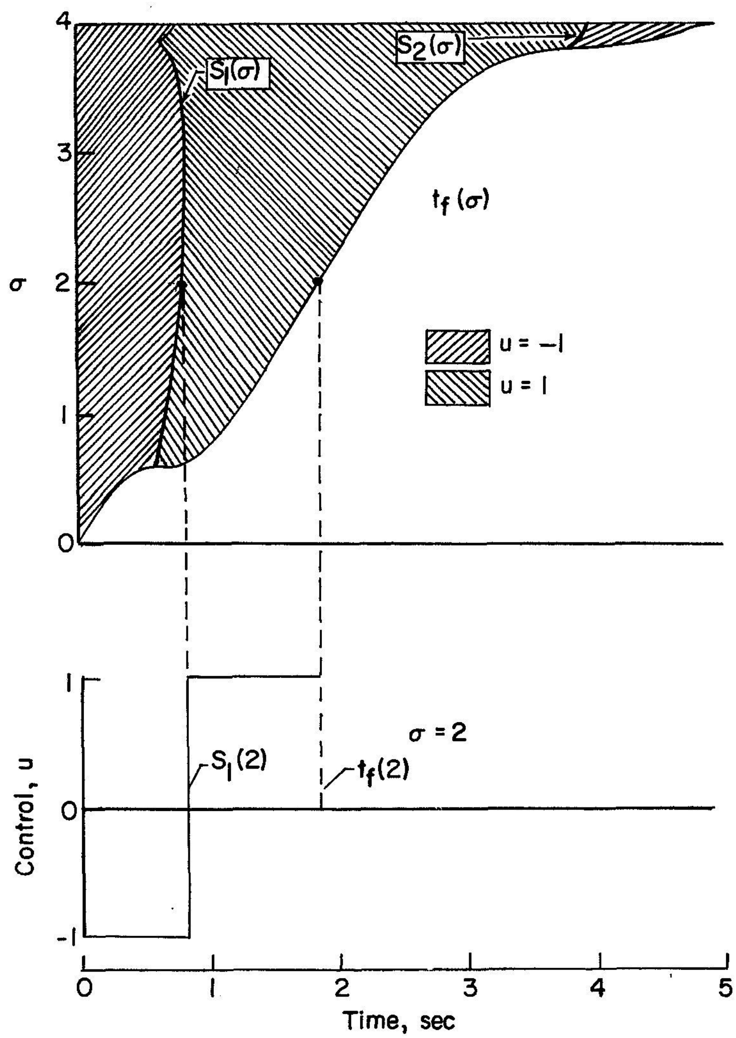


Figure 6.- Determination of control time history using t,σ plane.

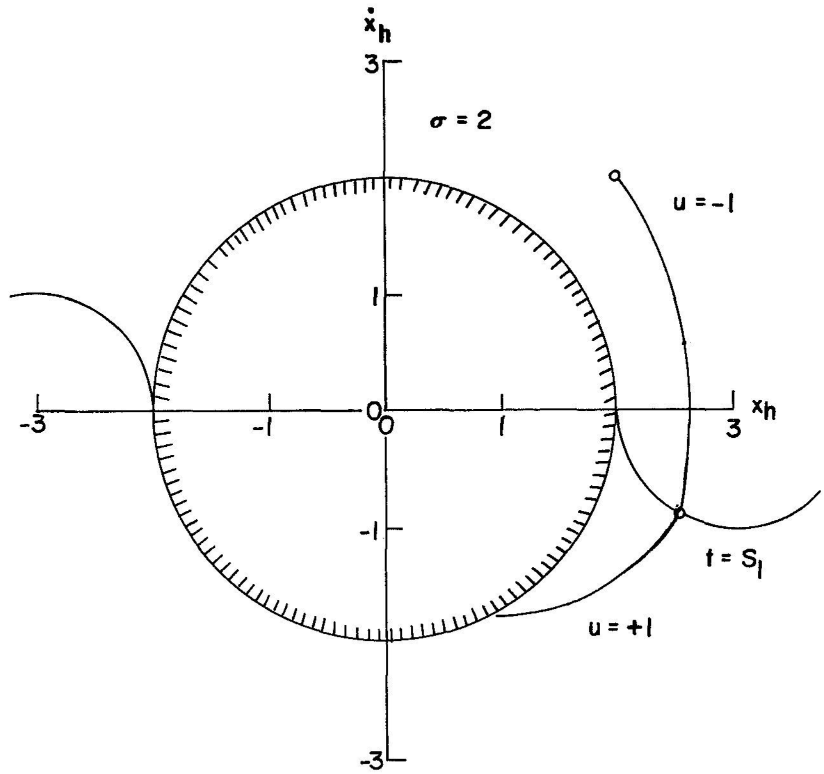


Figure 7.- Time-optimal solution to problem of harmonic oscillator for $\sigma = 2$.

increase in $K(t_f)$ from 0 to 1 being necessary. From equation (47) for terminal states where $x_2^f > 0$ the terminal control is $u^f = -1$ and for terminal states with $x_2^f < 0$ the terminal control is $u^f = 1$. Hence, at values of σ where the line $x_2^f = 0$ is crossed by the terminal state, a discontinuity in the control function is clearly indicated. Also, the time at which the discontinuity occurs is t_f which yields the initial condition $\lim_{\substack{\Delta \to 0 \\ \Delta > 0}} S_{K+1}(\sigma_1 + \Delta) = t_f(\sigma_1)$ for σ_1 corresponding to the singularity. The final state crosses the curve $x_2^f = 0$ again at $\sigma \approx 3.8$, and again, it is necessary to increase the value of $K(t_f)$ from 1 to 2 so as to allow the process to continue. Also indicated in figure 6 are the controls in the sectors of the t,σ plane for $0 < t < S_1$, $S_1 < t < S_2$, and $S_2 < t < t_f$. For this problem the value of σ corresponding to any given r can be obtained by using equations (33) and (43). This result is

$$\sigma = 4 - \frac{r^2}{2}$$

Figure 6 may be used to determine the control policy corresponding to time-optimal transition from the initial point (2,2) to the surface corresponding to a certain value of σ. For example, to transfer from (2,2) to a circle with $r = 2$ with the trace of constant

$\sigma = 2$, indicated by the dashed line in the t,σ plane of figure 6, yields the time-optimal control policy indicated on the lower curve in figure 6. When this control time history is employed in equation (40), the phase plane trace presented in figure 7 is obtained.

Applications to Nonlinear Analysis

For an application of terminal imbedding to nonlinear problems, consider a simplified model of an aerial attack in a horizontal plane where one airplane A attacks a target T. The geometry of the situation is illustrated in figure 8. The target T is

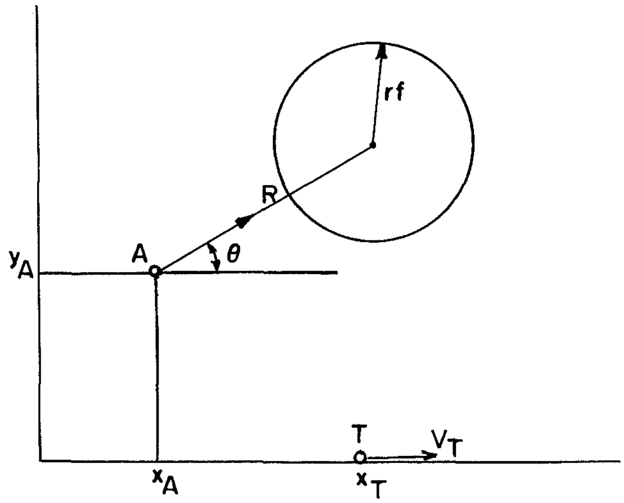


Figure 8.- Geometry of aerial attack.

constrained to fly along the X-axis at a constant speed V_T, and, initially, the target is assumed to be at the origin of the coordinate system. The equations of motion for the airplane are

$$\left.\begin{aligned} \dot{x}_A &= V \cos \theta \\ \dot{y}_A &= V \sin \theta \\ \dot{\theta} &= K_1 \frac{u_c}{V} \\ \dot{V} &= K_2\left(V_c^2 - V^2\right) \end{aligned}\right\} \tag{48}$$

In equations (48), the variables x_A and y_A represent the position of the airplane A in the horizontal plane; θ is the angle between the velocity vector and the X-axis, measured counterclockwise, and V is the speed of the airplane. The controls of the airplane are u_c corresponding to normal acceleration and V_c corresponding to thrust. The constant K_1 is a maximum normal acceleration so that $u_c \leqq 1$ and the constant K_2 depends on the drag of the airplane. For the airplane selected for the study at a speed of

277 meters/second and at an altitude of 3658 meters above sea level, the constant K_2 is approximately 4.54×10^{-5} meter^{-1}. For a maximum normal acceleration of five times the acceleration of gravity, the constant K_1 is approximately 49 meters/second^2. These constants have been used in this example computation.

Termination is assumed whenever the airplane maneuvers so that the target is within the circle indicated in figure 8. Mathematically, this termination condition is written in the form

$$J\left(x^f\right) = \frac{\left(x_A^f + R\cos\theta^f - x_T^f\right)^2 + \left(y_A^f + R\sin\theta^f\right)^2 - \left(r^f\right)^2}{2} = 0$$

wherein the state vector $\left(x^f\right)^T = \left(x_A^f, y_A^f, \theta^f, V^f, x_T^f\right)$ has been used.

The method of terminal imbedding is used to solve the problem of completing the attack in the shortest possible time. The family of terminal manifolds consists of

$$J\left(x^f,\sigma\right) = J\left(x^f\right) + (\sigma - 1)J(x^o) = 0$$

This is equivalent to selecting a termination zone radius large enough to insure instant termination for $\sigma = 0$ and gradually collapsing the radius until it becomes r^f at $\sigma = 1$. The radius of the termination zone corresponding to a value of σ is given by

$$r^2(\sigma) = \left(r^f\right)^2 + (1 - \sigma)J(x^o)$$

For this example r^f has been chosen to be 30 meters.

In order to apply the theory of terminal imbedding, the gradients of the terminal manifolds with respect to x^f are required. These gradients are

$$\nabla_x J\left(x^f\right) = \begin{bmatrix} x_A^f + R\cos\theta^f - x_T^f \\ y_A^f + R\sin\theta^f \\ R\left[y_A^f\cos\theta^f - \left(x_A^f - x_T^f\right)\sin\theta^f\right] \\ 0 \\ -\left(x_A^f + R\cos\theta^f - x_T^f\right) \end{bmatrix} \tag{49}$$

and

$$\nabla_{xx} J\left(x^f\right) = \begin{bmatrix} 1 & 0 & -R \sin \theta^f & 0 & -1 \\ 0 & 1 & R \cos \theta^f & 0 & 0 \\ -R \sin \theta^f & R \cos \theta^f & -R\left[x_A^f \sin \theta^f + \left(x_A^f - x_T^f\right) \cos \theta^f\right] & 0 & R \sin \theta^f \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & R \sin \theta^f & 0 & 1 \end{bmatrix}$$

For this problem the differential equations governing the transition matrix $\Phi\left(t,t_o,\sigma\right)$ can be integrated by quadrature. The transition matrix satisfies the equations

$$\frac{\partial \Phi}{\partial t}\left(t,t_o,\sigma\right) = A(t,\sigma)\ \Phi\left(t,t_o,\sigma\right)$$

$$\Phi\left(t_o,t_o,\sigma\right) \equiv I$$

where the matrix $A(t,\sigma)$, as defined by equation (10), is

$$A(t,\sigma) = \begin{bmatrix} 0 & 0 & -V \sin \theta & \cos \theta & 0 \\ 0 & 0 & V \cos \theta & \sin \theta & 0 \\ 0 & 0 & 0 & \dfrac{-K_1 u_c}{V^2} & 0 \\ 0 & 0 & 0 & -K_2 V & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{50}$$

and $B(t,\sigma)$, as defined by equation (11), is

$$B(t,\sigma) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{K_1}{V} & 0 \\ 0 & K_2 \\ 0 & 0 \end{bmatrix} \tag{51}$$

where the control vector $u^T = \left(u_c, V_c^2\right)$ has been used. The dependence of all variables in equations (50) and (51) on t and σ is understood. The matrix $\Phi\left(t, t_o, \sigma\right)$ is then

$$\Phi\left(t, t_o, \sigma\right) = \begin{bmatrix} 1 & 0 & y_A\left(t_o\right) - y_A(t) & \phi_{14} & 0 \\ 0 & 1 & x_A(t) - x_A\left(t_o\right) & \phi_{24} & 0 \\ 0 & 0 & 1 & \phi_{34} & 0 \\ 0 & 0 & 0 & \phi_{44} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{52}$$

where

$$\phi_{44} = \exp\left(-K_2 \int_{t_o}^{t} V(\tau)\, d\tau\right) \tag{53}$$

$$\phi_{34} = -\int_{t_o}^{t} \frac{K_1 u_c(\tau)}{V_c^2(\tau)} \exp\left(-K_2 \int_{t_o}^{\tau} V\left(\tau_1\right) d\tau_1\right) d\tau \tag{54}$$

$$\begin{aligned} \phi_{24} = & \int_{t_o}^{t} \sin \theta(\tau) \exp\left(-K_2 \int_{t_o}^{\tau} V\left(\tau_1\right) d\tau_1\right) d\tau \\ & - \int_{t_o}^{t} V(\tau) \cos \theta(\tau) \int_{t_o}^{\tau} \frac{K_1 u_c\left(\tau_1\right)}{V\left(\tau_1\right)} \exp\left(-K_2 \int_{t_o}^{\tau_1} V\left(\tau_2\right) d\tau_2\right) d\tau_1\, d\tau \end{aligned} \tag{55}$$

and

$$\begin{aligned} \phi_{14} = & \int_{t_o}^{t} \cos \theta(\tau) \exp\left(-K_2 \int_{t_o}^{\tau} V\left(\tau_1\right) d\tau_1\right) d\tau \\ & + \int_{t_o}^{t} V(\tau) \sin \theta(\tau) \int_{t_o}^{\tau} \frac{K_1 u_c\left(\tau_1\right)}{V\left(\tau_1\right)} \exp\left(-K_2 \int_{t_o}^{\tau_1} V\left(\tau_2\right) d\tau_2\right) d\tau_1\, d\tau \end{aligned} \tag{56}$$

In equations (52) to (56) the dependence of the functions on σ is also understood.

The terminal control condition requires that u^f must be selected so that it minimizes the function $G_1(x^f,u^f,\sigma)$ as defined in equation (19). For the time-optimal problem, this is equivalent to maximizing $\nabla_x^T J(x^f)\, f(x^f,u^f)$. From equations (48) and (49) the following equation is obtained:

$$\nabla_x^T J(x^f)\, f(x^f,u^f) = \alpha(x^f) + \frac{K_1 R}{V^f}\, \beta(x^f)\, u_c^f \tag{57}$$

where

$$\alpha(x^f) = \left(V^f \cos\theta^f - V_T\right)\left(x_A^f + R\cos\theta^f - x_T^f\right) + V^f \sin\theta^f\left(y_A^f + R\sin\theta^f\right)$$

$$\beta(x^f) = y_A^f \cos\theta^f - \left(x_A^f - x_T^f\right)\sin\theta^f$$

Hence,

$$u_c^f = \operatorname{sgn}\left(\beta(x^f)\right) \tag{58}$$

and is not defined when $\beta(x^f) = 0$. The speed control V_c^2 does not appear in equation (57). The procedure of extending the series representation of $\mu^T x^f(\sigma+\epsilon)$ to include second-order terms must be used to define the final speed control. To accomplish this, note that

$$G_2(x^f,u^f,\sigma) = \frac{dG_1(x^f,u^f,\sigma)}{d\sigma}$$

Thus,

$$G_2(x^f,u^f,\sigma) = \frac{1}{\left[\nabla_x^T J(x^f)\, f(x^f,u^f)\right]^2}\, \frac{\partial}{\partial V^f}\left[\nabla_x^T J(x^f)\, f(x^f,u^f)\right] + \gamma_1$$

where the term γ_1 is independent of V_c^2. Performing the indicated differentiation and using equation (58) gives

$$G_2(x^f,u^f,\sigma) = \frac{1}{\left[\nabla_x^T J(x^f)\, f(x^f,u^f)\right]^2}\, \delta(x^f)\, \frac{dV^f}{d\sigma} + \gamma_1$$

where

$$\delta(x^f) = \left(x_A^f - x_T^f + R\cos\theta^f\right)\cos\theta^f + \left(y_A^f + R\sin\theta^f\right)\sin\theta^f - \frac{2K_1 R\left|\beta(x^f)\right|}{\left(V^f\right)^2} \tag{59}$$

Since

$$\frac{dV^f}{d\sigma} = K_2\left[\left(V_c^f\right)^2 - \left(V^f\right)^2\right] + \gamma_2$$

where γ_2 is independent of V_c^2,

$$V_c^f = \begin{cases} V_{min} & \delta\left(x^f\right) < 0 \\ V_{max} & \delta\left(x^f\right) > 0 \end{cases} \tag{60}$$

where V_{min} is the minimum flying speed of the airplane and V_{max} is the maximum speed of the airplane.

Equations (58), (59), and (60) complete the definition of the final controls except for the singular control case. When $\beta\left(x^f\right)$ vanishes over an interval of finite measure in σ, the terminal control u_c^f is not defined by equation (58). Instead, one should employ the general point condition of theorem 2 over the time interval from $\left(t_f-\epsilon, t_f\right]$ with $\epsilon > 0$. The general point condition for this problem implies that

$$\nabla_x^T J\left(x^f\right) \Phi\left(t_f,t,\sigma\right) b_1(t,\sigma) = 0$$

where $b_1(t,\sigma)$ is the first column of the $B(t,\sigma)$ matrix of equation (51). Substituting equations (49), (51), and (52) into the last equation results in

$$\left(x_A^f - x_T^f + R\cos\theta^f\right)\left(y_A - y_A^f\right) + \left(y_A^f + R\sin\theta^f\right)\left(x_A^f - x_A\right) = 0 \tag{61}$$

where the side condition $\beta\left(x^f\right) = 0$ has been used. The last expression must hold over the time interval $\left(t_f-\epsilon, t_f\right]$ if the control u_c^f is not on the boundary of the control space. Hence, differentiating equation (61) with respect to time t and dividing by V gives

$$\left(x_A^f - x_T^f + R\cos\theta^f\right)\cos\theta - \left(y_A^f + R\sin\theta^f\right)\sin\theta = 0 \tag{62}$$

where the kinematic equations (48) have been used. Equation (62) is an identity in t over the interval $\left(t_f-\epsilon, t_f\right]$ and hence differentiating it with respect to time yields

$$\left[\left(x_A^f - x_T^f + R\cos\theta^f\right)\sin\theta + \left(y_A^f + R\sin\theta^f\right)\cos\theta\right]\frac{K_1 u_c}{V} = 0 \tag{63}$$

Now, since equation (62) holds, the bracketed term of equation (63) cannot vanish. It follows that the control u_c in equation (63) must vanish. Hence, from the definition of u^f as $\lim_{t\to t_f} u(t,\sigma)$, the general point condition infers that $u_c^f = 0$ for $\beta\left(x^f\right) = 0$ over an interval of finite measure in σ.

Now that the terminal control has been determined, one may form the set of equations (22), (24), and (31) for $K = 0$ to obtain

$$\left.\begin{aligned}
\frac{dx_A^f}{dt_f} &= V^f \cos \theta^f \\
\frac{dy_A^f}{dt_f} &= V^f \sin \theta^f \\
\frac{d\theta^f}{dt_f} &= \frac{K_1 u_c^f}{V^f} \\
\frac{dV^f}{dt_f} &= K_2\left[\left(V_c^f\right)^2 - \left(V^f\right)^2\right] \\
\frac{dx_T^f}{dt_f} &= V_T
\end{aligned}\right\} \tag{64}$$

with u_c^f defined according to equation (58) and V_c^f defined according to equation (60). Equation (64) may be integrated until either $\beta\left(x^f\right) = 0$ or $\delta\left(x^f\right) = 0$. For the purpose of this example, it is assumed that $\beta\left(x^f\right)$ vanishes first. This assumption is true for the example computation illustrated in figure 9. The initial conditions for the attack illustrated are

$$x_A^f = -304.8 \text{ meters}$$

$$y_A^o = 3048 \text{ meters}$$

$$\theta^o = 0^o$$

$$V^o = 304.8 \text{ meters/second}$$

$$x_T^o = 0 \text{ meter}$$

and the target speed is 304.8 meters/second. The maximum airplane speed is 457.2 meters/second for this example. The value of σ for which $\beta\left(x^f\right)$ vanishes has been computed to be 0.89037. For this example, $\beta(x^o) < 0$ and $\delta(x^o) > 0$ sets the controls in zone ξ_1 as $u_c = -1$ and $V_c = V_{c,max}$ as illustrated in figure 10.

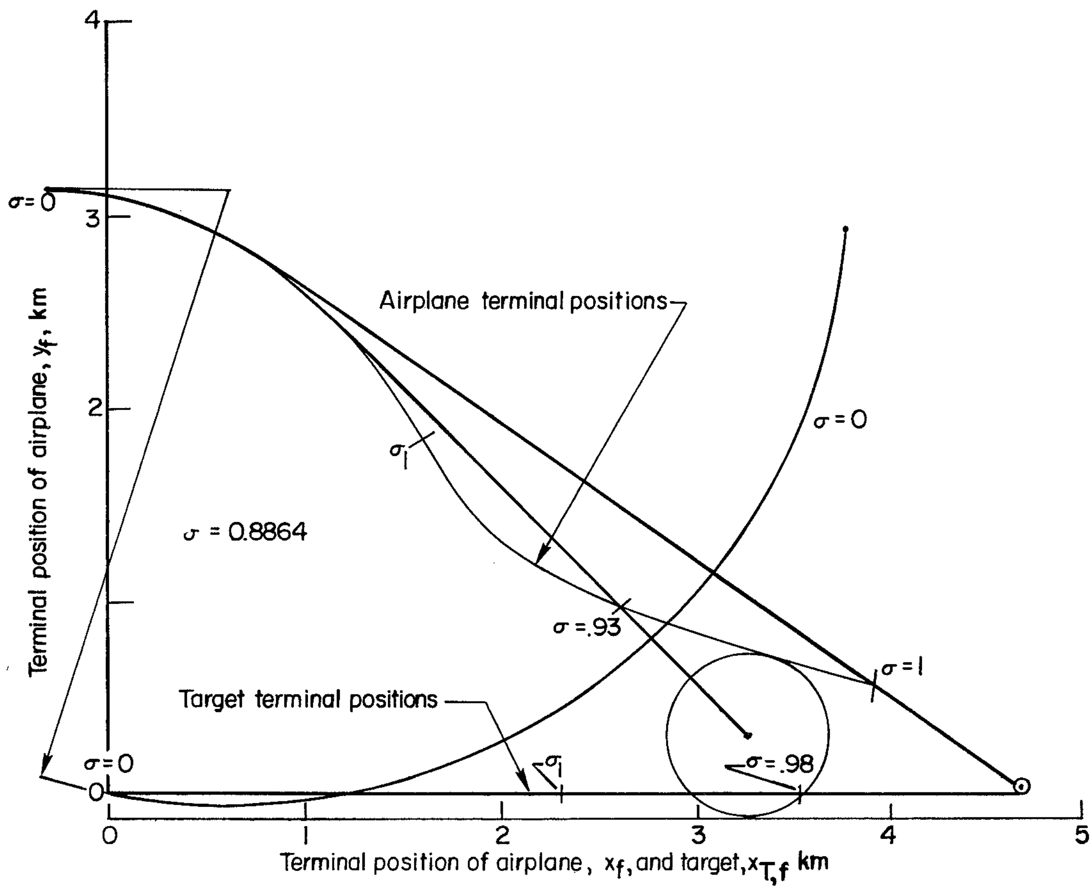


Figure 9.- Terminal positions of both airplane and target.

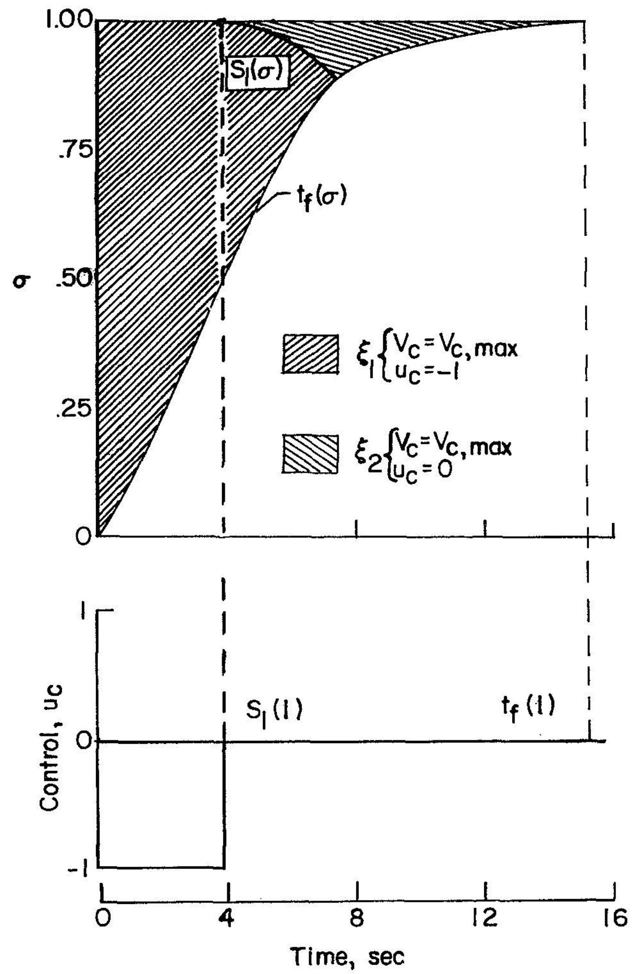


Figure 10.- Terminal time and switching time in t, σ plane for aerial attack.

For values of σ greater than 0.89037, the region ξ_2 in figure 10 must be included in the analysis and equations (22) and (31) must be formulated for $K = 1$ with the controls in ξ_2 being $u_c = 0$ and $V_c = V_{c,max}$. This set of equations after considerable reduction for this example is

$$
\left.
\begin{aligned}
&\frac{dx_A^f}{dt_f} + \frac{K_1\left(u_1^- - u_1^+\right)}{V^1}\left(y_A^f - y_A^1\right)\frac{dS_1}{dt_f} = V^f \cos \theta^f \\
&\frac{dy_A^f}{dt_f} - \frac{K_1\left(u_1^- - u_1^+\right)}{V^1}\left(x_A^f - x_A^1\right)\frac{dS_1}{dt_f} = V^f \sin \theta^f \\
&\frac{d\theta^f}{dt_f} - \frac{K_1\left(u_1^- - u_1^+\right)}{V^1}\frac{dS_1}{dt_f} = \frac{K_1 u_c^f}{V^f} \\
&\frac{dV^f}{dt_f} = K_2\left[\left(V_c^f\right)^2 - \left(V^f\right)^2\right] \\
&\frac{dx_T^f}{dt_f} = V_T \\
&a_1 \frac{dx_A^f}{dt_f} + a_2 \frac{dy_A^f}{dt_f} + a_3 \frac{d\theta^f}{dt_f} + a_4 \frac{dx_T^f}{dt_f} + a_5 \frac{dS_1}{dt_f} = a_6
\end{aligned}
\right\} \tag{65}
$$

where $u_1^- = -1$, $u_1^+ = 0$, $u_c^f = 0$, and

$$
\left.
\begin{aligned}
&a_1 = y_A^1 - y_A^f - R \sin \theta^f \\
&a_2 = x_A^f - x_A^1 + R \cos \theta^f \\
&a_3 = -R\left[y_A^1 \sin \theta^f + \left(x_A^1 - x_A^f\right) \sin \theta^f\right] \\
&a_4 = -a_1 \\
&a_5 = \left(x_A^f - x_T^f + R \cos \theta^f\right)\left[V^1 \sin \theta^1 + \left(x_A^1 - x_A^f\right)\frac{K_1\left(u_1^- - u_1^+\right)}{V^1}\right] \\
&\qquad - \left(y_A^f + R \sin \theta^f\right)\left[V^1 \cos \theta^1 - \left(y_A^1 - y_A^f\right)\frac{K_1\left(u_1^- - u_1^+\right)}{V^1}\right] \\
&a_6 = V^f\left[\left(x_A^f - x_T^f\right) \sin \theta^f - y_A^f \cos \theta^f\right]
\end{aligned}
\right\} \tag{66}
$$

and the notation $x_A^1 = x_A(S_1,\sigma)$, $y_A^1 = y_A(S_1,\sigma)$, $V^1 = V(S_1,\sigma)$ has been used. When integrated, the set of equations (65) yields the portion of the terminal position graphs in figure 9 from σ_1 to the final value of σ of unity corresponding to a bomb radius of 30.48 meters. The variation in terminal time and switching time obtained is indicated in figure 10. The optimal control time history is the trace on the t,σ plane at $\sigma = 1$. If, during the process of integrating equations (65) the function $\delta(x^f)$ had vanished, it would have been necessary to formulate equations (22) and (31) for $K = 2$ and introduce another region ξ_3 in the t,σ plane.

CONCLUDING REMARKS

Imbedding theory has been applied in this report to variational or optimal control problems. This application was called terminal imbedding. Terminal imbedding involves imbedding the optimization problem of interest in a family of problems parameterized by the terminal conditions. By collapsing the terminal conditions of the family of problems onto those of the original problem while continuously modifying the control function appropriately the solution to the original trajectory optimization problem may be obtained. Using this method, necessary conditions have been derived analogous to those of the variational calculus and the maximum principle of Pontryagin. This analogy is established in an appendix. In this work these conditions are called the switching point condition and the general point condition. An additional condition called the terminal control condition results from the terminal imbedding theory. This condition is not included in the maximum principle since it is necessary only if the optimal trajectory is imbedded in a continuous field of solutions; however, it is important in the construction of the field of optimal trajectories. With the use of the imbedding concept, the placement of singular subarcs in an optimal trajectory is determined. This is not easily accomplished by the maximum principle.

An important consideration in the practical use of terminal imbedding is the inclusion of discontinuities in the field of optimal trajectories. These arise at terminal manifolds where the continuability of the imbedding process is not assured or where the terminal control condition fails to define the terminal control. In applying the method of conversion to differential form for the solution of nonlinear algebraic equations, it was possible to derive a sufficiency condition which assures the continuability of the imbedding process (theorem 1). This condition has not been generalized to trajectory optimization problems and when obtaining solutions to these problems it is necessary to examine each singular situation as it arises to determine the continuability of the imbedding process. The nature of the singular points is that they may prevent continuation of the imbedding process or they may require the admission of another point of discontinuity in the field of optimal controls. For problems where $\frac{\partial u}{\partial \sigma}(t,\sigma) \equiv 0$ in regions of the t,σ plane where the control is free of discontinuities, the evolution of the points of discontinuity as a function of the imbedding parameter σ was obtained and was applied to a simplified model of an aerial attack as an illustrative example.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., February 24, 1971.

APPENDIX A

ANALOGY WITH THE MAXIMUM PRINCIPLE

Before demonstrating an analogy between the analysis presented in this report and the maximum principle of Pontryagin (ref. 4), a statement of the major results of the maximum principle as applied to the problem considered in this work is presented. Then some difficulties in applying the maximum principle to singular problems in control are discussed and the extraction of necessary conditions pertinent to singular control problems from the maximum principle using the approach of Johnson and Gibson (ref. 7) is outlined. Finally, an analogy between the results of the maximum principle and those of this report is demonstrated.

Presentation of the maximum principle is facilitated if a Hamiltonian function is first defined according to

$$H(x,\psi,u) \triangleq \psi^T f(x,u)$$

where ψ is an (n+1)-dimensional vector sometimes called the costate vector. The state and costate vectors satisfy the canonical equations

$$\dot{x} = \nabla_\psi H(x,\psi,u) = f(x,u)$$

$$\dot{\psi} = -\nabla_x H(x,\psi,u)$$

The basic results of the maximum principle are that any control u which minimizes $x_{n+1}(t_f)$ must satisfy the following conditions:

(1) It must maximize $H(x,\psi,u)$ over the entire trajectory

(2) At termination $H(x,\psi,u)\big|_{t=t_f} = 0$

(3) At termination the vector $\psi^n(t_f)$, the n-dimensional vector formed by the first n components of $\psi(t_f)$, is normal to the terminal manifold and $\psi_{n+1}(t_f)$ must be nonpositive and can be taken as -1

In the ideal application of the maximum principle, condition (1) is first used to define the optimal control in terms of the state and costate variables. That is, the optimal control u is related to x,ψ via the relation

$$u = \underset{v\in\Omega}{\operatorname{argmax}}\, H(x,\psi,v) = u(x,\psi)$$

This result, when substituted into the canonical equations, yields a complete set of $2n + 2$ ordinary differential equations

$$\dot{x} = f\big(x,u(x,\psi)\big)$$

$$\dot{\psi} = -\nabla_x H\big(x,\psi,u(x,\psi)\big)$$

At this point, conditions (2) and (3) of the maximum principle are used to provide a complete set of boundary conditions for the last set of differential equations. These boundary conditions are

$$\psi_{n+1}\left(t_f\right) = -1$$

$$H\big(x,\psi,u(x,\psi)\big)\Big|_{t=t_f} = 0$$

$$\psi^n\left(t_f\right) = k\nabla_x J\left(x^n\left(t_f\right)\right)$$

where k is a scalar. These conditions together with the initial conditions $x^n\left(t_o\right) = x^o$ and $x_{n+1}\left(t_o\right) = 0$ form $2n + 2$ two-point boundary conditions for the problem and thereb reduce the optimization problem to a two-point boundary value problem.

The terminology "singular control" refers to those special situations which arise in control theory where the idealized application of the maximum principle envisioned in the last paragraph cannot be realized. The basic difficulty is that the condition that u maximizes $H(x,\psi,u)$ at every point of the trajectory may not be adequate in itself to define the optimal control; thus, the function $u(x,\psi)$ cannot be defined, making the conversion of the optimization problem into a two-point boundary value problem impossible. One of the problems in which this difficulty arises is the so-called "linear optimization problem" for bounded control where the Hamiltonian function is a linear function of a scalar control u and can be written as

$$H(x,\psi,u) = \tilde{I}(x,\psi) + u\tilde{F}(x,\psi)$$

Over any interval of finite duration where $F(x,\psi)$ vanishes along an optimal trajectory the condition that u maximizes $H(x,\psi,u)$ obviously fails to define u. Pontryagin, Boltyanskii, Gamkrelidze, and Mishchenko (ref. 4) have shown that the condition that $H(x,\psi,u)\big|_{t=t_f} = 0$ can be strengthened for autonomous free end-time problems to infer that $H(x,\psi,u) \equiv 0$ along the entire optimal trajectory. Johnson and Gibson (ref. 7) have employed this condition over the singular arcs to yield the identities

$$\tilde{F}(x,\psi) = \dot{\tilde{F}}(x,\psi) = \ddot{\tilde{F}}(x,\psi) = \dots = 0$$

and

$$\tilde{I}(x,\psi) = \dot{\tilde{I}}(x,\psi) = \ddot{\tilde{I}}(x,\psi) = \dots = 0$$

which must be satisfied by any singular arc and can be used to determine the nature of the control along the singular arc. This procedure will be illustrated by a simple example.

Consider the boat steering problem whose nomenclature is given in figure 11.

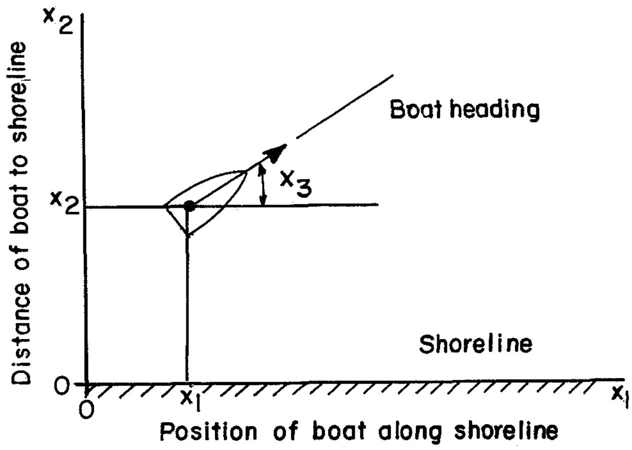


Figure 11.- Nomenclature for boat steering problem.

The boat travels with unit speed with a turning capability limited to 1 radian per second and the following kinematic equations are descriptive of the motion of the boat:

$$\dot{x}_1 = \cos x_3$$

$$\dot{x}_2 = \sin x_3$$

$$\dot{x}_3 = u$$

where the control u is such that $|u| \leq 1$. The problem considered here is to reach the shoreline (the x_1 axis) in the shortest possible time. The Hamiltonian function is

$$H(x,\psi,u) = \tilde{I}(x,\psi) + u\tilde{F}(x,\psi)$$

where

$$\tilde{I}(x,\psi) = \psi_1 \cos x_3 + \psi_2 \sin x_3 - 1$$

$$\tilde{F}(x,\psi) = \psi_3$$

The costate differential equations are

$$\dot{\psi}_1 = \dot{\psi}_2 = 0$$

$$\dot{\psi}_3 = \psi_1 \sin x_3 - \psi_2 \cos x_3$$

According to the maximum principle the optimal control must satisfy the equation $u = \operatorname{sgn} \psi_3$ provided that $\psi_3 \neq 0$ over an interval of finite duration. If $\psi_3 \equiv 0$ over an interval of finite duration, then the relations

$$\tilde{F}(x,\psi) = 0$$

$$\dot{\tilde{F}}(x,\psi) = 0$$

$$\ddot{\tilde{F}}(x,\psi) = 0$$

$$\vdots$$

yield the results

$$\psi_3 = 0$$

$$\psi_1 \sin x_3 - \psi_2 \cos x_3 = 0$$

$$\left[\psi_1 \cos x_3 + \psi_2 \sin x_3\right] u = 0$$

respectively, for ψ_1 and ψ_2 nontrivial constants. The last two equations are consistent only for $u \equiv 0$. The singular control can only be $u \equiv 0$. At this point it is noted that this is the isolated control action which is a straight course for the boat. All optimal voyages which are long enough will involve this singular arc and in many cases the singular arc will represent the major portion of the distance traveled. Therefore, construction of the singular arc cannot be discarded as a trivial problem. Optimal trajectories for this specific problem are segments of two types of trajectories: one represents full turning capability employed to direct the course of the boat perpendicular to the shoreline, and the other is the singular portion ($u = 0$), the straight line portion of the trajectories.

To verify an analogy with the maximum principle, define the vector $\psi(t,\sigma)$ according to the relation

$$\psi(t,\sigma) = -\Phi^T\left(t_f,t,\sigma\right) C^T\left(x^f,u^f,\sigma\right) \mu$$

where Φ and C are defined by equations (13) and (18), respectively. Note that the vector $\psi(t,\sigma)$ satisfies the relations

$$\frac{\partial\psi}{\partial t}(t,\sigma) = -A^T(t,\sigma)\ \psi(t,\sigma)$$

$$\psi\left(t_f,\sigma\right) = -C^T\left(x^f,u^f,\sigma\right)\mu$$

identically in σ.

APPENDIX A

Now consider the function $H \triangleq \psi^T(t,\sigma)\, f(x,u)$. Write $H_f \triangleq \psi^T(t_f,\sigma)\, f\left(x^f,u^f\right)$. From the definition of H_f,

$$H_f = -\mu^T\, C\left(x^f,u^f,\sigma\right) f\left(x^f,u^f\right)$$

so that from the definition of $C\left(x^f,u^f,\sigma\right)$

$$H_f = \mu^T \left[\frac{f\left(x^f,u^f\right) \nabla_x^T J\left(x^f,\sigma\right)}{\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right)} - I \right] f\left(x^f,u^f\right)$$

from which it immediately follows that

$$H_f = 0$$

provided $\nabla_x J\left(x^f,\sigma\right) f\left(x^f,u^f\right) \neq 0$.

At termination the vector $\psi(t,\sigma)$ satisfies the expression

$$\psi\left(t_f,\sigma\right) = -C^T\left(x^f,u^f,\sigma\right)\mu = -\mu + \frac{f_{n+1}\left(x^f,u^f\right)}{\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right)} \nabla_x J\left(x^f,\sigma\right)$$

Note that the $n+1$ component of $\psi(t_f,\sigma)$ is -1 and also that $\psi^n\left(t_f,\sigma\right)$ is normal to the terminal manifold generated by each value of σ.

Finally, note that $\nabla_u H$ is given by the equation

$$\nabla_u H = \psi^T(t,\sigma)\, \nabla_u f(x,u) = -\mu^T\, C\left(x^f,u^f,\sigma\right) \Phi\left(t_f,t,\sigma\right) B(t,\sigma)$$

Hence, the general point condition of theorem 2 implies that, for any admissible control variation, the component of $\nabla_u H$ must be nonpositive and it follows that $u(t,\sigma)$ must maximize H since all admissible changes in u must not increase it. Any trajectory which satisfies the maximum principle with the costate variables taken at $\psi(t,\sigma)$ also satisfies the conditions of theorem 2 of this report.

APPENDIX B

PROOF OF THEOREM 2

It is the purpose of this appendix to provide a rigorous justification of theorem 2 of this report. Some equations given in the discussion of theorem 2 in the main body of this report are needed in the proof of the theorem; therefore, these equations are restated in this appendix. Because of the requirement that all admissible trajectories satisfy the governing differential equation (eq. (9)) the field of optional trajectories must satisfy the partial differential equation

$$\frac{\partial x}{\partial t}(t,\sigma) = f\big(x(t,\sigma),u(t,\sigma)\big) \tag{8}$$

on every interval ξ_i; hence,

$$x(t,\sigma) = x^o + \int_{t_o}^{t} f\big(x(\tau,\sigma),u(\tau,\sigma)\big)\, d\tau$$

Because of the requirement that $x(t,\sigma)$ and $u(t,\sigma)$ must be differentiable in both t and σ on intervals ξ_i and that the functions $S_i(\sigma)$ be differentiable, it follows that

$$\frac{\partial x}{\partial \sigma}(t,\sigma) = \int_{t_o}^{t} \left[A(\tau,\sigma)\,\frac{\partial x}{\partial \sigma}(\tau,\sigma) + B(\tau,\sigma)\,\frac{\partial u}{\partial \sigma}(\tau,\sigma)\right] d\tau + \sum_{i=1}^{K(t)} \Delta f_i(\sigma)\,\frac{dS_i}{d\sigma}(\sigma) \tag{9}$$

for t,σ in ξ_{K+1}. In equation (9), $K(t)$ represents the number of $S_i(\sigma)$ functions contained in the interval $[t_o,t)$ and the following definitions have been used:

$$A(t,\sigma) \triangleq \nabla_x f\big(x(t,\sigma),u(t,\sigma)\big) \tag{10}$$

$$B(t,\sigma) \triangleq \nabla_u f\big(x(t,\sigma),u(t,\sigma)\big) \tag{11}$$

and

$$\Delta f_i(\sigma) \triangleq \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \left[f\big(x(S_i,\sigma),u(S_i-\epsilon,\sigma)\big) - f\big(x(S_i,\sigma),u(S_i+\epsilon,\sigma)\big)\right]$$

Equation (9) is a linear Volterra integral equation for $\frac{\partial x}{\partial \sigma}(t,\sigma)$. If the transformation

$$\frac{\partial x}{\partial \sigma}(t,\sigma) = \chi(t,\sigma) + \sum_{i=1}^{K(t)} \Delta f_i(\sigma)\,\frac{dS_i}{d\sigma}(\sigma)$$

is used in equation (9), the function $\chi(t,\sigma)$ satisfies the equation

$$\chi(t,\sigma) = \int_{t_o}^{t} A(\tau,\sigma)\ \chi(\tau,\sigma)\ d\tau + \int_{t_o}^{t} h(\tau,\sigma)\ d\tau$$

wherein

$$h(t,\sigma) \triangleq B(t,\sigma)\ \frac{\partial u}{\partial \sigma}(t,\sigma) + A(t,\sigma) \sum_{i=1}^{K(t)} \Delta f_i(\sigma)\ \frac{dS_i}{d\sigma}(\sigma) \tag{B1}$$

The unique solution for χ is

$$\chi(t,\sigma) = \int_{t_o}^{t} \Phi(t,\tau,\sigma)\ h(\tau,\sigma)\ d\tau$$

which can be verified by direct substitution where $\Phi\left(t,t_o,\sigma\right)$ satisfies the equations

$$\frac{\partial \Phi}{\partial t}\left(t,t_o,\sigma\right) = A(t,\sigma)\ \Phi\left(t,t_o,\sigma\right) \tag{13}$$

and

$$\Phi\left(t_o,t_o,\sigma\right) \equiv I \tag{14}$$

Here, I is the identity matrix and $\Phi\left(t,t_o,\sigma\right)$ is a (n+1)-square matrix function. Using the definitions of $\chi(t,\sigma)$ and $h(t,\sigma)$, the solution of $\frac{\partial x}{\partial \sigma}(t,\sigma)$ can be obtained. After some reduction $\frac{\partial x}{\partial \sigma}(t,\sigma)$, in turn, can be written as

$$\frac{\partial x}{\partial \sigma}(t,\sigma) = \sum_{i=1}^{K(t)} \Phi\left(t,S_i,\sigma\right) \Delta f_i(\sigma)\ \frac{dS_i}{d\sigma} + \int_{t_o}^{t} \Phi(t,\tau,\sigma)\ B(\tau,\sigma)\ \frac{\partial u}{\partial \sigma}(\tau,\sigma)\ d\tau \tag{12}$$

for all t,σ in ξ_{K+1}. Equations (8) and (12) may be used to evaluate the total derivative of $x(t,\sigma)$ with respect to σ along the line $t = t_f(\sigma)$. This result is

$$\frac{dx^f}{d\sigma}(\sigma) = \sum_{i=1}^{K(t_f)} \Phi\left(t_f,S_i,\sigma\right) \Delta f_i(\sigma)\ \frac{dS_i}{d\sigma}(\sigma) + \int_{t_o}^{t_f} \Phi\left(t_f,\tau,\sigma\right) B(\tau,\sigma)\ \frac{\partial u}{\partial \sigma}(\tau,\sigma)\ d\tau$$

$$+ f\left(x^f,u^f\right) \frac{dt_f}{d\sigma}(\sigma) \tag{15}$$

where the notations $x^f(\sigma) \triangleq x\left(t_f(\sigma),\sigma\right)$ and $u^f(\sigma) \triangleq u\left(t_f(\sigma),\sigma\right)$ have been used.

APPENDIX B

Each trajectory of the imbedding family must satisfy the terminal constraint

$$J\left(x^f,\sigma\right) \equiv 0$$

for all σ. Hence variations in x^f, given by equation (22), must satisfy

$$\frac{\partial J}{\partial \sigma}\left(x^f,\sigma\right) + \nabla_x^T J\left(x^f,\sigma\right) \frac{dx^f}{d\sigma}(\sigma) = 0 \tag{16}$$

identically in σ. An expression for the required variation in terminal time t_f can be obtained by substituting equation (15) into equation (16) and solving the resulting expression for $\frac{dt_f}{d\sigma}(\sigma)$. The result $dt_f/d\sigma$ can be inserted into equation (15) to obtain an expression for $dx^f/d\sigma$, which is required to satisfy the differential constraint equation (16). Thus, one obtains

$$\frac{dx^f}{d\sigma} = -\frac{\partial J}{\partial \sigma}\left(x^f,\sigma\right)\frac{f\left(x^f,u^f\right)}{\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right)} + C\left(x^f,u^f,\sigma\right)\left[\sum_{i=1}^{K(t_f)} \Phi\left(t_f,S_i,\sigma\right) \Delta f_i(\sigma) \frac{dS_i}{d\sigma}(\sigma) + \int_{t_o}^{t_f} \Phi\left(t_f,\tau,\sigma\right) B(\tau,\sigma) \frac{\partial u}{\partial \sigma}(\tau,\sigma)\, d\tau\right] \tag{17}$$

wherein

$$C\left(x^f,u^f,\sigma\right) \triangleq I - \frac{f\left(x^f,u^f\right) \nabla_x^T J\left(x^f,\sigma\right)}{\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right)} \tag{18}$$

An expression for the variation in the performance index x_{n+1}^f can be obtained from equation (17). This is accomplished by premultiplying the equation by an (n+1)-dimensional unit vector μ whose n+1 component is unity. The poles of $\frac{dx_{n+1}^f}{d\sigma}(\sigma)$ serve to define the boundaries of the ξ_i intervals, hence the functions $\frac{dS_i}{d\sigma}(\sigma)$. For other regular points where $\frac{dx_{n+1}^f}{d\sigma}(\sigma)$ is well defined and continuous one can use a Taylor's series expansion with a Lagrangian remainder to represent $x_{n+1}^f(\sigma)$. Hence,

$$x_{n+1}^f(\sigma+\epsilon) = x_{n+1}^f(\sigma) + \frac{dx_{n+1}^f}{d\sigma}\left(\sigma_1\right) \epsilon \tag{B2}$$

for $\epsilon > 0$ and some σ_1 satisfying the condition $\sigma < \sigma_1 < \sigma + \epsilon$. Equation (B2) holds so long as $\frac{dx^f_{n+1}}{d\sigma}(\sigma)$ is continuous in the interval $[\sigma, \sigma + \epsilon]$. Hence, if $x^f_{n+1}(\sigma)$ is a minimum, for $x^f_{n+1}(\sigma+\epsilon)$ to be a minimum with respect to ϵ then it is necessary that $\frac{dx^f_{n+1}}{d\sigma}(\sigma_1)$ be a minimum. Furthermore on considering the limiting form of equation (B2) as $\epsilon \to 0$ the continuity of $\frac{dx^f_{n+1}}{d\sigma}(\sigma)$ requires that $\frac{dx^f_{n+1}}{d\sigma}(\sigma)$ be a minimum. From equation (17) it is seen that $\frac{dx^f_{n+1}}{d\sigma}(\sigma)$ depends on $dS_i/d\sigma$ with $i = 1,2,\ldots,K(t_f)$, the function $\frac{\partial u}{\partial \sigma}(t,\sigma)$, and the terminal control u^f. The minimization of $\frac{dx^f_{n+1}}{d\sigma}(\sigma)$ must, therefore, be taken with respect to this set of quantities.

Next, consider minimization of $\mu^T \frac{dx^f}{d\sigma}(\sigma)$ with respect to the quantities mentioned in the preceding paragraph. To do this one should consider the integral term of the form

$$J\left[\frac{\partial u}{\partial \sigma}\right] = \int_{t_o}^{t_f} M^T(t,\sigma) \frac{\partial u}{\partial \sigma}(t,\sigma)\, dt \tag{B3}$$

wherein $M(t,\sigma)$ and $\frac{\partial u}{\partial \sigma}(t,\sigma)$ are m-dimensional vector functions of t and σ. The requirement of constraining $u(t,\sigma)$ to be in a manifold region Ω leads to directional constraints on $\frac{\partial u}{\partial \sigma}(t,\sigma)$. In other words, at any given time t when $u(t,\sigma)$ is on a boundary of Ω a change in σ at constant t must be consistent with the constraints and hence $\frac{\partial u}{\partial \sigma}(t,\sigma)$ cannot be directed out of Ω at that boundary point. Also if $u(t,\sigma)$ is not at a boundary point of Ω then there are no directional constraints and $\frac{\partial u}{\partial \sigma}(t,\sigma)$ is totally unconstrained.

Statement 1: A necessary condition for $\frac{\partial u}{\partial \sigma}(t,\sigma)$ to minimize $J[\cdot]$ is that over any interval of finite measure in t, say, ξ where the ith component of u satisfies $\alpha_i < u_i < \beta_i$ then the ith component of $M(t,\sigma)$ vanishes over ξ.

Proof by contradiction: Assume that $\partial u/\partial \sigma$ minimizes $J[\cdot]$ and that $\alpha_i < u_i < \beta_i$ over ξ when $M_i(t,\sigma) \neq 0$. Select a comparison arc $\overline{\partial u}/\partial \sigma$ such that $\left(\frac{\overline{\partial u}}{\partial \sigma}\right)_j \equiv \left(\frac{\partial u}{\partial \sigma}\right)_j$ for $j \neq i$ and $\left(\frac{\overline{\partial u}}{\partial \sigma}\right)_i \equiv \left(\frac{\partial u}{\partial \sigma}\right)_i$ except over the interval ξ where

$$\left(\frac{\overline{\partial u}}{\partial \sigma}\right)_i(t,\sigma) \equiv \left(\frac{\partial u}{\partial \sigma}\right)_i(t,\sigma) - \epsilon M_i(t,\sigma)$$

with $\epsilon > 0$. The value of the functional $J[\cdot]$ evaluated along this comparison arc is

$$J\left[\overline{\frac{\partial u}{\partial \sigma}}\right] = J\left[\frac{\partial u}{\partial \sigma}\right] - \epsilon \int_{\xi} M_i^2(t,\sigma)\, dt$$

so that $J\left[\overline{\partial u}/\partial\sigma\right] < J\left[\partial u/\partial\sigma\right]$. This contradicts the assumption that $\partial u/\partial\sigma$ minimizes $J[\cdot]$. Hence, $M_i(t,\sigma) = 0$ over the interval ξ.

Statement 2: A necessary condition for $\partial u/\partial\sigma$ to minimize $J[\cdot]$ is that over any interval of finite measure in t, say, ξ where $u_i \equiv \beta_i$ over ξ, $M_i(t,\sigma) \leqq 0$ and if $M_i(t,\sigma) < 0$ then $\left(\frac{\partial u}{\partial \sigma}\right)_i(t,\sigma) = 0$.

Proof by contradiction: First, the condition that $M_i(t,\sigma) \leqq 0$ over ξ is proved. Assume that over ξ, $M_i(t,\sigma) > 0$ and that $\frac{\partial u}{\partial \sigma}(t,\sigma)$ minimizes $J[\cdot]$. The constraint that $u_i = \beta_i$ implies that $\left(\frac{\partial u}{\partial \sigma}\right)_i(t,\sigma) \leqq 0$ over ξ. Select then a comparison curve $\overline{\frac{\partial u}{\partial \sigma}}(t,\sigma)$ such that $\left(\overline{\frac{\partial u}{\partial \sigma}}\right)_j(t,\sigma) \equiv \left(\frac{\partial u}{\partial \sigma}\right)_j(t,\sigma)$ for $j \neq i$ and $\left(\overline{\frac{\partial u}{\partial \sigma}}\right)_j(t,\sigma) \equiv \left(\frac{\partial u}{\partial \sigma}\right)_j(t,\sigma)$ except over the interval ξ where $\left(\overline{\frac{\partial u}{\partial \sigma}}\right)_i(t,\sigma) \equiv \left(\frac{\partial u}{\partial \sigma}\right)_i(t,\sigma) - \epsilon$ for some $\epsilon > 0$. This arc satisfies the directional constraints. The value of the functional $J[\cdot]$ evaluated along the comparison arc satisfies

$$J\left[\overline{\frac{\partial u}{\partial \sigma}}\right] = J\left[\frac{\partial u}{\partial \sigma}\right] - \epsilon \int_{\xi} M_i(t,\sigma)\, dt$$

Hence, if $M_i(t,\sigma) > 0$ over ξ then $J\left[\overline{\partial u}/\partial\sigma\right] < J\left[\partial u/\partial\sigma\right]$ which contradicts the assumption that $\frac{\partial u}{\partial \sigma}(t,\sigma)$ minimizes $J[\cdot]$. From this contradiction the conclusion that $M_i \leqq 0$ over ξ is obtained.

Next the fact that if $M_i(t,\sigma) < 0$ over ξ then $\left(\frac{\partial u}{\partial \sigma}\right)_i(t,\sigma) = 0$ over ξ is proved. Assume that $\partial u/\partial\sigma$ minimizes $J[\cdot]$ and that over ξ, $M_i(t,\sigma) < 0$ with $\left(\frac{\partial u}{\partial \sigma}\right)_i(t,\sigma) \neq 0$. The directional constraints on $\frac{\partial u}{\partial \sigma}(t,\sigma)$ require that $\left(\frac{\partial u}{\partial \sigma}\right)_i \leqq 0$. Hence only $\left(\frac{\partial u}{\partial \sigma}\right)_i < 0$ over ξ need be considered. If $\left(\frac{\partial u}{\partial \sigma}\right)_i < 0$ over ξ construct a comparison arc $\overline{\partial u}/\partial\sigma$ such that $\left(\overline{\frac{\partial u}{\partial \sigma}}\right)_j \equiv \left(\frac{\partial u}{\partial \sigma}\right)_j$ for $j \neq i$ and $\left(\overline{\frac{\partial u}{\partial \sigma}}\right)_i \equiv \left(\frac{\partial u}{\partial \sigma}\right)_i$ except over the interval ξ where $\overline{\frac{\partial u}{\partial \sigma}}(t,\sigma) \equiv \frac{\partial u}{\partial \sigma}(t,\sigma) + \epsilon < 0$ for $\epsilon > 0$ small enough. The value of the functional $J[\cdot]$ evaluated along the comparison arc satisfies

$$J\left[\overline{\frac{\partial u}{\partial \sigma}}\right] = J\left[\frac{\partial u}{\partial \sigma}\right] + \epsilon \int_{\xi} M_i(t,\sigma)\, dt$$

and if $M_i(t,\sigma) < 0$ over ξ then $J\left[\overline{\partial u}/\partial\sigma\right] < J\left[\partial u/\partial\sigma\right]$ which contradicts the assumption that $\partial u/\partial\sigma$ minimizes $J\left[\cdot\right]$. Hence $\frac{\partial u}{\partial\sigma}(t,\sigma)$ must vanish over any interval ξ in t where $M_i(t,\sigma) > 0$.

Statement 3: A necessary condition for $\partial u/\partial\sigma$ to minimize $J\left[\cdot\right]$ is that over any interval of finite measure in t, say, ξ where $u_i \equiv \alpha_i$ over ξ, $M_i(t,\sigma) \geqq 0$ and if $M_i(t,\sigma) > 0$ then $\left(\frac{\partial u}{\partial\sigma}\right)_i(t,\sigma) = 0$.

Proof: The proof of statement 3 is similar to that of statement 2 and is not presented.

Statement 4: A necessary condition for $\frac{\partial u}{\partial\sigma}(t,\sigma)$ to minimize $J\left[\cdot\right]$ is that $J\left[\partial u/\partial\sigma\right] = 0$.

Proof: The proof of statement 4 follows directly from statements 1, 2, and 3. If, over any interval ξ, a component of the control vector, say u_i, is not on the boundary of the control space then $M_i(t,\sigma) \equiv 0$ from statement 1. If the component is on the boundary of the control space and if $M_i(t,\sigma) \neq 0$ from statements 2 and 3, $\partial u/\partial\sigma$ must vanish. Hence over any interval ξ of finite measure in t the integrand $M^T(t,\sigma)\,\frac{\partial u}{\partial\sigma}(t,\sigma)$ of $J\left[\cdot\right]$ must vanish if $\frac{\partial u}{\partial\sigma}(t,\sigma)$ is to minimize $J\left[\cdot\right]$. Therefore if $\frac{\partial u}{\partial\sigma}(t,\sigma)$ minimizes $J\left[\cdot\right]$ then $J\left[\partial u/\partial\sigma\right] = 0$.

Statements 1 to 4 provide the basic tools for minimizing the integral term of $\mu^T \frac{dx^f}{d\sigma}(\sigma)$. Applying statement 4 with $M^T(t,\sigma)$ taken as $\mu^T\, C\left(x^f,u^f,\sigma\right)\,\Phi\left(t_f,t,\sigma\right)\,B(t,\sigma)$, the minimization of $\mu^T \frac{dx^f}{d\sigma}(\sigma)$ is seen to require that

$$\mu^T\, C\left(x^f,u^f,\sigma\right)\int_{t_o}^{t_f} \Phi\left(t_f,t,\sigma\right)\,B(t,\sigma)\,\frac{\partial u}{\partial\sigma}(t,\sigma)\,dt = 0 \tag{B4}$$

Now consider the remaining terms. Note that the functions $\frac{dS_i}{d\sigma}(\sigma)$, $i = 1,\ldots,K$, are totally unconstrained. Hence the coefficients preceding each of the terms $dS_i/d\sigma$ in $\mu^T \frac{dx^f}{d\sigma}(\sigma)$ must vanish independently so that any discontinuities must occur at times $S_i(\sigma)$ satisfying

$$\mu^T\, C\left(x^f,u^f,\sigma\right)\,\Phi\left(t_f,S_i,\sigma\right)\,\Delta f_i(\sigma) = 0 \tag{B5}$$

Equations (B4) and (B5) may be used to determine the variation in the performance index $x^f_{n+1}(\sigma)$ after substitution into equation (17) to obtain

$$\mu^T \frac{dx^f}{d\sigma} = G_1\left(x^f,u^f,\sigma\right) \triangleq - \frac{\partial J}{\partial \sigma}\left(x^f,\sigma\right) \frac{f_{n+1}\left(x^f,u^f\right)}{\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right)} \tag{B6}$$

From the definition of an admissible trajectory, $J(x,\sigma) > 0$ for x evaluated along an admissible trajectory and $J(x,\sigma) = 0$ only at termination. Hence, $\partial J/\partial t \leqq 0$ at termina tion and this implies that

$$\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right) \leqq 0$$

For a given parameterization, equation (B6) serves to define $\frac{dx^f_{n+1}}{d\sigma}(\sigma)$ as long as there is an admissible control u^f satisfying $\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right) < 0$. Since $\frac{dx^f_{n+1}}{d\sigma}(\sigma)$ is to be minimized it follows that u^f minimizes $G_1\left(x^f,u^f,\sigma\right)$ subject to the constraint $\nabla_x^T J\left(x^f,\sigma\right) f\left(x^f,u^f\right) < 0$.

At every point t,σ the control must either be on the boundary or in the interior of the control space Ω. If over any interval of finite measure in t a component u_i of the control vector is not on the boundary of Ω, statements 2 and 3 imply that

$$M_i(t,\sigma) = \left[\mu^T \; C\left(x^f,u^f,\sigma\right) \Phi\left(t_f,t,\sigma\right) B(t,\sigma)\right]_i = 0$$

Now, if over any interval of finite duration in t a component u_i of the control vector is at the boundary of the control space Ω, statements 2 and 3 infer that

$$M_i(t,\sigma) \geqq 0 \quad \text{if} \quad u_i = \alpha_i$$

$$M_i(t,\sigma) \leqq 0 \quad \text{if} \quad u_i = \beta_i$$

and

$$\left[\frac{\partial u}{\partial \sigma}\right]_i(t,\sigma) = 0 \quad \text{if} \quad M_i(t,\sigma) \neq 0$$

The results of the preceding paragraphs can be compiled into theorem 2 of the text.

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