NOTE ON ARITHMETIC CODES
AND ARITHMETIC DISTANCE

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Department of

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UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA

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ABSTRACT

This report presents some observations on cyclic arithmetic codes and their distance-properties. The main result is the demonstration that modular arithmetic weight is invariant to cyclic shifts of codewords. As a consequence of this result, it is shown that the minimum distance of a cyclic arithmetic code can be found by a search of less than one-sixth of the codewords. This result also permits a simple proof of a result due to Goto and Fukumura for computation of arithmetic weighs in terms of the residue classes modulo B, the number of codewords.

I. INTRODUCTION

It will be assumed that the reader is familiar with the concepts of arithmetic weight and distance, denoted AW and AD respectively, as described by Massey (6). We shall also require the following three definitions, the first due to Reitweisner (7), and the other two due to Garcia (3).

Definition 1 The <u>nonadjacent form</u> (NAF) of an integer I is the unique expression for I of the form $I = \sum_{i=o}^{n} a_i 2^i$ where $a_i \; \varepsilon \; \{-1,0,1\}$ and $a_i \; a_{i+1} = 0$, $\underline{i > 0}$.

The arithmetic weight of I is the number of non-zero terms in the NAF for I. Note that $AW(I) = AW(-I)$ since these two NAF differ only in the sign of their respective terms.

Definition 2 The <u>modular arithmetic weight</u> of an integer I, relative to the modulus m, denoted MAW(I) is defined as:

$$MAW(I) = \min \; [AW(I), \; AW(m-I)].$$

Example 1:   Let m = 31

$$MAW \; (21) = \min \; [AW(21), \; AW(31-21)]$$

$$= \min \; [AW(21), \; AW(10)]$$

$$= \min \; [3,2]$$

$$= 2$$

It should be noted that MAW does not necessarily satisfy the triangle inequality, as seen in the next example.

Example 2:   Let m = 35

$$MAW \; (3+19) = MAW \; (22) = 3 \text{ but } MAW \; (3) = MAW \; (19) = 1$$

Definition 3 The modular arithmetic distance (MAD) between integers I_1 and I_2 is:

$$MAD \; (I_1, I_2) = MAW \; (|I_1 - I_2|)$$

MAD is not in general a true metric since the triangle inequality fails for certain modulo's.

Example 3:   Let m = 35

MAD (0,22) = MAW (22) = 3

But MAD (0,3) = MAD (3,22) = 1

Garcia (3) has however shown that for modulo's of the form 2^n or 2^n-1 the triangle inequality does hold, and hence MAD is a true metric for these values of the modulus.

Definition 4 The arithmetic code with B codewords generated by the integer A is the set of integers $\{0,A,2A, \ldots (B-1)A\}$.

It is customary to think of the codeword A . N as resulting from the encoding of the information digit N, and arithmetic codes are often called AN codes for this reason.

An AN code is said to be cyclic if the n-place cyclic shift of the radix two form of every codeword is the radix two form of a codeword, where n is defined by $AB = 2^n-1$. Some of the more convenient properties of cyclic arithmetic codes are: 1) The codewords are closed under addition modulo $m=2^n-1$ and, in fact, form an ideal in the ring of integers modulo 2^n-1. 2) If I is a codeword then 2^n-1-I is also a codeword. Also 3) the minimum MAW and the minimum AW of the nonzero codewords coincide.

In the rest of this report unless otherwise mentioned the modulus m will be 2^n-1 for the appropriate n.

Let $[a,b]$ denote the set of integers I such that $a \leq I \leq b$ and $(a,b]$ denote the set of integers I such that $a < I \leq b$. The set $W = (0,2^n-1]$ where $AB = 2^n-1$ is of considerable interest in the theory of cyclic AN codes. The lower third (L3) of W is defined to be the set $(0, \frac{2^n}{3}]$, the middle third (M3) as the set $(\frac{2^n}{3}, \frac{2^{n+1}-1}{3}]$ and the upper third (U3) as the set $(\frac{2^{n+1}-1}{3}, 2^n-1]$.

If $[a_n\ a_{n-1}\ \cdot\ \cdot\ \cdot\ a_o]$ is the concatenation of the coefficients in the NAF of an integer $I \varepsilon W$ then:

I ε L3 if and only if $a_n = a_{n-1} = 0$

I ε M3 if and only if $a_n = 0$, $a_{n-1} = 1$

I ε U3 if and only if $a_n = 1$, $a_{n-1} = 0$

Five lemmas will be given that simplify the proofs of the subsequent theorems.

Lemma 1 For any integers I and J,

$$AW(I) - AW(J) \leq AW(I+J) \leq AW(I) + AW(J)$$

Proof: By the triangle inequality

$$AW(I+J) \leq AW(I) + AW(J)$$

again by the triangle inequality $AW(I) = AW(-J + (I+J) \leq AW(-J) + AW(I+J)$ or $AW(I) - AW(J) \leq AW(I+J)$

Lemma 2 For any integer I and any modulus m, MAW (I) = MAW $(m-I)$.

Proof: Follows directly from the definition of modular arithmetic weight.

The next lemma shows that for an integer I in the lower third of W, the modular arithmetic and arithmetic weighs coincide.

Lemma 3 For I ε L3 then

$$MAW\ (I) = AW\ (I)$$

Proof: $AW(2^n-I) = AW(I) + 1$ since the NAF of I may be written $\sum_{i=0}^{n-2} a_i 2^i$ so that $2^n - \sum_{i=0}^{n-2} a_i 2^i$ is already the NAF for 2^n-I.

thus $\quad AW(2^n-I-1) \geq AW(2^n-I) - AW(1)$

$$\geq AW(I) + 1 - 1$$

$$\geq AW(I).$$

and hence $MAW(I) = \min [AW(I), AW(2^n-1-I)] = AW(I)$

The next two lemmas relate the arithmetic weight of I and the arithmetic weight of I-1 according to the endings of I in NAF. If $\sum_{i=0}^{n} a_i 2^i$ is the NAF of I, we shall often represent this NAF as the concatenation of its coefficients in descending order letting P represent +1, and N represent -1. For instance, I=3 has the NAF $2^2 - 2^0$ which we shall denote by PON. The notation $OP(ON)^i$ denotes the sequence in which OP is followed by i repetitions of the subsequence ON. It should be noted that the cases in the two lemmas are just shifts of each other.

Lemma 4 For an odd integer I,

$$AW(I) - 1 \leq AW(I-1) \leq AW(I).$$

Proof: The only possible endings for I in NAF are $OP(ON)^i$ or $OO(ON)^j$ where $i \geq 0$, $j \geq 1$.

Subtracting one results in NAF's with endings

$$OO(PO)^i \text{ or } O(NO)(PO)^{j-1}.$$

In the first case $AW(I-1) = AW(I) - 1$.

In the second case $AW(I-1) = AW(I)$.

Lemma 5 For an even integer I,

$$AW(I) \leq AW(I-1) \leq AW(I) + 1.$$

Proof: The possible endings in NAF for I are

$$PO(NO)^i \text{ or } OO(NO)^i \text{ where } i \geq 0$$

Subtracting one results in the NAF's

$$OP(OP)^i \text{ or } ON(OP)^i.$$

In the first case   AW (I-1) = AW (I)

In the second case AW (I-1) = AW(I) + 1.

Theorem 1 For I ε W then

MAW (I) = AW (I)

if (i) I ε L3

(ii) I ε M3 and even

otherwise MAW (I) = AW (2^n-1-I)

Proof I ε L3 by Lemma 3 previously

I ε U3 by Lemmas 3 and 4 previously

I ε M3 and I even

AW (2^n-1) = AW (I)

AW (2^n-1-I) $\le$ AW (I) + 1

∴   MAW (I) = AW (I)

I ε M3 and I odd

AW (2^n-I) = AW (I)

AW (2^n-I-1) AW (2^n-I)

AW (I)

∴   MAW (I) = AW (2^n-1-I)

Definition 5 Let I ε W and T (I) be the integer whose radix two

form is the n-place cyclic shift of the radix two form of I.

Similarly, let T^i (I) be the integer corresponding to the i-th cyclic

shift. Note that T(I) = 2I if I ε $(0, 2^{n-1}-1]$ and

T(I) = 2I - 2^n+1 if I ε $(2^{n-1}-1, 2^n-1]$.

The following theorem shows that modualr arithmetic weight is

invariant to cyclic shifts.

<u>Theorem 2</u> For $I \in W$ $\text{MAW}(I) = \text{MAW} [T(I)]$.

<u>Proof</u> $I \in (0, 2^{n-1}-1]$

$I \in L3$ $T(I) \in L3$ or $T(I) \in M3$ and $T(I)$ even

$\text{MAW}(I) = \text{AW}(2I) = \text{AW}(I) = \text{MAW}(I)$

$I \in M3$ and I even $\text{MAW}(I) = \text{AW}(I)$

$T(I) \in U3$ and $\text{MAW}(2I) = \text{AW}(2^n-1-2I)$

but $\text{AW}(2^n-2I) = \text{AW}(I) -1$ since 2^n-2I ends in 00

$\text{AW}(2^{n}-2I-1) = \text{AW}(I) - 1 + 1 = \text{MAW}(I)$

$I \in M3$ and I odd $\text{MAW}(I) = \text{AW}(2^n-1-I)$

$T(I) \in U3$

$\text{MAW}(2I) = \text{AW}(2^n-1-2I)$

$\text{AW}(2^n-I) = \text{AW}(I)$

$\text{AW}(2^n-2I) = \text{AW}(I) -1$

and by lemmas 3 and 4

$\text{AW}(I) \leq \text{AW}(2^n-I-1) \leq \text{AW}(I) - 1$

$\text{AW}(I) \leq \text{AW}(2^n-2I-1) \leq \text{AW}(I) - 1$

and the equalities go together

$\therefore \text{MAW}(2I) = \text{MAW}(I)$

$I \in (2^{n-1}-1, 2^n-1]$

$\text{MAW}(I) = \text{MAW}(2^n-1-I)$

$\text{MAW}(T(I)) = \text{MAW}(2I-2^n+1) = \text{MAW}(2(2^n-1-I))$

but since from the previous parts of the theorems

$\text{MAW}(J) = \text{MAW}(2J)$ $J \in [0, 2^{n-1})$

then $\text{MAW}(I) = \text{MAW}[T(I)]$.

<u>Corollary 2</u> The minimum distance of a cyclic AN code is the minimum

of the arithmetic weights of its non-zero odd codewords in the lower

third.

<u>Proof:</u> It must be shown that the minimum of the arithmetic weighs of

the non-zero codewords is attained by an odd codeword in L3.   If

I is an even codeword then I is one or more cyclic shifts of an odd

codeword. The odd codewords in U3 and M3 are obtained by $T(I')$ where

I' is an even codeword.

The following development relates the arithmetic weight of integers

in L3 to the cyclic group of the powers of 2 modulo B and the cosets

of this cyclic group.

<u>Definition 6</u> The NAF of $I \ \varepsilon \ W$ is said to be <u>cyclic nonadjacent</u> if

$a_{n-1} \ a_0 = 0$.

<u>Example 4</u> Let I = 11

NAF of I = (PONON) is not cyclic nonadjacent if n=5 but is

cyclic nonadjacent if n>5.

Note that any number in L3 or in M3 and even, automatically is

cyclic nonadjacent.

<u>Definition 7</u> For $I \ \varepsilon$ L3, let $Z(I)$ be the integer whose NAF if the

n-place cyclic shift of the NAF for I. Similarly let $Z^i(I)$ be the

i^{th} cyclic shift. Note that $Z^i(I)$ may be negative.

<u>Example 5</u>   Let I = 11 and n = 6

$I = OPONON = 11$

$Z^1(I) = PONONO = 22$

$Z^2(I) = ONONOP = -19$

$Z^3(I) = NONOPO = -38$

$Z^4(I) = ONOPON = -13$

$Z^5(I) = NOPONO = -26$

$Z^6(I) = I \qquad = 11$

<u>Lemma 6</u> $Z^i(I) = T^i(I)$ if $Z^i > 0$, otherwise $Z^i(I) = T^i(I) - 2^n + 1$.

<u>Proof:</u>   It suffices to show that $T^i(I)$ and $Z^i(I)$ are either the same integer or differ by exactly $2^n - 1$. But since cyclic shifting always doubles the integers with perhaps the addition or subtraction of $2^n - 1$, it follows that $T^i(I) \equiv Z^i(I) \mod 2^n - 1$ all i and also $o < T^i(I) < 2^n$ and $-2^n < Z^i(I) < 2^n$ so the conclusion follows.

The following theorem gives us – simple counting procedure to find the AW of an integer in L3.

<u>Theorem 3</u>   For I ε L3

$$AW(I) = \#\{i : T^i(I)\varepsilon M3, i = 0,1...N-1\}$$

<u>Proof:</u>   AW(I) is just the number of non-zero terms in the NAF of I. But a_{n-1-i} is the leading term in the NAF of $Z^i(I)$ and hence is zero if and only if $|Z^i(I)|\varepsilon$L3 by lemma 6 this is equivalent to $T^i(I)\varepsilon$L3 or $(2^n-1) - T^i(I)\varepsilon$L3. But $2^n-1-T^i(I)\varepsilon$L3 is equivalent ot $T^i(I)\varepsilon$U3. Hence $a_{n-1-i} \neq o$ if and only if $T^i(I)\varepsilon$M3.

Let M3B the "middle third of B" be the set of integers I such that AIεM3. It is readily checked that M3B = $(\frac{B}{3},\frac{2B}{3}]$. We now have as a consequence the following corollary due originally to Goto and Fuhumura (4) and used by them to simplify the Barrows-Mandelbaum codes.

<u>Corollary 3</u>   The minimum distance of a cyclic AN code is given by

min $\#\{i : Z^iL \mod B\varepsilon M3B, i = 0,1...(n-1)\}$

L < B/3

L odd

<u>Proof:</u>   AW(AI) = $\#\{i : T^i(AI)\varepsilon M3 \ i = 0,1..(n-1)\}$ but $\#\{i : T^i(AI)\varepsilon M3\}$ is the same as $\#\{i : 2^i(AI) \mod AB\varepsilon M3\}$. And that is the same as $\#\{i : 2^iI \mod B\varepsilon M3B\}$

This corollary shows that the minimum arithmetic weight of the non-zero codewords of a cyclic arithmetic code can be obtained without ever actually constructing the codewords but simply by considering integers modulo B.

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