NECESSARY CONDITIONS FOR CONTINUOUS PARAMETER STOCHASTIC OPTIMIZATION PROBLEMS

by

H. J. Kushner+
Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island

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1. Introduction

This paper applies the abstract variational theory of Neustadt [1] to obtain a stochastic maximum principle. Since the papers of Kushner on the stochastic maximum principle [2], [3], a number of developments were reported in Brodeau [4], Baum [5], Fleming [6], Sworder [7] - [8]. The versatile mathematical programming ideas were not used explicitly in [2] - [8], and, with relative ease, we are able to handle greater varieties of state space constraints then treated in the references. A discrete parameter analog of the discrete maximum principle of Halkin [9] and Holtzman [10] appears in Kushner [11]. Even in the deterministic case, the ability to handle general constraints with relative ease gives the programming approach a distinct advantage over more direct approaches.

It is premature to assert that the stochastic maximum principle will be useful in providing any deep understanding of stochastic control problems. Nevertheless, it seems likely that the implicit geometric framework (at least in the programming approach) will suggest some useful approximation or numerical procedures. The results may serve as a departure point for a perturbation analysis as in the formal work [12], and the nature and interpretation of the random multipliers may shed additional light on the physical interpretation of the derivatives (weak or strong) of the minimum cost function which appears in the dynamic programming formulation for a fully Markovian problem. These various points are under current investigation for both the present work and [11]. Even for an initially Markovian problem, dynamic programming is not always applicable when there are state space constraints, and the alternative programming formulation may be useful to shed light on the control problem. For a discussion, for an elementary stochastic control problem of the relationship between randomized controls and 'singular arcs' see [13].

The problem formulation and mathematical background is given in Section 2. A required result of Neustadt is stated in Section 3, the linearized equations are discussed in Section 4. Section 5 derives a certain convex cone. The maximum principle is stated in Section 6. The development in Sections 4-6 is for the open loop case and extensions are discussed in Section 7.

2. Problem Formulation and Mathematical Background

A Remark on Notation.

Let $m(\cdot,\cdot)$ denote an arbitrary random function with values $m(\omega,t)$, $0 \leq t \leq T$. The notation will be simplified by omitting the ω variable. The term $m(t)$ will be used for both $m(\cdot,t)$ and $m(\omega,t)$, and either $m(\cdot)$ or m (depending on the context) will be used for the random function $m(\cdot,\cdot)$. A random variable $M(\cdot)$ with value $M(\omega)$ will be written simply as M.

R^n denotes an n-dimensional Euclidean space.

Assumptions.

Let[+] $z(\cdot) = (z_o(\cdot),\ldots,z_n(\cdot))'$, $0 \leq t \leq T$ be an $n+1$ dimensional normalized Brownian motion on the probability triple $(\Omega, P(\cdot), \mathscr{B})$, where Ω is the sample space, and $P(\cdot)$ the measure on the σ-algebra $\mathscr{B}$ on Ω. For any finite dimensional vector $\alpha = (\alpha_1,\ldots\alpha_r)$ and matrix $\Phi = \{\varphi_{ij};\ i,j = 1,\ldots,r\}$, define the Euclidean norms $|\alpha|^2 = \sum_i |\alpha_i|^2$, $|\Phi|^2 = \sum_{i,j} \varphi_{ij}^2$. The control is an n_1 dimensional random function whose properties are described in (I-1) below. Let $f(\cdot,\cdot,\cdot)$ denote an R^{n+1} valued function on $R^{n+1} \times R^{n_1} \times [0,T]$ and $\sigma(\cdot,\cdot)$ an $(n+1) \times (n+1)$ matrix valued function on $R^{n+1} \times [0,T]$. Further properties of $f(\cdot,\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are given in (I-2) below. The control system of concern is the $n+1$ dimensional stochastic differential equation (of the Itô type) (1) on the fixed time interval $[0,T]$.

$$dx(t) = f(x(t),u(t),t)dt + \sigma(x(t),t)dz(t) \tag{1}$$

$$x(t) = (x_o(t),\ldots,x_n(t))'.$$

The control $u(\cdot)$ and $x(0)$ satisfy (I-1) below. Write $\sigma(\alpha,t) = [\sigma_o(\alpha,t),\ldots,\sigma_n(\alpha,t)]$, where $\sigma_i(\alpha,t)$ is the i^{th} column of $\sigma(\alpha,t)$.

$$dx(t) = f(x(t),u(t),t)dt + \sum_i \sigma_i(x(t),t)dz_i(t). \tag{2}$$

[+]The ' denotes transpose.

Let $h(\cdot)$ be a real valued Borel function on R^{n+1} for which $Eh(x(T))$ exists for the $x(T)$ corresponding to any admissible control (see I-1 below). Let $t_o,\ldots,t_{k+1}$, for a fixed integer k, denote a sequence of fixed times satisfying $0 = t_o < t_1 < \cdots < t_{k+1} = T$. Let $a_o,\ldots,a_{k+1}$ and b_o and b_1 be given finite numbers. Let $\tilde{q}_i^j(\cdot,\cdot)$, $i = 0,\ldots,k+1$, $j = 1,\ldots,a_i$ and $\tilde{r}_o^j(\cdot,\cdot)$, $j = 1,\ldots,b_o$ and $\tilde{r}_T^j(\cdot,\cdot)$, $j = 1,\ldots,b_1$, be real valued Borel functions on $R^{n+1} \times R^{n+1}$ and define $\tilde{q}_i(\cdot,\cdot) = (\tilde{q}_i^1(\cdot,\cdot),\ldots,\tilde{q}_i^{a_i}(\cdot,\cdot))'$, and $\tilde{r}_o(\cdot,\cdot) = (\tilde{r}_o^1(\cdot,\cdot),\ldots,\tilde{r}_o^{b_o}(\cdot,\cdot))'$, $\tilde{r}_T(\cdot,\cdot) = (\tilde{r}_T^1(\cdot,\cdot),\ldots,\tilde{r}_T^{b_1}(\cdot,\cdot))'$. For any admissible control (see I-1), let the corresponding $x(t_o),\ldots,x(t_{k+1})$ satisfy $E|\tilde{q}_i(x(t_i),Ex(t_i))| < \infty$, $i = 0,\ldots,k+1$, and $E|\tilde{r}_i(x(t_i),Ex(t_i))| < \infty$, $i = 0,T$, (properties guaranteed by (I-4) below).

<u>The Problem.</u>

Assume (I1-2), and the above properties on $h,\tilde{q}_i,\tilde{r}_i$. Define the cost function

$$\varphi_o(x(\cdot)) \equiv Ex_o(T) + Eh(x(T)). \tag{3}$$

In the class of admissible controls for which the corresponding trajectories satisfy the constraints

$$\begin{aligned} q_i(x(t_i)) &\equiv E\tilde{q}_i(x(t_i),Ex(t_i)) \leq 0, \quad i = 0,\ldots,k+1 \\ r_i(x(t_i)) &\equiv E\tilde{r}_i(x(t_i),Ex(t_i)) = 0, \quad i = 0,T\ , \end{aligned} \tag{4}$$

assume that there is one, denoted by $\hat{u}$, for which the cost is minimized (or is no greater for any other control in the class). It is assumed that $q_o(x(\cdot)) = 0$ implies that $x_o(0) = 0$. As discussed below, more general constraints can be treated. Let $\hat{x}$ denote the corresponding optimal solution to (1).

Now, assume in addition, (I3-5), and find a necessary condition for $\hat{u}$ and $\hat{x}$.

Assumptions.

(I-1) Let $\mathscr{B}_t$, $T \geq t \geq 0$, denote a family of given σ-algebras which are non-anticipative with respect to the Wiener $z(\cdot)$ process. The $\mathscr{B}_t$ are the data σ-algebras. $x(0)$ is measurable on $\mathscr{B}_o$ and $E|x(0)|^2 < \infty$. Let $\mathscr{U}_t$ denote a sequence of given non-empty n_1 dimensional sets. The family of admissible controls, denoted by $\tilde{\mathscr{U}}$, is the collection of n_1 dimensional random functions $u(\cdot,\cdot)$, with values $u(\omega,t)$ in $\mathscr{U}_t$ at time t, and $u(\cdot,t)$ is measurable over $\mathscr{B}_t$. As noted above, we will write either u or $u(\cdot)$ for the function $u(\cdot,\cdot)$, and $u(t)$ for either $u(\omega,t)$ or $u(\cdot,t)$.

(I-2) The $f(\cdot,\cdot,\cdot)$ and $\sigma_i(\cdot,\cdot)$ are Borel functions of their arguments. $f(\cdot,\beta,t)$ and $\sigma_i(\cdot,t)$ are differentiable for each fixed β,t, and t, resp. Write $f_x(\alpha,\beta,t)$ and $\sigma_{i,x}(\alpha,t)$ for the matrices with i,k^{th} elements $\partial f_j(\alpha,\beta,t)/\partial\alpha_k$ and $\partial\sigma_{ij}(\alpha,t)/\partial\alpha_k$, resp., and suppose that both are uniformly bounded. Assume $|f(\alpha,\beta,t)|^2 \leq K_o(1+|\alpha|^2)$, $|\sigma_i(\alpha,t)|^2 \leq K_o(1+|\alpha|^2)$, uniformly in $\beta \in \mathscr{U}_t$ and $t \in [0,T]$. The function $f(\cdot,\beta,t)$ is continuous at each $\beta \in \mathscr{U}_t$ and

$t \in [0,T]$, uniformly in t.

(I-3) For each fixed $t \in (0,T]$ and $\mathscr{B}_t$ measurable and $\mathscr{U}_t$ valued <u>random variable</u> u_t, there is a $\delta(t) > 0$ so that for each $\delta < \delta(t)$ there is a <u>random variable</u> $\tilde{u}_{t-\delta}$ with the property that $\tilde{u}_{t-\delta}$ is measurable over each $\mathscr{B}_s$ and has values in each $\mathscr{U}_s$ where $s \in [t-\delta,t]$, and the sequence $\tilde{u}_{t-\delta}$ satisfies

$$(5) \qquad f(\hat{x}(t),u_t,t) - f(\hat{x}(t),\tilde{u}_{t-\delta},t) \to 0$$

in probability as $\delta \to 0$. Both $\tilde{u}_{t-\delta}$ and $\delta(t)$ may depend on u_t and t.

<u>Note</u>. The condition of the last paragraph is included since we will use piecewise constant and non-anticipative perturbations to the optimal control. Its intuitive meaning is simply that the effect of any random control u_t which can be used at time t can be approximated by some random control $\tilde{u}_{t-\delta}$ which can be used at any time in the small interval $[t-\delta,t]$.

(I-4) Assume that, for any R^{n+1} valued random variable v

$$|q_i(v)| \le K_o(1+E|v)|^2), \quad i = 0,1,\ldots,k+1$$
$$|r_i(v)| \le K_o(1+E|v)|^2), \quad i = 0,T.$$

The $\tilde{q}_i(\cdot,\cdot)$ and $\tilde{r}_i(\cdot,\cdot)$ and $h(\cdot)$ are vector valued (except for

$h(\cdot)$, which is real valued) Borel functions whose first derivatives with respect to each argument exist. Write $\hat{q}_{i,x}$, $\hat{q}_{i,e}$, $\hat{r}_{i,x}$, $\hat{r}_{i,e}$ for the matrices of first partial derivatives of $\tilde{q}_i(\alpha,\beta)$ and $\tilde{r}_i(\alpha,\beta)$ with respect to the first and second arguments (α and β) evaluated at $\alpha = \hat{x}(t_i)$, $\beta = E\hat{x}(t_i)$, and suppose that they are square integrable. Write $\hat{h}_x$ for the gradient of $h(\alpha)$ evaluated at $\alpha \equiv \hat{x}(T)$, and suppose that it is square integrable.

Define the linear vector valued operators Q_i, R_i, and scalar valued operator H, all on the space of square integrable $n + 1$ dimensional random variables, as follows

$$Q_i v = E[\hat{q}_{i,x}\cdot v + \hat{q}_{i,e}\cdot Ev]$$
$$R_i v = E[\hat{r}_{i,x}\cdot v + \hat{r}_{i,e}\cdot Ev]$$
$$Hv = E\hat{h}'_x\cdot v,$$

where v is an arbitrary $(n+1)$ vector valued random variable with square integrable components. Let Q_i^j be the i^{th} component of the vector valued functional Q_i; i.e., $Q_i^j v = E[(\hat{q}^j_{i,x})'\cdot v + (E\hat{q}^j_{i,e})'v]$, where $\hat{q}^j_{i,x}$ is the gradient of $q_i^j(\alpha,\beta)$ evaluated at $\alpha = \hat{x}(t_i)$, $\beta = E\hat{x}(t_i)$, and $Q_i v = \sum_j Q_i^j v$. For any square integrable R^{n+1} valued random variable v, and any sequence v_n for which $E|v_n - v|^2 \to 0$, let

$$\frac{1}{\epsilon} E[\tilde{q}_i(\hat{x}(t_i) + \epsilon v_n,\ E\hat{x}(t_i) + E\epsilon v_n) - \tilde{q}_i(\hat{x}(t_i),\ E\hat{x}(t_i))] \to Q_i\cdot v$$

as $\epsilon \to 0$. Assume that the components of the vector valued linear functional R_o are linearly independent, and similarly for those of R_T.

(I-5) For the inactive+ inequality constraints $q_i^j(o)$, suppose that there is some $\epsilon_i > 0$ so that

$$q_i^j(\hat{x}(t_i) + v) < 0$$

for $E|v|^2 < \epsilon_i$. For each i suppose that there is a square integrable random variable v_i so that for each active component $q_i^j(\cdot)$ of $q_i(\cdot)$, we have

$$Q_i^j \cdot v_i < 0.$$

3. A Variational Result of Neustadt

For future reference, we describe a variational result of Neustadt [1]. Let $\mathscr{T}$ denote a locally convex topological space, and let Q be a set in $\mathscr{T}$.

Definition. For any integer μ, let P^μ denote the set of vectors in R^μ, $\{\beta: \beta_i \geq 0, \sum_{i=1}^{\mu} \beta_i \leq 1\}$. Let K be a convex set in $\mathscr{T}$ which

+ $q_i^j(\cdot)$ is active at $\hat{x}(\cdot)$ if $q_i^j(\hat{x}(t_i)) = 0$. Otherwise $q_i^j(\hat{x}(t_i)) < 0$, and the constraint is said to be inactive.

contains the origin $\{0\}$ and some point other than $\{0\}$. For each μ points, $w_1,\ldots,w_\mu$, of K, and arbitrary neighborhood N of $\{0\}$, let there exist an $\epsilon_o > 0$ (depending on $w_1,\ldots,w_v$ and N) so that, for each ϵ in $(0,\epsilon_o]$, there is a continuous map $\zeta_\epsilon(\beta)$ from P^μ to $\mathscr{T}$ with the property

$$\zeta_\epsilon(\beta) \subset \{\epsilon(\sum_{i=1}^{\mu} \beta_i w_i + N)\} \cap Q.$$

Then K is said to be a <u>first order convex approximation to</u> Q.

A Basic Optimization Problem.

Let Q' be a set in $\mathscr{T}$. For some finite given integers μ and β, let $\varphi_i(\cdot)$, $-\beta \leq i \leq \mu$, be real valued functions on $\mathscr{T}$. Find an element $\hat{w}$ in Q' which minimizes $\varphi_o(\cdot)$. Among all the points $\hat{w}$ in $\mathscr{T}$ which satisfy the constraints $\varphi_i(w) = 0$, $i = 1,\ldots,\mu$, $\varphi_{-i}(w) \leq 0$, $i = 1,\ldots,\beta$, find the element w which minimizes $\varphi_o(w)$. More precisely, we say that $\hat{w}$ is a <u>local solution</u> to the optimization problem (or, more loosely, the <u>optimal solution</u>) if, for some neighborhood N of $\{0\}$, $\varphi_o(w) \geq \varphi_o(\hat{w})$ for all w in $\hat{w} + N$ which satisfy the constraints. Let $\hat{w}$ denote the optimal solution. The constraints $\varphi_{-i}(\cdot)$, $i > 0$, for which $\varphi_{-i}(\hat{w}) = 0$ are called the <u>active constraints</u>. Define the set of indices $J \doteq \{i\colon \varphi_{-i}(\hat{w}) = 0,\ i > 0\} \cup \{0\}$.

The Basic Necessary Condition for Optimality.

First we collect some assumptions

(II-1) The $\varphi_i(\cdot)$, $i \geq 1$, are continuous at $\hat{w}$. There are continuous and linearly independent functionals $\ell_1, \ldots, \ell_\mu$ with the following property. For any element $w \in \mathscr{T}$, and any bounded sequence w_n converging to w in $\mathscr{T}$, we have $[\varphi_i(\hat{w}+\epsilon w_n) - \varphi_i(\hat{w})]/\epsilon \to \ell_i(w)$ as $\epsilon \to 0$.

(II-2) There is a neighborhood N of $\{0\}$ in $\mathscr{T}$ so that $\varphi_{-i}(\hat{w}+w) < 0$ for $w \in N$, and all <u>inactive</u> constraints $\varphi_{-i}(\cdot)$.

(II-3) Let the active constraints and also $\varphi_o(\cdot)$ be continuous at $\hat{w}$. For the active constraints, let there exist continuous and convex functionals $c_i(\cdot)$ with the property that for any $w \in \mathscr{T}$, and any bounded sequence w_n converging to w in $\mathscr{T}$,

$$[\varphi_{-i}(\hat{w}+\epsilon w_n) - \varphi_{-i}(\hat{w})]/\epsilon \to c_i(w)$$

as $\epsilon \to 0$. Assume that there is some w and some $j \in J$ for which $c_j(w) > 0$. Let there be a w for which $c_j(w) < 0$ for all $j \in J$.

A case of particular importance is where the differentials $c_i(\cdot)$ are linear functionals. Then the next to last sentence of (II-3) is implied by the last sentence of (II-3).

We now have a particular case of (Neustadt [1], Theorem 4.2). The <u>local or optimal solution</u> here is called a <u>totally regular local solution</u> in [1].

<u>Theorem</u> 1. <u>Assume</u> (II-1 - II-3). <u>Define</u> $Q \equiv Q' - \hat{w}$. <u>Let</u> $\hat{w}$ <u>be a</u>

local solution to the optimization problem. Then there exists $\alpha_1,\ldots,\alpha_\mu,\alpha_o,\alpha_{-1},\ldots,\alpha_{-\beta}$ not all zero with $\alpha_{-i} \leq 0$ for $i \geq 0$, so that

$$\sum_{i=1}^{\mu} \alpha_i \ell_i(w) + \sum_{i\in J} \alpha_{-i} c_i(w) \leq 0$$

for all w in $\overline{K}$, where K is a first order convex approximation to Q, and $\overline{K}$ is the closure of K in $\mathscr{T}$.

Remark. Let $\varphi_i(\cdot) \equiv 0$, $i > 0$. If there is a $w \in K$ for which $c_j(w) < 0$ for all active j, then $\alpha_o < 0$, and we can get $\alpha_o = -1$.

Identification with the Stochastic Control Problem.

For the problems of the sequel we define $\mathscr{T}$ to be the locally convex linear topological space of $(n+1)$ dimensional random functions v with values $v(\omega,t)$, where $v_n \to 0$ in $\mathscr{T}$ if and only if

$$E|v_n(\omega,t)|^2 \to 0$$

for each t in $[0,T]$. The set Q' is defined to be the set of solutions+ $x(\cdot,\cdot)$, $0 \leq t \leq T$ to (1) for all controls and initial conditions satisfying (I-1). The (inequality constraint) functions $\{q_i^j\}$ are identified with the $\{\varphi_{-\ell}, \ell > 0\}$ and the (equality constraint) functions $\{r_i^j\}$ with the $\{\varphi_\ell, \ell > 0\}$ of Section 3. Also $\varphi(x) = Ex_o(T) + Eh(x(T))$. $\hat{x}$ is the optimal element of Q' and $Q \equiv Q' - \{\hat{x}\}$. Conditions (I-1) - (I-5) imply (II-1) - (II-3).

With the framework of constraints (4), we can include constraints such as $\int_o^{t_i} g_i(x(s))ds \leq d_i$ and can approximate constraints such as $P\{x(t) \in A\} \leq d_i$, where A has a smooth boundary. More general inequality constraints than (4) can be included, once the appropriate linear or convex differentials c_i (see II-3) are calculated.

4. The Linearized Equations

Consider the Itô stochastic differential equations (6) and (7), where $0 \leq \tau$ is fixed and t satisfies $\tau \leq t \leq T$, and $\Phi(t,\tau)$ is an $(n+1) \times (n+1)$ random matrix++

+It is easiest to work in the space of random functions $\mathcal{T}$, as it is described above. By (I-1), (I-2), we lose nothing by altering $\mathcal{T}$ so that $v_n \to 0$ if $E|v_n(\omega,t)|^p \to 0$ for any $p \geq 2$. In this case the quadratic estimates (I-4) on q_i and r_i can be replaced by $|q_i(x(t_i))| \leq K_o(1+E|x(t_i)|^p)$, etc. More general situations are obviously possible and, in particular, the Lipschitz and growth conditions on the zeroeth component of $f(\alpha,\beta,t)$ can be relaxed.

++$\hat{f}_x$ denotes the random matrix $f_x(\hat{x}(t),\hat{u}(t),t)$, and similarly for $\hat{\sigma}_{i,x}$.

(6) $$dy(t) = \hat{f}_x \cdot y(t)dt + \sum_i dz_i(t)\hat{\sigma}_{i,x}y(t),$$

(7) $$d\Phi(t,\tau) = \hat{f}_x\Phi(t,\tau)dt + \sum_i dz_i(t)\hat{\sigma}_{i,x}\Phi(t,\tau),$$

where, by assumption, $\Phi(\tau,\tau) = I$, the identity, and also by assumption, $E|y(\tau)|^2 < \infty$ and $y(\tau)$ is independent of $z(t) - z(s)$, for all $t \geq s \geq \tau$. Both (6) and (7) have unique continuous (in t) solutions w.p.1, with finite mean square values. We can suppose that the chosen continuous version of $\Phi(t,\tau)$ is measurable in (t,ω) for each τ. The uniqueness of the solution, to (6) implies that, for each $\tau \in [0,T]$, w.p.1,

$$\Phi(t,\tau)y(\tau) = y(t)$$

and, for $t > \tau_1 > \tau$, w.p.1,

$$\Phi(t,\tau_1)\Phi(\tau_1,\tau) = \Phi(t,\tau).$$

Furthermore, if we fix t and let τ vary in the range $0 \leq T \leq t$, $\Phi(t,\tau)$ is mean square continuous in τ, uniformly in $t \in [\tau,T]$. Indeed, we have w.p.1, that $\Phi(t,\tau+\epsilon)$ and $\Phi(t,\tau)$ are the solutions (w.p.1) of (7) which start at time $\tau + \epsilon$ with initial values I and $\Phi(\tau+\epsilon,\tau)$, resp. By known estimates for solutions of stochastic differential equations, for some real K_i,

$$E|\Phi(\tau+\epsilon,\tau) - I|^4 \le K_1\epsilon^2$$

and, hence, for t in $[\tau+\epsilon, T]$

$$E|\Phi(t,\tau+\epsilon) - \Phi(t,\tau)|^4 \le K_2\epsilon^2. \tag{8}$$

Equation (8) implies that there is a continuous version of $\Phi(T,\tau)$ (Proposition III.5.3 of [14]) (τ is the parameter, $0 \le \tau \le T$). Finally, if $E|y(0)|^2 = O(\epsilon^2)$ (or $o(\epsilon^2)$), then $E|y(t)|^2 = O(\epsilon^2)$ (or $o(\epsilon^2)$). This last fact will be used frequently in Theorem 2.

5. The Convex Cone K

We will require the following lemma.+

Lemma 1. Assume (I-1) - (I-2). Let $\phi(\cdot,\cdot)$ be a R^{n+1} valued measurable function with values $\phi(\omega,t)$, $0 \le t \le T$. Suppose that $\phi(\omega,\cdot)$ is Lebesgue integrable on $[0,T]$ for almost all fixed ω. Then the function $F(\cdot,\cdot)$ defined by

$$F(\omega,t) = \int_{t_o}^{t} \phi(\omega,s)ds$$

is differentiable with respect to t on an (ω,t) set of full measure with derivative $\phi(\omega,t)$. Thus, there is a null set

+The proof of Lemma 1 resulted from a discussion with W. Fleming.

$T_1 \subset (0,T)$ so that, at each $t \notin T_1$, $F(\omega,t)$ is differentiable with derivative $\phi(\omega,t)$, w.p.1. In particular, if we define $\phi(\cdot,\cdot)$ by $\phi(\omega,s) = f(\hat{x}(s),\hat{u}(s),s)$ and let α_1,α_2 be any scalars, we have

$$\frac{1}{\epsilon}\int_{t-\epsilon\alpha_1}^{t+\epsilon\alpha_2} f(\hat{x}(s),\hat{u}(s),s)ds - (\alpha_2+\alpha_1)f(\hat{x}(t),\hat{u}(t),t) \to 0$$

w.p.1, for any t not in some null set T_1.

There is a null set $T_2 \subset (0,T)$ so that for any $t \notin T_2$ and any R^{n_1} valued random variable v,

$$\frac{1}{\epsilon}\int_{t-\epsilon\alpha_1}^{t+\epsilon\alpha_2} f(\hat{x}(s),v,s)ds - (\alpha_2+\alpha_1)f(\hat{x}(t),v,t) \to 0$$

w.p.1.

Proof. For arbitrary scalars α_1,α_2, define the function $F_r(\cdot,\cdot)$

$$F_r(\omega,t) = \frac{1}{r}\int_{t-\alpha_1 r}^{t+\alpha_2 r} \phi(\omega,s)ds,$$

where r is rational in $[0,1]$. There is a null ω set N_o so that, for $\omega \notin N_o$, $\phi(\omega,\cdot)$ is Lebesgue integrable on $[0,T]$ and, hence, for $\omega \notin N_o$,

$$F_r(\omega,t) - (\alpha_1+\alpha_2)\phi(\omega,t) \to 0$$

for almost all t (the null t set depending on ω). Also $F_r(\omega,t)$

converges to $(\alpha_1+\alpha_2)\phi(\omega,t)$ on a measurable set $S \subset (\Omega-N_o) \times [0,T]$ as $r \to 0$. If $F_r(\omega,t)$ converges as $r \to 0$ through the rationals, it converges to the same limit as $r \to 0$ through any sequence.

The Lebesgue measure of the fixed ω sections of S (for $\omega \notin N_o$) is T. Hence by Fubini's theorem, the measurable set S has full measure. Thus, there is a null t set T_1 so that for $t \notin T_1$, $F_r(\omega,t) \to (\alpha_1+\alpha_o)\phi(\omega,t)$ w.p.1. The statements of the first paragraph of the lemma follow from this.

Let $g(\cdot,\cdot)$ with values $g(v,t)$ denote a Borel function from $R^m \times [0,T]$ to R^{n+1} which is continuous at each v, uniformly in t. Let $g(v(t),t)$ be integrable on $[0,T]$ for any R^m valued continuous function $v(t)$. Then there is a null set T_2 so that, for $t \notin T_2$ and any continuous $v(\cdot)$ function,

$$\frac{1}{\epsilon}\int_{t-\epsilon\alpha_1}^{t+\epsilon\alpha_2} g(v(s),s)ds - (\alpha_1+\alpha_2)g(v(t),t) \to 0$$

as $\epsilon \to 0$. The second paragraph of the lemma follows from this by defining $g(\cdot,\cdot)$ to be $f(\hat{x}(\omega,t),v(\omega),t) = g(v(\omega,t),t)$, where $g(v(\omega,t),t)$, where $v(\omega,t) = (\hat{x}(\omega,t),v(\omega))$ and noting the continuity (in (α,v)) properties of $f(\alpha,v,t)$ which were assumed in (I-2). Q.E.D.

<u>The Convex Cone K.</u>

For any fixed s in the set $[0,T] - \hat{T}$, where+ $\hat{T} = T_1 \cup T_2$, and any random variable u_s which is $\mathscr{B}_s$ measurable and has values in

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+The T_i are defined in Lemma 1.

$\mathscr{U}_s$, define the element $\delta x_{s,u_s}$ of $\mathscr{T}$. Following our usual notation, we use $\delta x_{s,u_s}(t)$ for either $\delta x_{s,u_s}(\cdot,t)$ or $\delta x_{s,u_s}(\omega,t)$

$$\begin{aligned}\delta x_{s,u_s}(t) &= 0, \qquad 0 \le t < s \le T \\ &= \Phi(t,s)[f(\hat{x}(s),u_s,s) - f(\hat{x}(s),\hat{u}(s),s)] \\ &\qquad\qquad T \ge t \ge s.\end{aligned}$$

Let K denote the set of convex finite combinations of points of the type $c_o\Phi(t,0)\delta x(0)$, where c_o is arbitrary in $[0,\infty)$ and $\delta x(0)$ is an arbitrary admissible condition, and points of the type $c\delta x_{s,u_s}$, where c is arbitrary in $[0,\infty)$ and s is arbitrary in $[0,T] - \hat{T}$, and u_s is an arbitrary $\mathscr{B}_s$ measurable, $\mathscr{U}_s$ valued random variable. Define

$$\delta x_o(t) = \Phi(t,0)\delta x(0).$$

By Theorem 1, K is a first order convex approximation to the set $Q' - \{\hat{x}\} \equiv Q$.

<u>Theorem</u> 2. <u>Assume</u> (I-1 - I-3). <u>Then</u> K <u>is a first order convex approximation to</u> $Q \equiv Q' - \{\hat{x}\}$.

<u>Proof</u>. Let m denote an arbitrary, but fixed, integer. Define the set $\Lambda \equiv \{\lambda = (\lambda_1,\ldots,\lambda_m)\colon \lambda_i \ge 0, \sum_i \lambda_i \le 1\}$. Let $\delta x^1,\ldots,\delta x^m$ denote any m elements of K. Then, there is an integer q, a set of

fixed times s_i, $i = 1,\ldots,q$, a set of $\mathscr{U}_{s_i}$ valued and $\mathscr{B}_{s_i}$ measurable random variables u_{s_i} (written u_i), $i = 1,\ldots,q$, and a set of $\tilde{\beta}_{ij} \geq 0$, $\beta_{ij} \geq 0$, and admissible initial conditions $\delta x_o^i(0)$, $i = 1,\ldots,q$, so that each δx^i has the representation

$$\delta x^i = \sum_{j=1}^{q} \beta_{ij} \delta x_{s_j,u_j} + \sum_{j=1}^{q} \tilde{\beta}_{ij} \delta x_o^i .$$

We assume that $s_i \leq s_{i+1}$. Any element in $\tilde{K}$, the convex hull of $(0,\delta x^1,\ldots,\delta x^m)$, corresponds to some $\lambda \in \Lambda$ (and conversely), and has the form

$$\delta x_\lambda = \sum_{i=1}^{m} \lambda_i \Big(\sum_{j=1}^{q} \beta_{ij} \delta x_{s_j,u_j} \Big) + \sum_{i=1}^{m} \lambda_i \Big(\sum_{j=1}^{q} \tilde{\beta}_{ij} \delta x_o^j \Big) = \sum_{j=1}^{q} \delta t_j(\lambda) \delta x_{s_j,u_j} + \sum_{j=1}^{q} \delta \tilde{t}_j(\lambda) \delta x_o^j$$

$$\delta t_j(\lambda) = \sum_{i=1}^{m} \beta_{ij} \lambda_i, \quad \delta \tilde{t}_j(\lambda) = \sum_{j=1}^{m} \tilde{\beta}_{ij} \lambda_i .$$

Note that $\epsilon \delta t_i(\lambda) = \delta t_i(\epsilon\lambda)$ for any scalar $\epsilon > 0$, and similarly for $\delta \tilde{t}_i(\lambda)$. Let $\tilde{O}(\epsilon^2)$ denote any random function v_ϵ for which $E|v_\epsilon(t)|^2 = O(\epsilon^2)$ for $t \in [0,T]$, and write $v_\epsilon = \tilde{o}(\epsilon^2)$ if $E|v_\epsilon(t)|^2 = o(\epsilon^2)$ for each $t \in [0,T]$. To prove the theorem we must show that there is an $\epsilon_o > 0$ so that, for each $\epsilon < \epsilon_o$, there is a continuous map $\zeta_\epsilon(\lambda)$ from Λ into $\mathscr{T}$ of the form

$$\zeta_\epsilon(\lambda) = \hat{x} + \epsilon \delta x_\lambda + \rho_{\epsilon,\lambda} \tag{9}$$

where $\rho_{\epsilon,\lambda} = \tilde{o}(\epsilon^2)$.

Next, a perturbed control and perturbed initial condition will be described. Suppose first that the s_i are distinct and $s_i \notin \hat{T} = T_1 \cup T_2$. Define $\tau \equiv \sup_{i,\lambda\in\Lambda} \delta t_i(\lambda)\cdot q$, and the set $I_i(\epsilon\lambda)$

$$I_i(\epsilon\lambda) \equiv \{t: s_i - \epsilon\delta t_i(\lambda) < t \leq s_i\}.$$

For each $u_{s_i} \equiv u_i$, let $\delta(s_i)$ be the interval which was defined in (I-3) (corresponding to u_{s_i}). There is an $\epsilon_o > 0$ so that for $\epsilon < \epsilon_o$ we have (i): the $I_i(\epsilon\lambda)$ are distinct, (ii): all $s_i - \epsilon\tau \geq 0$, (iii): $\epsilon\tau \leq \min [\delta(s_1),\ldots,\delta(s_q)]$. Define the perturbed control $u_{\epsilon\lambda}(t)$

$$
\begin{aligned}
(10) \qquad u_{\epsilon\lambda}(t) &= \hat{u}(t), \; t \notin \bigcup_i I_i(\epsilon\lambda) \\
&= \tilde{u}_{s_i-\epsilon\tau}, \; t \in I_i(\epsilon\lambda),
\end{aligned}
$$

where $\tilde{u}_{s_i-\epsilon\tau}$ corresponds to u_{s_i} by (I-3), and as $\epsilon \to 0$, (5) of (I-3) holds.

If the s_i are not distinct, we follow the method for the deterministic problem [16] and define τ_i by

$$\tau_i = \delta t_i(\lambda) + \cdots + \delta t_q(\lambda) \quad \text{if} \quad s_i = s_{i+1} \cdots = s_q$$

$$\tau_i = \delta t_i(\lambda) + \cdots + \delta t_r(\lambda) \quad \text{if} \quad s_i = s_{i+1} \cdots = s_r < s_{r+1}, \; r < q.$$

and $I_i(\epsilon\tau)$ by

$$I_i(\epsilon\lambda) = \{t: s_i - \epsilon\tau_i < t \leq s_i - \epsilon\tau_i + \epsilon\delta t_i(\lambda)\}$$

$$= \{t: s_i - \epsilon(\delta t_i(\lambda) + \cdots + \delta t_r(\lambda))$$

$$< t \leq s_i - \epsilon(\delta t_{i+1}(\lambda) + \cdots + \delta t_r(\lambda)).$$

Then define $u_{\epsilon\lambda}(t)$ as in (10). Thus, if some s_i are identical, the intervals are shifted to the left.

By (I-1) and (I-3), the perturbed control $u_{\epsilon\lambda}$ is admissible. Let $x_{\epsilon\lambda} \in \mathscr{T}$ denote the solution of (1) for control $u_{\epsilon\lambda}$ and initial condition

(11) $$\hat{x}(0) + \epsilon \sum_{j=1}^{q} \delta\tilde{t}_j(\lambda)\delta x_o^i(0) \equiv \hat{x}(0) + \epsilon\delta x_\lambda(0) \equiv x_{\epsilon\lambda}(0)$$

where we use

$$\epsilon\delta x_\lambda(0) = \delta x_{\epsilon\lambda}(0).$$

Define

(12) $$\zeta_\epsilon(\lambda) = x_{\epsilon\lambda}.$$

Fix ϵ in $(0,\epsilon_o)$. Let $\lambda(n) \to \lambda$ in Λ, as $n \to \infty$. Then $E|x_{\epsilon\lambda(n)}(0) - x_{\epsilon\lambda}(0)|^2 \to 0$, and the total length of the intervals on which $u_{\epsilon\lambda(n)}(t) \neq u_{\epsilon\lambda}(t)$ converges to zero. These facts imply that $E|x_{\epsilon\lambda(n)}(t) - x_{\epsilon\lambda}(t)|^2 \to 0$ for each t, which implies the continuity of $\zeta_\epsilon(\lambda)$ for each $\epsilon < \epsilon_o$. We need only prove the expansion (9), and this will be done in three parts.

1^o. Let K_i denote real numbers. We have the following relations

(13a) $$d\hat{x}(t) = f(\hat{x}(t), \hat{u}(t), t)dt + \sum_j dz_j(t)\sigma_j(\hat{x}(t), t)$$

(13b) $$dx_{\epsilon\lambda}(t) = f(x_{\epsilon\lambda}(t), u_{\epsilon\lambda}(t), t)dt + \sum_j dz_j(t)\sigma_j(x_{\epsilon\lambda}(t), t)$$

(13c) $$dy_{\epsilon\lambda}(t) = \hat{f}_x y_{\epsilon\lambda}(t)dt + [f(\hat{x}(t), u_{\epsilon\lambda}(t), t) - f(\hat{x}(t), \hat{u}(t), t)] + \sum_j dz_j(t)\hat{\sigma}_{j,x} y_{\epsilon\lambda}(t)$$

$$y_{\epsilon\lambda}(0) = \delta x_{\epsilon\lambda}(0) = \epsilon\delta x_\lambda(0).$$

Using standard estimates for solutions of Itô stochastic differential equations it can be shown that, for some $K_1 < \infty$,

(14) $$\max_{\epsilon<\epsilon_o, \lambda\epsilon\Lambda} E \max_{0\leq t\leq T} |x_{\epsilon\lambda}(t)|^2 \leq K_1.$$

Next, we define $\tilde{x}(t) \equiv \hat{x}(t) - x_{\epsilon\lambda}(t)$ and show that

(15) $$E|\tilde{x}(t)|^2 \equiv E|\hat{x}(t) - x_{\epsilon\lambda}(t)|^2 = O(\epsilon^2)$$

uniformly in t. Equation (15) holds for $t = 0$. Assume it holds for $t = t_o$, and that $u_{\epsilon\lambda}(t) = \hat{u}(t)$ for $t \in [t_o, t_1]$. We will show that (15) holds uniformly in $[t_o, t_1]$. Then, if (15) holds at $t = s_i - \epsilon\rho$, we show that it holds uniformly in $[s_i - \epsilon\rho, s_i]$, for any real ρ for which $s_i - \epsilon\rho \geq 0$. These two facts imply (15) as asserted. Let $\tilde{x}(t) \equiv x_{\epsilon\lambda}(t) - \hat{x}(t)$. Then,

$$\tilde{x}(t) = \tilde{x}(t_o) + \int_{t_o}^{t} [f(x_{\epsilon\lambda}(s), \hat{u}(s), s) - f(\hat{x}(s), \hat{u}(s), s)]ds$$
$$+ \int_{t_o}^{t} [\sigma(x_{\epsilon\lambda}(s), s) - \sigma(\hat{x}(s), s)]dz(s),$$

where $E|\tilde{x}(t_o)|^2 = O(\epsilon^2)$. By standard estimates, stochastic differential equations,

$$E|\tilde{x}(t)|^2 \leq K_2|\tilde{x}(0)|^2 + K_2 \int_{t_o}^{t} E|\tilde{x}(s)|^2 ds$$

which implies (15) in $[t_o, t_1]$. Next, write

$$\tilde{x}(t) = \tilde{x}(s_i - \epsilon\rho) + \int_{s_i - \epsilon\rho}^{t} [f(x_{\epsilon\lambda}(s), u_{\epsilon\lambda}(s), s) - f(\hat{x}(s), \hat{u}(s), s)]ds$$
$$+ \int_{s_i - \epsilon\rho}^{t} [\sigma(x_{\epsilon\lambda}(s), s) - \sigma(\hat{x}(s), s)]dz(s).$$

Using the Lipschitz condition on σ, and Schwarz's inequality on the drift term, gives

$$E|\tilde{x}(t)|^2 \le K_3 E|\tilde{x}(s_i-\epsilon\rho)|^2 + K_3 t \int_{s_i-\epsilon\rho}^{t} E[f(x_{\epsilon\lambda}(s),u_{\epsilon\lambda}(s),s) -$$

$$- f(\hat{x}(s),\hat{u}(s),s)]^2 ds + K_3 \int_{s_i-\epsilon\rho}^{t} |\tilde{x}(s)|^2 ds.$$

Using (14) and the growth condition $|f|^2 \le K_o(1+|x|^2)$ in (I-2) gives

$$E|\tilde{x}(t)|^2 \le K_3 E|\tilde{x}(s_i-\epsilon\rho)|^2 + K_4 t^2 + K_3 \int_{s_i-\epsilon\rho}^{t} E|\tilde{x}(s)|^2 ds$$

from which (15) follows in $[s_i-\epsilon\rho, s_i^-]$.

By reasoning close to the foregoing, it can be shown that

$$E|y_{\epsilon\lambda}(t)|^2 = O(\epsilon^2) \tag{16}$$

uniformly in $t \in [0,T]$.

2^o. Next, it will be shown that

$$E|x_{\epsilon\lambda}(t) - \hat{x}(t) - y_{\epsilon\lambda}(t)|^2 = o(\epsilon^2) \tag{17}$$

by the method used to show (15). Suppose $\hat{u}(t) = u_{\epsilon\lambda}(t)$ in $t \in [t_o,t_1]$ and (17) holds for $t = t_o$. Write $\tilde{y}(t) \equiv x_{\epsilon\lambda}(t) - \hat{x}(t) - y_{\epsilon\lambda}(t)$. Then, for $t \in [t_o,t_1]$,

$$\tilde{y}(t) = \tilde{y}(t_o) + \int_{t_o}^{t} [f(x_{\epsilon\lambda}(s),\hat{u}(s),s) - f(\hat{x}(s),\hat{u}(s),s) - \hat{f}_x y(s)]ds$$

$$+ \int_{t_o}^{t} \sum_j dz_j(s)[\sigma_j(x_{\epsilon\lambda}(s),s) - \sigma_j(\hat{x}(s),s) - \hat{\sigma}_{j,x} y_{\epsilon\lambda}(s)]$$

$$(18) \qquad = \tilde{y}(t_o) + \int_{t_o}^{t} \hat{f}_x \tilde{y}(s)ds + \int_{t_o}^{t} \sum_j dz_j(s)\hat{\sigma}_{j,x}\tilde{y}(s) + e_1(t) + e_2(t),$$

where, for $\tilde{x}(s) \equiv x(s) - \hat{x}(s)$, we define

$$e_1(t) = \int_{t_o}^{t} [f_x(\hat{x}(s) + \varphi(s)\tilde{x}(s),\hat{u}(s),s) - f_x(\hat{x}(s),\hat{u}(s),s)]\tilde{x}(s)ds$$

$$e_2(t) = \int_{t_o}^{t} \sum dz_j(s)[\sigma_{j,x}(\hat{x}(s) + \tilde{\varphi}(s)\tilde{x}(s),s) - \sigma_{j,x}(\hat{x}(s),s)]\tilde{x}(s)$$

where $\varphi(\cdot)$ and $\tilde{\varphi}(\cdot)$ are scalar valued random functions with values in $[0,1]$. By (15) and the continuity (in α) and boundedness properties of $f_x(\alpha,\beta,s)$ and $\sigma_{i,x}(\alpha,s)$,

$$E|e_i(t)|^2 = o(\epsilon^2)$$

uniformly in t. With this estimate (17), easily follows from (18) in $[t_o,t_1]$.

Next write $\delta t_i(\lambda) = \rho_i$ and let $E|\tilde{y}(s_i-\epsilon\tau_i)|^2 = o(\epsilon^2)$. For $t \in [s_i-\epsilon\tau_i, s_i-\epsilon\tau_i+\epsilon\rho_i]$ write

$$(19)\qquad \tilde{y}(t) = \tilde{y}(s_i-\epsilon\tau_i) + \int_{s_i-\epsilon\tau_i}^{t} [f(x_{\epsilon\lambda}(s),u_{\epsilon\lambda}(s),s) - f(\hat{x}(s),\hat{u}(s),s) - \hat{f}_x y_{\epsilon\lambda}(s)]ds$$

$$- \int_{s_i-\epsilon\tau_i}^{t} [f(\hat{x}(s),u_{\epsilon\lambda}(s),s) - f(\hat{x}(s),\hat{u}(s),s)]ds$$

$$+ \int_{s_i-\epsilon\tau_i}^{t} \sum_j dz_j(s)[\sigma_i(x_{\epsilon\lambda}(s),s) - \sigma_i(\hat{x}(s),s) - \hat{\sigma}_{i,x}y_{\epsilon\lambda}(s)]$$

(19) can be written as

$$(20)\qquad \tilde{y}(t) = \tilde{y}(s_i-\epsilon\tau_i) + \int_{s_i-\epsilon\tau_i}^{t} [f(x_{\epsilon\lambda}(s),u_{\epsilon\lambda}(s),s) - f(\hat{x}(s),u_{\epsilon\lambda}(s),s)]$$

$$+ \int_{s_i-\epsilon\tau_i}^{t} \sum_j dz_j(s)[\sigma_j(x_{\epsilon\lambda}(s),s) - \sigma_j(\hat{x}(s),s)] + e_3(t)$$

where we define

$$e_3(t) = -[\int_{s_i-\epsilon\tau_i}^{t} \hat{f}_x y_{\epsilon\lambda}(s)ds + \int_{s_i-\epsilon\tau_i}^{t} \sum_j dz_j(s)\hat{\sigma}_{j,x}y_{\epsilon\lambda}(s)].$$

Using $E|y_{\epsilon\lambda}(s)|^2 = O(\epsilon^2)$ uniformly in s we get, for t in the desired interval,

$$E\left|\int_{s_i-\epsilon\tau_i}^{t} \sum_j dz_j(s)\hat{\sigma}_{j,x}y_{\epsilon\lambda}(s)\right|^2 \leq K_5 \int_{s_i-\epsilon\tau_i}^{t} |y_{\epsilon\lambda}(s)|^2 ds = o(\epsilon^2),$$

and similarly for the first term of $e_3(t)$. Using this and the estimates for the two integrals in (20),

$$E\left|\int_{s_i-\epsilon\tau_i}^{t} \sum_j dz_j(s)[\sigma_j(x_{\epsilon\lambda}(s),s) - \sigma_j(\hat{x}(s),s)]\right|^2 +$$

$$E\left|\int_{s_i-\epsilon\tau_i}^{t} [f(x_{\epsilon\lambda}(s),u_{\epsilon\lambda}(s),s) - f(\hat{x}(s),u_{\epsilon\lambda}(s),s)]ds\right|^2$$

$$\leq K_5 \int_{s_i-\epsilon\tau_i}^{t} E|\tilde{x}(s)|^2 ds,$$

and (15), gives (17) in $[s_i-\epsilon\tau_i,\ s_i-\epsilon\tau_i + \epsilon\rho_i]$.

3^o. To complete the proof, it only remains to show that

(21) $$E|y_{\epsilon\lambda}(t) - \delta x_{\epsilon\lambda}(t)|^2 = o(\epsilon^2)$$

(21) holds for $t = 0$, and indeed, (21) is zero for $t \in [0,s_1-\epsilon\tau_1]$. If (21) holds at t_o, then it is true in $[t_o,t_1]$ if $u_{\epsilon\lambda}(t) = \hat{u}(t)$ in $[t_o,t_1]$, since w.p.1

(22) $$y_{\epsilon\lambda}(t_1) - \delta x_{\epsilon\lambda}(t_1) = \Phi(t_1,t_o)[y_{\epsilon\lambda}(t_o) - \delta x_{\epsilon\lambda}(t_o)].$$

Next, for $t \in [s_i - \epsilon\tau_i,\ s_i - \epsilon\tau_i + \epsilon\rho_i]$, where $\rho_i = \delta t_i(\lambda)$,

$$y_{\epsilon\lambda}(t) = y_{\epsilon\lambda}(s_i - \epsilon\tau_i) + J_i^{\epsilon}(t) + \tilde{o}(\epsilon^2) \tag{23}$$

where we define

$$J_i^{\epsilon}(t) = \int_{s_i-\epsilon\tau_i}^{t} [f(\hat{x}(s), u_{\epsilon\lambda}(s), s) - f(\hat{x}(s), \hat{u}(s), s)]ds$$

Let $J_i^{\epsilon} \equiv J_i^{\epsilon}(s_i - \epsilon\tau_i + \epsilon\rho_i)$. If $y_{\epsilon\lambda}^{1}(t_o) = y_{\epsilon\lambda}(t_o) + \tilde{o}(\epsilon^2)$, and $u_{\epsilon\lambda}(t) = \hat{u}(t)$, $t \in [t_o, t_1]$, then, w.p.1.,

$$\tilde{y}_{\epsilon\lambda}^{1}(t_1) = \Phi(t_1, t_o) y_{\epsilon\lambda}(t_o) + \tilde{o}(\epsilon^2).$$

Furthermore, $E|J_i^{\epsilon}(t)|^2 = O(\epsilon^2)$ uniformly in t, and $\Phi(t-\epsilon\varphi_1, \tau-\epsilon\varphi_2) \to \Phi(t,\tau)$ in probability as $\epsilon \to 0$, for any constants φ_1, φ_2.

The last paragraph implies that w.p.1., for $t \notin \cup I_i^{o}(\epsilon\lambda)$, where $I_i^{o}(\epsilon\lambda)$ is the interior of $I_i(\epsilon\lambda)$,

$$y_{\epsilon\lambda}(t) = \sum_{t>s_i} \Phi(t, s_i) J_i^{\epsilon} + \Phi(t,0)\delta x_{\epsilon\lambda}(0) + \tilde{o}(\epsilon^2). \tag{24}$$

Define

$$J_i = \int_{s_i-\epsilon\tau_i}^{s_i-\epsilon\tau_i+\epsilon\rho_i} [f(\hat{x}(s), u_i, s) - f(\hat{x}(s), \hat{u}(s), s)]ds.$$

Then (I-3) implies that $E|J_i^\epsilon - J_i|^2 = o(\epsilon^2)$. Thus (24) is valid for J_i replacing J_i^ϵ. By Lemma 1, (letting $\alpha_2 = -\tau_i + \rho_i, \alpha_1 = \tau_i$)

$$\frac{1}{\epsilon\rho_i} J_i \to f(\hat{x}(s_i),u_i,s_i) - f(\hat{x}(s_i),\hat{u}(s_i),s_i)$$

w.p.l. as $\epsilon \to 0$.

Thus, for $t \notin \cup I_i^o(\epsilon\lambda)$,

$$y_{\epsilon\lambda}(t) = \epsilon \sum_{t>s_i} \Phi(t,s_i)\delta t_i(\lambda)[f(\hat{x}(s_i),u_i,s_i) - f(\hat{x}(s_i),\hat{u}(s_i),s_i)] \tag{25}$$
$$+ \Phi(t,0)\delta x_{\epsilon\lambda}(0) + \tilde{o}(\epsilon^2)$$
$$= \delta x_{\epsilon\lambda}(t) + \tilde{o}(\epsilon^2).$$

Since the sets $I_i^o(\epsilon\lambda)$ decrease to the empty set as $\epsilon \to 0$, (25) holds for all $t \in [0,T]$. Q.E.D.

6. The Maximum Principle

Combining Theorems 1 and 2 we get Theorem 3. Define the $n + 1$ dimensional column vector $P \equiv (1,0,\ldots,0)'$. Theorem 3 reduces to the Pontriagin maximum principle, if the noise is absent $(\sigma \equiv 0)$.

Theorem 3. *Assume (I-1) - (I-5). There are continuous (in t) versions of $\Phi(T,\cdot)$, $\Phi(t_i,\cdot)$ (for $t \le T$ and $t \le t_i$, resp.). There is a* scalar $\theta \leq 0$, vectors $\alpha_i \leq 0$, $i = 0,1,\ldots,k+1$, (non-positive components α_i^j) where $\alpha_i^j = 0$ if $q_i^j(\hat{x}) < 0$, and vectors b_o, b_T, not all zero, and a null set $\tilde{T} \in [0,T]$ so that for all $t \notin \tilde{T}$, and all $\mathscr{B}_t$ measurable, $\mathscr{U}_t$ valued random variables u_t, and admissible $x(0)$, (26a and b) hold, w.p.1.

$$\text{(26a)} \quad \Big\{\theta E[P+h_x(\hat{x}(T))]'\Phi(T,t) + \sum_{i:t_i>t} \alpha_i' E[\hat{q}_{i,x}+E\hat{q}_{i,e}]\Phi(t_i,t) + b_T' E[\hat{r}_{T,x}+E\hat{r}_{T,e}]\Phi(T,t)\Big\} \cdot \Big\{f(\hat{x}(t),u_t,t) - f(\hat{x}(t),\hat{u}(t),t)\Big\} \leq 0$$

$$\text{(26b)} \quad \Big\{\theta E[P+h_x(\hat{x}_T)]'\Phi(T,0) + \sum_i \alpha_i' E[\hat{q}_{i,x}+E\hat{q}_{i,e}]\Phi(t_i,0) + b_T' E[\hat{r}_{T,x}+E\hat{r}_{T,e}]\Phi(T,0) + b_o' E[\hat{r}_{o,x}+E\hat{r}_{o,e}]\Big\}\delta x(0) \leq 0$$

(26b) implies that the term in braces in (26b) is zero. Define the vector $p(T)$ by its transpose (27)

$$\text{(27)} \quad p'(T) = \theta[P+h_x(\hat{x}(T))]' + b_T'[\hat{r}_{T,x}+E\hat{r}_{T,e}] + \alpha_{k+1}'[\hat{q}_{T,x}+E\hat{q}_{T,e}].$$

Define the $n+1$ dimensional random function $p(t)$, $t < T$ by its transpose (28).

$$\text{(28)} \quad \begin{aligned} p'(t) &= p'(T)\Phi(T,t), & t_k \leq t \leq t_{k+1} = T \\ p'(t_i^-) &= p'(t_i) + \alpha_i'[\hat{q}_{i,x}+E\hat{q}_{i,e}], & i = 1,\ldots,k \\ p'(t) &= p'(t_i^-)\Phi(t_i,t), & 0 = t_o \leq t_{i-1} \leq t < t_i \end{aligned}$$

with the use of (27-8), (26) can be written as

(29a) $$Ep'(t)[f(\hat{x}(t),u_t,t) - f(\hat{x}(t),\hat{u}(t),t)] \leq 0$$

(29b) $$E[p'(0) + b_o'(\hat{r}_{o,x}+E\hat{r}_{o,e})]\delta x(0) = 0.$$

Furthermore, w.p.1

(30a) $$E\{p'(t)[f(x(t),u(t),t) - f(x(t),u_t,t)]|\mathscr{B}_t\} \leq 0$$

(30b) $$E\{[p'(0) + \alpha_o'(\hat{r}_{o,x}+E\hat{r}_{o,e})]|\mathscr{B}_o\} = 0.$$

Proof. The proof of (26) follows from Theorem 1 using the appropriate identification of the (continuous in $\mathscr{T}$ by (I-4)) components of Q_i, R_i with the c_j, ℓ_j in Theorem 1, and the fact that K is a first order convex approximation to $Q' = Q - \hat{x}$ by Theorem 2. Also the (continuous in $\mathscr{T}$ by (I-4)) linear operator which acts on $\delta x(T)$ in $E[P+h_x(\hat{x}(T)]'\delta x(T)$, is identified with c_o. Equation (29) follows from (26) upon using the substitution (27), (28). To prove (30a) suppose that (30a) is violated on a $\mathscr{B}_t$ measurable set B_t with $P(B_t) > 0$. Define $\bar{u}_t = u_t$ on B_t, $\bar{u}_t = \hat{u}(t)$ on $\Omega - B_t$. Then (29a) is violated with the admissible $\bar{u}_t$ replacing the u_t there. A similar proof yields (30b). Q.E.D.

7. Extensions to Closed Loop Systems

Thus far the admissible controls are defined to be measurable on the a priori given σ-algebra $\mathscr{B}_t$. If the admissible controls are assumed to depend explicitly on the state - or on its past values, i.e., $u(t) = u(x(t),t)$ or $u(t) = u(x_s, s \leq t, t)$, then a very similar development can be carried out provided either the Lipschitz condition

$$(31) \qquad |u(\alpha,t) - u(\beta,t)| \leq K|\alpha-\beta|$$

or the generalized Lipschitz condition,

$$|u(x,t) - u(y,t)| \leq \int_0^t |x(t-s) - y(t-s)| dm(s)$$

for a bounded measure $m(\cdot)$, hold.[+] Indeed, with the use of the perturbed controls and a convex cone K of the type used in Theorem 2, we obtain Theorem 3, with the exception that the $\hat{f}_x$ terms in the $y_{\epsilon\lambda}(t)$ and $\Phi(t,\tau)$ equations are replaced by $\hat{f}_x + \hat{f}_u \cdot \hat{u}_x$. In particular, let the data available to the control at time t be $g(x(t),t)$, where $g(\cdot,\cdot)$ is a Borel function satisfying (31) with values in some Euclidean space and $|g(\alpha,t)|^2 \leq K_o(1+|\alpha|^2)$. Let the class of admissible controls $\tilde{\mathscr{U}}$ be the family of Borel functions $u(\cdot,\cdot)$ with values $u(g(x(t),t),t)$ and which satisfies (31), and which has values in $\mathscr{U}_t$ at time t. Let $x(0)$ satisfy the relevent parts of (I-1). Let (I-2) hold for admissible u. The convex cone is composed of elements with values (for almost all s)

[+]For more detail on the more general stochastic differential delay system, see [15].

$$\Phi(t,0)\delta x(0) + \Phi(t,s)[f(\hat{x}(s),u(g(\hat{x}(s),s),s),s) - f(\hat{x}(s),\hat{u}(g(\hat{x}(s),s),s),s)].$$

For each t and admissible u, it is supposed that there is a continuous function $\tilde{u}(\cdot,\cdot)$ (of g,t) satisfying (31) with $\tilde{u}(g,t) \in \mathscr{U}_t$ for all g and $|\tilde{u}(g,t)|^2 \leq K_0(1+|g|^2)$ such that $u(g,s) = \tilde{u}(g,s)$. (This is not a significant restriction.)

Let $\tilde{u}_i(g,t)$ be a function which satisfies the conditions on $\tilde{u}_i(g,t)$ above and reduces to $u(g,s_i)$ at $t = s_i$. In (10), let $u_{\epsilon\lambda},t = \tilde{u}_i(g(x(t),t),t)$ in $I_i(\epsilon\lambda)$. Then, under the additional conditions (I4-5), Theorem 3 holds with the conditioning on $\mathscr{B}_t$ replaced by conditioning on $g(\hat{x}(t),t)$. We have not given more details on the extensions to state dependent controls, since attempts to extend the method to a more general class of controls, whose members may be discontinuous in the state, have failed so far.

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