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Small-Pitch-Angle Synchrotron Radiation

by

Richard I. Epstein

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Office of Naval Research
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ABSTRACT

Synchrotron radiation from electrons with very small pitch angles is described and the intensity and polarization of this radiation are derived.

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It has recently been pointed out, by O'Dell and Sartori [1], that the conventional formulas for synchrotron radiation are inappropriate even for highly relativistic electrons if the pitch angles are very small. O'Dell and Sartori were concerned with the significance of this fact for theories of pulsars, but their point is of much wider significance. For instance, some theories of the acceleration of electrons in solar flares [2] lead to the acceleration of electrons with very small pitch angles. Small-pitch-angle effects may therefore be important for our understanding of some types of solar radio bursts.

In this report, the qualitative difference between "normal" and "small-pitch-angle" synchrotron radiation will be described, and the intensity and polarization of the radiation for the latter case will be derived. When the velocity of a relativistic electron is nearly parallel to a magnetic field, the power received by a fixed observer will be much greater than the power emitted by the moving electron. This effect and its consequence for the synchrotron radiation from a stationary ensemble of electrons will be reviewed. The changes in pitch angle and emissivity caused by radiation losses also will be discussed.

A. Normal and Small-Pitch-Angle Synchrotron Radiation

For normal synchrotron radiation, the pitch angle of the helical trajectory of the relativistic electron ξ is large when compared to $1/\gamma$, where γ is the Lorentz factor of the electron $\gamma = E/mc^2$ and E is the electron energy. Radiation from the electron is confined primarily to a narrow beam of half angle $\sim 1/\gamma$ in the direction of the electron velocity. Each time the electron completes an orbit, its velocity vector (and, thus, the beam of radiation) describes a cone of half angle ξ, as seen in Fig. 1. An observer located in the field of radiation will receive a pulse of

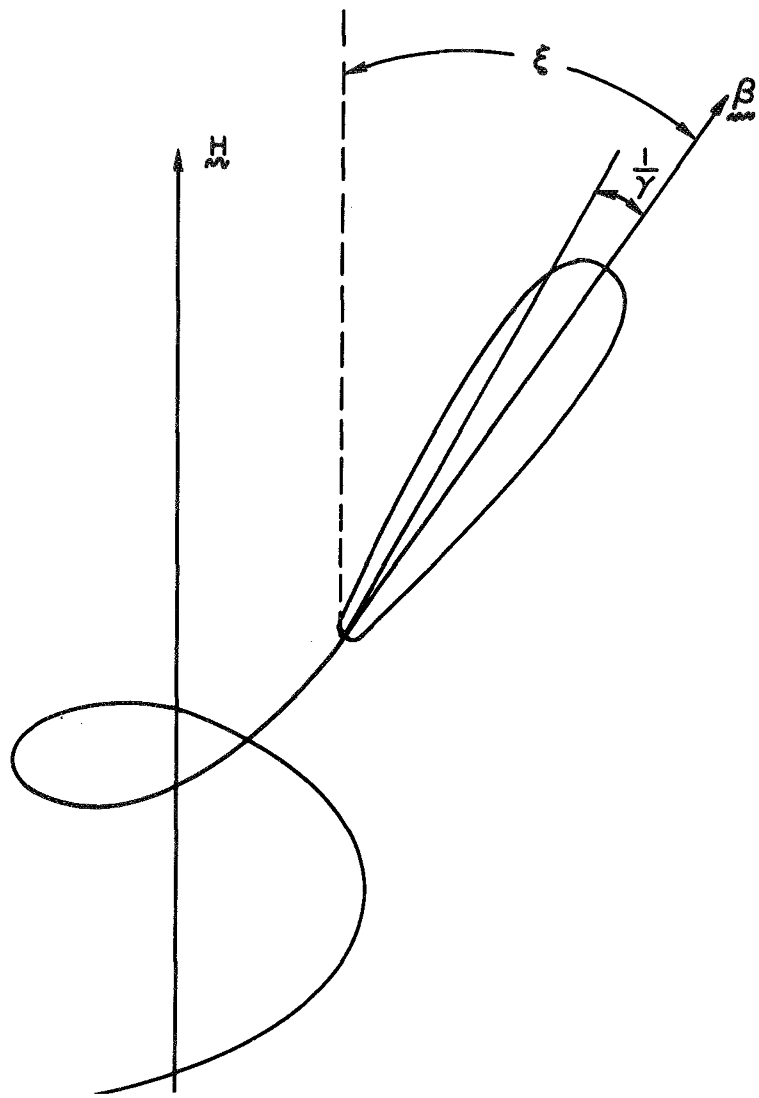

Fig. 1. RADIATION PATTERN OF A RELATIVISTIC ELECTRON SPIRALING WITH A LARGE PITCH ANGLE IN A MAGNETIC FIELD.

energy each time the beam sweeps over him, and the characteristic frequency of this radiation is roughly the reciprocal of the pulse width.

When the pitch angle ξ is small compared to $1/\gamma$, the radiation will be basically different from that described above. Now, the beam of radiation is large compared to the pitch angle; therefore, instead of sweeping around a large cone, the beam merly gyrates about a point near its center. An observer within the beam will receive a continuous stream of radiation rather than discrete bursts.

B. Derivation of Small-Pitch-Angle Synchrotron Radiation

To find the characteristics of radiation from an electron moving in a helical trajectory, the calculation has been divided into three parts: (1) transforming to an inertial frame in which the electron is in a circular orbit, (2) computing the electron emissivity in the "moving frame," and (3) transforming back to find the emissivity of the electron in the "laboratory frame." This type of calculation simplifies the mathematics and clarifies the physics. The distinction between radiation from electrons in circular and helical orbits is made by the choice of inertial frame: one man's circular orbit is another man's helical orbit. Conceptually, it is useful to separate "intrinsic radiation" from an electron in a circular orbit and the special relativistic effects caused by the constant velocity of the electron along the axis of the helix. Some care must be taken, however, to avoid confusion between the quantities that are measured in different frames and between the properties of radiation emitted by the moving electron and the radiation received by a stationary observer. The quantities in the moving frame (the frame in which the electron is in a circular orbit) will be denoted by a hat (^), and the properties of the emitted and received radiation will be distinguished by the subscripts e and r, respectively.

In the laboratory frame, the electron has a relativistic energy γmc^2 and a very small pitch angle ξ. To state the conditions in a more useful form, $\gamma = O(1/\epsilon)$ and $\gamma\xi = O(\epsilon)$ where $\epsilon \ll 1$.

Without loss of generality, the observer is taken to be in the $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ plane and the electron position and velocity (Fig. 2) to be

$$\underset{\sim}{r} = \frac{c\gamma\beta_\perp}{\omega_H}\left[\underset{\sim}{e}_1 \cos\left(\frac{\omega_H t}{\gamma}\right) + \underset{\sim}{e}_2 \sin\left(\frac{\omega_H t}{\gamma}\right)\right] + c\beta_{\parallel} t\underset{\sim}{e}_3 \tag{1}$$

$$\underset{\sim}{v}/c \equiv \underset{\sim}{\beta} = \beta_\perp\left[-\underset{\sim}{e}_1 \sin\left(\frac{\omega_H t}{\gamma}\right) + \underset{\sim}{e}_2 \cos\left(\frac{\omega_H t}{\gamma}\right)\right] + \beta_{\parallel}\underset{\sim}{e}_3 \tag{2}$$

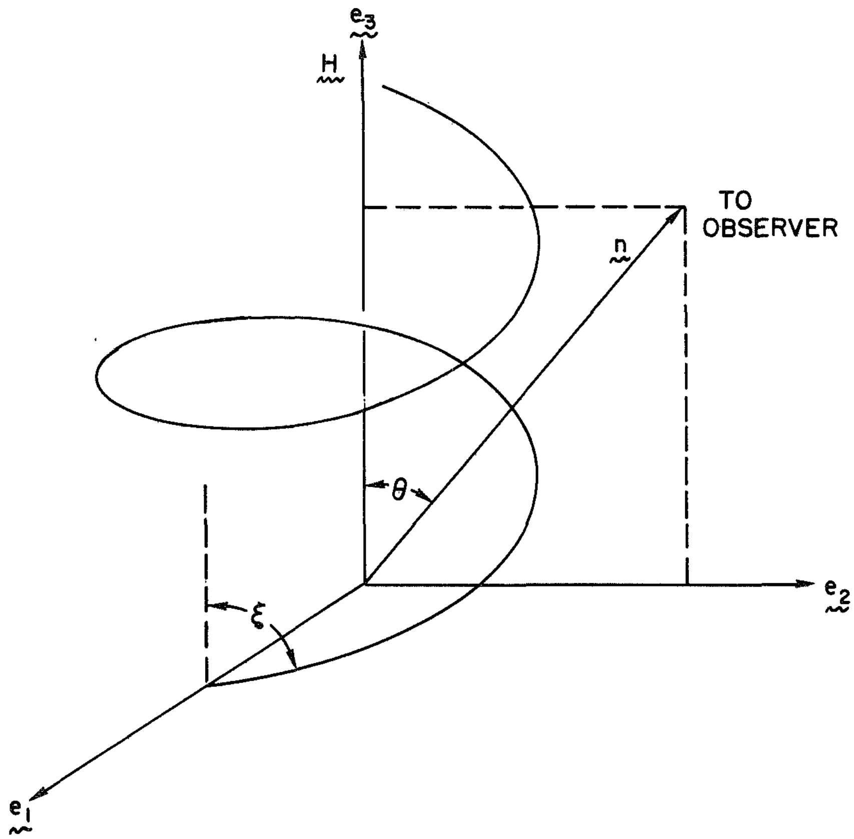

Fig. 2. ELECTRON TRAJECTORY IN THE LABORATORY FRAME.

where $\omega_H = eH/mc$, e is the absolute value of the electron charge, m is the electron mass, and H is the magnetic-field strength. The velocities in units of c parallel and perpendicular to the magnetic field are $\beta_{\parallel} = \beta \cos \xi$ and $\beta_\perp = \beta \sin \xi$, respectively, where $\beta = (1-\gamma^{-2})^{1/2}$.

Equations (1) and (2) describe the electron motion in the laboratory frame. In an inertial frame which is moving with a velocity

$\beta_{\parallel} \underset{\sim}{e}_3$ relative to the laboratory frame, the electron energy, position, and velocity (Fig. 3) are

$$\hat{E} = \hat{\gamma} mc^2 \tag{3}$$

$$\hat{\underset{\sim}{r}} = \frac{c\hat{\gamma}\hat{\beta}}{\omega_H}\left[\hat{\underset{\sim}{e}}_1 \cos\left(\frac{\omega_H \hat{t}}{\hat{\gamma}}\right) + \hat{\underset{\sim}{e}}_2 \sin\left(\frac{\omega_H \hat{t}}{\hat{\gamma}}\right)\right] \tag{4}$$

$$\hat{\underset{\sim}{v}} = \hat{\beta} c\left[-\hat{\underset{\sim}{e}}_1 \sin\left(\frac{\omega_H \hat{t}}{\hat{\gamma}}\right) + \hat{\underset{\sim}{e}}_2 \cos\left(\frac{\omega_H \hat{t}}{\hat{\gamma}}\right)\right] \tag{5}$$

where

$$\hat{\gamma} = (\gamma^2 \sin^2 \xi + \cos^2 \xi)^{1/2} = 1 + O(\epsilon^2) \tag{6}$$

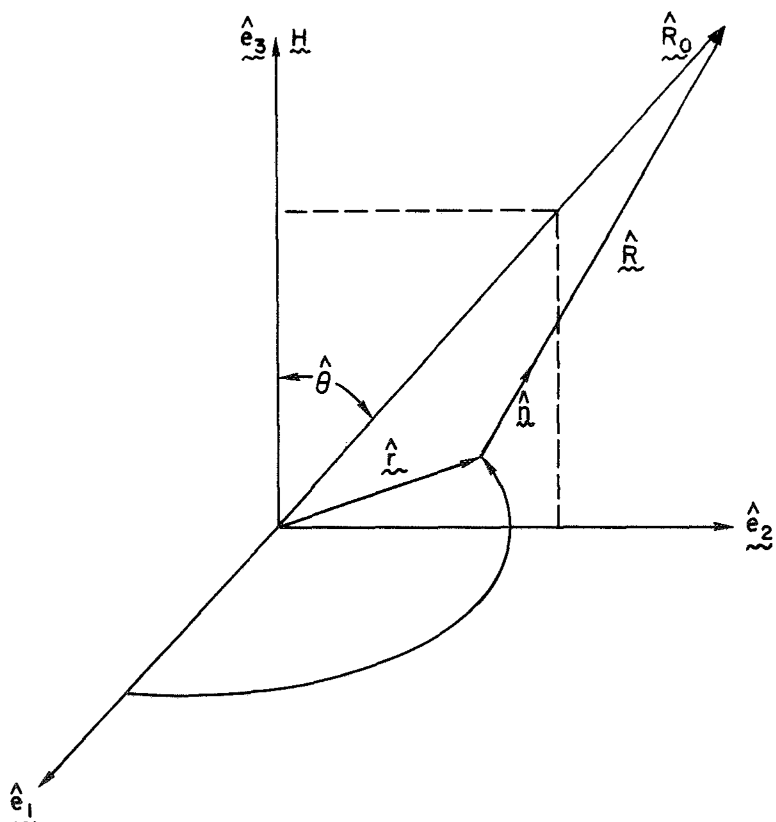

Fig. 3. ELECTRON TRAJECTORY IN THE MOVING FRAME.

$$\hat{\beta} = \frac{(\gamma^2-1)^{1/2} \sin \xi}{(\gamma^2 \sin^2 \xi + \cos^2 \xi)} = \gamma\xi + 0(\epsilon^3) \tag{7}$$

and

$$\hat{t} = \frac{\hat{\gamma}}{\gamma}\, t = \frac{t}{\gamma}\, [1 + 0(\epsilon^2)] \tag{8}$$

As O'Dell and Sartori noted, the electron is nonrelativistic in the inertial frame in which it is moving in a circular orbit; the radiation from this type of electron is simply cyclotron radiation (discussed in the Appendix) the emissivity of which is

$$\hat{\eta}_e(\hat{\nu},\hat{\theta}) = \frac{e^2\hat{\beta}^2\omega_H^2}{4c}\,(1 + \cos^2 \hat{\theta})\, \delta\left(\hat{\nu} - \frac{\omega_H}{2\pi}\right) \quad \text{erg sterad}^{-1}\ \text{sec}^{-1}\ \text{Hz}^{-1} \tag{9}$$

The degree of linear polarization in the direction perpendicular to the magnetic field is

$$\hat{\Pi}_\ell(\hat{\theta}) = \frac{\sin^2 \hat{\theta}}{1 + \cos^2 \hat{\theta}} \tag{10}$$

and the degree of circular polarization is

$$\hat{\Pi}_c(\hat{\theta}) = \frac{2 \cos \hat{\theta}}{1 + \cos^2 \hat{\theta}} \tag{11}$$

where $\hat{\Pi}_c > 0$ for right-hand polarization (the observer sees the electric vector rotate counter clockwise) and $\hat{\Pi}_c < 0$ for left-hand polarization. The total radiated power is

$$\hat{\eta}_T = -\frac{d\hat{E}}{d\hat{t}} = \int \hat{\eta}_e(\hat{\theta},\hat{\nu})\, d\hat{\Omega}\, d\hat{\nu} = \frac{2e^2\hat{\beta}^2\omega_H^2}{3c} \tag{12}$$

The problem of finding the characteristics of the radiation now reduces to finding the proper transformation for the quantities in Eqs. (7) through (12). The number of photons emitted in intervals of solid angle, time, and frequency is a scalar and is invariant under Lorentz transformations; therefore,

$$\frac{\eta_e(\nu,\theta)}{h\nu}\, d\Omega\, d\nu\, dt_e = \frac{\hat{\eta}_e(\hat{\nu},\hat{\theta})}{h\hat{\nu}}\, d\hat{\Omega}\, d\hat{\nu}\, d\hat{t}_e \tag{13}$$

Because

$$\frac{d\nu}{\nu} = \frac{d\hat{\nu}}{\hat{\nu}} \tag{14}$$

Eq. (13) yields

$$\eta_e(\nu,\theta) = \hat{\eta}_e(\hat{\nu},\hat{\theta})\, \frac{d\hat{\Omega}}{d\Omega}\, \frac{d\hat{t}_e}{dt} \tag{15}$$

The angles between the direction of propagation and the magnetic field, as measured in the moving and laboratory frames, are related by [3]

$$\tan\hat{\theta} = \frac{\sin\theta(1 - \beta_{\parallel}^2)^{1/2}}{\cos\theta - \beta_{\parallel}} \tag{16}$$

and

$$\frac{d\hat{\Omega}}{d\Omega} = \frac{d(\cos\hat{\theta})}{d(\cos\theta)} = \frac{1 - \beta_{\parallel}^2}{(1 - \beta_{\parallel}\cos\theta)^2} \tag{17}$$

To a relative accuracy of $O(\epsilon^2)$, this relationship is

$$\frac{d\hat{\Omega}}{d\Omega} = \frac{1}{\gamma^2(1 - \beta\cos\theta)^2} \tag{18}$$

For $\theta \gg 1/\gamma$, $d\hat{\Omega}/d\Omega$ decreases rapidly; therefore, it is sufficient to treat θ as a small quantity on the order of ϵ. The solid angle transformation then becomes

$$\frac{d\hat{\Omega}}{d\Omega} = \frac{4\gamma^2}{(1 + \gamma^2\theta^2)^2} \tag{19}$$

When special relativistic effects are significant (that is, when considering velocities comparable with the speed of light), temporal and spatial displacements are interrelated and both must be specified to perform a transformation. The time intervals dt_e and $d\hat{t}_e$ refer to the changes in the time coordinates of the laboratory and moving frames, respectively, which would be measured at the position of the electron. (The relationship between dt_r, the time interval as measured at the observer, and dt_e will be discussed in the next section.)

The radiation emitted in the interval $d\hat{t}_e$ in the moving frame is emitted during the interval dt_e when time is measured in the laboratory frame, and

$$\frac{d\hat{t}_e}{dt_e} = \left(1 - \beta_\parallel^2\right)^{1/2} = \frac{1}{\gamma} + 0(\epsilon^3) \tag{20}$$

Combining Eqs. (15), (19), and (20) yields

$$\eta_e(\theta, \nu) = \hat{\eta}_e(\hat{\theta}, \hat{\nu}) \frac{4\gamma}{\left(1 + \gamma^2\theta^2\right)^2} \tag{21}$$

The frequency of radiation in the moving frame is related to the frequency of radiation in the laboratory frame by [3]

$$\frac{\hat{\nu}}{\nu} = \frac{\left(1 - \beta_\parallel^2\right)^{1/2}}{\cos\theta - \beta_\parallel} = \frac{2\gamma}{1 + \theta^2\gamma^2} \cdot \left[1 + 0(\epsilon^2)\right] \tag{22}$$

From Eq. (16), the angular dependence of radiation can be obtained for small θ by

$$1 + \cos^2\hat{\theta} = \frac{2(1 + \gamma^4\theta^4)}{1 + \gamma^2\theta^2} \tag{23}$$

Combining Eqs. (7), (9), (21), (22), and (23) produces the electron emissivity in terms of the parameters of the laboratory frame

$$\eta_e(\theta, \nu) = \frac{2e^2\omega_H\gamma^3\xi^2}{c} \frac{\nu}{\nu_c} \left[2\left(\frac{\nu}{\nu_c}\right)^2 - \frac{2\nu}{\nu_c} + 1\right] \delta\left(\theta^2\gamma^2 - \frac{\nu_c}{\nu} + 1\right)$$

$$\text{erg sterad}^{-1}\ \text{sec}^{-1}\ \text{Hz}^{-1} \tag{24}$$

where $\nu_c = \gamma\omega_H/\pi$. The radiation in any direction θ will be monochromatic with $\nu = \nu_c/(1 + \theta^2\gamma^2)$. The frequency spectrum of radiation over all angles (Fig. 4) is

$$\eta_e(\nu) \equiv \int \eta_e(\nu,\theta)\, d\Omega = \begin{cases} \dfrac{2\pi e^2 \omega_H \gamma \xi^2}{c} \dfrac{\nu}{\nu_c} \left[2\left(\dfrac{\nu}{\nu_c}\right)^2 - \dfrac{2\nu}{\nu_c} + 1\right] & \nu < \nu_c \\ 0 & \nu > \nu_c \end{cases}$$

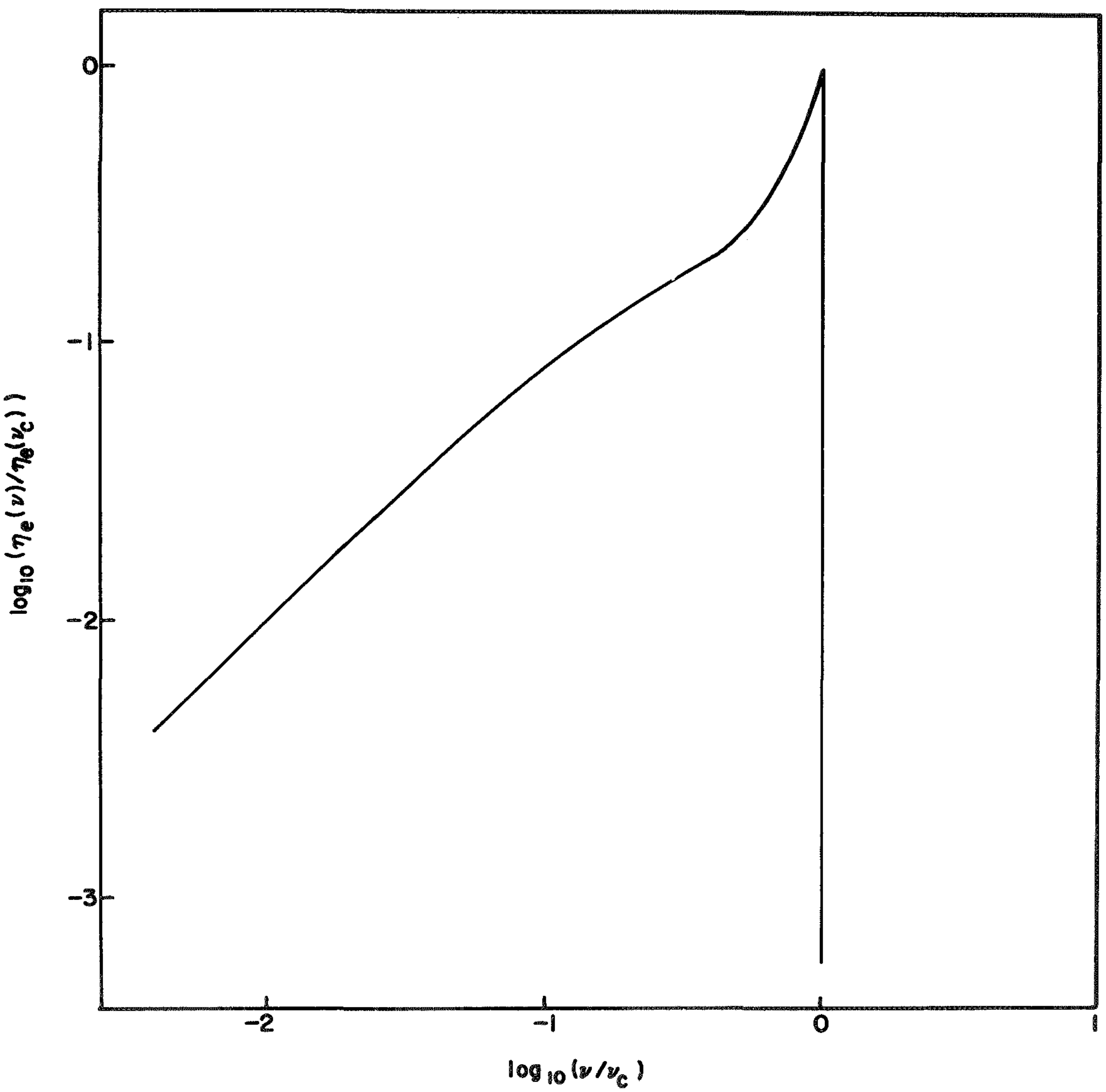

Fig. 4. FREQUENCY SPECTRUM OF THE SMALL-PITCH-ANGLE SYNCHROTRON EMISSION INTEGRATED OVER ALL ANGLES.

and the angular distribution of power over all frequencies (Fig. 5) is

$$\eta_e(\theta) = \int \eta_e(\nu,\theta)\, d\nu = \frac{4e^2\omega_H^2\gamma^4\xi^2}{c}\ \frac{1+\theta^4\gamma^4}{\left(1+\theta^2\gamma^2\right)^5}$$

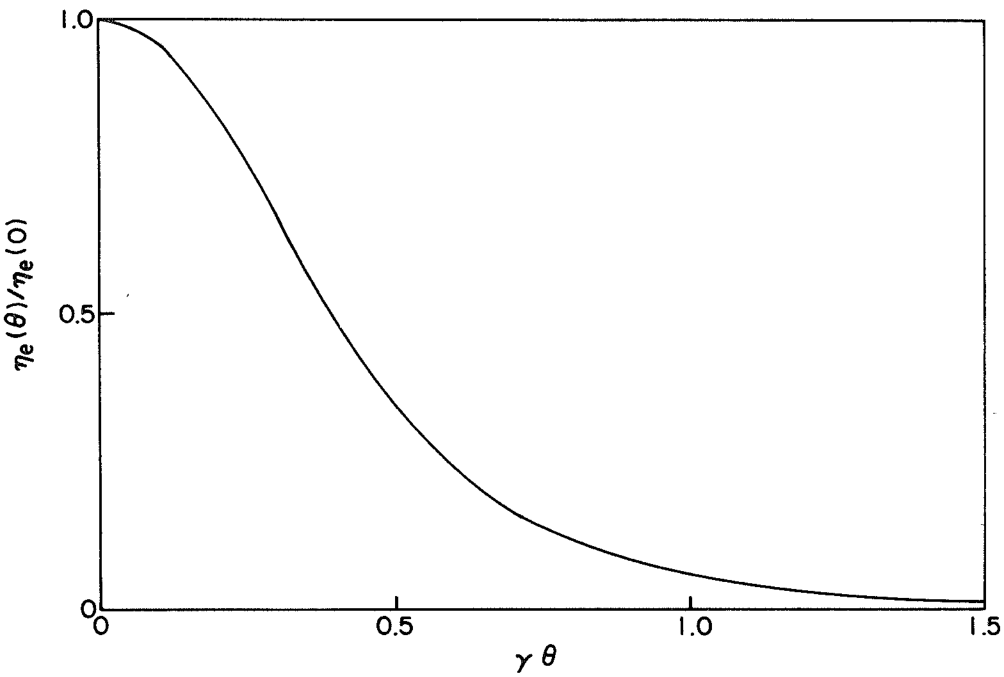

Fig. 5. ANGULAR DISTRIBUTION OF THE SMALL-PITCH-ANGLE SYNCHROTRON EMISSION.

The total power emitted over all frequencies and angles is

$$\eta_T = -\frac{dE}{dt} = \int \eta_e(\nu)\, d\nu = \int \eta_e(\theta)\, d\Omega = \frac{2e^2\omega_H^2\gamma^2\xi^2}{3c} \tag{25}$$

Inserting $\gamma\xi = \hat{\beta}$ into Eq. (25) shows that the total emission in the laboratory and moving frames [Eq. (12)] are the same because the particle energy E and the time interval dt_e have identical transformation properties; that is

$$\frac{d\hat{E}}{dE} = \frac{d\hat{t}}{dt} \tag{26}$$

The degrees of polarization are scalar quantities and thus are invariant:

$$\Pi_{\ell,c}(\theta) = \hat{\Pi}_{\ell,c}(\hat{\theta}) \tag{27}$$

Using Eqs. (10), (11), and (16) and remembering that $\theta = 0(\epsilon)$, we obtain

$$\Pi_{\ell} = \frac{2\theta^2\gamma^2}{1 + \theta^4\gamma^4} \tag{28}$$

for the degree of linear polarization perpendicular to the magnetic field and

$$\Pi_{c} = \frac{1 - \theta^4\gamma^4}{1 + \theta^4\gamma^4} \tag{29}$$

for the degree of circular polarization, where $\Pi_c > 0$ for right-hand polarization and $\Pi_c < 0$ for left-hand polarization.

C. Apparent Emissivity†

Because of their relative motion, the power received by a stationary observer can differ significantly from the power emitted by a moving electron. Examination of Fig. 6 reveals that the radiation emitted in a time interval dt_e will be detected in a time interval dt_r where

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†A good discussion of the material in this section with a somewhat different orientation can be found in the article by Ginzburg and Syrovatskii [4].

$$dt_r = dt_e - \beta_{\parallel} \cos\theta \, dt_e = dt_e(1 - \beta_{\parallel} \cos\theta)$$

$$= dt_e\left(\frac{1 + \gamma^2\theta^2}{2\gamma^2}\right) \cdot \left[1 + 0(\epsilon^2)\right] \tag{30}$$

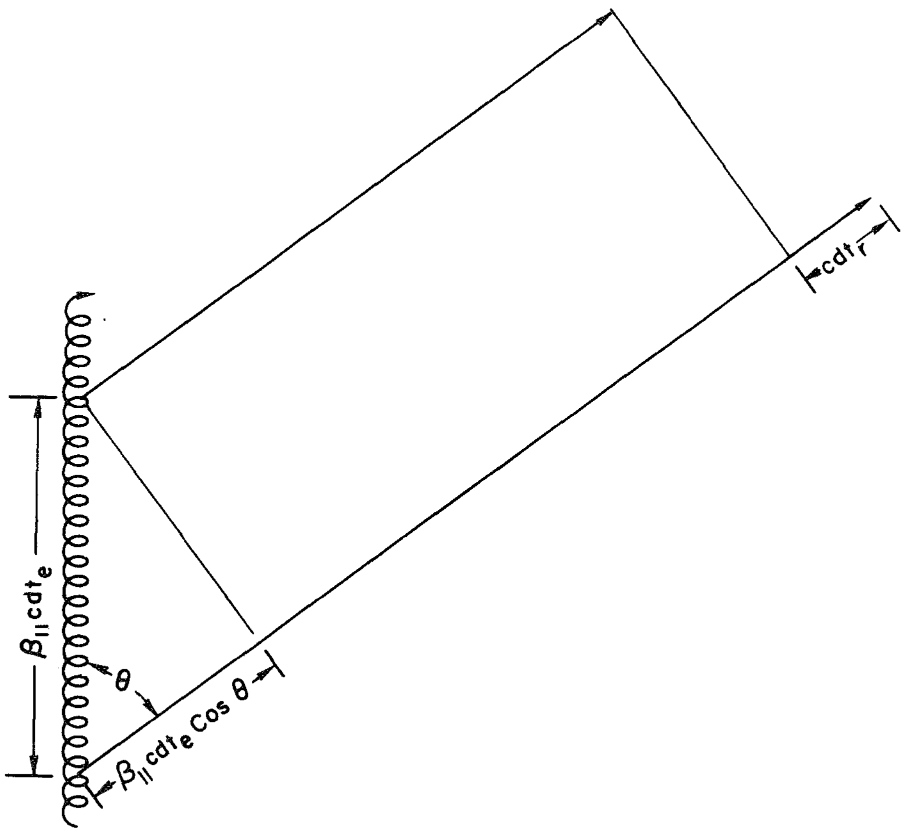

Fig. 6. DIAGRAM OF THE RELATIONSHIP BETWEEN dt_e AND dt_r.

The second term is the effect of decreasing the propagation time of the electron as the distance between the electron and the observer is shortened.

The relationship between the "real emissivity" of the electron (the energy lost by the electron in intervals of time, frequency, and solid angle) and the "apparent emissivity" (the energy received in intervals of time and frequency by an observer who subtends an element of solid angle) is

$$\eta_r(\theta, \nu)\ dt_r = \eta_e(\theta, \nu)\ dt_e \tag{31}$$

From Eqs. (30) and (31),

$$\eta_r(\theta, \nu) = \eta_e(\theta, \nu)\left(\frac{2\gamma^2}{1 + \gamma^2\theta^2}\right) \tag{32}$$

Stationary observers can measure only $\eta_r(\theta, \nu)$; therefore, in the absence of any knowledge concerning the physics of the source, observers could not distinguish between a stationary radiator that emits at a rate $\eta_r(\theta, \nu)$ and a moving one with a velocity $\beta_{\parallel} \underset{\sim}{e}_3$ that emits energy at a rate $\eta_e(\theta, \nu)$.

The power received from a single electron can be considerably greater than the power emitted; however, the electron will radiate in a given direction for only a finite time before it loses its energy or is deflected, and the time during which the observer receives the radiation is correspondingly shorter than the time in which the radiation is emitted, as shown by Eq. (31). In agreement with the conservation laws, the total energy that can be detected from an electron equals the total energy emitted by the electron.

D. Stationary Ensemble of Electrons

To calculate the volume emissivity of a time-independent ensemble of electrons, say, a stationary "cloud" or a steady stream, one needs to consider only the real emissivity of each particle. The apparent emissivity of an electron $\eta_r(\theta, \nu)$ is more compressed than its real emissivity (it is more intense and of shorter duration); however, when the electron distribution is stationary, the fluctuations of the individual electrons are unimportant and only the average emissivity is significant.

It is easy to show that the average apparent emissivity (averaged over the electron "lifetime") is equal to the real emissivity. If an electron radiates at a constant rate for a time T_e, an observer will detect the apparent emissivity $\eta_r(\theta, \nu)$ for a time $T_e \cdot (dt_r/dt_e)$. The average apparent emissivity is then

$$\eta_{\text{average}}(\theta, \nu) = \frac{1}{T_e}\left[\eta_r(\theta, \nu) T_e \frac{dt_r}{dt_e}\right] = \eta_r(\theta, \nu) \frac{dt_r}{dt_e} \tag{33}$$

and, by Eq. (31), this is

$$\eta_{\text{average}}(\theta, \nu) = \eta_e(\theta, \nu) \tag{34}$$

The volume emissivity of a time-independent ensemble of electrons, therefore, is given by

$$\epsilon(\underset{\sim}{r}, \theta, \nu) = \int f(\underset{\sim}{r}, \gamma, \xi)\ \eta_e(\theta, \nu)\ d\gamma\ d\xi \tag{35}$$

where $f(\underset{\sim}{r}, \gamma, \xi)$ is a distribution function for the electrons.

E. Pitch-Angle Relaxation

Equation (12) gives the time dependence of the electron velocity in the moving frame,

$$\frac{d\hat{E}}{d\hat{t}} = \frac{d}{d\hat{t}}\left(\frac{1}{2}\, mc^2\hat{\beta}^2\right) = -\frac{2e^2\omega_H^2\hat{\beta}^2}{3c} \tag{36}$$

Integrating Eq. (36), one obtains

$$\hat{\beta}(\hat{t}) = \hat{\beta}(0)\, e^{-\hat{t}/\hat{\tau}} \tag{37}$$

where

$$\hat{\tau} \equiv \frac{3mc^2}{2e^2\omega_H^2}$$

The pitch angle ξ is related to $\hat{\beta}$ by Eq. (7), $\xi = \hat{\beta}/\gamma$. In the laboratory frame, the electron velocity is principally in the $\underset{\sim}{e}_3$ direction, $\beta \approx \beta_{\parallel}$; therefore, the Lorentz factor γ is only negligibly dependent on $\hat{\beta}$. Combining Eqs. (7), (8), and (37) yields

$$\xi(t) = \xi(0)\, e^{-t/\tau} \tag{38}$$

where

$$\tau = \gamma\hat{\tau} = \frac{3mc^3\gamma}{2e^2\omega_H^2}$$

Using Eqs. (24) and (38), the time-dependent single-particle emissivity for an electron in a uniform magnetic field is found to be

$$\eta_e\left[\theta, \xi(t_e)\right] = \eta_e\left[\theta, \xi(0)\right]\, e^{-2t/\tau} \tag{39}$$

and the time-dependent apparent emissivity is obtained by including the effects of time contraction [Eq. (30)] and intensity enhancement [Eq. (32)],

$$\eta_r(\theta, t_r) = \frac{2\gamma^2}{1 + \theta^2\gamma^2}\, \eta_e[0, \xi(0)] \exp\left(\frac{-2\gamma^2}{1 + \theta^2\gamma^2}\, \frac{t_r}{\tau}\right)$$

As explained in the previous sections, the received pulse is more intense but of shorter duration than the emitted pulse, and the total radiation received in any direction is equal to the total radiation emitted in that direction.

Appendix

CYCLOTRON RADIATION FROM A NONRELATIVISTIC ELECTRON IN A CIRCULAR ORBIT

The electron is orbiting in a uniform magnetic field $\underset{\sim}{H} = H\underset{\sim}{e}_3$ and its position, velocity, and acceleration (Fig. 3) are

$$\hat{\underset{\sim}{r}} = \frac{c\hat{\beta}}{\omega_H}\left[\hat{\underset{\sim}{e}}_1 \cos(\omega_H \hat{t}) + \hat{\underset{\sim}{e}}_2 \sin(\omega_H \hat{t})\right] \tag{A.1}$$

$$\hat{\underset{\sim}{\beta}} = \hat{\beta}\left[-\hat{\underset{\sim}{e}}_1 \sin(\omega_H \hat{t}) + \hat{\underset{\sim}{e}}_2 \cos(\omega_H \hat{t})\right] \tag{A.2}$$

$$\dot{\hat{\underset{\sim}{\beta}}} = -\hat{\beta}\omega_H\left[\hat{\underset{\sim}{e}}_1 \cos(\omega_H \hat{t}) + \hat{\underset{\sim}{e}}_2 \sin(\omega_H \hat{t})\right] \tag{A.3}$$

where the dot denotes differentiation with respect to time. Because the electron is nonrelativistic, $\hat{\beta} \ll 1$ and $\hat{\gamma} \equiv (1 - \hat{\beta}^2)^{1/2} \approx 1$.

The electric field far from the electron [3] is

$$\hat{\underset{\sim}{E}}(t) = -\frac{e}{c}\left[\frac{\hat{\underset{\sim}{n}} \times (\hat{\underset{\sim}{n}} \times \dot{\hat{\underset{\sim}{\beta}}})}{\hat{R}}\right]_{ret} \tag{A.4}$$

where $\hat{\underset{\sim}{n}}$ is a unit vector in the direction of propagation, $\hat{R}$ is the distance from the electron to the observer, and the term in the brackets is evaluated at the retarded time

$$\hat{t}_{ret} = \frac{\hat{t} - \hat{R}}{c} \tag{A.5}$$

The vector from the origin to the observer is $\hat{\underset{\sim}{R}}_o$; therefore,

$$\hat{R} = |\hat{\underset{\sim}{R}}_o - \hat{\underset{\sim}{r}}| \approx \hat{R}_o - \frac{\hat{\underset{\sim}{R}}_o \cdot \hat{\underset{\sim}{r}}}{\hat{R}_o} \tag{A.6}$$

where $\hat{R}_o \equiv |\hat{\underset{\sim}{R}}_o|$ and $\hat{R}_o \gg |\hat{\underset{\sim}{r}}|$.

Using Eq. (A.1), it can be seen that the retarded time is

$$\hat{t}_{ret} = \frac{\hat{t} - \hat{R}_o}{c} + 0\left(\frac{\hat{\beta}}{\omega_H}\right) \tag{A.7}$$

Because the significant time scale for changes is $1/\omega_H$, terms of the order of $\hat{\beta}/\omega_H \ll 1/\omega_H$ will be neglected.

The unit vector $\hat{n}$ is taken to be (Fig. 3) $\hat{\underset{\sim}{n}} = \hat{\underset{\sim}{e}}_2 \sin\hat{\theta} + \hat{\underset{\sim}{e}}_3 \cos\hat{\theta}$. Equations (A.3) and (A.4) now give

$$\underset{\sim}{\hat{E}}(\hat{t}) = -\frac{e\hat{\beta}\omega_H}{c\hat{R}_o}\left[\underset{\sim}{e}_1 \cos\left(\omega_H\hat{t} - \frac{\hat{R}_o\omega_H}{c}\right) + \underset{\sim}{e}_2 \cos^2\hat{\theta}\sin\left(\omega_H\hat{t} - \frac{\hat{R}_o\omega_H}{c}\right)\right.$$
$$\left. + \underset{\sim}{e}_3 \sin\hat{\theta}\cos\hat{\theta}\sin\left(\omega_H\hat{t} - \frac{\hat{R}_o\omega_H}{c}\right)\right] \tag{A.8}$$

or, in exponential notation,

$$\underset{\sim}{\hat{E}}(\underset{\sim}{\hat{t}}) = -\frac{e\hat{\beta}\omega_H}{c\hat{R}_o}\left(\underset{\sim}{e}_1 + i\underset{\sim}{e}_2\cos^2\hat{\theta} + i\underset{\sim}{e}_3 \sin\hat{\theta}\cos\hat{\theta}\right)\exp\left[-i\left(\omega_H\hat{t} - \frac{\hat{R}_o\omega_H}{c}\right)\right] \tag{A.9}$$

where it is understood that only the real part of $\underset{\sim}{\hat{E}}(t)$ is physically significant.

From Eq. (A.9), the radiation intensity and polarization can be derived. The energy flow across a unit area is given by the time average of the Poynting vector

$$\langle S\rangle = \frac{c}{8\pi}\,\underset{\sim}{\hat{E}} \cdot \underset{\sim}{\hat{E}}^* \tag{A.10}$$

$$\langle S \rangle = \frac{e^2 \hat{\beta} \omega_H^2}{8\pi c \hat{R}_o^2} (1 + \cos^2 \theta) \qquad \text{erg cm}^{-2} \text{ sec}^{-1} \tag{A.11}$$

where $\hat{\underset{\sim}{E}}^*$ is the complex conjugate of $\hat{\underset{\sim}{E}}$.

Because radiation is monochromatic, a delta function is required to describe the spectral distribution of power. Multiplying by $\hat{R}_o^2$, one obtains the electron emissivity

$$\hat{\eta}_e(\hat{\theta}, \hat{\nu}) = \frac{e^2 \hat{\beta}^2 \omega_H^2}{4c} (1 + \cos^2 \hat{\theta}) \, \delta\left(\hat{\nu} - \frac{\omega_H}{2\pi}\right) \qquad \text{erg sterad}^{-1} \text{ sec}^{-1} \text{ Hz}^{-1} \tag{A.12}$$

The degree of polarization is the difference between the emission in independent modes of polarization relative to the total emission. Linear polarization is found easily by resolving the electric field along the triad of basis vectors

$$\begin{aligned} \hat{\underset{\sim}{\ell}}_1 &= \hat{\underset{\sim}{e}}_1 \\ \hat{\underset{\sim}{\ell}}_2 &= \hat{\underset{\sim}{e}}_1 \times \hat{\underset{\sim}{n}} \\ \hat{\underset{\sim}{\ell}}_3 &= \hat{\underset{\sim}{n}} \end{aligned} \tag{A.13}$$

Here, $\hat{\underset{\sim}{\ell}}_1$ is perpendicular to the magnetic field, $\hat{\underset{\sim}{\ell}}_2$ is parallel to the projection of the magnetic field on the plane perpendicular to the direction of propagation, and $\hat{\underset{\sim}{\ell}}_3$ is in the direction of propagation. The electric-field equation (A.9) is now

$$\hat{\underset{\sim}{E}}(t) = -\frac{e\hat{\beta}\omega_H}{c\hat{R}_o}\left(\hat{\underset{\sim}{\ell}}_1 + i\hat{\underset{\sim}{\ell}}_2 \cos\hat{\theta}\right) \exp\left[-i\left(\omega_H\hat{t} - \frac{\hat{R}_o\omega_H}{c}\right)\right] \equiv \hat{E}_{\ell_1}\hat{\underset{\sim}{\ell}}_1 + \hat{E}_{\ell_2}\hat{\underset{\sim}{\ell}}_2 \tag{A.14}$$

No electric field occurs in the direction of propagation, as required by Eq. (A.4). The degree of linear polarization in the $\hat{\underset{\sim}{\ell}}_1$ direction is

$$\hat{\Pi}_{\ell_1} = \frac{\hat{E}_{\ell_1}\hat{E}^*_{\ell_1} - \hat{E}_{\ell_2}\hat{E}^*_{\ell_2}}{\hat{\underset{\sim}{E}}\cdot\hat{\underset{\sim}{E}}^*} = \frac{1-\cos^2\hat{\theta}}{1+\cos^2\hat{\theta}} \tag{A.15}$$

Circular polarization is similarly found by resolving the electric vector along the triad of basis vectors

$$\begin{aligned}\hat{\underset{\sim}{c}}_1 &= \frac{1}{2}\left(\hat{\underset{\sim}{\ell}}_1 + i\hat{\underset{\sim}{\ell}}_2\right)\\ \hat{\underset{\sim}{c}}_2 &= \frac{1}{2}\left(\hat{\underset{\sim}{\ell}}_1 - i\hat{\underset{\sim}{\ell}}_2\right)\\ \hat{\underset{\sim}{c}}_3 &= \hat{\underset{\sim}{\ell}}_3\end{aligned} \tag{A.16}$$

from which one obtains

$$\hat{\underset{\sim}{E}}(\hat{t}) = -\frac{e\hat{\beta}\omega_H}{2c\hat{R}_o}\left[(1+\cos\hat{\theta})\,\hat{\underset{\sim}{c}}_1 + (1-\cos\hat{\theta})\,\hat{\underset{\sim}{c}}_2\right]$$

$$\exp\left[-i\left(\omega_H\hat{t} - \frac{\hat{R}_o\omega_H}{c}\right)\right] \equiv \hat{E}_{c_1}\hat{\underset{\sim}{c}}_1 + \hat{E}_{c_2}\hat{\underset{\sim}{c}}_2 \tag{A.17}$$

where $\hat{E}_{c_1}$ is the right-hand polarized component of the electric field (the observer sees the electric vector rotating counter clockwise), and

$\hat{E}_{c_2}$ is the left-hand polarized component. The degree of circular polarization is

$$\hat{\Pi}_c = \frac{\hat{E}_{c_1}\hat{E}^*_{c_1} - \hat{E}_{c_2}\hat{E}^*_{c_2}}{\hat{\underset{\sim}{E}} \cdot \hat{\underset{\sim}{E}}^*} = \frac{2 \cos \hat{\theta}}{1 + \cos^2 \hat{\theta}}$$

where $\hat{\Pi}_c > 0$ for right-hand polarization and $\hat{\Pi}_c < 0$ for left-hand polarization.

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Synchrotron radiation from electrons with very small pitch angles is described and the intensity and polarization of this radiation are derived.

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