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JOINT INSTITUTE FOR LABORATORY ASTROPHYSICS

REPORT

UNIVERSITY OF COLORADO

NATIONAL BUREAU OF STANDARDS

CLASSICAL PATH BROADENING FUNCTIONS

FOR A DEBYE-SHIELDED INTERACTION

NASA

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and

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JILA REPORT No. 107

University of Colorado

Boulder, Colorado

March 17, 1971

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JILA Report #107

Classical Path Broadening Functions

For a Debye-Shielded Interaction

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Abstract

Stark broadening calculations of isolated neutral atom lines in the classical path approximation usually involve an electron (moving along a straight line path) interacting directly with the atom. Correlations between the electrons are then taken into account by imposing a cutoff in the interaction, when the distance from the atom, ρ, exceeds the Debye length, ρ_D. A more consistent procedure for the correlation effects is to replace the direct interaction of the electron by a Debye-shielded interaction. The functions A, B, a and b which are required in the theory when the Debye-shielded interaction is used are considered in detail in this report. When $\rho/\rho_D < 0.1$, it is found that the a and b functions may be closely reproduced by using unshielded functions in conjunction with an upper cutoff of 0.68 ρ_D. In the appendix is a computer program to generate these functions written by U. Palmer of JILA.

1. Introduction

Calculations of the broadening by electrons in the impact approximation usually involve the evaluation of an operator (the ϕ operator) in which the classical path S-matrices for the electron collisions have been expanded to second order (see, for example, Refs. 1, 2 and 3). Higher order terms in the expansion are then accounted for by a lower cutoff in the integral over impact parameters, and the effect of electron correlations is included by an upper cutoff at an impact parameter at the order of the Debye length. The purpose of this report is to give an alternative numerical procedure to the use of the upper cutoff (although we will show in Sec. 6 that a judicious choice of upper cutoff will, under some circumstances, reproduce our results.

More recent theory[4,5] has shown that the broadening operator for the complete line profile which is correct to second order in the interaction potential V(t) can be expressed (for $|\alpha\rangle$, $|\alpha'\rangle$ matrix elements), using Eq. (49) of Ref. 4, as

$$\langle\alpha|\mathcal{L}(\Delta\omega)|\alpha'\rangle = -i\sum_k \langle\alpha|\int_0^\infty dt e^{i\Delta\omega_k t}\{V(t)|k\rangle\langle k|V(0)\}_{Av}|\alpha'\rangle \quad . \qquad (1.1)$$

Here intermediate states $|k\rangle$ have been explicitly inserted and $\Delta\omega_k$ is the frequency difference of the radiation from the intermediate state. This result was originally obtained by Baranger[6] from a wing expansion, rather than the more complete general theory. In particular this result has important consequences, since it shows [see Eq. (1.3) below] that, in the correct second-order treatment, functions like $A(z_1,z_2)$ as introduced in Ref. 3 corresponding to off-diagonal elements of the broadening operator are needed only for the simple case of $z_1=z_2$.

Putting the potential V(t) equal to the dipole interaction $-\vec{d}\cdot\vec{E}(t)$ where $\vec{E}(t)$ is the electric field due to all the electrons, we see that Eq. (1.1) involves the evaluation of the electric field autocorrelation function, namely:

$$g(\Delta\omega) = \int_0^\infty dt e^{i\Delta\omega t}\{\vec{E}(t)\vec{E}(0)\}_{Av} \tag{1.2}$$

This function has been examined in Refs. 7 and 8. In particular, it is possible to evaluate the electric field average as if each electron were an independent quasi-particle interacting with the radiating atom through its dynamically screened electric field. The full evaluation of the dynamically screened potential requires the use of the wave number and frequency-dependent dielectric constant $\varepsilon^+(\vec{k},\vec{k}\cdot\vec{v})$, however, the dominant contribution to $g(\Delta\omega)$ is from the region where $\Delta\omega \simeq \vec{k}\cdot\vec{v}$ [and in fact, $\vec{k}\cdot\vec{v}$ is set equal to $\Delta\omega$ for the real part of $g(\Delta\omega)$[8]]. In general, the full evaluation of $g(\Delta\omega)$ is quite complicated,[8,9] but, when $\Delta\omega << \omega_p$ (the plasma frequency) the dynamic dielectric constant can be approximated by the static dielectric constant and the shielded field is the Debye-screened field. When $\Delta\omega >> \omega_p$, unshielded fields must be used.

In this report, we consider the evaluation of the second-order terms using static Debye-screened fields. This is therefore only strictly correct when $\Delta\omega$ (the frequency separation to the intermediate state) is less than ω_p; however, correlation and shielding are most important in this region where the lines of the spectrum are overlapping or starting to overlap. In addition, when $\Delta\omega >> \omega_p$ the difference between using static Debye-screened fields and unshielded fields in Eq. (1.2) is negligible, and the value of Eq. (1.2) is quite small in any case. Finally, by comparison with the results of Ref. 8 [in particular for the real part of $g(\Delta\omega)$] errors in the region $\Delta\omega \simeq \omega_p$ from the use of Debye-screened fields are not expected to be large, provided $g(\Delta\omega)$ is not appreciably enhanced in this region due to instabilities and other non-thermal effects.

To relate Eq. (1.2) to the usual A and B functions, it is necessary to rewrite it slightly. Since the electrons act as independent quasi-particles the average in Eq. (1.2) may be written in terms of integrals over the frequency of collisions $d\nu$ (an integral essentially over velocities and impact parameters) and the time of closest approach t_o (see for example section 4B of Ref. 4 and Ref. 10).

Thus

$$\int_o^\infty e^{i\Delta\omega t}\ \{\vec{E}(t)\vec{E}(0)\}_{Av}\ dt$$

$$= \int_{-\infty}^\infty dt_o \int d\nu \int_o^\infty dt e^{i\Delta\omega t}\ \vec{E}_s(t+t_o)\vec{E}_s(t_o) \qquad \text{where } \vec{E}_s \text{ represents the shielded field}^8$$

$$= \int d\nu \int_{-\infty}^\infty dx_2 \int_{x_2}^\infty dx_1 e^{i\Delta\omega(x_1-x_2)}\vec{E}_s(x_1)\vec{E}_s(x_2) \quad \text{with } \begin{matrix} x_1=t+t_o \\ x_2=t_o \end{matrix}$$

$$= \int d\nu \int_\infty^\infty dx_1 \int_{-\infty}^{x_1} dx_2\ e^{i\Delta\omega(x_1-x_2)}\vec{E}_s(x_1)\vec{E}_s(x_2) \tag{1.3}$$

by using the Dirichlet integral formula [see Eq. (57), Ref. 4].

2. The Shielded Broadening Functions

Thus, using static Debye-screened fields instead of pure Coulomb fields, which amounts to multiplying the latter by $(1+r/\rho_D)\exp(-r/\rho_D)$, the usual second-order time integral[1-3] is replaced by

$$F(z,z';q) = A + iB =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{x} dx' \, (1+xx')g(x;q)g(x';q)e^{i(zx-z'x')} \quad , \qquad (2.1)$$

where

$$g(x;q) = \left[1+q(1+x^2)^{1/2}\right] (1+x^2)^{-3/2} e^{-q(1+x^2)^{1/2}} \quad , \qquad (2.2)$$

$z = \rho\omega/v$, $z' = \rho\omega'/v$, $q = \rho/\rho_D$, ρ is the impact parameter, $\rho_D = (kT/4\pi Ne^2)^{1/2}$ the Debye length, and v the electron velocity. It is easily seen that the new A and B functions possess the same symmetry properties as the unshielded ones, viz.[3]

$$A(z,z';q) = A(z',z;q) = A(-z,-z';q) \quad , \qquad (2.3)$$

$$B(z,z';q) = B(z',z;q) = -B(-z,-z';q) \quad . \qquad (2.4)$$

The general expression in Eq. (2.1) should be used in the impact theory[1,3] to compute the matrix elements of the broadening operator for lines with forbidden components (for an isolated line $z' = z$). <u>However, as stressed earlier, in the unified theory approach only "diagonal" functions with $z = z' = \rho\Delta\omega_k/v$ will occur.</u> Of course, no upper cutoff on impact parameters is required with the new functions.

Sticking to real integration variables in Eq. (2.1), simple parity considerations show that the A-function is given by

$$A = \int_0^{\infty} dx\; g(x;q)\cos(zx) \int_0^{\infty} dx'\; g(x';q)\cos(z'x')$$
$$+ \int_0^{\infty} dx\; xg(x;q)\sin(zx) \int_0^{\infty} dx'\; x'g(x';q)\sin(z'x') \quad . \qquad (2.5)$$

The necessary Fourier transforms are readily obtained from[11]

$$\int_0^{\infty} dx(\alpha^2+x^2)^{-1/2} e^{-q(\alpha^2+x^2)^{1/2}} \cos(zx) = K_0[\alpha(z^2+q^2)^{1/2}] \quad , \qquad (2.6)$$

$$\int_0^\infty dx\; x(\alpha^2+x^2)^{-1/2} e^{-q(\alpha^2+x^2)^{1/2}} \sin(zx) = \alpha z (z^2+q^2)^{-1/2} K_1[\alpha(z^2+q^2)^{1/2}], \quad (2.7)$$

by differentiation with respect to α. Here K_0 and K_1 are modified Bessel functions of the second kind. Hence:

$$A(z,z';q) = zz'K_0(R)K_0(R') + RR'K_1(R)K_1(R') \quad , \quad (2.8)$$

where we have put for brevity

$$R = (z^2+q^2)^{1/2}, \quad R' = (z'^2+q^2)^{1/2} \quad . \quad (2.9)$$

For the diagonal function we get

$$A(z,z;q) \equiv A(z;q) = z^2 K_0{}^2(R) + R^2 K_1{}^2(R) \quad (2.10)$$

Since z, z', and q are all proportional to ρ, the integration over the impact parameter (now from ρ_{min} to ∞) can be expressed in terms of

$$a(z,z';q) = \int_1^\infty \frac{d\lambda}{\lambda} A(\lambda z, \lambda z'; \lambda q) \quad , \quad (2.11)$$

which represents the natural generalization of the function a(z) of GBKO. To save space, we give here the results only for the case z = z':

$$a(z;q) = RK_0(R)K_1(R) - \frac{1}{2}q^2[K_1{}^2(R) - K_0{}^2(R)] \quad (2.12)$$

It is also convenient to introduce in Eqs. (2.10), (2.12), a new parameter $z_D = z/q = \rho_D \Delta\omega/v$, so that $R = \beta z$, with $\beta = (1+1/z_D{}^2)^{1/2}$. For $z_D \sim \Delta\omega/\omega_p >> 1$ one has $A(z;z/z_D) \cong A(z)$ and $a(z;z/z_D) \cong a(z)$. Therefore, when $\Delta\omega >> \omega_p$, using either shielded fields or unshielded fields in Eq. (1.2) gives the same results. As we have said before, the fact that we get the correct answer in the important limit of $\Delta\omega << \omega_p$ and that both shielded and unshielded fields give nearly identical results when $\Delta\omega >> \omega_p$ is the main justification for the functions presented here.

3. Dispersion Relations for Off-Diagonal Broadening Functions

A convenient way to calculate the B function, once A is known, is provided by the dispersion relation, which expresses B as a Hilbert transform of A. This method has been extensively used in the past to compute various diagonal B functions contributing to the broadening of isolated lines emitted by neutral atoms and positive ions.[1,12-14] In both cases the dispersion relations merely reflect the analyticity of the second-order time integral with respect to one of its parameters, considered as a complex variable.

The extension to off-diagonal functions must be done with some care. In this case it is better to introduce an auxiliary complex variable, while keeping all physical parameters real (the dispersion relations given recently[15] for the neutral functions $A(z,pz)$ and $B(z,pz)$, where $p = z'/z$ is a real fixed ratio, are wrong, since for $p \neq 1$ the time integral does not possess the required analytic properties when z is allowed to take complex values). For instance, from the structure of Eq. (2.1):

$$F(z,z';q) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} dx' \; f(x,x';q) \; e^{i(zx-z'x')} \quad , \tag{3.1}$$

it is apparent that for any real z, z', and $q \geq 0$, the function $\varphi(\zeta) = F(z+\zeta,z'+\zeta;q)$ is holomorphic in the upper half-plane $\mathrm{Im}\zeta>0$. Applying the usual analysis, based upon Cauchy's theorem and the well-known relation $1/(\zeta-i0) = P(1/\zeta) + i\pi\delta(\zeta)$, one readily gets

$$\varphi(0) = \frac{1}{\pi i} \, P \int_{-\infty}^{\infty} d\zeta \, \frac{\varphi(\zeta)}{\zeta} \quad , \tag{3.2}$$

whence

$$B(z,z';q) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{d\zeta}{\zeta} A(z+\zeta,z'+\zeta;q) \quad , \tag{3.3}$$

or alternatively

$$B(z,z';q) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{d\zeta}{\zeta-\frac{1}{2}(z+z')} A[\zeta+\tfrac{1}{2}(z-z'),\zeta-\tfrac{1}{2}(z-z');q] \quad . \tag{3.4}$$

Notice that Eq. (3.3) can be given also the equivalent form

$$B(z,z';q) = -\frac{1}{\pi} \int_{0}^{\infty} \frac{d\zeta}{\zeta} [A(z+\zeta,z'+\zeta;q)-A(z-\zeta,z'-\zeta;q)] \tag{3.5}$$

from which the singularity has been removed.

Although the above relations allow us in principle to compute B to the desired accuracy, in practice the limiting process inherent to all of them might become sometimes a serious source of trouble.

4. Complex Integration

We shall now apply to the shielded case the powerful contour integration method which led to closed-form expressions for both A and B in the non-shielding limit $q = 0$.[16] To this end we assume $z \geq z' > 0$, and make the change of variables $x = \sinh u$, $x' = \sinh u'$. Equation (2.1) may then be rewritten as

$$F = F_1(z,z';q) + F_2(z,z';q) \quad , \tag{4.1}$$

where

$$F_1 = \frac{1}{2} \int_{-\infty}^{\infty} du \sinh u \, \frac{1+q \cosh u}{\cosh^2 u} \, e^{-q \cosh u + iz \sinh u} \, \phi_1(u) , \tag{4.2}$$

with

$$\phi_1(u) = \int_{-\infty}^{u} du' \sinh u' \frac{1+q \cosh u'}{\cosh^2 u'} e^{-q \cosh u' - iz' \sinh u'} , \quad (4.3)$$

and

$$F_2 = \frac{1}{2} \int_{-\infty}^{\infty} du \frac{1+q \cosh u}{\cosh^2 u} e^{-q \cosh u + iz \sinh u} \phi_2(u) , \quad (4.4)$$

with

$$\phi_2(u) = \int_{-\infty}^{u} du' \frac{1+q \cosh u'}{\cosh^2 u'} e^{-q \cosh u' - iz' \sinh u'} . \quad (4.5)$$

Let us first evaluate F_1. Since

$$\sinh x \frac{1+q \cosh x}{\cosh^2 x} e^{-q \cosh x} = - \frac{d}{dx} \left(\frac{e^{-q \cosh x}}{\cosh x} \right) , \quad (4.6)$$

integration by parts gives

$$\phi_1(u) = -(1/\cosh u) e^{-q \cosh u - iz' \sinh u} - iz' \psi_1(u) , \quad (4.7)$$

where

$$\psi_1(u) = \int_{-\infty}^{u} du' e^{-q \cosh u' - iz' \sinh u'} , \quad (4.8)$$

and further

$$2F_1 = - \int_{-\infty}^{\infty} du \sinh u \frac{1+q \cosh u}{\cosh^3 u} e^{-2q \cosh u + i\eta \sinh u}$$

$$- iz' \int_{-\infty}^{\infty} (du/\cosh u) e^{-2q \cosh u + i\eta \sinh u} + zz' G_1 , \quad (4.9)$$

where $\eta = z-z' \geq 0$, and

$$G_1 = \int_{-\infty}^{\infty} du e^{-q \cosh u + iz \sinh u} \int_{-\infty}^{u} du' e^{-q \cosh u' - iz' \sinh u'} . \quad (4.10)$$

Defining

$$\tan \alpha = q/z, \quad \tan \alpha' = q/z', \quad (0 \leq \alpha \leq \alpha' \leq \pi/2) \quad , \tag{4.11}$$

we may rewrite G_1 as

$$G_1 = \int_{-\infty}^{\infty} du \; e^{iR \sinh (u+i\alpha)} \int_{-\infty}^{u} du' \; e^{-iR' \sinh (u'-i\alpha')} \tag{4.12}$$

with R,R' given by Eq. (2.9) and $R \geq R'$.

We now consider u and u' as complex variables and transform Eq. (4.12) into a repeated contour integral $G_1 = \int_\Gamma du \ldots \int_{\Gamma_u} du' \ldots$, with the contours shown in Fig. 1. It is easily seen that the contribution of the

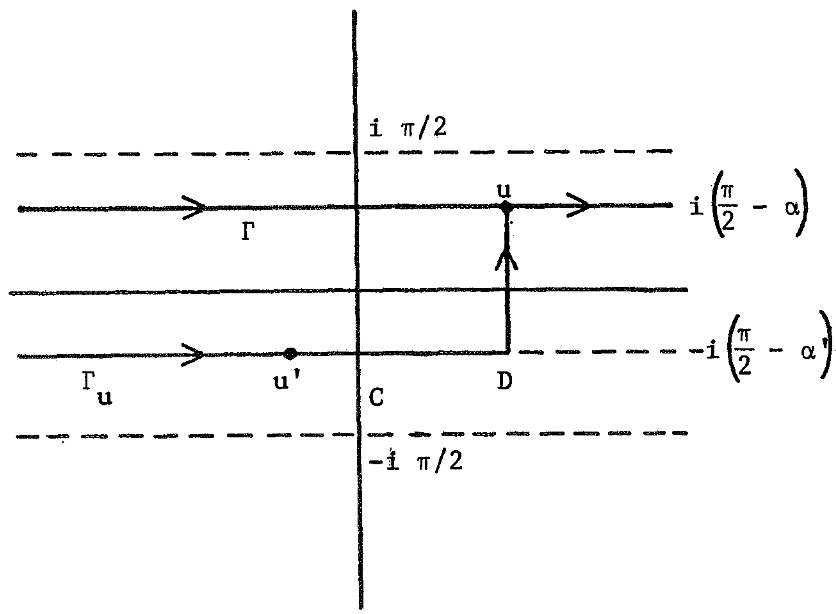


Figure 1

segment CD vanishes, since $\int_0^u du' \ldots$ is an odd function of u, and therefore

$$G_1 = \int_{-\infty}^{\infty} du\, e^{-R\cosh u} \left[\int_{-\infty}^{0} du' e^{-R'\cosh u'} + i \int_0^{\pi-\alpha-\alpha'} d\theta\, e^{-R'\cosh(u+i\theta)} \right] =$$

$$= 2\, K_0(R) K_0(R') + i\, M_1 \quad , \tag{4.13}$$

where

$$M_1 = \int_0^{\pi-\alpha-\alpha'} d\theta \int_{-\infty}^{\infty} du\, e^{-R\cosh u - R'\cosh(u+i\theta)} \quad . \tag{4.14}$$

After shifting the u integration from the real axis onto the line $\mathrm{Im}\, u = -\chi (u \to u - i\chi)$, with χ given by $\tan\chi = R'\sin\theta/(R + R'\cos\theta) > 0$, and changing θ in $\pi - 2\theta$, one gets

$$M_1 = 4 \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta\, K_0[S(\theta)] \quad , \tag{4.15}$$

where

$$S(\theta) = (R^2 + R'^2 - 2RR' \cos 2\theta)^{\frac{1}{2}} \quad . \tag{4.16}$$

The second term F_2 in Eq. (4.1) can be treated in a similar manner, but the calculation is slightly more cumbersome. Starting from

$$\frac{1+q\cosh x}{\cosh^2 x}\, e^{-q\cosh x} = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\, e^{-q\cosh x}\right) + q\cosh x\, e^{-q\cosh x} \quad , \tag{4.17}$$

one first obtains by partial integration

$$2F_2 = \int_{-\infty}^{\infty} du \sinh u \, \frac{1+q \cosh u}{\cosh^3 u} \, e^{-2q \cosh u + i\eta \sinh u}$$

$$- \int_{-\infty}^{\infty} du \tanh u \, (q \cosh u + iz' \sinh u) \, e^{-2q \cosh u + i\eta \sinh u} + RR'G_2 \quad , \tag{4.18}$$

where

$$G_2 = \int_{-\infty}^{\infty} du \sinh (u+i\alpha) \, e^{iR \sinh (u+i\alpha)} \int_{-\infty}^{u} du' \sinh(u'-i\alpha') e^{-iR' \sinh(u'-i\alpha')} \quad . \tag{4.19}$$

By contour integration this can be reduced further to

$$G_2 = 2 \, K_1(R)K_1(R') + i \, M_2 \quad , \tag{4.20}$$

where

$$M_2 = 4 \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta \, \sin^2\theta \, K_0(S) + 4 \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta \, K_1(S)/S$$

$$-4(R-R')^2 \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta \, \cos^2\theta \, K_2(S)/S^2 \quad . \tag{4.21}$$

With F_1 determined from Eqs. (4.9), (4.13) and (4.15), and F_2 from Eqs. (4.18), (4.20) and (4.21), we go back to Eq. (4.1) and separate the real and imaginary parts of F. This yields for A the closed form expression already found in Sec. 2:

$$A(z,z';q) = z \, z'K_0(R)K_0(R') + R \, R'K_1(R)K_1(R') \quad , \tag{4.22}$$

and for B the integral representation

$$B(z,z';q) = -q(z+z')K_1(S_0)/S_0 + 2zz' \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta\ K_0(S)$$

$$+ 2RR' \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta\ \sin^2\theta\ K_0(S) + 2RR' \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta\ K_1(S)/S$$

$$- 2RR'(R-R')^2 \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta\ \cos^2\theta\ K_2(S)/S^2 \quad , \qquad (4.23)$$

where $S_0 = (\eta^2+4q^2)^{\frac{1}{2}}$, and S is given by Eq. (4.16).

The above form of B is not quite satisfactory for numerical computation, since it contains terms which diverge individually under certain circumstances, although the net result will always be finite (for instance, in the diagonal case $z'=z$ the first term behaves like $1/q$ when $q \to 0$, and so does the fourth). This difficulty is, however, easily resolved by noticing that S_0 is nothing but the value of S at the lower limit of integration $\theta = \frac{1}{2}(\alpha+\alpha')$. Developing the right-hand side of the identity

$$-q(z+z')K_1(S_0)/S_0 = RR' \int_{(\alpha+\alpha')/2}^{\pi/2} d\left[\sin\ 2\theta\ K_1(S)/S\right] \qquad (4.24)$$

and substituting into Eq. (4.23) then enables us to eliminate the unpleasant terms and eventually results in the new integral representation

$$B(z,z';q) = 2zz' \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta\ K_0(S) - 2RR' \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta\ \cos\ 2\theta\ K_0(S) \quad , \qquad (4.25)$$

which is the best we can get.

In view of the subsequent integration over the impact parameter we define, by analogy with Eq. (2.11), the function

$$b(z,z';q) = \int_1^\infty \frac{d\lambda}{\lambda} B(\lambda z,\lambda z';\lambda q) \quad , \tag{4.26}$$

which generalizes the function $b(z)$ of GBKO. Substituting from Eq. (4.25) and noticing that α and α' do not depend on λ, one readily gets

$$b(z,z';q) = 2zz' \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta\ K_1(S)/S - 2RR' \int_{(\alpha+\alpha')/2}^{\pi/2} d\theta \cos 2\theta\ K_1(S)/S \quad . \tag{4.27}$$

In the diagonal case, Eqs. (4.25) and (4.27) simplify respectively to

$$B(z;q) = 2z^2 \int_\alpha^{\pi/2} d\theta\ K_0(2R\sin\theta) - 2R^2 \int_\alpha^{\pi/2} d\theta \cos 2\theta\ K_0(2R\sin\theta) \quad , \tag{4.28}$$

and

$$b(z;q) = (z^2/R) \int_\alpha^{\pi/2} d\theta\ K_1(2R\sin\theta)/\sin\theta - R \int_\alpha^{\pi/2} d\theta \cos 2\theta\ K_1(2R\sin\theta)/\sin\theta \quad . \tag{4.29}$$

If we let here $q \to 0$ we obtain compact integral representations for the unshielded shift functions, viz.

$$B(z) = 4z^2 \int_0^{\pi/2} d\theta \sin^2\theta\ K_0(2z\sin\theta) \quad , \tag{4.30}$$

$$b(z) = 2z \int_0^{\pi/2} d\theta \sin\theta\ K_1(2z\sin\theta) \quad . \tag{4.31}$$

The connection with the closed form expressions reported earlier[16] is provided by Nicholson's formula[17]

$$I_n(z)K_\nu(z) = (-)^n(2/\pi) \int_0^{\pi/2} d\theta \cos(n+\nu)\theta\ K_{\nu-n}(2z\cos\theta) \quad , \tag{4.32}$$

valid when n is an integer, and $|\mathrm{Re}(\nu-n)| < 1$. Corresponding results are not likely to hold for the shielded functions, which are represented by incomplete integrals with a variable limit. However, the latter are much easier to handle numerically than the principal value integrals in Sec. 3.

5. Approximate formulae

In this section we collect various useful approximate expressions of the shielded broadening functions, restricting ourselves to the "diagonal" case, which is the most important in practice.

We begin with asymptotic formulae for A and a, valid when z or/and q are large. These are obtained simply by substituting the standard expansions of K_0 and K_1 [18] into Eqs. (2.10) and (2.12). The leading terms are respectively

$$A(z;q) \sim (\pi/2)(1+z^2/R^2)\ R\ e^{-2R} \quad , \tag{5.1}$$

$$a(z;q) \sim (\pi/2)(1-q^2/2R^2)\ e^{-2R} \quad . \tag{5.2}$$

In particular, if one is interested in the asymptotic behavior for $\rho \to \infty$, then one must let both z and q tend to infinity, while keeping their ratio $z/q = z_D$ finite. Thus, when $z \to \infty$:

$$A(z;z/z_D) \sim (\pi/2)(\beta+1/\beta)\ z\ e^{-2\beta z} \quad , \tag{5.3}$$

$$a(z;z/z_D) \sim (\pi/2\beta)(\beta+1/\beta)\ e^{-2\beta z} \quad , \tag{5.4}$$

where $\beta = (1+1/z_D{}^2)^{\frac{1}{2}}$.

On the other hand, for $\rho \to 0$ one has

$$a(z;z/z_D) \cong \log(0.68 z_D/z), \quad \text{for } z_D \ll 1 \quad , \tag{5.5}$$

whereas $A(z;z/z_D) \to 1$.

The corresponding formulas for B and b may be derived from the integral representations given in Sec. 4. Let us assume first that only $z \to \infty$, while q remains finite. After changing the variable in Eq. (4.29) to $u = 2\,R \sin\theta$, we expand it as follows:

$$B = \frac{z^2-R^2}{R} \int_{2q}^{2R} du\, K_0(u) + \frac{3R^2+z^2}{8R^3} \int_{2q}^{2R} du\, u^2 K_0(u)$$

$$+ \frac{5R^2+3z^2}{128R^5} \int_{2q}^{2R} du\, u^4 K_0(u) + \ldots \tag{5.6}$$

This is readily transformed into an asymptotic expansion by developing the coefficients in powers of $1/z$ and extending all the integrations to infinity. Eventually

$$B(z;q) \sim C_1(q)/z + C_3(q)/z^3 + \ldots\,, \qquad z \to \infty \tag{5.7}$$

where

$$C_1(q) = \left(\frac{1}{2} - q^2\right) Ki_1(2q) + 2q^2 K_1(2q) + qK_0(2q) \quad ,$$

$$C_3(q) = \left(\frac{9}{16} - \frac{3q^2}{8} + \frac{q^4}{2}\right) Ki_1(2q) + \left(\frac{9}{4} - \frac{q^2}{2}\right) q^2 K_1(2q)$$

$$+ \left(\frac{9}{8} + \frac{3q^2}{4}\right) qK_0(2q) \quad , \tag{5.8}$$

with the new function Ki_1 defined by[18]

$$Ki_1(x) = \int_x^\infty du\, K_0(u) \quad . \tag{5.9}$$

Since $Ki_1(0) = \pi/2$, in the limit $q \to 0$ one has $C_1(0) = \pi/4$, $C_3(0) = 9\pi/32$, and we recover the well-known asymptotic expansion of $B(z)$.

Similarly, from Eq. (4.29) one obtains

$$b(z;q) \sim c_1(q)/z + c_3(q)/z^3 + \ldots , \qquad z \to \infty \tag{5.10}$$

where

$$c_1(q) = \left(\frac{1}{2} + q^2\right) Ki_1(2q) - q^2 K_1(2q) + qK_0(2q) \quad ,$$

$$c_3(q) = \left(\frac{3}{16} - \frac{3q^2}{8} - \frac{q^4}{2}\right) Ki_1(2q) + \left(\frac{3}{4} + \frac{q^2}{2}\right) q^2 K_1(2q)$$

$$+ \left(\frac{3}{8} - \frac{q^2}{4}\right) qK_0(2q) \tag{5.11}$$

In the limit $q \to 0$ we have $c_1(0) = \pi/4$, $c_3(0) = 3\pi/32$, and Eq. (5.10) reduces to the asymptotic expansion of $b(z)$.

The above procedure clearly breaks down when $q \to \infty$, since all the terms in Eq. (5.6) are then of the same order. If z is kept finite an asymptotic estimate of B is, however, readily obtained from Eq. (4.28) by noticing that the integration interval shrinks as q is increased ($\pi/2 - \alpha \sim z/q$). This allows us to write

$$B(z;q) \cong (2z^2 - 2R^2 \cos 2\alpha)\; K_0(2\; R \sin\alpha)(\pi/2 - \alpha)$$

$$\sim 2zqK_0(2q) \quad , \quad q \to \infty \quad . \tag{5.12}$$

Similarly, from Eq. (4.30) we get

$$b(z;q) \sim zK_1(2q) \quad , \qquad q \to \infty \quad . \tag{5.13}$$

Hence, when $z \ll q$ both B and b become exponentially small and depend linearly on z.

Let us consider now the case when z and q tend together to infinity so that their ratio $z/q = z_D$ remains finite. In this case no shrinkage occurs, but the integrals in Eqs. (4.28) and (4.29) may be evaluated asymptotically on integrating by parts.[19] Neglecting higher order contributions one finds

$$B(z;z/z_D) \sim (z/z_D^2)\ Ki_1(2z/z_D), \quad z \to \infty \tag{5.14}$$

and

$$b(z;z/z_D) \sim (1/2z_D)\ K_0(2z/z_D), \quad z \to \infty \quad . \tag{5.15}$$

At the opposite limit, as $\rho \to 0$, from Eq. (4.28) one gets $B(z;z/z_D) \to 0$ and Eq. (4.29) yields

$$b(z;z/z_D) \to \arctan z_D - \tfrac{1}{2}z_D/(1+z_D^2), \quad z \to 0 \tag{5.16}$$

The last limit is $\cong \pi/2$ for $z_D >> 1$, and $\cong z_D/2$ for $z_D << 1$.

6. Numerical results

Numerical results are not presented here in tabular form for $A(z,q)$ and $a(z,z_D)$ since the Bessel functions on Eqs. (2.10) and (2.12) are so simple to calculate (see Ref. 18, formulae 9.8.5, 9.8.6, 9.8.7 and 9.8.8). Tables 1 and 2 show calculated values of $B(z,q)$ and $b(z,z_D)$. Where asymptotic forms could not be used, these functions were calculated both from the formulae of Eqs. (4.28) and (4.29) and from the Hilbert transform Eq. (3.5) [with direct integration of Eq. (4.26) to give $b(z,z_D)$]. The first procedure was by far the simpler, however, overall agreement between the two methods of better than 2% was obtained. For values of z greater than those in the table, sufficient accuracy can be obtained by using the first terms of Eqs. (5.7) and (5.10). Notice that $B(z,q=0) = \pi z^2[K_0(z)I_0(z)-K_1(z)I_1(z)]$ and $b(z,z_D=\infty) = \pi/2 - \pi z K_0(z)I_1(z)$ are the straight line results. $B(z,q>10)$ and $b(z,z_D<.002)$ are for all effective purposes negligible.

$A(z,q)$, $B(z,q)$ and $b(z,q)$ are plotted in Figs. 2, 3 and 4 for various values of $q(= \rho/\rho_D = z/z_D)$; $a(z,q)$ was not plotted, but it diverges logarithmically at small z and q [Eq. (5.5)]. Notice in particular that the functions get small rapidly when $q > 1$. This is expected since Eqs. (5.1) (5.2) (5.12) and (5.13) all predict an exponential fall off when $q >> 1$. This rapid cutoff when $\rho \gtrsim \rho_D$, certainly to some extent justifies the usual procedure[2,12] for treating shielding by a cutoff. To further test the validity of these cutoff procedures, in Fig. 5 a function $F(z,q)$ is plotted. $F(z,q)$ is essentially[12] $[b(z) - b(z_{max})]$ where $z_{max} = \rho_{max}\omega/v$. Two cases are considered, firstly $\rho_{max} = \rho_D$ and secondly $\rho_{max} = 0.68\ \rho_D$. In both cases the agreement between $F(z,q)$ and $b(z,q)$ of Fig. 4 is surprisingly good. (Notice, however, that $F(z,q) = 0$ for $\rho \geq \rho_{max}$.) In fact, when $\rho/\rho_D < 0.1$ the agreement for the $\rho_{max} = 0.68\ \rho_D$ case is better than 5%. This value ($0.68\ \rho_D$) was chosen as the cutoff since its use in $[a(z) - a(z_{max})]$ exactly reproduces $a(z,z_D)$ for small z and z_D [see Eq. (5.5)], as has been noted in Ref. 7. Actually, for $q = \rho/\rho_D < 0.1$ the agreement for all values of z between $[a(z) - a(z_{max})]$ and $a(z,z_D)$ is again always better than about 4%; and for $q < 0.1$ for both $B(z,q)$ and $A(z,q)$ the difference between these functions and the unshielded ones ($q = 0$) is also very small (see Figs. 2 and 3). We therefore conclude that the usual cutoff procedures should certainly be adequate when simplicity is desired and when $\rho/\rho_D < 0.1$ (which is true in most cases of physical interest). However, we believe that the functions presented here are of even greater utility, especially if ρ/ρ_D should get large.

TABLE 1

B(z,q)

z \ q	.00	.01	.10	.20	.50	1.00	2.00	5.00	10.00
.001	2.063E-05	8.165E-05	3.507E-04	4.459E-04	4.210E-04	2.278E-04	4.464E-05	1.778E-07	1.148E-11
.002	7.327E-05	1.670E-04	7.013E-04	8.917E-04	8.421E-04	4.556E-04	8.928E-05	3.556E-07	2.296E-11
.003	1.534E-04	2.609E-04	1.052E-03	1.338E-03	1.263E-03	6.834E-04	1.339E-04	5.334E-07	3.445E-11
.004	2.582E-04	3.666E-04	1.404E-03	1.784E-03	1.684E-03	9.112E-04	1.786E-04	7.112E-07	4.593E-11
.005	3.860E-04	4.866E-04	1.756E-03	2.230E-03	2.105E-03	1.139E-03	2.232E-04	8.890E-07	5.741E-11
.006	5.352E-04	6.227E-04	2.109E-03	2.677E-03	2.526E-03	1.367E-03	2.678E-04	1.067E-06	6.889E-11
.007	7.047E-04	7.761E-04	2.464E-03	3.124E-03	2.947E-03	1.595E-03	3.125E-04	1.245E-06	8.038E-11
.008	8.936E-04	9.474E-04	2.819E-03	3.571E-03	3.369E-03	1.822E-03	3.571E-04	1.422E-06	9.186E-11
.009	1.101E-03	1.137E-03	3.176E-03	4.018E-03	3.790E-03	2.050E-03	4.017E-04	1.600E-06	1.033E-10
.010	1.326E-03	1.344E-03	3.534E-03	4.466E-03	4.211E-03	2.278E-03	4.464E-04	1.778E-06	1.148E-10
.050	2.056E-02	2.034E-02	2.067E-02	2.328E-02	2.116E-02	1.139E-02	2.231E-03	8.888E-06	5.740E-10
.100	6.094E-02	6.068E-02	5.370E-02	5.156E-02	4.294E-02	2.282E-02	4.458E-03	1.776E-05	1.148E-09
.200	1.621E-01	1.619E-01	1.467E-01	1.274E-01	9.003E-02	4.584E-02	8.883E-03	3.542E-05	2.291E-09
.400	3.588E-01	3.586E-01	3.402E-01	3.029E-01	1.965E-01	9.236E-02	1.749E-02	6.999E-05	4.550E-09
.600	4.981E-01	4.978E-01	4.805E-01	4.407E-01	2.972E-01	1.369E-01	2.552E-02	1.029E-04	6.747E-09
.800	5.759E-01	5.758E-01	5.603E-01	5.230E-01	3.707E-01	1.747E-01	3.264E-02	1.335E-04	8.851E-09
1.000	6.059E-01	6.058E-01	5.922E-01	5.587E-01	4.122E-01	2.022E-01	3.854E-02	1.610E-04	1.084E-08
1.200	6.031E-01	6.030E-01	5.912E-01	5.615E-01	4.268E-01	2.183E-01	4.302E-02	1.851E-04	1.268E-08
1.400	5.801E-01	5.799E-01	5.696E-01	5.436E-01	4.221E-01	2.242E-01	4.599E-02	2.053E-04	1.436E-08
1.600	5.459E-01	5.458E-01	5.367E-01	5.136E-01	4.052E-01	2.221E-01	4.753E-02	2.215E-04	1.586E-08
1.800	5.068E-01	5.067E-01	4.987E-01	4.782E-01	3.815E-01	2.144E-01	4.780E-02	2.337E-04	1.718E-08
2.000	4.669E-01	4.668E-01	4.596E-01	4.414E-01	3.548E-01	2.033E-01	4.704E-02	2.419E-04	1.830E-08
3.000	3.054E-01	3.054E-01	3.008E-01	2.893E-01	2.353E-01	1.402E-01	3.621E-02	2.361E-04	2.105E-08
4.000	2.158E-01	2.158E-01	2.124E-01	2.041E-01	1.654E-01	9.828E-02	2.580E-02	1.916E-04	2.003E-08
5.000	1.663E-01	1.662E-01	1.636E-01	1.570E-01	1.267E-01	7.471E-02	1.938E-02	1.476E-04	1.719E-08
6.000	1.358E-01	1.358E-01	1.336E-01	1.282E-01	1.032E-01	6.051E-02	1.552E-02	1.161E-04	1.412E-08
8.000	1.001E-01	1.000E-01	9.842E-02	9.439E-02	7.582E-02	4.427E-02	1.123E-02	8.097E-05	9.683E-09
10.000	7.947E-02	7.945E-02	7.816E-02	7.495E-02	6.016E-02	3.506E-02	8.858E-03	6.293E-05	7.300E-09
12.000	6.598E-02	6.597E-02	6.489E-02	6.222E-02	4.992E-02	2.907E-02	7.330E-03	5.174E-05	5.914E-09
14.000	5.643E-02	5.642E-02	5.550E-02	5.321E-02	4.268E-02	2.484E-02	6.258E-03	4.401E-05	4.996E-09
16.000	4.931E-02	4.930E-02	4.849E-02	4.649E-02	3.729E-02	2.170E-02	5.461E-03	3.833E-05	4.334E-09
18.000	4.378E-02	4.378E-02	4.306E-02	4.129E-02	3.311E-02	1.926E-02	4.846E-03	3.396E-05	3.830E-09
20.000	3.938E-02	3.937E-02	3.873E-02	3.713E-02	2.977E-02	1.732E-02	4.356E-03	3.050E-05	3.434E-09
25.000	3.147E-02	3.146E-02	3.094E-02	2.967E-02	2.379E-02	1.384E-02	3.478E-03	2.432E-05	2.731E-09

TABLE 2

$b(z, z_D)$

z \ z_D	.01	.02	.1	.2	.5	1.0	2.0	5.0	10.0	100.0	∞
.000	.005	.010	.050	.101	.264	.535	.907	1.277	1.422	1.556	1.571
.001	.005	.010	.050	.101	.263	.535	.907	1.277	1.422	1.556	1.571
.002	.004	.010	.050	.101	.263	.535	.907	1.277	1.422	1.556	1.571
.003	.004	.009	.050	.101	.262	.535	.907	1.277	1.422	1.556	1.571
.004	.003	.009	.050	.100	.262	.535	.907	1.277	1.422	1.556	1.571
.005	.003	.008	.050	.100	.262	.535	.907	1.277	1.422	1.556	1.571
.006	.003	.008	.050	.100	.262	.534	.907	1.277	1.422	1.556	1.571
.007	.002	.007	.049	.100	.262	.534	.907	1.277	1.421	1.556	1.570
.008	.002	.007	.049	.100	.262	.534	.907	1.277	1.421	1.555	1.570
.009	.002	.007	.049	.099	.262	.534	.906	1.277	1.421	1.555	1.570
.010	.001	.006	.048	.099	.262	.534	.906	1.277	1.421	1.555	1.570
.020	.000	.003	.044	.096	.260	.532	.905	1.275	1.419	1.553	1.568
.030	.000	.001	.040	.092	.257	.530	.902	1.273	1.417	1.551	1.566
.040	.000	.001	.035	.088	.253	.526	.899	1.269	1.413	1.547	1.562
.050	.000	.000	.031	.083	.249	.523	.895	1.265	1.410	1.544	1.559
.060	.000	.000	.026	.078	.245	.519	.891	1.261	1.405	1.539	1.554
.070	.000	.000	.023	.074	.240	.514	.886	1.256	1.401	1.535	1.549
.080	.000	.000	.020	.069	.236	.509	.881	1.251	1.395	1.529	1.544
.090	.000	.000	.017	.064	.231	.505	.876	1.246	1.390	1.524	1.539
.100	.000	.000	.014	.060	.226	.499	.870	1.240	1.384	1.518	1.533
.200	.000	.000	.003	.028	.174	.440	.804	1.169	1.312	1.445	1.460
.300	.000	.000	.000	.012	.128	.376	.728	1.085	1.227	1.360	1.375
.400	.000	.000	.000	.005	.092	.316	.651	.999	1.138	1.270	1.285
.500	.000	.000	.000	.002	.065	.263	.578	.913	1.050	1.182	1.196
.600	.000	.000	.000	.001	.045	.217	.509	.832	.966	1.096	1.111
.700	.000	.000	.000	.000	.031	.177	.447	.756	.887	1.016	1.031
.800	.000	.000	.000	.000	.021	.145	.391	.685	.813	.941	.956
.900	.000	.000	.000	.000	.015	.117	.341	.621	.746	.872	.887
1.000	.000	.000	.000	.000	.010	.095	.297	.562	.684	.809	.823
2.000	.000	.000	.000	.000	.000	.011	.073	.216	.306	.418	.433
3.000	.000	.000	.000	.000	.000	.001	.019	.095	.162	.262	.276
4.000	.000	.000	.000	.000	.000	.000	.005	.047	.097	.188	.202
5.000	.000	.000	.000	.000	.000	.000	.002	.025	.063	.145	.160
6.000	.000	.000	.000	.000	.000	.000	.001	.014	.043	.118	.132
7.000	.000	.000	.000	.000	.000	.000	.000	.008	.031	.099	.113
8.000	.000	.000	.000	.000	.000	.000	.000	.005	.022	.085	.099
9.000	.000	.000	.000	.000	.000	.000	.000	.003	.016	.074	.088
10.000	.000	.000	.000	.000	.000	.000	.000	.002	.012	.065	.079

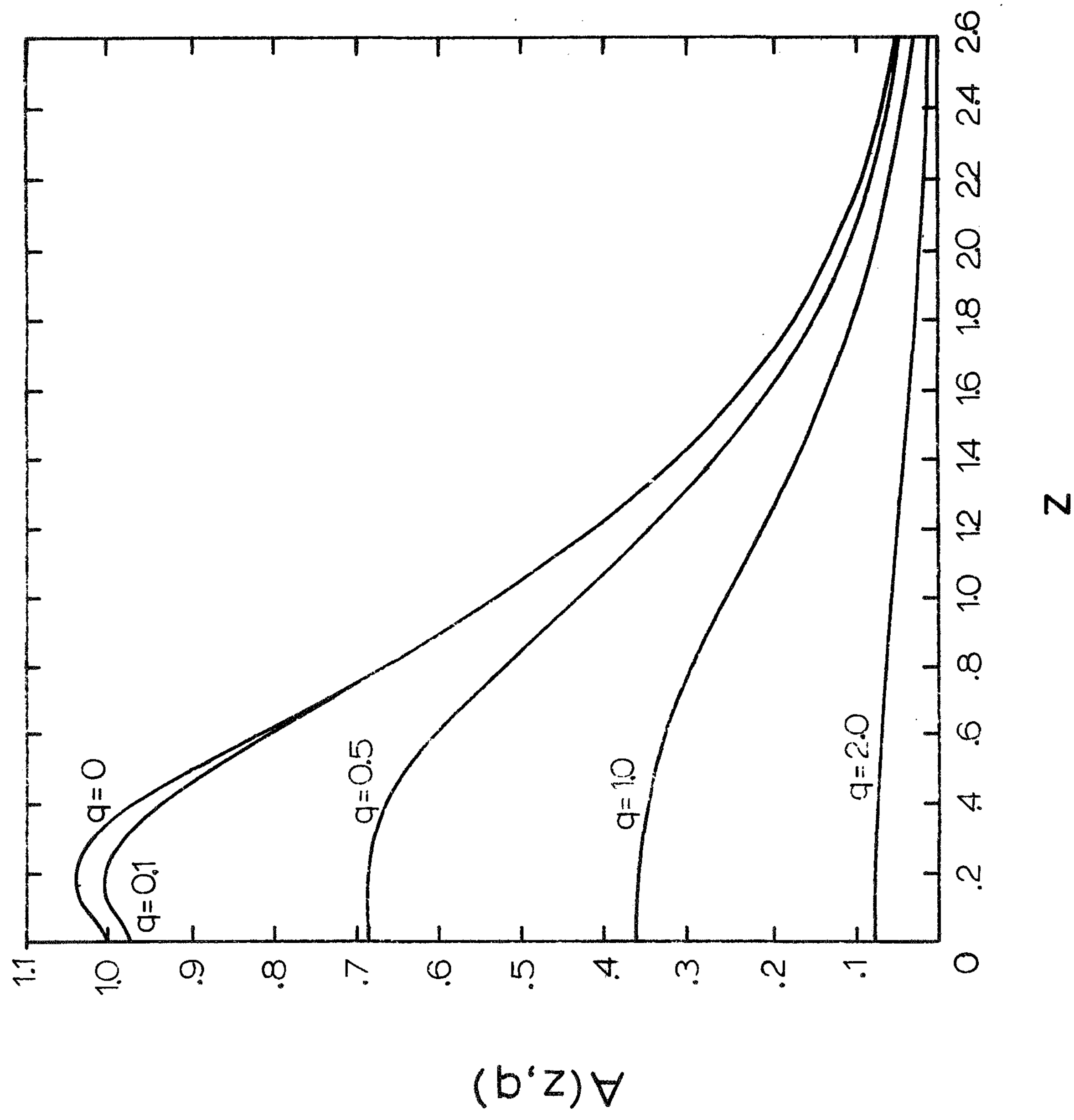

A(z,q)
Z
q=0
q=0.1
q=0.5
q=1.0
q=2.0
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6

Figure 2

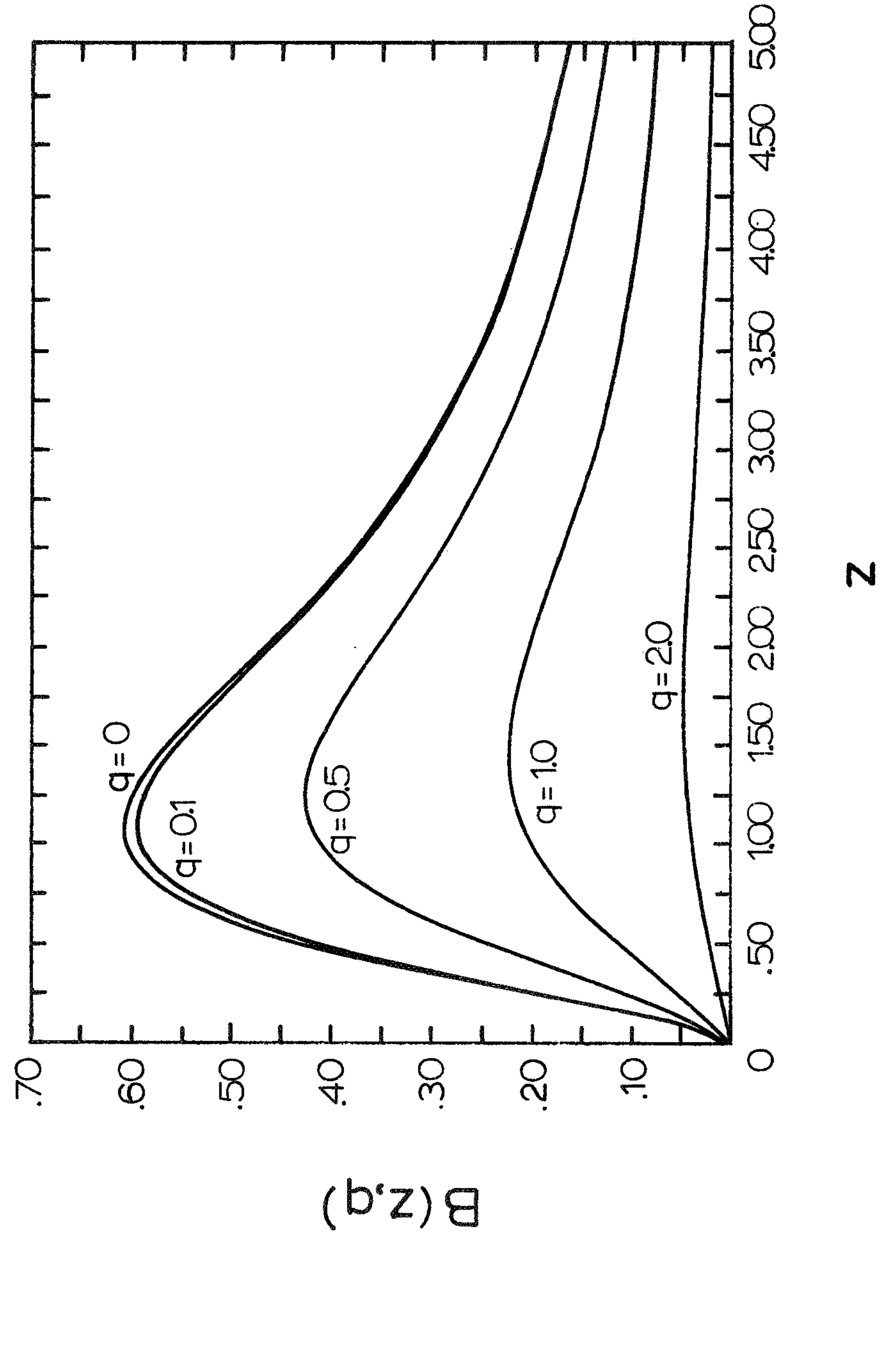
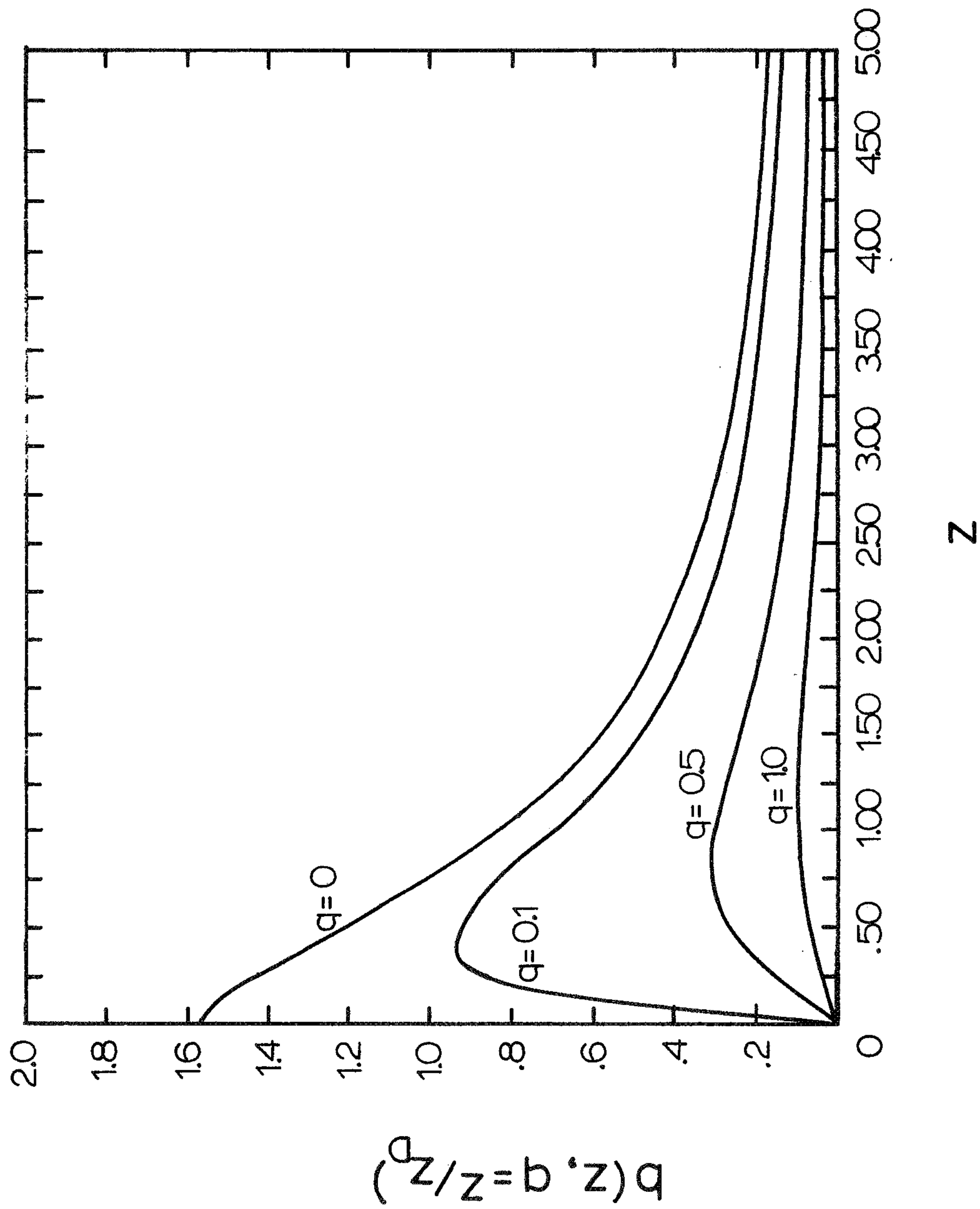

B (z,q)
.70
.60
.50
.40
.30
.20
.10
0
.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
5.00
Z
q=0
q=0.1
q=0.5
q=1.0
q=2.0

Figure 3



b(z, q=z/z_a)
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
5.00
z
q=0
q=0.1
q=0.5
q=1.0

Figure 4

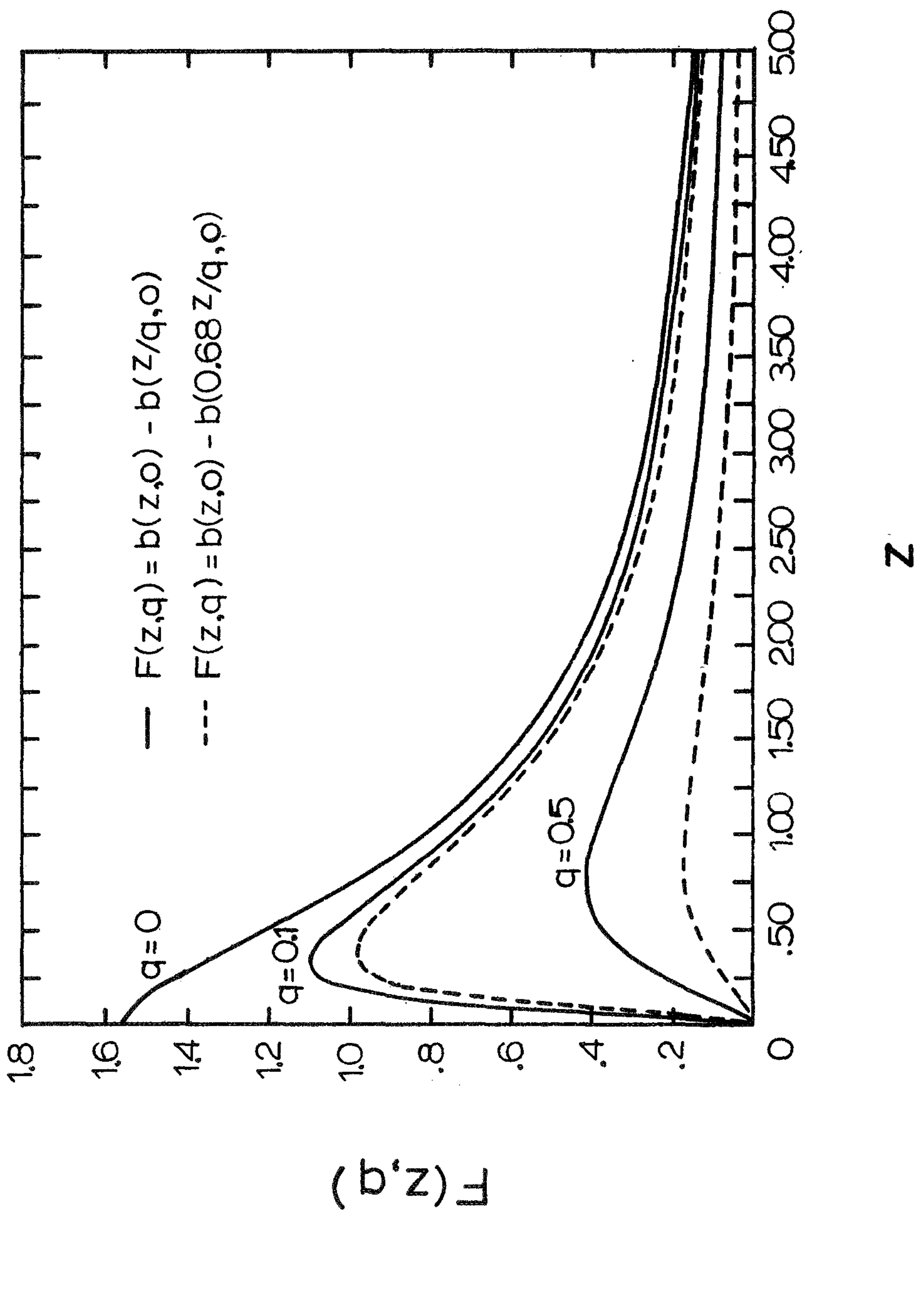

F(z,q) = b(z,o) - b(z/q,o)
F(z,q) = b(z,o) - b(0.68 z/q,o)
q=0
q=0.1
q=0.5
F(z,q)
Z
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
5.00

Figure 5

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APPENDIX

FORTRAN Programs for $B(z,q)$ and $b(z,z_D)$

(Written by U. Palmer, Joint Institute for Laboratory Astrophysics)

$b(z,z_D)$

$b(z,z_D)$ is calculated by FUNCTION FOFZZD(Z,ZD)

A sample FORTRAIN CALL: B = FOFZZD(Z,ZD)

FUNCTION FOFZZD requires the following seven routines and a data deck marked $b(z,z_D)$

1. FUNCTION B2(Z)
2. FUNCTION FACT(N)
3. SUBROUTINE KI1(X,CAY)
4. SUBROUTINE INTERP(NX,NY,X,Y,Z,R,S,T)
5. SUBROUTINE LAGRANG(NPTS,X,Y,NP,LAP,ISTART,IEND,LEND,R,V)
6. SUBROUTINE BESSK(X,CKE,EI)
7. SUBROUTINE BESSI(X,EI)
8. data deck $b(z,z_D)$

$B(z,q)$

$B(z,q)$ is calculated by FUNCTION FOFZQ(ZZ,QQ)

A sample FORTRAN CALL: B = FOFZQ(Z,Q)

The following routines are needed by FUNCTION FOFZQ(ZZ,QQ)

A. SUBROUTINE INTPB

B. FUNCTION BQASYM(Z,Q)

C. ROUTINES numbered 2-7 on the list for $b(z,z_D)$

D. data deck marked B(Z,Q)

Use of FUNCTIONS FOFZQ(ZZ,QQ) and FOFZZD(Z,ZD)

$b(z,z_D)$ and $B(z,q)$ are calculated by interpolating in tables B(Z,ZD) and B(Z,Q) respectively. Accuracy within the $B(z,q)$ table exceeds 2%. When $b(z,z_D)$ exceeds 0.01 the accuracy is also better than 2%, between 0.01 and 0.001 the first non-zero digit is definitely significant and, when less than 0.001, $b(z,z_D) = 0$ will be returned. The tables are provided by the data decks marked $b(z,z_D)$ and $B(z,q)$.

The first call to either routine will cause the reading of its own data deck. When inserting the routines into an already existing program, care must be taken in arranging the data. Therefore, it would be best to make dummy calls to FOFZZD and FOFZQ in the beginning of the main program and at the same time placing the data decks at the start of the program's whole data set.

For example:

PROGRAM: PROGRAM MAIN

DUMMY = FOFZQ(2.0,2.0) causes reading of table B(Z,Q) and stores B(2.0,2.0) into DUMMY

DUMMI = FOFZZD(2.0,2.0) causes reading of table $b(z,z_D)$ and stores b(2.0,2.0) into DUMMI

rest of program

END

DATA: Table B(Z,Q)

Table $b(z,z_D)$

rest of data for Program Main

```
      PROGRAM   TESTBS (INPUT,OUTPUT)
      DIMENSION ZZZ(25),QQQ(25)
      DUMMY = FOFZQ(2.0,2.0)
      DUMMI = FOFZZD(2.0,2.0)
      PRINT 600
   10 CONTINUE
      READ 500,NZZZ,NQQQ
      IF(NZZZ.LT.0) CALL EXIT
      READ 501,(ZZZ(I),I=1,NZZZ)
      READ 501,(QQQ(I),I=1,NQQQ)
      DO 400 J = 1,NQQQ
      PRINT 670
  670 FORMAT(1H0)
      Q = QQQ(J)
      DO 400 I  = 1,NZZZ
      Z = ZZZ(I)
      BZQ = FOFZQ(Z,Q)
      BZZD = FOFZZD(Z,Q)
      PRINT 601,Z,Q,BZQ,BZZD
  400 CONTINUE
  500 FORMAT(20I4)
  501 FORMAT(5E15.0)
  600 FORMAT(1H1,18X,*Z*,13X,*Q OR ZD*,13X,*B (Z,Q)*,12X,*B (Z,ZD)*,//)
  601 FORMAT(4F20.8)
      END
```

```
      FUNCTION FOFZZD(Z,ZD)
      COMMON/SETBZZD/B(38,21),ZE(38),ZEDE(21),ZLIM(14),Y(5),NZ,NZD,NZLIM
      DIMENSION CK(3),CEI(3)
      DATA(LSWITCH = 1)
      GO TO (1,2),LSWITCH
    1 LSWITCH = 2
C
C     READ B(Z,ZD) DATA DECK
C
      READ 500,NZ,NZD,NZLIM
      READ 501,(ZE(I),I=1,NZ)
      DO 110 J = 1,NZD
      READ 501,ZEDE(J)
      READ 501,(B(I,J),I=1,NZ)
  110 CONTINUE
      PRINT 600,(ZEDE(IU),IU = 1,10)
      PRINT 601,(ZE(IU),(B(IU,LU),LU=1,10),IU = 1,NZ)
      PRINT 603,(ZEDE(IU),IU = 11,NZD)
      PRINT 604,(ZE(IU),(B(IU,LU),LU=11,NZD),IU = 1,NZ)
      READ 501,(ZLIM(I) , I= 1,NZLIM)
      DO 120 I = 1,5
      Y(I) = ALOG10(ZEDE(I+16))
  120 CONTINUE
C
C     CALCULATE B(Z,ZD)
C
    2 IF(ZD.GE.0.001) GO TO 10
    5 FOFZZD = 0.0
      RETURN
   10 IF( (ZD.LT.2.0).AND.(Z.GT.10.0) ) GO TO 5
      DO 20 J = 2,NZLIM
      IF((ZD.LE.ZEDE(J) ).AND.(Z.GT.ZLIM(J) ))5,20
   20 CONTINUE
      IF((ZD.LT..1) .OR.(Z.GT.10.0) ) GO TO 30
      IF(ZD.GT.20000.0) GO TO 40
      IF( (ZD.GE.10.0).AND.(ZD.LE.2000.0) )  GO TO 50
C
C     LAG. INTERP
C
      CALL INTERP(NZD,NZ,ZEDE,ZE,B,ZD,Z,TEMP,4)
      FOFZZD = TEMP
      GO TO 200
C
C     B(Z,ZD) ASYM
C
   30 ZZD = Z/ZD
      TZZD = 2.0*ZZD
      CALL BESSK(TZZD,CK,CEI)
      FK0 = CK(1)
      FK1 = CK(2)
      IF(ZD.LT.0.1) GO TO 60
      CALL KI1(TZZD,FKI1)
      TZZDSQ = TZZD * ZZD
      FOFZZD = (1./(2.*Z))*(FKI1*(1.+TZZDSQ)-TZZDSQ*FK1+TZZD*FK0)
      GO TO 200
C
C     B2(Z)
C
   40 FOFZZD = B2(Z)
      GO TO 200
```

```
C
C     LOG INTERP
C
   50 CALL INTERP(5,NZ,Y,ZE,B(1,17),ALOG10(ZD),Z,TEMP,2)
      FOFZZD = TEMP
      GO TO 200
   60 FOFZZD = Z * FK1
  200 IF(FOFZZD.LT.0.001) FOFZZD = 0.0
      RETURN
  500 FORMAT(20I4)
  501 FORMAT(10F8.4)
  600 FORMAT(1H1,*B(Z,ZD)*,//,*   ZD*,3X,10F10.3,//,*   Z*,//)
  601 FORMAT(F8.4,10F10.5)
  603 FORMAT(1H1,*B(Z,ZD)*,//,*   ZD*,3X,11F10.3,//,*   Z*,//)
  604 FORMAT(F8.4,11F10.5)
      END
```

```
      FUNCTION B2(Z)
      DIMENSION C(3),EI(3)
      PI = 3.141592654
      CALL BESSK (Z,C,EI)
      B2 = 0.5 * PI - PI * Z * C(1) * EI(2)
      RETURN
      END
```

```
      FUNCTION FACT (N)
      DOUBLE F
      F = 1.0
      IF(N .GE. 0) GO TO 10
      PRINT 600
  600 FORMAT(1H0,*NEGATIVE FACTORIAL*)
      CALL EXIT
   10 FACT = 1.0
      IF(N.GT. 1) GOTO 20
      RETURN
   20 DO 30 I = 1,N
      F = F * I
   30 CONTINUE
      FACT = F
      RETURN
      END
```

```
      SUBROUTINE KI1(X,CAY)
      DIMENSION EK(7)
      DATA(PI=3.141592654),(EK=1.25331414,0.11190289,0.02576646,
     1          0.00933994,0.00417454,0.00163271,0.00033934)
      DATA(EULER= 0.5772156649)
      PI2 = PI/2.0
      IF(X.NE.0.0) GO TO 10
      CAY = PI2
      RETURN
   10 IF(X.GT.7.0) GO TO 200
      XT = X/2.0
      EPS = 1.0E-9
      COEF = -(EULER+ALOG(XT))*X
      SUM = 0.0       $       SUMA=0.0
      SUMB = 0.0      $       SUMC = 0.0
      KB = 0
      DO 100 K = KB,100
      TM = 2*K+1
      TK = FACT(K)**2
      TN = XT**(2*K)
      TA = TN/(TK*TM)
      TB = TN / (TK*TM**2)
      SUM = SUM + 1.0 / (K+1.0)
      TC = (XT**(2*(K+1))/(FACT(K+1)**2*(2*(K+1)+1)))*SUM
      SUMA = SUMA + TA
      SUMB = SUMB + TB
      SUMC = SUMC + TC
      IF(ABS (TA) .GT. EPS   *ABS (SUMA)) GO TO 100
      IF(ABS (TB) .GT. EPS   *ABS (SUMB)) GO TO 100
      IF(ABS (TC) .GT. EPS   *ABS (SUMC)) GO TO 100
      GO TO 150
  100 CONTINUE
C     PRINT 600,TA,TB,TC,SUM,SUMA,SUMB,SUMC,K
  600 FORMAT(40X,7E13.5,I5)
      CALL EXIT
  150 CAY = PI2 - COEF*SUMA - X*SUMB -X *SUMC
C     PRINT 600,TA,TB,TC,SUM,SUMA,SUMB,SUMC,K
      RETURN
  200 X7 = X/7.0          $       SUMD = 0.0
      DO 250 M=1,7
      SUMD = SUMD +     (-1.0)**(M-1) * EK(M) / X7**(M-1)
  250 CONTINUE
      CAY = SUMD/(SQRT(X) * EXP(X))
      RETURN
      END
```

```
      SUBROUTINE INTERP(NX,NY,X,Y,Z,R,S,T,NP)
      DIMENSION X(NX),Y(NY),Z(NY,NX)
      DIMENSION DIN(20),STORE(20)
      NPTS = 4
      DO 10 N = 2,NX
      IF(R.GT.X(N)) 10,20
   10 CONTINUE
   15 PRINT 600,R,S,X(NX),Y(NY)
      CALL EXIT
  600 FORMAT(*0R,S,X(NX),Y(NY)*,4E20.9,//)
   20 NR = N-1
      DO 30 N = 2,NY
      IF(S.GT.Y(N)) 30,40
   30 CONTINUE
      GO TO 15
   40 NS = N  - 1
      IF(NS.EQ.1) NS = NS + 1
      IF(NR.EQ.1) NR = NR + 1
      IF(NS.EQ.(NY-1) ) NS = NS - 1
      IF(NR.EQ.(NX-1) ) NR = NR - 1
      DO 100 I = 1,4
      II = NR-2+I
      DO 50 J = 1,4
      JJ = NS-2+J
      DIN(J) = Z(JJ,II)
   50 CONTINUE
      CALL LAGRANG(4,Y  (NS-1),DIN,NPTS,NPTS-1,1,1,LEND,S    ,STORE(I) )
  100 CONTINUE
      CALL LAGRANG(4,X  (NR-1),STORE,NPTS,NPTS-1,1,1,LEND,R,T)
      RETURN
      END
```

```
      SUBROUTINE LAGRANG  (NPTS,X,Y,NP,LAP,ISTART,IEND,LEND,R,V)
      DIMENSION X(NPTS),Y(NPTS),R(IEND),V(IEND),DN(15),DD(15)
      DO 100 I = 1,15
      DN(I) = 1.
  100 DD(I) = 1.
      LT = NP/2 - 1
      NPMLAP = NP - LAP
      NI = 1
      NE = NP
      NTEMP = 0
      IR = ISTART
  103 IF(R(IR) - X(NE-LT) ) 102,102,101
  101 IF((NE.EQ.NPTS).AND.(R(IR).LE.X(NE)))GO TO 102
      NI = NI + NPMLAP
      NE = NE + NPMLAP
      NTEMP = NI - 1
      IF(NE - NPTS)103,103,104
  104 LEND = IR - 1
      RETURN
  102 DO 110 K = 1,NP
      KK = K + NTEMP
      DO 110 I = 1,NP
      II = I + NTEMP
      IF(K-I)108,110,108
  108 DD(K) = DD(K) * (X(KK) - X(II) )
  110 CONTINUE
  112 V(IR) = 0.0
      DO 120 IT = NI,NE
      IF(R(IR) - X(IT))120,111,120
  111 V(IR) = Y(IT)
      GO TO 149
  120 CONTINUE
      DO 130 K = 1,NP
      KK = K + NTEMP
      DO 140 I = 1,NP
      II = I + NTEMP
      IF(K-I)141,140,141
  141 DN(K) = DN(K) *(R(IR) - X(II))
  140 CONTINUE
      V(IR)     = V(IR) + ( DN(K) * Y(KK)/ DD(K) )
  130 CONTINUE
  149 IF(IR - IEND)150,151,151
  150 IR = IR + 1
      DO 170 MZ = 1,NP
  170 DN(MZ) = 1.0
      IF(R(IR) - X(NE - LT))112,112,161
  161 IF(NE .LT. NPTS) GO TO 162
      IF(R(IR) .LE. X(NPTS) ) GO TO 112
  162 DO 180 MZ = 1,NP
  180 DD(MZ) = 1.0
      GO TO 101
  151 LEND = IEND
      RETURN
      END
```

```
      SUBROUTINE BESSK (X,CKE,EI)
      DIMENSION FIRST(4),EI(3),COEF(4),CKE(3),A(10,4)
      DATA (A =  0.42278420,   .23069756,   .03488590,
     1            .00262698,   .00010750,   .00000740, 3(0.0), 6.0,
     2            .15443144,  -.67278579,  -.18156897,
     3           -.01919402,  -.00110404,  -.00004686, 3(0.0), 6.0,
     4           -.07832358,   .02189568,  -.01062446,
     5            .00587872,  -.00251540,   .00053208, 3(0.0), 6.0,
     6            .23498619,  -.03655620,   .01504268,
     7           -.00780353,   .00325614,  -.00068245, 3(0.0), 6.0   )
      CALL BESSI (X,EI)
      IF(X .LT. 2.0) 10,20
   10 T = X / 2.0
      XP =  ALOG(T)
      FIRST(1) = -XP * EI(1) - 0.57721566
      FIRST(2) =  X * XP * EI(2)  + 1.0
      FACTOR = T * T
      COEF(1) = 1.0
      COEF(2) = 1.0 / X
      JJ = 1
      GO TO 50
   20 T = 2.0 / X
      FIRST(3) = FIRST(4)  = 1.25331414
      JJ = 3
      COEF(3) = COEF(4) = 1.0 / (SQRT (X) * EXP (X) )
      FACTOR = T
   50 JEND = JJ + 1
      I = 0
      DO 200 J = JJ,JEND
      I = I + 1
      PROD = 1.0
      SUM = 0.0
      KEND = A(10,J) + 0.000001
      DO 100 K = 1,KEND
      PROD = PROD * FACTOR
      SUM = SUM + PROD * A(K,J)
  100 CONTINUE
      CKE(I) = COEF(J) * (FIRST(J) + SUM)
  200 CONTINUE
      CKE(3) = (2.0/X) * CKE(2) + CKE(1)
      RETURN
      END
```

```
      SUBROUTINE BESSI (X,EI)
      DIMENSION A(10,4),FIRST(4),COEF(4),EI(3)
      DATA(FIRST = 1.0,0.5,2(0.39894228)),
     1     (A=  3.5156229,  3.0899424, 1.2067492,
     2           .2659732,   .0360768,  .0045813,  3(0.0), 6.0,
     3          .87890594,  .51498869, .15084934,
     4          .02658733,  .00301532, .00032411,  3(0.0), 6.0,
     5          .01328592,  .00225319,-.00157565,
     6          .00916281, -.02057706, .02635537,
     7         -.01647633,  .00392377,               0.0 , 8.0,
     8         -.03988024, -.00362018, .00163801,
     9         -.01031555,  .02282967,-.02895312,
     1          .01787654, -.00420059,               0.0 , 8.0     )

      T = X / 3.75
      COEF(1) = 1.0
      COEF(2) = X
      COEF(3) = COEF(4) = EXP (X) / SQRT (X)
      IF(X .LT. 3.75) 10,20
   10 FACTOR = T * T
      JJ = 1
      GO TO 50
   20 FACTOR = 1.0 / T
      JJ = 3
   50 JEND = JJ + 1
      I = 0
      DO 200 J = JJ,JEND
      I = I + 1
      PROD = 1.0
      SUM = 0.0
      KEND = A(10,J) + .000001
      DO 100 K = 1,KEND
      PROD = PROD * FACTOR
      SUM = SUM + PROD * A(K,J)
  100 CONTINUE
      EI(I) = COEF(J) * ( FIRST(J) + SUM )
  200 CONTINUE

      EI(3) = (-2.0/X) * EI(2) + EI(1)
      RETURN
      END
```

DATA DECK FOR $b(z,z_D)$

38	21	14							
0.	0.001	.002	.003	.004	.005	.006	.007	.008	.009
.01	.02	.03	.04	.05	.06	.07	.08	.09	.1
.2	.3	.4	.5	.6	.7	.8	.9	1.	2.
3.	4.	5.	6.	7.	8.	9.	10.		
.0010									
.0005	.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
.0050									
.0025	.0022	.0017	.0013	.0010	.0007	.0005	.0004	.0003	.0002
.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
.01									
.0050	.0048	.0044	.0039	.0034	.0030	.0026	.0022	.0019	.0016
.0014	.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
.02									
.0100	.0099	.0096	.0092	.0087	.0083	.0078	.0074	.0069	.0065
.0060	.0028	.0012	.0005	.0002	.0001	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
.0500									
.0250	.0250	.0248	.0246	.0243	.0240	.0236	.0232	.0228	.0224
.0219	.0173	.0131	.0097	.0071	.0051	.0036	.0026	.0018	.0013
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
.1									
.0502	.0505	.0504	.0503	.0501	.0499	.0496	.0493	.0490	.0487
.0483	.0442	.0396	.0350	.0306	.0265	.0229	.0197	.0169	.0144
.0025	.0004	.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
.1500									
.0755	.0764	.0763	.0762	.0760	.0759	.0757	.0754	.0752	.0749
.0746	.0711	.0668	.0622	.0574	.0528	.0483	.0441	.0401	.0364
.0121	.0037	.0011	.0003	.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
.2									
.1012	.1008	.1007	.1006	.1005	.1003	.1002	.1000	.0998	.0995
.0993	.0962	.0922	.0878	.0831	.0783	.0736	.0689	.0643	.0599
.0276	.0118	.0048	.0019	.0008	.0003	.0001	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
.3000									
.1538	.1531	.1531	.1530	.1529	.1528	.1526	.1525	.1523	.1521
.1519	.1493	.1459	.1419	.1375	.1329	.1281	.1232	.1183	.1134
.0703	.0410	.0230	.0127	.0069	.0037	.0020	.0011	.0006	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
.5									
.2637	.2626	.2626	.2625	.2625	.2624	.2623	.2621	.2620	.2619
.2617	.2596	.2567	.2532	.2493	.2450	.2405	.2357	.2307	.2256
.1738	.1280	.0918	.0647	.0451	.0312	.0214	.0146	.0099	.0002
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
.7500									
.4035	.4026	.4025	.4025	.4024	.4023	.4022	.4021	.4020	.4019
.4018	.3999	.3973	.3941	.3904	.3862	.3818	.3770	.3721	.3669
.3104	.2537	.2031	.1604	.1254	.0974	.0752	.0578	.0443	.0028
.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		

1.									
.5354	.5347	.5346	.5346	.5345	.5345	.5344	.5343	.5342	.5340
.5339	.5321	.5296	.5265	.5228	.5187	.5143	.5095	.5045	.4992
.4396	.3762	.3164	.2629	.2167	.1774	.1446	.1174	.0950	.0105
.0011	.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
1.5000									
.7520	.7516	.7516	.7515	.7515	.7514	.7513	.7512	.7511	.7510
.7509	.7491	.7466	.7435	.7399	.7357	.7312	.7263	.7212	.7157
.6518	.5804	.5096	.4430	.3826	.3287	.2813	.2400	.2043	.0388
.0074	.0015	.0003	.0001	0.0000	0.0000	0.0000	0.0000		
2.									
.9072	.9070	.9070	.9070	.9069	.9068	.9067	.9067	.9065	.9064
.9063	.9046	.9020	.8989	.8952	.8910	.8864	.8814	.8761	.8705
.8040	.7282	.6513	.5777	.5093	.4470	.3911	.3413	.2974	.0730
.0187	.0052	.0015	.0005	.0002	.0001	0.0000	0.0000		
3.0000									
1.0990	1.0992	1.0992	1.0992	1.0991	1.0990	1.0990	1.0989	1.0988	1.0986
1.0985	1.0967	1.0942	1.0910	1.0873	1.0830	1.0783	1.0732	1.0678	1.0620
.9928	.9125	.8298	.7492	.6731	.6026	.5381	.4796	.4271	.1346
.0463	.0177	.0073	.0032	.0014	.0007	.0003	.0001		
5.									
1.2773	1.2775	1.2775	1.2775	1.2774	1.2773	1.2772	1.2771	1.2770	1.2769
1.2768	1.2750	1.2725	1.2692	1.2654	1.2611	1.2563	1.2512	1.2457	1.2398
1.1687	1.0854	.9988	.9135	.8320	.7558	.6854	.6209	.5622	.2163
.0947	.0468	.0251	.0142	.0083	.0049	.0030	.0018		
10.									
1.4216	1.4218	1.4218	1.4217	1.4217	1.4216	1.4215	1.4214	1.4213	1.4212
1.4211	1.4193	1.4167	1.4135	1.4096	1.4053	1.4005	1.3953	1.3897	1.3838
1.3117	1.2269	1.1382	1.0504	.9662	.8870	.8134	.7456	.6836	.3064
.1615	.0971	.0632	.0433	.0306	.0221	.0162	.0121		
100.									
1.5558	1.5559	1.5558	1.5558	1.5557	1.5556	1.5556	1.5555	1.5554	1.5552
1.5551	1.5533	1.5507	1.5475	1.5436	1.5393	1.5345	1.5292	1.5236	1.5177
1.4453	1.3598	1.2703	1.1816	1.0964	1.0160	.9411	.8720	.8087	.4180
.2616	.1875	.1454	.1181	.0990	.0849	.0739	.0652		
1000.									
1.5693	1.5692	1.5692	1.5692	1.5691	1.5690	1.5690	1.5689	1.5688	1.5686
1.5685	1.5667	1.5641	1.5609	1.5570	1.5527	1.5478	1.5426	1.5370	1.5311
1.4587	1.3732	1.2837	1.1950	1.1097	1.0293	.9544	.8853	.8219	.4312
.2746	.2003	.1581	.1307	.1114	.0971	.0861	.0772		
10000.									
1.5707	1.5701	1.5701	1.5700	1.5700	1.5699	1.5698	1.5697	1.5696	1.5695
1.5694	1.5676	1.5650	1.5617	1.5579	1.5535	1.5487	1.5435	1.5379	1.5319
1.4595	1.3741	1.2845	1.1958	1.1106	1.0301	.9553	.8862	.8228	.4320
.2754	.2012	.1589	.1315	.1123	.0980	.0869	.0781		
20000.									
1.5707	1.5701	1.5701	1.5700	1.5700	1.5699	1.5698	1.5697	1.5696	1.5695
1.5694	1.5676	1.5650	1.5617	1.5579	1.5535	1.5487	1.5435	1.5379	1.5319
1.4595	1.3741	1.2846	1.1958	1.1106	1.0301	.9553	.8862	.8228	.4320
.2754	.2012	.1589	.1315	.1123	.0980	.0869	.0781		
.001	.01	.02	.06	.1	.4	.6	.8	1.	2.
3.	4.	6.	8.						

```
      FUNCTION FOFZQ(ZZ,QQ)
      COMMON /DAT/ Z(45),Q(17),B(45,17),NZ,NQ,NPTS
      COMMON/ORIG/BB(45,17),PI
      COMMON/NSM/  BNSM(45,17)
      DIMENSION EI(3),CKE(3)
      DATA(LSWITCH = 1)
      DATA(PI = 3.141592654)
      GO TO (100,200),LSWITCH
  100 LSWITCH = 2
C
C     READ DATA DECK B(Z,Q) AND GENERATE SMOOTHED TABLES FOR B(Z,Q)
C
      CALL INTPB
C
C     CALCULATE B(Z,Q)
C
  200 IF(QQ.GT.0.0) GO TO 201
      CALL BESSK(ZZ,CKE,EI)
      FOFZQ = PI * ZZ**2 * (CKE(1)*EI(1) - CKE(2) * EI(2) )
      RETURN
  201 IF(QQ.LT.10.0) GO TO 10
    5 FOFZQ = 0.0
      RETURN
   10 TEMPB = BQASYM(ZZ,QQ)
      IF(ZZ.LT.Z(1) ) GO TO 5
      IF(ZZ.LE.0.05) GO TO30
      IF(ZZ.LE.25.0) GO TO 40
      FOFZQ = TEMPB
      RETURN
   30 CALL INTERP(NQ,NZ,Q,Z,BNSM,QQ,ZZ,TEMP,NPTS)
      TZ = ZZ**2
      TQ = 2.0 * QQ
      RTZTQ = SQRT(TZ+TQ**2)
      CALL BESSK(RTZTQ,CKE,EI)
      DENUM = PI*ZZ*SQRT(TZ+(TQ/PI)**2) * CKE(1)
      FOFZQ = TEMP * DENUM
      RETURN
   40 CALL INTERP(NQ,NZ,Q,Z,B    ,QQ,ZZ,TEMP,NPTS)
      CALL LAGRANG (NZ,Z,BB(1,1),4,3,1,1,LEND,ZZ,BBB)
      TB = TEMP * BBB  * (4.0*ZZ/PI) * TEMPB
      FOFZQ = TB
      RETURN
      END
```

```
      SUBROUTINE INTPB
      COMMON /DAT/ Z(45),Q(17),B(45,17),NZ,NQ,NPTS
      COMMON/ORIG/ P(45,17),PI
      COMMON/NSM/  BNSM(45,17)
      DIMENSION C(3),EI(3)
      NZ = 45
      NQ = 17
      PI = 3.141592654
      NPTS = 4
C
C     READ DATA DECK FOR B(Z,Q)
C
    5 READ 501,(Z(I),I=1,NZ)
      DO 10 J = 1,NQ
      READ 502,Q(J)
      READ 502, (P(I,J),I=1,NZ)
   10 CONTINUE
C
C     PRINT TABLE B(Z,Q)
C
      PRINT 612,(Q(I),I=1,9)
      PRINT 613
      PRINT 611,(Z(I),(P(I,J),J=1,9),I=1,NZ)
      PRINT 612,(Q(I),I=10,NQ)
      PRINT 613
      DO 1509 I = 1,NZ
      PRINT 611,(Z(I),(P(I,J),J=10,NQ))
 1509 CONTINUE
      DO 15 I = 1,NZ
C
C     SMOOTH TABLE B(Z,Q) FOR Z .LE. 0.05
C
      B(I,1) = 1.0
      TEMP = 4.0*Z(I)/PI
      DO 13 IY = 1,NQ
      TBNSM = SQRT(Z(I)**2+(2.0*Q(IY))**2)
      CALL BESSK(TBNSM,C,EI)
      DENEW = PI*Z(I) * SQRT(Z(I)**2+(2.0*Q(IY)/PI)**2)*C(1)
      BNSM(I,IY) = P(I,IY)/DENEW
   13 CONTINUE
C
C     SMOOTH TABLE B(Z,Q) FOR Z .GT. 0.05 AND Z .LE. 25.0
C
      DO 15 J = 2,NQ
      B(I,J) = (P(I,J)/P(I,1)) /(TEMP*BQASYM(Z(I),Q(J)))
   15 CONTINUE
  501 FORMAT(10F8.0)
  502 FORMAT(6E12.4)
  611  FORMAT(F10.3,9E14.4)
  612 FORMAT(1H1,*B(Z,Q)*,//,7X,*Q*,2X,9F14.3,/)
  613 FORMAT(1H ,*  Z*,/)
      RETURN
      END
```

```
FUNCTION BQASYM(Z,Q)
DIMENSION CK(3),CEI(3)
TWOQ = 2.0 * Q
CALL BESSK(TWOQ,CK,CEI)
CALL KI1(TWOQ,CAY)
QSQ = Q**2
BQASYM = (1.0/Z)*((0.5-QSQ)*CAY+Q*CK(1)     + 2.0*QSQ * CK(2) )
RETURN
END
```

DATA DECK FOR B(z,q)

```
 .001      .002      .003      .004      .005      .006      .007      .008      .009      .010
 .020      .030      .040      .050      .060      .070      .080      .090      .100      .200
 .300      .400      .500      .600      .700      .800      .900     1.000     1.200     1.400
1.600     1.800     2.000     3.000     4.000     5.000     6.000     8.000    10.000    12.000
14.000    16.000    18.000    20.000    25.000
1.0000E-08
2.0495E-05  7.3269E-05  1.5339E-04  2.5824E-04  3.8597E-04  5.3519E-04
7.0472E-04  8.9362E-04  1.1010E-03  1.3262E-03  4.4350E-03  8.8363E-03
1.4272E-02  2.0564E-02  2.7577E-02  3.5205E-02  4.3356E-02  5.1957E-02
6.0942E-02  1.6214E-01  2.6576E-01  3.5884E-01  4.3662E-01  4.9806E-01
5.4397E-01  5.7594E-01  5.9594E-01  6.0592E-01  6.0312E-01  5.8005E-01
5.4588E-01  5.0684E-01  4.6690E-01  3.0545E-01  2.1582E-01  1.6625E-01
1.3578E-01  1.0006E-01  7.9469E-02  6.5978E-02  5.6429E-02  4.9306E-02
4.3785E-02  3.9379E-02  3.1470E-02
5.0000E-03
4.8939E-05  1.0761E-04  1.8318E-04  2.7951E-04  3.9791E-04  5.3832E-04
7.0001E-04  8.8208E-04  1.0836E-03  1.3037E-03  4.3862E-03  8.7779E-03
1.4209E-02  2.0498E-02  2.7510E-02  3.5136E-02  4.3286E-02  5.1887E-02
6.0871E-02  1.6207E-01  2.6569E-01  3.5878E-01  4.3656E-01  4.9801E-01
5.4391E-01  5.7590E-01  5.9589E-01  6.0588E-01  6.0309E-01  5.8002E-01
5.4586E-01  5.0682E-01  4.6688E-01  3.0544E-01  2.1581E-01  1.6624E-01
1.3578E-01  1.0005E-01  7.9466E-02  6.5976E-02  5.6427E-02  4.9304E-02
4.3784E-02  3.9379E-02  3.1471E-02
1.0000E-02
8.1654E-05  1.6695E-04  2.6092E-04  3.6659E-04  4.8655E-04  6.2265E-04
7.7606E-04  9.4739E-04  1.1368E-03  1.3444E-03  4.3253E-03  8.6646E-03
1.4067E-02  2.0338E-02  2.7338E-02  3.4956E-02  4.3101E-02  5.1697E-02
6.0678E-02  1.6187E-01  2.6550E-01  3.5859E-01  4.3639E-01  4.9785E-01
5.4377E-01  5.7576E-01  5.9577E-01  6.0576E-01  6.0298E-01  5.7993E-01
5.4578E-01  5.0675E-01  4.6682E-01  3.0540E-01  2.1578E-01  1.6622E-01
1.3576E-01  1.0004E-01  7.9454E-02  6.5965E-02  5.6418E-02  4.9296E-02
4.3776E-02  3.9372E-02  3.1463E-02
5.0000E-02
2.4279E-04  4.8608E-04  7.3038E-04  9.7618E-04  1.2240E-03  1.4742E-03
1.7275E-03  1.9841E-03  2.2446E-03  2.5093E-03  5.4762E-03  9.2111E-03
1.3840E-02  1.9355E-02  2.5681E-02  3.2723E-02  4.0382E-02  4.8571E-02
5.7211E-02  1.5688E-01  2.6024E-01  3.5343E-01  4.3144E-01  4.9318E-01
5.3939E-01  5.7167E-01  5.9195E-01  6.0221E-01  5.9989E-01  5.7723E-01
5.4340E-01  5.0464E-01  4.6494E-01  3.0420E-01  2.1490E-01  1.6553E-01
1.3519E-01  9.9613E-02  7.9116E-02  6.5684E-02  5.6177E-02  4.9086E-02
4.3590E-02  3.9204E-02  3.1330E-02
1.0000E-01
3.5072E-04  7.0131E-04  1.0524E-03  1.4040E-03  1.7563E-03  2.1094E-03
2.4636E-03  2.8190E-03  3.1757E-03  3.5339E-03  7.2351E-03  1.1253E-02
1.5704E-02  2.0668E-02  2.6186E-02  3.2270E-02  3.8906E-02  4.6061E-02
5.3697E-02  1.4668E-01  2.4764E-01  3.4018E-01  4.1830E-01  4.8050E-01
5.2733E-01  5.6030E-01  5.8128E-01  5.9222E-01  5.9116E-01  5.6959E-01
5.3667E-01  4.9868E-01  4.5961E-01  3.0083E-01  2.1245E-01  1.6360E-01
1.3359E-01  9.8420E-02  7.8164E-02  6.4892E-02  5.5498E-02  4.8491E-02
4.3060E-02  3.8726E-02  3.0945E-02
1.5000E-01
4.1175E-04  8.2359E-04  1.2356E-03  1.6479E-03  2.0605E-03  2.4735E-03
2.8870E-03  3.3012E-03  3.7160E-03  4.1316E-03  8.3473E-03  1.2726E-02
1.7340E-02  2.2246E-02  2.7490E-02  3.3102E-02  3.9100E-02  4.5489E-02
5.2264E-02  1.3635E-01  2.3218E-01  3.2253E-01  4.0002E-01  4.6239E-01
5.0982E-01  5.4359E-01  5.6548E-01  5.7734E-01  5.7806E-01  5.5808E-01
5.2653E-01  4.8969E-01  4.5159E-01  2.9576E-01  2.0877E-01  1.6070E-01
1.3119E-01  9.6638E-02  7.6743E-02  6.3710E-02  5.4487E-02  4.7608E-02
4.2277E-02  3.8023E-02  3.0387E-02
2.0000E-01
```

4.4588E-04	8.9169E-04	1.3377E-03	1.7838E-03	2.2301E-03	2.6766E-03
3.1235E-03	3.5707E-03	4.0183E-03	4.4663E-03	8.9816E-03	1.3593E-02
1.8345E-02	2.3275E-02	2.8419E-02	3.3802E-02	3.9446E-02	4.5363E-02
5.1561E-02	1.2737E-01	2.1635E-01	3.0292E-01	3.7876E-01	4.4073E-01
4.8848E-01	5.2298E-01	5.4580E-01	5.5870E-01	5.6152E-01	5.4348E-01
5.1363E-01	4.7825E-01	4.4138E-01	2.8933E-01	2.0413E-01	1.5705E-01
1.2817E-01	9.4392E-02	7.4952E-02	6.2221E-02	5.3212E-02	4.6494E-02
4.1287E-02	3.7132E-02	2.9672E-02			
3.5000E-01					
4.6237E-04	9.2475E-04	1.3872E-03	1.8496E-03	2.3121E-03	2.7747E-03
3.2374E-03	3.7001E-03	4.1630E-03	4.6260E-03	9.2662E-03	1.3935E-02
1.8645E-02	2.3409E-02	2.8241E-02	3.3150E-02	3.8147E-02	4.3240E-02
4.8438E-02	1.0680E-01	1.7469E-01	2.4499E-01	3.1070E-01	3.6751E-01
4.1354E-01	4.4859E-01	4.7336E-01	4.8904E-01	4.9851E-01	4.8728E-01
4.6371E-01	4.3384E-01	4.0170E-01	2.6451E-01	1.8626E-01	1.4304E-01
1.1662E-01	8.5800E-02	6.8105E-02	5.6526E-02	4.8336E-02	4.2230E-02
3.7499E-02	3.3724E-02	2.6949E-02			
5.0000E-01					
4.2102E-04	8.4206E-04	1.2631E-03	1.6842E-03	2.1052E-03	2.5263E-03
2.9475E-03	3.3686E-03	3.7899E-03	4.2111E-03	8.4275E-03	1.2654E-02
1.6897E-02	2.1160E-02	2.5449E-02	2.9767E-02	3.4119E-02	3.8510E-02
4.2942E-02	9.0035E-02	1.4209E-01	1.9649E-01	2.4933E-01	2.9721E-01
3.3797E-01	3.7065E-01	3.9521E-01	4.1217E-01	4.2677E-01	4.2209E-01
4.0517E-01	3.8145E-01	3.5477E-01	2.3525E-01	1.6536E-01	1.2671E-01
1.0317E-01	7.5820E-02	6.0157E-02	4.9918E-02	4.2680E-02	3.7285E-02
3.3106E-02	2.9773E-02	2.3792E-02			
7.5000E-01					
3.2071E-04	6.4142E-04	9.6213E-04	1.2828E-03	1.6036E-03	1.9243E-03
2.2450E-03	2.5658E-03	2.8865E-03	3.2073E-03	6.4158E-03	9.6266E-03
1.2841E-02	1.6060E-02	1.9285E-02	2.2517E-02	2.5757E-02	2.9006E-02
3.2264E-02	6.5532E-02	1.0014E-01	1.3551E-01	1.7038E-01	2.0325E-01
2.3279E-01	2.5806E-01	2.7858E-01	2.9424E-01	3.1200E-01	3.1482E-01
3.0705E-01	2.9265E-01	2.7468E-01	1.8525E-01	1.2993E-01	9.9167E-02
8.0542E-02	5.9064E-02	4.6823E-02	3.8838E-02	3.3199E-02	2.8998E-02
2.5746E-02	2.3152E-02	1.8498E-02			
1.0000E+00					
2.2778E-04	4.5558E-04	6.8336E-04	9.1115E-04	1.1389E-03	1.3667E-03
1.5945E-03	1.8223E-03	2.0501E-03	2.2779E-03	4.5561E-03	6.8347E-03
9.1141E-03	1.1394E-02	1.3676E-02	1.5959E-02	1.8244E-02	2.0530E-02
2.2818E-02	4.5838E-02	6.9092E-02	9.2358E-02	1.1517E-01	1.3691E-01
1.5695E-01	1.7474E-01	1.8991E-01	2.0223E-01	2.1834E-01	2.2422E-01
2.2210E-01	2.1442E-01	2.0333E-01	1.4020E-01	9.8277E-02	7.4710E-02
6.0513E-02	4.4270E-02	3.5065E-02	2.9072E-02	2.4845E-02	2.1698E-02
1.9262E-02	1.7320E-02	1.3836E-02			
1.5000E+00					
1.0422E-04	2.0844E-04	3.1266E-04	4.1687E-04	5.2109E-04	6.2531E-04
7.2953E-04	8.3374E-04	9.3796E-04	1.0422E-03	2.0843E-03	3.1264E-03
4.1683E-03	5.2100E-03	6.2515E-03	7.2928E-03	8.3337E-03	9.3743E-03
1.0414E-02	2.0782E-02	3.1040E-02	4.1102E-02	5.0856E-02	6.0165E-02
6.8884E-02	7.6871E-02	8.4003E-02	9.0186E-02	9.9499E-02	1.0476E-01
1.0641E-01	1.0513E-01	1.0173E-01	7.3999E-02	5.2109E-02	3.9332E-02
3.1674E-02	2.3051E-02	1.8223E-02	1.5095E-02	1.2893E-02	1.1256E-02
9.9905E-03	8.9818E-03	7.1737E-03			
2.0000E+00					
4.4639E-05	8.9277E-05	1.3392E-04	1.7855E-04	2.2319E-04	2.6783E-04
3.1247E-04	3.5711E-04	4.0174E-04	4.4638E-04	8.9273E-04	1.3390E-03
1.7852E-03	2.2312E-03	2.6771E-03	3.1228E-03	3.5683E-03	4.0135E-03
4.4584E-03	8.8835E-03	1.3241E-02	1.7493E-02	2.1600E-02	2.5522E-02
2.9215E-02	3.2640E-02	3.5760E-02	3.8543E-02	4.3020E-02	4.5994E-02
4.7528E-02	4.7797E-02	4.7041E-02	3.6207E-02	2.5802E-02	1.9379E-02

1.5516E-02	1.1228E-02	8.8578E-03	7.3302E-03	6.2576E-03	5.4613E-03
4.8461E-03	4.3561E-03	3.4784E-03			
3.5000E+00					
2.9736E-06	5.9471E-06	8.9207E-06	1.1894E-05	1.4868E-05	1.7841E-05
2.0815E-05	2.3788E-05	2.6762E-05	2.9735E-05	5.9468E-05	8.9197E-05
1.1892E-04	1.4863E-04	1.7834E-04	2.0803E-04	2.3770E-04	2.6736E-04
2.9699E-04	5.9182E-04	8.8232E-04	1.1664E-03	1.4420E-03	1.7072E-03
1.9603E-03	2.1994E-03	2.4232E-03	2.6302E-03	2.9904E-03	3.2741E-03
3.4800E-03	3.6102E-03	3.6706E-03	3.2557E-03	2.4672E-03	1.8574E-03
1.4666E-03	1.0415E-03	8.1605E-04	6.7323E-04	5.7375E-04	5.0022E-04
4.4356E-04	3.9852E-04	3.1799E-04			
5.0000E+00					
1.7780E-07	3.5560E-07	5.3340E-07	7.1120E-07	8.8900E-07	1.0668E-06
1.2446E-06	1.4224E-06	1.6002E-06	1.7780E-06	3.5559E-06	5.3335E-06
7.1109E-06	8.8878E-06	1.0664E-05	1.2440E-05	1.4215E-05	1.5989E-05
1.7762E-05	3.5418E-05	5.2863E-05	6.9994E-05	8.6712E-05	1.0292E-04
1.1854E-04	1.3347E-04	1.4766E-04	1.6102E-04	1.8508E-04	2.0530E-04
2.2151E-04	2.3368E-04	2.4193E-04	2.3614E-04	1.9155E-04	1.4759E-04
1.1608E-04	8.0969E-05	6.2926E-05	5.1737E-05	4.4013E-05	3.8331E-05
3.3965E-05	3.0501E-05	2.4319E-05			
7.5000E+00					
1.4729E-09	2.9459E-09	4.4188E-09	5.8917E-09	7.3646E-09	8.8376E-09
1.0310E-08	1.1783E-08	1.3256E-08	1.4729E-08	2.9458E-08	4.4185E-08
5.8910E-08	7.3633E-08	8.8352E-08	1.0307E-07	1.1778E-07	1.3248E-07
1.4718E-07	2.9372E-07	4.3895E-07	5.8224E-07	7.2299E-07	8.6058E-07
9.9446E-07	1.1241E-06	1.2490E-06	1.3686E-06	1.5908E-06	1.7877E-06
1.9576E-06	2.0993E-06	2.2124E-06	2.3888E-06	2.1259E-06	1.7304E-06
1.3816E-06	9.4617E-07	7.2403E-07	5.9132E-07	5.0140E-07	4.3584E-07
3.8572E-07	3.4609E-07	2.7560E-07			
1.0000E+01					
1.1483E-11	2.2965E-11	3.4447E-11	4.5930E-11	5.7412E-11	6.8895E-11
8.0377E-11	9.1859E-11	1.0334E-10	1.1482E-10	2.2964E-10	3.4446E-10
4.5926E-10	5.7404E-10	6.8880E-10	8.0354E-10	9.1826E-10	1.0329E-09
1.1476E-09	2.2911E-09	3.4267E-09	4.5504E-09	5.6583E-09	6.7467E-09
7.8120E-09	8.8509E-09	9.8600E-09	1.0836E-08	1.2679E-08	1.4361E-08
1.5865E-08	1.7181E-08	1.8302E-08	2.1049E-08	2.0028E-08	1.7190E-08
1.4124E-08	9.6835E-09	7.2997E-09	5.9143E-09	4.9961E-09	4.3338E-09
3.8305E-09	3.4338E-09	2.7310E-09			