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NUMERICAL ANALYSIS OF A PAIR OF SINGULAR INTEGRAL EQUATIONS WITH A PRINCIPAL VALUE

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NUMERICAL ANALYSIS OF A PAIR OF SINGULAR INTEGRAL EQUATIONS WITH A PRINCIPAL VALUE

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SUMMARY

This report describes a numerical analysis of a pair of integral equations. One equation is of the Volterra type and has a singularity which requires special handling. The other equation is of the Fredholm type with a principal value.

The technique of integration of the principal value integral is described and tested on an improper integral with a known solution. To attain convergence of the iterative solution by the method of Picard it was necessary to take two actions to overcome instability near the singularity: (1) reduce the relaxation parameter, and (2) constrain one function to behave smoothly near the singularity.

INTRODUCTION

Research on the shape of a magnetically balanced arc reported in reference 1 required the solution of a pair of integral equations. Work done on the numerical solution of this problem has two features of interest to numerical analysts with similar problems: (1) the technique of handling the principal value of an integral, and (2) the method of achieving a satisfactory convergence of the iterative solution of the simultaneous integral equations despite the presence of a singularity.

Because numerical analysts occasionally have need to solve problems involving principal values of integrals or to deal with integral equations having singular points, the experience described in this report should prove helpful.

SYMBOLS

a_1, b_1, c_1	coefficients in definition of f_1, eq. (20)
a_2, b_2, c_2	coefficients in definition of f_2, eq. (21)
$D(s)$	difference in old and new θ functions, eq. (5b)
f_1, f_2	quadratic approximations used in eq. (19)
$g(s)$	known function of s used as test of accuracy of numerical method used to solve for $\theta(s)$
I_1	integral to left of principal value integral
I_2	principal value integral
I_3	integral to right of principal value integral
$J(s)$	approximating function to numerator integral in eq. (3)
j	index number
j_{max}	maximum index for S_j
K	refers to the iteration number
N	number of points in interval $0 \leq s \leq 1.0$ such that $\Delta s = 1/(N - 1)$
P.V.	principal value
q	dummy variable of integration
S_j	integral sum symmetric about $q = s$ (eq. (16))
s	parameter
T_j	τ function j steps to left of $s = 1$
U_K	function θ on K^{th} iteration if there were no relaxation
x	dummy variable of integration in eq. (19)
α_1, α_2	limits of integration in eq. (19)
β	relaxation parameter
ϵ	variable approaching zero in defining $\theta(s)$ as principal value of integral
$\theta(s)$	one of two unknown functions of s
$\theta_K(s)$	function $\theta(s)$ at K^{th} iteration
$\theta_o(s)$	initial guess at function $\theta(s)$
$\tau(s)$	one of two unknown functions of s

$\tau^*(q)$ function τ obtained by expanding $\tau_K(q)$ in first three terms of its Taylor series about $q = s$

$\tau'(s)$ derivative of τ with respect to s

$\tau''(s)$ second derivative of τ with respect to s

$\tau_a(q)$ function $\tau(q)$ obtained by replacing τ^* by its finite difference analog

$\tau_K(s)$ function $\tau(s)$ on K^{th} iteration

$\tau'_K(s)$ derivative of τ with respect to s on K^{th} iteration

$\tau''_K(s)$ second derivative of τ with respect to s on K^{th} iteration

STATEMENT OF THE PROBLEM

The equations to be solved are

$$\left(\frac{1 - \sqrt{1 - s^2}}{s}\right)^3 e^{3\tau(s)} = \frac{\int_0^s \sin\left[\theta(q)\right] q \, dq}{\int_0^1 \sin\left[\theta(q)\right] q \, dq} \tag{1}$$

and

$$\theta(s) = \text{P. V.}\left[\frac{1}{\pi}\int_{-1}^{1} \frac{\tau(q)}{q - s} dq\right] \tag{2}$$

where P.V. is the principal value, and the right side of equation (2) is defined as

$$\lim_{\epsilon \to 0}\left[\frac{1}{\pi}\int_{-1}^{s-\epsilon} \frac{\tau(q)dq}{q - s} + \frac{1}{\pi}\int_{s+\epsilon}^{1} \frac{\tau(q)dq}{q - s}\right]$$

The quantities $\tau(s)$ and $\theta(s)$ are the unknown functions over the interval $0 \leq s \leq 1$. $\tau(-s) = \tau(s)$ defines τ in the interval $-1 \leq s \leq 0$. Equation (1) is nonlinear and of the

Volterra type (ref. 2). If taken as an equation for $\tau(s)$, it has a singularity at $s = 0$.

Equation (2) is linear and of the Fredholm type. It has a singularity at $q = s$.

The solution of the problem depends on the proper handling of these two singularities. The given conditions on $\tau(s)$ and $\theta(s)$ are

$$\tau(s) = \tau(-s) \text{ with } \tau(1) = 0$$

$$\theta(s) = -\theta(-s) \text{ with } \theta(0) = 0$$

where $\tau(s)$ is an even function of s, and $\theta(s)$ is an odd function of s.

METHOD OF SOLUTION

The numerical solution is obtained by using Picard's method of iteration. Equations (1) and (2) are rewritten as

$$e^{3\tau_K(s)} = \left(\frac{s}{1 - \sqrt{1 - s^2}}\right)^3 \frac{\int_0^s q \sin\left[\theta_{K-1}(q)\right] dq}{\int_0^1 q \sin\left[\theta_{K-1}(q)\right] dq} \tag{3}$$

$$U_K(s) = \text{P.V.}\left[\frac{1}{\pi}\int_{-1}^{1} \frac{\tau_K(q)dq}{q - s}\right] \tag{4}$$

The next estimate θ_K is defined by

$$\theta_K(s) = \theta_{K-1}(s) + \beta D(s) \tag{5a}$$

where D is the suggested correction to θ_{K-1} defined by

$$D(s) = U_K(s) - \theta_{K-1}(s) \tag{5b}$$

and β is the relaxation parameter.

The three steps of the iterative loop are briefly

(1) Solve equation (3) for $\tau_K(s)$.
(2) Solve equation (4) for $U_K(s)$.
(3) Combine θ_{K-1} and U_K by equations (5a) and (5b) to get a new estimate θ_K.
The starting estimate used was

$$\theta_o(s) = \frac{-\pi s}{4} \tag{6}$$

Two aspects of the iterative process will be discussed in greater detail in the following sections:

(1) In the section INTEGRATION TECHNIQUE, the principal value
(2) In the section ACHIEVEMENT OF CONVERGENCE, convergence of the iterative process by
  (a) Proper handling of the singularity in equation (3)
  (b) Proper choice of the relaxation parameter β of equation (5a)

INTEGRATION TECHNIQUE

To evaluate U_K from equation (4) requires a technique for dealing with the principal value of an integral. Since the singularity is at $q = s$, and τ_K is known at all s points, -1, $-1 + \Delta s$, . . . , $1 - \Delta s$, and 1, the integral is split up into at most three parts, with the second, I_2, containing the singularity:

$$\begin{aligned} U_K &= I_1 + I_2 + I_3 \\ &= \int_{-1}^{s-\Delta s} + \int_{s-\Delta s}^{s+\Delta s} + \int_{s+\Delta s}^{1.00} \end{aligned} \tag{7}$$

The I_1 and I_3 integrals may be evaluated by Simpson's rule. The I_2 integral is treated as follows:

For $s = 0$, Δs, $2\Delta s$, . . . , and $1 - \Delta s$ expand τ_K in a Taylor series around $q = s$ and truncate after three terms to get the approximation τ^*:

$$\tau^*(q) = \tau_K(s) + \tau_K'(s)(q - s) + \tau_K''(s)\frac{(q - s)^2}{2} \tag{8}$$

Replace equation (8) by its finite difference analog to get the approximation $\tau_a(q)$:

$$\tau_a(q) = \tau_K(s) + \frac{\tau_K(s + \Delta s) - \tau_K(s - \Delta s)}{2\Delta s} (q - s)$$

$$+ \frac{\tau_K(s + \Delta s) - 2\tau_K(s) + \tau_K(s - \Delta s)}{\Delta s^2} \frac{(q - s)^2}{2} \tag{9}$$

Replace $\tau_K(q)$ in I_2 by $\tau_a(q)$ to get

$$I_2(s) = \frac{\tau_K(s)}{\pi} \int_{s-\Delta s}^{s+\Delta s} \frac{dq}{q - s} + \frac{\tau_K(s + \Delta s) - \tau_K(s - \Delta s)}{2\Delta s \pi} \int_{s-\Delta s}^{s+\Delta s} dq$$

$$+ \frac{\tau_K(s + \Delta s) - 2\tau_K(s) + \tau_K(s - \Delta s)}{\pi \Delta s^2} \int_{s-\Delta s}^{s+\Delta s} \frac{q - s}{2} dq \tag{10}$$

The principal value of the first integral is zero. The second integral is $2\Delta s$, and the third integral vanishes to yield

$$I_2(s) = \frac{\tau_K(s + \Delta s) - \tau_K(s - \Delta s)}{\pi} \quad \text{for} \quad s = 0,\ \Delta s,\ \ldots,\ 1 - \Delta s \tag{11}$$

At $s = 1$, I_2 reduces to

$$\int_{1-\Delta s}^{1} \frac{\tau_K(q) dq}{q - 1}$$

For this point, all derivatives appearing in the Taylor series expansion of $\tau_K(q)$ about the point $q = 1$ are one-sided. Use T_j to denote $\tau_K(1 - j\Delta s)$; then, since $T_o = 0$, the equations for the finite difference analogs of τ' and τ'' become

$$\tau_K'(1) = \frac{-T_1}{\Delta s} + 0.5 \left(-\frac{T_1}{\Delta s} - \frac{T_1 - T_2}{\Delta s} \right) = \frac{T_2 - 4T_1}{2\Delta s} \tag{12}$$

and

$$\tau_K''(1) = \frac{-2T_1 + T_2}{\Delta s^2} + \left(\frac{-2T_1 + T_2}{\Delta s^2} - \frac{T_1 - 2T_2 + T_3}{\Delta s^2}\right)$$

$$= \frac{-5T_1 + 4T_2 - T_3}{\Delta s^2} \tag{13}$$

Substitute equations (12) and (13) in equation (8) for $s = 1$ to get the approximation $\tau_a(q)$:

$$\tau_a(q) = \frac{T_2 - 4T_1}{2\Delta s}(q - 1) + \frac{-5T_1 + 4T_2 - T_3}{\Delta s^2}\,\frac{(q - 1)^2}{2} \tag{14}$$

This formula is used in the equation for

$$I_2(1) = \frac{1}{\pi}\int_{1-\Delta s}^{1} \frac{\tau_a(q)dq}{q - 1} \tag{15a}$$

to yield

$$I_2(1) = \frac{-3T_1 - 2T_2 + T_3}{4\pi}$$

$$= \frac{-3\tau_K(1 - \Delta s) - 2\tau_K(1 - 2\Delta s) + \tau_K(1 - 3\Delta s)}{4\pi} \tag{15b}$$

The evaluation of I_1 and I_3 was handled first by Simpson's rule formulas. Later a second method was applied. Both methods yielded sufficient accuracy for this problem and for the test case. But other problems involving principal value integration may benefit from the more careful technique of the second method. Therefore, its derivation is given here.

Note that the I_1 integrand $\tau_K(q)/(q - s)$ is large and of one sign near $s - \Delta s$, whereas the I_3 integrand is large and of opposite sign near $s + \Delta s$. Using differences of large numbers may increase the truncation error. Combining symmetric portions of

I_1 and I_3 should reduce the truncation error.

Let S_j be the integral sum

$$S_j = \int_{s-(j+1)\Delta s}^{s-j\Delta s} \frac{\tau_K(q)dq}{q - s} + \int_{s+j\Delta s}^{s+(j+1)\Delta s} \frac{\tau_K(q)dq}{q - s} \tag{16}$$

where $j = 1, 2, 3, \ldots, j_{max}$ and

$$j_{max} = N - i \tag{17}$$

where i is such that $s = (i - 1)\Delta s$ and N is the number of points in $0 \leq s \leq 1$ such that $\Delta s = 1/(N - 1)$. Then the sum of I_1 and I_3 becomes

$$I_1(s) + I_3(s) = \int_{-1}^{2s-1} \frac{\tau_K(q)dq}{q - s} + \sum_{j=1}^{j_{max}} S_j \tag{18}$$

where much of the important cancellation of large numbers occurs in the computing of S_j for low j.

The S_j are computed as follows. Make use of a change of variables to obtain for equation (16)

$$S_j = \int_{-\alpha_2}^{-\alpha_1} \frac{f_1(x)dx}{x} + \int_{\alpha_1}^{\alpha_2} \frac{f_2(x)dx}{x} \tag{19}$$

where $\alpha_1 < \alpha_2$.

Fit a parabola through the three values of f_1 at $x = -\alpha_2$, $-\alpha_1$, and $\alpha_2 - 2\alpha_1$ to get

$$f_1(x) = a_1 + b_1x + c_1x^2 \tag{20}$$

and fit a parabola through the three values of f_2 at $x = 2\alpha_1 - \alpha_2$, α_1, and α_2 to get

$$f_2(x) = a_2 + b_2x + c_2x^2 \tag{21}$$

Then S_j may be integrated in closed form to get

$$S_j = a_1 \ln \frac{\alpha_1}{\alpha_2} + b_1(\alpha_2 - \alpha_1) + \frac{c_1}{2}\left(\alpha_1^2 - \alpha_2^2\right)$$

$$+ a_2 \ln \frac{\alpha_2}{\alpha_1} + b_2(\alpha_2 - \alpha_1) + \frac{c_2}{2}\left(\alpha_2^2 - \alpha_1^2\right) \tag{22}$$

or

$$S_j = (a_2 - a_1) \ln \frac{\alpha_2}{\alpha_1} + (b_1 + b_2)(\alpha_2 - \alpha_1) + \frac{(c_2 - c_1)}{2}\left(\alpha_2^2 - \alpha_1^2\right) \tag{23}$$

As a test, this integration process was applied to the function

$$g(s) = \int_{-1}^{1} \frac{(1 - q^2)dq}{q - s} \tag{24}$$

The numerator of the integrand in equation (24) is like τ in being even and zero at $q = 1$. The exact value of $g(s)$ is

$$g(s) = (1 - s^2)\ln \frac{1 - s}{1 + s} - 2s \tag{25}$$

Note that $g(1) = -2$.

The following table gives the integral of equation (24) for values of s together with the correct value of $g(s)$. The integration technique is clearly accurate.

Distance parameter, s	Integral of eq. (24)	
	Numerical value	Correct value
0.1	-0.39866259	-0.39866398
.2	-.78924440	-.78924650
.3	-1.1633240	-1.1633257
.4	-1.5117280	-1.5117302
.5	-1.8239566	-1.8239592
.6	-2.0872253	-2.0872283
.7	-2.2846440	-2.2846465
.8	-2.3909990	-2.3910008
.9	-2.3594423	-2.3594434
.99	-2.0853368	-2.0853368

ACHIEVEMENT OF CONVERGENCE

One action necessary to achieve convergence is to handle properly the singularity in the equation for τ at $s = 0$. In equation (3), the term in brackets can be written as

$$\frac{s}{1 - \sqrt{1 - s^2}} \sim \frac{s}{1 - \left(1 - \frac{1}{2}s^2 - \frac{1}{8}s^4\right)} = \frac{2}{s\left(1 + \frac{s^2}{4}\right)} \tag{26}$$

Its cube behaves like s^{-3} near $s = 0$.

Since the upper integral on the right side of equation (3) behaves like

$$J(s) = \int_0^s q \sin(Kq)dq \sim \int_0^s Kq^2\, dq = \frac{Ks^3}{3} \tag{27}$$

it becomes obvious that the right side of equation (3) behaves like s^3/s^3 in the vicinity of zero.

Because of the singularity at zero, the numeric difficulties become more severe as s approaches zero. Any errors generated in the evaluation of the upper integral are magnified when divided by s^3 and lead to instability. Since τ is known to be an even function of s with zero derivative at $s = 0$, it seems logical to impose proper behavior in the neighborhood of zero numerically.

Another necessary action to achieve convergence was to choose the proper relaxation parameter β. (See eq. (5a) for its definition.)

The use of $\beta = 1.0$ as the coefficient of the suggested change led to an oscillation of about 5 percent in the value of $\tau(0)$ (see fig. 1). Figure 2 shows how the oscillation drops to less than 3 percent when β is 0.9. Figure 3 shows how the oscillation drops to less than 0.5 percent when β is 0.7. Figure 4 shows that with $\beta = 0.5$ the oscillation disappears, and convergence is very good after 18 iterations.

The converged shape of the $\tau(s)$ function is shown in figure 5. The converged shape of $\theta(s)$ is shown in figure 6.

CONCLUDING REMARKS

This report shows that there were two necessary steps required to achieve the solution of the original system of two integral equations. The steps taken were

(1) The $\tau_K(s)$ function was isolated in the neighborhood of $s = 0$, and a smoothing process was applied to remove the erratic behavior of τ_K near $s = 0$.

(2) A relaxation factor equal to about 0.5 was used to remove the oscillations of the $\tau_K(s)$ curve.

This method, then, provided an accurate solution to the system of two integral equations, as is shown in the section of the report on accuracy.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, February 10, 1971,
129-04.

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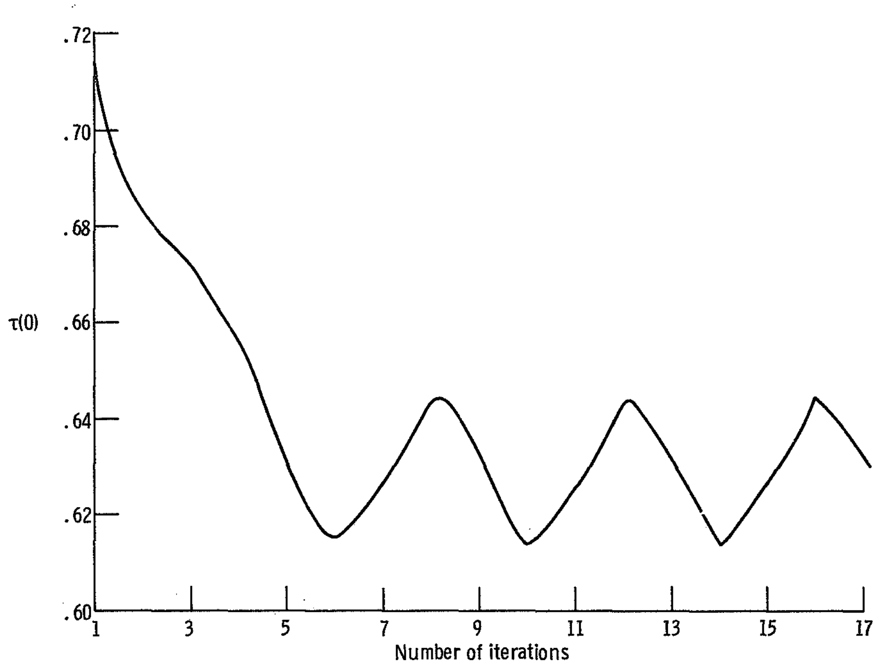

Figure 1. - Behavior of τ(0) value with successive iterations with relaxation parameter $\beta = 1$.

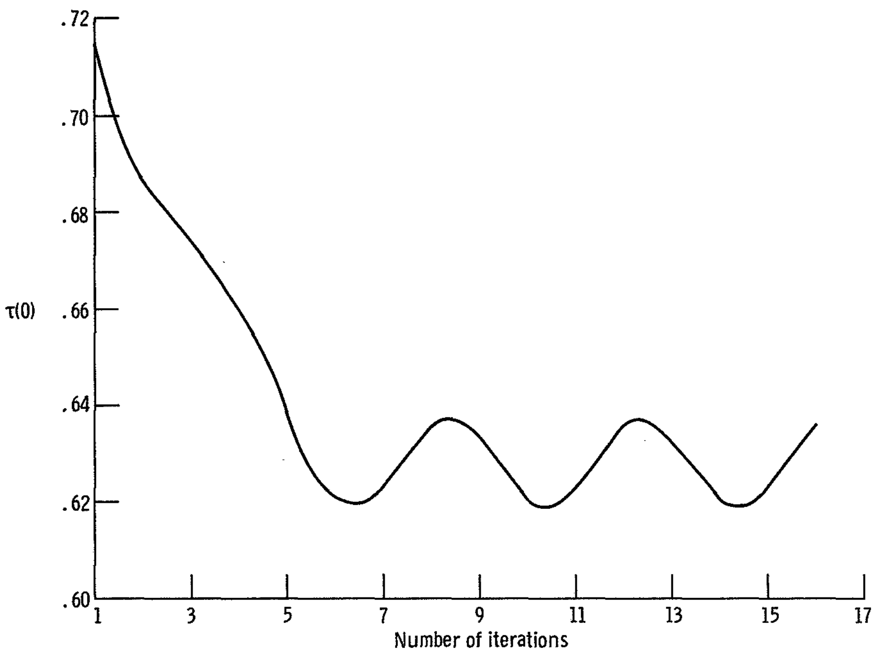

Figure 2. - Behavior of τ(0) value with successive iterations with relaxation parameter $\beta = 0.9$.

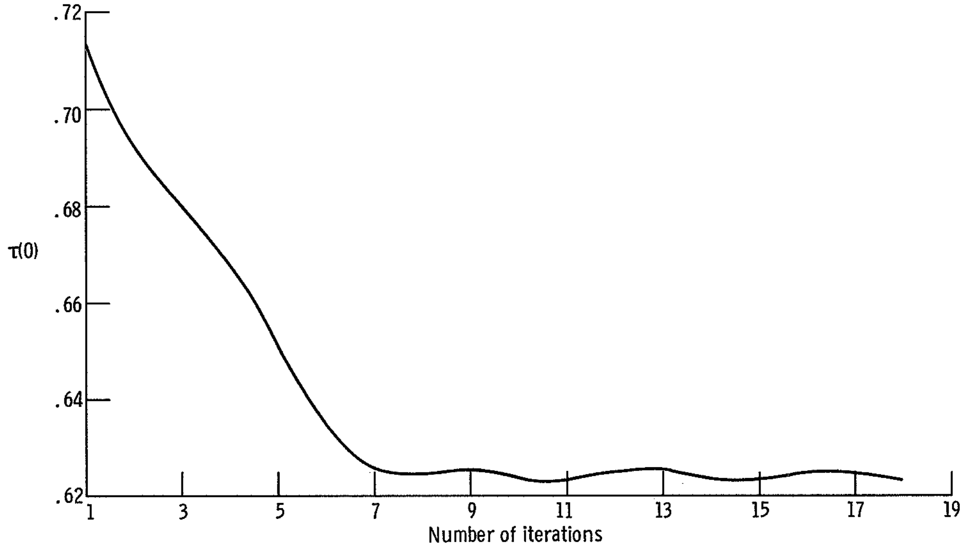

Figure 3. - Behavior of $\tau(0)$ value with successive iterations with relaxation parameter $\beta = 0.7$.

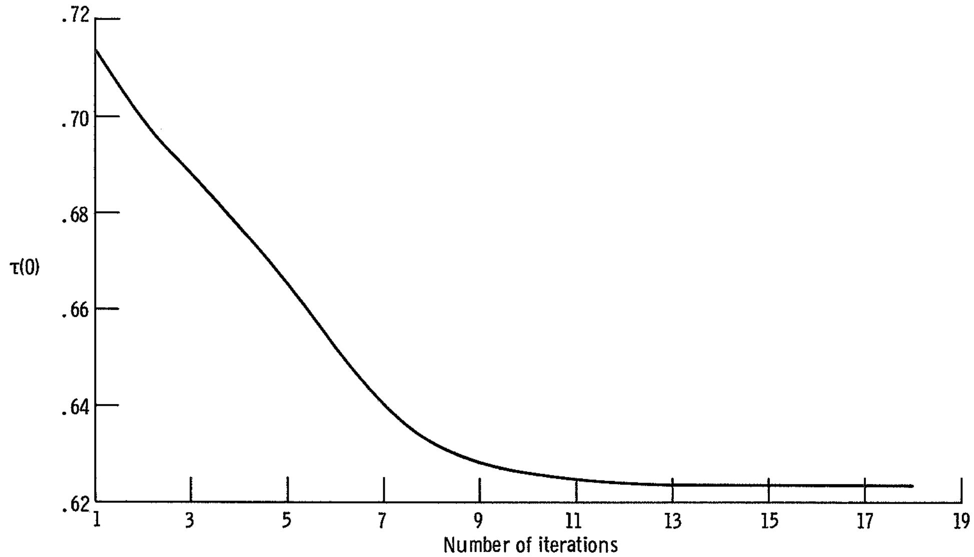

Figure 4. - Behavior of $\tau(0)$ value with successive iterations with relaxation parameter $\beta = 0.5$.

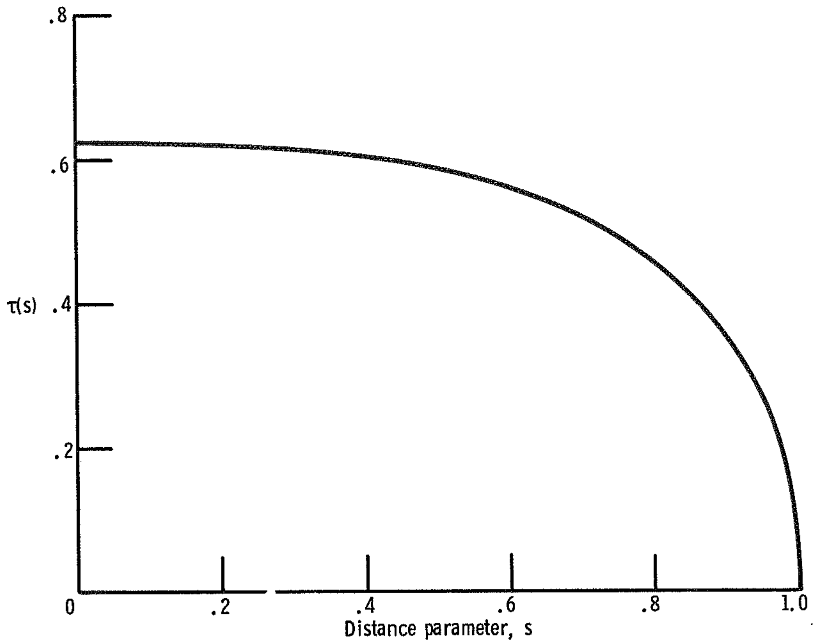

Figure 5. - Plot of converged τ as function of distance parameter s.

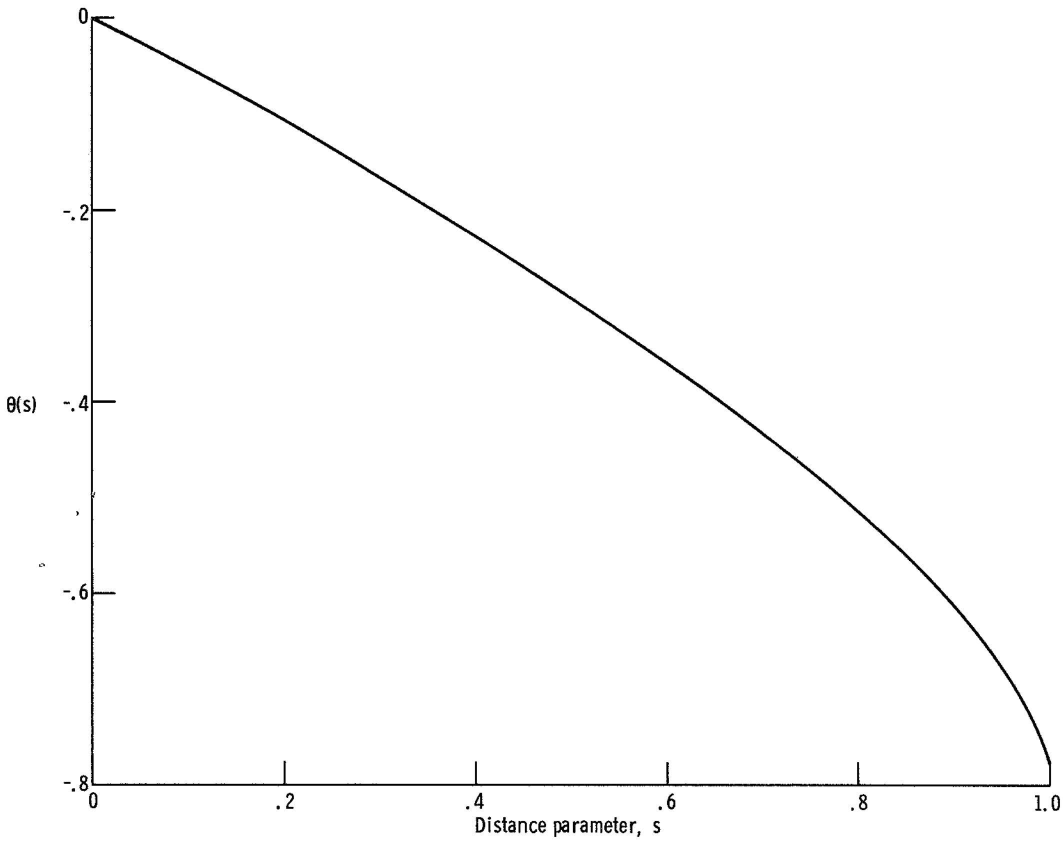

Figure 6. - Plot of converged θ as function of distance parameter s.

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—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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