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# ADIABATIC INVARIANCE

## THE EVOLUTION OF A CONCEPT

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# Adiabatic Invariance

## The Evolution of a Concept

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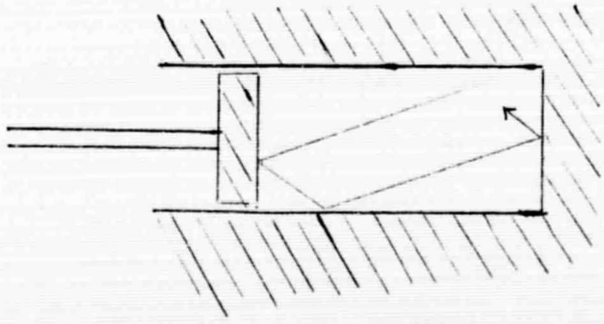
I would like to present here today a case history in physics, a review of the evolution of a concept. As you will see, the story begins with matters that are now old history -- and yet it is not a finished affair, there still remain unresolved problems and unexplored avenues, which is the way a physicist likes his problems to be.

One difference between adiabatic invariance and other concepts in physics is that here is an idea that started with a quantum problem and ended in classical mechanics -- or at least this is true for those aspects of it which will be discussed today. Back around the turn of the century physicists were surprised by the discrete character of energy exchange between matter and electromagnetic radiation. In black-body equilibrium, say, or in the photo-electric effect, it appeared that radiation of frequency $\nu$ tranferred its energy only in amounts that satisfied

$$E / \nu = h$$

The question arose, what did all this mean?

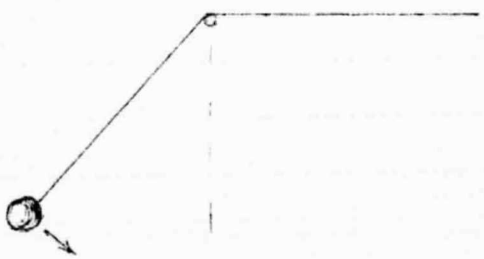
Now one possible clue that was explored was the way radiation changes its energy <u>without</u> interaction with matter. Suppose we have a perfectly reflecting enclosure with a perfectly reflecting piston at one end, filled with electro-magnetic radiation:



As the piston is pushed forward, it compresses the radiation gas adiabatically and two things change: the <u>energy</u> changes, since the piston does work against the radiation pressure, and the <u>frequency distribution</u> changes, due to the Doppler effect by reflection from a moving piston. This is how Wien in 1893 derived his displacement law, for radiation in equilibrium with a given temperature.

A few years after that Rayleigh noted that it may be convenient to assume here that the cavity is rectangular and to treat the standing modes of electromagnetic waves in it, and this led him to the Rayleigh-Jeans formula (actually, it was Rayleigh who derived it -- Jeans only pointed out that it included an unnecessary factor of 8 ). Ehrenfest around 1910 examined the effect of adiabatic compression in <u>this</u> case and found that for each wave mode, for infinitely slow compression, the ratio (energy)/(frequency) stayed constant.

Now Rayleigh, in 1902, had also examined a mechanical system with rather similar properties (earlier, Boltzmann and others had already done some work in this direction). Suppose we have a pendulum on a string that is being slowly drawn up through a hole in the ceiling:



As the string is shortened, two things occur: the frequency increases, because the shorter a pendulum is, the shorter its period -- and so does the energy, since work is being done against the centrifugal force of the oscillation.

In this process, the ratio (energy)/(frequency) is only an approximate constant, but it has the following interesting property. Suppose for the sake of definiteness that the string is shortened to half its length. Then we can make the variation of the ratio during the entire process arbitrarily small by making the shortening process sufficiently slow -- that is, by stretching it out over a sufficiently long time.

Since this resembled a property of the adiabatically compressed radiation gas, Ehrenfest called this type of conservation adiabatic invariance. I will call this the "old" definition of adiabatic invariance, since -- as you will see -- there also exist other definitions. Before going further, let me say a few words about what exactly is happening in this particular example.

If the length of the pendulum is kept constant, the motion is periodic. The energy $E$, the angular frequency $\omega$ and their ratio -- which we will denote by $J$ -- are all exact constants of the motion, and that's all.

On the other hand, if the string is drawn upwards at an appreciable rate, the motion is no longer exactly periodic, and small changes must be made in the equations of motion to take this factor into account. We then say that we have a perturbed pendulum motion. There exist approximate methods for treating such a motion -- so-called perturbation methods -- which can be used provided the difference from pure pendulum motion is never very great. In the present case this reduces to the requirement that the relative change in the string's length per oscillation is small.

Now perturbation methods usually express such requirements by means of some constant $\varepsilon$, which has to be much less than unity in order for the method to work:

$$\varepsilon \ll 1$$

In this case

let $\tau$ be the oscillation period

let $T$ be the time required to pull up the string by one half (or to $1/e$ of its length, etc. )

Then one may take

$$\varepsilon = \tau / T$$

Let us furthermore denote by small $\delta$ changes over a <u>single</u> <u>period</u> and by capital $\Delta$ changes effected over the entire drawing-up process.

In a single oscillation, the string length $L$ changes by an amount of order $\varepsilon L$ , and similar changes occur in $E$ , $\omega$ and $J$ :

$$\delta L = O(\varepsilon L)$$

$$\delta E = O(\varepsilon E)$$

$$\delta\omega = O(\varepsilon\omega)$$

and

$$\delta J = O(\varepsilon J)$$

There exists one important difference, however, between $\delta J$ and the other small deltas: $\delta L$ is always negative (the string steadily gets shorter), $\delta E$ and $\delta\omega$ are always positive -- but it may be shown that $\delta J$ , to the first approximation, oscillates with the pendulum motion and has zero average, or anyway, an average of higher order:

$$\langle \delta J \rangle = O( \varepsilon^2 J )$$

The entire drawing-up process contains $\varepsilon^{-1}$ periods, which is a large number. In calculating the accumulated change in $J$ during this time, we may replace $\delta J$ by its oscillation-average, leading to

$$\Delta J = \overbrace{\varepsilon^{-1}}^{\text{number of periods}} \langle \delta J \rangle = O(\varepsilon J)$$

Thus $\Delta J$ may be made arbitrarily small by making $\varepsilon$ small enough, that is, by pulling the string slowly enough. Please note that this is a property of $J$ alone, not of $E$ , $\omega$ or of any other constants of the pure pendulum motion: only $J$ gives us the extra factor of $\varepsilon$ , because only it is conserved <u>on the average</u> to one order in $\varepsilon$ better than it is conserved <u>instantaneously.</u>

Before continuing let me point out that this is a rather remarkable result: we get an approximate constant of the perturbed motion <u>without ever having to know what the perturbation is</u>. All that is required is that the motion be periodic and that the perturbation be slow and not resonate with the basic periodicity. If you think about it you will realize that this is a remarkable bargain -- you almost get something for nothing. I know of no other theory that is so generous.

Ehrenfest guessed -- and so did Lorentz and Einstein, who were also involved in this -- that quantised variables were those that in the classical limit were adiabatically conserved. The problem now became how to identify such adiabatic invariants in other periodic mechanical systems. In order to examine this point in more detail, I must now step back and review what classical mechanics was doing at that time, and in particular trace the evolution of celestial mechanics which -- as you will see -- is quite relevant here.

The first general theory of mechanics was due to Newton and was based on the concept of force. Combining it with the law of gravitation, Newton was able to account for Kepler's laws of planetary motion and ever since that time, much of classical mechanics was directed towards analyzing the motion of celestial bodies, because here was a problem in which all factors were known and all were easily observed.

An ambitious test of Newton's theory was undertaken in 1705 by one of his contemporaries, the astronomer Edmund Halley. Halley guessed that the comets of 1531, 1607 and 1682 were all the same object and he confidently predicted its return in 1758. In calculating the orbit of the comet, Halley took into account the attractions of Jupiter and Saturn and thus was the first to use perturbation theory in celestial mechanics. He died in 1742 and thus did not see the event which he predicted; neither did he realize that he had made an error in his perturbation calculation and that therefore the comet was behind his schedule -- it was first seen on Christmas night, 1758, and passed perihelion only in March 1759.

Perturbation methods improved steadily, however, and even before the predicted return of Halley's comet, Clairaut had already calculated its perihelion time within a month of the correct date.

The real advances, though, came in the 19th century, after William Rowan Hamilton reformulated Newton's mechanics, basing it on the concept of <u>energy</u> rather than force. He showed that there exists, for a large class of motions, a function — the Hamiltonian -- which not only can be identified with the energy, but which also contains in it all the information about the evolution of the mechanical system.

The arguments of this function are generalized momenta and generalized coordinates, usually denoted by p-s and q-s ; I won't elaborate on this, since you all are probably familiar with Hamiltonians. The Hamiltonian of, say, planetary motion, might have a form like

$$H = H^{(0)}(\underline{p}, \underline{q}) + \varepsilon H^{(1)}(\underline{p}, \underline{q}) + \varepsilon^2 H^{(2)} + \dots$$

where $H^{(0)}$ represents the planet's Kepler motion around the sun, the correction proportional to $\varepsilon$ represents perturbations due to other planets, and there may exist terms of higher orders as well (underlined quantities are vectors lumping together the p-s and the q-s, the so-called <u>canonical variables</u>) .

Based on the Hamiltonian, Jacobi devised a partial differential equation, the Hamilton-Jacobi equation, which when solved gives the complete evolution of the system. Specifically, it combines the canonical variables into new quantities, which are either constants of the motion or grow linearly in time.

In practice one soon finds out that if you can solve a problem by elementary methods, the Hamilton-Jacobi equation can be solved, too -- but if not, nothing helps. The method therefore is not an all-purpose shortcut to a solution: its real usefulness is as a good starting point for perturbation schemes. One such scheme is due to Poincare and Von Zeipel[1] and requires the basic system to have a periodic character -- a property that is satisfied by the Kepler motion. It goes as follows.

First let us choose the initial variables so that each pair $(p_i, q_i)$ corresponds to a different periodicity of $H^{(0)}$. If $H^{(0)}$ does not have the maximum number of periodicities, it may lack the corresponding variables: for instance, $H^{(0)}$ for the Kepler problem contains only one pair of canonical variables, because the motion represented by it has only one periodicity.

Next, solve the Hamilton-Jacobi equation for the unperturbed part $H^{(0)}$ alone. This gives the problem a new Hamiltonian formulation, with new canonical variables -- we will denote them by capital P-s and Q-s -- that are either constant or linear in time. We choose the new variables so that the $Q_i$ are the only variables that may be linear in time, and that they all be angles, of the sort that enters the problem only as the argument of periodic functions -- say, of sines and cosines. With this choice, even though the $Q_i$ may grow without limit, all observable quantities merely oscillate: by contrast, a steadily increasing variable that is not an angle usually spells trouble.

It turns out that this is not an unreasonable demand and can be met. The new momenta conjugate to these "angle variables" are usually denoted by capital J-s (rather then P-s) and have the form

$$J_i = \oint p_i \, dq_i$$

where the integration is over a complete period of the appropriate degree of freedom. The $J_i$ are called <u>action variables</u>.

<u>Now</u> we turn our attention to the <u>entire</u> Hamiltonian, including correction terms of order $\varepsilon$ and higher, and re-formulate it in terms of the action and angle variables. After this there exists a rather straightforward procedure for cranking out, order by order, a near-identity transformation to yet another set of canonical variables, which we will distinguish by asterisks,

$$J_i^* = J_i + \varepsilon J_i^{*(1)} + \varepsilon^2 J_2^{*(2)} + \ldots$$

$$Q_i^* = Q_i + \varepsilon Q_i^{*(1)} + \varepsilon^2 Q_i^{*(2)} + \ldots$$

such that for the real motion -- perturbation included -- the $J_i^*$ are constant while the $Q_i^*$ may evolve linearly in time. In other words, we solve the Hamilton-Jacobi equation <u>in two steps</u>, first for $H^{(0)}$ alone -- this gives us $J_i$ and $Q_i$ -- and then for the entire $H$, giving the asterisk-marked variables as solutions.

This is standard operating procedure in celestial mechanics, and scientists have carried it to as many orders in $\varepsilon$ as their patience could stand. More recently computers have been programmed to do the algebra, which saves wear and tear on the nerves.

Now let us return to adiabatic invariants, where Ehrenfest and his colleagues were trying to extend results derived for the drawn-up pendulum to general periodic systems. They found and proved that in such systems, the action variables

$$J_i = \oint p_i \, dq_i$$

are adiabatically conserved. This led to the Bohr-Sommerfeld theory of the atom, in which such integrals were quantised, and as you know this theory gave quite useful results for one-electron systems but completely failed for the Helium atom. For details of all this -- including perturbation theory, the drawn-up pendulum, adiabatic invariance and so forth -- I recommend Born's book "Mechanics of the Atom" written in 1924 (a 1960 edition exists).[2] This was probably the last attempt to attack quantum problems by using classical adiabatic invariance.

Soon after the book appeared the real breakthrough occured, when Schrödinger and others decided to look not at the action variables but at the Hamilton-Jacobi equation that generated them. They found that if that equation is regarded as the limiting equation describing the "geometrical optics" of a wave, a consistent quantum theory could be derived, and you all know that this theory has been very successful indeed. Among other things, quantum theory has its counterpart of the adiabatic invariance properties described earlier (explaining among other things the Bohr-Sommerfeld atom), but this talk is concerned with classical motion and therefore we will not continue in this direction any more.

Instead, let us look further into the significance of classical adiabatic invariance. In celestial mechanics we found that the action variable $J$ was the first term in a series giving the true invariant $J^*$

$$J^* = J + \varepsilon J^{*(1)} + \varepsilon^2 J^{*(2)} + \dots$$

We now ask: is the action variable of the drawn-up pendulum also merely the first term in an infinite series giving an exact invariant -- and is the omission of higher order terms the reason why it is only approximately conserved?

The answer is, yes. But it is not quite so simple.

Yes, there indeed exists such a series, giving (if it converges) an exact invariant $J^*$, which is what people nowadays usually mean by adiabatic invariant (however, some stick with the old definition and there also exists a third definition, all of which causes occasional confusion).

Furthermore, it turns out that the first correction term $J^{*(1)}$ is, to the lowest approximation, a purely oscillating quantity with zero average. If one averages over the oscillation one therefore gets

$$J^* = \langle J \rangle + O(\varepsilon^2)$$

Since $J^*$ is a constant, this shows that $J$ is conserved, on the average, to order $\varepsilon^2$, with the consequences that have already been described.

But this is not the entire story, because this is not the same kind of perturbation as one finds in celestial mechanics. In celestial mechanics, say we have a planet circling the sun and perturbed by Jupiter. Then $H^{(0)}$ represents the energy of its motion around the sun while the correction term $\varepsilon H^{(1)}$ represents the influence of Jupiter. Since the attraction of Jupiter may be a thousand times smaller than that of the sun, at all times, the total energy $H$ is always close to $H^{(0)}$. We say that the perturbation is small.

For the pendulum, the perturbation is <u>not</u> small: if the string is shortened by one half, the energy changes considerably. In this case, the perturbation is not contained in small terms added to H but in an explicit time dependence of H , and this dependence must be "<u>slow</u>" , that is

$$\frac{\partial}{\partial t} H ( q, p, t ) = O ( \varepsilon [ H/\tau ] )$$

(the period $\tau$ is included here to make the dimensionality correct)

It would take me too long to describe how such <u>adiabatic perturbations</u> are treated. Let me just state that a theory can be developed for them(3) that exactly parallels the theory for <u>small</u> perturbations. The slow dependence, by the way, may be on the time t or also on some of the canonical p-s and q-s , but I won't elaborate on this. The end result is that J is indeed the first term of a series for an exact invariant $J^*$ , just as in celestial mechanics.

During the 1930-s, classical adiabatic invariance was regarded as little more than in interesting problem of historic interest. But then it suddenly reappeared from quite a different direction.

In the 40-s a Swedish scientist by the name of Hannes Alfven -- you may recall that he shared the Nobel Prize last year -- got interested in the motion of charged particles in a magnetic field. His main interest was in the motion of particles causing the polar aurora, which is a subject that comes quite naturally to someone living in Sweden, where auroras are often seen.

Alfvén found that particles of low energy tended to follow magnetic field lines while spiralling around them. You can easily prove that this happens if the magnetic field is constant in direction and magnitude: the momentum component $\underline{p}_\perp$ orthogonal to field lines has a constant magnitude and rotates at a constant rate, so that the particle spirals around its guiding field line along a helix of circular cross section. The radius of this helix, the so-called gyration radius $\rho$, gets smaller and smaller as the particle's energy decreases.

If the field is slightly inhomogeneous -- field lines curve or converge slightly -- the magnitude of $\underline{p}_\perp$ is no longer constant. Alfvén however showed that the quantity

$$M = p_\perp^{2} / B$$

(with B the field intensity) is an approximate constant of the motion. He called it (or rather, a quantity proportional to it) the magnetic moment of the particle, since if you replace the particle circling the field by a small wire loop carrying the same amount of current, M is proportional to the magnetic moment of this loop.

At first Alfvén did not apparently realize that he was dealing with an adiabatic invariant: in his book "Cosmical Electrodynamics" that appeared in 1950, the term is never mentioned.[4] I don't know who first realized the connection: the earliest reference I know of is the 1951 English edition of "The Classical Theory of Fields" by Landau and Lifshitz,[5] where this is given, of all things, as an excercise for the student!

Briefly, what happens is the following. The slightly inhomogeneous magnetic field, with field lines slowly converging or curving, may be regarded as a perturbed version of the homogeneous field, where the particle gyrates around field lines with strict periodicity, while sliding along them with constant velocity. In a homogeneous field these two motions may be separated, and one finds the Hamiltonian for the gyration to be very much like that of a harmonic oscillator, leading to an adiabatic invariant that turns out to be proportional to $M$ . The perturbed motion is termed guiding center motion and has been the subject of much research.

If all the forces are magnetic, the energy $E$ is conserved, for magnetic forces are always orthogonal to the velocity and can do no work. The total momentum $p$ is then also conserved. Because a component of a vector cannot exceed the magnitude of the vector itself, we get an inequality

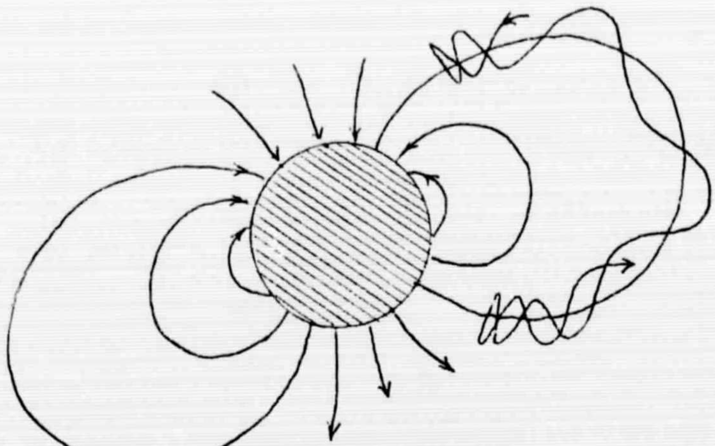
$$M = \frac{p_{\perp}^2}{B} \leq \frac{p^2}{B}$$

or

$$B \leq p^2/M = B_{max}$$

That is, if $M$ is conserved there exists for each particle a maximum field intensity $B_m$ (we abbreviate the subscript!) beyond which it cannot penetrate. If in its motion along a field line the particle approaches fields exceeding $B_m$, its advance is slowed down and finally stopped and reflected back at the point where $B$ equals $B_m$.

Consider now a radiation-belt particle in the earth's magnetic field. To a good approximation this field resembles the field of a dipole, with field lines arching out, from one hemisphere to the other :



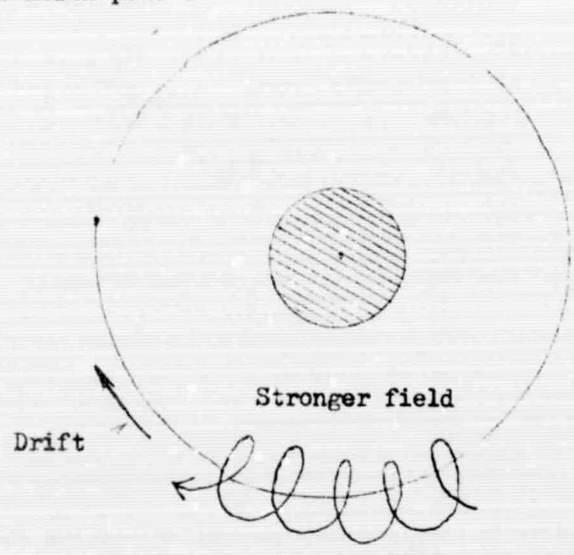
On any dipole field line the field intensity is highest near the surface of the earth and weakest in the equatorial region, where the distance from the earth is greatest. A particle of suitable M can be trapped on such a field line, bouncing back and forth between regions of higher field intensity, as illustrated here.

It was the plasma physicist Marshall Rosenbluth who first realized,[6] in the early 50-s , that since this motion is periodic, it too ought to have an adiabatic invariant associated with it, namely

$$J = \oint p_{//} \, ds$$

Here $p_{//}$ is the momentum component parallel to the field and the integration is along the guiding field line over one entire "bounce period". Usually J is called the second invariant or the longitudonal invariant, and in a moment we will see that it is indeed a most useful concept.

There exists yet another periodicity and to see its cause, let us pick an orbit that stays fairly close to the equator and observe it from above -- say, from the north pole :




As said before, the particle spirals around field lines, which are perpendicular to the picture. Since the field is stronger closer to the earth, the inner portion of the orbit will curve a little more tightly than the outer part (the drawing greatly exaggerates this effect). The net result, as you can see, will be a so-called drift motion sideways, ultimately carrying the orbit all the way around the earth.

If the field is axisymmetrical, the drift motion is axisymmetrical too. In actual fact the earth's field is only approximately symmetrical and has appreciable non-dipole components. If a particle then starts from a given field line in this field, it becomes a real problem to determine onto which of the adjacent field lines it latches on next.

The answer, however, is easily guessed if one knows about adiabatic invariants: the particle moves to that field line on which the value of $J$, for a particle that is reflected at the same field intensity $B_m$, remains unchanged. The answer is in general unique and it explains, perhaps, why NASA spends a great deal of computer time on numerical integrations along geomagnetic field lines.

Now the drift motion ultimately carries the particle all the way around the earth, so you get a third periodicity and — you guessed it — a third invariant. That's its usual name -- "the third invariant" -- it was introduced by Northrop and Teller in 1960[(6)] and is quite useful for handling time-dependent field perturbations, but I do not have the time to describe the details.

To give you some quantitative feeling for this motion, let me add that a typical 1 Mev electron about 5000 Km. from the earth --

- gyrates around its guiding field line about a million times each second
- makes about 10 back-and-forth bounces per second, and
- takes over half an hour to drift once around the earth

during which time it actually covers about 500,000,000 Km., since a 1 Mev electron is rather relativistic. On July 9th, 1962, the U.S. Air Force exploded a hydrogen bomb over the Pacific, creating an intense belt of trapped electrons. Three minutes later a radio observatory in Peru detected synchrotron radiation from this belt,[(7)] peaking about 6 minutes after the explosion, and this agrees fairly well with the calculated drift times of such particles.

I may add that this artificial belt required over 5 years to decay, during which time some of its particles must have covered several light years. This is a rather surprising amount of stability for a motion described by the first term of a series that may or may not converge. I would like to devote the rest of my talk to these questions of stability and convergence, but let me first warn you that some of what I am going to say may be more speculation than fact.

To make things simple, let us concentrate on the magnetic moment $M$. As was said earlier, $M$ is the first term of a series

$$M^* = M + \varepsilon M^{(1)} + \varepsilon^2 M^{(2)} + \dots$$

By now, the first-order correction $M^{(1)}$ has been derived and the second one is known for some special kinds of field, though I am not aware of anyone needing these corrections for a practical purpose. One may ask what $\varepsilon$ represents here. In the present case, the slow dependence is on <u>location</u>, not on time (though slow dependence on time could also be added), so we require that the <u>scale length</u> over which the field varies is much larger than the gyration radius $\rho$. In figures

$$\partial B_i / \partial x_j \ll B / \rho \qquad \text{(any i, j)}$$

This is called <u>Alfvén's criterion</u>. One might extract an $\varepsilon$ of sorts from this, but it would take me too long to discuss the full implications.

Now suppose that the series <u>does</u> converge. It may then be expected to have a certain — quite finite — radius of convergence in $\varepsilon$. If so, as one increases $\varepsilon$ -- say, by increasing the energy of the particle, which makes $\rho$ grow -- at a certain point the series begins diverging and suddenly everything breaks down. Even the first term M then is no longer an approximation to a constant of the motion. This is called <u>breakdown of adiabatic invariance</u> : it has been observed (if this is the word) in computer simulations of particle motion,[8][9] and it indeed happens rather suddenly.

Next we consider a different point. I have promised to tell you about the 3rd definition of adiabatic invariance, and here perhaps is a good place to do so. Consider again a case of slow time dependence -- such as our pendulum -- with invariant

$$J^* = J + \varepsilon J^{(1)} + \varepsilon^2 J^{(2)} + \dots$$

Suppose that the system starts from one unperturbed state, the perturbation begins smoothly, it carries the system to a different state, switches off smoothly and finally leaves the system undisturbed in its new state. Then by the definition of Chandrasekhar,[10] Lenard,[11] Littlewood[12] etc. an <u>adiabatic invariant of order n</u> is a quantity undergoing only variations of order $\varepsilon^n$ in such a transition.

At first glance it seems that J does not change at all. In the initial state there is no time dependence, so $\varepsilon$ is zero and $J = J^*$ . The same holds true for the final state and since $J^*$ stays constant all the time, J must have the same value before and after .

Nevertheless Lenard et al. did <u>not</u> claim that J was conserved, but instead they called it <u>"adiabatically invariant to all orders."</u> There still exists some confusion about this phrase, especially with people who have a different definition of adiabatic invariance in mind, but the main implication appears to be that these scientists did not believe that their series converged. In fact, they called it an asymptotic series. To understand the reason for this lack of faith, we must once more turn back to celestial mechanics.

Newton solved the gravitational motion of two bodies, but the motion of three or more bodies -- say, of the sun and two planets perturbing each other -- turned out to be quite a different preposition. One can try to solve it by a series of expansion in $\varepsilon$, similar to the series described earlier, and in fact the first few terms of such expansions give quite usable results. However, try as they would, the astronomers who derived such series could not prove their convergence, or even that the quantities which they represented remained bounded.

A great deal of work and frustration was spent on this topic during the 19th century and King Oscar the 2nd of Sweden even offered a prize to whoever came up with a solution. The prize went in 1889 to Poincaré who proved that the series did not converge.

The proof, briefly, is as follows (you can find it in Whittaker's text on mechanics[13]). It may be shown that there exist initial conditions under which the expansion breaks down, due to vanishing denominators in some higher terms. Furthermore, there are infinitely many such cases, in a way that requires them to have a point of accumulation. Poincaré showed that under these conditions the expanded function cannot be analytic.

Now even if a series does not converge the quantity which it represents may remain bounded. Let us take an example closer to our subject -- a particle trapped in a dipole field. If then the series for $M^*$ does not converge, this does not mean that the point at which the particle gets turned back wanders without restriction -- so that sooner or later it wanders into the atmosphere and the particle gets absorbed, as some people have suggested. It may well be that the excursions of this point are limited -- except perhaps for some singular orbits -- in which case one says that the motion is <u>stable</u>.

A great deal of high-powered math has gone into this problem and I am glad to report that Jürgen Moser of the Courant Institute at the University of New York proved certain types of 3-body motions to be stable, for which the U.S. National Academy of Sciences awarded him the James Craig Watson Medal in April 1969. I am also glad to report that only last year Martin Braun, a student of Moser, proved the stability of charged particle motion in a dipole field,[14] and also in a magnetic mirror configuration, if you know what that means. Thus if the radiation belt comes down, it won't be due to a deterioration of adiabatic invariance.

Let me now speculate a bit: it could actually be that the series <u>does</u> converge, in many cases of adiabatic invariance. Poincaré's example -- two planets around the sun -- has two independent frequencies, namely those of each planet's motion considered separately, and any time you have two different frequencies in a system, you may get a resonance, at least in some higher harmonics. This need not necessarily happen with the drawn-up pendulum, or with its mathematical idealization, the slowly perturbed harmonic oscillator.

One possible avenue to explore in this connection is a remarkable result found by Ralph Lewis of Los Alamos several years ago.[15] He showed that for the perturbed harmonic oscillator, with the Hamiltonian

$$H = (1/2m) \left[ p^2 + m^2 \omega^2(t)\, q^2 \right]$$

the quantity

$$I = m^2 \frac{q^2}{\rho^2} + \left\{ \rho p - m \frac{d\rho}{dt} q \right\}^2$$

is a constant of the motion, provided $\rho$ satisfies the equation

$$d^2\rho / dt^2 + \omega^2(t)\,\rho - \rho^{-3} = 0$$

It may be shown that $I$ equals the adiabatic invariant in this case. The equation may be solved by series expansion, and this seems to be the fastest route yet for deriving the adiabatic invariant of the perturbed harmonic oscillator to high orders. There exists however another advantage, which may be more important: with this approach, one may dispense altogether with the series expansion. Instead, one now looks at existence theorems for solutions of $\rho$, and with suitable choices of $\omega$ it may be that these solutions can be extended to arbitrarily large values of $t$, in which case the invariant $I$ exists for all times. Professor Keith Symon of the University of Wisconsin at Madison is working on this approach and on extending Lewis' method to more general cases, and I wish him success.

In conclusion, we have ended up where we started — with the drawn-up pendulum and with an intriguing problem that is not completely solved. The nice thing about physics, even old-style classical physics, is that you never seem to run out of intriguing problems.

## References and Comments


(1) This is basically the classical perturbation theory described in texts on analytical mechanics. See, for instance, H.C.Corben and P. Stehle, Classical Mechanics, 2nd. Ed., Section 75 (John Wiley and Sons, 1960) or D. Ter Haar, Elements of Hamiltonian Mechanics (North Holland Publ. Co. 1961), chapter 7 .

(2) Max Born, The Mechanics of the Atom, republication of translated edition by Frederick Ungar Publishing Co., New York 1960.

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