NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1503

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Prepared Under Contract No. NAS 7-100
National Aeronautics and Space Administration

Preface

The work described in this report was performed by the Telecommunications Division of the Jet Propulsion Laboratory.

Contents

Abstract

An analysis based on geometrical optics for a dual-reflector antenna system with two arbitrarily shaped reflectors is carried out. Formulas for the phase and amplitude distribution in the aperture of the second reflector are obtained when the source function and the reflector surfaces are given. A design technique based on the derived formulas is also discussed.

Arbitrarily Shaped Dual-Reflector Antennas

I. Introduction

The design of dual-reflector antennas based on the technique described by Galindo (Ref. 1) has been rather successful. The surfaces of these reflectors are surfaces of revolution about an axis on which the feed horn is located. Galindo derived a pair of first-order nonlinear differential equations according to the laws of geometrical optics. Upon solving this set of equations one obtains the shapes of the dual reflectors for a given phase and amplitude distribution in the aperture of the second reflector. Although Galindo's technique may be extended, in principle, to treat the case in which the feed horn may be located away from the axis of the reflectors, the resultant equations are a set of nonlinear *partial* differential equations which are very difficult (if not impossible) to solve.

In an attempt to circumvent this difficulty it is proposed that one consider the following problem: Assuming that the shapes of the two reflector surfaces are given, find the aperture field distribution in the aperture of the second reflector. To obtain the desired aperture field distribution a trial-and-error method is used to find the appropriate shapes of the dual reflectors. For the initial trial the shapes of the reflectors are assumed to be those obtained according to Galindo's method for a symmetrical, on-axis feed, dual-reflector system. The purpose of the present work is to analyze this problem.

II. Analysis

A point source is assumed to be located at a point A (x_0, y_0, z_0). A ray originating from the point source at A is assumed to be reflected at the point B (x_1, y_1, z_1) on a doubly curved reflecting surface called *surface 1*. The reflected ray from surface 1 is again reflected at the point C (x_2, y_2, z_2) on another doubly curved reflecting surface called *surface 2*. This reflected ray from surface 2 finally arrives at the field point D (x_3, y_3, z_3). The geometry of this problem is given in Fig. 1. It is assumed that surface 1 may be described by the equation $z = f_1(x, y)$, and surface 2 by $z = f_2(x, y)$.

The techniques of geometrical optics will be used to treat the present problem. The geometrical optics analysis assumes that at each point on the reflector the incident ray is reflected by the tangent plane according to the laws of reflection. The intensity of the reflected wave in a given direction is obtained by applying the principle of the conservation of energy to the total power contained in an incident cone of rays and the total power contained in the associated reflected pencil of rays. The use of the laws of reflection assumes that (1) the reflector 1 or 2 can be regarded locally as a plane surface, and (2) the incident wavefront can be regarded locally as a plane wave. In other words, the radii of curvature of reflector 1 or 2 and of the incident wavefront must be large compared with the wavelength. Condition (2) may be assured by the fact that the reflectors are in the far-zone field of the source.

The phase distribution across the aperture of reflecting surface 2 will depend on the path length and phase change upon reflection. The phase length is easily calculated if the point of reflection on the surface is known and if the point of reflection is uniquely defined by the Snell law, which states that the angle of incidence is equal to the angle of reflection. For example (refer to Fig. 1), the total path length of a ray travelling from point A to D is $(L_1 + L_2 + L_3)$, with

$$L_1 = [(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]^{1/2}$$

$$L_2 = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$$

$$L_3 = [(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2]^{1/2}$$

A procedure for finding the locations of (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is given in Appendix A.

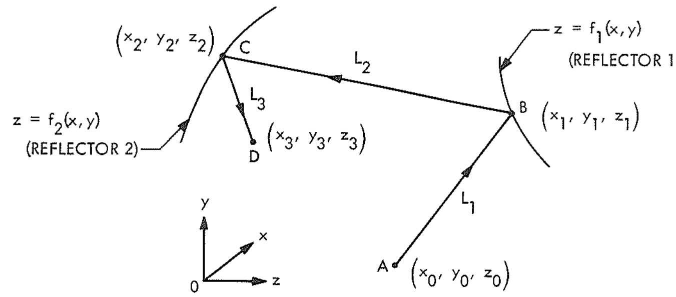
Upon considering the dispersive effect of the curved reflecting surfaces and the transformation of the polarization on reflection, one obtains the following expression for the electric field at point D[3]:

$$\mathbf{E}_D = [(\mathbf{n}_C \cdot \mathbf{E}_{Ci})\,\mathbf{n}_C - (\mathbf{n}_C \times \mathbf{E}_{Ci}) \times \mathbf{n}_C] \times \left| \frac{R_1^C R_2^C}{(R_1^C + L_3)(R_2^C + L_3)} \right|^{1/2} e^{-ikL_3} \qquad (1)$$

with

$$\mathbf{E}_{Ci} = [(\mathbf{n}_B \cdot \mathbf{E}_{Bi})\,\mathbf{n}_B - (\mathbf{n}_B \times \mathbf{E}_{Bi}) \times \mathbf{n}_B] \times \left| \frac{R_1^B R_2^B}{(R_1^B + L_2)(R_2^B + L_2)} \right|^{1/2} e^{-ikL_2} \qquad (2)$$

$$\mathbf{E}_{Bi} = \frac{\mathbf{E}_{Ai}\, e^{-ikL_1}}{kL_1} \qquad (3)$$



A IS SOURCE POINT
D IS FIELD POINT

Fig. 1. The geometry of the problem

where

$\mathbf{E}_{Ai}$ = electric field at point A

$\mathbf{E}_{Bi}$ = incident electric field at point B

$\mathbf{E}_{Ci}$ = incident electric field at point C

$\mathbf{E}_D$ = electric field at point D

$\mathbf{n}_B$ = unit normal vector at point B

$\mathbf{n}_C$ = unit normal vector at point C

k = free-space wave number

R_1^B, R_2^B = principal radii of curvature of the reflected wavefront at point B

R_1^C, R_2^C = principal radii of curvature of the reflected wavefront at point C

The principal radii of curvature of the reflected wavefront can be found by considering the principle of conservation of energy in an incident cone of rays. Detailed derivation is given in Appendix B. The R_1^B and R_2^B can be obtained from the following equation

$$(R_{1,2}^B)^{-2}\,[E^B G^B - (F^B)^2] - (R_{1,2}^B)^{-1}\,[E^B N^B + G^B L^B - 2F^B M^B] + [L^B N^B - (M^B)^2] = 0, \qquad (4)$$

where

$$E^B = 1, \qquad F^B = 0, \qquad G^B = \cos^2 i^B$$

$$L^B = 2a^B \cos i^B - \frac{1}{L_1}$$

$$M^B = -2c^B \cos i^B$$

$$N^B = 2b^B \cos i^B - \frac{1}{L_1} \cos^2 i^B$$

$$a^{B} = -\left[\frac{\cos^2 \theta_2^B}{R_\xi^B \sin^2 i^B} + \frac{\cos^2 \theta_1^B}{R_\eta^B \sin^2 i^B}\right]$$

$$b^{B} = -\left[\frac{\cos^2 \theta_1^B}{R_\xi^B \sin^2 i^B} + \frac{\cos^2 \theta_2^B}{R_\eta^B \sin^2 i^B}\right]$$

$$c^{B} = -\frac{\cos \theta_1^B \cos \theta_2^B}{\sin^2 i^B}\left(\frac{1}{R_\xi^B} - \frac{1}{R_\eta^B}\right)$$

The symbol i^B denotes the angle between the incident ray and the unit normal $\mathbf{n}_B$ at point B. The symbols θ_1^B and θ_2^B denote, respectively, the angles between the incident ray and the principal axes of the reflector ξ^B and η^B. The symbols R_ξ^B and R_η^B denote the principal radii of curvature of the reflector surface at point B. According to the theory of differential geometry (Ref. 4), if the equation of a surface is given by

$$z = f_1(x, y) \tag{5}$$

the principal radii of curvature of the surface at point B can be found from the equation

$$[r^B t^B - (s^B)^2](R_{\xi,\eta}^B)^2 - [(1 + (q^B)^2) r^B - 2 p^B q^B s^B + (1 - (p^B)^2) t^B] (1 + (p^B)^2 + (q^B)^2)^{1/2} R_{\xi,\eta}^B + [1 + (p^B)^2 + (q^B)^2]^2 = 0 \tag{6}$$

where

$$p^B = \left.\frac{\partial f_1}{\partial x}\right|_{\text{at B}}, \qquad q^B = \left.\frac{\partial f_1}{\partial y}\right|_{\text{at B}}, \qquad r^B = \left.\frac{\partial^2 f_1}{\partial x^2}\right|_{\text{at B}}, \qquad s^B = \left.\frac{\partial^2 f_1}{\partial x \partial y}\right|_{\text{at B}}, \qquad t^B = \left.\frac{\partial^2 f_1}{\partial y^2}\right|_{\text{at B}}$$

The principal radii of curvature of the reflected wavefront at point C, $R_{1,2}^C$ can be obtained in a similar fashion as that shown for $R_{1,2}^B$ by replacing the B by C, the f_1 by f_2, and the L_1 by L_2.

III. Conclusions

Since to achieve maximum gain requires uniform phase and amplitude distribution across the aperture, it is desirable to design the reflector surfaces so that the requirement on phase and amplitude may be met. The formula given (i.e., Eq. 1) will enable us to calculate the aperture field distribution when the source function and the reflector surfaces are given.

Appendix A

Phase Length Computation

It is assumed that the phase center of the feed horn is located at (x_0, y_0, z_0) and that the direction cosine of a ray starting from this point is (ℓ_0, m_0, n_0). Furthermore, the equations for the two reflecting surfaces are given by $z = f_1(x, y)$ and $z = f_2(x, y)$. The geometry of this problem is shown in Fig. 1. The following procedures will be followed to compute the phase length of each ray:

(1) Ray 1, originating from (x_0, y_0, z_0) with direction cosines (ℓ_0, m_0, n_0), satisfies the equation

$$\frac{x - x_0}{\ell_0} = \frac{y - y_0}{m_0} = \frac{z - z_0}{n_0} \tag{A-1}$$

This ray intercepts the first reflecting surface $z = f_1(x, y)$ at (x_1, y_1, z_1). The point (x_1, y_1, z_1) is found by solving the following set of equations:

if $m_0 \neq 0$, then

$$\frac{n_0}{m_0}(y_1 - y_0) + z_0 = f_1\left(\frac{\ell_0}{m_0}(y_1 - y_0) + x_0, y_1\right) \tag{A-2}$$

$$x_1 = \frac{\ell_0}{m_0}(y_1 - y_0) + x_0 \tag{A-3}$$

$$z_1 = f_1(x_1, y_1) \tag{A-4}$$

if $\ell_0 \neq 0$, then

$$\frac{n_0}{\ell_0}(x_1 - x_0) + z_0 = f_1\left(x_1, \frac{m_0}{\ell_0}(x_1 - x_0) + y_0\right) \tag{A-5}$$

$$y_1 = \frac{m_0}{\ell_0}(x_1 - x_0) + y_0 \tag{A-6}$$

$$z_1 = f_1(x_1, y_1) \tag{A-7}$$

The length of ray 1 is then

$$L_1 = [(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]^{1/2}$$

(2) From the Snell law, one obtains the unit vector for the reflected ray (ray 2),

$$\mathbf{T}_1 = -\mathbf{I}_1 + 2(\mathbf{I}_1 \cdot \mathbf{N}_1)\mathbf{N}_1 \tag{A-8}$$

where the direction cosines of the unit vectors $\mathbf{I}_1$ and $\mathbf{N}_1$ are, respectively,

$$\mathbf{I}_1 = (\ell_0, m_0, n_0) \tag{A-9}$$

$$\mathbf{N}_1 = (\ell_{N1}, m_{N1}, n_{N1}) \tag{A-10}$$

with

$$\left.\begin{aligned} \ell_{N1} &= \frac{-\partial f_1/\partial x}{\Delta_1} \\ m_{N1} &= \frac{-\partial f_1/\partial y}{\Delta_1} \\ n_{N1} &= \frac{1}{\Delta_1} \\ \Delta_1 &= \left[1 + \left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2\right]^{1/2} \end{aligned}\right\} \tag{A-11}$$

So, the direction cosines for ray 2 are

$$\mathbf{T}_1 = (\ell_1, m_1, n_1) \tag{A-12}$$

with

$$\ell_1 = \ell_0(2\ell_{N1}^2 - 1)$$

$$m_1 = m_0(2m_{N1}^2 - 1)$$

$$n_1 = n_0(2n_{N1}^2 - 1)$$

Hence, the equation for ray 2 is

$$\frac{x - x_1}{\ell_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \tag{A-13}$$

(3) Now, ray 2 intercepts the second reflecting surface $z = f_2(x, y)$ at (x_2, y_2, z_2). The point (x_2, y_2, z_2) can be found in a manner similar to that described in (1). Only the subscripts 0 and 1 in Eqs. (A-2) through (A-7) need be replaced, respectively, by the

subscripts 1 and 2. The length of ray 2 is then $L_2 = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$. From the Snell law and following the procedure given in (2), we can obtain the unit vector for the reflected ray (ray 3):

$$\mathbf{T}_2 = (\ell_2, m_2, n_2) \tag{A-14}$$

where

$$\ell_2 = \ell_1 (2\ell_{N2}^2 - 1)$$

$$m_2 = m_1 (2m_{N2}^2 - 1)$$

$$n_2 = n_1 (2n_{N2}^2 - 1)$$

$$\ell_{N2} = \frac{\partial f_2/\partial x}{\Delta_2}$$

$$m_{N2} = \frac{\partial f_2/\partial y}{\Delta_2}$$

$$n_{N2} = \frac{-1}{\Delta_2}$$

$$\Delta_2 = \left[1 + \left(\frac{\partial f_2}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2\right]^{1/2}$$

Hence, the equation for ray 3 is

$$\frac{x - x_2}{\ell_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} \tag{A-15}$$

(4) Ray 3 intercepts the surface $z = f_3(x, y)$ at (x_3, y_3, z_3). The point (x_3, y_3, z_3) can be found in the same way as described in (1). Only the subscripts 0 and 1 in Eqs. (A-2) through (A-7) need be replaced, respectively, by the subscripts 2 and 3. The length of ray 3 is then

$$L_3 = [(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2]^{1/2}$$

Appendix B

Derivation of the Amplitude Reflection Coefficient

The magnitude of the reflected wave will be derived according to the laws of geometrical optics and the principle of the conservation of energy (Ref. 2). Consider a tube of rays that cuts out elements dA_1 and dA_2 on the two wave surfaces S_1 and S_2 in free space. The principle of conservation of energy requires that

$$P_1 dA_1 = P_2 dA_2 \tag{B-1}$$

where $P_1 = \sqrt{\epsilon/\mu}\,|E_1|^2$ is the power flow per unit area at the wave surface S_1, and $P_2 = \sqrt{\epsilon/\mu}\,|E_2|^2$ is the power flow per unit area at the wave surface S_2. The symbols (ϵ, μ) are the constitutive parameters of free space. The absolute values $|E_1|$ and $|E_2|$ are, respectively, the magnitudes of the fields at S_1 and S_2. Let R_1 and R_2 be the two principal radii of curvatures at a point in dA_1 on the wave surface S_1, and let p be the length of rays between the wavefront surfaces S_1 and S_2; then one can easily show by consideration of simple geometrical relations that the relation between the cross sections of the tube of rays is

$$dA_2 = \left|\frac{(p + R_1)(p + R_2)}{R_1 R_2}\right| dA_1 \tag{B-2}$$

Hence, we have

$$|E_2| = |E_1| \left(\left|\frac{R_1 R_2}{(p + R_1)(p + R_2)}\right|\right)^{1/2} \tag{B-3}$$

Referring now to the problem of reflection for a conducting surface and according to the laws of reflection and the conservation of energy, one has

$$|E_i| = |E_r| \tag{B-4}$$

where $|E_i|$ is the magnitude of the incident wave and $|E_r|$ is the magnitude of the reflected wave. So the magnitude of the field amplitude $|E_p|$ at a distance p along the reflected ray is

$$|E_p| = |E_i| \left|\frac{R_1 R_2}{(p + R_1)(p + R_2)}\right|^{1/2} \tag{B-5}$$

Here R_1 and R_2 refer to the radii of curvature of the reflected wavefront. For computational purposes it is much better to express R_1 and R_2 in terms of the radii of curvature of the reflecting surface R_ξ and R_η. The values for R_ξ and R_η can be found according to Eq. (6) in the text.

Assuming that (u, v, w) are the coordinates of a point on the reflected wavefront and (x, y, z) are the coordinates of the point on the reflector for which the reflected ray passes through a given point (u, v, w) on the wavefront, the radii of curvature of the reflected wavefront (R_1, R_2) can be found from the following equation (Ref. 4):

$$R_{1,2}^2 (EG - F^2) - R_{1,2}(EN + GL - 2FM) + (LN - M^2) = 0 \tag{B-6}$$

where the reflector surface is assumed to be denoted by the equation

$$z = f(x, y) \tag{B-7}$$

and

$$E = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \tag{B-8}$$

$$F = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{B-9}$$

$$G = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \tag{B-10}$$

$$L = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} \dfrac{\partial^2 u}{\partial x^2} & \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial^2 v}{\partial x^2} & \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \\ \dfrac{\partial^2 w}{\partial x^2} & \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{vmatrix} \tag{B-11}$$

$$M = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} \dfrac{\partial^2 u}{\partial x \partial y} & \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial^2 v}{\partial x \partial y} & \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \\ \dfrac{\partial^2 w}{\partial x \partial y} & \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{vmatrix} \tag{B-12}$$

$$N = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} \frac{\partial^2 u}{\partial y^2} & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial^2 v}{\partial y^2} & \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial^2 w}{\partial y^2} & \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} \tag{B-13}$$

with

$$u = u(x, y), \qquad v = v(x, y), \qquad w = w(x, y) \tag{B-14}$$

These relationships between (u, v, w) and (x, y, z) are determined according to the law of reflection (Ref. 2). In the neighborhood of the point of reflection P, the equation for the reflector surface is describable by the expression

$$z \simeq \frac{1}{2} a x^2 + \frac{1}{2} b y^2 - c x y \tag{B-15}$$

with

$$a = -\left(\frac{\cos^2 \omega}{R_\xi} + \frac{\sin^2 \omega}{R_\eta}\right)$$

$$b = -\left(\frac{\sin^2 \omega}{R_\xi} + \frac{\cos^2 \omega}{R_\eta}\right)$$

$$c = -\sin \omega \cos \omega \left(\frac{1}{R_\xi} - \frac{1}{R_\eta}\right)$$

where ω is the angle between the plane of incidence and the principal plane of curvature of the reflector at P. (At P, we have $x = y = z = 0$, and $u = v = w = 0$). The surface of the reflected wave in the neighborhood of P may be represented by the following equations:

$$u = x + G_1(x, y)(r_0 - r) \tag{B-16}$$

$$v = y + G_2(x, y)(r_0 - r) \tag{B-17}$$

$$w = z + G_3(x, y)(r_0 - r) \tag{B-18}$$

with

$$G_1(x, y) = \frac{x}{r} + 2d \frac{\partial z}{\partial x} \tag{B-19}$$

$$G_2(x, y) = \frac{y - r_0 \sin i}{r} + 2d \frac{\partial z}{\partial y} \tag{B-20}$$

$$G_3(x, y) = \frac{z - r_0 \cos i}{r} - 2d \tag{B-21}$$

$$d = -\frac{x}{r\Delta^2}\frac{\partial z}{\partial x} - \frac{1}{r\Delta^2}\frac{\partial z}{\partial y}(y - r_0 \sin i) + \frac{1}{r\Delta^2}(z - r_0 \cos i) \tag{B-22}$$

$$\Delta = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2} \tag{B-23}$$

The symbol r_0 denotes the distance between the originating point of the incident ray and P, the point of reflection on the surface; and r is the distance between the originating point and an arbitrary point (x, y, z) on the reflector, i.e.:

$$r = [x^2 + y^2 + z^2 + r_0^2 - 2r_0(y \sin i + z \cos i)]^{1/2} \tag{B-24}$$

The symbol i denotes the angle between the incident ray and the outward normal at P on the reflector surface.

Carrying out the tedious computation and taking the limit as $x \to 0$ and $y \to 0$, one has

$$\frac{\partial u}{\partial x} = 1 + (r_0 - r)\frac{\partial G_1}{\partial x} + G_1\left(-\frac{\partial r}{\partial x}\right)\Bigg|_{x, y \to 0} = 1 \tag{B-25}$$

$$\frac{\partial^2 u}{\partial x^2} = (r_0 - r)\frac{\partial^2 G_1}{\partial x^2} + 2\frac{\partial G_1}{\partial x}\left(-\frac{\partial r}{\partial x}\right) + G_1\left(-\frac{\partial^2 r}{\partial x^2}\right)\Bigg|_{x, y \to 0} = 0 \tag{B-26}$$

$$\frac{\partial^2 u}{\partial x \partial y} = (r_0 - r)\frac{\partial^2 G_1}{\partial x \partial y} + \frac{\partial G_1}{\partial x}\left(-\frac{\partial r}{\partial y}\right) + \frac{\partial G_1}{\partial y}\left(-\frac{\partial r}{\partial x}\right) + G_1\left(-\frac{\partial^2 r}{\partial x \partial y}\right)\Bigg|_{x, y \to 0} = \frac{\sin i}{r_0} - 2a \sin i \cos i \tag{B-27}$$

$$\frac{\partial u}{\partial y} = \frac{\partial G_1}{\partial y}(r_0 - r) + G_1\left(-\frac{\partial r}{\partial y}\right)\Bigg|_{x,y\to 0} = 0 \tag{B-28}$$

$$\frac{\partial^2 u}{\partial y^2} = (r_0 - r)\frac{\partial^2 G_1}{\partial y^2} + 2\frac{\partial G_1}{\partial y}\left(-\frac{\partial r}{\partial y}\right) + G_1\left(-\frac{\partial^2 r}{\partial y^2}\right)\Bigg|_{x,y\to 0} = 4\,c\cos i\sin i \tag{B-29}$$

$$\frac{\partial v}{\partial x} = (r_0 - r)\frac{\partial G_2}{\partial x} + G_2\left(-\frac{\partial r}{\partial x}\right)\Bigg|_{x,y\to 0} = 0 \tag{B-30}$$

$$\frac{\partial^2 v}{\partial x^2} = (r_0 - r)\frac{\partial^2 G_2}{\partial x^2} + 2\frac{\partial G_2}{\partial x}\left(-\frac{\partial r}{\partial x}\right) + G_2\left(-\frac{\partial^2 r}{\partial x^2}\right)\Bigg|_{x,y\to 0} = -a\sin i\cos i + \frac{\sin i}{r_0} \tag{B-31}$$

$$\frac{\partial^2 v}{\partial x\partial y} = (r_0 - r)\frac{\partial^2 G_2}{\partial x\partial y} + \frac{\partial G_2}{\partial x}\left(-\frac{\partial r}{\partial y}\right) - \left(\frac{\partial G_2}{\partial y}\right)\left(-\frac{\partial r}{\partial x}\right) + G_2\left(-\frac{\partial^2 r}{\partial x\partial y}\right)\Bigg|_{x,y\to 0} = 3\,c\cos i\sin i \tag{B-32}$$

$$\frac{\partial v}{\partial y} = 1 + \frac{\partial G_2}{\partial y}(r_0 - r) + G_2\left(-\frac{\partial r}{\partial y}\right)\Bigg|_{x,y\to 0} = \cos^2 i \tag{B-33}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 G_2}{\partial y^2}(r_0 - r) + 2\frac{\partial G_2}{\partial y}\left(-\frac{\partial r}{\partial y}\right) + G_2\left(-\frac{\partial^2 r}{\partial y^2}\right)\Bigg|_{x,y\to 0} = -5\,b\sin i\cos i + \frac{1}{r}(3\cos^2 i\sin i) \tag{B-34}$$

$$\frac{\partial w}{\partial x} = \frac{\partial z}{\partial x} + (r_0 - r)\frac{\partial G_3}{\partial x} + G_3\left(-\frac{\partial r}{\partial x}\right)\Bigg|_{x,y\to 0} = 0 \tag{B-35}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} + (r_0 - r)\frac{\partial^2 G_3}{\partial x^2} + 2\frac{\partial G_3}{\partial x}\left(-\frac{\partial r}{\partial x}\right) + G_3\left(-\frac{\partial^2 r}{\partial x^2}\right)\Bigg|_{x,y\to 0} = a(1 + \cos^2 i) - \frac{\cos i}{r_0} \tag{B-36}$$

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial^2 z}{\partial x\partial y} + (r_0 - r)\frac{\partial^2 G_3}{\partial x\partial y} + \frac{\partial G_3}{\partial x}\left(-\frac{\partial r}{\partial y}\right) + G_3\left(-\frac{\partial^2 r}{\partial x\partial y}\right)\Bigg|_{x,y\to 0} = c(-1 + 2\sin^2 i - \cos^2 i) \tag{B-37}$$

$$\frac{\partial w}{\partial y} = \frac{\partial z}{\partial y} + (r_0 - r)\frac{\partial G_3}{\partial y} + G_3\left(-\frac{\partial r}{\partial y}\right)\Bigg|_{x,y\to 0} = \cos i\sin i \tag{B-38}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 z}{\partial y^2} + (r_0 - r)\frac{\partial^2 G_3}{\partial y^2} + 2\frac{\partial G_3}{\partial y}\left(-\frac{\partial r}{\partial y}\right) + G_3\left(-\frac{\partial^2 r}{\partial y^2}\right)\Bigg|_{x,y\to 0} = b(1 - 4\sin^2 i + \cos^2 i) + \frac{1}{r_0}\cos i(2\sin^2 i - \cos^2 i) \tag{B-39}$$

It can also be shown that as $x, y \to 0$,

$$\frac{\partial r}{\partial x} \to 0$$

$$\frac{\partial r}{\partial y} \to -\sin i$$

$$\frac{\partial^2 r}{\partial x^2} \to \frac{1}{r_0}(1 - r_0\, a \cos i)$$

$$\frac{\partial^2 r}{\partial x \partial y} \to c \cos i$$

$$\frac{\partial^2 r}{\partial y^2} \to \frac{1}{r_0}(\cos^2 i - r_0\, b \cos i)$$

$$G_1 \to 0$$

$$G_2 \to -\sin i$$

$$G_3 \to \cos i$$

$$\frac{\partial G_1}{\partial x} \to \frac{1}{r_0} + 2\,a\,(-\cos i)$$

$$\frac{\partial G_1}{\partial y} \to 2\,c \cos i$$

$$\frac{\partial G_2}{\partial x} \to 2\,c \cos i$$

$$\frac{\partial G_3}{\partial x} \to 2\,c \sin i$$

$$\frac{\partial G_3}{\partial y} \to \frac{\cos i \sin i}{r_0} - 2\,b \sin i$$

Substituting Eqs. (B-25–B-39) into Eqs. (B-8–B-13) gives

$$E = 1, \qquad F = 0$$

$$G = \cos^2 i$$

$$EG - F^2 = \cos^2 i$$

$$L = 2\,a \cos i - \frac{1}{r_0}$$

$$M = -2\,c \cos i$$

$$N = 2\,b \cos i - \frac{\cos^2 i}{r_0}$$

Using Eq. (B-40) and the definition for a, b, and c given in Eq. (B-15), one can readily derive Eq. (4) in the text.

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