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SAMPLED-DATA SYSTEMS AND GENERATING FUNCTIONS

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16. ABSTRACT

The Z-transform has been known in the past as the Borel transform, the generating function, the power series, and the infinite degree polynomial. Mathematicians prefer to call it "the formal power series" because is is distinguished from power series by coefficients and radix belonging to distinct mathematical systems. Physical systems described by Z-transforms are fundamentally pulse amplitude modulation devices and their demodulators. Thus, the subject is beyond time invariant electrical network theory and requires additional preparation before relevant physical systems can be studied with ease.

This report presents a lengthy and formal acquaintance with Z-transforms. A formal study of Z-transforms pays off by relegating physical applications to very easy corollary status.

Physical applications studied are:

1. The numerical transform, the natural transition from analog to discrete systems.
2. Sampled-data feedback devices.
3. Multiplexing and demultiplexing.
4. Periodically reverse switched capacitor.
5. Pulse code modulation decoders.
6. The sampled-data servo.
7. Proper and subharmonic digital filters.

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INTRODUCTION

The Z-transform [1] has been known in the past as the Borel transform, the generating function, the power series, and the infinite degree polynomial. Mathematicians [2] prefer to call it "the formal power series" because it is distinguished from power series by coefficients and radix belonging to distinct mathematical systems. Physical systems described by Z-transforms are fundamentally pulse amplitude modulation (PAM) devices and their demodulators. Thus, the subject is beyond time invariant electrical network theory and requires additional preparation before relevant physical systems can be studied with ease.

This report presents a lengthy and formal acquaintance with Z-transforms. Many a hurried engineer has bypassed this formal study only to acquire a numbed mental state. On the other hand, a formal study of Z-transforms pays off by relegating physical applications to very easy corollary status.

Physical applications studied are:

1. The numerical transform, the natural transition from analog to discrete systems.

2. Sampled-data feedback devices.

3. Multiplexing and demultiplexing.

4. Periodically reverse switched capacitor.

5. Pulse code modulation (PCM) decoders.

6. The sampled-data servo.

7. Proper and subharmonic digital filters.

THE Z-TRANSFORM

We presume that the reader is familiar with the uses of Laplace transforms to speed solution of differential equations generated by linear physical systems such as electrical networks. However, a different notation will be required before venturing into the Z-transform which is, in all reality, a Laplace transform. This new notation is motivated by the fact that linear operators can occur in Laplace transforms and that upper case letters will designate linear operators while lower case letters will designate operands or functions.

The properties of a linear operator, L, are:

0. If $f=f(t)$ and $Lf=g$, then $g=g(t)$ and $t \epsilon [0^+, \infty]$;

1. $L(f+g)=Lf + Lg$; and

2. $L(fg)=fLg$ only if f=constant.

A linear transform has similar properties except that property 0 is modified to

(0') if $f=f(t)$ and $\mathcal{L}f=\overline{f}(s)$.

In order to drop parentheses after one is familiar with the variables t and s, we can introduce an overbar:

(0'') if $f=f(t)$ and $\mathcal{L}f=\overline{f}$, then $\overline{f}=\overline{f}(s)$.

Note that a linear transform is designated by a script capital letter.

Laplace transforms are first employed in physical systems (those describable by ordinary differential equations) because they algebraicize

$(f+g)$ and $\int_0^t f(t)\, g(t-\tau)\, d\tau \stackrel{\Delta}{=} f*g$ into $(\overline{f}+\overline{g})$ and $\overline{f}\,\overline{g}$. Adjoin the multiplication operation, and $f+g$, $f*g$, and fg transform into $\overline{f}+\overline{g}$, $\overline{f}\,\overline{g}$, and

$\frac{1}{2\pi j} \oint \overline{f}(\lambda)\, \overline{g}(s-\lambda)\, d\lambda$. The last integral is not tractable unless we fix one factor in fg; the fixed factor f can be considered a linear operator resulting in Fg, and the complex convolution integral can be shortened to $F\overline{g}$. Thus, by adhering to traditional operator notation, a compact notation is available for general mathematical systems within which lies the Z-transform as a special case.

The Z-transform is obtained by choosing the fixed factor to be the periodic delta function, $\sum_{n=0}^{\infty} \delta(t-nT)$, $0 < T < \infty$. The resulting complex convolution integral and its shortened form are

$$\frac{1}{2\pi j} \oint \overline{g}(\lambda)\, d\lambda / (1-e^{-T(s-\lambda)}) \triangleq Z\overline{g} \quad . \tag{1}$$

Note that we began with $\overline{g}(s)$ and ended with a function of s; hence, an operator (not transform) notation must be used and it is the upper case Z.

It is natural to ask, "What does Zg mean?" Note that precision requires use of a different operator symbol for the fixed factor $\sum_{n=0}^{\infty} \delta(t-nT)$, but unfortunately no unique symbol has been assigned to the periodic delta function. Therefore, our fundamental viewpoint enables interpretation of Z in Zg to be the fixed sampling factor or, alternatively, the sampling operator.

The accepted standard notation, $\mathcal{L}f=F(s)$, is definitely inadmissible in our notation because F(s) is a function of the variable s that has been arbitrarily given a contradictory operator symbol, the upper case F, which would lead us to believe that a function of time, t, is at hand.

A sight-readable, stripped-down notation has been constructed and adequate reasons for departing from standard notation have been given. The next concern will be to give meaning to these symbols. The reader can note that f and Zf, the set of sampled values of f(t), are indistinguishable if T is small enough and f(t) is a continuous function. Later, or other, physical situations will require fairly large sampling intervals.

THE WORKING Z-TRANSFORM

The complex convolution integral is the reality behind the symbol $Z\overline{g}$. It has several forms, but we shall direct attention to the generating function form in the following lemma.

Lemma 1. If $g(t)$ is a continuous function on $(0, \infty]$ and has a single jump discontinuity at $t=0^+$, then

$$Z\overline{g} = \sum_{n=0}^{\infty} g(nT)\ e^{-nTs}$$

and

$$g(0) = \lim_{\epsilon \to 0} g(\epsilon) \ ,$$

where $\epsilon \geq 0$. This leads us to the important causality condition:

$$\text{if} \quad t \in [-\infty, 0), \quad \text{then} \quad g(t) = 0 \quad .$$

The delayor, e^{-Ts}, occurs so frequently that it will be useful to shorten it to

$$e^{-Ts} \triangleq z \quad . \tag{2}$$

However, z remains a function of s and is not a separately existing entity. Another parenthesis dropping situation is possible when we define g_n to be

$$g_n \triangleq g(nT) \quad . \tag{3}$$

In this way, the stripped down form $Z\overline{g} = \sum_{n=0}^{\infty} g_n z^n$ is obtained. Lemma 1 can be extended to functions with denumerably infinite jump discontinuities by employing the case $t=0$ as an example.

Note that coefficients of $Z\overline{g}$ are real numbers, but z is a delay operator and is definitely not in the number field. Indeed, the next lemma discloses another standard inconsistency.

Lemma 2. The inverse of the delay operator, the predictor, does not exist. Proof depends on the fact that only one-to-one transformations can have inverse transformations. In addition, causality can be considered almost synonymous with nonexistence of predictors. Proof will be given in the next section.

It will be advantageous to work entirely in the Laplace transform domain. It can be shown that

$$Z(\overline{f}+\overline{g}) = Z\overline{f}+Z\overline{g} \quad ;$$

$$(Z\overline{f})\ (Z\overline{g}) = \sum_{n=0}^{\infty} h_n z^n , \quad \text{where } h_n = \sum_{k=0}^{n} f_k g_{n-k} \; ;$$

$$Z[\overline{f}\overline{g}] \neq (Z\overline{f})\ (Z\overline{g}) \quad ;$$

and

$$Z[\overline{f}\ Z\overline{g}] = (Z\overline{f})\ (Z\overline{g}) ,$$ the Ragazzini-Zadeh (R-Z) identity.

Some simple time functions yield

$$Z(1/s) = 1/(1-z) \quad ;$$

$$Z(1/s+a) = 1/(1-ze^{-aT}) \quad ;$$

$$Z(1/s-j\omega) = 1/(1-ze^{j\omega T}) , \quad j^2 = -1 \quad ;$$

and

$$Z(1/s+k\omega) = [1/1-z] + (k\omega) \quad [Tz/(1-z)^2] \; , \quad k^2 = 0 \quad .$$

NONEXISTENCE OF THE PREDICTOR

Previously the delayor, e^{-Ts}, was shortened to z. However, the literature tacitly approves a fallacy when it defines $z=e^{Ts}$ and makes amends afterward by using $e^{-Ts}=z^{-1}$. This sham can be dismissed by proving the nonexistence of the predictor e^{Ts} as follows:

1. Given a sequence of nonzero numbers $\{f_0, f_1, f_2, f_3, \ldots f_n, \ldots\}$;

2. A prediction, e^{Ts}, followed by the delayor yields $\{0, f_1, f_2, f_3, \ldots f_n \ldots\}$;

3. A delay followed by a prediction yields $\{f_0, f_1, f_2, f_3, \ldots, f_n, \ldots\}$;

4. Note that only one of two possible sequences of 2 and 3 gives the correct result 3;

5. If the inverse of an operator exists, then the inverse is both a left inverse and a right inverse;

6. After close scrutiny, it is the predictor in conjunction with causality that is the cause of the contradiction in 3 that nonzero f_0 equals zero;

7. Finally, one can physically construct delay lines but no engineer can fashion a predictor.

This result justifies our symbolization of the delayor with z and discarding the predictor. If the engineer is puzzled with this viewpoint, it is suggested that the difficulty lies in unwarranted association of the root field of a polynomial in z (i.e., the complex number field) with the polynomial radix z.

DIGITAL COMPUTATION

It is imperative that early acquaintance be made with the close association of the Z-transform with digital computation. The algebraic symbolism must be converted from a high level language used by an investigator to a low level language, that of recurrence relations (difference equations and all), for the digital computer. Most of the mental work on a particular physical problem is executed in the high level language and the arithmetic work is executed by the digital computer in the low level language. Z-transform manipulations are not exercises in integration of functions of a complex variable, but are an extension of ordinary arithmetic. Thus, discrete convolution,

$(Z\bar{f})\ (Z\bar{g}) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f_k\, g_{n-k} \right) z^n = Z\bar{h}$, is a carry-less multiplication of two sequences. We shall reinforce this arithmetic viewpoint shortly.

The digital computer is a mass production device and the chief mechanism for effecting mass production is the recurrence relation. Unfortunately, discrete convolution, $h_n = \sum_{k=0}^{n} f_k g_{n-k}$, is not a recurrence relation, in general, because h_{n+1} cannot be made a linear combination of h_n and/or h_{n-1}. We must search for a recurrence relation situation in Z-transforms.

Perhaps the simplest method of producing a Z-transform is to form a ratio of two relatively prime (i.e., no common factors) polynomials in the variable z. Familiar long division uncovers the fact that division leads to multistep recurrence relations and the number of steps (i.e., starting values) is equal to the degree of the denominator. Long division calculations can be swiftly made with short division (the usual nomenclature, synthetic division, does not relate properly to long division). But, an even swifter conversion process is short-short division whose rules are:

$$Z\overline{g} \rightarrow g_n \quad ,$$

$$\alpha \; z^k \; Z\overline{g} \rightarrow \alpha \; g_{n-k} \quad ,$$

and

$$g_{-1} = 0 \quad .$$

These rules are self-evident but an added proof reinforces them as follows:

1. Let $Z\overline{g} = \dfrac{P(z)}{Q(z)} = \dfrac{\sum_{k=0}^{\alpha} p_k z^k}{\sum_{k=0}^{\beta} q_k z^k}$, where $Z\overline{g}$ is unknown and

$P(z)$ and $Q(z)$ are known;

2. Since $\dfrac{1}{Q(z)} P(z)$ is void of recurrence relations, multiply both sides by $Q(z)$;

3. $\left(\sum_{k=0}^{\beta} q_k z^k\right) (Z\overline{g}) = \sum_{k=0}^{\alpha} p_k z^k$;

4. The nth terms on each side lead to $\sum_{k=0}^{n} q_k \, g_{n-k} = p_n$;

5. $q_0 \, g_n + \sum_{k=1}^{n} q_k \, g_{n-k} = p_n$ and $q_k = 0$, if $k > \beta$;

6. The required recurrence relation is $g_r = \frac{p_r}{q_0} - \frac{1}{q_0} \sum_{k=1}^{r} q_k \, g_{r-k}$;

7. This formula is the same as that obtained by long division, short division, and short-short division rules;

8. Use short-short division rules on $Q(z) \; Z\overline{g} = P(z)$ to obtain $\sum_{k=0}^{n} q_k \, g_{n-k} = p_n$, and the rest is similar to procedures 5 and 6.

Simplicity of the short-short division rule will be appreciated only when one becomes highly involved in a physical problem.

This association of the Z-transform with a digital computation process should be learned and then relegated to the subconscious. From here on, the high level algebraic language will be the focus of attention.

THE RAGAZZINI-ZADEH IDENTITY

The Ragazzini-Zadeh [3] identity $Z(\overline{f} \, \overline{Zg}) = (Z\overline{f}) \; (Z\overline{g})$ is the gem rock of the Z-transform while the inequation $Z(\overline{fg}) \neq (Z\overline{f}) \; (Z\overline{g})$ is the rock of the Z-transform. However, the R-Z identity requires polishing for practical problems and must be extended to be useful. Nonrealization of this has decoyed much past effort into premature use of the modified Z-transform. The rock can also be processed into a semiprecious stone, the numerical transform. However, the immediate purpose of this section is to prove the R-Z identity.

The proof that $Z(\overline{f} \, \overline{Zg}) = (Z\overline{f}) \; (Z\overline{g})$ proceeds as follows:

1. $\mathcal{L}^{-1} [\overline{f} \overline{Zg}] = \sum_{n=0}^{\infty} g_n \, f(t-nT)$;

2. $Z(\overline{f} \, \overline{Zg}) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} g_n \, f(rT-nT) \, z^r$;

3. Causality of f(t) implies $f(rT-nT) = 0$ for $r = 0, 1, 2, \ldots, n-1$;

4. $$Z(\overline{f}\, Z\overline{g}) = \sum_{r=n}^{\infty} \sum_{n=0}^{\infty} g_n f_{r-n} z^{r-n} z^n \;;$$

5. Redefine $r-n = \alpha$;

6. $$Z(\overline{f}\, Z\overline{g}) = \sum_{\alpha=0}^{\infty} \sum_{n=0}^{\infty} g_n f_\alpha z^\alpha z^n = \left(\sum_{\alpha=0}^{\infty} f_\alpha z^\alpha\right)\left(\sum_{n=0}^{\infty} g_n z^n\right) = (Z\overline{f})(Z\overline{g}) \;;$$

7. End of proof.

This elementary proof is the core for the more general slow sampler and its R-Z identity to be discussed later. Indeed, we can only then write about Z-transforms.

THE NUMERICAL TRANSFORM

The numerical transform calculates $Z(\overline{f}\overline{g})$ approximately. There exist many possible cases that serve as points of departure.

Case 0. Both $\overline{f}$ and $\overline{g}$ are known functions of the variable s and are of exponential order. If the denominators of $\overline{f}$ and $\overline{g}$ can be factored and if $\overline{f}\overline{g}$ is a proper fraction, then a partial fraction expansion can be executed and $Z(\overline{f}\overline{g})$ can be formed exactly.

Case 1. If $\overline{f}$ is known but $g(t)$ is known with time domain processing intended, again a partial fraction expansion on the poles of $\overline{f}$ can be executed and converted in convolution integrals, and, finally, any one of a large number of known quadrature formulas can be employed. For instance, a typical additive term might be

$$Z\left(\frac{1}{s+a}\,\overline{g}\right) \triangleq Z\left(\int_0^t e^{-a(t-\tau)} g(\tau)\, d\tau\right) = Z\left[e^{-at}\int_0^t e^{a\tau} g(\tau)\, d\tau\right] \quad . \; (4)$$

Cases 0 and 1 use polynomial factorization routines which can be time consuming and sensitive calculations.

Case 2. $Z(\overline{f}\overline{g})$ is to be determined by root-free methods. We shall emphasize a root-free method in this section because cases 0 and 1 can be snarled quite effectively by employing functions of mixed variables s and z. These occur quite naturally whenever LC lines have inductors and capacitors imbedded in them. Moreover, a particularly simple mixed variable case is given by the phase-lock loop (PLL) whose linearized output is $\left(\frac{10z}{s+10z}\right)\overline{g}$. In the PLL situation, cases 0 and 1 must be fortified with Padé approximants and extensive tests for goodness of approximation, whereas root-free methods are directly applicable.

The numerical transform procedure to be presented here is based on trapezoidal convolution. Although it has only modest accuracy performance, it boasts of other virtures such as stability of accuracy versus sampling interval size; computer realization is minimal in both size and execution time requirements. The minimal execution time requirement is especially important in the exploratory phase of an investigation where unknown parameters need to be identified quickly and without expending a fraction of the gross national product on digital computers. After widened experience requires it, then more accurate Runge-Kutta and predictor-corrector methods can be used. Many engineering situations will not require more accurate procedures.

The basis of our numerical transform (and there exist many others) is lemma 3.

Lemma 3. If f(t) and g(t) are analytic and continuous on $[0, \infty]$ except for jump discontinuities at $t=0$, then $Z(\overline{f}\overline{g}) \doteq T(Z\overline{f})(Z\overline{g}) - 0.5T(f_0\, Z\overline{g} + g_0\, Z\overline{f})$ and the greatest lower bound on the error order is the same as that on trapezoidal quadratures (order=2). The above formula is called "trapezoidal convolution."

Another requirement for the solution of ordinary time-invariant differential equations is a table of Z-transforms of $1/s^n$. This table is presented in the appendix and a proof can be found in Reference 4. A third requirement is the R-Z identity, and the fourth requirement is short-short division.

An example of finding the ideal phase-lock loop response, $\frac{10z^3}{s+10z^3}\overline{g}$, will knit these requirements together:

1. $\overline{f} = \frac{10z^3}{s+10z^3}\overline{g}$, $\overline{g}$ and $f_0=0$ are known,and $Z\overline{f}$ is to be found;

2. Multiply both sides by $s+10z^3$;

3. $(s+10z^3)\bar{f}=10z^3\bar{g}$;

4. Divide by s ;

5. $\left(1+\frac{10z^3}{s}\right)\bar{f}=\frac{10z^3}{s}\bar{g}$;

6. Sample both sides and use the R-Z identity to obtain

7. $Z\bar{f}+10z^3\,Z\left(\frac{1}{s}\bar{f}\right)=10z^3\,Z\left(\frac{1}{s}\bar{g}\right)$;

8. Trapezoidal convolution gives the equation

$$Z\left(\frac{1}{s}\bar{f}\right)\doteq\frac{T}{2}\,\frac{1+z}{1-z}\,Z\bar{f}-\frac{Tf_0}{2(1-z)} \quad ;$$

9. $Z\bar{f}+10z^3\left[\frac{T}{2}\,\frac{1+z}{1-z}\,Zf-\frac{Tf_0}{2(1-z)}\right]=10z^3\,\frac{T}{2}\,\frac{1+z}{1-z}\,Z\bar{g}-\frac{Tg_0}{2(1-z)}$;

10. $Z\bar{f}+5Tz^3\,\frac{(1+z)}{(1-z)}\,Z\bar{f}=5Tz^3\left[\frac{(1+z)}{(1-z)}\,Z\bar{g}-\frac{g_0}{(1-z)}\right]$ because $f_0=0$;

11. Short-short division is now being readied;

12. Multiply both sides by $(1-z)$;

13. $(1-z+5Tz^3+5Tz^4)\,Z\bar{f}=(5Tz^3+5Tz^4)\,Z\bar{g}-5Tg_0\,z^3$;

14. $f_n-f_{n-1}+5Tf_{n-3}+5Tf_{n-4}=5Tg_{n-3}+5Tg_{n-4}-x_n$;

15. $\{x_n\}=\{0,0,0,-5Tg_0,\underline{0,0,0}\}$;

16. The underlined zeros are repeated indefinitely;

17. Solving for f_n yields the discrete solution.

Certain implicit features need emphasis. Note that the characteristic polynomial of the transfer function is not readily factored and yet this factor-free method finds a discrete solution. If the initial condition on $f(t)$ was not zero, then this would be introduced in step 3 and reintroduced in step 9. Finally, if only the recurrence relation of step 14 is placed on the digital computer, then computation execution is swift.

The problem has been fashioned so that an integer power of z was present. In general there is a decision problem about this integer which is designated by m. If $e^{-\tau s}$ has the smallest time delay, τ, then set $\tau = mT$ where m is an integer and $T \epsilon (0, \infty)$. The integer property has priority and one picks an arbitrary integer for m; one can then determine $T = \tau/m$. Suppose the next largest time delay is ν (i.e., $\tau < \nu$); then, either $\nu = n(mT)$ or $\nu = (n+\alpha)\, mT$ where n is an integer and α is a fraction. The former case poses no difficulties and the latter case must be treated separately and introduces the modified Z-transform. This lengthy but simple procedure will be called "The product rule."

THE MODIFIED Z-TRANSFORM

The modified Z-transform approximates the sampling of functions that have fractional delay; that is, given an $\alpha \epsilon (0,1)$ and an f(t), find $Z(z^{\alpha}\bar{f})$ approximately. The collocation of a prescribed number of sampled values by a polynomial form is the approximation process.

Lemma 4. If f(t) is a continuous function of time and $\alpha \epsilon (0,1)$, then

$$Z(z^{\alpha}\bar{f}) \doteq [(1-\alpha(1-z)]\, Z\bar{f} \text{ in the sense of linear interpolation}$$

and

$$Z(z^{\alpha}\bar{f}) \doteq [1-\alpha(1-z) + 0.5\alpha\ (\alpha-1)(1-z)^2]Z\bar{f} \qquad (5)$$

in the sense of quadratic interpolation.

The following proof of equation (5) is indicative of general Newton interpolation:

1. Unknown constants p, q, r are subject to three collocations,

$$f_n = p + qn + rn(n+1) \quad ,$$

$$f_{n-1} = p + q(n-1) + r(n-1)n \quad ,$$

and

$$f_{n-2} = p + q(n-2) + r(n-2)(n-1) \quad ,$$

where f_n, f_{n-1}, f_{n-2}, and n are known ;

2. Successive backward differences, Δ and Δ^2, result in matrix triangularization,

$$\begin{bmatrix} f_n \\ \Delta f_n \\ \Delta^2 f_n \end{bmatrix} = \begin{bmatrix} 1 & n & n(n+1) \\ 0 & 1 & 2n \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad ;$$

3. Solving for (p, q, r) yields $r = \Delta^2 f_n/2$, $q = \Delta f_n - n\Delta^2 f_n$, and $p = f_n - n\Delta f_n + \frac{1}{2} n(n-1)\Delta^2 f_n$;

4. Calculate $f_{n-\alpha} = p+q(n-\alpha)+r(n-\alpha)(n-\alpha+1)$;

5. $f_{n-\alpha} = f_n - \alpha(\Delta f_n) + \frac{1}{2}\alpha(\alpha-1)\Delta^2 f_n$, the Newton interpolation formula;

6. $$\sum_{n=0}^{\infty} f_{n-\alpha} z^n = \sum_{n=0}^{\infty} f_n z^n - \alpha \sum_{n=0}^{\infty} (f_n - f_{n-1}) z^n + \frac{1}{2}\alpha(\alpha-1) \sum_{n=0}^{\infty} (f_n - 2f_{n-1} + f_{n-2}) z^n \quad ;$$

7. $$\sum_{n=0}^{\infty} f_{n-\alpha} z^n = \left[1 - \alpha(1-z) + \frac{1}{2}\alpha(\alpha-1)(1-z)^2\right] Z\bar{f} \doteq Z(z^{\alpha}\bar{f}) \quad ;$$

8. End of proof.

Some explanation of the exclusive use of backward differences is required. One of the purposes of Z-transforms is to solve differential equations, and this can be accomplished only by employing past sampled values to find the present sampled value. This automatically excludes forward differences and central differences, except that central differences and trapezoidal quadrature yield identical results when solving a second-order linear differential equation.

Numerical transforms are not too far from these Newton interpolation formulas. It is necessary to enlarge the space of continuous functions to include a jump discontinuity of magnitude f_0 at the time origin which also coincides with a sampling time. Trapezoidal quadrature is obtained by integration with respect to αT, $\alpha \in [0, 1]$, T=constant and partial summation, $1/(1-z)$, in

$$\frac{T}{1-z} \left\{ \int_0^1 [1-\alpha(1-z)]\, d\alpha\, Z\overline{f} \right\} = \frac{T}{2} \frac{1+z}{1-z} Z\overline{f} \quad . \tag{6}$$

However, this procedure is risky because the initial jump is assumed zero. This risk can be removed by starting with the discrete values $[f_n]$ and ending with Z-transforms. The correct procedure is easy with trapezoidal quadrature and yields

$$Z\left(\frac{1}{s}\overline{f}\right) \doteq \frac{1}{2} \frac{1+z}{1-z} Z\overline{f} - \frac{Tf_0}{2(1-z)} \quad .$$

The derivation of a quadrature process, the Simpson Rule, from equation (5) is now presented:

$$\begin{aligned} \int_0^2 f_{n-\alpha}\, d(\alpha T) &= T \int_0^2 \left[f_n - \alpha \Delta f_n + \frac{1}{2} \alpha(\alpha-1) \Delta^2 f_n \right] d\alpha \\ &= T\left(2f_n - 2\Delta f_n + \frac{1}{3} \Delta^2 f_n \right) = \frac{T}{3}\left(f_n + 4f_{n-1} + f_{n-2} \right) . \end{aligned} \tag{7}$$

However, Simpson's Rule is risky in solving a differential equation. The correct utilization is given in the next section.

BEYOND TRAPEZOIDAL QUADRATURE

The fast Z-transform language is directly applicable to the first three members of the Newton-Cotes quadrature family [5]; these are the left-Riemann sum, the right-Riemann sum, and trapezoidal quadrature cases. This success is a coincidence but the naked Simpson Rule, which is the next member of the Newton-Cotes family, cannot duplicate this success. The correct utilization of Newton interpolation formula to achieve an integration formula is presented as follows:

1. Consider the uniform sampling times, nT, n=0, 1, 2, 3, . . . , and $0 < T < 1$;

2. The incremental integral sequence's first element must be 0;

3. Its second element can be formed only from samples f_0 and f_1, and hence the trapezoidal quadrature $\frac{T(f_0+f_1)}{2}$ is entered here;

4. Hereafter, even-numbered indices, n, will utilize

$$\int_0^1 [1-\alpha(1-z) + 0.5\alpha(\alpha-1)(1-z)^2]\, d(\alpha T) = T\left(f_n - \frac{1}{2}\Delta f_n - \frac{T}{12}\Delta^2 f_n\right)$$

$$= \left(\frac{T}{12}\right)(5f_n + 8f_{n-1} - f_{n-2});$$

5. Odd-numbered indices, n, will utilize

$$\int_1^2 [1-\alpha(1-z) + 0.5\alpha(\alpha-1)(1-z)^2]\, d(\alpha T) = T\left(f_n - \frac{3}{2}\Delta f_n + \frac{5}{12}\Delta^2 f_n\right)$$

$$= \left(\frac{T}{12}\right)(-f_n + 8f_{n-1} + 5f_{n-2});$$

6. The incremental integral sequence is

$$T\left\{0,\ f_1 - \frac{1}{2}\Delta f_1,\ \ f_2 - \frac{1}{2}\Delta f_2 - \frac{1}{12}\Delta^2 f_2,\ \ f_3 - \frac{3}{2}\Delta f_3 + \frac{5}{12}\Delta^2 f_3, \ldots\right\} \quad ;$$

7. Subtraction of the known trapezoidal portion yields the sequence

$$T\left\{0,\ 0,\ \left(-\frac{1}{12}\Delta^2 f_2\right),\ \left(-\Delta f_3 + \frac{5}{12}\Delta^2 f_3\right),\ \left(-\frac{1}{12}\Delta^2 f_4\right),\left(-\Delta f_5 + \frac{5}{12}\Delta^2 f_5\right),\ \ldots\right\};$$

8. Sequence 7 decomposes additively into

$$-T\left\{0,\ 0,\ 0,\ \Delta f_3,\ 0,\ \Delta f_5,\ 0,\ \Delta f_7, \ldots\right\}$$

and

$$\left(\frac{T}{12}\right)\left\{0,\ 0,\ -\Delta^2 f_2,\ 5\Delta^2 f_3,\ -\Delta^2 f_4,\ 5\Delta^2 f_5,\ -\Delta^2 f_6,\ 5\Delta^2 f_7, \ldots\right\} \quad ;$$

9. The Z-transforms of these two sequences yield

$$-T\left[(\Delta f_3)\, z^3 + (\Delta f_5)\, z^5 + (\Delta f_7)\, z^7 + \ldots\right]$$

$$-\frac{T}{12}\left[(1-z)^2 Zf - (f_1 - 2f_0)\, z - f_0\right] +$$

$$\frac{6T}{12}\left[(\Delta^2 f_3)\, z^3 + (\Delta^2 f_5)\, z^5 + (\Delta^2 f_7)\, z^7 + \ldots\right] \quad ;$$

10. A breakdown occurs because expressions $\left[(\Delta^2 f_3)\, z^3 + (\Delta^2 f_5)\, z^5 + (\Delta^2 f_7)\, z^7 + \ldots\right]$ and $\left[(\Delta f_3)\, z^3 + (\Delta f_5)\, z^5 + (\Delta f_7)\, z^7 + \ldots\right]$ cannot be described by the algebraic Z-transform language employed to this moment;

11. Later in the algebraic description of multiplexing, a notation for the delayed sampler will be developed and the result will be stated abruptly in the next two steps;

12. $$\frac{6T}{12}\left[(\Delta^2 f_3)\, z^3 + (\Delta^2 f_5)\, z^5 + (\Delta^2 f_7)\, z^7 + \ldots\right] = \frac{6T}{12}\left[Z_2{}^{(1)}\bar{f} - 2zZ_2{}^{(0)}\bar{f} + z^2 Z_2{}^{(1)}\bar{f} - (f_1 - 2f_0)\, z\right] \quad ;$$

13. $-T\left[(\Delta f_3)z^3+(\Delta f_5)z^5+(\Delta f_7)z^7+\ldots\right] = -T\left[Z_2^{(1)}\bar{f}-zZ_2^{(0)}\bar{f}-(f_1-f_0)z\right]$;

14. $Zf = Z_1^{(0)}\bar{f}$;

15. The desired result is obtained after applying partial summation, $1/(1-z)$, to step 9 and an added trapezoidal quadrature expression;

16. The desired result is

$$Z\left(\frac{1}{s}f\right) \doteq \left[\frac{T}{2}\;\frac{1+z}{1-z} - \frac{T}{12}\;(1-z)\right] Z\bar{f} - \left[\frac{5Tf_0}{12(1-z)}\right] + \left[\frac{T(7f_1-2f_0)z}{12(1-z)}\right]$$
$$- \left[\frac{T(1+z)}{2}\right] Z_2^{(1)}\bar{f} \quad . \tag{8}$$

This quadrature expression issues from a correct application of Newton's interpolation formula and does not produce anything similar to previous attempts. In effect, the apparatus behind Simpson's Rule motivates this result and the Newton-Cotes quadrature problem has been breached.

The appearance of delayed samplers, $Z_2^{(1)}\bar{f}$, demonstrates that it is not always easy to convert from recurrence relations to fast Z-transforms. However, the reverse procedure is easy except when the sampled values of a continuous convolution enmesh the unknown function.

FAST AND SLOW SAMPLERS

The Z-transform needs enrichment with the slow sampler before control devices with samplers can be analyzed effectively.

Definition. If m is any integer and $0 < T < \infty$, then $Z_m\bar{f} = \sum_{n=0}^{\infty} f(nmT)\,z^{nm}$.

Furthermore if $m=1$, then Z_1 is called a fast sampler; and if $m \geq 2$, then Z_m is called a slow sampler.

The following lemmas disclose interrelations necessary for analysis of control devices:

Lemma 5. If $Z_1\bar{f} = \bar{g}(z;T)$, then $Z_m\bar{f} \neq \bar{g}(z^m; mT)$ in general; equality holds only for undelayed unit polynomials and undelayed exponential functions.

Lemma 6. If m is any integer, then $Z_m(Z_1\bar{f}) = Z_m\bar{f}$.

Lemma 7. If m is any integer, then $Z_m(\bar{f}Z_m\bar{g}) = (Z_m\bar{f})(Z_m\bar{g})$.

Lemma 8. If m is any integer, then $Z_1(\bar{f}Z_m\bar{g}) = (Z_1\bar{f})(Z_m\bar{g})$.

Lemma 9. If m is any integer, then $Z_m(\bar{f}Z_1\bar{g}) = Z_m\left[(Z_1\bar{f})(Z_1\bar{g})\right]$.

Assertions of lemmas 6 and 7 are clear. Proofs of lemmas 7 and 8 follow the same pattern as that of the R-Z identity. The proof of lemma 9 can be obtained as follows:

$$Z_m(\bar{f}Z_1\bar{g}) = Z_m\left[Z_1(\bar{f}Z_1\bar{g})\right] = Z_m\left[(Z_1\bar{f})(Z_1\bar{g})\right] .$$

Further decomposition is not possible unless the equality condition of lemma 5 holds.

Sampled data control systems generate piecewise-continuous functions over relatively large intervallengths, T; these piecewise continuous functions are then passed through ordinary filters. We observe ruefully that the residue method for evaluating the inverse Laplace transform is no longer an effective calculation procedure. Lemmas 6 through 10 and short-short division regain effective computation. This is best demonstrated by the sampled-data feedback devices of the following sections.

Exercises are quite convincing. Use the residue method to find the inverse Laplace transform of $\frac{1}{1+s} \cdot \frac{1}{3} \cdot \left(\frac{1-z^3}{s}\right)^2$ and $\frac{1}{1+2s+2s^2+s^3} \cdot \frac{1}{3} \cdot \left(\frac{1-z^3}{s}\right)^2 \cdot \left(\frac{1}{1-z^6}\right)$ and $(1/s)(1/s+z^3)$. The interval length, T, is equal to 1 second.

SINGLE LOOP FEEDBACK DEVICES WITH SAMPLERS

It is not necessary to venture beyond single loop feedback devices to apply our knowledge of samplers and to discover new properties. This section will study the single loop feedback devices shown in Figures 1, 2, and 3 and show the particular effectiveness of lemma 8.

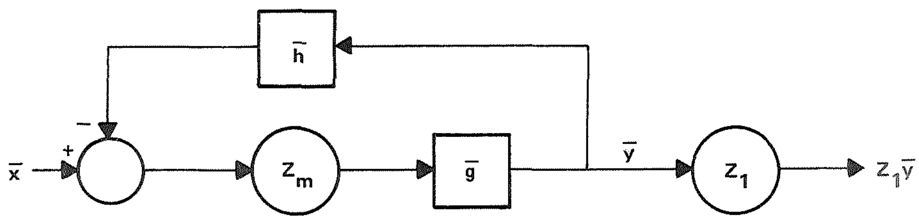

Figure 1. A single loop feedback device with a slow sampler in the feedthrough link.

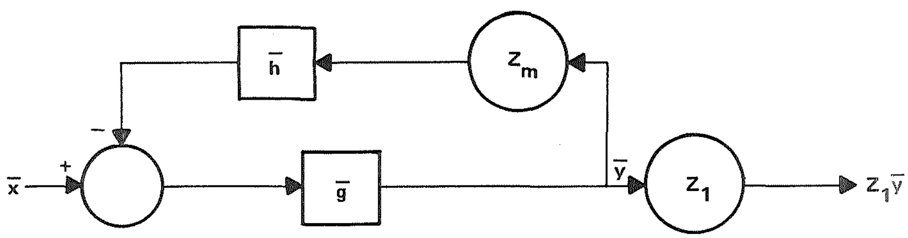

Figure 2. A single loop feedback device with a slow sampler in the feedback link.

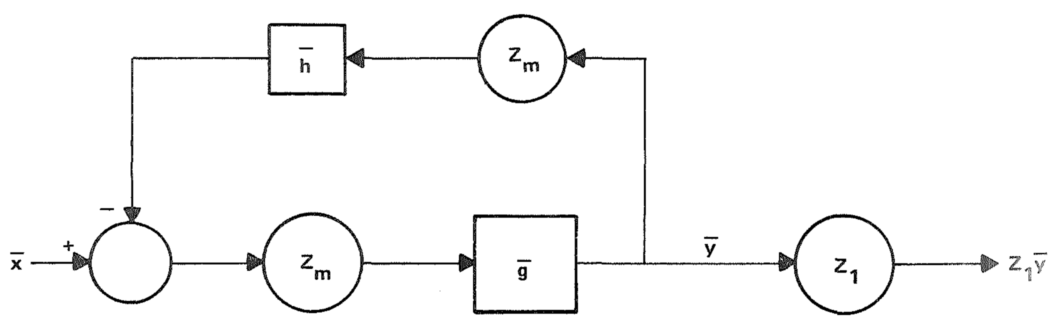

Figure 3. A single loop feedback device with a sampler in the feedthrough link and a sampler in the feedback link.

Lemma 10. The single loop feedback device of Figure 1 with a slow sampler in the feedthrough link and whose control equation is $\bar{g}Z_m(\bar{x}-\bar{h}\bar{y})=\bar{y}$ has an output response $\bar{y} = \dfrac{(\bar{g}Z_m\bar{x})}{[1+Z_m(\bar{g}\bar{h})]}$, and this output response is effectively computed with the formula $Z_1\bar{y} = \dfrac{(Z_1\bar{g})(Z_m\bar{x})}{1+Z_m(\bar{g}\bar{h})}$.

Lemma 11. The single loop feedback device of Figure 2 with a slow sampler in the feedback link and whose control equation is $\bar{g}(\bar{x}-\bar{h}Z_m\bar{y})=\bar{y}$ has an output response $\bar{y} = \dfrac{\bar{g}\bar{x}}{1+Z_m(\bar{g}\bar{h})} + \dfrac{(\bar{g}\bar{x})Z_m(\bar{g}\bar{h})-(\bar{g}\bar{h})Z_m(\bar{g}\bar{x})}{1+Z_m(\bar{g}\bar{h})}$ and is effectively computed with the formula $Z_1\bar{y}=Z_1(\bar{g}\bar{x}) - \dfrac{Z_1(\bar{g}\bar{h})Z_m(\bar{g}\bar{x})}{1+Z_m(\bar{g}\bar{h})}$. (Note the non-existence of transfer functions.)

The response of the feedback device in Figure 2 gives a clue to the existence of seminull functions. A function of time which has a zero or zero-crossing at every sampling time, $n(mT)$, is called a seminull function. Thus, there exists an $\bar{f}$ not identically zero such that $Z_m\bar{f}=0$.

The next constructive lemma is from Kim and Kranc [6].

Lemma 12. If $\bar{a}$ and $\bar{b}$ are any two distinct functions exclusive of improper impulse functions, then the canonical form of a seminull function is $\bar{a}Z_m\bar{b} - \bar{b}Z_m\bar{a}$. An immediate corollary to lemma 12 is the existence of semi-identities. The only proper impulse function is the delta function.

Definition. A function, $\bar{g}(s) \neq 1$, is a semi-identity if for a particular function, $\bar{f}(s)$, the equality $Z(\bar{f}\,\bar{g}) = Z\bar{f}$ is true.

Lemma 13. If $p(s)$ is a polynomial of finite degree and $\bar{n}(s)$ is a seminull function, then $Z\left[\dfrac{1}{p(s)} + \bar{n}(s)\right] = Z\left[\dfrac{1}{p(s)}\right]$ implies that the semi-identity is of the form $\bar{g} = 1+p\bar{n}$. There is a semi-identity example in the following problem. Find mT such that

$$Z_m\left[\frac{1}{s^2}\,\frac{1+\tau s}{1+\tau s+(\tau s)^2}\right] = Z_m\left[\frac{1}{s^2}\right] \quad . \tag{9}$$

The complete solution of these feedback devices hinges on a twofold application of extended R-Z identities of lemmas 7 and 8. Lemma 8 is a discrete interpolation lemma and this nomenclature explains its usage. Other detailed procedures to determine responses of these feedback devices can be found in texts on sampled data control devices.

Previous feedback devices contained $Z_m(\overline{g}\,\overline{h})$ in their responses. If both $\overline{g}$ and $\overline{h}$ are known, then partial fraction expansions can determine $Z_m(\overline{g}\,\overline{h})$ exactly or trapezoidal convolution can determine $Z_m(\overline{g}\overline{h})$ approximately. The feedback device of Figure 3 eases this problem slightly. This situation is presented in the next lemma and exercise.

<u>Lemma 14.</u> The single loop feedback device with the same samplers in the feedthrough link and in the feedback link and whose control equation is $\overline{g}Z_m(\overline{x}-\overline{h}Z_m\overline{y})=\overline{y}$ has response $\overline{y} = \dfrac{\overline{g}}{1+[Z_m\overline{g}]\,[Z_m\overline{h}]}\, Z_m\overline{x}$, and this response is effectively computed by $Z_1\overline{y} = \dfrac{Z_1\overline{g}}{1+(Z_m\overline{g})\,(Z_m\overline{h})}\, Z_m\overline{x}$.

For an exercise, use dissimilar Z_m and Z_1 samplers in Figure 3 and analyze the feedback device for its response. This situation is similar to that in lemma 11. Another exercise is to find the response of the feedback device in Figure 4.

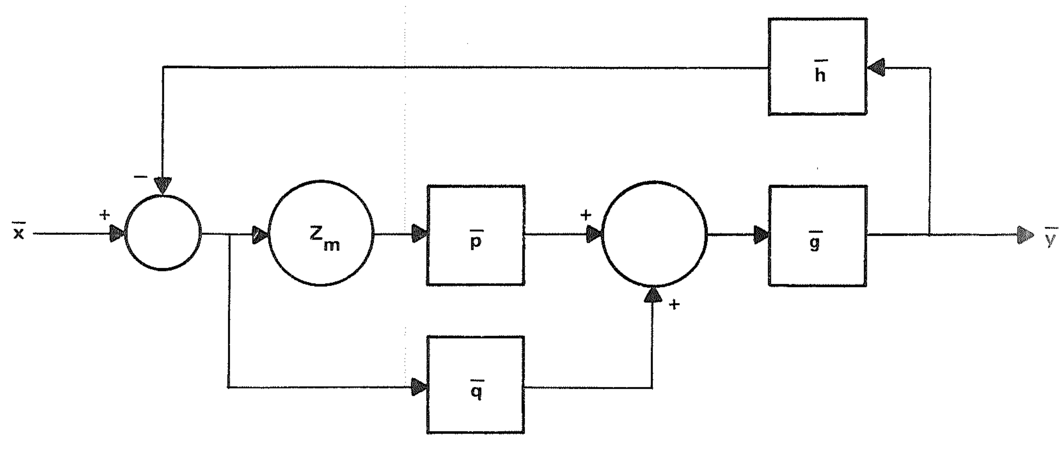

Figure 4. Single loop feedback device with a feedthrough link.

INTERPOLATORS

Interpolators are convolution type operators (i.e., transfer functions) that fill in points between two given sampled amplitudes (i.e., delta functions) separated in time. Note that nothing is implied about faithful reconstruction of the original sampled function in between the two given sampled amplitudes. Examples of full interpolators are the constant interpolator,

$$(1-e^{-Ts})/s$$

and the delayed linear interpolator,

$$(1-e^{-Ts})^2/Ts^2 \quad .$$

Fractional interpolators can be obtained from full interpolators by replacing T with αT where $0 < \alpha < 1$.

Often an interpolator will fulfill its purpose between two sampled values and will also produce tails beyond the largest of the two sampling instants. An example is the filter $\frac{T}{n} \sum_{k=1}^{n} \left[\frac{1}{1+(Ts/n)} \right]^k$ for any integer n. This filter is an approximant of the constant interpolator. These tails are effectively accounted for by extended R-Z identities.

Interpolation may produce only a finite number of fill-in points and a discrete interpolator results. One should investigate semi-identity and semi-delayor properties of interpolators.

INTERPOLATORS IN FEEDBACK DEVICES

The previous discussion did not include value judgments about the two interpolators. The linear interpolator is definitely superior to the constant interpolator. The next lemma shows that feedback can transform a constant interpolator into a linear interpolator.

Lemma 15. If $\overline{\gamma}(s)$ is an integrator-free proper rational function and the single loop feedback device of Figure 1 is used so that $\overline{h} = 1$ and

$\bar{g}(s) = \left(\frac{1-z^m}{s}\right)\left(\frac{\bar{\gamma}}{s}\right)$ (an extra integrator $1/s$ is adjoined), then the output response is linearly interpolated prior to the action of the transfer function $\bar{\gamma}(s)$.

The proof of this lemma is easy to execute by first specializing $\bar{\gamma}(s) = 10/(s+1)$. After executing an analysis, it will become apparent that the conclusion of this lemma is independent of the partial fraction expansion of a rational $\bar{\gamma}(s)$.

DELAYED SAMPLERS

A further generalization of Z-transforms is the delayed sampler, $Z_m^{(p)}$, in $Z_m^{(p)} f \triangleq \sum_{n=0}^{\infty} f[p+nm)T]z^{p+nm}$; m and n are integers and $0 \leq p \leq m-1$. The upper bound on the sampling delay real number p is not required, but this upper bound is useful in avoiding a "modulo m" suffix in succeeding formulas.

The delayed sampler can be used to study:

1. Sampler synchronization errors in feedback control devices.

2. Multiplexing and demultiplexing of many independent input signals.

3. Multiplexing of comparator signals in multiple control devices.

4. The polyphase periodically reverse switched capacitor.

5. The proper formulation of Simpson's Rule and higher order Newton-Cotes quadratures in numerical transforms.

6. The nonuniform periodic sampler.

Surely the wealth of physical devices has increased greatly.

Let us restrict m to be greater than one and begin with table construction.

Lemma 16. If $f_n(t) = \frac{t^{n-1}}{(n-1)!} u(t-pT)$, then $\bar{f}_n(s) = \frac{z^p}{s^n} \sum_{k=0}^{n-1} (pTs)^k/k!$.

Lemma 17. $Z_m^{(p)}\bar{f}$ is calculated in the following direct manner:

1. Start with $f(t)$;
2. Form $f(t) \cdot u(t-pT)$;
3. $\mathcal{L}[f(t) \cdot u(t-pT)] = z^p\bar{g}(s)$;
4. $\bar{g}(s)$ is given by step 3 (see lemma 15) ;
5. Calculate $Z_1^{(0)}\bar{g} = P(z)/Q(z)$, a rational function;
6. $Z_m^{(p)}\bar{f} = Z_m^{(p)}\left[Z_1^{(0)}(z^p\bar{g})\right]$, Q.E.D.

Note that this proof applies to both integral and fractional sampling delay numbers p.

Lemmas 15 and 16 and a table of $Z(1/s^n)$ (see the appendix) can be used to compute $Z_m^{(p)}(1/s^n)$. Other table entries are initiated when s is replaced by $(s+a)$ and $(s-j\omega)$ in lemmas 15 and 16.

Partial constant interpolation yields forms such as $Z_{mr}^{(p)}\left(\frac{1-z^r}{s}\bar{f}\right)$. The new problem is an integer form of the modified Z-transform which can be formulated exactly in the next lemma.

Lemma 18. If $f(t)$ is of sufficiently long duration, then $Z_{mr}^{(p)}(z^r\bar{f}) = z^r Z_{mr}^{(p \dot{-} r)}\bar{f}$ where

$$p \dot{-} r \equiv \begin{Bmatrix} p-r & \text{if } r \le p \\ mr+p-r & \text{if } r < p \end{Bmatrix} .$$

This special subtraction is equivalent to modulo mr subtraction, but the constraint, $p \le mr-1$, simplifies notation. If fractional delayors occur, then an approximation is given by the Newton interpolation formula as in the modified Z-transform. There exist a counterpart of the R-Z identity for delayed samplers [7].

Lemma 19. $Z_m^{(p)}\left[\bar{f}\, Z_m^{(q)}\bar{g}\right] = \left[Z_m^{(p \dot{-} q)}\bar{f}\right]\left[Z_m^{(q)}\bar{g}\right]$, where

$$p \dot{-} q \equiv \begin{Bmatrix} p-q & \text{if } q \le p \\ m+p-q & \text{if } q > r \end{Bmatrix}.$$

Slight generalizations of lemmas 6 and 8 are presented in lemmas 20 and 21, respectively.

Lemma 20. If m and p are integers, then $Z_m^{(p)}\,(Z_1\bar{f}) = \left(Z_m^{(p)}\bar{f}\right)$.

Lemma 21. If m and p are integers, then $Z_1\left(\bar{f}Z_m^{(p)}\bar{g}\right) = (Z_1\bar{f})\left(Z_m^{(p)}\bar{g}\right)$.

Lemma 20 is a general discrete interpolation lemma. An easy result appears next.

Lemma 22. If m and p are integers, then $\sum_{p=0}^{m-1} Z_m^{(p)}\bar{f} = Z_1^{(0)}\bar{f}$. This in turn leads to the nonuniform periodic sampler $\sum_{k=0}^{m-1} \alpha_k Z_m^{(k)}$, the skip sampler, where $\alpha_k = 0$ or 1.

Delayed samplers easily describe multiplexing and demultiplexing.

Lemma 23. If $\bar{x}$ is an $(m-1)$ vector of sampled input signals, which are time-division multiplexed and then passed through a single channel transfer function, $\bar{g}(s)$, then the demultiplexed output $(m-1)$ vector, $\bar{y}$, is given by the matrix-vector equation, $\bar{y} = M\bar{x}$, where the ijth elements of M are $\mu_{ij} = \left[Z_m^{(i \dot{-} j)}\bar{g}\right]$, $i, j = 1, 2, 3, \ldots, m-1$.

The main physical conclusions of this lemma are that a long tail on the impulse response of $\bar{g}(s)$ produces crosstalk, and cascading with M^{-1} will diagonalize the multiplex system.

PERIODICALLY REVERSE SWITCHED CAPACITOR

Delayed samplers are required in the analysis of the polyphase periodically reverse switched (PRS) capacitor [8, 9]. The surprising element is that the mathematics of sampling is applicable without any prior clues. The crucial observation here is that the delta function acts in another role as the carrier of the initial condition in the solution of a first order differential equation.

Consider two stationary half-circle commutator segments with zero thickness electrical insulation and a 1-farad capacitor rotating inside these commutator segments at a constant angular velocity such that it takes mT seconds to make a half revolution assuming it started pT seconds from reversal. The PRS capacitor will behave like an ordinary capacitor until polarity reversal occurs and the negative of the final value becomes the initial value for the next half revolution. Recall that the voltage-current description of an ordinary capacitor is $\bar{i} = s\bar{e} - e_0$. To reverse the polarity of the initial condition, $2Z_m^{(p)}\bar{e} + e_0 k(p)$ must replace e_0 resulting in

$$\bar{i} = s\bar{e} - 2Z_m^{(p)}\bar{e} - e_0 k(p) \quad , \quad \text{where } k(p) = \begin{Bmatrix} -1 & \text{if} & p=0 \\ +1 & \text{if} & p\geq 1 \end{Bmatrix} \quad . \qquad (10)$$

This formula shows the nonexistence of the impedance concept in this physical domain. The inverse operation will make this nonexistence apparent.

PCM DECODERS

The pulse code modulation decoder is the transfer function $1/(s+a)$ followed by a sampler. PCM is an elementary form of finite sampling in that $\sum_{k=0}^{m-1} \alpha_k z^k$, $\alpha_k = 0, 1$, a finite degree binary polynomial, is the signal. The division portion of Z-transforms executed on $\sum_{k=0}^{m-1} \alpha_k z^k/(1-0.5z)$ is sufficient to describe the decoder. The main results are:

1. The input fractional binary coder is reversed and m bits long.

2. The constant in $1/(s+a)$ is given by $a = (\ln 2)/T$.

3. The output of the demodulator, $1/(s+a)$, should be sampled at mT seconds to reproduce the original fractional amplitude.

The Shannon-Rack decoder [10] is an application of semi-identities to achieve local constancy at the sampling intervals, $0, T, 2T, \ldots, (m-1)T$ seconds. Such local constancy frees the final sampler from small pulse jitter effects. Simple calculations show that $\mathcal{L}^{-1}[1/(s+a)]$ and $\mathcal{L}^{-1}\{a/[(s+a)^2 + (2\pi/T)^2]\}$ have similar and opposite signed slopes at nT seconds. Hence their sum,

$$\frac{1}{s+a} + \frac{a}{(s+a)^2+(2\pi/T)^2} = \frac{1}{s+a} \cdot \frac{(s+a)(s+2a)+(2\pi/T)^2}{(s+a)^2+(2\pi/T)^2} , \tag{11}$$

gives the required transfer function for the Shannon-Rack decoder; the bi-quadratic rational function is the semi-identity for $1/(s+a)$. This overall transfer function can be synthesized by a single operational amplifier with RLC networks. PCM decoders depend on several linear disciplines such as passive network synthesis, pulse amplitude modulation, and binary arithmetic.

The reader is invited to study delta modulation [11] at this point. The relationship of PCM to delta modulation is obtained by contrast of efficient counting in the binary code with inefficient counting in the caveman code. In the caveman code, counting is achieved by entering n ONES.

THE SAMPLED DATA SERVO

The first problem in sampled data control devices is the construction of a servo. A servo is a device whose output will asymptotically follow or track its input. Attention will be focused on constructing the goal which we shall call "the linear servo admissibility problem."

The ideal servo has an identity transfer function, but this is definitely not attainable with motors that possess inertia and resistance. Hence, the exact identity needs to be replaced with an "approximate identity." Formality is enhanced by names, and "approximate identity" will be shortened to "aidentity." The servo admissibility problem is then the conversion of stable transfer functions into stable aidentities.

Classical feedback specialists were catalogers of physical devices and by the time this task was done, not much energy was left for admissibility studies, which were lumped unconsciously with performance studies. This was then followed by stability studies which were initiated by Cauchy about the year 1820. Thus, one becomes aware of the limited maneuvering space within confines of the three walls of servo study: admissibility, performance, and stability.

In 1953, King [12] initiated the admissibility problem by stating that servo transfer functions should have the form

$$\frac{a+bs+cs^2}{a+bs+cs^2+ds^3},$$

where the above is a proper fraction, and that input function space should be the unit polynomials $t^n/n!$. Unity of these polynomials stems from the fact that their nth derivatives are equal to one. The unit step function results when $n=0$. It is apparent that forced completion of the equality of coefficients in numerator and denominator of corresponding powers of s will achieve an exact identity.

The number of successive equal coefficient pairs is the aidentity order. The previous rational function has an aidentity order of three and its physical importance is that position, velocity, and acceleration error coefficients are all zero.

King's auspicious beginning was either unnoticed or barely accepted as the "Butterworth sense" approximation of an identity.

A procedure is now presented without proofs that extends but does not complete the King admissibility program for time invariant sampled-data rational functions.

Lemma 24. If β is any positive constant such that $-\frac{1}{2} < \beta < \infty$, then the transfer function $1/[1+\beta(1-z)]$ is a proper positive real function and an aidentity of order one.

Positive realness implies stability and a method is needed that will produce stable high order aidentities. Higher order aidentities can be constructed iteratively by a Newton process for approximating the $\sqrt{1}$ [13].

The next stable rational function satisfying King's conditions is that

$$\frac{1}{(1/2)\,[f+(1/f)]} = \frac{2+2\beta(1-z)}{2+2\beta(1-z)+\beta^2(1-z)^2}$$

and that the aidentity order is now two. Note that in the sampled data case, s is replaced by $(1-z)$ in determining aidentity order. Indicated operations preserve positive realness and hence all iterations will be stable. The general case is as follows.

Lemma 25. If A is a proper p.r.f. and an aidentity and B_n is defined by $\frac{1-B_n}{1+B_n} = \left(\frac{1-A}{1+A}\right)^{2n}$, $n=1,2,3,\ldots$, then each B_n is a closed form solution of the nth iterated Newton process, each B_n is a proper positive real function, and the aidentity order of B_n is strictly greater than the aidentity order of B_{n-1}.

The even power $2n$ can also be replaced by an odd power $(2n+1)$ and similar conclusions follow, except that this involves the closed form for a Halley process for finding the $\sqrt{1}$. Thus, a trivial method in numerical analysis yields effective procedures in servo transfer function admissibility.

Of course, there exist many arbitrary procedures for constructing stable approximate identities outside the realm of proper positive real functions. A Hurwitz polynomial can be placed in the denominator, and the numerator is formed by deletion of an appropriate number of coefficients to form the required aidentity order. Such an aidentity can be called a "Hurwitz aidentity."

An exercise will show that if $A=1/(1+s)$ in Lemma 25, then B_n can be realized by a unity feedback device and the feedthrough transfer function is a linear combination of single and multiple integrators. If s is replaced by $(1-z)$, then a similar statement can be made for discrete feedback devices, except that integrators must be replaced by partial summers. The point here is that continuous or discrete servos need not be unity feedback devices.

A proper positive real function has poles and zeros in the left-half plane only. An improper positive real function (Foster reactance function) has poles and zeros on the boundary between the left-half plane and right-half plane exclusive of s and $1/s$. Similar statements apply to functions of z, except that the stability boundary is the unit circle and the exceptional functions are $(1-z)$ and $1/(1-z)$. The next lemma characterizes the Hurwitz aidentity functions in a control context.

Lemma 26. If A is a proper Hurwitz aidentity and B_n is defined by $\frac{1 - B_n}{1 + B_n} = \left(\frac{1 - A}{1 + A}\right)^n$, then any B_n is a closed form solution of either a Newton or Halley process, a finite number of these will be stable aidentities, and the remainder will be unstable transfer functions.

The reader can prove that the Halley process aidentities are realized by model reference feedback devices.

On can foresee that semi-identity functions will enable existence of positive real functions of z whereas semidelayors will be an obstacle to utilization of positive real function semi-identities. In the latter case, one must resort to successive coefficient pair equalities followed by stability determination.

PROPER DIGITAL FILTER

The proper digital filter computes the periodic output of a stable linear time-invariant differential-recurrence system devoid of the transient, and the input signal is periodic. This definition casts the proper digital filter in the role of a discrete counterpart of classical $j\omega$ analysis [14].

Two lemmas characterize the discrete periodic input signal.

Lemma 27. A bounded periodic discrete function, $Z\bar{f}$, of period mT seconds is represented by $Z\bar{f} = B(z)/(1 - z^m)$, where $B(z)$ is a real coefficient polynomial whose degree is strictly less than the integer m and $B(z)$ and $(1 - z^m)$ are relatively prime. The coefficients of $B(z)$ are the sampled amplitudes of the periodic function.

An immediate consequence of this lemma eliminates the need for originating proper digital filters with the Fourier integral.

Lemma 28. If $Z\bar{f}$ is a bounded discrete periodic function of period mT seconds, then $Z\bar{f}$ can be represented by

$$\frac{1}{m} \sum_{k=1}^{m} \frac{B(e^{j2\pi k/m})}{1 - z\, e^{-j2\pi k/m}} \;, \tag{12}$$

a finite discrete Fourier sum.

Proof. 1. $B(z)/(1 - z^m)$ is a proper rational function;

2. All factors of $(1 - z^m)$ are distinct;

3. Heaviside's partial fraction expansion can derive the finite discrete Fourier sum of m terms;

4. Computation shows that $\frac{1}{m} \sum_{k=1}^{m} e^{j2\pi kp/m}\, e^{-j2\pi kq/m} = \delta_{pq}$,

the Kronecker delta.

The discrete output of a stable linear time-invariant differential-recurrence system subject to a periodic input is approximated by

$$Z\overline{y} \doteq \frac{P(z, y_0, f_0)}{Q(z)\,(1 - z^m)} \;, \tag{13}$$

where, unlike a transfer function, the numerator polynomial is a function of initial conditions in general.

Stability implies that $Q(z)$ and $(1 - z^m)$ are relatively prime. Furthermore, if $Z\overline{y}$ is a proper rational function, then straightforward partial fraction expansions on only the mth roots of unity yield the proper digital filter. The literature gives this device the name, "The discrete Fourier transform."

If $Z\overline{y}$ is not a proper rational function, then division must be initiated to eliminate a finite time transient. This yields a new numerator and the calculations of the previous paragraph follow.

Comb filters and Bessel, Butterworth, Chebyshev, and Cauer low-pass filters of classical filter theory and their bandpass, band-elimination, high-pass counterparts can be source material for digital filters, proper or not.

SUBHARMONIC DIGITAL FILTER

The digital computer enables extrapolation of classical filter networks by linear methods into the heretofore nonlinear phenomenon of subharmonic oscillations. In addition to a periodic input signal, one can imagine a periodic linear system yielding the output form

$$Z\overline{y} \doteq \frac{P(z)}{Q(z)(1-z^m)(1-z^n)} \quad . \tag{14}$$

This problem can be solved by the same digital filter methods of the previous section, and the essential material will be abstracted from the given form.

Observe that $1/(1-z^m)(1-z^n)$ has at least double poles on the unit circle for arbitrary integers m and n. The resultant unboundedness can be eliminated by reducing double poles on the unit circle; this results in $(1-z)/(1-z^m)(1-z^n)$ and a restriction that m and n be relatively prime integers. A linear subharmonic filter is defined by the form

$$Z\overline{y} = \frac{P(z)(1-z)}{Q(z)(1-z^m)(1-z^n)} \quad , \tag{15}$$

where m and n are relatively prime integers and (P,Q), $[P,(1-z^m)(1-z^n)]$, and $[Q,(1-z^m)(1-z^n)]$ are relatively prime polynomial pairs.

The rough lemma is: If m and n are relatively prime integers, then

$$\frac{(1-z)}{(1-z^m)(1-z^n)} = \frac{\prod^{K-1} F_k(z)}{(1-z^{mn})} \quad , \tag{16}$$

where the numerator is a product of cyclotomic polynomials whose total degree is less than the product mn and K is the number of distinct prime factors of the product mn.

The proof depends on a definition of the cyclotomic polynomial [15, 16],

$$F(z;pq) = \frac{(1-z^{pq})(1-z)}{(1-z^p)(1-z^q)} \quad ; \tag{17}$$

p and q are prime integers and Euler's factorization of an integer with prime factors. It is apparent that periods mT and nT seconds have been transformed into the period mnT seconds, hence the term "subharmonic digital filter."

Our terminology has not included the term "proper subharmonic digital filter" because there exists a binary choice of computing the output with or without the transient component.

The subharmonic digital filter is a concrete example of the arithmetic aspects of Z-transform applications. Periodic discrete functions are realized by circulating registers and the subharmonic filter is a natural extension.

CONCLUSIONS

Fundamentals of fast samplers with applications to numerical transforms and digital filters and delayed slow samplers with application to sampled-data controls, multiplexing, and polyphase periodically reverse switched capacitors have been presented.

There exist similarities to time-invariant systems and also striking dissimilarities. Indeed, both must be attended to and, thereby, one encompasses electrical networks and modulation.

It is hoped that the revised notation will gain acceptance. There exists a need for such words as semi-identity, semidelayor, and seminull functions. The former two words and the latter word are subtle variants of controllability and nonobservability concepts, respectively. Relation to ring theory at the scalar level without matrices is apparent.

APPENDIX: TABLE OF $Z(1/s^n)$

The given table presents coefficients of the n-2 degree polynomial $A_n(z)$ for calculating

$$Z\left(\frac{1}{s^n}\right) = \frac{T^{n-1}}{(n-1)!} \frac{z A_n(z)}{(1-z)^n} \tag{A-1}$$

for $n = 1, 2, 3, \ldots, 14, 15.$

The table can be extended with a coefficient recurrence relation

$$A(n,p) = (p+1)A(n-1,p) + (n-p-1)A(n-1,p-1) \quad , \tag{A-2}$$

where n and p are integers and

$$A_n(z) = \sum_{p=0}^{n-2} A(n,p) z^p \quad .$$

COEFFICIENTS: A(n, p) IN $A_n(z) = \sum_{p=-1}^{n-2} A(n,p)\, z^p$ FOR $Z(1/s^n) = \frac{T^{n-1} z}{(n-1)!} \cdot \frac{A_n(z)}{(1-z)^n}$

n \ p	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1														
2		1													
3		1	1												
4		1	4	1											
5		1	11	11	1										
6		1	26	66	26	1									
7		1	57	302	302	57	1								
8		1	120	1 191	2 416	1 191	120	1							
9		1	247	4 293	15 619	15 619	4 293	247	1						
10		1	502	14 608	88 234	156 190	88 234	14 608	502	1					
11		1	1 013	47 840	455 192	1 310 354	1 310 354	455 192	47 840	1 013	1				
12		1	2 036	152 637	2 203 488	9 738 114	15 724 248	9 738 114	2 203 488	152 637	2 036	1			
13		1	4 083	478 271	10 187 685	66 318 474	162 512 286	162 512 286	66 318 474	10 187 685	478 271	4 083	1		
14		1	8 178	1 479 726	45 533 450	423 281 535	1 505 621 508	2 275 172 004	1 505 621 508	423 281 535	45 533 450	1 479 726	8 178	1	
15		1	16 369	4 537 314	198 410 786	2 571 742 175	12 843 262 863	27 971 176 092	27 971 176 092	12 843 262 863	2 571 742 175	198 410 786	4 537 314	16 369	1

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