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Abstract

The steady state heat conduction equation representing the diurnal variations in the upper atmosphere are numerically integrated with extreme ultraviolet radiation as the only source of energy input. Two models of vertical flow are assummed and horizontal flow is neglected. The results yield a maximum at 1800 local time. Other dynamical effects are discussed.

INTRODUCTION

The diurnal variation of the upper atmosphere at heights above 200 km has been well established by satellite drag observations. During the morning, the density increases until a maximum is reached at about 14:00 hours local time. Then it decreases rather rapidly in the afternoon and evening, followed by a slower rate of decrease during the night till about 04:00 hours local time. This is observed for all levels of solar activity.

Models of this variation have been constructed by various authors (Jacchia (1965), Harris and Priester (1962)) relating temperature variations and density variations (through the hydrostatic law). In all these models, the temperature and density variations are in phase above 200 km.

The model of Harris and Priester (1962) was a time dependent one which produced the observed variations by considerations of the effects heat conduction, solar extreme ultraviolet heating and an adjustable artificial heat source. The time dependent heat conduction equation in one dimension was integrated assuming hydrostatic equilibrium at every instant. Their theory also included a model of vertical convective heat transport. Horizontal conduction and convective transport was assumed to be negligible. It was found necessary to introduce a second artificial heat source in order to obtain a maximum at 14:00 hours local time. With E. U. V. heating alone the maximum would appear at about 18:00 hours local time with an amplitude too large (with the ratio of the diurnal maximum to minimum in temperature greater than two, instead of the model produced ones of 1.3 or 1.5). Lagos and Mahoney (1967) has obtained similar results with E. U. V. heating (though with a smaller variation). They also calculated latitudinal variations, by allowing for zenith angle variations with latitude, but still only integrating the one dimension time varying heat conduction equation.

In this paper, the three dimensional heat conduction equation is integrated. Essentially similiar results are obtained for the latitudinal and longitudinal variations as those derived by integrating the time dependent one dimensional equation without the extra heat source. It is concluded that dynamical effects are sizeable, so that their inclusion will be necessary (and hopefully sufficient) to bring about agreement with observation.

Steady State Heat Conduction Equation

The time dependent heat conduction equation is

$$\rho C_p \left(\partial T/\partial t + \underline{u} \cdot \nabla T \right) = \nabla \cdot K \nabla T + Dp/Dt + Q \,, \tag{1}$$

where Q is the heat sources and sinks, $\underline{u}$ the convective flow velocity, C_p the specific heat at constant pressure, K the thermal conductivity and ρ the density. The diurnal variation appears to be a steady state phenonemon, though its amplitude and mean value may vary due to varying levels of solar activity. Thus in a frame of reference fixed with respect to the sun the atmosphere appears to have a bulge displaced about thirty degrees from the earth-sun line towards the evening side, though the atmosphere may rotate rigidly with the earth. Thus in the steady state, we may take the term on the left side as zero, and we are left with the three dimensional heat conduction equation. But to an observer on the earth, there appears to be dynamical variations. To obtain the appropriate form of the heat conduction equation for an observer on the earth, we must first transform equation (1) to a frame rotating with the earth and then take the limit to steady state. Let ϕ be the longitude in the fixed frame and ϕ' the longitude in the rotating frame, and let t' be the time in the rotating frame. Then the transformation equations are

$$\phi' = \phi - \omega t \quad ,$$
$$t' = t \quad ,$$

where ω is the angular velocity of the earth.

The partial derivative with respect to longitude becomes

$$\partial/\partial\phi = \partial/\partial\phi' \quad ;$$

and the partial derivative with respect to time is

$$\partial/\partial t = \partial/\partial t' - \omega\, \partial/\partial\phi'$$

In the steady state we can let the partial derivative with respect to time go to zero and thus obtain the following relationship between the local time variations and the longitudinal variations for an observer on the earth

$$\partial/\partial t' = \omega \cdot \partial/\partial \phi'$$

(vector quantities may similiary be transformed (see Eshenazi (1967)).

Once one has obtained a solution as a function of t', one can obtain the general solution by replacing the argument t' with $\omega t' + \phi'$. Care must be exercised when this relationship is used; in particular, one must not try to describe transient phenomena using equations based upon this relationship. Boundary conditions imposed upon the equations must be consistent with the assumption of steady state if such a relationship is used (see Volland (1966). The heat conduction equation, upon the assumption of steady state, becomes an elliptic type with the periodicity in the longitudinal coordinate (or local time) as an essential boundary condition. The solution will be completely determined by imposing boundary conditions at the lower and upper boundaries in altitude, together with the knowledge of the heat sources and sinks.

It is to be noted that the solutions of the tidal equations (Chapman and Lindzen (1970)) that would correspond to steady state are those for which the parameters f and s in tidal theory have the ratio one half.

For the heating due to the extreme ultra-violet radiation, we take

$$Q_{EUV}(r,\theta,\phi) = \sum_i \epsilon_i n_i(r,\theta,\phi) \int_0^\infty d\lambda\, F(\lambda)\, \sigma_i(\lambda) \exp(-\tau(r,\theta,\phi)) \quad , (2)$$

where τ is the optical depth.

$$\tau(r,\theta,t) = \sum_i \int_r^\infty \sigma_i(\tau)\, n_i(r',\theta',t)\, dr'\, r'/\sqrt{r^2 - r'^2 \sin^2\theta}$$

As we are interested in along latitudinal variations, the plane earth approximation used by Harris and Priester is no longer valid. Care must be exercised here in the calculation of the optical depth so that one obtains the correct phase of the heating source as a function of latitude and local time. After numerous tests of various methods, the following approach was adopted for the calculations of the optical depth. Consideration had to be given to the speed of calculation and memory size available in the computer. The technique yielded an accuracy of about 2% in the calculation of the optical path for critical values of the optical path (values between 3 and .002. This gives yielded a maximum numerical error of about 7 % in the calculation of the heating. Variations of the density along the optical path due to local time variations could be ignored (test showed that this caused an error of at most a fraction of a percent. Thus densities corresponding to the local time and latitude at which the heating was being calculated were used in the calculation of the optical path. The technique is at follow: In the numerical integration of the heat conduction equation, altitude steps of one km were used. For the calculation of the optical depth, a seven point Gauss-Laquerre type integration formula was used, using as a scaling factor the scale height of the constituent. Linear interpolation was performed for the densities when the abcissas of the integration formula fell between the altitude steps. As the scale heights ranged in values from 25 km and up no lost in accuracy was obtained by this procedure. For zenith angles greater than 88 degrees, the procedure was modified by breaking the integral into two parts. The first part covering a range of altitudes from the point at which the absorption is being calculated) to a height of one scale height was calculated by a Gauss-Chebyshev type integration formula, and then the remainder by Gauss-Laquerre type.

The cooling due to the reradiation on the infra-red from atomic oxygen was included. It is sufficient for this type of calculation to take the entire extreme ultraviolet flux as composed of two flux lines (see Mahoney (1966), Harris and Priester (1962)) with the following cross sections; 1.5×10^{-17} cm^2 for nitrogen and molecular oxygen, 1.2×10^{-17} cm^2 for atomic oxygen. The total flux used in the EUV range was 1.2 ergs/cm^2 sec and 0.5 ergs/cm^2 sec of ratiation in the Shuman-Runge region. The initial number densities at 120 km were 4×10^{11}/cm^3, 7.5×10^{10}/cm^3 and 7.6×10^{10}/cm^3 for nitrogen, molecular oxygen and atomic oxygen respectively. The calculation of the thermal conduction coefficients and specific heats was as performed by Harris and Priester (1962).

For a complete solution of the problem one should also include the Navier-Stokes equation together with the continuity equation and the equation of state. This will not be attempted here. Instead, we shall use a phenomological approach, making reasonable physical assumptions and see what conclusions can follow. As the approach to hydrostatic equilibrium is rather rapid compared to the time scale of the phenomenon we shall be interested in (see Thomas (1969)), we shall assume hydrostatic equilibrium in the steady state. We are interested in the steady state solution so we shall follow the procedure discussed by replacing time derivatives with longitudinal derivatives or vice versa, in a system rotating with the earth.

In this work, we shall not deal with horizontal convective transport of heat. For the vertical convective transport of heat, we may consider several models. In the construction of these models, one should take care that the interpretation in the rotating frame and fixed frame are consistent. The first one we shall consider is the following. In an earlier paper (Harris and Priester (1965)), it was assumed that the diurnal motion of the atmosphere was such that any parcel of the atmosphere moved in such a manner that the total pressure above it was constant (see Thomas and Chins (1969)). This may be expressed as follows

$$DP/Dt = 0 \quad ,$$

where D /Dt is the total convective derivative, i.e. we have

$$\partial P/\partial t + \underset{\sim}{u} \cdot \nabla P = 0 \tag{3}$$

where $\underset{\sim}{u}$ is the flow velocity. Ignoring horizontal flow and making use of hydrostatic equilibrium

$$\partial P/\partial r = - \rho g$$

we have (where W is the vertical flow velocity and g the acceleration of gravity and ρ the density.

$$\partial P/\partial t = W \rho g \quad ,$$

or

$$W = (RT/g)\, \partial/\partial t\, \ln P \quad ,$$

where R is gas constant. Making use of the hydrostatic law again, we have from

$$\ln P = -\int_{r_0}^{r} (1/H)\, dr + C \quad ,$$

where H is the pressure scale height and r_0 the lower boundary and r the altitude we are at, we finally have

$$W = -H\, \partial/\partial t \left(\int dz/H \right) \quad . \tag{4}$$

Thus we have the final form for the heat conduction equation

$$\rho C_p \left(\partial T/\partial t + W\, \partial T/\partial r \right) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 K\, \partial T/\partial r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{K}{r} \frac{\partial T}{\partial \theta} \right)$$

$$+ \frac{1}{r\omega \sin\theta} \frac{\partial}{\partial t}\left(\frac{K}{r\omega \sin\theta} \frac{\partial T}{\partial t} \right) + Q \quad . \tag{5}$$

Here C_p is the specific heat at constant pressure, r the vertical distance from the center of the earth, K the thermal conductivity, θ the colatitude, t the local time or longitude, ω the angular velocity of the earth. The calculations of the heat sources and sinks have been discussed. The construction of the finite difference equation and other numerical procedures are given in the appendix.

An alternate model of the vertical flow can be derived from the following considerations. It has been convenient in studies of fluid flow to assume incompressibility and its implication diverence-free flow. In the upper atmosphere the incompressibility condition cannot be utilized; but it may be, because of frictional effects (ion drag, viscosity), that the steady state of the fluid can be such where the flow may be regarded as divergence-free, i.e.

$$\nabla \cdot \underset{\sim}{u} = 0 \quad .$$

We shall explore the results of this assumption together with hydrostatic equilibrium in the steady state (again ignoring horizontal transport).

From the continuity equation

$$\partial \rho / \partial t \quad + \nabla \cdot (\rho \underset{\sim}{u}) = 0$$

we have

$$W_D = - (\partial \rho / \partial t) / (\partial \rho / \partial z) .$$

where W_D is now the vertical flow velocity. From the perfect gas law we have

$$W_D = \left(-\partial P/\partial t - \frac{P}{T}\frac{\partial T}{\partial t}\right) \Big/ \left(\partial P/\partial z - P/T \, \partial T/\partial z\right). \quad (6a)$$

Making use of hydrostatic equilibrium and using the expression for W, we obtain (where H is the scale height)

$$W_D = \left(W - H/T \, \partial T/\partial z\right) \Big/ \left(1 + H/T \, \partial T/\partial z\right). \quad (6b)$$

Thus the heat conductivity equation becomes

$$\rho C_v \left(\partial T/\partial t + W_D \, \partial T/\partial r\right) = \nabla \cdot (K \nabla T) + Q, \quad (7)$$

where C_v is the specific heat at constant volume (as the flow is divergence-free). One may expect different solutions from this form as compared to the earlier form as now the local time derivative has a coefficient down by a ratio of the specific heats; and thus the solutions should follow the local time variations of the heat source more closely than the previous ones.

The two solutions are compared in Figure 1; where the equatorial exospheric temperature is plotted for the same heat input at the equinoxes. It is seen that the diverence free solution has a greater amplitude (as expected because of the smaller heat capacity), but the maximum is only slightly shifted with respect to the solution of the first equation (5). In both cases the maximum appears close to sunset not at fourteen hours local time.

In Figure 2, there is given the solution corresponding to the first case, where the convective derivative of the pressure is zero. Curves of constant exospheric temperature are plotted for latitude versus local time. It is seen that there is decided latitude variation (as constrasted to the drag results given by Jacchia and Slowey (1967)). This is mainly due to the variation of zenith angle as Lagos and Mahoney (1967) has found by integrating the time dependent one dimensional equation. There appears to be a shift at high latitudes towards noon where the heat input is low.

In Figure 3, the solution is given for the solstice case (in all solutions the incident heat flux is the same). Thus these solutions give a decided seasonal variation which again is not observed from the satellite drag measurements.

Discussion of Horizontal Flow:

The complete solution for the heat transport problem in the upper atmosphere would involve the solution of the Navier-Stokes equation for the horizontal flow velocities, together with the heat conductivity equation. In addition, any coupling with the lower atmosphere has to be considered. Horizontal flow has been calculated by several researchers from the diurnal variation as given by various atmospheric models. In these calculations gross simplifications of the Navier-Stokes equations have been made. The importance some of these simplifications can be estimated.

The Navier-Stokes equation in the form we shall consider in the rotating frame is

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} + \underline{\omega}\times(\underline{\omega}\times\underline{r}) + 2\underline{\omega}\times\underline{u} = -\nabla p/\rho + \underline{g} + \underline{F_I} + \underline{F_v} , \qquad (8)$$

where ρ is the density, p the pressure, g the acceleration due to gravity, F_I the ion drag and F_v the viscous drag.

We shall assume steady state and to make estimates of the order of magnitude of various terms, we shall consider the horizontal flow velocity along the equator during equinox conditions. Furthermore, we shall assume that the vertical flow velocity is given by W.

Letting v be the horizontal flow velocity at the equator the Navier-Stokes equation becomes under these assumptions.

$$\frac{\partial v}{\partial t} + \frac{v}{r\omega}\frac{\partial v}{\partial t} = -\left(\frac{1}{r\rho\omega}\right)\partial p/\partial t + F_I + F_v . \qquad (9)$$

At the equator we may ignore Coriolios and the centrifugal term. We shall consider what velocities may be obtained from a linearized treatment ignoring drag terms, i.e. when we have

$$\frac{\partial v}{\partial t} = -\frac{1}{r\rho\omega}\,\partial p/\partial t = -\frac{1}{r\omega}\,\frac{RT}{g}\; g\,\partial/\partial t\,(\ln p) .$$

In order to make an estimate we may replace RT/g by some mean value of the scale height $\bar{H}$ along the equator and integrate to obtain

$$v - v_0 = g/\omega r \; \bar{H} \; \ln\left(p/p_0 \right) ,$$

where v_0, p_0 are some reference velocity and pressure at some point on the equator. As an estimate, we may set ln (p/p) to unity, ωr equal to 450 m/sec and $\bar{H}$ to 50 km. Then v - v_o is about 1000 m/sec, much greater than that given by the earth's angular velocity. With such velocities, it is difficult to understand how a steady state condition could be obtained.

If we were to include the non-linear term, i.e. $v/r\omega \frac{\partial v}{\partial t}$ then the solution would be

$$v - v_0 = r\omega \left(-1 \pm \sqrt{1 - \left(2g/\omega r \right) \bar{H} \ln\left(p/p_0 \right)} \right) , \tag{10}$$

where we would choose the positive square roots in order to obtain the linearized result for small pressure gradients. This then implies that for a steady state to exist we must have

$$\frac{2g}{\omega r} \; \bar{H} \; \ln\left(p/p_0 \right) < \omega r \tag{11}$$

Thus to obtain steady state conditions in the upper atmosphere, with the pressure gradients that exist, we must include the other terms neglected, i.e. ion drag, viscosity etc.

For the form of the ion drag, we may take

$$\underline{F_I} = - \rho_i \nu_i / \rho \cdot \underline{u} \tag{12}$$

(see Geissler (1966)), where ρ_i is the ion density which we shall assume to be fixed to the field lines, ν_i the effective collision frequency. Geissler uses a value for ν_i/N (where N is the number density) of 5.2×10^{-10} cm^3/sec. Stubbe (1968) has calculated an improved value for the case of atomic oxygen of about a factor two greater, but we shall use the smaller value in order to underestimate the effect of ion drag. For the ion density, we shall use 2×10^{-5} cm^3 a rather low value. For an estimate of the velocity, let us choose a value given by the earth's angular velocity, for it is only magnitudes comparable to this value that can cause appreciable heat transport to effect the diurnal variation, i.e. $u = \omega r$. We shall calculate the ratio of the ion drag to the driving force, the pressure gradients. The pressure gradient can be approximated by

$$(1/\omega r)\, \bar{H} g\, \partial/\partial t\, (\ln P)$$

For $\partial/\partial t \ln P$ we may use a value of $1/4 \times 10^{-4}$, assuming that the logarithm of the pressure changes by unity in a half of a day (this is an overestimate). $\bar{H}$ may be taken to be 50 km. Then the ratio of the ion drag to the pressure gradients

$$(\rho_i \nu_i / \rho \cdot u) / (1/\omega r \cdot \bar{H} g\, \partial/\partial t \ln P) \tag{13}$$

becomes $4.5/3 > 1$.

It is to be noted that the above ratio is an underestimate. Thus for velocities comparable to the earth's rotational velocity, the ion drag force is comparable to the pressure gradients. Thus velocities much smaller than that given by the earth's rotational velocity are to be expected. But for the convective horizontal transport to be effective, it must have a value comparable to the value of the local time derivative in the heat conduction equation.

$$\underset{\sim}{u}_H \cdot \nabla_H T \approx \partial T / \partial t \tag{14}$$

Here $\underset{\sim}{u}_H$ is the horizontal velocity, ∇_H the horizontal gradient of the temperature. For the horizontal gradient of the temperature (at the equator) we may take it to be (again assuming steady state and for equatorial conditions at the equinox) given by the local time variations and the lefthand side of the above equation becomes

$$u/\omega r \cdot \partial T/\partial t \quad ,$$

which as seen implies velocities comparable to the earth's angular velocity in order for the above expression to be comparable to the left hand side of (14).

Another source of energy input into the upper atmosphere are gravity waves. It has been suggested that these may propagate upward with sufficient energy that it can be comparable to the E. U. V. heating. This implies a dynamical coupling between the upper atmosphere and the mesosphere. In the steady state, these motions must occur repeatedly. Such an energy input is not inconsistent with the assumption of steady state and hydrostatic equilibrium. We can incorporate this possibility in the method utilized here.

Let us again ignore horizontal flow and assume hydrostatic equilibrium, and let U be the vertical upward velocity. Using the following expression for W

$$W = \frac{1}{\rho g} \, \partial p / \partial t \quad ,$$

then the total corrective derivative of the pressure may be expressed as

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + U \frac{\partial p}{\partial r} = g\rho\,(W - U) \quad .$$

The heat conductivity equation becomes

$$\rho C_p \left[\frac{\partial T}{\partial t} + \left((U - W) + W \right) \frac{\partial T}{\partial r} \right] = g\rho\,(W - U) + \nabla \cdot k \nabla T + Q,$$

where we can regard the terms containg $W-U$ as a correction to equation (5). The expression for W identically vanishes at the lower boundary (120 km) and the value of U at this lower boundary will be determined by the propagation of tidal motions from the lower atmosphere to this altitude.

An estimate of this additional term to the right handside of the heat conductivity equation can be made as follows. If we assume that U does not change appreciably over one scale height, so that we may take the difference $W-U$ as a constant U_c , then we may integrate this term over the altitude and obtain an estimate of the peak flux due to tidal motions from the lower atmosphere into the upper atmosphere.

Thus we obtain

$$\omega_0 \int_{z_0}^{\infty} \rho g \, dz \simeq u_0 p \quad ,$$

where p is the pressure at the lower boundary about equal to 2.6×10^{-2} dynes /cm^2. Thus a decimal varying velocity at the lower boundary of less than 1m/sec would represent an energy input comparable to the E.U.V. flux. Harris and Priester (1962) have shown that an additional heat source of this magnitude can be adjusted so as to obtain agreement with satellite drag observations.

Lindzen and Blake (1970) have shown that the semi-decimal component of the lower atmospheric tides may propagate to the upper atmosphere with a flux in agreement with the above estimate.

For a fuller description of the steady state of the upper atmosphere one needs a model of the tidal motions at the lower boundary together with the integration of the Navier-Stokes equation for horizontal flow. Though longitudinal flow may be decreased by ion-drag, the numerical results presented here (which have large latitudinal temperature gradients) indicate that convective transport towards the poles)for which ion drag is less important can be significant. But such a flow is strongly coupled to the longitudinal flow due to the non-linear and Coriolis terms in the Navier-Stokes equation. The calculations are more involved and work toward developing a numerical procedure is in progress. In such a procedure, it is a prerequisite the successful integration of the three dimensional heat conduction equation, e.g. by a method described in this paper.

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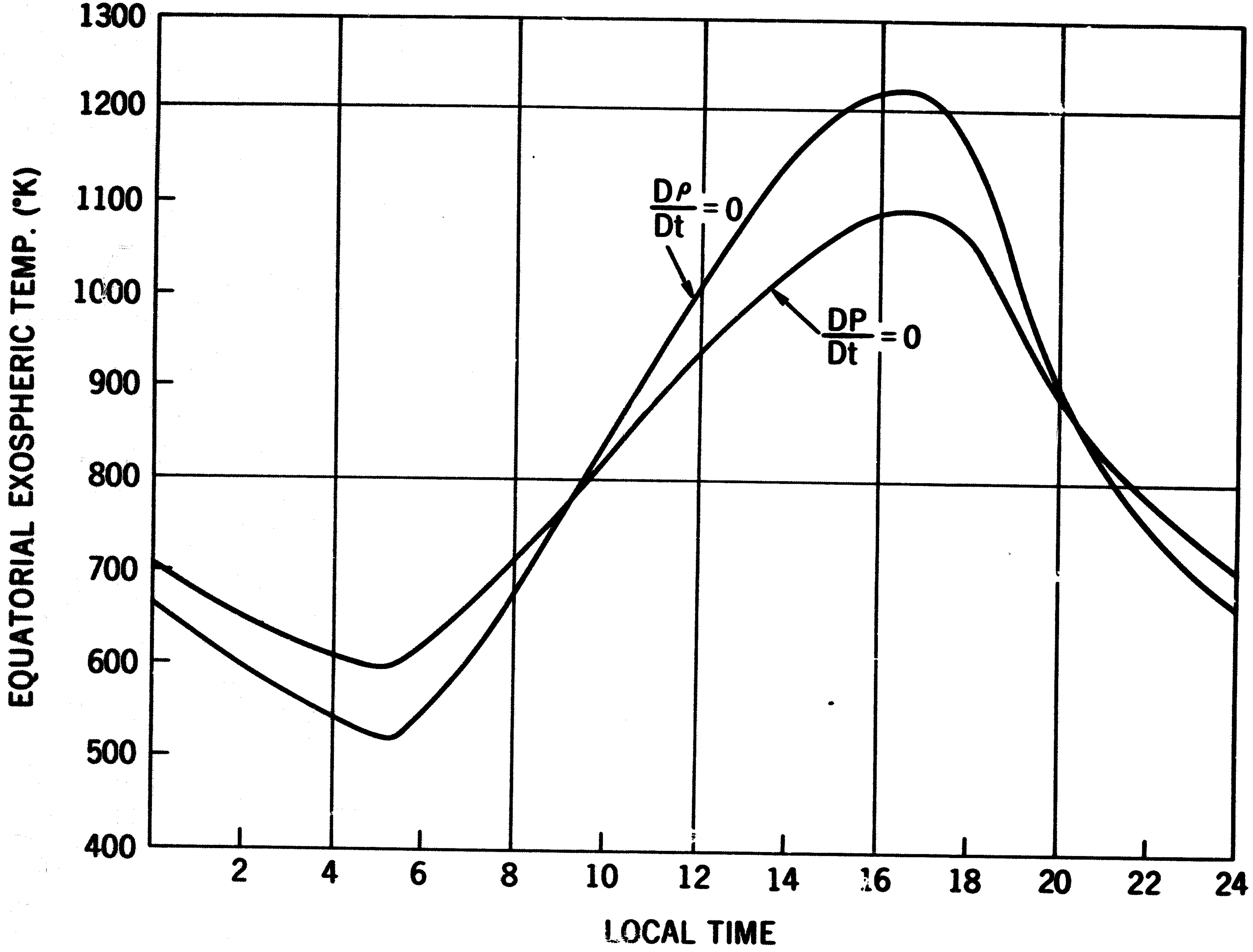
Figure Captions:

Fig. 1. Equatorial exospheric temperatures for two models of vertical flow.

Fig. 2. Latitudinal and local time variations of the exospheric temperatures for equinox conditions.

Fig. 3. Latitudinal and local time variations of the exorpheric temperatures for solstice conditions.



1300
1200
1100
1000
900
800
700
600
500
400
EQUATORIAL EXOSPHERIC TEMP. (°K)
$\frac{D\rho}{Dt} = 0$
$\frac{DP}{Dt} = 0$
2
4
6
8
10
12
14
16
18
20
22
24
LOCAL TIME

Fig 1

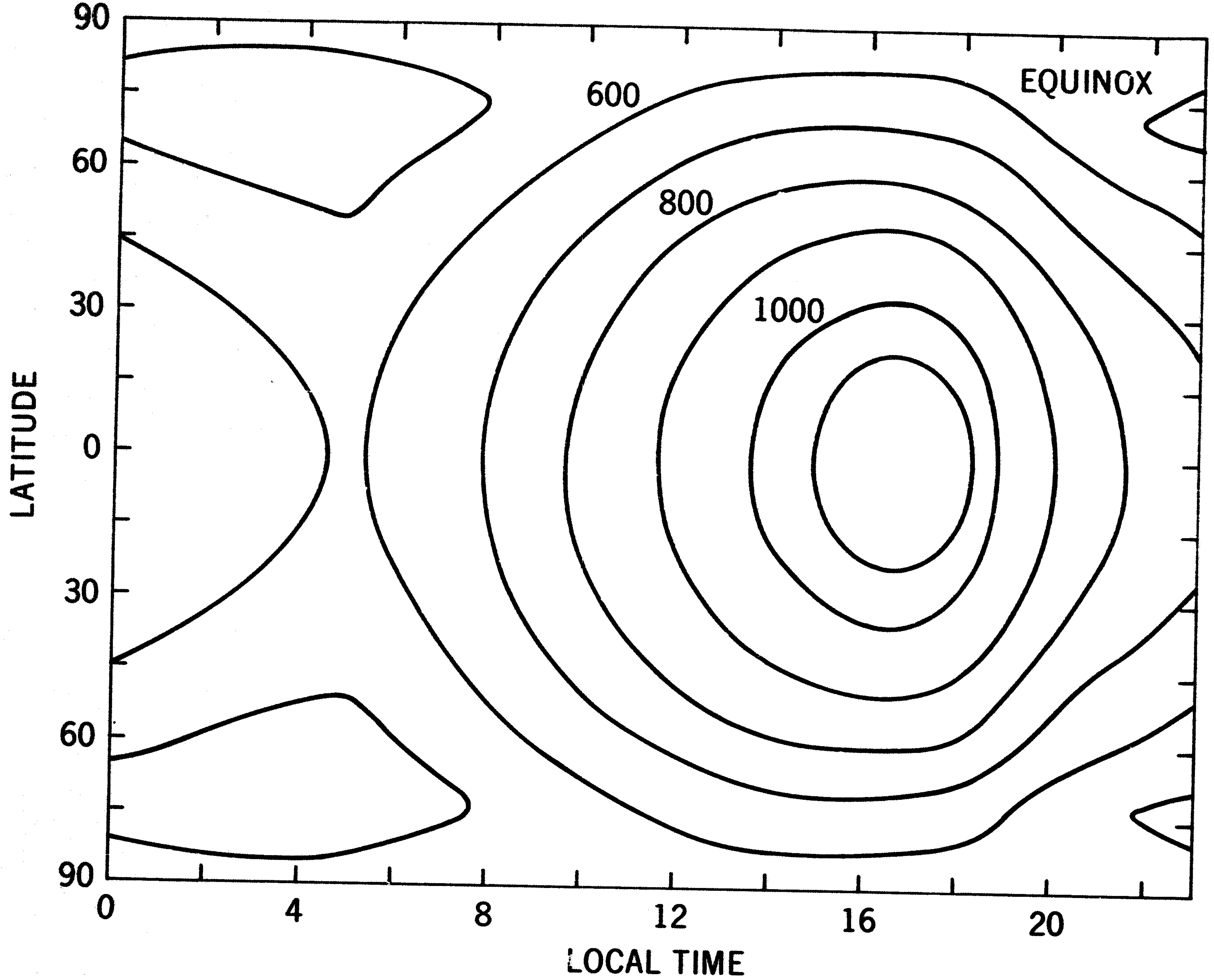

EQUINOX
600
800
1000
LATITUDE
90
60
30
0
30
60
90
0
4
8
12
16
20
LOCAL TIME

Fig. 2

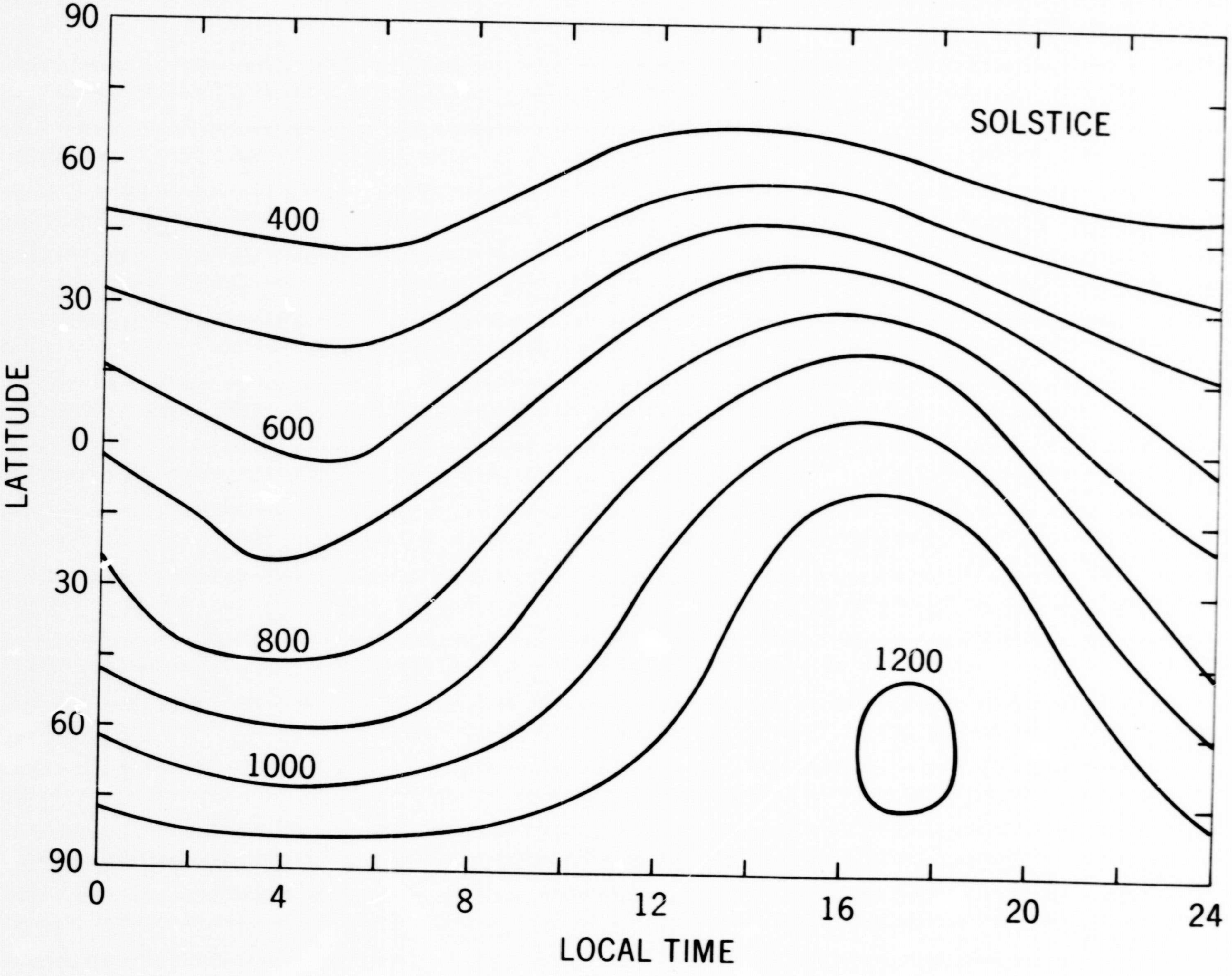

SOLSTICE
400
600
800
1000
1200
90
60
30
0
30
60
90
LATITUDE
0
4
8
12
16
20
24
LOCAL TIME

Fig. 3

APPENDIX

Derivation of the Difference Equations

The steady state heat conduction equation in three dimensions in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 K \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta K \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(K \frac{\partial T}{\partial \phi}\right)$$

$$+ Q = \rho C_p \left(\omega \frac{\partial T}{\partial \phi} + W \frac{\partial T}{\partial r}\right)$$

Here r is the radius vector, θ the colatitude, ϕ the longitude, Q the heat sources and sinks as given by equation two, W the vertical flow velocity as given by equation (4) or (6b), ρ the mass density, K the thermal conductivity.

The development of the difference equations will start will the procedure given in Varga (1962) on the derivation of difference equations for elliptic equations in more than one dimension. We divide the three dimensional space to increments Δr, $\Delta\theta$, $\Delta\phi$ and denote the various values of r by the subscript j, of θ by the subscript k, and of ϕ (or local time) by the subscript or super-subsrcipt i. For points not on the boundary let us consider the elementary volume which is a six sided parallapiped centered around the point (r_j, θ_k, ϕ_i) whose corners pass through the points given by the set of values ($r_j + \Delta r/2$, $\theta_k + \Delta\theta/2$, $\phi_i + \Delta\phi_i/2$). The term involving the thermal conductivity may be integrated over the elementary volume and expressed as surface integrals by means of Greens theorem. Then the surface integrals are performed by replacing the values of the various quantities by their values at the center of the surfaces. The term containing $\partial T/\partial\phi$ is integrated by evaluating its coefficients at the center of the elementary volume. All other terms are integrated by assuming their values at the center of the elementary volume. Then the first derivitives of the temperature with respect to r, θ, at $j \pm \frac{1}{2}, k \pm \frac{1}{2}$ are replaced by central difference formulae. The values of the physical quantities at $j \pm \frac{1}{2}, k \pm \frac{1}{2}$ are calculated by taking their mean vlaues at $j \pm 1$, $k \pm 1$ appropriately. Aside from the matrix elements that arise from the condition that the temperature should be periodic in longitude the matrix of the coefficients of the difference equation can be arrange to that one would have tridiagonal matrix whose elements are matrices. Also the term $\partial T/\partial\phi$ should dominate over the term $\partial/\partial\phi(k\, \partial T/\partial\phi)$; if the latter term weren't present the heat conduction equation in steady state would be of parabolic type. Thus an iterative method in terms of discrete approximation to a parabolic differential equation, (Varga (1962)) where we iterate until a periodic solution in longitude is obtained would appropriate. To obtain such a method we express the temperature at the point $i+\frac{1}{2}$ in terms of points at $i, i-1, i-2$, i.e. (writing now the index i as an superscript)

$$T_{j,k}^{i+1/2} = 1/8\left(15T_{j,k}^{i} - 10T_{j,k}^{i-1} + 3T_{j,k}^{i-2}\right) \quad ,$$

and the longitudinal derivative of the temperature

$$\left(\partial T/\partial\phi\right)_{j,k}^{i+1/2} = \frac{1}{\Delta\phi}\left(2T_{j,k}^{i} - 3T_{j,k}^{i-1} + T_{j,k}^{i-2}\right) \quad .$$

Finally one obtains the following difference equations for points not on the boundary.

$$a_{j,k}^{i}T_{j-1,k}^{i} + b_{j,k}^{i}T_{j+1,k}^{i} + u_{j,k}^{i}T_{j,k-1}^{i} + \ell_{j,k}^{i}T_{j,k+1}^{i} + d_{j,k}^{i}T_{j,k}^{i} = R_{j,k}^{i} \quad , \tag{A1}$$

where

$$a_{j,k}^{i} = \sin\theta_k K_{j-1/2,k}^{i}\, r_{j-1/2}^{2}/(\Delta r)^2 , \qquad b_{j,k}^{i} = a_{j+1,k}^{i} \quad , \tag{A2}$$

$$u_{j,k}^{i} = K_{j,k-1/2}^{i}\sin\theta_{k-1/2}/(\Delta\theta)^2 , \qquad \ell_{j,k}^{i} = u_{j,k+1}^{i} \quad , \tag{A3}$$

$$-d_{j,k}^{i} = a_{j,k}^{i} + b_{j,k}^{i} + u_{j,k}^{i} + \ell_{j,k}^{i} - K_{j,k}^{i}\left[1 + \tfrac{3}{4}\left(T_{j,k}^{i-1} - T_{j,k}^{i-2}\right)/TT_{j,k}^{i}\right]/(\sin\theta_k\Delta\phi^2)$$

$$+ 11/8\ \omega(\rho c_p)_{j,k}^{i}\, r_j^2 \sin\theta_k/\Delta\phi \quad , \tag{A4}$$

$$R_{j,k}^{i} = -Q_{j,k}^{i}\sin\theta_k r_j^2 - 1/(\sin\theta_k\Delta\phi^2)\cdot K_{j,k}^{i}\left[-T_{j,k}^{i-1}\left(2 + \frac{T^{i-1} - T^{i-2}}{TT^{i}}\right)\right.$$

$$\left. + T_{j,k}^{i-2}\left(1 + \frac{1}{4}\,\frac{T_{j,k}^{i-1} - T_{j,k}^{i-2}}{T\,T_{j,k}^{i}}\right)\right] + (\omega\rho c_p)_{j,k}^{i}\, r_j^2 \sin\theta_k \cdot \tag{A5}$$

$$(1/8\Delta\phi)\left[-14\,T_{j,k}^{i-1} + 3\,T_{j,k}^{i-2}\right] \quad ,$$

where

$$TT^{i}_{j,k} = 2T^{i-1}_{j,k} - T^{i-2}_{j,k} \quad ,$$

and we have taken the thermal conductivity (so as to linearize the equations at the point i to be given by

$$K(T^{i}_{j,k}) = K(TT^{i}_{j,k})$$

Also we have made use of the fact that the major dependence of the thermal conductivity on temperature is of the form proportional to the square root of the temperature, so we may make use of this to obtain the following approximation

$$K(T^{i+1/2}_{j,k}) = K(TT^{i}_{j,k})\left(1 + \frac{1}{4}\,\frac{T^{i}_{j,k} - T^{i-2}_{j,k}}{TT^{i}_{j,k}}\right)$$

Similary the terms $\rho(T^{i}_{jk})$, C_p are evaluated by using the projected values of the temperatures $TT^{i}_{j,k}$. Thus only a slight error is introduced by not iterating the ab ...ations in order to obtain the values of K, ρ, and C_p.

For the boundary condition on r we used that at $j = JN$, the maximum value of j, $\partial T/\partial r = qd$, and at $j = 1$, $T(r,\theta) = T_1$ for all latitudes. The upper boundary of the elementary volume for the point $j = JN$ is taken to pass through r_{JN} and integrating as before one obtains

$$b^{i}_{JN,K} = 0 \quad , \quad R^{i}_{JN,K} = (R^{i}_{j,K})_{j=JN} - K_{JN}\, r^2_{JN} \sin\theta_k \cdot qd/\Delta r_j$$

where by $(r^{i}_{j,k})_{j=JN}$ we mean expression (A5)is evaluated with $j = JN$.

Similiary one obtains from the boundary conditions at the lower boundary

$$a^{i}_{1,k} = 0 \qquad R^{i}_{1,k} = (R^{i}_{j,k})_{j=1} - (a^{i}_{j,k})_{j=1} T_1$$

For the boundary conditions on latitude one may take the temperatures at the poles to be the average over boundary of the temperatures at the latitude immediately adjacent to the poles. Alternatively one may obtain the temperatures at the poles by integrating the one dimenaional heat conduction in altitude at the poles. Both procedures were used. Both procedures yielded temperatures at the poles within ten percent of each other and less than a couple of percent at all other latitudes. The term representing vertical flow was evalutated explicitily in term of $T^{i-1}_{j,k}$ and $T^{i-2}_{j,k}$ and added to $R^{i}_{j,k}$. Calculations were performed with and without this term and at most fifty degrees difference in the exospheric temperatures where obtained. There was no shift in the peak with local time of the exospheric temperatures.

Care had to excerised in choosing the proper increment in longitude and latitude, otherwise numerical instability would developed, for the latitude nearest the pole. This instability can attributed to the term proportional to $\Delta\phi^{-2}$ in $R^{i}_{j,k}$, which would change sign unexpectedly, due to the low value of Q . It was found that this method of deriving difference equations tended to yield difference equation which were more stable against this type of instabitlity than straight forward differencing would yield.

The matrix of the coefficients of equation A1 (including their values at the boundary points) can be structured into block form in which the sublockc are labeled by the index j, and the rows and columns within the blocks by the index k. Then the matrix considered in block form is tridiagonal and direct inversion was performed to obtain a solution (Varga (1962)).

Calculations were performed at nine and eleven latitude increments with an increment in local time of fifteen and thirty minutes. The calculations with nine and eleven latitudes agreed well with each other. The results for fifteen minutes local time increment raise the peak exospheric temperature slightly and slightly narrowed the peak as expected. The results presented are for fifteen minutes local time and eleven latitudes. The increment in altitude chosen was one kilometer.

The initial temperatures profiles required for the iterations procedures, $T_{j,k}^{i-1}$ and $T_{j,k}^{i-2}$, were set equal to the lower boundary (120 km) value of 335°k for all values of j and k. Rapid convergence was obtained, i.e. $T_{j,k}^{i+96} - T_{j,k}^{i}$, decreasing by roughly $1/2$, where 96 corresponds to the number of increments of longtitude in twentyfour hours, when a fifteen minute integration step was used in local time. To speed up the calculation one can used one hour integration steps for two days and then reduced it to fifteen minutes. Widely different initial profiles were used and they all converged to the same results.