NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Memorandum 33-451

Design and Implementation of Models for the Double Precision Trajectory Program (DPTRAJ)

Gerd W. Spier

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
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Preface

The work described in this report was performed by the Systems Division of the Jet Propulsion Laboratory.

Acknowledgment

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Abstract

A common requirement for all lunar and planetary missions is the extremely accurate determination of the trajectory of a spacecraft. The Double Precision Trajectory Program (DPTRAJ) developed by JPL has proved to be a very accurate and dependable tool for the computation of interplanetary trajectories during the *Mariner* missions in 1969. This report describes the mathematical models that are used in DPTRAJ at present, with emphasis on the development of the equations.

Design and Implementation of Models for the Double Precision Trajectory Program (DPTRAJ)

I. Introduction

An important factor in determining high-precision interplanetary trajectories is the computation and subsequent integration of the acceleration of a spacecraft that is moving in the solar system and is subject to a variety of forces. The forces acting upon the spacecraft determine its acceleration according to Newton's second law; therefore, knowledge of the forces implies knowledge of the acceleration of the spacecraft. Integration of the total acceleration in some convenient frame of reference establishes the ephemeris of the spacecraft, and hence its trajectory.

The only known method for describing the abovementioned forces is that of mathematical models; i.e., one or more equations describing certain physical phenomena. It should be clear that every model reflects reality only to a certain degree. Many of the forces are not well known at present (e.g., the effect of the harmonic coefficients of Mars) and others are so small that they are negligible; therefore, only a relatively small number of models describing forces acting upon the spacecraft exist at the present time, and most of these are subject to improvement because of new knowledge acquired in various fields of the physical sciences.

Because the total acceleration of a spacecraft cannot be integrated in closed form, recourse must be taken to numerical methods. At present, the equations of motion of a spacecraft are integrated by a so-called second-sum numerical-integration scheme relative to some central body (Cowell method).

The equations of motion are solved for the spacecraft only, and ignore the negligible perturbations of the spacecraft on celestial bodies (i.e., on the sun, moon, and planets); hence, it is sufficient to obtain positions and velocities of these bodies in the form of planetary and lunar ephemerides in some convenient reference frame. The coordinates have been traditionally referred to the Cartesian system, based on the earth mean equator and equinox of 1950.0; thus, the ephemerides of the spacecraft and the bodies are uniformly expressed in this system. The collection of the ephemerides is usually done on a magnetic tape—the so-called *ephemeris tape.*

To obtain information about the spacecraft or any of the bodies in some other reference frame, an appropriate transformation must be applied to the 1950.0 frame. It should be noted that many of the numerical values of angles and related information given in Section V are subject to revision and should, therefore, not be considered final.

II. Time and Coordinate Transformations in General

This section describes time transformations and the transformation of the input spacecraft initial conditions (injection conditions)—which may be expressed in one of many systems—to the earth mean equator and equinox of 1950.0 Cartesian coordinates. Three types of coordinate transformations will be discussed: conversion, rotation, and translation.

A. Coordinate Conversions

There are five basic types of coordinate conversions, as follows:

From	To
Spherical $(R,\phi,\theta,V,\gamma,\sigma)$	Cartesian $(X,Y,Z,\dot{X},\dot{Y},\dot{Z})$
Cartesian $(X,Y,Z,\dot{X},\dot{Y},\dot{Z})$	Spherical $(R,\phi,\theta,V,\gamma,\sigma)$
Classical orbital $(a,e,i,\omega,\Omega,\Delta t)$	Cartesian $(X,Y,Z,\dot{X},\dot{Y},\dot{Z})$
Cartesian $(X,Y,Z,\dot{X},\dot{Y},\dot{Z})$	Classical orbital $(a,e,i,\omega,\Omega,\Delta t)$
Asymptotic $(\Sigma_L,R,\Gamma,C_3,\Phi_s,H_s)$	Cartesian $(X,Y,Z,\dot{X},\dot{Y},\dot{Z})$
Pseudo-asymptotic $(\Sigma_L,R,\Gamma,C_3,\Phi_s,H_s,R_{max})$	Cartesian $(X,Y,Z,\dot{X},\dot{Y},\dot{Z})$

B. Rotations

The Cartesian elements can be expressed in any of the coordinate systems described below, and can be transformed from one system to another.[1]

1. Space-fixed. The x-axis is in the direction of the ascending node between the orbit of the planet and its equatorial plane. The equinox can be that of 1950.0 or any later time. The z-axis is normal to the specified plane in the same direction as the angular-momentum vector. The y-axis completes the right-handed coordinate system.

[1]Witt, J. W., JPL internal document, Oct. 20, 1968.

The coordinate systems are:

(1) Earth mean or true equator.

(2) Earth mean orbit (defined by the ascending node of the earth orbit on the earth mean equatorial plane).

(3) Earth true orbit (defined by the ascending node of the earth orbit on the earth true equatorial plane).

(4) Mars mean and true orbits.

(5) Mars mean equator (computed from Mars mean orbit).

(6) Mars true equator (computed from Mars true orbit).

(7) Moon true equator.

2. Body-fixed. The xy-plane is the true of date equatorial plane of the planet. The x-axis points toward the prime meridian of the planet, the z-axis points to the north celestial pole, and the y-axis completes the right-handed coordinate system. Three body-fixed systems will be described—the systems for earth, Mars, and the moon.

The transformations will be discussed in the following sequence:

(1) Time transformations.

(2) Coordinate types of transformation (i.e., spherical to Cartesian, etc.).

(3) Earth-related transformations.

(4) Mars-related transformations.

(5) Moon-related transformations.

(6) Translation of centers.

III. Systems of Time

The familiar time that we keep on our clocks and adjust occasionally by means of signals sent out from Naval observatories is known to be subject to irregular changes; i.e., time, which is generally considered the uniform argument in all applications, is actually not a single-invariant quantity. This fact may be disregarded for most purposes of measuring time; in orbit and trajectory computations, however, the nonuniformity of time must be taken into account. In fact, to determine any orbit from earth-based observations, it is necessary to have at least two independent and quite distinct forms of time; namely, sidereal time for the observer and ephemeris time for the calculation of ephemerides.

A. Tropical Year and Ephemeris Time[2]

The standard for time measurements is the tropical year, which is the time required by the sun to make an apparent revolution of the ecliptic from vernal equinox to vernal equinox; however, this time interval is not constant because of the precession of the equinoxes. Therefore, the reference year was arbitrarily chosen as the instantaneous tropical year at 1900; in practice, it is defined in terms of the angular rate of the mean sun of 1900.0 as determined by observation. The adopted value of this rate is 129,602,768.13 s/Julian century of 36,525 days. In this context, the day is expressed in terms of an independent parameter that appears in the theories of the motion of bodies in the solar system. This parameter is called ephemeris time (E.T.). The number of ephemeris days in the tropical year 1900.0 is

$$\begin{aligned} 1 \text{ tropical year} &= 360 \times 60 \times 60 \times \frac{36{,}525}{129{,}602{,}768.13} \\ &= 365.24219879 \text{ ephemeris days} \end{aligned} \tag{1}$$

Ephemeris time is the uniform measure of time that is the independent variable for the equations of motion, and hence it is the argument for the ephemerides of the planets, the moon, and the spacecraft.

B. Atomic Time

Atomic time (A.1) (Ref. 1, p. 36) is obtained from oscillations of the U.S. Frequency Standard located at Boulder, Colo. The value of A.1 was set equal to UT2 (see below) on January 1, 1958, at $0^h0^m0^s$ UT2. Atomic time increases at the rate of 1 s/9,192,631,770 cycles of the cesium atom, which is the best current estimate of the length of the ephemeris second.

C. Universal Time

Universal Time (UT) is the precise measure of time that is used as the basis for all civil time-keeping, and is defined (Ref. 2, p. 73) as 12 h plus the Greenwich hour angle of a point on the true equator whose right ascension measured from the mean equinox of date is

$$R_U(\text{UT}) = 18^h38^m45^s.836 + 8{,}640{,}184^s.542\, T_U + 0^s.0929\, T_U^2 \tag{2}$$

where T_U is the number of Julian centuries of 36,525 days of UT elasped since 1900 Jan 0, 12 h UT. The hour angle of this point is θ_M (Greenwich hour angle of mean equinox of date). Hence, UT is a function only of θ_M:

$$\theta_M = \text{UT} + R_U(\text{UT}) + 12\text{ h}\, 0 \leq \theta_M, \qquad \text{UT} \leq 24\text{ h}$$

Universal Time is obtained from meridian transits of stars by the U.S. Naval Observatory. At the instant of observation, the right ascension of the observing station is equal to that of the observed star relative to the true equator and equinox of date. Subtraction of the east longitude of the observing station gives the true Greenwich sidereal time θ at the instant of observation:

$$\theta = \text{true Greenwich sidereal time (Greenwich hour angle of true equinox of date)} \tag{3}$$

Each observing station has a nominal value of longitude used for computing UT; if this nominal value is used, the resulting UT is labeled UT0. Because of wandering of the pole, the latitude and longitude of a fixed point on the earth are a function of time. If the true longitude of the observing station at the observation time is used, the resulting UT is labeled UT1. When the predictable seasonal fluctuations of UT1 are removed, the resulting time is labeled UT2.

D. Transformation Between Time Scales[3]

The most common problem is to find the E.T. for ephemeris consultation. The transformation between A.1 time and E.T. is given by

$$\begin{aligned} \text{E.T.} - \text{A.1} &= \Delta T_{1958} - (T - T_{58}) \frac{\Delta f_{\text{cesium}}}{9{,}192{,}631{,}770} \\ &\quad + 0.829\,(1 + \alpha) 10^{-3} \sin E \end{aligned} \tag{4}$$

[2]Witt, J. W., JPL internal document, Oct. 20, 1968.

[3]Witt, J. W., JPL internal document, Oct. 20, 1968.

where

ΔT_{1958} = time difference (in seconds) between A.1 and E.T. at 1958.0 (solve-for parameter)

T = A.1 or E.T., in seconds, past January 1, 1950, $0^h0^m0^s$

T_{58} = 252 460 800.0 (i.e., January 1, 1958, $0^h0^m0^s$ in seconds past January 1, 1950, $0^h0^m0^s$)

Δf_{cesium} = change in cesium frequency (a solve-for parameter)

The last term of Eq. (4) accounts for general relativistic effects:

α = 1 or -1 depending on whether or not it is desired to include general relativistic effects of the rotation of the geocenter about the earth–moon barycenter

E = eccentric anomaly of the heliocentric orbit of the earth–moon barycenter

For an accuracy of 10^{-6} s in the value of the periodic term (see Ref. 1, p. 37), the eccentric anomaly E may be computed from the following approximate solution to Kepler's equation:

$$E \approx M + e \sin M$$

where

e = eccentricity of heliocentric orbit of earth–moon barycenter (0.01672)

$M = 328°28'32\overset{''}{.}77 + 129{,}596{,}579\overset{''}{.}10\, T$

where T is the number of Julian centuries of 36,525 days of E.T. elapsed since 1900 Jan 0, 12 h E.T.

The first term of Eq. (4) arises because A.1 was set equal to UT2 at the beginning of 1958. The second term accounts for the difference between the lengths of the E.T. and A.1 seconds (if Δf_{cesium} is nonzero). The periodic term arises, as mentioned above, from general relativity. It accounts for the fact that A.1 time is a measure of *proper time* observed on earth, and that E.T. is a measure of *coordinate time* in the heliocentric (strictly barycentric) space–time frame of reference.

The remaining transformations between the various time scales are specified by linear or quadratic functions of $(T - t)$:

$$\text{A.1} - \text{UTC} = d + e(T - t) \tag{5}$$

$$\text{A.1} - \text{UT1} = f + g(T - t) + h(T - t)^2 \tag{6}$$

where

d,e,f,g,h,t = given parameters

UTC = time scale (see Glossary)

IV. Coordinate Type Transformations

A. Spherical to Cartesian Coordinates Transformation[4]

In spherical coordinates, position and velocity are given by the triples R,ϕ,θ and V,γ,σ, respectively. Given these two triples, it is required to compute X,Y,Z and $\dot{X},\dot{Y},\dot{Z}$. From Fig. 1 and elementary trigonometry, it follows that

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} R\cos\phi\cos\theta \\ R\cos\phi\sin\theta \\ R\sin\phi \end{pmatrix} \tag{7}$$

[4]Witt, J. W., JPL internal document, Oct. 20, 1968.

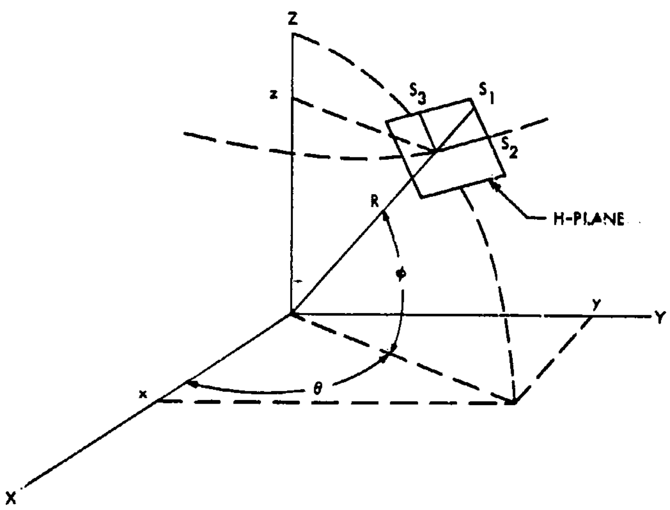

Fig. 1. Spherical coordinates

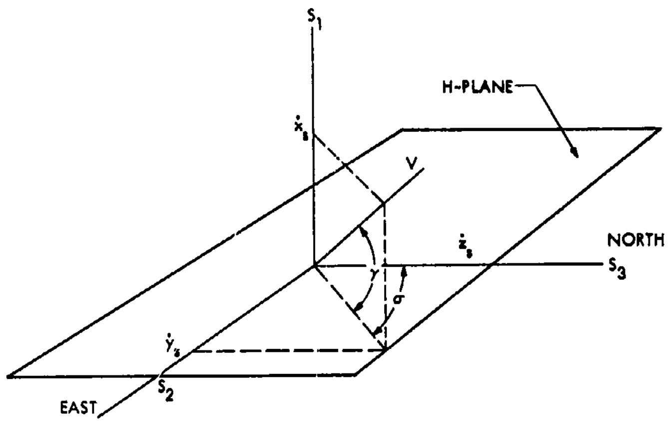

Fig. 2. Enlargement of the *H*-plane

If the velocity in the S_1,S_2,S_3 reference frame (Fig. 2) is computed, the velocity components $\dot{X}_s,\dot{Y}_s,\dot{Z}_s$ are given by

$$\begin{pmatrix} \dot{X}_s \\ \dot{Y}_s \\ \dot{Z}_s \end{pmatrix} = \begin{pmatrix} V\sin\gamma \\ V\cos\gamma\sin\sigma \\ V\cos\gamma\cos\sigma \end{pmatrix} \tag{8}$$

The rotation matrix relating the XYZ system to the S_1,S_2,S_3 system consists of the product of two rotation matrices:

$$\mathbf{C} = \mathbf{C}_2\mathbf{C}_1$$

$$= \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \\ -\sin\theta & \cos\theta & 0 \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \end{pmatrix} \tag{9}$$

where

$\mathbf{C}_1$ = matrix that rotates XYZ frame about Z-axis by angle θ; this yields $X'Y'Z$ system

$\mathbf{C}_2$ = matrix that rotates $X'Y'Z$ frame about Y'-axis by angle ϕ; this yields S_1,S_2,S_3 system

Hence, the velocity components $\dot{X},\dot{Y},\dot{Z}$ in the XYZ system are given by

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \mathbf{C}^{-1}\begin{pmatrix} \dot{X}_s \\ \dot{Y}_s \\ \dot{Z}_s \end{pmatrix} \tag{10}$$

Because $\mathbf{C}$ is an orthogonal matrix, $\mathbf{C}^{-1} = \mathbf{C}^T$; and from Eqs. (9) and (10), it follows that

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta & -\sin\theta & -\sin\phi\cos\theta \\ \cos\phi\sin\theta & \cos\theta & -\sin\phi\sin\theta \\ \sin\phi & 0 & \cos\phi \end{pmatrix}\begin{pmatrix} V\sin\gamma \\ V\cos\gamma\sin\sigma \\ V\cos\gamma\cos\sigma \end{pmatrix} \tag{11}$$

Equations (7) and (11) give position (X,Y,Z) and velocity $(\dot{X},\dot{Y},\dot{Z})$ in Cartesian coordinates.

B. Cartesian to Spherical Coordinates Transformation

Given X,Y,Z and $\dot{X},\dot{Y},\dot{Z}$, the problem is to compute the spherical coordinates R,ϕ,θ and V,γ,σ (this is the inverse of the transformation described in Section IV-A). From Fig. 1, it is easily seen that

$$\left.\begin{aligned} R &= (X^2+Y^2+Z^2)^{1/2} \\ \phi &= \sin^{-1}\left(\frac{Z}{R}\right), \qquad -90\text{ deg} \le \phi \le 90\text{ deg} \\ \theta &= \tan^{-1}\left(\frac{Y}{X}\right), \qquad 0\text{ deg} \le \theta < 360\text{ deg} \end{aligned}\right\} \tag{12}$$

From Eq. (10), there immediately follows

$$\begin{pmatrix} \dot{X}_s \\ \dot{Y}_s \\ \dot{Z}_s \end{pmatrix} = \mathbf{C} \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} \tag{13}$$

where **C** is given by Eq. (9). Then (see Fig. 2),

$$\left.\begin{aligned} V &= (\dot{X}_s^2 + \dot{Y}_s^2 + \dot{Z}_s^2)^{1/2} \\ \gamma &= \sin^{-1}\left(\frac{\dot{X}_s}{V}\right), \quad -90 \text{ deg} \leq \gamma \leq 90 \text{ deg} \\ \sigma &= \tan^{-1}\left(\frac{\dot{Y}_s}{\dot{Z}_s}\right), \quad 0 \text{ deg} \leq \sigma < 360 \text{ deg} \end{aligned}\right\} \tag{14}$$

Equations (12) through (14) express R,ϕ,θ and V,γ,σ in terms of X,Y,Z and $\dot{X},\dot{Y},\dot{Z}$.

C. Classical to Cartesian Coordinates Transformation

Given Δt, i, Ω, ω, a, e, and μ, where

$(t - T) = \Delta t$ = time of epoch minus time of perifocal passage

i = angle of inclination

Ω = longitude of ascending node

ω = argument of perifocus

a = semimajor axis (in case of a parabolic orbit, pericentron distance q must be supplied instead of a)

e = eccentricity

μ = gravitational constant

the problem is to find the inertial coordinates X,Y,Z and the velocity components $\dot{X},\dot{Y},\dot{Z}$ as a function of time for either a hyperbolic, an elliptic, or a parabolic orbit. The orbital elements are shown in Fig. 3.

Kepler's third law states that the orbital period P is computed according to

$$P = 2\pi\left(\frac{a^3}{\mu}\right)^{1/2} \text{ sec} \tag{15}$$

The mean motion n is then given by

$$n = \frac{2\pi}{P} = \left(\frac{\mu}{a^3}\right)^{1/2} \text{ rad/sec} \tag{16}$$

Kepler's equation is given for elliptic orbits by

$$M = n\Delta t = E - e \sin E \tag{17}$$

where

M = mean anomaly

e = eccentricity

E = eccentric anomaly

Differentiating M with respect to time, one obtains

$$\dot{M} = n \tag{18}$$

The rotation matrix $\mathbf{S} = \mathbf{S}_3 \cdot \mathbf{S}_2 \cdot \mathbf{S}_1$ for rotating the XYZ system into the orbital plane is

$$\mathbf{S} = \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} (\cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega) & (\cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega) & (\sin\omega\sin i) \\ (-\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega) & (-\sin\omega\sin\Omega + \cos\Omega\cos\omega\cos i) & (\cos\omega\sin i) \\ (\sin\Omega\sin i) & (-\sin i\cos\Omega) & (\cos i) \end{bmatrix} \tag{19}$$

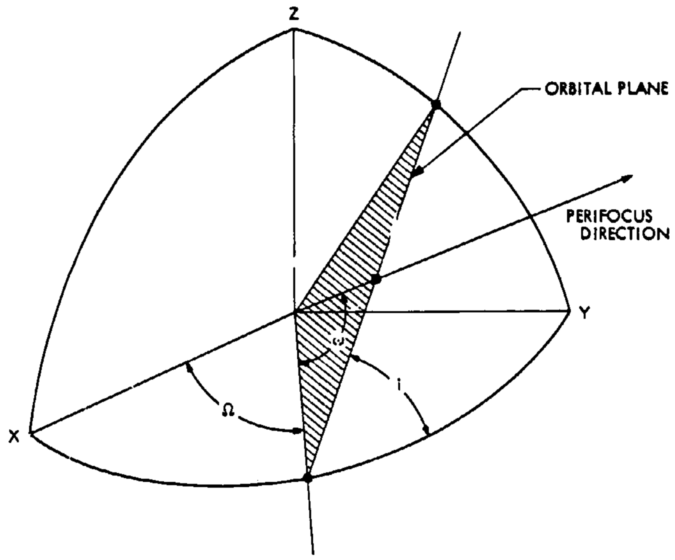

Fig. 3. Orbital elements

where

$\mathbf{S}_3$ = matrix for rotating through angle ω

$\mathbf{S}_2$ = matrix for rotating through angle i

$\mathbf{S}_1$ = matrix for rotating through angle Ω

The orbital rectangular coordinates $(x_\omega, y_\omega, z_\omega)$ are then transformed to Cartesian position and velocity coordinates (X,Y,Z) and $(\dot{X},\dot{Y},\dot{Z})$, respectively, in the reference plane by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{S}^T \begin{pmatrix} x_\omega \\ y_\omega \\ z_\omega \end{pmatrix} \tag{20}$$

and

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \mathbf{S}^T \begin{pmatrix} \dot{x}_\omega \\ \dot{y}_\omega \\ \dot{z}_\omega \end{pmatrix} \tag{21}$$

where $\mathbf{S}^T$ is the transpose of matrix $\mathbf{S}$ given by Eq. (19).

Because in all three cases (elliptic, hyperbolic, and parabolic) the orbit is planar, it is clear that

$$z_\omega = 0$$

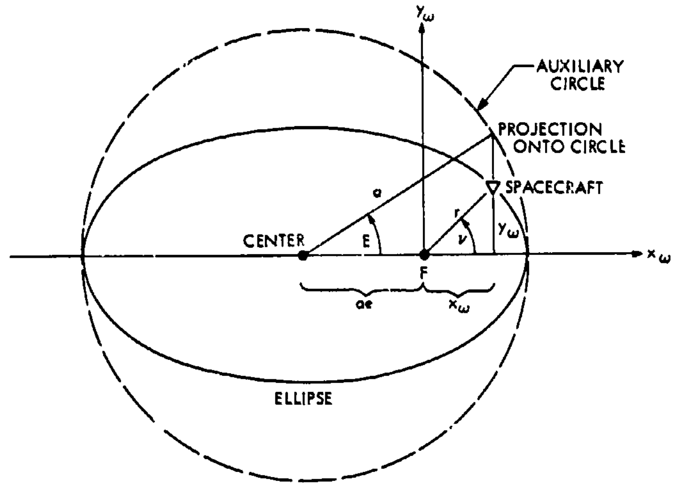

Fig. 4. Eccentric and true anomalies

and, therefore, also

$$\dot{z}_\omega = 0$$

Hence, it suffices to compute x_ω, y_ω, $\dot{x}_\omega$, and $\dot{y}_\omega$.

1. Elliptic orbit. From the geometry of Fig. 4, it is obvious that

$$\begin{aligned} r\cos\nu &= x_\omega \\ &= a\cos E - ae \end{aligned}$$

that is,

$$x_\omega = a(\cos E - e) \tag{22}$$

where E is the eccentric anomaly, which is defined in Fig. 4.

Now, in polar coordinates, the equation of an ellipse is given by

$$r = a(1 - e\cos E)$$

where e is the eccentricity. Then

$$\begin{aligned} y_\omega^2 &= r^2 - x_\omega^2 \\ &= a^2(1 - e^2)\sin^2 E \end{aligned}$$

and

$$
\begin{aligned}
y_\omega &= a\,(1 - e^2)^{1/2} \sin E \\
&= r \sin \nu
\end{aligned}
\tag{23}
$$

where ν is the true anomaly (see Fig. 4).

From Eqs. (22) and (23), it follows that

$$
\begin{aligned}
\dot{x}_\omega &= -a\dot{E} \sin E \\
\dot{y}_\omega &= a\,(1 - e^2)^{1/2}\, \dot{E} \cos E
\end{aligned}
\tag{24}
$$

The value of E is found from

$$
M = E - e \sin E \tag{25}
$$

by use of some iterative procedure; e.g., regula falsi (see Appendix C).

The time derivative of Eq. (25) yields

$$
\begin{aligned}
\dot{E} &= \frac{\dot{M}}{1 - e \cos E} \\
&= \frac{n}{1 - e \cos E}
\end{aligned}
\tag{26}
$$

by use of Eq. (18). Then

$$
\begin{aligned}
\dot{x}_\omega &= \frac{-an \sin E}{1 - e \cos E} \\
\dot{y}_\omega &= \frac{an\,(1 - e^2)^{1/2} \cos E}{1 - e \cos E}
\end{aligned}
\tag{27}
$$

2. ***Hyperbolic orbit.*** To find x_ω and y_ω for a hyperbolic path, recourse must be taken to hyperbolic functions because a assumes negative values. In analogy to the eccentric anomaly E of the ellipse, it is possible to define a new variable for hyperbolic motion as

$$
F = \frac{2 \times \text{area SPC}}{a^2}
$$

where the area SPC is defined in Fig. 5. Corresponding to the auxiliary circle in the case of an ellipse, there is an auxiliary hyperbola which is rectangular with the

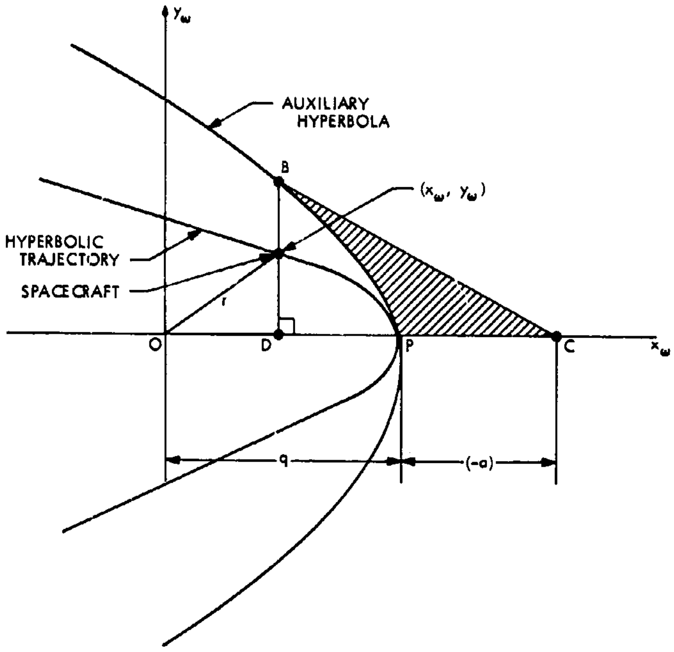

Fig. 5. A hyperbolic orbit

same semimajor axis and the same center as the hyperbolic trajectory. From the definition of hyperbolic functions,

$$
DC = -a \cosh F
$$

so that

$$
\begin{aligned}
x_\omega = OD &= q - (-a \cosh F + a) \\
&= a\,(1 - e) + a \cosh F - a
\end{aligned}
$$

or

$$
x_\omega = a\,(\cosh F - e) \tag{28}
$$

Also,

$$
x_\omega = r \cos \nu
$$

(see Fig. 5); and, by substitution of this result into the general equation of a conic,

$$
r = \frac{p}{1 + e \cos \nu}
$$

it follows that

$$
\begin{aligned}
r + e x_\omega &= p \\
&= a\,(1 - e^2)
\end{aligned}
$$

(see Eq. B-7, Appendix B).

Substituting for x_ω from Eq. (28), one obtains

$$r = a(1 - e \cosh F)$$

Because

$$r^2 = x_\omega^2 + y_\omega^2$$

y_ω is given by the expression

$$y_\omega = -a(e^2 - 1)^{1/2} \sinh F \tag{29}$$

The mean anomaly for hyperbolic motion is defined by

$$M_H = e \sinh F - F \tag{30}$$

with

$$M_H = n(t - T)$$

hence,

$$\dot{M}_H = n$$

F is computed from Eq. (30) by use of an iterative procedure; e.g., regula falsi (see Appendix C).

Differentiating Eqs. (28) and (29) with respect to time, one obtains

$$\begin{aligned} \dot{x}_\omega &= a \dot{F} \sinh F \\ \dot{y}_\omega &= -a(e^2 - 1)^{1/2} \dot{F} \cosh F \end{aligned} \tag{31}$$

$\dot{F}$ is computed from Eq. (30) as

$$\begin{aligned} \dot{F} &= \frac{\dot{M}_H}{e \cosh F - 1} \\ &= \frac{n}{e \cosh F - 1} \end{aligned}$$

Hence,

$$\dot{x}_\omega = \frac{an \sinh F}{e \cosh F - 1} \tag{32}$$

$$\dot{y}_\omega = \frac{-an(e^2 - 1)^{1/2} \cosh F}{e \cosh F - 1} \tag{33}$$

Equations (28) and (29) and Eqs. (32) and (33) give the position and velocity, respectively, of a spacecraft on a hyperbolic orbit.

*3. **Parabolic orbit.*** In the case of parabolic orbits (Fig. 6), $a = \infty$, $e = 1$, and Kepler's equation is indeterminate. Therefore, a new relationship between position and time must be sought. The equation of a conic, specialized to the parabola, is given by

$$r = \frac{p}{1 - e \cos \nu} = \frac{2q}{1 - \cos \nu} \tag{34}$$

because $p = q(1 + e) = 2q$ for the parabola, where ν is the true anomaly.

By the half-angle formulas, Eq. (34) becomes

$$r = q\left[1 + \tan^2\left(\frac{\nu}{2}\right)\right] = q\left[\sec^2\left(\frac{\nu}{2}\right)\right] \tag{35}$$

From a derivation in Ref. 3 (pp. 112–113),

$$p = \frac{r^4 \dot{\nu}^2}{\mu}$$

where μ is the gravitational coefficient of the reference body, km^3/s^2. Thus,

$$(\mu p)^{1/2} = (2\mu q)^{1/2} = r^2 \dot{\nu} = q^2\left[1 + \tan^2\left(\frac{\nu}{2}\right)\right]\left[\sec^2\left(\frac{\nu}{2}\right)\right]\frac{d\nu}{dt}$$

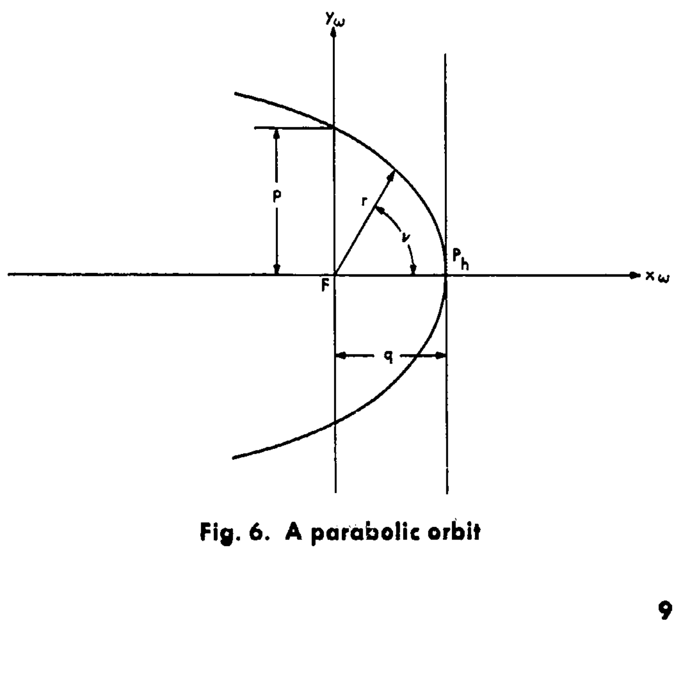

Fig. 6. A parabolic orbit

Upon simplification:

$$\left(\frac{\mu}{2q^3}\right)^{1/2} dt = \left[1 + \tan^2\left(\frac{\nu}{2}\right)\right]\left[\sec^2\left(\frac{\nu}{2}\right)\right] d\left(\frac{\nu}{2}\right) \tag{36}$$

Integration of Eq. (36) between T and t yields

$$\left(\frac{\mu}{2q^3}\right)^{1/2} (t - T) = \left(\frac{\mu}{2q^3}\right)^{1/2} \Delta t = \tan\left(\frac{\nu}{2}\right) + \frac{1}{3}\tan^3\left(\frac{\nu}{2}\right) \tag{37}$$

Rearrangement of Eq. (37) yields

$$\mu^{1/2}\Delta t = q\left[(2q)^{1/2}\tan\left(\frac{\nu}{2}\right)\right] + \frac{1}{6}\left[(2q)^{1/2}\tan\left(\frac{\nu}{2}\right)\right]^3 \tag{38}$$

Letting $D = (2q)^{1/2}\tan(\nu/2)$, one obtains from Eq. (38)

$$\frac{1}{6}D^3 + qD = \mu^{1/2}\Delta t \qquad \text{(Barker's equation)} \tag{39}$$

Equation (39) may be solved for D. Then

$$x_\omega = q - \frac{D^2}{2} \tag{40}$$

$$y_\omega = (2q)^{1/2} D \tag{41}$$

and

$$\dot{x}_\omega = -D\dot{D} = -D\frac{\mu^{1/2}}{q + \frac{D^2}{2}} \tag{42}$$

where $\dot{D}$ is computed from Eq. (39)

$$\dot{y}_\omega = (2q)^{1/2}\dot{D} = \frac{(2q\mu)^{1/2}}{q + \frac{D^2}{2}} \tag{43}$$

D. Cartesian to Classical Coordinates Transformation

Given the inertial coordinates X,Y,Z and velocity components $\dot{X},\dot{Y},\dot{Z}$ as a function of time for either an elliptic, hyperbolic, or parabolic orbit, the problem is to find a, e, i, Ω, ω, and $\Delta t = t - T$, where the last six quantities are defined as in Section IV-C.

Given the vis-viva integral

$$v^2 = \mu\left(\frac{2}{R} - \frac{1}{a}\right) \tag{44}$$

there follows for the semimajor axis of the conic

$$a = \frac{\mu R}{2\mu - Rv^2} \tag{45}$$

where

$$R = (X^2 + Y^2 + Z^2)^{1/2}$$
$$v = (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)^{1/2}$$

The eccentricity e of the conic is given by the standard formula

$$e = \left(1 - \frac{p}{a}\right)^{1/2} \tag{46}$$

where p is the semilatus rectum (see Eq. 476, Section X-B).

Now p is given by the formula (see Appendix B)

$$p = \frac{\mathbf{h}'\cdot\mathbf{h}'}{\mu} = \frac{h^2}{\mu}$$

where $\mathbf{h}$ is the angular-momentum vector per unit mass and $h = |\mathbf{h}|$. Because $\mathbf{h}$ is in the direction of the vector $\mathbf{R}\times\mathbf{v}$, the components of $\mathbf{h}$ are as follows:

$$h_x = Y\dot{Z} - Z\dot{Y}$$
$$h_y = Z\dot{X} - X\dot{Z}$$
$$h_z = X\dot{Y} - Y\dot{X}$$

Since

$$h^2 = R^2v^2 - (\mathbf{R}\cdot\mathbf{v})^2$$

one obtains

$$p = \frac{R^2 v^2 - (\mathbf{R} \cdot \mathbf{v})^2}{\mu} \tag{47}$$

By use of Eq. (44), Eq. (47) becomes

$$p = R\left(2 - \frac{R}{a}\right) - \frac{(\mathbf{R} \cdot \mathbf{v})^2}{\mu} \tag{48}$$

Substitution of this expression for p into Eq. (46) yields

$$e = \left[1 + \frac{R}{a}\left(\frac{R}{a} - 2\right) + \frac{(\mathbf{R} \cdot \mathbf{v})^2}{\mu a}\right]^{1/2} \tag{49}$$

Figure 7 shows the orbit of a spacecraft. On this figure, it should be noted that $\hat{\mathbf{W}}$ is normal to the orbital plane. The unit vectors $\hat{\mathbf{N}}$, $\hat{\mathbf{P}}$, $\hat{\mathbf{U}}$, $\hat{\mathbf{M}}$, $\hat{\mathbf{Q}}$, and $\hat{\mathbf{V}}$ are in the orbital plane and

$$\hat{\mathbf{N}} \perp \hat{\mathbf{M}}, \hat{\mathbf{P}} \perp \hat{\mathbf{Q}}, \hat{\mathbf{U}} \perp \hat{\mathbf{V}}$$

where $\hat{\mathbf{U}} = \mathbf{R}/R$. From the definition of $\mathbf{h}$, and from Fig. 7, it follows that

$$\hat{\mathbf{W}} = \frac{\mathbf{h}}{|\mathbf{h}|} = \frac{\mathbf{h}}{h} \tag{50}$$

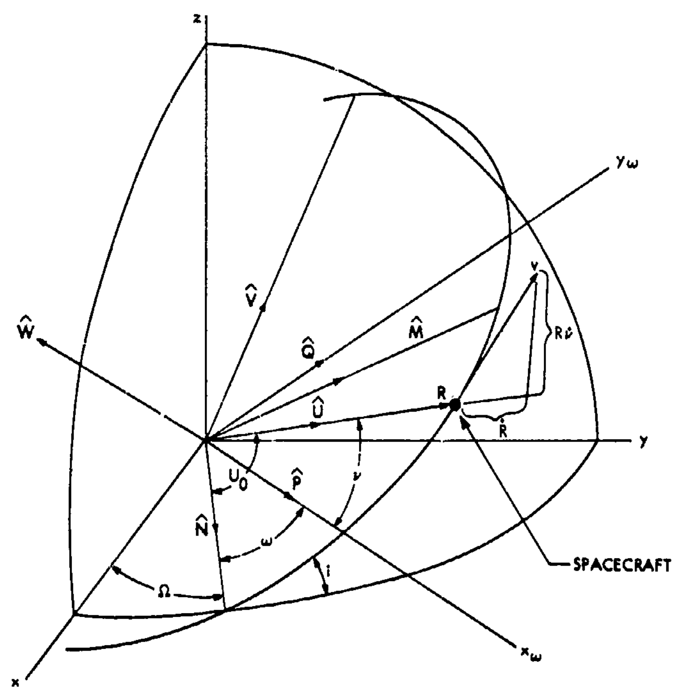

Fig. 7. Orbit of a spacecraft

Now it can easily be seen that the inclination of the orbital plane to the reference plane is

$$i = \cos^{-1}(W_z)$$

Hence,

$$i = \cos^{-1}\left(\frac{h_z}{h}\right), \qquad 0 \text{ deg} \leq i \leq 180 \text{ deg} \tag{51}$$

The unit vector $\hat{\mathbf{N}}$ along the line of the ascending node is given by its components

$$N_x = -\frac{W_y}{(W_x^2 + W_y^2)^{1/2}}$$

$$N_y = \frac{W_x}{(W_x^2 + W_y^2)^{1/2}}$$

$$N_z = 0$$

The longitude Ω of the ascending node is, therefore,

$$\Omega = \tan^{-1}\left(\frac{N_y}{N_x}\right), \qquad 0 \text{ deg} \leq \Omega \leq 360 \text{ deg} \tag{52}$$

The unit vector $\hat{\mathbf{U}}$, which points in the direction of the spacecraft, is given by

$$\hat{\mathbf{U}} = \frac{\mathbf{R}}{R}$$

To determine ω (the argument of perifocus), the argument of the latitude u_0 and the true anomaly ν are needed (i.e., u_0 is the angle between $\hat{\mathbf{N}}$ and $\hat{\mathbf{U}}$ and ν is the angle between $\mathbf{P}$ and $\hat{\mathbf{U}}$). To compute u_0, the right-hand set for $\hat{\mathbf{W}}$ and $\hat{\mathbf{N}}$ is completed by the cross product

$$\hat{\mathbf{M}} = \hat{\mathbf{W}} \times \hat{\mathbf{N}}$$

Since

$$\hat{\mathbf{U}} \cdot \hat{\mathbf{M}} = \cos(90 - u_0) = \sin u_0$$

and

$$\hat{\mathbf{U}} \cdot \hat{\mathbf{N}} = \cos u_0$$

it follows that

$$u_0 = \tan^{-1}\left(\frac{\hat{\mathbf{U}} \cdot \hat{\mathbf{M}}}{\hat{\mathbf{U}} \cdot \hat{\mathbf{N}}}\right), \qquad 0 \text{ deg} \leq u_0 < 360 \text{ deg} \tag{53}$$

The argument of perifocus is, of course, not defined for $e = 0$; in this case, let $\nu = u_0$.

If $e \neq 0$, one obtains from the general equation of a conic

$$R = \frac{p}{1 + e\cos\nu}$$

the expression

$$e\cos\nu = \frac{p}{R} - 1$$

and, hence,

$$\cos\nu = \frac{1}{e}\left(\frac{p}{R} - 1\right) \tag{54}$$

Direct differentiation of Eq. (54) with ν and R as variables yields

$$\sin\nu = \frac{1}{e}\,\frac{p\dot{R}}{R^2\dot{\nu}} = \frac{1}{e}\,\dot{R}\left(\frac{p}{\mu}\right)^{1/2} \tag{55}$$

because

$$p\mu = (R^2\dot{\nu})^2$$

From Eqs. (54) and (55), it follows that

$$\nu = \tan^{-1}\left[\frac{\dot{R}\left(\frac{p}{\mu}\right)^{1/2}}{\frac{p}{R} - 1}\right] \tag{56}$$

where $\dot{R}$ may be computed by differentiating

$$R^2 = RR = X^2 + Y^2 + Z^2 \tag{57}$$

to obtain

$$R\dot{R} = X\dot{X} + Y\dot{Y} + Z\dot{Z}$$

so that

$$\dot{R} = \frac{X\dot{X} + Y\dot{Y} + Z\dot{Z}}{R} = \hat{\mathbf{U}} \cdot \mathbf{v}$$

The argument of perifocus is then obtained as

$$\omega = u_0 - \nu, \qquad 0 \leq \omega \leq 2\pi \tag{58}$$

where u_0 and ν are given by Eqs. (53) and (56), respectively.

The vector $\mathbf{v}$ is given by

$$\mathbf{v} = R\dot{\nu}\hat{\mathbf{V}} + \dot{R}\hat{\mathbf{U}}$$

Hence,

$$\hat{\mathbf{V}} = \frac{1}{R\dot{\nu}}\mathbf{v} - \frac{\dot{R}}{R\dot{\nu}}\hat{\mathbf{U}} \tag{59}$$

Because $h = R^2\dot{\nu}$, one obtains from Eq. (59)

$$\hat{\mathbf{V}} = \frac{R}{h}\mathbf{v} - \frac{\dot{R}}{h}R\hat{\mathbf{U}}$$

$$= \frac{R}{h}\mathbf{v} - \frac{\dot{R}}{h}\mathbf{R}$$

The vector $\hat{\mathbf{P}}$ is then given by

$$\hat{\mathbf{P}} = \cos\nu\,\hat{\mathbf{U}} - \sin\nu\,\hat{\mathbf{V}} \tag{60}$$

and, hence,

$$\hat{\mathbf{Q}} = \sin\nu\,\hat{\mathbf{U}} + \cos\nu\,\hat{\mathbf{V}} \tag{61}$$

It remains to determine Δt, which is the epoch time minus the time of perifocal passage. The computation of this quantity depends upon the type of conic described.

If $1/a > 0$, then the orbit is an ellipse or a circle. If the orbit is a circle ($e = 0$), then $\omega = 0$ and T is taken as the time of nodal passage, which is given by

$$\Delta t = \frac{u_0}{\left(\frac{\mu}{a^3}\right)^{1/2}} \tag{62}$$

If the orbit is an ellipse, the eccentric anomaly E is given by

$$E = \tan^{-1}\frac{R\dot{R}a}{(a - R)(\mu a)^{1/2}} \tag{63}$$

Kepler's equation is

$$M = E - e \sin E \tag{64}$$

where

$$M = \text{mean anomaly}$$

$$= \left(\frac{\mu}{a^3}\right)^{1/2} \Delta t$$

(see Eqs. 16 and 17). Hence,

$$\Delta t = \frac{M}{\left(\frac{\mu}{a^3}\right)^{1/2}} \tag{65}$$

where M is given by Eq. (64).

Equation (63) is obtained from the standard equations for the eccentric anomaly (Ref. 4, p. 118):

$$\sin E = \frac{(1 - e^2)^{1/2} \sin \nu}{1 + e \cos \nu}$$

$$\cos E = \frac{e + \cos \nu}{1 + e \cos \nu}$$

and noting that

$$1 - e^2 = \frac{p}{a}$$

$$\sin \nu = \frac{1}{e} \dot{R} \left(\frac{p}{\mu}\right)^{1/2} \qquad \text{(see Eq. 55)}$$

$$\cos \nu = \frac{1}{e}\left(\frac{p}{R} - 1\right) \qquad \text{(see Eq. 54)}$$

Then

$$\frac{\sin E}{\cos E} = \tan E = \frac{(1 - e^2)^{1/2} \sin \nu}{e + \cos \nu}$$

$$= \frac{\left(\frac{p}{a}\right)^{1/2} \frac{1}{e} \dot{R} \left(\frac{p}{\mu}\right)^{1/2}}{e + \frac{1}{e}\left(\frac{p}{R} - 1\right)} = \frac{p\dot{R}\left(\frac{1}{\mu a}\right)^{1/2}}{\left(e^2 - 1 + \frac{p}{R}\right)}$$

$$= \frac{p\dot{R}\left(\frac{1}{\mu a}\right)^{1/2}}{p\left(\frac{1}{R} - \frac{1}{a}\right)}$$

$$= \frac{R \dot{R} a}{(a - R)(\mu a)^{1/2}}$$

Thus,

$$E = \tan^{-1}\left[\frac{R \dot{R} a}{(a - R)(\mu a)^{1/2}}\right] \tag{66}$$

If $1/a < 0$, the orbit is hyperbolic. The mean anomaly of a hyperbola is

$$M_H = e \sinh F - F \qquad \text{(see Eq. 30)}$$

with

$$M_H = n \Delta t$$

or

$$M_H = \left(\frac{\mu}{-a^3}\right)^{1/2} \Delta t$$

(by definition of n, the mean motion; see Eq. 16). Hence,

$$\Delta t = \frac{M_H}{\left(\frac{\mu}{-a^3}\right)^{1/2}} \tag{67}$$

The eccentric anomaly F of a hyperbola may be computed as shown below. From Eq. (29),

$$\sinh F = \frac{y_\omega}{-a(e^2 - 1)^{1/2}}$$

$$= \frac{R \sin \nu}{[a^2(e^2 - 1)]^{1/2}}$$

$$= \frac{R \frac{1}{e} \dot{R} \left(\frac{p}{\mu}\right)^{1/2}}{[a^2(e^2 - 1)]^{1/2}} \qquad \text{(by use of Eq. 55)}$$

$$= \frac{R\dot{R}}{e\left[\frac{a^2(e^2 - 1)\mu}{p}\right]^{1/2}}$$

$$= \frac{R\dot{R}}{e(-a\mu)^{1/2}}$$

because, for a hyperbola,

$$\frac{a}{p} = \frac{-1}{e^2 - 1}$$

(see Eq. 476, Section X-B). Thus,

$$\sinh F = \frac{R\dot{R}}{e(-a\mu)^{1/2}}$$

and, therefore,

$$F = \sinh^{-1}\left[\frac{R\dot{R}}{e(-a\mu)^{1/2}}\right]$$

$$= \sinh^{-1} B$$

or

$$F = \log\left[B + (B^2 + 1)^{1/2}\right] \tag{68}$$

by use of a well-known identity for inverse hyperbolic functions, where

$$B = \frac{R\dot{R}}{e(-a\mu)^{1/2}} \tag{69}$$

In case

$$\frac{1}{a} \approx 0$$

the orbit is considered parabolic. From Eq. (39), one obtains

$$\Delta t = \frac{qD}{(\mu)^{1/2}} + \frac{D^3}{6(\mu)^{1/2}} \qquad \text{(Barker's equation)} \tag{70}$$

where

$$D = (2q)^{1/2} \tan\left(\frac{\nu}{2}\right) \tag{71}$$

E. Pseudo-Asymptote and Asymptote Coordinates to Cartesian Coordinates Transformation[5]

The hyperbolic excess-velocity vector $\mathbf{V}_\infty$ at launch is important because it tells the direction in which the spacecraft must be traveling relative to the launch planet when the spacecraft is on the point of leaving the gravitational influence of that planet. Although an infinite number of escape trajectories (all hyperbolas) can have the same excess-velocity vector, only a portion are practicable when related to existing launch sites and boost-vehicle constraints.

The following assumptions are made to obtain the solution of the escape phase of motion:

(1) The spacecraft is acted upon only by the gravitational force of the launch planet.

(2) The oblateness effects of the launch planet are disregarded.

The direction of the asymptote of the escape hyperbola is found by normalizing the hyperbolic excess-velocity vector $\mathbf{V}_\infty$. The injection energy C_3 of the escape hyperbola is found by squaring the hyperbolic excess speed, or

$$C_3 = V_\infty^2 \tag{72}$$

where $V_\infty = |\mathbf{V}_\infty|$ (see Eqs. 481–483, Section X-B).

As previously stated, not all of the infinite number of escape trajectories are practicable. Two of the practical aspects of a set of trajectories are the size and shape of the hyperbolas. Size is basically determined by C_3 (which, in turn, is a function of the boost-vehicle capability).

The shape of the hyperbola is determined by its eccentricity, which is a function of both C_3 and the perifocal distance according to

$$e = 1 + \frac{R_p C_3}{\mu} \tag{73}$$

where

R_p = perifocal distance

$\mu = GM$ = universal gravitational constant times mass of launch planet

Equation (73) may be derived as described below. From the general equation of a conic

$$R = \frac{p}{1 + e\cos\nu}$$

[5]Reference 5.

there follows for the perifocal distance R_p ($\nu = 0$)

$$R_p = \frac{p}{1+e}$$

$$= \frac{a(1-e^2)}{1+e}$$

$$= a(1-e) \tag{74}$$

or, to solve for e,

$$e = 1 - \frac{R_p}{a}$$

and because

$$C_3 = \frac{-\mu}{a}$$

(see Eq. 481, Section X-B). Since for a hyperbola, $a < 0$, Eq. (73) follows.

From Eq. (73) it can be seen that, for a fixed R_p, the eccentricity increases linearly with the energy.

Therefore, both the size and the shape are essentially determined by the energy alone, which is obtained from Eq. (72).

Given the size and shape of the escape hyperbola, its planar orientation must be determined. This can be done by consideration of two vectors: (1) the direction of the hyperbolic-excess vector, denoted by the unit vector $\hat{S}$, and (2) a unit vector $\hat{R}_L$ directed from the center of the launch planet to the launch site. The flight plane of the spacecraft will essentially be determined by these two vectors, as is shown in Fig. 8.

If the launch date and flight time of a mission are specified, the ascending asymptote vector **S** as well as the energy C_3 of the near-earth conic become known quantities (C_3 is actually twice the energy per unit mass; i.e., the vis-viva integral). This follows from the fact that the four defining quantities of an interplanetary (or lunar) trajectory are launch date, right ascension and declination of the ascending asymptote, and C_3. If it is assumed that the overall mission has been specified, then **S** and C_3 are constants of the problem.

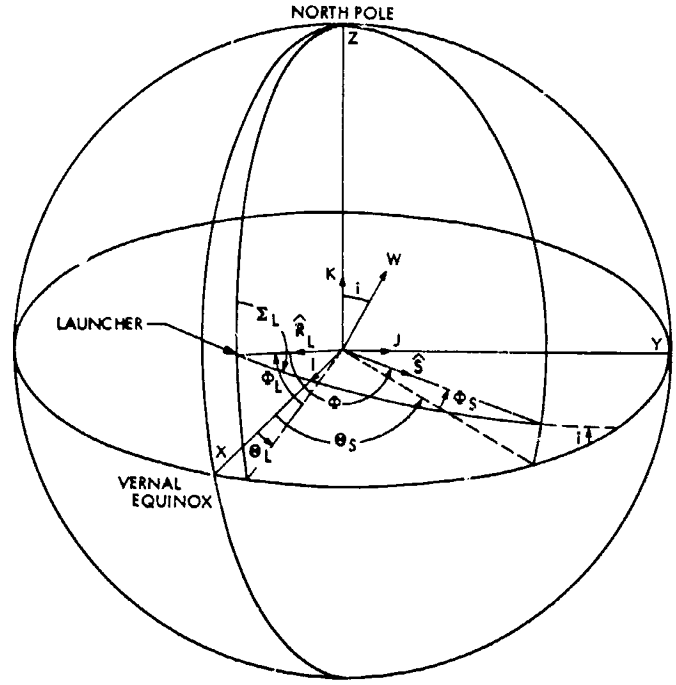

Fig. 8. Spacecraft flight plane

The problem of transforming pseudo-asymptote and asymptote coordinates to Cartesian coordinates may be formulated as described below. Given

Σ_L = azimuth of launch

Φ_L = latitude of launch site

R = injection radius

Γ = flight-path angle (Fig. 9)

C_3 = energy per unit mass at launch

Θ_s = right ascension of ascending asymptote

Φ_s = declination of ascending asymptote

$\mu = GM$ = gravitational constant of launch planet

it is desired to compute

$\mathbf{R} = (X,Y,Z)$ = radius vector at injection

$\mathbf{V} = (\dot{X},\dot{Y},\dot{Z})$ = velocity vector at injection

From Fig. 8, it is readily found that

$$W_z = \cos i$$

$$= \sin \Sigma_L \cos \Phi_L \tag{75}$$

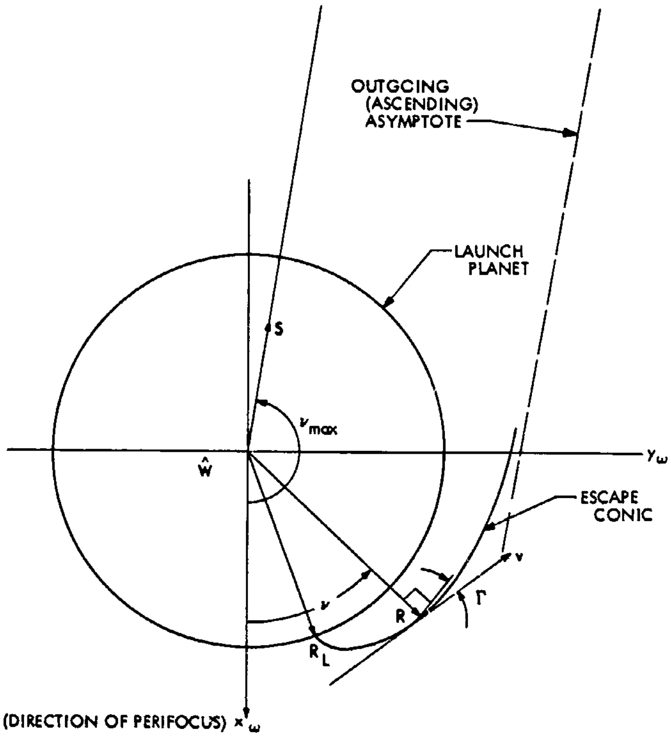

Fig. 9. Launch geometry

where

$$\Sigma_L = \text{azimuth of launch}$$
$$\Phi_L = \text{latitude of launch site}$$

It should be recalled that **S** points in the direction of the hyperbolic excess-velocity vector; hence, **S** points in the direction of the ascending asymptote.

Thus, the unit vector $\hat{\mathbf{S}}$ may be represented by

$$\hat{\mathbf{S}} = S_X\hat{\mathbf{I}} + S_Y\hat{\mathbf{J}} + S_Z\hat{\mathbf{K}} \tag{76}$$

where $\hat{\mathbf{I}},\hat{\mathbf{J}},\hat{\mathbf{K}}$ are unit vectors in the X,Y,Z directions, respectively.

Equation (76) may be rewritten as

$$\hat{\mathbf{S}} = \cos\Theta_S \cos\Phi_S \hat{\mathbf{I}} + \sin\Theta_S \cos\Phi_S \hat{\mathbf{J}} + \sin\Phi_S \hat{\mathbf{K}} = (S_X, S_Y, S_Z) \tag{77}$$

where

$$\Theta_S = \text{right ascension of } \hat{\mathbf{S}}$$
$$\Phi_S = \text{declination of } \hat{\mathbf{S}}$$

Equation (77) follows directly from inspection of Fig. 8.

Now $\hat{\mathbf{S}}$ is in the orbital plane; thus,

$$\hat{\mathbf{W}} \cdot \hat{\mathbf{S}} = 0$$

Also,

$$\hat{\mathbf{W}} \cdot \hat{\mathbf{W}} = 1$$

Hence,

$$\begin{aligned} W_X S_X + W_Y S_Y + W_Z S_Z &= 0 \\ W_X^2 + W_Y^2 + W_Z^2 &= 1 \end{aligned} \tag{78}$$

The system of equations represented by Eq. (78) may be solved simultaneously to yield

$$\begin{aligned} W_Y &= \frac{-W_Z S_Y S_Z \pm S_X (1 - S_Z^2 - W_Z^2)^{1/2}}{S_X^2 + S_Y^2} \\ &= \frac{-W_Z \sin\Theta_S \sin\Phi_S \pm \cos\Theta_S (\cos^2\Phi_S - W_Z^2)^{1/2}}{\cos\Phi_S} \end{aligned} \tag{79}$$

$$W_X = \frac{-(W_Y S_Y + W_Z S_Z)}{S_X} \tag{80}$$

where S_X, S_Y, S_Z are given by Eq. (77) (Ref. 6, p. 5).

The injection velocity is given by

$$v = \left(C_3 + \frac{2\mu}{R}\right)^{1/2} \tag{81}$$

where R is the injection radius. Equation (81) follows directly from the vis-viva integral (Eq. 44).

The (unit) angular momentum h is computed as follows:

$$h = |\mathbf{R} \times \mathbf{v}| = Rv\cos\Gamma \tag{82}$$

where Γ is the flight-path angle (Figs. 9 and 10).

The eccentricity of a conic is

$$e = \left(1 - \frac{p}{a}\right)^{1/2}$$

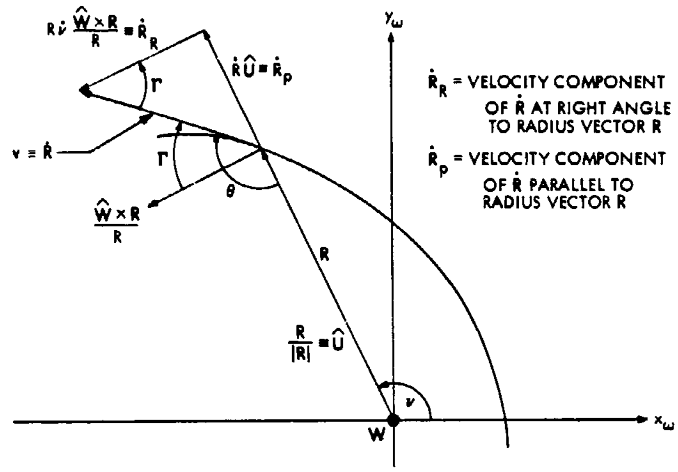

Fig. 10. The flight-path angle Γ

Because

$$(p\mu)^{1/2} = h$$

and

$$C_3 = -\frac{\mu}{a}$$

it follows that

$$e = \left(1 + \frac{h^2 C_3}{\mu^2}\right)^{1/2} \tag{83}$$

Case 1: $0 < e < 1$. The trajectory is an ellipse, and the semilatus rectum p is given by

$$p = \frac{\mu}{C_3}(e^2 - 1) \tag{84}$$

The true anomaly ν is computed from the basic formula of a conic

$$R = \frac{p}{1 + e\cos\nu}$$

Hence,

$$\nu = \cos^{-1}\left(\frac{p - R}{eR}\right), \qquad 0 \text{ deg} \leq \nu \leq 180 \text{ deg} \tag{85}$$

An ellipse does not have an asymptote. To enable use of the energy concept even in this case, the quantity ν_{max} = maximum true anomaly is defined as the angle between **P** (a vector pointing toward perigee) and a given radius

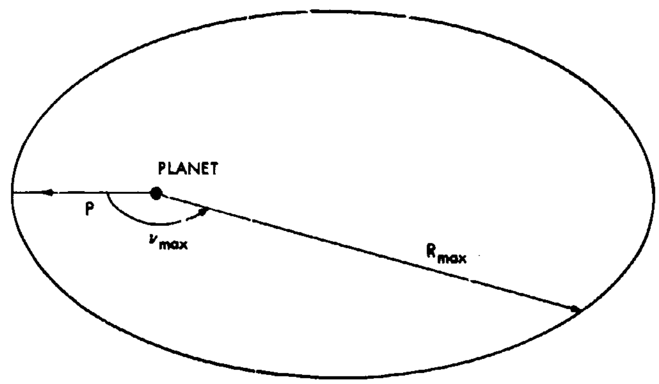

Fig. 11. The radius vector $\mathbf{R}_{max}$

$\mathbf{R}_{max}$ (Fig. 11), which is a "pseudo-asymptote" (e.g., $\mathbf{R}_{max}$ may be the vector from earth to moon). Then, from Eq. (85), one has for ν_{max}

$$\nu_{max} = \cos^{-1}\left(\frac{p - R_{max}}{eR_{max}}\right) \tag{86}$$

where $R_{max} = |\mathbf{R}_{max}|$.

Case 2: $e > 1$. The trajectory is a hyperbola, and the true anomaly (at injection) is computed from

$$\sin\Gamma = e\sin(\nu - \Gamma), \qquad -90 \text{ deg} \leq \nu - \Gamma \leq 90 \text{ deg} \tag{87}$$

so that

$$\nu = \Gamma + \sin^{-1}\left(\frac{\sin\Gamma}{e}\right) \tag{88}$$

A derivation of Eq. (87) is given at the end of this section.

To obtain ν_{max}, one should consider Eq. (85); i.e.,

$$\nu = \cos^{-1}\left(\frac{p - R}{eR}\right)$$

$$= \cos^{-1}\left(\frac{\frac{p}{R}}{e} - \frac{1}{e}\right)$$

Letting $R \to \infty$, one finds that

$$\nu_{max} = \cos^{-1}\left(-\frac{1}{e}\right), \qquad 90 \text{ deg} \leq \nu_{max} \leq 180 \text{ deg} \tag{89}$$

The radius vector to injection may now be computed, and it is as follows (see Fig. 9):

$$\mathbf{R} = R\,[\cos(\nu_{\max} - \nu)\,\hat{\mathbf{S}} + \hat{\mathbf{S}} \times \hat{\mathbf{W}} \sin(\nu_{\max} - \nu)] \tag{90}$$

The velocity vector is then given by

$$\mathbf{V} = V\left(\cos\Gamma \frac{\hat{\mathbf{W}} \times \mathbf{R}}{R} + \sin\Gamma \frac{\mathbf{R}}{R}\right) \tag{91}$$

(see Fig. 9).

To derive the equation $\sin\Gamma = e \sin(\nu - \Gamma)$, one observes that the angular-momentum vector $\mathbf{h}$ is given by

$$\mathbf{h} = \mathbf{R} \times \dot{\mathbf{R}} = \mathbf{R} \times \mathbf{v}$$

Thus,

$$\begin{aligned} h = |\mathbf{h}| = |\mathbf{R} \times \dot{\mathbf{R}}| &= |\mathbf{R}|\,|\dot{\mathbf{R}}| \sin\theta \\ &= Rv \sin\left(\frac{\pi}{2} + \Gamma\right) \\ &= Rv\cos\Gamma \end{aligned} \tag{92}$$

(see Eq. 82). Now

$$\hat{\mathbf{W}} = \frac{\mathbf{h}}{|\mathbf{h}|} = \frac{\mathbf{h}}{h}$$

thus,

$$\begin{aligned} \mathbf{h} = h\hat{\mathbf{W}} = \mathbf{R} \times \dot{\mathbf{R}} = \mathbf{R} \times (\dot{\mathbf{R}}_R + \dot{\mathbf{R}}_P) &= \\ \mathbf{R} \times \dot{\mathbf{R}}_R = R R\dot{\nu}\left(\sin\frac{\pi}{2}\right)\hat{\mathbf{W}} & \\ = R^2\dot{\nu}\,\hat{\mathbf{W}} & \end{aligned} \tag{93}$$

From Eqs. (92) and (93), one obtains

$$R^2\dot{\nu} = Rv\cos\Gamma \tag{94}$$

Because

$$v = [(R\dot{\nu})^2 + (\dot{R})^2]^{1/2}$$

Eq. (94) may be rewritten as

$$R^2\dot{\nu} = R\,[(R\dot{\nu})^2 + (\dot{R})^2]^{1/2}\cos\Gamma$$

or

$$R\dot{\nu} = [(R\dot{\nu})^2 + (\dot{R})^2]^{1/2}\cos\Gamma$$

so that

$$1 = \left[1 + \frac{1}{R^2}\left(\frac{\dot{R}}{\dot{\nu}}\right)^2\right]^{1/2} \cos\Gamma \tag{95}$$

But

$$\frac{\dot{R}}{\dot{\nu}} = \frac{dR}{d\nu}$$

and because

$$R = \frac{p}{1 + e\cos\nu}$$

one obtains

$$\frac{dR}{d\nu} = \frac{pe\sin\nu}{(1 + e\cos\nu)^2} = \frac{Re\sin\nu}{1 + e\cos\nu} = \frac{\dot{R}}{\dot{\nu}} \tag{96}$$

Substitution of Eq. (96) into Eq. (95) yields

$$\begin{aligned} 1 &= \left[1 + \left(\frac{e\sin\nu}{1 + e\cos\nu}\right)^2\right]^{1/2} \cos\Gamma \\ &= \frac{(1 + 2e\cos\nu + e^2)^{1/2}}{1 + e\cos\nu}\cos\Gamma \end{aligned}$$

or

$$\cos\Gamma = \frac{1 + e\cos\nu}{(1 + 2e\cos\nu + e^2)^{1/2}} \tag{97}$$

Hence,

$$\sin\Gamma = (1 - \cos^2\Gamma)^{1/2} = \frac{e\sin\nu}{(1 + 2e\cos\nu + e^2)^{1/2}} \tag{98}$$

From Eqs. (97) and (98), one obtains

$$\frac{\sin\Gamma}{\cos\Gamma} = \frac{e\sin\nu}{1 + e\cos\nu}$$

and so

$$\sin\Gamma + e\sin\Gamma\cos\nu = e\sin\nu\cos\Gamma$$

or

$$\sin\Gamma = e\sin(\nu - \Gamma)$$

V. Rotations of Coordinate Systems

The fundamental coordinate system for reference of the equations of motion is the Cartesian frame formed by the earth mean equator and equinox of 1950.0—the Julian ephemeris date (JED) 2433282.423; the position of the mean equator of the earth and the ascending node of the mean orbit of the sun on that equator, taken at the beginning of the Besselian year 1950, serve as the definition. The X-axis is directed along the node, the Z-axis northward above the equator, and the Y-axis in a direction to complete the usual right-handed coordinate system.

A. Earth-Related Transformations

The direction of the rotational axis of the earth is not fixed in space. The actions of the sun and the moon on the equatorial bulge cause variations in the orientation of the equatorial plane, whereas the perturbative effects of the planets produce a variation in the ecliptic. Once a fundamental inertial reference system is specified, it would be sufficient to tabulate the direction cosines of the rotational axis to the coordinate axes. The problem is not treated in this way for historical and practical reasons.

In practice, the motions of the ecliptic and equator are both explicitly computed as a matter of observational necessity. Furthermore, the long-term motions that can be treated as though they are secular (precession) are separated from the short-period motions (nutation). The fictitious equator, ecliptic, and equinox, which are defined as being represented by the precessional motions only, are called *mean*; those affected by both precession and nutation are called *true*. Values fixed at the time corresponding to a fundamental reference are values *at the epoch*, whereas those referring to instantaneous moments are the values *of date*.

B. Rotation From Mean Earth Equator of 1950.0 to Mean Equator of Date

The *rates* of precessional motions (general, planetary, lunisolar, in right ascension, and in declination) must be distinguished from both the accumulated *amounts* of the motions over an extended interval of time and the consequent displacements of the coordinate systems produced by precessional motions.

The amount of the precession in right ascension during the interval from t_0 to t is $(\zeta_0 + Z)$, where $(90 \text{ deg} - \zeta_0)$ is the right ascension of the ascending node of the mean equator at time t on the mean equator of t_0 reckoned from the mean equinox of t_0, and $(90 \text{ deg} + Z)$ is the right ascension of the node reckoned from the mean equinox of t.

The amount of the precession in declination is the inclination θ of the mean equator at time t to the mean equator of t_0 (in what follows, $t_0 = 1950.0$). Thus, the general precession of the terrestrial equator and the consequent retrograde motion of the equinox on the ecliptic may be represented by a rotation matrix **A**. Obtained by composing three rotations, matrix **A** rotates the mean equator of 1950.0 to the mean equator of date.[6]

The first rotation (Fig. 12) is about the Z-axis from the mean equinox of 1950.0 to the ascending node of the mean equator of date on the mean equator of 1950.0, with the matrix of rotation given by

$$\mathbf{A}_1 = \begin{pmatrix} \cos(90-\zeta_0) & \sin(90-\zeta_0) & 0 \\ -\sin(90-\zeta_0) & \cos(90-\zeta_0) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\zeta_0 & \cos\zeta_0 & 0 \\ -\cos\zeta_0 & \sin\zeta_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{99}$$

The second rotation is about the X''-axis through the angle θ (Fig. 13), where the matrix of rotation is

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{100}$$

The third rotation is a left-handed rotation about the Z'-axis to the mean equinox of date (Fig. 14). The matrix for this rotation is

$$\mathbf{A}_3 = \begin{pmatrix} \cos(90\text{ deg}+Z) & -\sin(90\text{ deg}+Z) & 0 \\ \sin(90\text{ deg}+Z) & \cos(90\text{ deg}+Z) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin Z & -\cos Z & 0 \\ \cos Z & -\sin Z & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{101}$$

[6]Witt, J. W., JPL internal document, Oct. 20, 1968.

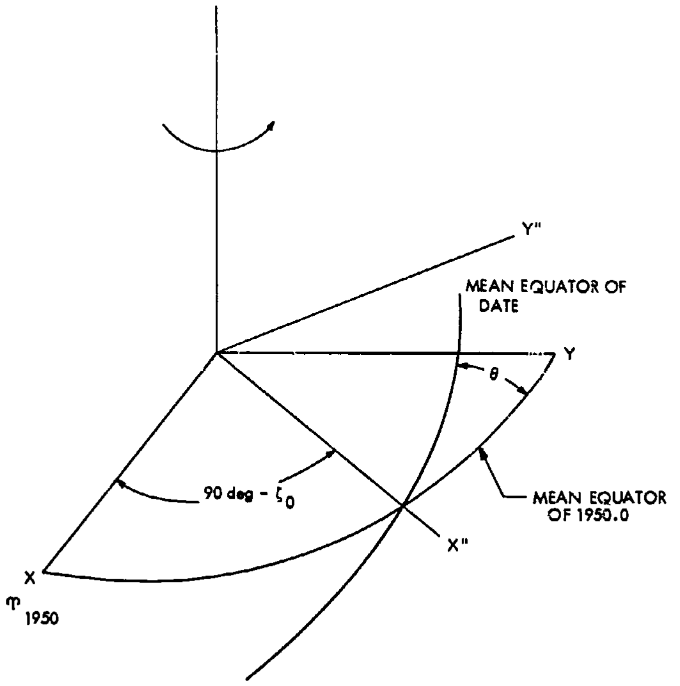

Fig. 12. Mean equator of 1950.0 to mean equator of date

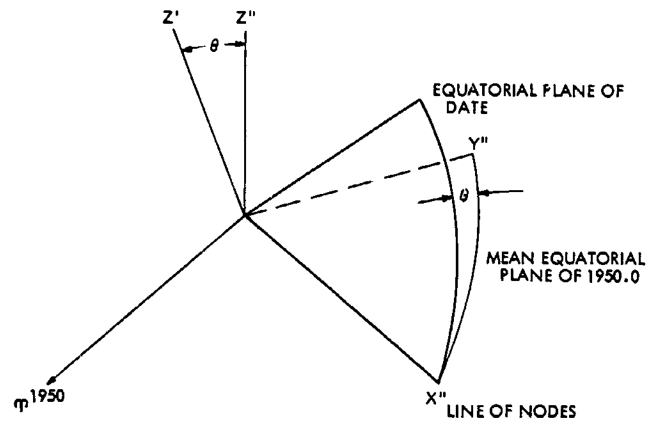

Fig. 13. Rotation about X"-axis

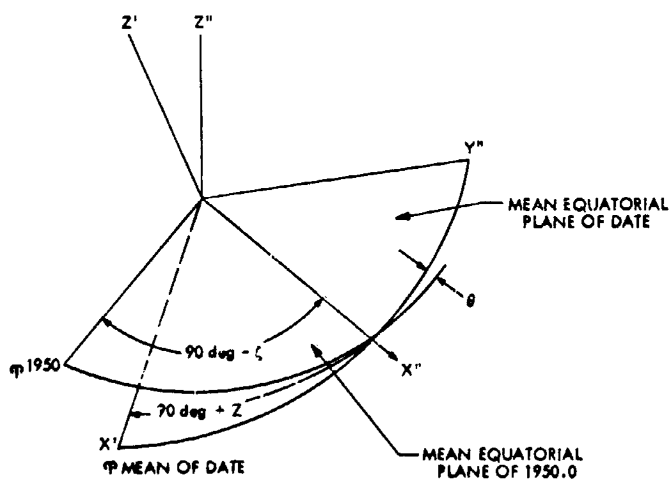

Fig. 14. Rotation about Z'-axis

The rotation from the mean equator of 1950.0 to the mean equator of date is then obtained by composing the three rotations to yield the precession-rotation matrix

$$\mathbf{A} = \mathbf{A}_3\,\mathbf{A}_2\,\mathbf{A}_1 = \begin{pmatrix} (-\sin Z \sin \zeta_0 + \cos Z \cos \theta \cos \zeta_0) & (-\sin Z \cos \zeta_0 - \cos Z \cos \theta \sin \zeta_0) & (-\cos Z \sin \theta) \\ (\cos Z \sin \zeta_0 + \sin Z \cos \theta \cos \zeta_0) & (\cos Z \cos \zeta_0 - \sin Z \cos \theta \sin \zeta_0) & (-\sin Z \sin \theta) \\ (\sin \theta \cos \zeta_0) & (-\sin \theta \sin \zeta_0) & (\cos \theta) \end{pmatrix} \tag{102}$$

The angles ζ_0, θ, and Z are[7]

$$\zeta_0 = 2304''\!.952\,T + 0''\!.3022\,T^2 + 0''\!.0180\,T^3$$

$$\theta = 2004''\!.257\,T - 0''\!.4268\,T^2 - 0''\!.0418\,T^3$$

$$Z = 2304''\!.952\,T + 1''\!.0951\,T^2 + 0''\!.0183\,T^3$$

where T is measured from 1950.0 in tropical centuries,

$$T = \frac{T_0 - 2433282.423357}{36524.21988}$$

where T_0 is the Julian date of the epoch in ephemeris time.

By use of the conversion factor

$$Q = \frac{36525}{36524.21988}, \quad \frac{\text{Julian century}}{\text{tropical century}} \tag{103}$$

the coefficients for ζ_0, θ, and Z may be converted so that T is measured in Julian centuries from 1950.0. Multiplication of the converted coefficients for ζ_0, θ, and Z by another conversion factor

$$Q' = \frac{0.17453292519943\,29}{360}, \quad \frac{\text{rad}}{\text{arc-sec}} \tag{104}$$

yields coefficients for ζ_0, θ, and Z in radians:

$$\left.\begin{aligned} \zeta_0 &= a_1 T + b_1 T^2 + c_1 T^3 \\ \theta &= a_2 T + b_2 T^2 + c_2 T^3 \\ Z &= a_3 T + b_3 T^3 + c_3 T^3 \end{aligned}\right\} \tag{105}$$

Given the Julian date of an epoch T_0 (in ephemeris time), the T required in Eqs. (105) is

$$T = \frac{T_0 - 2433282.423357}{36525}$$

in Julian centuries past 1950.0 E.T.

The time derivatives of ζ_0, θ, and Z are computed from Eq. (105):[8]

$$\left.\begin{aligned} \dot{\zeta}_0 &= a_1 + 2b_1 T + 3c_1 T^2, \text{ rad/Julian century} \\ \dot{\theta} &= a_2 + 2b_2 T + 3c_2 T^2, \text{ rad/Julian century} \\ \dot{Z} &= a_3 + 2b_3 T + 3c_3 T^2, \text{ rad/Julian century} \end{aligned}\right\} \tag{106}$$

The time derivative $\dot{\mathbf{A}}$ of the matrix $\mathbf{A}$ is given by

$$\frac{d}{dt} A_{ij} = \dot{A}_{ij}, \qquad i,j = 1,2,3 \tag{107}$$

where

$$\dot{A}_{11} = -\cos \zeta_0 (d_1 \sin Z + d_2) - d_3 \sin \zeta_0 \cos Z$$

$$\dot{A}_{12} = \sin \zeta_0 (d_1 \sin Z + d_2) - d_3 \cos \zeta_0 \cos Z$$

$$\dot{A}_{13} = \dot{Z} \sin \theta \sin Z - \dot{\theta} \cos \theta \cos Z$$

$$\dot{A}_{21} = \cos \zeta_0 (d_1 \cos Z - d_4) - d_3 \sin \zeta_0 \sin Z$$

$$\dot{A}_{22} = -\sin \zeta_0 (d_1 \cos Z - d_4) - d_3 \cos \zeta_0 \sin Z$$

$$\dot{A}_{23} = -\dot{Z} \sin \theta \cos Z - \dot{\theta} \cos \theta \sin Z$$

$$\dot{A}_{31} = \dot{\theta} \cos \zeta_0 \cos \theta - \dot{\zeta}_0 \sin \xi \sin \theta$$

$$\dot{A}_{32} = -\dot{\theta} \sin \zeta_0 \cos \theta - \dot{\zeta}_0 \cos \zeta_0 \sin \theta$$

$$\dot{A}_{33} = -\dot{\theta} \sin \theta$$

[7]Given by Khatib, A. R., JPL internal document, Jan. 10, 1969.

[8]Witt, J. W., JPL internal document, Oct. 20, 1968.

where

$$d_1 = (\dot{Z} \cos\theta + \dot{\zeta}_0)$$

$$d_2 = \dot{\theta} \sin\theta \cos Z$$

$$d_3 = (\dot{Z} + \dot{\zeta}_0 \cos\theta)$$

$$d_4 = \dot{\theta} \sin\theta \sin Z$$

Each component of $\mathbf{A}$ is divided by 0.315576×10^{10} s/century to yield the amount in radians per second.

The primed and unprimed coordinate sets are then related to each other by

$$\left.\begin{aligned} \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} &= \mathbf{A} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \\ \begin{pmatrix} \dot{X}' \\ \dot{Y}' \\ \dot{Z}' \end{pmatrix} &= \frac{d}{dt}\left[\mathbf{A} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\right] \\ &= \mathbf{A} \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} + \dot{\mathbf{A}} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \end{aligned}\right\} \qquad (108)$$

C. Mean Obliquity of the Ecliptic and Its Time Derivative

The mean obliquity of the ecliptic $\bar{\epsilon}$ is the angle between the mean equatorial plane of the earth and the ecliptic plane (Fig. 15), and is computed from the following expression:*

$$\bar{\epsilon} = 84404''.84 - 46.850\, T_1 - 0.0034\, T_1^2 + 0.0018\, T_1^3 \qquad (109)$$

where T_1 is the number of tropical centuries past 1950.0,

$$T_1 = \frac{T_0 - 2433282.423357}{36524.21988}$$

where T_0 is the Julian date of the epoch in ephemeris time.

*Khatib, A. R., JPL internal document, Jan. 10, 1969.

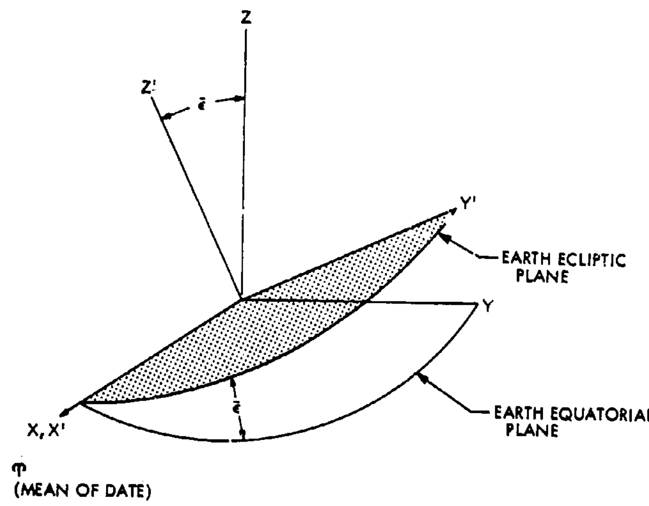

Fig. 15. Mean obliquity of the ecliptic $\bar{\epsilon}$

If it is desired to obtain $\bar{\epsilon}$ in radians, use may be made of the conversion factor Q' (see Eq. 104). Then

$$A = 84404''.84\, Q'$$

$$B = -46''.850\, Q'$$

$$C = -0''.0034\, Q'$$

$$D = 0''.0018\, Q'$$

in radians, so that

$$\bar{\epsilon} = A + BT + CT^2 + DT^3 \qquad (110)$$

The time derivative of $\bar{\epsilon}$ is then

$$\dot{\bar{\epsilon}} = \frac{B + 2CT + 3DT^2}{36525 \times 86400} \qquad (111)$$

in radians per second.

D. Earth Mean or True Equatorial Coordinates to Ecliptic Coordinates Rotation

Let it be assumed that the X-Y plane is the mean or true equator of the earth, with the X-axis in the direction of the mean or true vernal equinox. The ecliptic coordinate system (X', Y', Z') is obtained by rotating about the X-axis

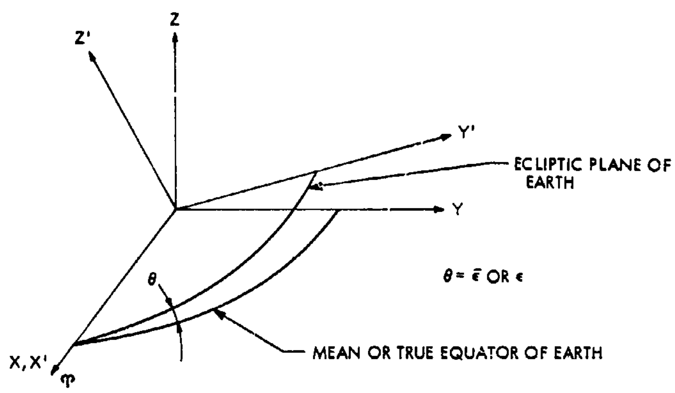

Fig. 16. True or mean obliquity

by the angle $\bar{\epsilon}$ or ϵ—the mean or true obliquity (Fig. 16). The rotation matrix, denoted by K, is then

$$\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{112}$$

where

$$\theta = \epsilon \text{ or } \bar{\epsilon} \tag{113}$$

Then

$$\dot{\mathbf{K}} = -\dot{\theta}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin\theta & -\cos\theta \\ 0 & \cos\theta & \sin\theta \end{pmatrix} \tag{114}$$

where

$$\theta = \epsilon \text{ or } \bar{\epsilon}$$
$$\dot{\theta} = \dot{\epsilon} \text{ or } \dot{\bar{\epsilon}}$$

The two coordinate sets are related by K and $\dot{K}$ as follows:

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{K}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{115}$$

$$\begin{pmatrix} \dot{X}' \\ \dot{Y}' \\ \dot{Z}' \end{pmatrix} = \mathbf{K}\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} + \dot{\mathbf{K}}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{116}$$

Thus, a position vector $\mathbf{r}$ in the unprimed system becomes

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{K}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{Kr} \tag{117}$$

and a velocity vector $\dot{\mathbf{r}}$ in the unprimed system becomes

$$\dot{\mathbf{r}}' = \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix} = \frac{d}{dt}\left[\mathbf{K}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \dot{\mathbf{K}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \mathbf{K}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$
$$= \dot{\mathbf{K}}\mathbf{r} + \mathbf{K}\dot{\mathbf{r}} \tag{118}$$

E. Mean Earth Equator and Equinox of Date Coordinates to True Earth Equator and Equinox of Date Coordinates Rotations

Nutation represents the difference between the position of the true celestial pole (rotational axis of the earth) and the mean celestial pole. Because it is entirely composed of the short-period effects caused by the actions of sun and moon on the figure of the earth, nutation affects only the equatorial plane, not the ecliptic. For this reason, it is most convenient to apply nutation to ecliptic coordinates, in which the vernal equinox is shifted from its mean position in the mean ecliptic of date to its true position, which is in the same plane. That is, the true ecliptic of date is also the mean ecliptic of date. The true equator of date differs from the mean equator of date by two increments:

(1) $\delta\psi$ = nutation in longitude, which is the true longitude of date of the mean equinox of date.

(2) $\delta\epsilon$ = nutation in obliquity

The mean obliquity of the ecliptic is defined as

$$\bar{\epsilon} = \epsilon + \delta\epsilon$$

where ϵ is the true obliquity of the ecliptic.

The transformation of Cartesian position and velocity coordinates from the mean earth equator and equinox of date to the true earth equator and equinox of date requires three rotations.

The XY-plane is the plane of the mean equator of the earth, with the X-axis in the direction of the mean vernal equinox. The relation between the true earth equatorial coordinate system (X', Y', Z') and the mean earth equa-

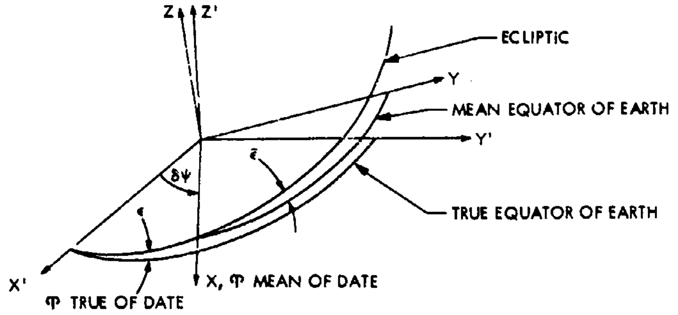

Fig. 17. Mean and true equinox of date

torial coordinate system (X,Y,Z) is shown in Fig. 17. The nutation in longitude $\delta\psi$ is measured from the true vernal equinox at the X'-axis to the mean vernal equinox at the X-axis. To rotate from the mean equator and equinox of date to the true equator and equinox of date, one first rotates from the mean equator to the ecliptic about the X-axis with

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\bar\epsilon & \sin\bar\epsilon \\ 0 & -\sin\bar\epsilon & \cos\bar\epsilon \end{pmatrix}$$

A negative rotation of the result through the nutation in longitude about the Z-axis to the true vernal equinox is accomplished by

$$\mathbf{B} = \begin{pmatrix} \cos\delta\psi & -\sin\delta\psi & 0 \\ \sin\delta\psi & \cos\delta\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A negative rotation about the X'-axis to the true equator is obtained by application of the matrix

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & -\sin\epsilon \\ 0 & \sin\epsilon & \cos\epsilon \end{pmatrix}$$

Thus, the primed and unprimed coordinate systems are related by

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{CBA} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{119}$$

where the primed system corresponds to the true equator and equinox of date and the unprimed system corresponds to the mean equator and equinox of date. If one lets $\mathbf{N} = \mathbf{CBA}$, then[10]

$$\mathbf{N} = \begin{bmatrix} (\cos\delta\psi) & (-\sin\delta\psi\cos\bar\epsilon) & (-\sin\delta\psi\sin\bar\epsilon) \\ (\sin\delta\psi\cos\epsilon) & (\cos\delta\psi\cos\epsilon\cos\bar\epsilon + \sin\epsilon\sin\bar\epsilon) & (\cos\delta\psi\cos\epsilon\sin\bar\epsilon - \sin\epsilon\cos\bar\epsilon) \\ (\sin\delta\psi\sin\epsilon) & (\cos\delta\psi\sin\epsilon\cos\bar\epsilon - \cos\epsilon\sin\bar\epsilon) & (\cos\delta\psi\sin\epsilon\sin\bar\epsilon + \cos\epsilon\cos\bar\epsilon) \end{bmatrix} \tag{120}$$

The time derivative of N; $d(\mathbf{N})/dt = \dot{\mathbf{N}}$, is given by

$$\begin{aligned}
\dot N(1,1) &= -\dot{\delta\psi}\sin\delta\psi \\
\dot N(1,2) &= \dot{\bar\epsilon}\sin\delta\psi\sin\bar\epsilon - \dot{\delta\psi}\cos\delta\psi\cos\bar\epsilon \\
\dot N(1,3) &= -\dot{\bar\epsilon}\sin\delta\psi\cos\bar\epsilon - \dot{\delta\psi}\cos\delta\psi\sin\bar\epsilon \\
\dot N(2,1) &= -\dot\epsilon\sin\delta\psi\sin\epsilon + \dot{\delta\psi}\cos\delta\psi\cos\epsilon \\
\dot N(2,2) &= d_1\sin\bar\epsilon\cos\epsilon + d_2\sin\bar\epsilon\cos\epsilon - \dot{\delta\psi}\cos\epsilon\cos\bar\epsilon\sin\delta\psi \\
\dot N(2,3) &= d_1\sin\bar\epsilon\sin\epsilon - d_2\cos\bar\epsilon\cos\epsilon - \dot{\delta\psi}\cos\epsilon\sin\bar\epsilon\sin\delta\psi \\
\dot N(3,1) &= \dot\epsilon\sin\delta\psi\cos\epsilon + \dot{\delta\psi}\sin\epsilon\cos\delta\psi \\
\dot N(3,2) &= d_2\sin\bar\epsilon\sin\epsilon - d_1\cos\epsilon\cos\bar\epsilon - \dot{\delta\psi}\sin\epsilon\cos\epsilon\sin\delta\psi \\
\dot N(3,3) &= -d_1\cos\epsilon\sin\bar\epsilon - d_2\cos\bar\epsilon\sin\epsilon - \dot{\delta\psi}\sin\epsilon\sin\bar\epsilon\sin\delta\psi
\end{aligned}$$

[10]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

where

$$d_1 = \dot{\bar{\epsilon}} - \dot{\epsilon}\cos\delta\psi$$

$$d_2 = \dot{\epsilon} - \dot{\bar{\epsilon}}\cos\delta\psi$$

Therefore, the position and velocity coordinates in the true equator and equinox of date reference system (the primed system) are obtained as follows:

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{N}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{121}$$

and

$$\begin{pmatrix} \dot{X}' \\ \dot{Y}' \\ \dot{Z}' \end{pmatrix} = \mathbf{N}\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} + \dot{\mathbf{N}}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{122}$$

F. Rotation Transforming Earth-Centered True Equatorial of Date, Space-Fixed Coordinates to Earth-Fixed Coordinates

The XY-plane is the plane of the true equator of the earth, with the X-axis in the direction of the true vernal equinox. The earth-fixed coordinate system $(X'Y'Z')$ is obtained by rotation about the Z-axis by the angle $\gamma(T)$, which is the Greenwich hour angle (GHA) (Fig. 18).

The two coordinate systems are related to each other by the rotation matrix E, given by

$$\mathbf{E} = \begin{pmatrix} \cos\gamma(T) & \sin\gamma(T) & 0 \\ -\sin\gamma(T) & \cos\gamma(T) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{123}$$

The time derivative of E is

$$\dot{\mathbf{E}} = -\dot{\gamma}(T)\begin{pmatrix} \sin\gamma(T) & -\cos\gamma(T) & 0 \\ \cos\gamma(T) & \sin\gamma(T) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{124}$$

Therefore, E and $\dot{\mathbf{E}}$ relate the two coordinate sets as follows:

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{E}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{125}$$

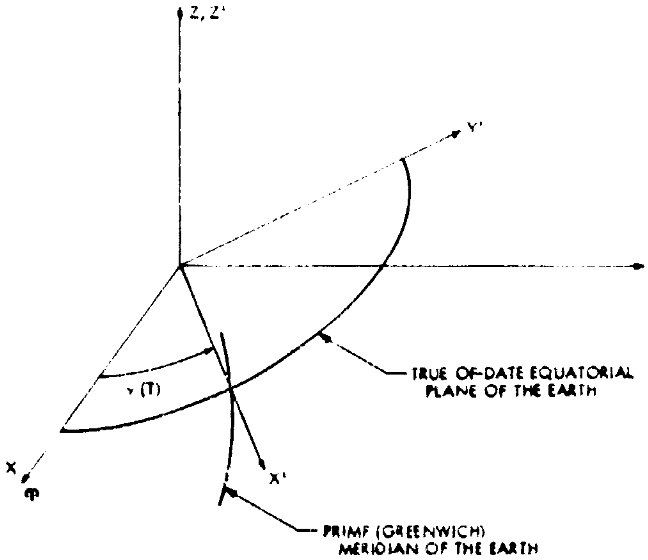

Fig. 18. Prime meridian of the earth

and

$$\begin{pmatrix} \dot{X}' \\ \dot{Y}' \\ \dot{Z}' \end{pmatrix} = \mathbf{E}\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} + \dot{\mathbf{E}}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{126}$$

The Greenwich hour angle $\gamma(T)$, which is defined as the angle between the vernal equinox of the earth and the Greenwich meridian (Fig. 19), is given by

$$\gamma(T) = \gamma_M(T) + \delta\alpha \tag{127}$$

where

$\delta\alpha = \delta\psi\cos\epsilon$

$= \text{nutation in right ascension}$

$\delta\psi = \text{nutation in longitude (Fig. 20)}$

$\epsilon = \text{true obliquity of ecliptic}$

$\gamma_M(T) = \text{Greenwich hour angle of mean equinox of date}$

$\gamma_M(T)$ is given by (see Ref. 2, p. 75)

$$\gamma_M(T) = UT1 + 23925^{\mathrm{s}}.836 + 8640184^{\mathrm{s}}.542\,T_u + 0^{\mathrm{s}}.0929\,T_u^2 \tag{128}$$

where T_u is the number of Julian centuries of 36,525 days of universal time since 12 h, January 0, 1900 UT.

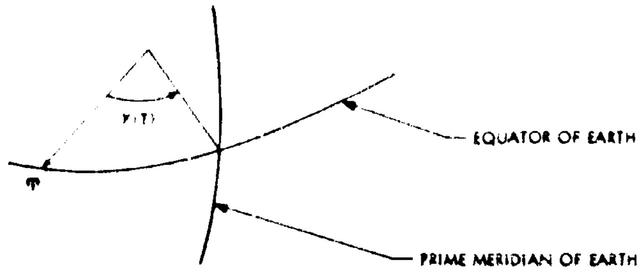

Fig. 19. Greenwich hour angle

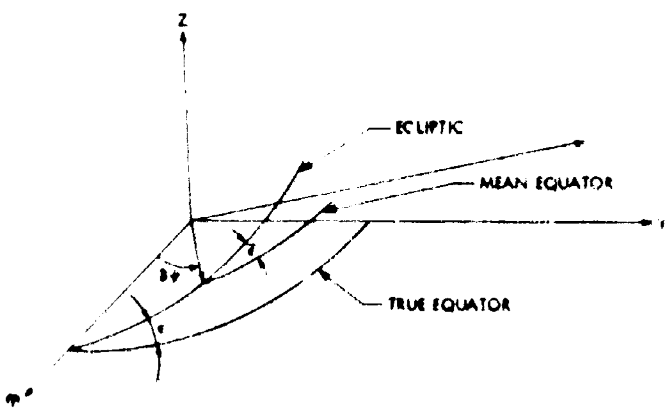

Fig. 20. Nutation in longitude, true and mean obliquity

$UT1$ is given by

$$UT1 = ET - (ET - A.1) - (A.1 - UT1) \tag{129}$$

where ET is the number of ephemeris seconds elapsed since 0 h ephemeris time on the current day and $(ET - A.1)$ and $(A.1 - UT1)$ are time transformations (see Section III-E).

Because $\gamma_M(T)$ is usually desired in radians, the terms on the right side of Eq. (128) must be converted into radians. This conversion is effected as follows:

$$A = 23925^s.836 f$$

$$B = 8640184^s.542 f$$

$$C = 0^s.0929 f$$

in radians, where

$$f = \frac{0.01745329251994329}{240} \tag{130}$$

in radians divided by ephemeris seconds. The conversion factor f is determined as follows:

$$1 \text{ deg} = 0.01745329251994329$$

$$24 \text{ h} = 24 \times 3600 \text{ s} = 360 \text{ deg}$$

$$= 360 \times 0.01745329251994329$$

in radians; thus,

$$f = \frac{360 \times 0.01745329251994329}{24 \times 3600} = \frac{0.01745329251994329}{240}$$

in radians divided by ephemeris seconds. Given the Julian date of an epoch T_u in seconds past January 1, 1950, 0 h $UT1$, T_t may be computed as follows:[11]

$$T_t = \frac{T_u - 2415020}{36525}$$

where 2415020 is the Julian date of 12 h, January 0, 1900 UT.

An equivalent expression for $UT1$ in Eq. (128) is

$$\overline{UT1} = T_f 2\pi \tag{131}$$

in radians, where $T_f = [T_u - 0.5]$ decimal part (Ref. 7, p. 37). Therefore, the GHA in radians is

$$\gamma(T) = \overline{UT1} + A + BT_t + CT_t^2 + \delta\psi \cos\epsilon \quad (\text{modulo } 2\pi) \tag{132}$$

The derivative of $\gamma(T)$ with respect to ephemeris time is given by

$$\dot{\gamma}(T) = \dot{\gamma}_M(T) + \delta\dot{\psi} \cos\epsilon - \dot{\epsilon}\,\delta\psi \sin\epsilon \tag{133}$$

in radians divided by ephemeris seconds, where $\dot{\gamma}_M(T)$ is computed from Eq. (128) (see Ref. 7, p. 37):

$$\dot{\gamma}_M(T) = \frac{dUT1}{dET}\left(1 + \frac{8640184^s.542 + 0^s.1858\,T_t}{36525 \times 86400}\right)\frac{2\pi}{86400} \tag{134}$$

in radians divided by ephemeris seconds.

[11] Warner, M. R., et al., JPL internal document, Oct. 30, 1969.

From Eq. (129), and from Eqs. (4) and (6), it follows that

$$\frac{dUT1}{dET} = 1 + \frac{\Delta f_{\text{cesium}}}{9{,}192{,}631{,}770} - g - 2ht \qquad (135)$$

where t is the number of seconds past the start of the time block containing the given parameters f, g, and h.

G. Mars-Related Transformations

For purposes of defining Mars-related coordinate systems, four reference planes are chosen:

(1) Mean equator of date.

(2) True equator of date.

(3) Mean orbit of date.

(4) True orbit of date.

The reference directions are the ascending node of the orbit on the equator (both mean or both true) and the intersection of the Martian prime meridian with the true equator. The former is called the Martian (vernal) equinox of date (mean or true), and the angle between the prime meridian and the true equinox of date is called the "Mars hour angle" (see Ref. 2, p. 334–335, for the definition of the prime meridian of Mars).

Five coordinate systems[12] result:

(1) Mean equinox and orbit of date.

(2) Mean equinox and equator of date.

(3) True equinox and orbit of date.

(4) True equinox and equator of date (space-fixed).

(5) Prime meridian and true equator of date (body-fixed).

Only the precession (therefore, only the mean equator) of Mars is well known at present. Until nutation is also well known (and, with it, the true equator), the mean and true equators will be considered coincident; therefore, the mean and true equinoxes are the same. Because of this fact, it suffices to consider only three coordinate systems:

(1) Mean (=true) equinox and orbit of date.

(2) Mean (=true) equinox and equator of date (space-fixed).

(3) Prime meridian and equator of date (body-fixed).

The Martian equator and equinox are shown in Fig. 21.

The elements of the mean orbit of date with respect to the ecliptic and mean equinox of date (see Ref. 2, p. 113) are given by

$$\bar{\Omega} = 48°47'11''.19 + 2775''.57\,T - 0''.005\,T^2 - 0''.0192\,T^3 \qquad (136)$$

$$\bar{i} = 1°51'01''.20 - 2''.430\,T + 0''.0454\,T^2 \qquad (137)$$

where T is the number of Julian centuries of 36,525 ephemeris days elapsed since 1900, January 0, 12 h E.T. (JD = 2415020.0).

The adopted position of the north pole of Mars[13] is

$$\alpha = 316.55 + \frac{1.62024}{240}\,T \qquad (138)$$

$$\delta = 52.85 + \frac{12.528}{3600}\cdot T \qquad (139)$$

[12] Witt, J. W., JPL internal document, Oct. 20, 1968.

[13] Khatib, A. R., JPL internal document, Jan. 10, 1969.

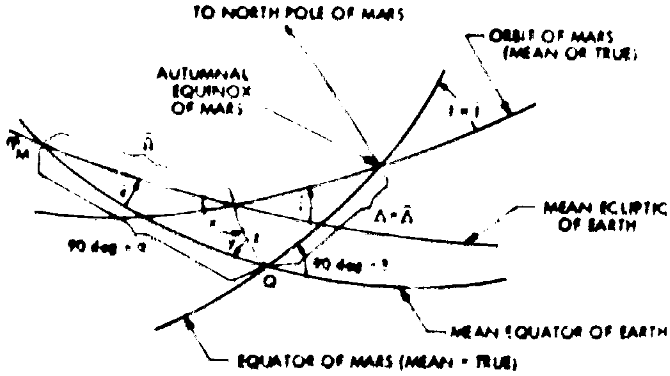

Fig. 21. Martian equator and equinox

where

α, δ = right ascension and declination of pole with respect to mean earth equator and equinox of date

T = number of tropical centuries past 1905.0

These angles are used for both the mean and true Martian equators because the nutation is unknown at present.

With the above angles, it is now possible to locate the mean (= true) equinox for Mars. First the auxiliary angles x, y, and z are computed (consider the spherical triangle shown in Fig. 22). Then, from spherical trigonometry,

$$\frac{\sin x}{\sin(90+\alpha)} = \frac{\sin\bar{\epsilon}}{\sin z}$$

or

$$\sin z \sin x = \sin\bar{\epsilon}\cos\alpha \tag{140}$$

$$\begin{aligned}\sin z\cos x &= \cos(90+\alpha)\sin\bar{\Omega} - \sin(90+\alpha)\cos\bar{\Omega}\cos\bar{\epsilon}\\ &= -\sin\alpha\sin\bar{\Omega} - \cos\alpha\cos\bar{\Omega}\cos\bar{\epsilon}\end{aligned} \tag{141}$$

$$\begin{aligned}\cos z &= \cos(90+\alpha)\cos\bar{\Omega} + \sin(90+\alpha)\sin\bar{\Omega}\cos\bar{\epsilon}\\ &= -\sin\alpha\cos\bar{\Omega} + \cos\alpha\sin\bar{\Omega}\cos\bar{\epsilon}\end{aligned} \tag{142}$$

Also,

$$\frac{\sin\bar{\epsilon}}{\sin z} = \frac{\sin y}{\sin\bar{\Omega}}$$

hence,

$$\sin z\sin y = \sin\bar{\epsilon}\sin\bar{\Omega} \tag{143}$$

$$\begin{aligned}\sin z\cos y &= \cos\bar{\Omega}\sin(90+\alpha) - \sin\bar{\Omega}\cos(90+\alpha)\cos\bar{\epsilon}\\ &= \cos\bar{\Omega}\cos\alpha + \sin\bar{\Omega}\sin\alpha\cos\bar{\epsilon}\end{aligned} \tag{144}$$

Then $\bar{I}$ is computed from

$$\begin{aligned}\cos\bar{I} = {} & \cos[180-(x-\bar{i})]\cos[180-(y+90-\delta)]\\ & + \sin[180-(x-\bar{i})]\\ & \times\sin[180-(y+90-\delta)]\cos z\end{aligned}$$

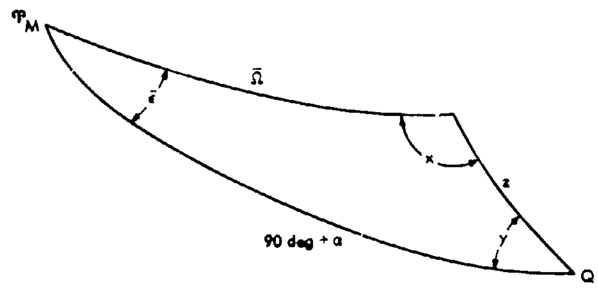

Fig. 22. Spherical triangle 1

or

$$\cos\bar{I} = \cos(x-\bar{i})\sin(y-\delta) + \sin(x-\bar{i})\cos(y-\delta)\cos z \tag{145}$$

where $\bar{I}$ is in the first quadrant.

Another necessary angle is the arc between the ascending node of the Martian equator on the mean equator of earth and the Martian vernal equinox $\Delta + 180$ deg; Δ (Fig. 23) is obtained from two additional equations,

$$\frac{\sin\Delta}{\sin[180-(x-\bar{i})]} = \frac{\sin z}{\sin\bar{I}}$$

Hence,

$$\sin\bar{I}\sin\Delta = \sin z\sin(x-\bar{i}) \tag{146}$$

$$\begin{aligned}\sin\bar{I}\cos\Delta = {} & \cos[180-(x-\bar{i})]\sin[90-(y-\delta)]\\ & -\sin[180-(x-\bar{i})]\cos[90-(y-\delta)]\cos z\end{aligned}$$

or

$$\begin{aligned}\sin\bar{I}\cos\Delta = {} & -\cos(x-\bar{i})\cos(y-\delta)\\ & +\sin(x-\bar{i})\sin(y-\delta)\cos z\end{aligned} \tag{147}$$

The value of Δ may then be computed from Eqs. (146) and (147).

The hour angle of the mean equinox of Mars[11] is given by

$$V = 149.475 + 350.891962\ (JD - 2418322.0) \tag{148}$$

where 2418322.0 is the Julian date of 0 h, January 15, 1909.

These are all of the angles necessary to perform the required rotations.

[11]Khatib, A. R., JPL internal document, Jan. 10, 1969.

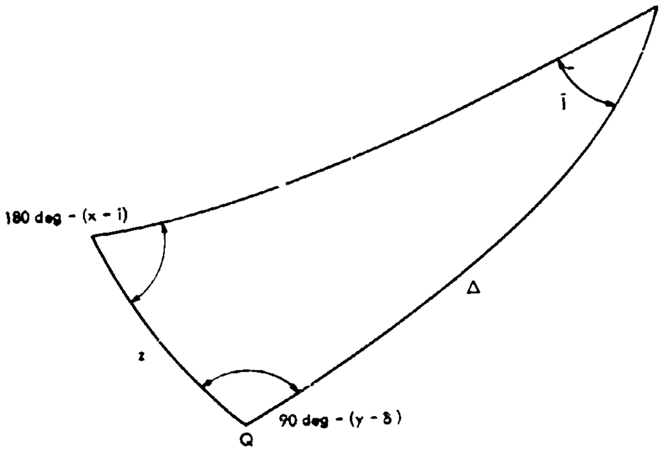

Fig. 23. Spherical triangle 2

1. Rotation matrices for position vectors. If the three coordinate systems listed above Eq. (136) are denoted by (X_i, Y_i, Z_i), $i = 1, 2, 3$, the rotation from body-fixed to space-fixed coordinates is then

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \cos V & -\sin V & 0 \\ \sin V & \cos V & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_3 \\ Y_3 \\ Z_3 \end{pmatrix}$$

or, in vector notation,

$$\mathbf{r}_2 = \mathbf{R}_V \mathbf{r}_3 \tag{149}$$

i.e., a left-handed rotation about the spin axis. Then the rotation to the mean (= true) equinox and orbit of date is

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \bar{I} & \sin \bar{I} \\ 0 & -\sin \bar{I} & \cos \bar{I} \end{pmatrix} \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} \tag{150}$$

or, in vector notation,

$$\mathbf{r}_1 = \mathbf{R}_{\bar{I}} \mathbf{r}_2 \tag{151}$$

The rotation from the space-fixed system to the 1950.0 earth-equatorial system[15] is given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{1950} = \mathbf{A}^T \mathbf{R}_3 \mathbf{R}_2 \mathbf{R}_1 \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} \tag{152}$$

or, in vector notation,

$$\mathbf{r}_{50} = \mathbf{A}^T \mathbf{R}_3 \mathbf{R}_2 \mathbf{R}_1 \mathbf{r}_2 \tag{153}$$

where

$\mathbf{A}$ = precession–rotation matrix (see Eq. 102)

$$\mathbf{R}_3 = \begin{bmatrix} \cos(\Delta + 180\,\mathrm{deg}) & -\sin(\Delta + 180\,\mathrm{deg}) & 0 \\ \sin(\Delta + 180\,\mathrm{deg}) & \cos(\Delta + 180\,\mathrm{deg}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \cos\Delta & \sin\Delta & 0 \\ \sin\Delta & \cos\Delta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(90\,\mathrm{deg} - \delta) & \sin(90\,\mathrm{deg} - \delta) \\ 0 & \sin(90\,\mathrm{deg} - \delta) & \cos(90\,\mathrm{deg} - \delta) \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\delta & -\cos\delta \\ 0 & \cos\delta & \sin\delta \end{pmatrix}$$

$$\mathbf{R}_1 = \begin{bmatrix} \cos(90\,\mathrm{deg} + \alpha) & -\sin(90\,\mathrm{deg} + \alpha) & 0 \\ \sin(90\,\mathrm{deg} + \alpha) & \cos(90\,\mathrm{deg} + \alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} -\sin\alpha & -\cos\alpha & 0 \\ \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[15] Witt, J. W., JPL internal document, Oct. 20, 1969.

2. *Rotation matrices for velocity vectors.* From Eq. (149), it follows immediately that

$$\dot{\mathbf{r}}_2 = \mathbf{R}_V \dot{\mathbf{r}}_3 + \dot{\mathbf{R}}_V \mathbf{r}_3 \tag{154}$$

where

$$\dot{\mathbf{R}}_V = \dot{V}\begin{pmatrix} -\sin V & -\cos V & 0 \\ \cos V & -\sin V & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$\dot{V} = 350.891962 \text{ deg/day} \qquad \text{(from Eq. 148)}$$

From Eq. (151),

$$\dot{\mathbf{r}}_1 = \mathbf{R}_I \dot{\mathbf{r}}_2 + \dot{\mathbf{R}}_I \mathbf{r}_2 \tag{155}$$

where

$$\dot{\mathbf{R}}_I = \dot{\bar{I}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin \bar{I} & \cos \bar{I} \\ 0 & -\cos \bar{I} & -\sin \bar{I} \end{pmatrix} \tag{156}$$

From Eq. (153),

$$\dot{\mathbf{r}}_{50} = \mathbf{A}^T \mathbf{R}_3 \mathbf{R}_2 \mathbf{R}_1 \dot{\mathbf{r}}_2 + (\dot{\mathbf{A}}^T \mathbf{R}_3 \mathbf{R}_2 \mathbf{R}_1 + \mathbf{A}^T \dot{\mathbf{R}}_3 \mathbf{R}_2 \mathbf{R}_1 + \mathbf{A}^T \mathbf{R}_3 \dot{\mathbf{R}}_2 \mathbf{R}_1 + \mathbf{A}^T \mathbf{R}_3 \mathbf{R}_2 \dot{\mathbf{R}}_1)\mathbf{r}_2 \tag{157}$$

where

$\dot{\mathbf{A}}$ = derivative of precession matrix (see Eq. 107)

$$\dot{\mathbf{R}}_3 = \dot{\alpha}\begin{pmatrix} -\cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & -\cos \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{158}$$

$$\dot{\mathbf{R}}_2 = \dot{\delta}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos \delta & \sin \delta \\ 0 & -\sin \delta & \cos \delta \end{pmatrix} \tag{159}$$

$$\dot{\mathbf{R}}_1 = \dot{\Delta}\begin{pmatrix} \sin \Delta & \cos \Delta & 0 \\ -\cos \Delta & \sin \Delta & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{160}$$

The derivatives of the various angles in Eqs. (156) through (160) are obtained as described below. From Eqs. (138) and (139),

$$\dot{\alpha} = \frac{1.62024}{240} \text{ deg/tropical century}$$

$$\dot{\delta} = \frac{12.528}{3600} \text{ deg/tropical century}$$

From Eqs. (136) and (137),

$$\dot{\bar{\Omega}} = \frac{2775.57}{3600}\,\frac{1}{36525} \text{ deg/day}$$

$$\dot{\bar{I}} = \frac{-2.430}{3600}\,\frac{1}{36525} \text{ deg/day}$$

Differentiation of Eq. (145) yields

$$\begin{aligned} -\dot{\bar{I}}\sin \bar{I} = {} & (\dot{y} - \dot{\delta})\left[\cos(x - \bar{\Omega})\cos(y - \delta) - \sin(x - \bar{\Omega})\sin(y - \delta)\cos z\right] \\ & + (\dot{x} - \dot{\bar{\Omega}})\left[-\sin(x - \bar{\Omega})\sin(y - \delta) + \cos(x - \bar{\Omega})\cos(y - \delta)\cos z\right] \\ & + \dot{z}\left[-\sin(x - \bar{\Omega})\cos(y - \delta)\sin z\right] \end{aligned} \tag{161}$$

Equation (161) may then be solved for $\dot{\bar{I}}$.

Differentiation of Eq. (146) gives

$$\dot{\Delta}\sin I\cos\Delta = -\dot{\bar{I}}\cos\bar{I}\sin\Delta + \dot{z}\cos z\sin(x-\bar{i}) + (\dot{x}-\dot{\bar{i}})\sin z\cos(x-\bar{i}) \qquad (162)$$

and Eq. (162) may be solved for $\dot{\Delta}$.

From Eq. (142),

$$-\dot{z}\sin z = \dot{c}\,[-\cos\bar{\epsilon}\sin\bar{\Omega}\sin\alpha - \cos\bar{\Omega}\cos\alpha] + \dot{\bar{\Omega}}\,[\cos\bar{\epsilon}\cos\bar{\Omega}\cos\alpha + \sin\bar{\Omega}\sin\alpha] - \dot{\bar{\epsilon}}\,[\sin\bar{\epsilon}\sin\bar{\Omega}\cos\alpha] \qquad (163)$$

where $\dot{\bar{\epsilon}}$ is given by Eq. (111).

From Eq. (140),

$$\dot{x}\sin z\cos x = -\dot{z}\cos z\sin x - \dot{\alpha}\sin\bar{\epsilon}\sin\alpha + \dot{\bar{\epsilon}}\cos\bar{\epsilon}\cos\alpha \qquad (164)$$

From Eq. (143),

$$\dot{y}\sin z\cos y = -\dot{z}\cos z\sin y + \dot{\bar{\Omega}}\sin\bar{\epsilon}\cos\bar{\Omega} + \dot{\bar{\epsilon}}\cos\bar{\epsilon}\sin\bar{\Omega} \qquad (165)$$

Thus, Eqs. (163) through (165) yield $\dot{z}$, $\dot{x}$, and $\dot{y}$.

H. Moon-Related Transformations

Figure 24 illustrates the geometry of the lunar equator and orbit, and defines the required angles. The angle Ω on the mean ecliptic of the earth between the mean vernal equinox of the earth and the mean lunar equator is given in Ref. 2 (p. 107) as

$$\Omega = 933{,}059''.79 - 6{,}962{,}911''.23\,T + 7''.48\,T^2 + 0''.008\,T^3 \qquad (166)$$

where, by use of the factor Q' (see Eq. 104),

$$Q' = 4.84813681 \times 10^{-6}, \quad \frac{\text{rad}}{\text{arc-sec}} \qquad (167)$$

the above coefficients may be converted into radians. In Eq. (166), and in all following equations, T is the number of Julian centuries (36,525 ephemeris days) elapsed since 12 h, January 0, 1900 E.T., Julian date 2415020.

The angle of inclination I of the mean lunar equator to the mean ecliptic of the earth is given in Ref. 2 (p. 108) as

$$I = 1°32'.1 \qquad (168)$$

By use of Eq. (167), I may be converted into radians.

The mean longitude of the moon ☾ is measured in the ecliptic from the mean equinox of date to the mean ascending node of the lunar orbit, and then along the orbit (Fig. 25). The quantity ☾ is derived from lunar theory, and is given as a polynomial in time (see Ref. 2, p. 107):

$$☾ = 973{,}562''.99 + 1{,}732{,}564{,}379''.31\,T - 4''.08\,T^2 + 0''.0068\,T^3 \qquad (169)$$

Another quantity needed is the mean longitude of the lunar perigee Γ', which is measured in the ecliptic from the mean equinox of date to the mean ascending node of the lunar orbit, and then along the orbit. The quantity Γ' is given in Ref. 2 (p. 107) as

$$\Gamma' = 1{,}203{,}586''.40 + 14{,}648{,}522''.52\,T - 37''.17\,T^2 - 0''.045\,T^3 \qquad (170)$$

By use of Eq. (167), the coefficients of Eqs. (169) and (170) may be converted into radians.

Perturbations in the mean values Ω, I, and ☾ are the physical librations σ, ρ, and τ, respectively; that is, the

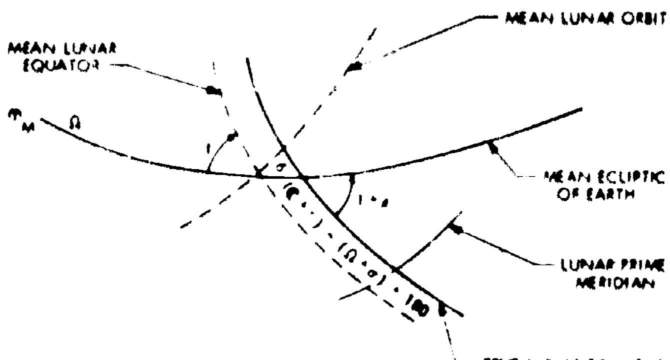

Fig. 24. Lunar equator and orbit

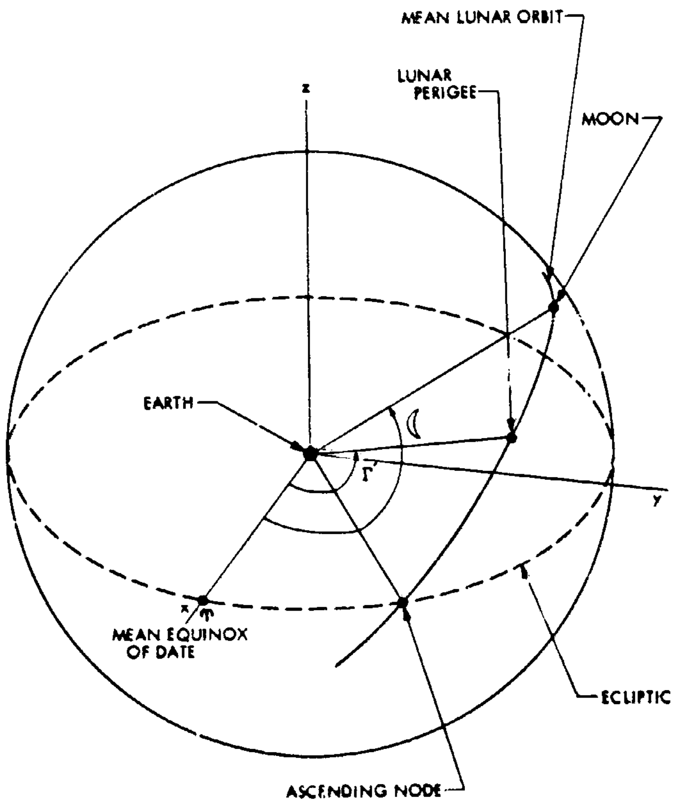

Fig. 25. The orbit of the moon relative to earth

actual angle of inclination and the angle between the mean vernal equinox and the descending node of the lunar equator on the ecliptic are $I + \rho$ and $\Omega + \sigma$, respectively, and the angular distance from the descending node of the lunar equator to the prime meridian is $[180 \text{ deg} + (☾ + \tau) - (\Omega + \sigma)]$.[16]

The physical librations (as opposed to the optical librations) are the results of the moon being a triaxial ellipsoid and not a sphere; the longest diameter is directed toward the earth and the shortest along the axis of rotation.

The librations in node, inclination, and longitude σ, ρ, and τ are computed from the following equations:[17]

$$\sigma I = I\tau - 100''\!.7 \sin(g) + 23''\!.8 \sin(g + 2\omega) - 10''\!.6 \sin(2g + 2\omega) \tag{171a}$$

$$\rho = -98''\!.5 \cos(g) + 23''\!.9 \cos(g + 2\omega) - 11''\!.0 \cos(2g + 2\omega) \tag{171b}$$

$$\tau = -16''\!.9 \sin(g) + 91''\!.7 \sin(g^1) - 15''\!.3 \sin(2\omega) + 18''\!.7 \sin(3^R\!.8983813 + 217^R\!.9812\,T) \tag{171c}$$

where

$g = ☾ - \Gamma' =$ mean anomaly of moon

$\omega = \Gamma' - \Omega =$ argument of perigee of moon

$g^1 =$ mean anomaly of sun[18]

$$= 1{,}290{,}513''\!.04 + 129{,}596{,}579''\!.10\,T - 0''\!.54\,T^2 - 0''\!.012\,T^3 \tag{172}$$

If it is assumed that the coefficients of Eqs. (166) and (168) through (172) have been converted into radians by using Eq. (167), then

$$\Omega = A_1 + B_1 T + C_1 T^2 + D_1 T^3 \tag{173a}$$

$$☾ = A_2 + B_2 T + C_2 T^2 + D_2 T^3 \tag{173b}$$

$$\Gamma' = A_3 + B_3 T + C_3 T^2 + D_3 T^3 \tag{173c}$$

$$g^1 = A_4 + B_4 T + C_4 T^2 + D_4 T^3 \tag{173d}$$

$$\sigma I = I\tau + B_5 \sin(g) + C_5 \sin(g + 2\omega) + D_5 \sin(2g + 2\omega) \tag{173e}$$

$$\rho = B_6 \cos(g) + C_6 \cos(g + 2\omega) + D_6 \sin(2g + 2\omega) \tag{173f}$$

$$\tau = A_7 \sin(g) + B_7 \sin(g^1) + C_7 \sin(2\omega) + D_7 \sin(E_7 + F_7 T) \tag{173g}$$

in radians, and so[18]

$$\dot{\Omega} = \frac{B_1 + 2C_1 T + 3D_1 T^2}{f} \tag{174a}$$

$$\dot{☾} = \frac{B_2 + 2C_2 T + 3D_2 T^2}{f} \tag{174b}$$

$$\dot{\Gamma}' = \frac{B_3 + 2C_3 T + 3D_3 T^2}{f} \tag{174c}$$

$$\dot{g}^1 = \frac{B_4 + 2C_4 T + 3D_4 T^2}{f} \tag{174d}$$

[16] Warner, M. R., et al., JPL internal document, Oct. 30, 1968.
[17] Khatib, A. R., JPL internal document, Jan. 10, 1968.
[18] Witt, J. W., JPL internal document, Oct. 20, 1968.

in radians per second, where

$$f = (36{,}525 \text{ days/century})\ (86{,}400 \text{ s/day}) = 3{,}155{,}760{,}000 \text{ s/century}$$

Also,[19]

$$\dot{\sigma} I = I\dot{\tau} + \dot{g}\,[B_5 \cos(g) + C_5 \cos(g + 2\omega) + 2D_5 \cos(2g + 2\omega)] + 2\dot{\omega}\,[C_5 \cos(g + 2\omega) + D_5 \cos(2g + 2\omega)] \tag{175a}$$

$$\dot{\rho} = \dot{g}\,[-B_6 \sin(g) - C_6 \sin(g + 2\omega) + 2D_6 \cos(2g + 2\omega)] + 2\dot{\omega}\,[-C_6 \sin(g + 2\omega) + D_6 \cos(2g + 2\omega)] \tag{175b}$$

$$\dot{\tau} = A_7\, \dot{g} \cos(g) + B_7\, \dot{g}^1 \cos(g^1) + 2C_7 \dot{\omega} \cos(2\omega) + \frac{F_7}{3155760000} D_7 \cos(E_7 + F_7 T) \tag{175c}$$

in radians per second, where

$$\begin{aligned} \dot{g} &= \dot{☾} - \dot{\Gamma}' \\ \dot{\omega} &= \dot{\Gamma}' - \dot{\Omega} \end{aligned} \tag{176}$$

Hence, to perform the rotation through the descending node of the lunar equator on the ecliptic, the angle $\Omega + \sigma$ and its derivative are used. To perform the rotation through the inclination of the lunar equator to the ecliptic, the angle $I + \rho$ and its derivative are used. A rotation from the descending node of the lunar equator to the prime meridian may be performed by using the third angle $[180 \deg + (☾ + \tau) - (\Omega + \sigma)]$ and its derivative.

I. Rotation From Earth Mean Ecliptic to Moon True Equator Coordinate System

In what follows, a 3×3 rotation matrix and its time derivative will be computed. This matrix transforms Cartesian position and velocity components expressed in an earth mean ecliptic and equinox of date coordinate system to components expressed in a moon true equator and equinox of date coordinate system.

The XY-plane is the mean ecliptic of the earth, with the X-axis in the direction of the mean equinox of date. Figure 26 illustrates the relation between the earth mean ecliptic coordinate system (X, Y, Z) and the moon true equatorial coordinate system (X', Y', Z'). Let

$$\alpha = \Omega + \sigma \tag{177}$$

$$\beta = I + \rho \tag{178}$$

Then

$$\dot{\alpha} = \dot{\Omega} + \dot{\sigma} \tag{179}$$

$$\dot{\beta} = \dot{I} + \dot{\rho} \tag{180}$$

where Ω, σ, I, ρ, $\dot{\Omega}$, $\dot{\sigma}$, $\dot{I}$, and $\dot{\rho}$ are obtained from Eqs. (168), (171), (173), (174), and (175).

A rotation about the Z-axis through the angle α moves the X-axis from the mean equinox of date of the earth to the descending node of the true lunar equator on the ecliptic. The matrix of rotation $\mathbf{M}_1$ is then

$$\mathbf{M}_1 = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

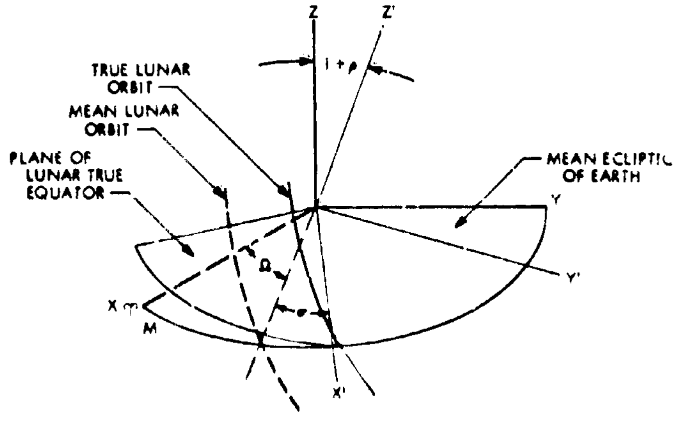

Fig. 26. Earth mean ecliptic and moon true equatorial coordinate systems

[19] Khatib, A. R., JPL internal document, Jan. 10, 1969.

A negative rotation about the X'-axis through the angle β moves the $X'Y'$-plane to the true lunar equator of date; i.e., space-fixed. The matrix of rotation $\mathbf{M}_2$ is

$$\mathbf{M}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}$$

Let

$$\mathbf{M} = \mathbf{M}_2 \mathbf{M}_1$$

that is,

$$\mathbf{M} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\cos\beta\ \sin\alpha & \cos\beta\ \cos\alpha & -\sin\beta \\ -\sin\beta\ \sin\alpha & \sin\beta\ \cos\alpha & \cos\beta \end{pmatrix} \tag{181}$$

Then

$$\dot{\mathbf{M}} = \begin{bmatrix} -\dot{\alpha}\sin\alpha & \dot{\alpha}\cos\alpha & 0 \\ (-\dot{\alpha}\cos\beta\cos\alpha + \dot{\beta}\sin\beta\sin\alpha) & (-\dot{\beta}\sin\beta\cos\alpha - \dot{\alpha}\cos\beta\sin\alpha) & -\dot{\beta}\cos\beta \\ (-\dot{\alpha}\sin\beta\cos\alpha - \dot{\beta}\sin\alpha\cos\beta) & (-\dot{\alpha}\sin\beta\sin\alpha + \dot{\beta}\cos\beta\cos\alpha) & -\dot{\beta}\sin\beta \end{bmatrix} \tag{182}$$

The rotation matrices $\mathbf{M}$ and $\dot{\mathbf{M}}$ relate the primed and unprimed coordinate systems as follows:

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{M} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{183}$$

$$\begin{pmatrix} \dot{X}' \\ \dot{Y}' \\ \dot{Z}' \end{pmatrix} = \mathbf{M} \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} + \dot{\mathbf{M}} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{184}$$

J. Rotation From Moon True Equator and Equinox of Date Coordinates to Moon-Fixed Coordinates

The transformation of Cartesian position and velocity components, expressed in a moon true equator and lunar equinox of date coordinate system, to components expressed in a moon-fixed coordinate system is achieved by one rotation. Figure 27 illustrates the relation between the moon true equator coordinate system (X, Y, Z), where the XY-plane is the plane of the true lunar equator with the X-axis pointing in the direction of the true lunar equinox, and the moon-fixed coordinate system (X', Y', Z').

A rotation about the Z-axis through the angle γ (γ is defined in Fig. 27) moves the X-axis from the true lunar equinox to the lunar prime meridian on the true lunar equator. The matrix of rotation $\mathbf{B}$ is

$$\mathbf{B} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{185}$$

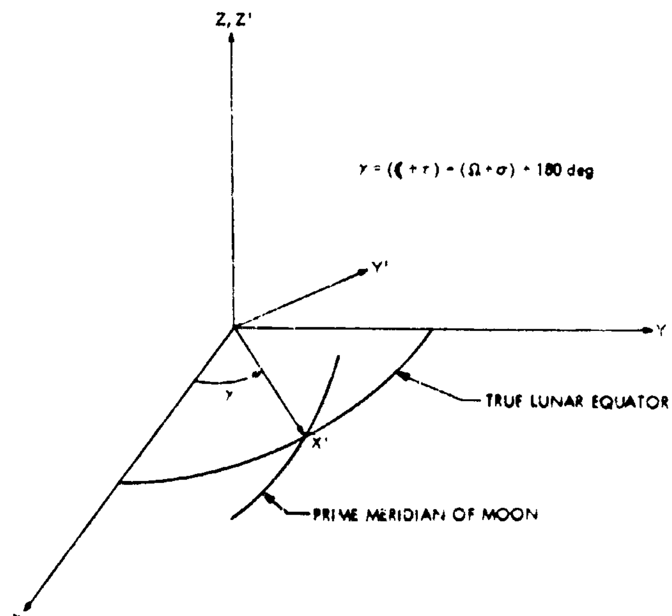

Fig. 27. Space- and body-fixed lunar coordinates

The time derivative of $\mathbf{B}$ is

$$\dot{\mathbf{B}} = \dot{\gamma} \begin{pmatrix} -\sin\gamma & \cos\gamma & 0 \\ -\cos\gamma & -\sin\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{186}$$

Rotation matrices $\mathbf{B}$ and $\dot{\mathbf{B}}$ relate the two coordinate systems as follows:

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{B} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{187}$$

$$\begin{pmatrix} \dot{X}' \\ \dot{Y}' \\ \dot{Z}' \end{pmatrix} = \mathbf{B} \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} + \dot{\mathbf{B}} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{188}$$

VI. Translation of Centers

It may be assumed that the positions and velocities of the planet are available (e.g., from an ephemeris tape) in the earth equatorial 1950.0 system. At a change in center (e.g., during integration), the position and velocity of the spacecraft in 1950.0 coordinates, relative to the old center of integration, are incremented by the position and velocity, respectively, at the old center relative to the new center (Fig. 28). The translated position vector is then

$$\mathbf{r}' = \mathbf{r} - \mathbf{r}_d \tag{189}$$

and the translated velocity vector is given by

$$\dot{\mathbf{r}}' = \dot{\mathbf{r}} - \dot{\mathbf{r}}_d \tag{190}$$

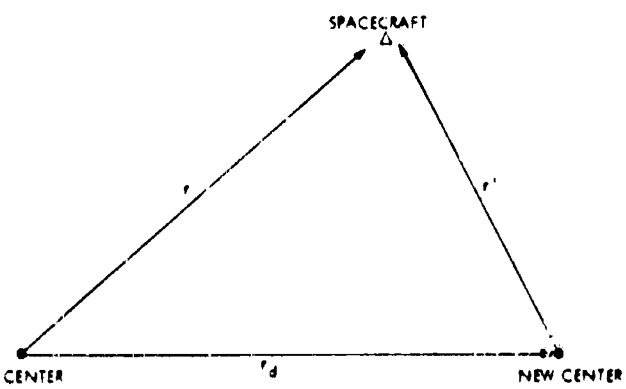

Fig. 28. Translation of centers

The same change of centers is employed when it is necessary to transform the initial-condition coordinates to the integration center that is to be used at the start of the trajectory.

VII. Equations of Motion of a Spacecraft

This section describes the differential equations of motion of spacecraft that are integrated numerically in a rectangular coordinate system to give the spacecraft ephemeris with E.T. as the independent variable. The X-axis is directed along the mean equinox of 1950.0; the Z-axis is normal to the mean earth equator of 1950.0, directed north; and the Y-axis completes the right-handed system. The center of integration is located at the center of mass of the sun, the moon, or one of the nine planets. It may be specified as one of these bodies or it may be allowed to change as the spacecraft passes through the sphere of influence of a planet (relative to the sun) or the moon (relative to the earth). In this case, the center of integration will be the body within whose sphere of influence the spacecraft lies. At a change in the center of integration, the position and velocity of the spacecraft relative to the old center of integration are incremented by the position and velocity, respectively, of the old center relative to the new center. The injection position and velocity components may be referred to any body (not necessarily the center of integration; see Section VII-D). The injection epoch may be specified in the UT1, A.1, or E.T. time scales and must be transformed to ephemeris seconds past Jan. 1, 1950, 0^h.

The acceleration of the spacecraft consists of:

(1) Newtonian point-mass acceleration relative to the center of integration.

(2) Direct oblate acceleration caused by a nearby planet or proximity of the earth and the moon.

(3) Acceleration caused by solar radiation pressure and low-thrust acceleration forces, such as operation of the attitude-control system and gas leaks.

(4) Acceleration caused by motor burns. (A motor burn of short duration or a spring separation may be represented alternatively as a discontinuity of the spacecraft trajectory.)

(5) Acceleration caused by indirect oblateness.

(6) Acceleration caused by general relativity.

Thus,

$$\ddot{\mathbf{r}} = -\mu_c \frac{\mathbf{r}}{r^3} + \sum_i \mu_i \left[\frac{\mathbf{r}_{ic}}{r_{ic}^3} - \frac{\mathbf{r}_{ip}}{r_{ip}^3} \right] + \ddot{\mathbf{r}}(OBL) + \ddot{\mathbf{r}}(SRP,AC)$$
$$+ \ddot{\mathbf{r}}(MB) + \ddot{\mathbf{r}}(IOBL) + \ddot{\mathbf{r}}(GR) \tag{191}$$

where

$\ddot{\mathbf{r}}$ = acceleration of spacecraft

μ_c = gravitational constant of the center of integration, km³/sec

μ_i = gravitational constant of body i, km³/s²

$\mathbf{r}$ = position of spacecraft relative to center of integration in 1950.0 earth equatorial rectangular coordinates

$\mathbf{r}_{ic}$ = position of body i relative to center of integration in 1950.0 rectangular coordinates

$\mathbf{r}_{ip}$ = position of spacecraft relative to body i in 1950.0 rectangular coordinates.

It may be assumed that the precomputed position and velocity ephemerides for the celestial bodies within the solar system are available. These consist of the heliocentric ephemerides of eight planets and the earth–moon barycenter and the geocentric lunar ephemeris. The heliocentric ephemerides (with the exception of those for Mercury and Neptune) are obtained from a separate numerical integration for each body, with epoch values chosen to obtain a least-squares fit to source positions. The source positions, which represent astronomical observations, are obtained from an evaluation of certain general perturbation theories for the four inner planets; from a simultaneous integration of the equations of motion of the five outer planets (corrected for the motion of the inner planets); and from an evaluation of the Brown improved lunar theory (Ref. 8, pp. 374–375). The acceleration caused by each perturbing body is computed from either the source position or the position from the fitted ephemeris for the perturbing body (if previously computed). The ephemerides for Mercury, Neptune, and the moon are obtained directly from the source positions, with velocity obtained by numerical differentiation (see Ref. 7, p. 24).

A. Newtonian Point-Mass Acceleration

1. Center of mass and invariable plane. Newton's law of universal gravitation states that

$$F_{12} = \frac{k^2 m_1 m_2}{r_{12}^2} \tag{192}$$

that is, two bodies attract each other with a force F_{12} directly proportional to their masses m_1, m_2 and inversely proportional to the square of the distance between them r_{12}, where one defines

$$r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$$

(hence, $r_{12} = r_{21}$). The constant of gravitation is defined as k^2. This constant is defined as equal to Gm, which is the product of the universal gravitational constant and the first mass. This is done because k can be determined to much greater accuracy than can G. It should be noted that m_2 is actually (despite its denotation) the mass ratio of m_2/m_1; that is, m_2 is normalized with respect to m_1 (see Ref. 4, p. 33).

From Fig. 29 it is easy to see that the force in the x direction between bodies one and two is given by

$$F_{12_x} = F_{12} \cos \psi = F_{12} \frac{x_2 - x_1}{r_{12}} \tag{193}$$

where

$$r_{12} = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2}$$

Using Eq. (192), one can write

$$F_{12_x} = \frac{k^2 m_1 m_2}{r_{12}^2} \frac{x_2 - x_1}{r_{12}}$$

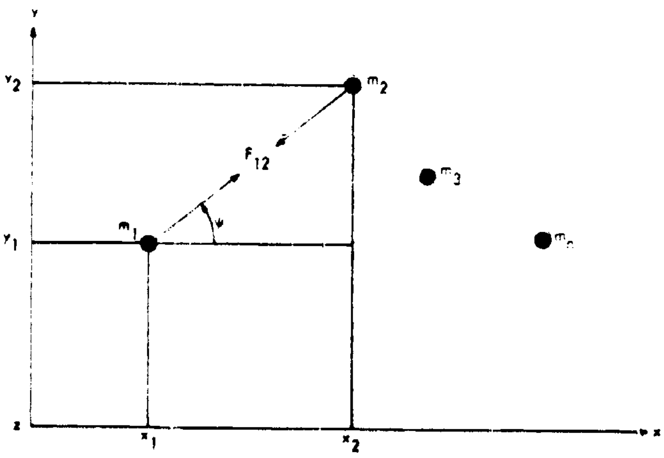

Fig. 29. A system of bodies in an inertial coordinate frame

or

$$F_{12_x} = \frac{k^2 m_1 m_2}{r_{12}^3}(x_2 - x_1) \tag{194}$$

In exactly the same manner, the component of attraction between body one and body n in the x-direction is derived as

$$F_{1n_x} = \frac{k^2 m_1 m_n}{r_{1n}^3}(x_n - x_1)$$

so that the total force on body one in the x-direction caused by n bodies is

$$F_{1x} = F_{12_x} + \cdots + F_{1n_x}$$

or

$$F_{1x} = k^2 \sum_{j=2}^{n} m_1 m_j \frac{x_j - x_1}{r_{1j}^3}$$

It follows that the force in the x-direction upon an arbitrary body i with mass m_i is

$$F_{ix} = k^2 \sum_{\substack{j=1 \\ j \neq i}}^{n} m_i m_j \frac{(x_j - x_i)}{r_{ij}^3}$$

By Newton's second law,

$$F_{ix} = m_i \frac{d^2 x_i}{dt^2}$$

So

$$m_i \frac{d^2 x_i}{dt^2} = k^2 m_i \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \frac{(x_j - x_i)}{r_{ij}^3}$$

Similarly,

$$m_i \frac{d^2 y_i}{dt^2} = k^2 m_i \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \frac{(y_j - y_i)}{r_{ij}^3}$$

and

$$m_i \frac{d^2 z_i}{dt^2} = k^2 m_i \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \frac{(z_j - z_i)}{r_{ij}^3}$$

Hence, in vector notation,

$$m_i \frac{d^2 \mathbf{r}_i}{dt^2} = k^2 m_i \sum_{j=1}^{n} m_j \frac{(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \tag{195}$$

Here the summation excludes $j = i$, and this case will automatically be excluded from future summations where it would result in the vanishing of a denominator. For a complete solution of this so-called n-body problem, $6n$ constants of integration are needed; actually, only 10 are known.

When all of the equations of the form of Eq. (195) are added, the terms on the right side cancel, yielding

$$\sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i = 0$$

This equation may be integrated at once to give

$$\sum_{i=1}^{n} m_i \mathbf{r}_i = \mathbf{a}t + \mathbf{b} \tag{196}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. This means that the center of mass (c.m.) of the system moves, with respect to the (inertial) system of reference, in a straight line with constant speed. The origin can, therefore, be set at the c.m.; then

$$\sum_{i=1}^{n} m_i \mathbf{r}_i = 0$$

and Eq. (195) remains valid.

Multiplication of Eq. (195) vectorially by $\mathbf{r}_i \times$, and addition of the resulting n equations, yields (since all terms on the right side cancel)

$$\sum_{i=1}^{n} m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = \mathbf{0}$$

or, by integration,

$$\sum_{i=1}^{n} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \mathbf{h} \tag{197}$$

where $\mathbf{h}$ is a constant vector.

The plane through the c.m., perpendicular to $\mathbf{h}$, is called the *invariable plane* of the system. Care must be used when one applies this plane rigorously to a physical system. The angular-momentum integral (Eq. 197) is the result of the absence of external forces acting on the system, and it expresses the fact that the *total* angular momentum of the system is constant; this is made up of the angular momentum arising from orbital revolutions and from axial rotations. If all of the bodies are unconnected, rigid, spherical bodies, whose concentric layers are homogeneous, then the axial rotations will remain constant, as will the orbital angular momentum. In this case, the system will have an invariable plane perpendicular to the orbital angular-momentum vector. If these conditions do not hold, however, precessional movements and the effects of tidal friction will result in an interchange between the orbital and rotational parts of the total angular momentum, and the invariable plane defined by $\mathbf{h}$ in Eq. (197) will not be constant. Because the conditions very nearly hold for a planetary system, it is justified in practice to speak of the invariable plane of the solar system; its elements are, approximately, $\Omega = 107$ deg and $i = 1$ deg, 35 min (Ref. 9, p. 206).

2. ***Force function.*** If the force function of the system U (see Ref. 9, p. 206) is defined by

$$U = k^2 \sum_{j=1}^{n} \sum_{i=1}^{j-1} \frac{m_i m_j}{r_{ij}} \tag{198}$$

then

$$\frac{\partial U}{\partial x_i} = k^2 m_i \frac{\partial}{\partial x_i} \left[\sum_{j=1}^{n} \frac{m_j}{r_{ij}} \right]$$

$$= -k^2 m_i \sum_{j=1}^{n} m_j \frac{(x_i - x_j)}{r_{ij}^3}$$

Therefore, Eq. (195) can be rewritten as

$$m_i \ddot{\mathbf{r}}_i = \nabla_i U \tag{199}$$

where

$$\nabla_i = \hat{\mathbf{i}} \frac{\partial}{\partial x_i} + \hat{\mathbf{j}} \frac{\partial}{\partial y_i} + \hat{\mathbf{k}} \frac{\partial}{\partial z_i}$$

3. ***Transfer of origin and perturbing forces.*** Let it be supposed, as is the case with the sun in the solar system, that one mass (say, m_n) is dominant. If the origin is transferred to m_n, and the position vector of m_i with respect to m_n is $\mathbf{r}'_i$, then

$$\mathbf{r}_i = \mathbf{r}'_i + \mathbf{r}_n$$

The r_{ij} elements are not affected, and $\partial/\partial x'_i = \partial/\partial x_i$, etc. Now,

$$U = k^2 m_n \sum_{j=1}^{n-1} \frac{m_j}{r_{nj}} + k^2 \sum_{i<j} \sum_{=1}^{n-1} \frac{m_i m_j}{r_{ij}}$$

$$= k^2 m_n \sum_{j=1}^{n-1} \frac{m_j}{r_{nj}} + U'$$

So

$$\nabla_i U = \nabla_i U' - k^2 m_n m_i \frac{\mathbf{r}'_i}{r_{in}^3}, \qquad i \neq n$$

Also, from the equation of motion of m_n,

$$\ddot{\mathbf{r}}_n = k^2 \sum_{j=1}^{n-1} m_j \frac{\mathbf{r}_n - \mathbf{r}_j}{r_{nj}^3}$$

$$= k^2 \sum_{j=1}^{n-1} m_j \frac{\mathbf{r}'_j}{r_{nj}^3}$$

The equation of motion of m_i is

$$\ddot{\mathbf{r}}'_i + \ddot{\mathbf{r}}_n = \ddot{\mathbf{r}}_i = \frac{1}{m_i} \nabla_i U$$

or

$$\ddot{\mathbf{r}}'_i + k^2 \sum_{j=1}^{n-1} m_j \frac{\mathbf{r}'_j}{r_{nj}^3} = \frac{1}{m_i} \left(\nabla_i U' - k^2 m_n m_i \frac{\mathbf{r}'_i}{r_{in}^3} \right)$$

Removal of the ith term in the summation on the left side yields

$$\ddot{\mathbf{r}}'_i + k^2 m_i \frac{\mathbf{r}'_i}{r_{in}^3} + k^2 \sum_{\substack{j=1 \\ j \neq i}}^{n-1} m_j \frac{\mathbf{r}'_j}{r_{nj}^3} = \frac{1}{m_i} \left(\nabla_i U' - k^2 m_n m_i \frac{\mathbf{r}'_i}{r_{in}^3} \right)$$

or, if the primes are dropped (since the transfer to the new origin is complete, and this can be done without ambiguity),

$$\ddot{\mathbf{r}}_i + k^2 (m_n + m_i) \frac{\mathbf{r}_i}{r_{in}^3} = \frac{1}{m_i} \nabla_i U' - k^2 \sum_{\substack{j=1 \\ j \neq i}}^{n-1} m_j \frac{\mathbf{r}_j}{r_{jn}^3} \tag{200}$$

Now if

$$R_{ij} = k^2\left(\frac{1}{r_{ij}} - \frac{\mathbf{r}_i \cdot \mathbf{r}_j}{r_{jn}^3}\right)$$

then

$$m_j \nabla_i R_{ij} = k^2 \nabla_i\left(\frac{m_j}{r_{ij}}\right) - k^2 m_j \frac{\mathbf{r}_j}{r_{jn}^3}$$

and

$$\sum_{j=1}^{n-1} m_j \nabla_i R_{ij} = \frac{k^2}{m_i}\sum_{j=1}^{n-1} \nabla_i\left(\frac{m_i m_j}{r_{ij}}\right) - k^2 \sum_{j=1}^{n-1} m_j \frac{\mathbf{r}_j}{r_{jn}^3}$$
$$= \frac{1}{m_i}\nabla_i U' - k^2 \sum_{j=1}^{n-1} m_j \frac{\mathbf{r}_j}{r_{jn}^3}$$

Combination of this result with Eq. (200) yields

$$\ddot{\mathbf{r}}_i + k^2(m_n + m_i)\frac{\mathbf{r}_i}{r_{in}^3} = \sum_{j=1}^{n-1} m_j \nabla_i R_{ij} \qquad (201)$$

The equations represented by Eq. (201) are fundamental. If the R_{ij} terms are zero, that leaves the equation of motion of two bodies; therefore, it is the R_{ij} terms that cause the departures (or perturbations) from Keplerian motion; they are called perturbative functions. Equation (201) may be rewritten

$$\ddot{\mathbf{r}}_i + k^2(m_n + m_i)\frac{\mathbf{r}_i}{r_{in}^3} = k^2 \sum_{\substack{j=1 \\ j\neq i}}^{n-1} m_j \left(\frac{\mathbf{r}_j - \mathbf{r}_i}{r_{ij}^3} - \frac{\mathbf{r}_j}{r_{jn}^3}\right) \qquad (202)$$

or

$$\ddot{\mathbf{r}}_i + k^2(m_n + m_i)\frac{\mathbf{r}_i}{r_{in}^3} = \sum_{\substack{j=1 \\ j\neq i}}^{n-1} \mu_j \left(\frac{\mathbf{r}_j - \mathbf{r}_i}{r_{ij}^3} - \frac{\mathbf{r}_j}{r_{jn}^3}\right) \qquad (203)$$

where $\mu_j = k^2 m_j$.

The first terms on the right side of Eq. (202) are the direct attractions on m_i caused by the perturbing bodies; the second terms are the indirect terms. If the ith body is identified with the spacecraft, and it is noted that $m_n + m_i \approx m_n$ [hence, $k^2(m_n + m_i) \approx k^2 m_n = \mu_n$], it can then easily be seen that Eq. (203) is equivalent to the first term of the right side of Eq. (191).

B. Acceleration Caused by an Oblate Body

This section develops expressions for the acceleration of a spacecraft produced by a nonspherical central body. The equatorial bulge of a planet or the moon is responsible for a deformation in the gravitational field of that planet or moon from that which would be produced by a point mass or spherical symmetrical body. These deformations, which are especially important near the surface of the body, produce conspicuous perturbations in the orbits of low-altitude spacecraft. Thus, an aspherical or nonsymmetric body produces a noncentral force field. If it is desired to write the equations of motion of a spacecraft in a noncentral force field, an aspherical potential must be determined. The generalized potential function U[20] for a planet or the moon, which allows the derivation of the direct acceleration of a spacecraft by the oblateness of a nonspherical body, is given by

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{r}\right)^n P_n^m(\sin\phi)\left(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda\right)\right] \qquad (204)$$

where

μ = gravitational constant of body, km³/s²

r, ϕ, λ = body-centered (planet or moon) radius, latitude, and longitude (positive east of prime meridian) of spacecraft

a_e = mean equatorial radius of body (an adopted constant used for U)

$P_n^m(\sin\phi)$ = associated Legendre function of first kind (the argument $\sin\phi$ will be omitted here)

$C_{n,m}, S_{n,m}$ = numerical coefficients (tesseral harmonic and sectorial harmonic coefficients)

The associated Legendre function P_n^m is defined as

$$P_n^m = \cos^m\phi \frac{d^m}{d(\sin\phi)^m} P_n$$

where P_n is the Legendre polynomial of degree n in $\sin\phi$.

The zonal harmonic coefficient J_n is defined as

$$J_n = -C_{n,0}$$

[20]Adopted by the International Astronomical Union in 1961 (Ref. 16, p. 2).

Equation (204) may be written in three terms—corresponding to the potential of a point mass, zonal harmonics, tesseral and sectorial harmonics (see Ref. 7, p. 27)—in the form

$$U = \frac{\mu}{r} - \frac{\mu}{r}\sum_{n=2}^{\infty} J_n \left(\frac{a_p}{r}\right)^n P_n + \frac{\mu}{r}\sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{a_p}{r}\right)^n P_n^m \left(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda\right) \tag{205}$$

with each term defined as

$$U = \frac{\mu}{r} + U(J) + U(C,S)$$

Because the coefficients are obtained from satellite orbit observations, the center of coordinates is taken as the dynamical c.m. of the particular body; in this case, the first-degree ($n = 1$) harmonics are zero. Therefore, the summation over n in Eq. (205) begins with 2. At the present time, harmonics are known only for the earth and the moon, these have been determined up to $n = 8$, $m = 8$. It should be noted, however, that the values of the higher-degree coefficients are very uncertain, even as to the sign of the value (see Ref. 10, p. 2). The order of magnitude of the tesseral harmonics for the earth is approximately 10^{-6}.

The inertial acceleration of the spacecraft is computed in a rectangular coordinate system (x',y',z'), with the x'-axis directed outward along the instantaneous radius to the spacecraft, the y'-axis directed east, and the z'-axis directed north, as shown in Fig. 30.

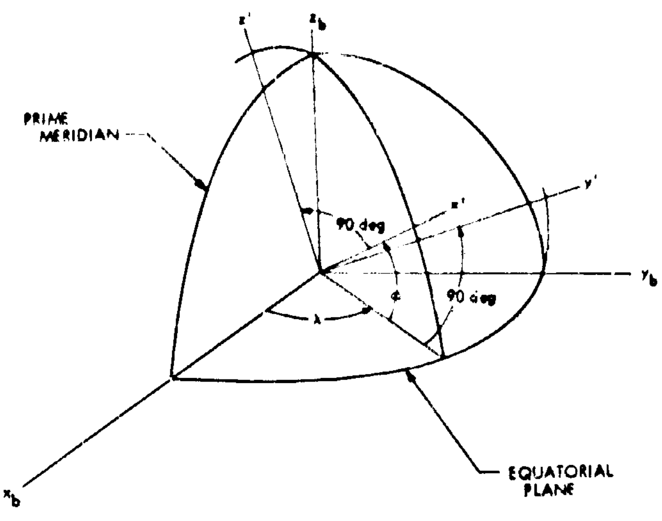

Fig. 30. Rectangular coördinate system axes x', y', and z' relative to body-fixed axes x_b, y_b, z_b

Figure 30 shows these axes relative to body-fixed axes x_b, y_b, z_b, where x_b is along the intersection of the prime meridian and equator of the body, z_b is directed north along the axis of rotation of the body, and y_b completes the right-handed system. The transformation from body-fixed coordinates $\mathbf{r}_b = (x_b, y_b, z_b)$ to $\mathbf{r}' = (x', y', z')$ coordinates is given by

$$\mathbf{r}' = \mathbf{R}\,\mathbf{r}_b$$

where

$$\mathbf{R} = \begin{bmatrix} \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & \sin\phi\sin\lambda & \cos\phi \end{bmatrix}$$

The position of the spacecraft relative to the body in rectangular coordinates, referred to the mean earth equator and equinox of 1950.0, is given by $\mathbf{r} - \mathbf{r}_i$, where $\mathbf{r}$ is the position of the spacecraft relative to the center of integration in 1950.0 earth equatorial rectangular coordinates and $\mathbf{r}_i$ is the position of body i relative to the center of integration in 1950.0 rectangular coordinates.

The transformation of these inertial coordinates to body-fixed coordinates $\mathbf{r}_b$ is defined as

$$\mathbf{r}_b = \mathbf{T}^T(\mathbf{r} - \mathbf{r}_i) \tag{206}$$

where $\mathbf{T}$ is the transformation matrix that rotates from body-fixed coordinates to the mean earth equator and equinox of 1950.0 ($\mathbf{T}$ is actually a product of rotation matrices wherein each factor has been specified in Section V). For instance, rotation from Mars-fixed coordinates to the mean earth equator and equinox of 1950.0 requires the following product to be formed:

$$\mathbf{T} = \mathbf{A}^T\mathbf{N}^T\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1\tilde{\mathbf{R}}_1$$

where each matrix appearing in the product is described in Sections V-B, -E, and -G.

The overall transformation from $\mathbf{r} - \mathbf{r}_i$ to $\mathbf{r}'$ is thus

$$\mathbf{r}' = \mathbf{G}(\mathbf{r} - \mathbf{r}_i) = \mathbf{R}\mathbf{T}^T(\mathbf{r} - \mathbf{r}_i)$$

and the inverse transformation is

$$\mathbf{r} - \mathbf{r}_i = \mathbf{T}\mathbf{R}^T\mathbf{r}' = \mathbf{G}^T\mathbf{r}'$$

If the body-fixed coordinates from Eq. (206) are used, trigonometric functions of ϕ and λ are given by

$$\sin\phi = \frac{z_b}{r}$$

$$\cos\phi = \frac{(x_b^2 + y_b^2)^{1/2}}{r}$$

$$\sin\lambda = \frac{y_b}{(x_b^2 + y_b^2)^{1/2}}$$

$$\cos\lambda = \frac{x_b}{(x_b^2 + y_b^2)^{1/2}}$$

An expression will be developed for the inertial acceleration (i.e., acceleration with respect to the mean earth equator and equinox of 1950.0 coordinate system) of a spacecraft caused by oblateness of any body, with rectangular components along the instantaneous directions of the x', y', z' axes, denoted by $\ddot{\mathbf{r}}'$. This acceleration can be broken down into $\ddot{\mathbf{r}}'(J)$ caused by zonal harmonics and $\ddot{\mathbf{r}}'(C,S)$ caused by tesseral harmonics. With these terms, the contribution to the spacecraft acceleration $\ddot{\mathbf{r}}$ relative to the center of integration in earth equatorial rectangular coordinates caused by the oblateness of any body is

$$\begin{aligned}\ddot{\mathbf{r}}(OBL) &= G^T \ddot{\mathbf{r}}' \\ &= G^T\{\ddot{\mathbf{r}}'(J) + \ddot{\mathbf{r}}'(C,S)\}\end{aligned}$$

It should be noted that $\ddot{\mathbf{r}}'$ does not represent the components of the acceleration relative to the rotating (x',y',z') coordinate system, but simply the components of the inertial acceleration $\ddot{\mathbf{r}}$ taken along the instantaneous x',y',z' axes.

The components of $\ddot{\mathbf{r}}'$ are computed from

$$\ddot{x}' = \frac{\partial U}{\partial r}$$

$$\ddot{y}' = \frac{1}{r\cos\phi}\frac{\partial U}{\partial\lambda}$$

$$\ddot{z}' = \frac{1}{r}\frac{\partial U}{\partial\phi}$$

where the point-mass term of U, which has been accounted for in Section VII-A, is here ignored.

The carrying out of these derivatives yields

$$\ddot{\mathbf{r}}'(J) = \frac{\mu}{r^2}\sum_{n=2}^{n_1} J_n \left(\frac{a_p}{r}\right)^n \begin{bmatrix} (n+1)P_n \\ 0 \\ -\cos\phi\, P_n' \end{bmatrix} \tag{207}$$

$$\ddot{\mathbf{r}}'(C,S) = \frac{\mu}{r^2}\sum_{n=1}^{n_2}\sum_{m=1}^{n}\left(\frac{a_p}{r}\right)^n \begin{bmatrix} -(n+1)P_n^m\{C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda\} \\ m\sec\phi\, P_n^m\{-C_{n,m}\sin m\lambda + S_{n,m}\cos m\lambda\} \\ \cos\phi\, P_n^{m\prime}\{C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda\}\end{bmatrix} \tag{208}$$

where, for practical computations, n_1 and n_2 must be predetermined.

Legendre polynomials, which appear in Eqs. (207) and (208), are "nice" functions in the sense that they can be defined recursively. Thus, the nth Legendre polynomial P_n is computed from

$$P_n = \frac{2n-1}{n}\sin\phi\, P_{n-1} - \left(\frac{n-1}{n}\right)P_{n-2}$$

starting with $P_0 = 1$ and $P_1 = \sin\phi$. The first derivative of P_n with respect to $\sin\phi$, denoted P_n', is given by

$$P_n' = \sin\phi\, P_{n-1}' + n\,P_{n-1}$$

starting with $P_1' = 1$.

The function $\sec\phi\, P_n^m$ is computed by first generating

$$\sec\phi\, P_m^m = (2m-1)\cos\phi\,[\sec\phi\, P_{m-1}^{m-1}]$$

starting with $\sec\phi\, P_1^1 = 1$, and then generating

$$\sec\phi\, P_n^m = \left(\frac{2n-1}{n-m}\right)\sin\phi\,[\sec\phi\, P_{n-1}^m] - \left(\frac{n+m-1}{n-m}\right)[\sec\phi\, P_{n-2}^m]$$

For each value of m between 1 and n_a, n is varied from $m+1$ to n_a. The general term P_a^b is zero if $b > a$.

The function $\cos\phi\, P_n^{m\prime}$, where $P_n^{m\prime}$ is the derivative of P_n^m with respect to $\sin\phi$, is computed from

$$\cos\phi\, P_n^{m\prime} = -n\sin\phi\,[\sec\phi\, P_n^m] + (n+m)\,[\sec\phi\, P_{n-1}^m]$$

C. Acceleration Caused by Solar Radiation Pressure and Operation of Attitude-Control System

A nongravitational force that acts upon a spacecraft is the pressure of the radiation from the sun. If J is the intensity of the solar radiation—i.e. the energy of the radiation (in ergs) falling per second on an area of 1 cm² perpendicular to the direction of the radiation—and c is the velocity of light (in cm/s), the pressure P (in dyn/cm²) exerted on a completely absorbing surface is given by

$$P = \frac{J}{c} \tag{209}$$

Equation (209) is derived from Einstein's relationship concerning the equivalence of mass and energy: a mass M is equivalent to energy Mc^2. If M is the mass of radiation photons falling on a unit area in a unit time, then the energy is equivalent to J; thus, $J = Mc^2$. Because photons travel at the speed of light, their momentum is equal to Mc, and hence to J/c. If the photons of total mass M are absorbed by unit area in unit time, the rate of change of momentum—i.e., the force per unit area, which is equal to the pressure P—is thus J/c. For a perfect reflector of the radiation, the rate of change of momentum is $2J/c$ because the photons strike the surface with momentum J/c and are reflected with equal momentum in the opposite direction; thus, $P = 2J/c$ if the surface is a perfect reflector.

At 1 AU, the order of magnitude of the pressure is approximately 10^{-7} g/cm² for a highly reflective surface exposed to the sun (Ref. 11, p. 77). The solar pressure acceleration for *Mariner IV* was 2.2×10^{-5} cm/s² at injection (Ref. 12, p. 8).

The acceleration of a spacecraft from solar radiation pressure and small forces (such as gas leaks from the attitude control system, noncoupled attitude-control jets, etc.) is represented by[21]

$$\begin{aligned}\ddot{\mathbf{r}}(SRP,AC) &= \ddot{\mathbf{r}}(SRP) + \ddot{\mathbf{r}}(AC)\\ &= [\ddot{\mathbf{r}}(SRP)\cdot\hat{\mathbf{U}}_{RP} + \ddot{\mathbf{r}}(AC)\cdot\hat{\mathbf{U}}_{RP}]\,\hat{\mathbf{U}}_{RP} + [\ddot{\mathbf{r}}(SRP)\cdot\hat{\mathbf{X}}^* + \ddot{\mathbf{r}}(AC)\cdot\hat{\mathbf{X}}^*]\,\hat{\mathbf{X}}^*\\ &\quad + [\ddot{\mathbf{r}}(SRP)\cdot\hat{\mathbf{Y}}^* + \ddot{\mathbf{r}}(AC)\cdot\hat{\mathbf{Y}}^*]\,\hat{\mathbf{Y}}^*\end{aligned} \tag{210}$$

or

$$\ddot{\mathbf{r}}(SRP,AC) = (\ddot{\mathbf{r}}_b\cdot\hat{\mathbf{U}}_{RP})\,\hat{\mathbf{U}}_{RP} + (\ddot{\mathbf{r}}_b\cdot\hat{\mathbf{X}}^*)\,\hat{\mathbf{X}}^* + (\ddot{\mathbf{r}}_b\cdot\hat{\mathbf{Y}}^*)\,\hat{\mathbf{Y}}^* \tag{211}$$

where

$$\ddot{\mathbf{r}}_b = [\mathbf{a} + \mathbf{b}\,(t - T_{AC1}) + \mathbf{c}\,(t - T_{AC1})^2]\,[u\,(t - T_{AC1}) - u\,(t - T_{AC2})] + \Delta\mathbf{a} + \frac{c_1 A_p}{m r_{RP}^2}[\mathbf{G} + \mathbf{G}'(\measuredangle EPS) + \Delta\mathbf{G}]\,u^*\,(t - T_{SRP}) \tag{212}$$

[21]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

or, for the components,

$$\ddot{r}_{b_r} = [a_r + b_r(t - T_{AC1}) + c_r(t - T_{AC1})^2]\,[u(t - T_{AC1}) - u(t - T_{AC2})]$$
$$+ \Delta a_r + \frac{c_1 A_p}{m r_{SP}^2}[G_r + G'_r(\measuredangle EPS) + \Delta G_r]\,u^*(t - T_{SRP})$$

$$\ddot{r}_{b_x} = [a_x + b_x(t - T_{AC1}) + c_x(t - T_{AC1})^2]\,[u(t - T_{AC1}) - u(t - T_{AC2})]$$
$$+ \Delta a_x + \frac{c_1 A_p}{m r_{SP}^2}[G_x + G'_x(\measuredangle EPS) + \Delta G_x]\,u^*(t - T_{SRP})$$

$$\ddot{r}_{b_y} = [a_y + b_y(t - T_{AC1}) + c_y(t - T_{AC1})^2]\,[u(t - T_{AC1}) - u(t - T_{AC2})]$$
$$+ \Delta a_y + \frac{c_1 A_p}{m r_{SP}^2}[G_y + G'_y(\measuredangle EPS) + \Delta G_y]\,u^*(t - T_{SRP})$$

The terms in these equations are defined as follows:

$\hat{\mathbf{U}}_{SP}$ = a unit vector directed from sun to spacecraft (spacecraft roll axis)

$\hat{\mathbf{X}}^*$ = a unit vector in spacecraft $+X$-axis direction (spacecraft pitch axis)

$\hat{\mathbf{Y}}^*$ = a unit vector in spacecraft $+Y$-axis direction (spacecraft yaw axis)

$\hat{\mathbf{U}}_{SP}, \hat{\mathbf{X}}^*, \hat{\mathbf{Y}}^*$ forms a right-handed, orthogonal, spacecraft-fixed coordinate system; thus, $\hat{\mathbf{U}}_{SP} = \hat{\mathbf{X}}^* \times \hat{\mathbf{Y}}^*$

a_i, b_i, c_i (where $i = r,x,y$) = solve-for coefficients of low-thrust acceleration polynomials, km/s², km/s³, km/s⁴

t = ephemeris time

T_{AC1}, T_{AC2} = epochs at which attitude-control acceleration polynomials are turned on and off, respectively; epochs may be specified in UTC, ST, or A.1 time scales (not E.T.)

$$u(t - T_{AC1}) = \begin{cases} 1 \text{ for } t \geq T_{AC1} \\ 0 \text{ for } t < T_{AC1} \end{cases} \qquad T_{AC1} \to T_{AC2}$$

$\Delta\mathbf{a} = (\Delta a_r, \Delta a_x, \Delta a_y)$ = input (*a priori*) acceleration, km/s² (value of each Δa_i, $i = r,x,y$, will be obtained by linear interpolation between input points on any time scale)

$$c_1 = \frac{J A_E^2}{c}\,\frac{1\ \text{km}^2}{10^6\ \text{m}^2} = 1.031 \times 10^{14}\ \frac{\text{km}^3\,\text{kg}}{\text{s}^2\,\text{m}^2}$$

where

J = solar radiation constant

$= 1.383 \times 10^{9}$ W/km²

$= 1.383 \times 10^{9}$ kg/s³

A_E = 1.496×10^{8} km (mean distance earth–sun = 1 AU)

c = 2.997925×10^{5} km/s (speed of light)

A_p = nominal area of spacecraft projected onto plane normal to sun–spacecraft line, m²

m = instantaneous mass of spacecraft

r_{SP} = distance from sun to spacecraft

T_{SRP} = epoch at which acceleration from solar radiation pressure is turned on (becomes effective); epoch may be specified in UTC, ST, or A.1 time scales (see Glossary for time scales UTC and ST)

$$u^*(t - T_{SRP}) = \begin{cases} 1 \text{ for } t \geq T_{SRP} \text{ and if spacecraft is in sunlight} \\ 0 \text{ for } t < T_{SRP} \text{ or if spacecraft is in shadow} \end{cases}$$

G_r = solve-for effective area of acceleration of spacecraft in radial direction from solar radiation pressure divided by nominal area A_p

G_x = solve-for effective area of acceleration of spacecraft in direction of its positive x-axis (along $\mathbf{X}^*$ vector) divided by A_p

G_y = solve-for effective area of acceleration of spacecraft in direction of its positive y-axis (along $\mathbf{Y}^*$ vector) divided by A_p

$\measuredangle EPS$ = earth–spacecraft–sun angle, rad

G'_r, G'_x, G'_y = solve-for derivatives of G_r, G_x, and G_y with respect to earth–spacecraft–sun angle

$\Delta G_r, \Delta G_x, \Delta G_y$ = increments to G_r, G_x, and G_y obtained by linear interpolation of input points specified in any time scale

The unit sun–spacecraft vector $\hat{\mathbf{U}}_{SP}$ is computed from

$$\hat{\mathbf{U}}_{SP} = \frac{\mathbf{r} - \mathbf{r}_S^{(C)}}{|\mathbf{r} - \mathbf{r}_S^{(C)}|} \tag{213}$$

where

$\mathbf{r}$ = rectangular coordinates of spacecraft relative to center of integration, referred to mean earth equator and equinox of 1950.0

$\mathbf{r}_S^{(C)}$ = rectangular coordinates of sun relative to center of integration C, referred to mean earth equator and equinox of 1950.0

The spacecraft $\hat{\mathbf{X}}^*$ and $\hat{\mathbf{Y}}^*$ unit vectors are obtained as a rotation of the tangential $\hat{\mathbf{T}}$ and normal $\hat{\mathbf{N}}$ vectors through the angle K (Fig. 31); i.e.,

$$\begin{pmatrix} \hat{\mathbf{X}}^* \\ \hat{\mathbf{Y}}^* \end{pmatrix} = \begin{pmatrix} \cos K & \sin K \\ -\sin K & \cos K \end{pmatrix} \begin{pmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}} \end{pmatrix} \tag{214}$$

The angle K is a given constant; that is, not solved for. Computation of the unit vectors $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ requires the unit vector $\hat{\mathbf{U}}_R$, which is a unit vector from the spacecraft to a reference body that orients the spacecraft about the roll axis (sun–spacecraft line). The reference body may be a star, a planet, or the moon. If the reference body is a star, then

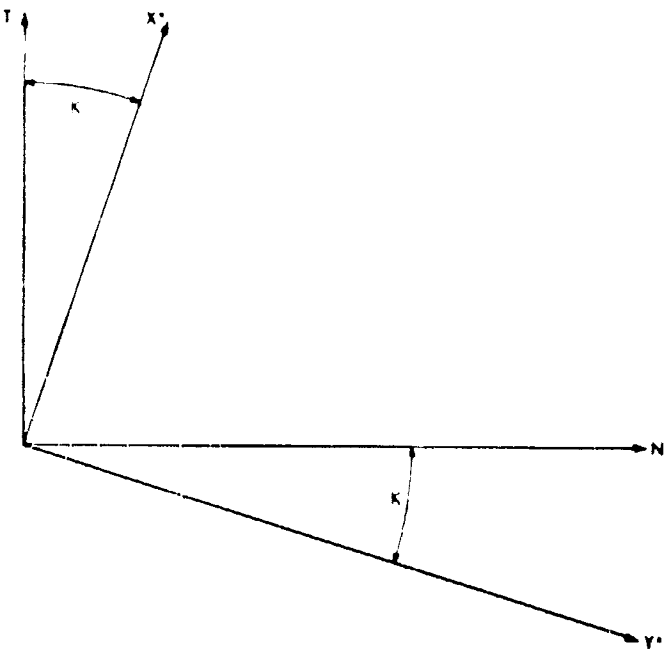

Fig. 31. The angle K

$$\hat{\mathbf{U}}_R = \begin{bmatrix} \cos\delta \ \cos\alpha \\ \cos\delta \ \sin\alpha \\ \sin\delta \end{bmatrix} \tag{215}$$

where the right ascension α and declination δ of the star are referred to the mean earth equator and equinox of 1950.0. If the reference body is a planet or the moon (normally the earth),

$$\hat{\mathbf{U}}_R = \frac{\mathbf{r}_B^{(C)} - \mathbf{r}}{|\mathbf{r}_B^{(C)} - \mathbf{r}|} \tag{216}$$

where $\mathbf{r}_B^{(C)}$ represents the rectangular coordinates of reference body B relative to the center of integration C, referred to the mean earth equator and equinox of 1950.0.

The unit normal vector $\hat{\mathbf{N}}$, normal to the sun–spacecraft–reference-body plane (Fig. 32), is computed from

$$\hat{\mathbf{N}} = \frac{\hat{\mathbf{U}}_R \times \hat{\mathbf{U}}_{SP}}{|\hat{\mathbf{U}}_R \times \hat{\mathbf{U}}_{SP}|} \tag{217}$$

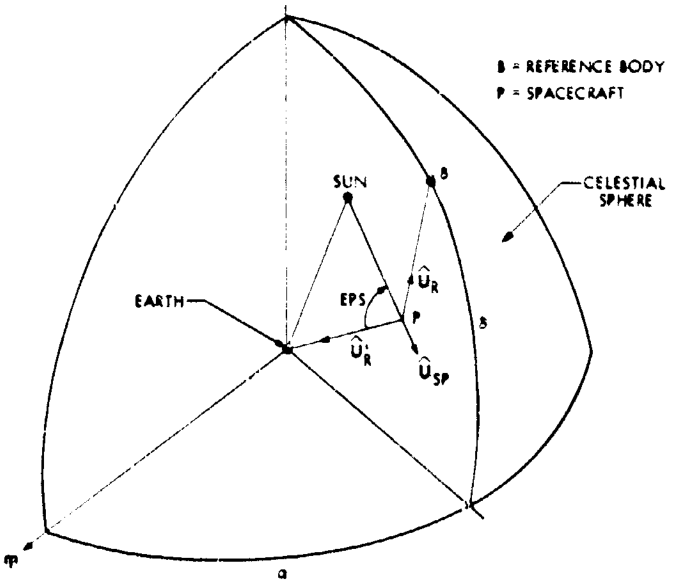

Fig. 32. Orientation of a spacecraft relative to sun, earth, and reference body

The unit tangential vector $\hat{\mathbf{T}}$, tangent to the sun–spacecraft–reference-body plane, is

$$\hat{\mathbf{T}} = \hat{\mathbf{N}} \times \hat{\mathbf{U}}_{sr}$$

The EPS angle may be computed from

$$\cos \measuredangle\, EPS = -\hat{\mathbf{U}}_{sP} \cdot \hat{\mathbf{U}}'_{R}, \qquad 0\ \mathrm{deg} < \measuredangle EPS < 180\ \mathrm{deg}$$

where $\hat{\mathbf{U}}'_R$ is computed from Eq. (216) using B = earth.

The acceleration of a spacecraft due to solar radiation pressure is computed whenever the spacecraft is in the sunlight and its solar panels have unfolded. The spacecraft is considered to be in the sunlight whenever the physical central body (PCB) is the sun. When the PCB is the moon or a planet, the spacecraft is considered to be in the sunlight if it is not in the shadow of the PCB, which is defined by the shaded region in Fig. 33.

To determine when the spacecraft enters and leaves the shadow of the PCB, the following quantity is computed:

$$D = \left| \frac{\mathbf{r}_{S\text{-}PCB}}{|\mathbf{r}_{S\text{-}PCB}|} \times \mathbf{r}_{PCB\text{-}P} \right| - r' \qquad (218)$$

where

$$r' = |\mathbf{r}'| = \text{radius of PCB}$$

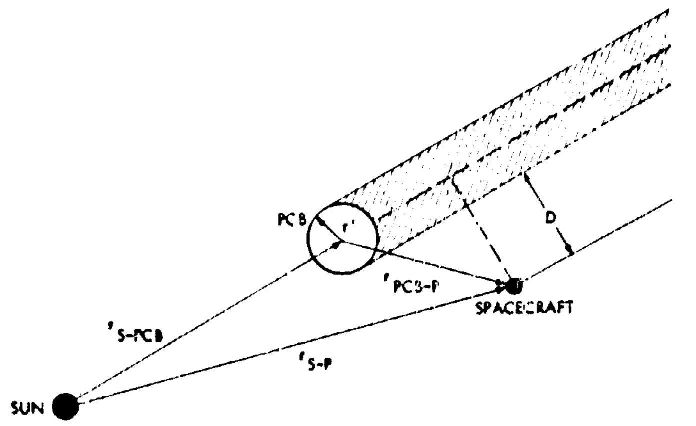

Fig. 33. The parameter D

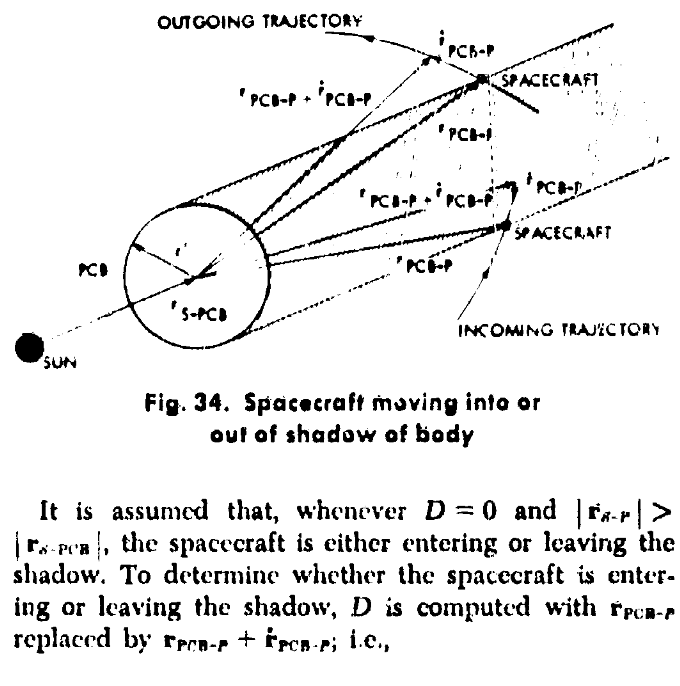

Fig. 34. Spacecraft moving into or out of shadow of body

It is assumed that, whenever $D = 0$ and $|\mathbf{r}_{S\text{-}P}| > |\mathbf{r}_{S\text{-}PCB}|$, the spacecraft is either entering or leaving the shadow. To determine whether the spacecraft is entering or leaving the shadow, D is computed with $\mathbf{r}_{PCB\text{-}P}$ replaced by $\mathbf{r}_{PCB\text{-}P} + \dot{\mathbf{r}}_{PCB\text{-}P}$; i.e.,

$$D' = \left| \frac{\mathbf{r}_{S\text{-}PCB}}{|\mathbf{r}_{S\text{-}PCB}|} \times (\mathbf{r}_{PCB\text{-}P} + \dot{\mathbf{r}}_{PCB\text{-}P}) \right| - r' \qquad (219)$$

When (1) $D' \geq 0$, the spacecraft is leaving the shadow of the PCB (if the direction of integration is forward), as shown by the "outgoing trajectory" in Fig. 34; when (2) $D' < 0$, the spacecraft is entering the shadow of the PCB (if the direction of integration is forward), as shown by the "incoming trajectory" in Fig. 34.

D. Acceleration Caused by Finite Motor Burns

The acceleration of the spacecraft relative to the barycenter of the solar system in 1950.0 rectangular coordinates caused by a finite motor burn, as opposed to an "instantaneous" motor burn (see below), is given in Ref. 7, (p. 30) as

$$\ddot{\mathbf{r}}(MB) = a\,\hat{\mathbf{U}}\,\{\mu(t - T_o) - \mu(t - T_f)\} \qquad (220)$$

where

a = magnitude of $\ddot{\mathbf{r}}(MB)$ vs time

$\hat{\mathbf{U}}$ = unit vector in direction of $\ddot{\mathbf{r}}(MB)$ vs time

T_o = effective start time of motor, E.T. value of the solve-for UTC epoch, T_0 (UTC)

T_f = effective stop time of motor, E.T.

t = ephemeris time

$$\mu(t - T_o) = \begin{cases} 1 \text{ for } t \geq T_o \\ 0 \text{ for } t < T_o \end{cases} \qquad T_o \rightarrow T_f$$

The effective stop time T_f is given by

$$T_f = T_o + T \qquad (221)$$

where T is the only solve-for burn time of the motor in ephemeris time.

The acceleration magnitude a (in km/s²) is given by

$$a = \frac{F(t)}{m(t)}\,C$$

$$= \frac{F_0 + F_1 t + F_2 t^2 + F_3 t^3 + F_4 t^4}{m_0 - \dot{M}_0 t - \frac{1}{2}\dot{M}_1 t^2 - \frac{1}{3}\dot{M}_2 t^3 - \frac{1}{4}\dot{M}_3 t^4}\,C \qquad (222)$$

where[22]

$F(t)$ = magnitude of thrust at time t (polynomial coefficients F_0, F_1, F_2, F_3, and F_4 are solve-for parameters)

$m(t)$ = spacecraft mass at time t

C = 0.001 for F in newtons and mass in kilograms

m_0 = mass of spacecraft at T_0 in ephemeris time

$\dot{M}_0, \dot{M}_1, \dot{M}_2, \dot{M}_3$ = polynomial coefficients for propellant mass flow rate (positive) at time t, $\dot{M}(t) = \dot{M}_0 + \dot{M}_1 t + \dot{M}_2 t^2 + \dot{M}_3 t^3$ (coefficients are not solve-for parameters, but must be supplied)

t = E.T. $- T_0$ (E.T.), s

where E.T. is seconds of ephemeris time from January 1, 0 h E.T., 1950.

The unit vector $\hat{\mathbf{U}}$ of thrust is given by

$$\hat{\mathbf{U}} = \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = \begin{pmatrix} \cos\delta\ \cos\alpha \\ \cos\delta\ \sin\alpha \\ \sin\delta \end{pmatrix} \qquad (223)$$

where

α = right ascension of $\hat{\mathbf{U}}$

δ = declination of $\hat{\mathbf{U}}$

given by

$$\left.\begin{aligned} \alpha &= \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4 \\ \delta &= \delta_0 + \delta_1 t + \delta_2 t^2 + \delta_3 t^3 + \delta_4 t^4 \end{aligned}\right\} \qquad (224)$$

where the polynomial coefficients are solve-for parameters. As an example of how these coefficients may be solved for, a midcourse maneuver will be considered. In this case, an impulsive velocity increment $\Delta\dot{\mathbf{r}}$ is assumed; a midcourse maneuver program computes the roll turns and burn time so that the computed $\Delta\dot{\mathbf{r}}$ is obtained by the spacecraft. Given

$$\Delta\dot{\mathbf{r}} = \begin{pmatrix} \Delta\dot{x} \\ \Delta\dot{y} \\ \Delta\dot{z} \end{pmatrix} \qquad (225)$$

[22]It should be noted that the coefficients

$$\dot{M}_0,\ \frac{1}{2}\dot{M}_1,\ \frac{1}{3}\dot{M}_2,\ \frac{1}{4}\dot{M}_3$$

are actually the coefficients of a Taylor series; that is,

$$\dot{M}_0 = \dot{M}_0,\quad \frac{1}{2}\dot{M}_1 = \frac{1}{2!}\ddot{M}_0,\quad \frac{1}{3}\dot{M}_2 = \frac{1}{3!}\dddot{M}_0,\quad \frac{1}{4}\dot{M}_3 = \frac{1}{4!}\ddddot{M}_0$$

The coefficients of the polynomial (Eq. 222) appear as in the first of these equations because they are supplied in this form by the Propulsion Division at JPL.

from the midcourse maneuver program, *a priori* values of α_o and δ_o for use in Eq. (224) may be computed from

$$\cos \alpha_o = \frac{\Delta\dot{x}}{[(\Delta\dot{x})^2 + (\Delta\dot{y})^2]^{1/2}}$$

$$\sin \alpha_o = \frac{\Delta\dot{y}}{[(\Delta\dot{x})^2 + (\Delta\dot{y})^2]^{1/2}}$$

$$\sin \delta_o = \frac{\Delta\dot{z}}{\Delta\dot{s}}$$

where

$$\Delta\dot{s} = [(\Delta\dot{x})^2 + (\Delta\dot{y})^2 + (\Delta\dot{z})^2]^{1/2}$$

The vector $\Delta\dot{\mathbf{r}}$ obtained from the midcourse program is referred to the true equator and equinox of date, and must be rotated to the mean equator and equinox of 1950.0 before α_o and δ_o are computed:

$$\Delta\dot{\mathbf{r}}_{1950.0} = (\mathbf{NA})^T \Delta\dot{\mathbf{r}}_{true}$$

where **N** and **A** are rotation matrices given in Sections V-E and -B. The remaining coefficients of Eq. (224) will be put at zero and solved for.[23]

It should be noted that the acceleration caused by solar radiation pressure must be computed from m_o before a maneuver, from $m(t)$ during a maneuver, and from $m(T_f)$ after a maneuver. The value of m_o for a given maneuver is $m(T_f)$ from the previous maneuver.

If no data are taken during the maneuver, and the duration of the maneuver is very short, it may be represented by *instantaneous* changes in $\Delta\mathbf{r}$ and $\Delta\dot{\mathbf{r}}$ of the spacecraft. A convenient set of maneuver parameters to represent the change in state $\Delta\mathbf{r}$, $\Delta\dot{\mathbf{r}}$ of the spacecraft is

$$\Delta\dot{\mathbf{r}} = \begin{pmatrix} \Delta\dot{x} \\ \Delta\dot{y} \\ \Delta\dot{z} \end{pmatrix} \tag{226}$$

(see Eq. 225), and t_b (the duration of the maneuver). The change in position $\Delta\mathbf{r}$ in terms of these parameters is

$$\Delta\mathbf{r} = \frac{1}{2}\Delta\dot{\mathbf{r}}t_b \tag{227}$$

[23]Moyer, T. D., JPL internal document, Sept. 11, 1964.

where $t_b = 0$ for a spring-separation maneuver.[24]

At an instantaneous maneuver, the program has the capability of reducing the area of the spacecraft by a specified amount ΔA. This area change would simulate, for example, the expulsion of protective shrouds during a spring separation. This reduced area in turn will affect the acceleration due to solar radiation pressure.

E. Acceleration Caused by Indirect Oblateness

The indirect acceleration of the center of integration caused by the oblateness of a perturbing body is generally ignored because the planets are separated by such large distances that the nonspherical effect is negligible. However, for the case where the earth is the center of integration and the moon is the disturbing body (or *vice versa*), an expression has been derived for the indirect acceleration; by using first-order oblateness terms, this expression accounts approximately for the oblateness of each body (see Ref. 8).[25]

1. Basic equations. In this subsection, an inertial Cartesian coordinate system $R(X,Y,Z)$, in which the axes are parallel to the 1950.0 mean earth equator and equinox coordinate system, shall be defined. If one lets X_o, Y_o, and Z_o be the coordinates of a spacecraft of mass M_o, and lets X_i, Y_i, and Z_i ($i = 1, \cdots, n$) be the coordinates of n bodies of mass M_i, the indices 1 and 2 will refer to the earth and moon, respectively. The force potential between any two bodies shall be denoted by U_{ij} so that the components of a force M_i caused by M_j are given by

$$F_{X_{ij}} = \frac{\partial U_{ij}}{\partial X_i}, \qquad X \to Y,Z \tag{228}$$

Then, according to Newton's second law,

$$\ddot{X}_i = \frac{1}{M_i}\sum_{\substack{j=0 \\ j\neq i}}^{n} \frac{\partial U_{ij}}{\partial X_i}, \qquad (i = 0, 1, \cdots, n)\; X \to Y,Z \tag{229}$$

represents the equations of motion of the $n + 1$ bodies in the inertial coordinate system.

If a parallel coordinate system $r(x,y,z)$ is defined as centered in one of the bodies (say, M_c), then

$$\mathbf{r}_j = \mathbf{r}_{cj} = \mathbf{R}_j - \mathbf{R}_c \tag{230}$$

[24]Moyer, T. D., JPL internal document, Sept. 11, 1964.

[25]Sturms, F. M., JPL internal documents, Aug. 10, 1964; Oct. 29, 1965, and Mar. 18, 1969.

represents the radius vector from the central body to the jth body. For the spacecraft, then, by use of Eqs. (229) and (230),

$$\ddot{x}_0 = \ddot{X}_0 - \ddot{X}_c$$

$$= \frac{1}{M_0}\sum_{j=1}^{n}\frac{\partial U_{0j}}{\partial X_0} - \frac{1}{M_c}\sum_{\substack{j=0\\ j\neq c}}^{n}\frac{\partial U_{cj}}{\partial X_c} \tag{231}$$

The first term in the second summation is

$$\frac{1}{M_c}\frac{\partial U_{c0}}{\partial X_c}, \qquad \underline{X_c \rightarrow Y_c, Z_c}$$

and represents the acceleration of the body on the central body. This term may be neglected, as it is very small. Thus,

$$\ddot{x}_0 = \frac{1}{M_0}\sum_{j=1}^{n}\frac{\partial U_{0j}}{\partial X_0} - \frac{1}{M_c}\sum_{\substack{j=1\\ j\neq c}}^{n}\frac{\partial U_{cj}}{\partial X_c}$$

$$= \frac{1}{M_0}\frac{\partial U_{0c}}{\partial X_0} + \sum_{\substack{j=1\\ j\neq c}}^{n}\left(\frac{1}{M_0}\frac{\partial U_{0j}}{\partial X_0} - \frac{1}{M_c}\frac{\partial U_{cj}}{\partial X_c}\right)$$

$$x_0 \rightarrow y_0, z_0 \tag{232}$$

Because

$$F_{X_{ij}} = -F_{X_{ji}}$$

$$\frac{\partial U_{ij}}{\partial X_i} = -\frac{\partial U_{ij}}{\partial X_j}$$

$$U_{ij} = U_{ji}$$

one obtains from Eq. (232), for the acceleration of the spacecraft,

$$\ddot{x}_0 = \frac{1}{M_0}\frac{\partial U_{0c}}{\partial X_0} + \sum_{\substack{j=1\\ j\neq c}}^{n}\left(\frac{1}{M_0}\frac{\partial U_{0j}}{\partial X_0} + \frac{1}{M_c}\frac{\partial U_{cj}}{\partial X_j}\right)$$

$$x_0 \rightarrow y_0, z_0 \tag{233}$$

Now the potential can be expressed as the point-mass term plus a nonspherical term

$$U_{ij} = \frac{GM_iM_j}{R_{ij}} + U'_{ij} \tag{234}$$

where

$$r_{ij} = R_{ij} = [(X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2]^{1/2}$$

It is desirable to include the effect of planetary oblateness on the spacecraft when it is within a specified distance from the body; thus, U'_{0j} will be zero or nonzero depending upon the distance from the body M_j. For the present, the nonspherical term will be taken to be zero for all pairs of planets except the earth and moon:

$$U_{12} = \frac{GM_1M_2}{R_{12}} + U'_{12} \tag{235}$$

In practice, the values of X_i, Y_i, and Z_i are not used; because they always appear as differences, the following relations are substituted:

$$X_i - X_j = x_i - x_j$$

$$\frac{\partial U_{ij}}{\partial X_i} = \frac{\partial U_{ij}}{\partial x_i}$$

$$r_{ij} = R_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2}$$

Equation (233) then takes on the form

$$\ddot{x}_0 = -\left[GM_c\frac{x_0}{r_0^3} + \sum_{\substack{j=1\\ j\neq c}}^{n} GM_j\left(\frac{x_0 - x_j}{r_{0j}^3} + \frac{x_j}{r_j^3}\right)\right]$$

$$+ \frac{1}{M_0}\left[\frac{\partial U'_{0c}}{\partial x_0} + \sum_{\substack{j=1\\ j\neq c}}^{n}\frac{\partial U'_{0j}}{\partial x_0}\right]$$

$$\left. + \frac{1}{M_i}\frac{\partial U'_{12}}{\partial x_i}\right\}$$

$i = 1$ if central body is moon
$i = 2$ if central body is earth
(this term is zero if central body is neither earth or moon)

(236)

The terms in the first pair of brackets in Eq. (236) correspond to the Newtonian point-mass acceleration (see Eq. 203); the terms in the second pair of brackets represent the (direct) oblateness perturbations. The last term is from the mutual attraction of the earth and moon; the derivation of this term will be discussed next.

2. *Derivation of term U'_{12}.* If one lets dM_1 and dM_2 be differential elements of the mass of the earth and the moon, respectively, one must define parallel coordinate systems (ξ,η,ζ) and (ξ',η',ζ') centered in the earth and moon, respectively, so that the c.m. of the moon is located

thus,

$$U_{12}^{(2)} = \int_{M_2}\int_{M_1} \frac{G}{r_{12}} \frac{1}{2} (3q^2 - 1)\alpha^2 \, dM_1 dM_2$$

$$= \int_{M_2}\int_{M_1} \frac{G}{r_{12}^3} \left\{ \frac{3}{2} (\xi - \xi')^2 - \frac{1}{2}[(\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2] \right\} dM_1 dM_2$$

$$= \frac{G}{r_{12}^3} \int_{M_2}\int_{M_1} \left\{ \left(\xi^2 - \frac{1}{2}\eta^2 - \frac{1}{2}\zeta^2 \right) + \left(\xi'^2 - \frac{1}{2}\eta'^2 - \frac{1}{2}\zeta'^2 \right) - 2\xi\xi' + \eta\eta' + \zeta\zeta' \right\} dM_1 dM_2 \quad (245)$$

The integrals of the product terms $-2\xi\xi' + \eta\eta' + \zeta\zeta'$ are zero because the coordinate systems are at the centers of mass. Regrouping of the remaining terms yields

$$U_{12}^{(2)} = \frac{GM_2}{r_{12}^3} \int_{M_1} \left[\xi^2 + \eta^2 + \zeta^2 - \frac{3}{2}(\eta^2 + \zeta^2) \right] dM_1 + \frac{GM_1}{r_{12}^3} \int_{M_2} \left[\xi'^2 + \eta'^2 + \zeta'^2 - \frac{3}{2}(\eta'^2 + \zeta'^2) \right] dM_2 \quad (246)$$

If one defines

$$\int_{M_1} (\xi^2 + \eta^2 + \zeta^2) \, dM_1 = \frac{1}{2}(A + B + C) \quad (247a)$$

$$\int_{M_2} (\xi'^2 + \eta'^2 + \zeta'^2) \, dM_2 = \frac{1}{2}(A' + B' + C') \quad (247b)$$

$$\int_{M_1} (\eta^2 + \zeta^2) \, dM_1 = I \quad (247c)$$

$$\int_{M_2} (\eta'^2 + \zeta'^2) \, dM_2 = I' \quad (247d)$$

where A, B, and C are the moments of inertia of the earth about the principal axes and I is the moment of inertia of the earth about the ξ-axis (similarly for the moon, by use of the primed rotations), then the indirect potential, if the earth and moon are taken to be triaxial ellipsoids, is given by

$$U'_{12} = U_{12}^{(2)} = \frac{GM_1M_2}{r_{12}^3} \left[\frac{(A + B + C - 3I)}{2M_1} + \frac{(A' + B' + C' - 3I')}{2M_2} \right] \quad (248)$$

This formulation of U'_{12} (i.e., in terms of moments of inertia) has the disadvantage that it leads to loss of significant figures because of small differences of large numbers; however, by formulation of U'_{12} in terms of spherical harmonics, this problem can be eliminated. In spherical harmonic form, Eq. (248) becomes (Ref. 13, p. 153)[26]

$$U'_{12} = \frac{GM_1M_2}{r_{12}^3} \{ R_1^2[-J_2P_2(\sin\phi) + C_{22}^* P_{22}(\sin\phi)\cos 2\theta] + R_2^2[-J'_2 P_2(\sin\beta) + C'^*_{22} P_{22}(\sin\beta)\cos 2\lambda] \} \quad (249)$$

[26]Sturns, F. M., JPL internal document, Oct. 29, 1965.

on the ξ-axis at a distance r_{12}. The distance of the mass elements is then given by

$$d^2 = (r_{12} + \xi' - \xi)^2 + (\eta' - \eta)^2 + (\zeta' - \zeta)^2$$

$$= r_{12}^2 \left[1 - \frac{2(\xi - \xi')}{r_{12}} + \frac{(\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2}{r_{12}^2} \right] \quad (237)$$

Letting

$$q = \frac{\xi - \xi'}{r_{12}\alpha} \quad (238)$$

where

$$\alpha^2 = \frac{(\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2}{r_{12}^2} \quad (239)$$

then Eq. (237) becomes

$$d^2 = r_{12}^2 (1 - 2q\alpha + \alpha^2) \quad (240)$$

By Newton's law of gravity, the element of force potential is

$$dU_{12} = \frac{GdM_1dM_2}{d}$$

and, by use of Eq. (240), this equation becomes

$$dU_{12} = \frac{GdM_1dM_1}{r_{12}} (1 - 2q\alpha + \alpha^2)^{-1/2} \quad (241)$$

The quantity raised to the $-1/2$ power is the generating function of a power series in α, with coefficients consisting of Legendre polynomials in q, $P_i(q)$; that is,

$$(1 - 2q\alpha + \alpha^2)^{-1/2} = 1 + P_1(q)\alpha + P_2(q)\alpha^2 + P_3(q)\alpha^3 + \cdots$$

Thus,

$$dU_{12} = \frac{GdM_1dM_2}{r_{12}} [1 + P_1(q)\alpha + P_2(q)\alpha^2 + \cdots]$$

Clearly, $\alpha < 1$, and, because the series is convergent, one integrate term by term so that

$$U_{12} = \int_{M_2}\int_{M_1} G\frac{dM_1dM_2}{r_{12}} [1 + P_1(q)\alpha + P_2(q)\alpha^2 + \cdots]$$

$$= U_{12}^{(0)} + U_{12}^{(1)} + \cdots \quad (242)$$

For $U_{12}^{(0)}$, one obviously has

$$U_{12}^{(0)} = \int_{M_2}\int_{M_1} G\frac{dM_1dM_2}{r_{12}} = G\frac{M_1M_2}{r_{12}} \quad (243)$$

which is just the point-mass term (see Eq. 235). Because

$$P_1(q) = q$$

and

$$q = \frac{\xi - \xi'}{r_{12}\alpha}$$

(Eq. 238), it follows that

$$U_{12}^{(1)} = \int_{M_2}\int_{M_1} G\alpha q \frac{dM_1dM_2}{r_{12}}$$

$$= \int_{M_2}\int_{M_1} G\frac{\xi - \xi'}{r_{12}^2} dM_1dM_2$$

However,

$$\int_{M_1} \xi dM_1 = 0$$

and

$$\int_{M_2} \xi' dM_2 = 0$$

because ξ and ξ' are measured from the centers of mass. Therefore,

$$U_{12}^{(1)} = 0 \quad (244)$$

Now

$$P_2(q) = \frac{1}{2}(3q^2 - 1)$$

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thus,

$$U_{12}^{(2)} = \int_{M_2}\int_{M_1} \frac{G}{r_{12}} \frac{1}{2}(3q^2 - 1)\alpha^2\, dM_1 dM_2$$

$$= \int_{M_2}\int_{M_1} \frac{G}{r_{12}^3} \left\{\frac{3}{2}(\xi - \xi')^2 - \frac{1}{2}[(\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2]\right\} dM_1 dM_2$$

$$= \frac{G}{r_{12}^3}\int_{M_2}\int_{M_1} \left\{\left(\xi^2 - \frac{1}{2}\eta^2 - \frac{1}{2}\zeta^2\right) + \left(\xi'^2 - \frac{1}{2}\eta'^2 - \frac{1}{2}\zeta'^2\right) - 2\xi\xi' + \eta\eta' + \zeta\zeta'\right\} dM_1 dM_2 \qquad (245)$$

The integrals of the product terms $-2\xi\xi' + \eta\eta' + \zeta\zeta'$ are zero because the coordinate systems are at the centers of mass. Regrouping of the remaining terms yields

$$U_{12}^{(2)} = \frac{GM_2}{r_{12}^3}\int_{M_1}\left[\xi^2 + \eta^2 + \zeta^2 - \frac{3}{2}(\eta^2 + \zeta^2)\right] dM_1 + \frac{GM_1}{r_{12}^3}\int_{M_2}\left[\xi'^2 + \eta'^2 + \zeta'^2 - \frac{3}{2}(\eta'^2 + \zeta'^2)\right] dM_2 \qquad (246)$$

If one defines

$$\int_{M_1} (\xi^2 + \eta^2 + \zeta^2)\, dM_1 = \frac{1}{2}(A + B + C) \qquad (247a)$$

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$$\int_{M_1} (\eta^2 + \zeta^2)\, dM_1 = I \qquad (247c)$$

$$\int_{M_2} (\eta'^2 + \zeta'^2)\, dM_2 = I' \qquad (247d)$$

where A, B, and C are the moments of inertia of the earth about the principal axes and I is the moment of inertia of the earth about the ξ-axis (similarly for the moon, by use of the primed rotations), then the indirect potential, if the earth and moon are taken to be triaxial ellipsoids, is given by

$$U'_{12} = U_{12}^{(2)} = \frac{GM_1M_2}{r_{12}^3}\left[\frac{(A + B + C - 3I)}{2M_1} + \frac{(A' + B' + C' - 3I')}{2M_2}\right] \qquad (248)$$

This formulation of U'_{12} (i.e., in terms of moments of inertia) has the disadvantage that it leads to loss of significant figures because of small differences of large numbers; however, by formulation of U'_{12} in terms of spherical harmonics, this problem can be eliminated. In spherical harmonic form, Eq. (248) becomes (Ref. 13, p. 153)[26]

$$U'_{12} = \frac{GM_1M_2}{r_{12}^3}\{R_1^2[-J_2P_2(\sin\phi) + C_{22}^* P_{22}(\sin\phi)\cos 2\theta] + R_2^2[-J'_2 P_2(\sin\beta) + C'^*_{22} P_{22}(\sin\beta)\cos 2\lambda]\} \qquad (249)$$

[26]Sturns, F. M., JPL internal document, Oct. 29, 1965.

where

$$C_{22}^{*} = (C_{22}^{2} + S_{22}^{2})^{1/2}$$

$$C_{22}'^{*} = (C_{22}'^{2} + S_{22}'^{2})^{1/2}$$

J_2, C_{22}, S_{22} = oblateness parameters for earth

J_2', C_{22}', S_{22}' = oblateness parameters for moon

R_1 = mean radius of earth

R_2 = mean radius of moon

r_{12} = magnitude of the vector $\mathbf{R}$ from the central body to the other body

where

$\mathbf{R} = \mathbf{X}_{\oplus\mathbb{D}1950.0}$ if earth is central body

$\mathbf{R} = -\mathbf{X}_{\oplus\mathbb{D}1950.0}$ if moon is central body

where $\mathbf{X}_{\oplus\mathbb{D}1950.0}$ is the earth–moon position vector in mean earth equator and equinox of 1950.0 coordinates. The vector $\mathbf{R}$ has spherical coordinates

ϕ = geocentric latitude

θ = geocentric longitude $-\alpha_1$

β = selenographic latitude

λ = selenographic longitude $-\alpha_2$

where α_1 and α_2 are defined by

$$\sin 2\alpha_1 = \frac{S_{22}}{(S_{22}^2 + C_{22}^2)^{1/2}}$$

$$\cos 2\alpha_1 = \frac{C_{22}}{(S_{22}^2 + C_{22}^2)^{1/2}}$$

hence,

$$\alpha_1 = \frac{1}{2}\tan^{-1}\left(\frac{S_{22}}{C_{22}}\right)$$

$$\sin 2\alpha_2 = \frac{S_{22}'}{(S_{22}'^2 + C_{22}'^2)^{1/2}}$$

$$\cos 2\alpha_2 = \frac{C_{22}'}{(S_{22}'^2 + C_{22}'^2)^{1/2}}$$

hence,

$$\alpha_2 = \frac{1}{2}\tan^{-1}\left(\frac{S_{22}'}{C_{22}'}\right)$$

The angles ϕ, θ, β, and λ and the earth–moon distance r_{12} are computed from $\mathbf{R}$ as follows:

$$\begin{pmatrix} r_{12} \\ \phi \\ \theta + \alpha_1 \end{pmatrix} = \mathbf{SENAR} \qquad (250a)$$

$$\begin{pmatrix} r_{12} \\ \beta \\ \lambda + \alpha_2 \end{pmatrix} = \mathbf{SBMKAR} \qquad (250b)$$

These are the spherical coordinates of $\mathbf{R}$ in earth-fixed and moon-fixed coordinates, respectively, where

$\mathbf{A}$ = precession–rotation matrix (Section V-B)

$\mathbf{N}$ = rotation matrix (Section V-E)

$\mathbf{E}$ = earth-fixed matrix (Section V-F)

$\mathbf{K}$ = mean obliquity matrix (Section V-D)

$\mathbf{M}$ = moon rotation matrix (Section V-I)

$\mathbf{B}$ = moon-fixed matrix (Section V-J)

$\mathbf{S}$ = transformation to spherical coordinates (Section IV-B)

The indirect oblateness acceleration in 1950.0 coordinates is then given by

$$\ddot{\mathbf{r}}_{50}(IOBL) = (\mathbf{ENA})^T \mathbf{L}_1 \ddot{\mathbf{r}}_1 + (\mathbf{BMKA})^T \mathbf{L}_2 \ddot{\mathbf{r}}_2 \qquad (251)$$

where the L matrices rotate to the body-fixed coordinate axes

$$\mathbf{L}_1 = \begin{pmatrix} \cos\phi\,\cos(\theta + \alpha_1) & -\sin(\theta + \alpha_1) & -\sin\phi\,\cos(\theta + \alpha_1) \\ \cos\phi\,\sin(\theta + \alpha_1) & \cos(\theta + \alpha_1) & -\sin\phi\,\sin(\theta + \alpha_1) \\ \sin\phi & 0 & \cos\theta \end{pmatrix}$$

and $\mathbf{L}_2$ is the same, with β replacing ϕ and $\lambda + \alpha_2$ replacing $\theta + \alpha_1$. The transposed matrices continue the rotations from the body-fixed to the 1950.0 coordinate system. The components of $\ddot{\mathbf{r}}_1$ are computed from

$$\ddot{x}_1 = \frac{\partial U'_{12}}{\partial r_{12}} \qquad \ddot{y}_1 = \frac{1}{r_{12}\cos\phi}\frac{\partial U'_{12}}{\partial \theta} \qquad \ddot{z}_1 = \frac{1}{r_{12}}\frac{\partial U'_{12}}{\partial \phi}$$

yielding

$$\ddot{\mathbf{r}}_1 = \frac{GM_i}{r_{12}^4} R_1^2 \left\{ J_2 \begin{bmatrix} \frac{9}{2}\sin^2\phi - \frac{3}{2} \\ 0 \\ -3\sin\phi\cos\phi \end{bmatrix} + C_{22}^* \begin{bmatrix} -9\cos^2\phi\cos 2\theta \\ -6\cos\phi\sin 2\theta \\ -6\sin\phi\cos\phi\ \cos 2\theta \end{bmatrix} \right\} \tag{252}$$

In a completely analogous way, one obtains

$$\ddot{\mathbf{r}}_2 = \frac{GM_i}{r_{12}^4} R_2^2 \left\{ J'_2 \begin{bmatrix} \frac{9}{2}\sin^2\beta - \frac{3}{2} \\ 0 \\ -3\sin\beta\cos\beta \end{bmatrix} + C'^*_{22} \begin{bmatrix} -9\cos^2\beta\cos 2\lambda \\ -6\cos\beta\sin 2\lambda \\ -6\sin\beta\cos\beta\ \cos 2\lambda \end{bmatrix} \right\} \tag{253}$$

where $i = 1$ if the moon is the central body and $i = 2$ if the earth is the central body. The expressions in Eqs. (252) and (253) are in local "up–north–east" coordinates (Fig. 35).

It should be noted that, if the moon is the central body, the spacecraft acceleration M_1/M_2 is as big (and opposite in direction) as it would be if the earth were the central body. The magnitude is determined by Eqs. (252) and (253); the reversal in sign occurs in the L_1, L_2 matrices, and enters through the $\mathbf{R}$ vector in Eq. (250).

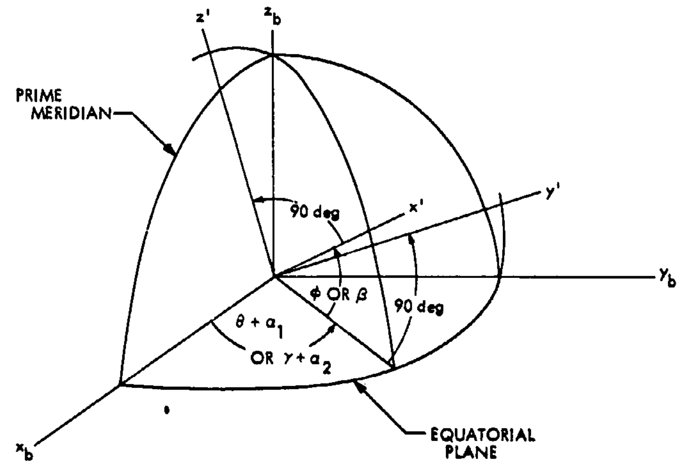

Fig. 35. Up–north–east coordinate system

F. Acceleration Caused by General Relativity

In 1915, Albert Einstein proposed the general theory of relativity, of which his earlier theory of 1905 was a special case. Basically, the general theory of relativity is a theory of gravitation that supersedes the classical Newtonian theory. In the great majority of cases of interest, the two theories lead to essentially the same results because the few predictions the theory makes about observable phenomena require an almost impossible precision for any decisive measurement. In his *Sourcebook on the Space Sciences*, Glasstone notes that such precision has been realized for only three experiments: (1) analysis of the orbit of the planet Mercury for a small relativistic precession of the perihelion of the orbit, (2) gravitational bending of starlight passing by the sun, and (3) the red shift of spectral lines emitted and observed at two different gravitational potentials (see Ref. 11, p. 854):

> In oversimplified terms, the theory of gravitation based on relativity involves two concepts. The first is the principle of equivalence which asserts that the observable effects of

inertia, i.e., the property of matter that causes it to resist any change in its motion, and of gravity are indistinguishable. Einstein illustrated this equivalence by considering an elevator falling freely in space, and supposing that a passenger in the elevator releases a mass that he has been holding. Since the elevator and the mass are falling at the same rate, the mass will not drop to the floor but will remain suspended. If the elevator is completely closed and the passenger is unaware of his surroundings, he will be under the impression that there is no gravitational force acting on the mass. Suppose, however, that a constant upward force is applied to the elevator, e.g., by means of a rocket, the mass which has been suspended in mid-air will drop to the elevator floor just as if it had been attracted by gravity. But the effect is actually due to the inertia of the mass and not to gravity.

The second concept is based on the postulate that all bodies are located in a space–time medium (or continuum); this medium has four dimensions, three of conventional geometrical space and one of time. As a result of its inertia, a body will move on a geodesic (or geodesic line) that is the shortest distance that can be drawn between two points on a three-dimensional surface in the four-dimensional continuum. The presence of any mass causes a distortion or curvature of space–time and consequently distorts the geodesics in its vicinity.

Let us consider two masses, and suppose for simplicity that one is fixed, whereas the other is free to move. The curvature of space–time by the fixed mass causes the other mass to travel along a geodesic that moves it in the direction of the first mass. To an observer, it would appear, therefore, as if the fixed mass is attracting the movable one by the force of gravity. What the moving mass does, however, is determined by its inertia and by the curvature of space–time, and not by the gravitational attraction exerted by the fixed mass. A physical analogy is provided by a sheet of rubber stretched across a frame, with a mass placed in the center of the sheet. The mass will then distort the sheet, and together they may be regarded as representing the curvature of space–time by matter (Fig. 36).

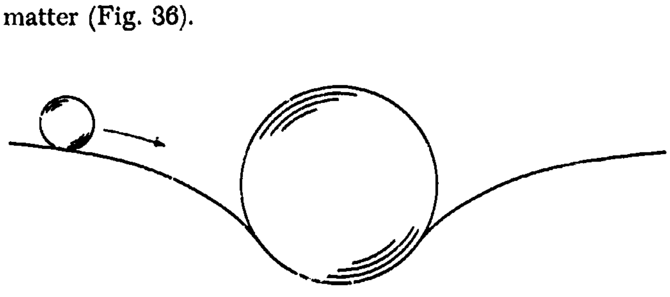

Fig. 36. Simplified schematic representation of gravity in space–time system

Another mass placed in the vicinity of the central mass will move toward the latter, not as a result of attraction, but rather because of the distortion of the medium in which it is constrained to travel.

In the classical Newtonian theory of gravitation, the mutual attraction of two bodies is the same regardless of whether they are stationary or in motion relative to one another. In relativity theory, however, there is a difference, and the magnitude of this difference increases as the velocity of motion approaches that of light (see Ref. 11, p. 855).

1. Relativistic equations of motion. This subsection gives the relativistic n-body equations of motion that may be used to generate the ephemeris for any celestial body or spacecraft within the solar system (or to correct ephemerides that were obtained without accounting for relativity).[27]

In what follows, the term *relativistic acceleration* means the perturbative inertial acceleration caused by general relativity, which is added to the Newtonian inertial acceleration; the term *inertial acceleration* indicates that the acceleration is relative to the mean earth equator and equinox of 1950.0 coordinate system.

The relativistic acceleration of a body relative to the barycenter of the solar system consists of the acceleration computed from Newton's equations of motion plus terms of order $1/c^2$ caused by each perturbing body, where c is the speed of light. The relativistic acceleration terms caused by the sun affect the motion of bodies throughout the solar system. However, the terms caused by a planet or the moon are significant only in a "small" region (small in relation to the scale of the solar system) surrounding the body, which is called the relativity sphere; its center is located at the c.m. of the body. The significant relativistic acceleration of the spacecraft is caused by the sun and any "near" bodies (where "near" implies being within the relativity sphere of a body). The radii of the relativity spheres are given in Table 1.

Table 1 gives the theoretical spheres for each body within which the acceleration due to relativity caused by that body is significant and hence should be computed. However, for programming efficiency, a body either contributes relativistically to the acceleration of the spacecraft for the entire trajectory or not at all. The bodies

[27]The equations were taken from Moyer, T. D., JPL internal document, Jan 4, 1968.

which are to be treated relativistically are specified by input. This manner of treating relativity spheres eliminates any discontinuities in the integration of the equations of motion due to relativity.

In the early formulation of the general theory of relativity, the equations of motion for a massless particle moving in the gravitational field of other bodies were taken to be the equations of a geodesic. That is, the motion of a particle was obtained by solving the field equations for the metric tensor, which describes the geometrical properties of space and time, and by assuming that the particle follows a geodesic curve in this geometry. The actual method for determining the motion of a system of n heavy bodies directly from the field equations was obtained for the first time by Einstein, Infeld, and Hoffmann in 1938. This method, which is referred to as the EIH approximation method, is, according to Bazański (Ref. 14, pp. 13–29), in principle, the only tool for obtaining an approximate solution to the problem of the motion of n heavy bodies in the general theory of relativity.

Table 1. Radii of relativity spheres[a]

Celestial body	Mean distance from sun a_p, AU	Sun–planet mass ratio μ_S/μ_P	Radius of relativity sphere r_p, km × 10^6
Mercury	0.387	6,000,000	2
Venus	0.723	408,500	7
Earth	1.000	333,000	9
Mars	1.524	3,100,000	4
Jupiter	5.20	1,047	400
Saturn	9.55	3,502	300
Uranus	19.20	22,900	200
Neptune	30.1	19,300	200
Pluto	39.5	360,000	50
Moon	1.000	27,100,000	1

[a]Moyer, T. D., JPL internal document, Jan. 4, 1968.

From Infeld's equations of motion, after some computations and simplifications,[28] the resultant equation for the relativistic acceleration of body i "due to body j," denoted by $\ddot{\mathbf{r}}_i(j)$, is

$$\ddot{\mathbf{r}}_i(j) = \frac{\mu_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}\left\{-\frac{4}{c^2}\phi_i - \frac{1}{c^2}\phi_j + \left(\frac{\dot{s}_i}{c}\right)^2 + 2\left(\frac{\dot{s}_j}{c}\right)^2 - \frac{4}{c^2}\dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_j - \frac{3}{2c^2}\left[\frac{(\mathbf{r}_i - \mathbf{r}_j)\bullet\dot{\mathbf{r}}_j}{r_{ij}}\right]^2 + \frac{1}{2c^2}(\mathbf{r}_j - \mathbf{r}_i)\bullet\ddot{\mathbf{r}}_j\right\}$$

$$+ \frac{1}{c^2}\frac{\mu_j}{r_{ij}^3}\left[(\mathbf{r}_i - \mathbf{r}_j)\bullet(4\dot{\mathbf{r}}_i - 3\dot{\mathbf{r}}_j)\right](\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) + \frac{7}{2c^2}\frac{\mu_j\ddot{\mathbf{r}}_j}{r_{ij}} \tag{254}$$

where

r_{ij} = coordinate distance between bodies i and j

$(\dot{s}_i)^2, (\dot{s}_j)^2$ = square of velocity of bodies i and j, respectively

ϕ_i = Newtonian potential at body i

ϕ_j = Newtonian potential at body j

and

$$\ddot{\mathbf{r}}_j = \sum_{m\neq j}\frac{\mu_m(\mathbf{r}_m - \mathbf{r}_j)}{r_{mj}^3}$$

The acceleration of body i "due to body j" is a function of the position and velocity of bodies i and j and the positions of all other bodies, which contribute to the Newtonian potential at bodies i and j and affect the acceleration of body j (terms 7 and 9 of Eq. 254). Although the effects of other bodies are included, all terms are proportional to μ_j and hence are attributable to the presence of body j. The effect of the mass of body i on its own acceleration is contained in term 2 (its contribution to the Newtonian potential at body j) and in its contribution to the acceleration of body j (terms 7 and 9).

[28]Moyer, T. D., JPL internal document, Jan. 4, 1968, and Khatib, A. R., JPL internal document, Feb. 11, 1969.

When it is desired to determine the relativistic inertial acceleration of any body caused by the sun, a number of terms in Eq. (191) are insignificant; it can be shown[29] that the significant inertial acceleration of any body i caused by the sun is given by

$$\ddot{\mathbf{r}} = \frac{\mu_S}{c^2 r^3}\left[(4\phi - \dot{s}^2)\mathbf{r} + 4(\mathbf{r}\cdot\dot{\mathbf{r}})\dot{\mathbf{r}}\right] \tag{255}$$

where

μ_S = gravitational constant of sun, km³/s²

c = speed of light, km/s

$\mathbf{r}, \dot{\mathbf{r}}$ = heliocentric position and velocity vector of body i (with rectangular components referred to a nonrotating coordinate system), km, km/s

r = magnitude of $\mathbf{r}$

$\dot{s}$ = magnitude of $\dot{\mathbf{r}}$

ϕ = Newtonian potential

If body i is a spacecraft, then

$$\phi = \phi_i = \frac{\mu_S}{r} + \sum_j \frac{\mu_j}{r_{ij}} \tag{256}$$

where the second term is the contribution to the Newtonian potential caused by any body which is relativistically turned on for the trajectory.

If body i is a planet P other than the earth,

$$\phi_P = \frac{\mu_S}{r} \tag{257}$$

If body i is the earth E or the moon M, then

$$\phi_E = \frac{\mu_S}{r} + \frac{\mu_M}{r_{EM}} \tag{258}$$

$$\phi_M = \frac{\mu_S}{r} + \frac{\mu_E}{r_{EM}} \tag{259}$$

2. ***Heliocentric ephemeris of a planet (other than earth).*** The relativistic acceleration of a planet P (other than the earth) relative to the sun is given by

$$\ddot{\mathbf{r}}_P^S = \ddot{\mathbf{r}}_P - \ddot{\mathbf{r}}_S \tag{260}$$

It has been shown[30] that, because of the uncertainty to which the value of the astronomical unit is known, Eq. (260) simplifies to

$$\ddot{\mathbf{r}}_P^S \approx \ddot{\mathbf{r}}_P \tag{261}$$

where $\ddot{\mathbf{r}}_P$ is given by Eqs. (255) and (257).

3. ***Heliocentric ephemeris of earth–moon barycenter.*** The relativistic acceleration of the earth–moon barycenter B relative to the sun S is given by

$$\ddot{\mathbf{r}}_B^S = \frac{\mu}{1+\mu}\ddot{\mathbf{r}}_E(S) + \frac{\mu}{1+\mu}\ddot{\mathbf{r}}_E(M) + \frac{1}{1+\mu}\ddot{\mathbf{r}}_M(S) + \frac{1}{1+\mu}\ddot{\mathbf{r}}_M(E) - \ddot{\mathbf{r}}_S \tag{262}$$

where E and M indicate the earth and moon, respectively; $\ddot{\mathbf{r}}_i(j)$ is the inertial relativistic acceleration of body i caused by body j; and

$$\mu \equiv \frac{\mu_E}{\mu_M} \tag{263}$$

where μ_E and μ_M are the gravitational constants for the earth and moon, respectively, in km³/s². The accelerations caused by the sun are given by Eqs. (255) and (258) or (259); those caused by the earth and moon are computed from Eqs. (254), (258), and (259). Again, it has been shown[31] that Eq. (262) simplifies to

$$\ddot{\mathbf{r}}_B^S = \frac{\mu}{1+\mu}\ddot{\mathbf{r}}_E(S) + \frac{1}{1+\mu}\ddot{\mathbf{r}}_M(S) \tag{264}$$

where $\ddot{\mathbf{r}}_E(S)$ is given by Eqs. (255) and (258) and $\ddot{\mathbf{r}}_M(S)$ is given by Eqs. (255) and (259).

The relativistic acceleration of the earth–moon barycenter could be computed directly from Eqs. (255) and (257) in terms of the heliocentric position and velocity of the barycenter. However, the third significant figure of the acceleration would be affected; therefore, it is recommended that Eq. (264) be used.

4. ***Geocentric ephemeris of moon.*** The relativistic acceleration of the moon relative to the earth is given by

$$\ddot{\mathbf{r}}_M^E = \ddot{\mathbf{r}}_M(S) - \ddot{\mathbf{r}}_E(S) + \ddot{\mathbf{r}}_M(E) - \ddot{\mathbf{r}}_E(M) \tag{265}$$

[29] Moyer, T. D., JPL internal document, Jan. 4, 1968.

[30,31] Moyer, T. D., JPL internal document, Jan. 4, 1968.

where the first two terms are computed from Eqs. (255), (258), and (259) and the last two terms are computed from Eqs. (254), (258), and (259).

The accelerations in Eqs. (261), (264), and (265) are numerically integrated to correct the basic ephemeris (as obtained from, e.g., an ephemeris tape) of a planet, the moon, or the earth–moon barycenter.

5. Equations of motion for generation of a spacecraft ephemeris. The acceleration of a spacecraft p relative to the center of integration C (a planet, the moon, or the sun), which is integrated to give the spacecraft ephemeris, is the sum of the usual Newtonian acceleration and the following relativistic acceleration:

$$\ddot{\mathbf{r}}_p^C = \ddot{\mathbf{r}}_p(S) - \ddot{\mathbf{r}}_C(S) + \sum_j \ddot{\mathbf{r}}_p(j) - \ddot{\mathbf{r}}_C(n) \qquad (266)$$

The first two terms are the accelerations of the spacecraft and center of integration caused by the sun, computed from Eq. (255). (The second term is zero if the center of integration is the sun.) The third term is the acceleration of the spacecraft caused by each "near body" j, computed from Eq. (254). The j-summation, if it exists, will include:

(1) A single planet.

(2) The earth and the moon.

(3) The planets Jupiter and Saturn.

The last term of Eq. (266) is the acceleration of the center of integration caused by a near body n, computed from Eq. (254). It is nonzero only when the center of integration is the earth or the moon, in which case the near body is the other of these two bodies. The Newtonian potentials appearing in Eqs. (254) and (255) are evaluated from Eqs. (256) through (259), as appropriate.

When the center of integration is the sun, its relativistic acceleration caused by the planets and the moon (3.5×10^{-18} km/s^2) is ignored.

VIII. Numerical Integration of Equations of Motion of a Spacecraft

As is the case with most differential equations arising in practical applications, the equations of motion of a spacecraft cannot be integrated in closed form. One reason for this is that to integrate the Newtonian point-mass acceleration (Section VII-A) in closed form, $6n$ constants are required, but only 10 are known. Therefore, discrete methods must be employed for solving Eq. (191). In a discrete method, the solution of a differential equation is computed at a discrete point t_n. To advance the solution from t_n to t_{n+1}, if only information in the interval $[t_n, t_{n+1}]$ is used, the method is a one-step method. If information from steps preceding t_n is used to advance the solution, the method is a multistep method.

The most obvious disadvantage of one-step methods is that no use is made of past information on the solution, and many derivative evaluations over $[t_n, t_{n+1}]$ are necessary if high accuracy is desired. Multistep methods have the advantage of using already computed values for most of their information; hence, the computational effort is reduced. A disadvantage of the multistep method is that, before it can be applied, some points to the left or right of the starting point (say, t_0) must be computed by some other method (for example, by use of a Taylor series expansion).

A. Solution Method

The algorithm used to solve numerically the second-order differential equations of motion is a multistep method in summed form. It is closely related to the familiar Adams-type method. The algorithm may be regarded as consisting of two parts:

(1) A starting procedure to produce the solution values at the time points $t_{-1}, t_{-2}, \cdots, t_{-m}$.

(2) A stepping procedure of the predictor–corrector type to advance the solution one time step, making use of the solution at the m immediately preceding points.

Before (1) and (2) are described in more detail, the subject of backward differences will be discussed.

B. Backward Differences

If a function $f(t)$ and a spacing h are given, the backward difference operator ∇ is defined by

$$\nabla f(t) = f(t) - f(t-h) \qquad (267)$$

Applying ∇ to $f(t_n) \equiv f_n$, one obtains

$$\nabla f_n = f_n - f_{n-1} \qquad (268)$$

Then

$$\nabla^r f_n = \nabla(\nabla^{r-1} f_n) \tag{269}$$

and it is easy to verify that

$$\begin{aligned}\nabla^r f_n &= f_n - \binom{r}{1} f_{n-1} + \binom{r}{2} f_{n-2} + \cdots \\ &\quad + (-1)^{r-1}\binom{r}{1} f_{n-r+1} + (-1)^r f_{n-r} \\ &= \sum_{i=0}^{r} (-1)^i \binom{r}{i} f_{n-i}\end{aligned} \tag{270}$$

(with $\nabla^0 f_n = f_n$).

The backward difference operator may be used symbolically as a number or variable (Ref. 15, p. 128) because it formally satisfies the laws of algebra; i.e.,

$$\nabla(f_n + f_m) = \nabla f_n + \nabla f_m = \nabla f_m + \nabla f_n$$

$$\nabla c f_n = c \nabla f_n$$

$$\nabla^i(\nabla^j f_n) = \nabla^{i+j} f_n$$

By making use of these properties, it is possible to express the differences of a function f in terms of its successive differences.

Let us consider the Taylor expansion of $f(t+h)$ about t:

$$f(t+h) = f(t) + \frac{h}{1!}\dot{f}(t) + \frac{h^2}{2!}\ddot{f}(t) + \frac{h^3}{3!}\dddot{f}(t) + \cdots \tag{271}$$

If one defines the differential operator

$$\begin{aligned} D &\equiv \frac{d}{dt} \\ D^r &\equiv \frac{d^r}{dt^r} \end{aligned} \tag{272}$$

its use in Eq. (271) yields

$$f(t+h) = f(t) + \frac{h}{1!}Df(t) + \frac{h^2}{2!}D^2 f(t) + \frac{h^3}{3!}D^3 f(t) + \cdots$$

or

$$f(t+h) = \left(1 + \frac{h}{1!}D + \frac{h^2}{2!}D^2 + \frac{h^3}{3!}D^3 + \cdots\right) f(t) \tag{273}$$

By means of the series expansion for $e^{\pm u}$,

$$e^{\pm u} = 1 \pm \frac{u}{1} + \frac{u^2}{2!} \pm \frac{u^3}{3!} + \cdots$$

the differential operator on the right side of Eq. (273) may be written formally

$$1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots = e^{hD}$$

and, hence, f_{n+1} may be written symbolically as

$$f_{n+1} = e^{hD} f_n$$

Change of h to $-h$ then yields

$$f(t-h) = e^{-hD} f(t)$$

or

$$f_{n-1} = e^{-hD} f_n$$

Equation (268) may then be written as

$$\nabla f_n = [1 - e^{-hD}] f_n \tag{274}$$

In purely operational form, Eq. (274) takes the form

$$\nabla = 1 - e^{-hD}$$

or

$$e^{-hD} = 1 - \nabla \equiv E^{-1} \tag{275}$$

where E is the shift operator defined by $E^k f_n = f(t_n + kh)$.

From Eq. (275), it follows that

$$D^{-1} = \frac{-h}{\ln(1 - \nabla)} \tag{276}$$

C. Derivation of Predictor–Corrector Formulas

The acceleration vector $\ddot{\mathbf{r}}$ in Eq. (191) has three components,

$$\ddot{\mathbf{r}} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}$$

and, in all of the following sections, the equations will be given for the variable x only. (Identical formulas would be used for y and z.)

The following definitions will be used:

$$\ddot{x}_i = \ddot{x}(t_i)$$

$$\dot{x}_i = \dot{x}(t_i)$$

$$\dot{x}_{i-1} = \dot{x}(t_i - h)$$

Now

$$\begin{aligned} \dot{x}_n - \dot{x}_{n-1} &= \int_{t_{n-1}}^{t_n} \ddot{x}(t)dt = \nabla(D^{-1}\ddot{x}_n) \\ &= \nabla\left(\frac{-h}{\ln(1-\nabla)}\right)\ddot{x}_n \\ &= h\left(\frac{\nabla}{\sum_{i=1}^{\infty}\frac{\nabla^i}{i}}\right)\ddot{x}_n \end{aligned} \qquad (277)$$

since

$$\ln(1 \pm u) = \pm u - \frac{u^2}{2} \pm \frac{u^3}{3} - \frac{u^4}{4} \pm \frac{u^5}{5} - \cdots$$

or

$$\begin{aligned} \dot{x}_n - \dot{x}_{n-1} &= h\left(\sum_{i=0}^{\infty} b_{i-1}\nabla^i\right)\ddot{x}_n \\ &= h\left(\nabla^0 - \frac{1}{2}\nabla^1 - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \cdots\right)\ddot{x}_n \end{aligned} \qquad (278)$$

To derive the so-called *summed* form of this formula, one writes the left side of Eq. (278) as $\nabla\dot{x}_n$ and then applies the improper operator ∇^{-1} to both sides of the equation:

$$\begin{aligned} \dot{x}_n &= h\left(\frac{-1}{\ln(1-\nabla)}\right)\ddot{x}_n \\ &= h\left(\sum_{i=-1}^{\infty} b_i\nabla^i\right)\ddot{x}_n \\ &= h\left(\nabla^{-1} - \frac{1}{2}\nabla^0 - \frac{1}{12}\nabla^1 - \frac{1}{24}\nabla^2 - \cdots\right)\ddot{x}_n \end{aligned} \qquad (279)$$

The expression $\nabla^{-1}\ddot{x}_n$, which appears in Eq. (279), is called a first sum of $\ddot{x}_n$ and satisfies the equation

$$\nabla^{-1}\ddot{x}_n - \nabla^{-1}\ddot{x}_{n-1} = \ddot{x}_n \qquad (280)$$

So far, $\nabla^{-1}\ddot{x}$ has only been defined to within an arbitrary additive constant. The practical use of Eqs. (279) and (280) requires that an initial value of $\nabla^{-1}\ddot{x}$ be derived from the given initial value of $\dot{x}$ (see Section VIII-E).

A generalization of Eq. (279) that gives $\dot{x}_{n-s}$ for arbitrary values of s is obtained as follows:

$$\dot{x}_{n-s} = E^{-s}\dot{x}_n = (1-\nabla)^s\dot{x}_n \qquad (281)$$

where E^{-1} is defined by Eq. (275), or

$$\begin{aligned} \dot{x}_{n-s} &= h\left[\frac{-(1-\nabla)^s}{\ln(1-\nabla)}\right]\ddot{x}_n \\ &= h\left[\sum_{i=-1}^{\infty} b_i(s)\nabla^i\right]\ddot{x}_n \end{aligned} \qquad (282)$$

With $s = -1$, this is a predictor formula; i.e., the summed form of the Adams–Bashforth formula:

$$\dot{x}_{n+1} = h\left(\nabla^{-1} + \frac{1}{2}\nabla^0 + \frac{5}{12}\nabla^1 + \frac{9}{24}\nabla^2 + \cdots\right)\ddot{x}_n$$

With $0 < s < 1$, Eq. (282) may be used for interpolation of values of $\dot{x}$ in the interval between t_{n-1} and t_n. Equation (282) will be used with s equal to a negative integer or zero as part of the starting algorithm described in Section VIII-F.

A similar formula giving x in terms of the backward difference line at $\dot{x}_n$ will also be needed. Reducing by one the order of derivatives in Eq. (282), one obtains

$$x_{n-s} = \left[h \frac{-(1-\nabla)^s}{\ln(1-\nabla)}\right]\dot{x}_n \tag{283}$$

Replacing $\dot{x}_n$ in Eq. (283) by the right side of Eq. (279), one obtains

$$\begin{aligned} x_{n-s} &= h^2 \left\{\frac{(1-\nabla)^s}{[\ln(1-\nabla)]^2}\right\}\ddot{x}_n \\ &= h^2 \sum_{i=-2}^{\infty} a_i(s)\, \nabla^i \ddot{x}_n \end{aligned} \tag{284}$$

As with Eq. (282), the use of appropriate values of s in Eq. (284) provides for prediction, correction, interpolation, and starting. In particular, the predictor and corrector formulas are, respectively,

$$x_{n+1} = h^2\left(\nabla^{-2} + 0\cdot\nabla^{-1} + \frac{1}{12}\nabla^0 + \frac{1}{12}\nabla^1 + \frac{19}{240}\nabla^2 + \cdots\right)\ddot{x}_n \tag{285}$$

and

$$x_n = h^2\left(\nabla^{-2} - \nabla^{-1} + \frac{1}{12}\nabla^0 + 0\cdot\nabla^1 - \frac{1}{240}\nabla^2 + \cdots\right)\ddot{x}_n \tag{286}$$

The expression $\nabla^{2}\ddot{x}$, which appears in Eq. (284), is called a second sum of $\ddot{x}$; it satisfies

$$\nabla^{-2}\ddot{x}_n - \nabla^{-2}\ddot{x}_{n-1} = \nabla^{-1}\ddot{x}_n \tag{287}$$

and its initial value must be determined from the given initial value of x.

To advance the solution one time step, Eqs. (282) and (284) are used with $s = -1$ for prediction and $s = 0$ for correction. If s has such values that $s = 1, 2, \cdots, +m$, equations are provided for computing past values of x in terms of a difference line at t_n; the method is said to be of order m if the highest-order difference used is $\nabla^m\ddot{x}$.

Thus, the basic formulas for integration, interpolation, or differentiation are:

$$x(t-sh) = h^2 \sum_{i=-2}^{m} a_i(s)\,\nabla^i \ddot{x}(t) \tag{288}$$

$$\dot{x}(t-sh) = h \sum_{i=-1}^{m} b_i(s)\,\nabla^i \ddot{x}(t) \tag{289}$$

$$\ddot{x}(t-sh) = \sum_{i=0}^{m} c_i(s)\,\nabla^i \ddot{x}(t) \tag{290}$$

$$\dddot{x}(t-sh) = h^{-1} \sum_{i=1}^{m} d_i(s)\,\nabla^i \ddot{x}(t) \tag{291}$$

where $-1 \le s \le m$, and $\nabla^m\ddot{x}$ is the highest-order difference retained. Formulas for computing $a_i(s)$, $b_i(s)$, $c_i(s)$, and $d_i(s)$ are given in Section VIII-D. The array $D_{t_n,h} = [\nabla^i\ddot{x}_n]$, $i = -2, -1, \cdots, m$ is known as the backward difference line of $\ddot{x}$ at t_n based on a step size (or mesh size) of h. It is said to space an interval

$$t_n - mh \le t \le t_n + h$$

because, given only h and $D_{t_n,h}$, one can compute $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ in that interval, by use of Eqs. (288) through (291). The solution is advanced from time t_n to t_{n+1} as follows: given the difference line at time t_n and a step size h, a new difference line is computed at time

$$t_{n+1} = t_n + h \tag{292}$$

The value x at t_{n+1} (i.e., x_{n+1}) is found by use of Eq. (284) with $s = -1$:

$$x_{n+1} = h^2 \sum_{i=-2}^{m} a_i(-1)\,\nabla^i \ddot{x}_n \tag{293}$$

and $\dot{x}_{n+1}$ is computed from Eq. (282) with $s = -1$:

$$\dot{x}_{n+1} = h \sum_{i=-1}^{m} b_i(-1)\,\nabla^i \ddot{x}_n \tag{294}$$

If the solution method uses the predictor only, the difference line at t_{n+1} is formed,

$$\nabla^i \ddot{x}_{n+1} = \nabla^{i-1} \ddot{x}_{n+1} - \nabla^{i-1} \ddot{x}_n, \qquad i = 1, 2, \cdots, m+1 \tag{295a}$$

$$\nabla^{-1} \ddot{x}_{n+1} = \nabla^{-1} \ddot{x}_n + \ddot{x}_{n+1} \tag{295b}$$

$$\nabla^{-2} \ddot{x}_{n+1} = \nabla^{-2} \ddot{x}_n + \nabla^{-1} \ddot{x}_{n+1} \tag{295c}$$

and the solution is advanced from t_{n+1} to t_{n+2} in an analogous way, as described above.

The $(m+1)$ backward difference is computed to obtain an estimate on the truncation error (see Section VIII-F-2); the $(m+1)$ difference in Eq. (299), below, is computed for the same reason.

If a predictor–corrector method is desired, corrector formulas are derived from Eqs. (284) and (282), with $s = 0$ and n replaced by $n+1$:

$$x_{n+1} = h^2 \sum_{i=-2}^{m} a_i(0) \nabla^i \ddot{x}_{n+1} \tag{296}$$

$$\dot{x}_{n+1} = h \sum_{i=-1}^{m} b_i(0) \nabla^i \ddot{x}_{n+1} \tag{297}$$

In using predictor–corrector formulas, one distinguishes between two possibilities. If the corrector formulas are applied to x_{n+1} and $\dot{x}_{n+1}$, and not to the difference line ("pseudopredictor–corrector"), the computation stops with Eq. (297). If the corrector formulas are used to correct x_{n+1}, $\dot{x}_{n+1}$, *and* the difference line (full predictor–corrector), Eqs. (298) and (299), below, are required.

If c_{n+1} is the difference between the corrected value $\ddot{x}^{(c)}_{n+1}$ and the predicted value $\ddot{x}_{n+1}$,

$$c_{n+1} = \ddot{x}^{(c)}_{n+1} - \ddot{x}_{n+1} \tag{298}$$

then one forms the corrected difference line

$$\nabla^i \ddot{x}^{(c)}_{n+1} = \nabla^i \ddot{x}_{n+1} + c_{n+1}, \qquad i = 1, 2, \cdots, m+1 \tag{299a}$$

$$\nabla^{-1} \ddot{x}^{(c)}_{n+1} = \nabla^{-1} \ddot{x}^{(c)}_n + \ddot{x}^{(c)}_{n+1} \tag{299b}$$

$$\nabla^{-2} \ddot{x}^{(c)}_{n+1} = \nabla^{-2} \ddot{x}^{(c)}_n + \nabla^{-1} \ddot{x}^{(c)}_{n+1} \tag{299c}$$

D. Computation of Coefficients $a_i(s)$, $b_i(s)$, $c_i(s)$, $d_i(s)$

In this section, the following conventions will be adopted:

(1) m denotes the highest order of the backward differences retained.

(2) $\binom{s}{n} = 0$ if s (an integer) is less than n.

(3) $s \geq -1$.

From Eq. (284), one finds that

$$\frac{1}{[\ln(1-\nabla)]^2}(1-\nabla)^s = \sum_{i=-2}^{\infty} a_i(s) \nabla^i \tag{300}$$

If ∇^2 is applied to both sides of Eq. (300), it follows that this equation is equivalent to

$$\left[\frac{\nabla}{\ln(1-\nabla)}\right]^2 (1-\nabla)^s = \sum_{i=0}^{\infty} a'_i(s) \nabla^i \tag{301}$$

From

$$\begin{aligned}
\frac{d}{dz}[\ln(1-z)]^2 &= -\frac{2}{1-z}\ln(1-z) \\
&= 2(1 + z + z^2 + z^3 + \cdots)\left(z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots\right) \\
&= 2(s_1 z + s_2 z^2 + s_3 z^3 + \cdots)
\end{aligned}$$

where

$$s_r = 1 + \frac{1}{2} + \cdots + \frac{1}{r}$$

denotes the rth partial sum of the harmonic series; it follows that

$$\left[\frac{\ln(1-z)}{z}\right]^2 = 1 + \frac{2}{3}s_2 z + \frac{2}{4}s_3 z^2 + \frac{2}{5}s_4 z^3 + \cdots \tag{302}$$

By multiplying both sides of Eq. (301) by $[\ln(1-\nabla)^2/\nabla]^2$ and using Eq. (302), one obtains

$$\left(1 + \frac{2}{3}s_2\nabla + \frac{2}{4}s_3\nabla^2 + \frac{2}{5}s_4\nabla^3 + \cdots\right)(a_0'(s) + a_1'(s)\nabla + a_2'(s)\nabla^2 + \cdots) = (1-\nabla)^s$$

or

$$a_0'(s) + \left[a_1'(s) + \frac{2}{3}s_2 a_0'(s)\right]\nabla + \left[a_2'(s) + \frac{2}{3}s_2 a_1'(s) + \frac{2}{4}s_3 a_0'(s)\right]\nabla^2$$

$$+ \cdots + \left[a_n'(s) + \frac{2}{3}s_2 a_{n-1}'(s) + \frac{2}{4}s_3 a_{n-2}'(s) + \cdots + \frac{2}{n+2}s_{n+1}a_0'(s)\right]\nabla^n + \cdots =$$

$$1 - \binom{s}{1}\nabla + \binom{s}{2}\nabla^2 - \cdots + (-1)^n\binom{s}{n}\nabla^n + \cdots$$

If coefficients of the same powers of ∇ are compared, it follows that

$$a_0'(s) = 1 \qquad a_1'(s) = -\binom{s}{1} - \frac{2}{3}s_2 a_0'(s) \qquad a_2'(s) = \binom{s}{2} - \frac{2}{3}s_2 a_1'(s) - \frac{2}{4}s_3 a_0'(s)$$

and hence one arrives at the recurrence relation

$$a_n'(s) = (-1)^n\binom{s}{n} - \frac{2}{3}s_2 a_{n-1}'(s) - \frac{2}{4}s_3 a_{n-2}'(s) - \cdots - \frac{2}{n+2}s_{n+1}a_0'(s), \qquad s \geq 0, n > 0 \tag{303}$$

A special case arises when $s = -1$. Then the right side of Eq. (302) is of the form

$$1 + \nabla + \nabla^2 + \nabla^3 + \cdots + \nabla^n + \cdots$$

and the recurrence relation (Eq. 303) becomes

$$a_n'(-1) = 1 - \frac{2}{3}s_2 a_{n-1}'(-1) - \frac{2}{4}s_3 a_{n-2}'(-1) - \cdots - \frac{2}{n+2}s_{n+1}a_0'(-1) \tag{304}$$

that is, the first term on the right side of Eq. (303) is replaced by 1. Changing back to unprimed notation, for the coefficients $a_i(s)$ in Eq. (300) one has the following expressions:

$$a_{-2}(s) = 1, \qquad \text{for all } s \tag{305}$$

$$a_{n-2}(s) = (-1)^n \binom{s}{n} - \frac{2}{3} s_2 a_{n-3}(s) - \frac{2}{4} s_3 a_{n-4}(s) - \cdots$$
$$- \frac{2}{n+2} S_{n+1} a_{-2}(s), \qquad 1 \le n \le m+2, \qquad s \ge 0 \tag{306}$$

and, when $s = -1$,

$$a_{n-2}(-1) = 1 - \frac{2}{3} s_2 a_{n-3}(-1) - \frac{2}{4} s_3 a_{n-4}(-1)$$
$$- \frac{2}{5} s_4 a_{n-5}(-1)$$
$$- \cdots - \frac{2}{n+2} s_{n+1} a_{-2}(-1),$$
$$1 \le n \le m+2 \tag{307}$$

Thus, coefficients $a_i(s)$ of Eq. (288) are given by Eqs. (305) through (307).

To compute coefficients $b_i(s)$, one should consider Eq. (282), from which it follows that

$$\left[\frac{-1}{\ln(1-\nabla)}\right](1-\nabla)^s = \sum_{i=-1}^{\infty} b_i(s) \nabla^i \tag{308}$$

Operation with ∇^1 on both sides of Eq. (308) shows that this equation is equivalent to

$$\left[\frac{-\nabla}{\ln(1-\nabla)}\right](1-\nabla)^s = \sum_{i=0}^{\infty} b'_i(s) \nabla^i \tag{309}$$

Because

$$\frac{\ln(1-z)}{-z} = \frac{-z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots}{-z}$$
$$= 1 + \frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{4} + \cdots$$

it follows, if Eq. (309) is multiplied by $\left[\frac{\ln(1-\nabla)}{-\nabla}\right]$,

$$[b'_0(s) + b'_1(s)\nabla + b'_2(s)\nabla^2 + \cdots]\left(1 + \frac{1}{2}\nabla + \frac{1}{3}\nabla^2 + \frac{1}{4}\nabla^3 + \cdots\right) =$$
$$1 - \binom{s}{1}\nabla + \binom{s}{2}\nabla^2 - \cdots + (-1)^n \binom{s}{n}\nabla^n + \cdots, \qquad s \ge 0$$

or

$$b'_0(s) + \left[\frac{1}{2} b'_0(s) + b'_1(s)\right]\nabla + \left[\frac{1}{3} b'_0(s) + \frac{1}{2} b'_1(s) + b'_2(s)\right]\nabla^2 + \cdots$$
$$+ \left[\frac{1}{n+1} b'_0(s) + \frac{1}{n} b'_1(s) + \cdots + \frac{1}{2} b'_{n-1}(s) + b'_n(s)\right]\nabla^n + \cdots =$$
$$1 - \binom{s}{1}\nabla + \binom{s}{2}\nabla^2 - \cdots + (-1)^n \binom{s}{n}\nabla^n + \cdots, \qquad s \ge 0 \tag{310}$$

Comparing coefficients of equal powers of ∇, one finds that

$$b'_0(s) = 1, \qquad \text{for all } s \tag{311a}$$

$$b'_1(s) = -\binom{s}{1} - \frac{1}{2} b'_0(s) \tag{311b}$$

and

$$b'_2(s) = \binom{s}{2} - \frac{1}{2}b'_1(s) - \frac{1}{3}b'_0(s) \qquad (311c)$$

for $s \geq 0$;

$$b'_n(s) = (-1)^n \binom{s}{n} - \frac{1}{2}b'_{n-1}(s) - \cdots - \frac{1}{n+1}b'_0(s)$$

$$s \geq 0, n > 0 \qquad (312)$$

When $s = -1$, the first term on the right side of Eq. (312) is replaced by 1, so that

$$b'_n(-1) = 1 - \frac{1}{2}b'_{n-1}(-1) - \cdots - \frac{1}{n+1}b'_0(-1)$$

$$n > 0 \qquad (313)$$

Changing back to the unprimed notation, one obtains

$$b_{-1}(s) = 1, \qquad \text{for all } s \qquad (314)$$

$$b_{n-1}(s) = (-1)^n \binom{s}{n} - \frac{1}{2}b_{n-2}(s) - \cdots - \frac{1}{n+1}b_{-1}(s)$$

$$1 \leq n \leq m+1, \qquad s \geq 0 \qquad (315)$$

$$b_{n-1}(-1) = 1 - \frac{1}{2}b_{n-2}(-1) - \cdots - \frac{1}{n+1}b_{-1}(-1)$$

$$1 \leq n \leq m+1 \qquad (316)$$

Coefficients $b_i(s)$ in Eq. (289) are then given by Eqs. (314) through (316).[32]

The coefficients $c_i(s)$ are computed by the method that follows. Increasing the order of the derivative in Eq. (281), one obtains

$$\ddot{x}_{n-s} = (1 - \nabla)^s \ddot{x}_n$$

$$= \sum_{i=0}^{m} (-1)^i \binom{s}{i} \nabla^i \ddot{x}_n, \qquad s \geq 0 \qquad (317)$$

[32]Witt, J. W., JPL internal document, Oct. 20, 1968.

Thus, the coefficients $c_i(s)$ are given by

$$c_n(s) = (-1)^n \binom{s}{n}, \qquad s \geq 0, \qquad 0 \leq n \leq m \qquad (318)$$

If $s = -1$, then

$$\ddot{x}_{n-s} = \frac{1}{1-\nabla}\ddot{x}_n$$

$$= \left(\sum_{i=0}^{\infty} \nabla^i\right)\ddot{x}_n \qquad (319)$$

so that

$$c_n(-1) = 1 \qquad (320)$$

for all n $(0 \leq n \leq m)$. A recursive formula for computing $c_n(s)$ is given[33] by

$$c_n(s) = \frac{n-s-1}{n}c_{n-1}(s), \qquad s \geq -1, \qquad 1 \leq n \leq m \qquad (321)$$

where

$$c_0(s) = 1 \qquad (322)$$

for all s.

It remains to find the coefficients $d_i(s)$. If the order of derivatives of Eq. (283) is raised by two, it follows that

$$\ddot{x}_{n-s} = h\left[\frac{-(1-\nabla)^s}{\ln(1-\nabla)}\right]\dddot{x}_n$$

Replacing first n by $n + s$, and then replacing s by $-s$, one obtains

$$\dddot{x}_{n-s} = h^{-1}[-(1-\nabla)^s \ln(1-\nabla)]\ddot{x}_n \qquad (323)$$

[33]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

Clearly,

$$-(1-\nabla)^s \ln(1-\nabla) = \nabla + \left[\frac{1}{2} - \binom{s}{1}\right]\nabla^2 + \left[\frac{1}{3} - \frac{1}{2}\binom{s}{1} + \binom{s}{2}\right]\nabla^3$$
$$+ \left[\frac{1}{n} - \frac{1}{n-1}\binom{s}{1} + \cdots + (-1)^j \frac{1}{n-j}\binom{s}{j} + \cdots + (-1)^{n-1}\binom{s}{n-1}\right]\nabla^n + \cdots$$

Thus,

$$d_1(s) = 1, \qquad \text{for all } s$$

$$d_2(s) = \frac{1}{2} - \binom{s}{1}$$

$$d_3(s) = \frac{1}{3} - \frac{1}{2}\binom{s}{1} + \binom{s}{2}$$

and, in general,

$$d_n(s) = \frac{1}{n} - \frac{1}{n-1}\binom{s}{1} + \cdots + (-1)^j \frac{1}{n-j}\binom{s}{j} + \cdots + (-1)^{n-1}\binom{s}{n-1}$$
$$= \sum_{j=0}^{n-1} \frac{1}{n-j} c_j(s) \qquad (324)$$

where the $c_j(s)$ terms are given by Eqs. (318) and (320) or by Eq. (321). Equation (324) is the desired formula for computing the $d_i(s)$ terms in Eq. (291).

E. Starting Procedures

The computation of the solution at $m+1$ time points to the left of the starting point t_0 requires a special starting procedure; this can be achieved by two methods:

(1) Taylor series expansion.

(2) Extrapolation.

1. Start by Taylor series expansion. The Taylor series starting procedure, used to generate a first approximation to the solution of

$$\ddot{x} = f(x, \dot{x}, t), \qquad x \to y, z \qquad (325)$$

near t_0, specifically assumes that f has the structure

$$f(x, \dot{x}, t) = -x\frac{\mu}{r^3} + g(x, \dot{x}, t), \qquad x \to y, z \qquad (326)$$

where

$\mu = GM$ (gravitational constant of central body)
$r = (x^2 + y^2 + z^2)^{1/2}$

and

$$|f| \gg |g| \qquad (327)$$

that is, a two-body approximation to the actual equations of motion is being used. Therefore, difficulties may be anticipated when g is not small compared to f; in such a case, the starting procedure described below may converge only very slowly, or may not converge at all. This problem occurs, for example, at a change of phase because it is then attempted to solve what is essentially a three-body problem as if it were a two-body problem; the assumption made in the above inequality is then no longer valid.

Let it be assumed now that inequality (Eq. 327) holds, and that initial conditions x_0 and $\dot{x}_0$ are given at time t_0; the formulas to be used are of index m (i.e., ∇^m is the highest-order backward difference used). The goal of the starting procedure may be regarded as the establishment of the backward difference line at $\ddot{x}_0$. This requires the determination of $m+3$ quantities; namely, $\nabla^i \ddot{x}_0$, for $i = -2, -1, 0, 1, \cdots, m$. An equivalent set of $m+3$ quantities is $\ddot{x}_0, \ddot{x}_{-1}, \cdots, \ddot{x}_{-m}$, $\nabla^{-1}\ddot{x}_0$, and $\nabla^{-2}\ddot{x}_0$. The quantity $\ddot{x}_0$ is immediately available because $\ddot{x}_0 = f(x_0)$.

The remaining $m+2$ quantities must satisfy the $m+2$ simultaneous equations obtained by using Eq. (282) with $n = 0$, $s = 0$. To compute $\dot{x}_0$, which is used for determining $\nabla^{-1}\ddot{x}_0$, see Eq. (281) and Eq. (284) with $n = 0$, $s = 0, 1, \cdots, m$ (it should be noted that this system of $m+2$ equations is nonlinear if f is nonlinear; hence, it must generally be solved by some iterative method).

The specific organization of the starting procedure is as described below (Ref. 16, p. 35). To obtain the starting algorithm, proceed as follows:

(1) Obtain a step size h (see Section VIII-F).

(2) Assume two-body motion, and compute Taylor series coefficients $x_0^{(j)}/j!$ at t_0 (for the actual computation of these coefficients, see Section VIII-E-2).

(3) Compute an initial estimate for x_i and $\dot{x}_i, i = -1, \cdots, -m$ by use of the Taylor series approximation

$$x_i = x(t_0 + \Delta t_i) = \sum_{j=0}^{6} \frac{x_0^{(j)}}{j!} (\Delta t_i)^j$$

$$\dot{x}_i = \dot{x}(t_0 + \Delta t_i) = \sum_{j=0}^{6} \frac{x_0^{(j)}}{j!} j(\Delta t_i)^{j-1}$$

$$= \sum_{j=0}^{6} \frac{x_0^{(j)}}{(j-1)!} (\Delta t_i)^{j-1}$$

where $\Delta t_i = ih\,(i = -1, \cdots, -m)$ and $x_0^{(j)}$ is the jth derivative of x at t_0. The quantity $x_0^{(j)}/j!$ is given by Eq. (368).

Also, compute $\ddot{x}_0 = f(x_0)$; let x_i^ν and $\dot{x}_i^\nu$ denote the νth estimate of x_i and of $\dot{x}_i$ (thus, initially, $\nu = 0$).

(4) Compute the accelerations

$$\ddot{x}_i^\nu = f(x_i^\nu), \qquad i = -1, -2, \cdots, -m$$

for points $t_{-1}, t_{-2}, \cdots, t_{-m}$.

(5) Compute a backward difference line at t_0.

(6) Compute $\nabla^{-1}f(x_0) = \nabla^{-1}\ddot{x}_0$ using Eq. (282), with $s = 0$, to obtain

$$\nabla^{-1}f(x_n) = \nabla^{-1}\ddot{x}_n = h^{-1}\dot{x}_n - \sum_{l=0}^{m} b_l(0) \nabla^l \ddot{x}_n \tag{328}$$

Hence, with $n = 0$,

$$\nabla^{-1}f(x_0) = \nabla^{-1}\ddot{x}_0 = h^{-1}\dot{x}_0 - \sum_{l=0}^{m} b_l(0) \nabla^l \ddot{x}_0 \tag{329}$$

(7) Compute $\nabla^{-2}f(x_0) = \nabla^{-2}\ddot{x}_0$ using Eq. (284), with $s = 0$,

$$\nabla^{-2}f(x_n) = \nabla^{-2}\ddot{x}_n = h^{-2}x_n - \sum_{l=-1}^{m} a_l(0) \nabla^l \ddot{x}_n \tag{330}$$

so that, with $n = 0$,

$$\nabla^{-2}f(x_0) = \nabla^{-2}\ddot{x}_0 = h^{-2}x_0 - \sum_{l=-1}^{m} a_l(0) \nabla^l \ddot{x}_0 \tag{331}$$

(8) Compute new estimates of x_{-i}^ν, $i = 1, \cdots, m$, using Eq. (284), with $s = 1, 2, \cdots, m$ and $n = 0$, to obtain $x_{-i}^{\nu+1}$, $i = 1, \cdots, m$; i.e.,

$$x_i^{\nu+1} = h^2 \sum_{l=-2}^{m} a_l(s) \nabla^l \ddot{x}_i^\nu \tag{332}$$

Similarly, compute $\dot{x}_{-i}^{\nu+1}$, $i = 1, \cdots, m$ (the new estimates of x_{-i} using Eq. (282), with $s = 1, 2, \cdots, m$, and $n = 0$, so that

$$\dot{x}_{-i}^{\nu+1} = h \sum_{l=-1}^{m} b_l(s) \nabla^l \ddot{x}_0^\nu \tag{333}$$

(9) Go back to step (4).

The algorithm is complete when the difference between two successively computed difference lines is "small;" i.e., when E_ν, the maximum relative error at the νth iteration,

$$E_\nu = \max E_\nu(\kappa), \qquad \kappa = x,y,z$$

where

$$E_\nu(x) = \frac{\max\limits_{-2 \le l \le m} \left|(\nabla^l \ddot{x}_0^\nu) - (\nabla^l \ddot{x}_0^{\nu-1})\right|}{\max\limits_{-2 \le l \le m} \left|(\nabla^l \ddot{x}_0^{\nu-1})\right|}, \qquad x \to y,z \tag{334}$$

is less than some specified tolerance ϵ.

In general, a maximum number of η iterations is allowed. If η iterations have not yielded convergence, appropriate steps must be taken; e.g., reduction of the mesh size h (see Section VIII-F-2).

The starting method described above is used at the very beginning of a trajectory and whenever a discontinuity is encountered (e.g., when the spacecraft enters or leaves the shadow of a body, when a finite motor burn occurs, etc.). An exception is the case of an instantaneous maneuver (motor burn or spring separation), when an extrapolative restart method, which is described in the next section, is used.

Let it be supposed that the integration is stepping from t_{n-1} to t_n, and that, in this time interval (say, at t'), a discontinuity occurs to which the Taylor series starting procedure is applicable. The present integration step is then completed, a new step size h' is computed (see Section VIII-F-2), and a new solution is started at the discontinuity (or in the near neighborhood of the discontinuity). The initial values x' and $\dot{x}'$ corresponding to t' are obtained by interpolation, by use of Eqs. (288) and (289), as described below. Let

$$s' = \frac{t_n - t'}{h} \tag{335}$$

(it should be noted that, in general, s' is not an integer). Then

$$x' = x(t') = h^2 \sum_{i=-2}^{m} a_i(s') \nabla^i \ddot{x}_n \tag{336}$$

$$\dot{x}' = \dot{x}(t') = h^2 \sum_{i=-1}^{m} b_i(s') \nabla^i \ddot{x}_n \tag{337}$$

where the coefficients $a_i(s')$ and $b_i(s')$ are computed[34] as follows:

$$a_i(s') = \sum_{j=-2}^{i} a_j(0)\, c_{i-j}(s'), \qquad i = -2, -1, 0, 1, \cdots, m \tag{338}$$

where

$$\left.\begin{aligned} c_n(s') &= \frac{n - s' - 1}{n} c_{n-1}(s') \\ c_0(s') &= 1 \end{aligned}\right\} \tag{339}$$

for all s' (compare with Eqs. 321 and 322), and $a_j(0)$, $j = -2, \cdots, i$, are given by Eqs. (305) and (306), with $s = 0$; also,

$$b_i(s') = \sum_{j=-1}^{i} b_j(0)\, c_{i-j}(s') \tag{340}$$

[34]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

where the $b_j(0)$ terms are given by Eqs. (314) and (315), with $s = 0$, and the $c_n(s')$ terms are given by Eq. (339).

Defining $h \equiv h'$, $t_0 \equiv t'$, $x_0 \equiv x'$, and $\dot{x}_0 \equiv \dot{x}'$, one should go back to step (2) of the algorithm described above to start a new solution at t'.

If the discontinuity is time-dependent (e.g., a finite motor burn), it is easy to determine t. However, if the discontinuity is position-dependent (e.g., when the spacecraft is entering the shadow of a body), the time of the occurrence of the discontinuity must then be computed (usually by some iterative method).

It should be noted that some of the mathematically treated discontinuities are not genuine discontinuities in the physical sense. For example, consider a phase change; the actual spacecraft trajectory is smooth as the spacecraft leaves the sphere of influence of one body and enters that of another. However, mathematically speaking, the spacecraft has encountered a discontinuity: a new force is acting upon the spacecraft beginning at a certain time t. Another imaginary discontinuity occurs when the oblateness of a planet or the moon becomes a significant term in the total acceleration of the spacecraft. At some point in time, oblateness is "turned on"—a discontinuity in the mathematical description of the motion of the spacecraft; from the physical point of view, however, oblateness has been contributing to the acceleration of the spacecraft at all previous time points.

2. ***Computation of Taylor series coefficients.*** Starting the integration process by the method of a Taylor series expansion requires the computation of the first n coefficients of this infinite series. It was stated above that, if the Taylor series starting method is used, the actual differential equation

$$\ddot{x} = -x\frac{\mu}{r^3} + g(x, \dot{x}, t), \qquad x \rightarrow y, z \tag{341}$$

is approximated by the two-body equation

$$\ddot{x} = -x\frac{\mu}{r^3}, \qquad x \rightarrow y,z, \qquad r = (x^2 + y^2 + z^2)^{1/2} \tag{342}$$

Moreover, it may be assumed that the initial values x_0 and $\dot{x}_0$ are given at the time t_0.

For short intervals of time, the solution of Eq. (342) can be expanded in a Taylor series about t_0; i.e.,

$$x(t) = x_0 + \dot{x}_0 (t - t_0) + \frac{1}{2!}\ddot{x}_0(t - t_0)^2 + \cdots$$

$$+ \frac{1}{n!} x_0^{(n)} (t - t_0)^n + \cdots \tag{343}$$

Because Eq. (342) may be written

$$\ddot{x} = Fx \tag{344}$$

where

$$F = -\frac{\mu}{r^3} \tag{345}$$

(and $\ddot{x}_0 = F_0 x_0$, which defines F_0), by successively differentiating Eq. (344) one obtains

$$\dddot{x} = \dot{F}x + F\dot{x} \tag{346a}$$

$$x^{(4)} = \ddot{F}x + 2\dot{F}\dot{x} + F\ddot{x} = (\ddot{F} + F^2)\, x + 2\dot{F}\dot{x} \tag{346b}$$

$$x^{(5)} = (\dddot{F} + 4F\dot{F})\, x + (3\ddot{F} + F^2)\, \dot{x} \tag{346c}$$

$$\begin{aligned} x^{(6)} &= (F^{(4)} + 4F\ddot{F} + 4\dot{F}^2 + 3F\ddot{F} + F^3)\, x \\ &\quad + (\dddot{F} + 4F\dot{F} + 3\ddot{F} + 2F\dot{F})\, \ddot{x} \\ &= (F^{(4)} + 7F\ddot{F} + 4\dot{F}^2 + F^3)\, x + (4\dddot{F} + 6F\dot{F})\, \dot{x} \end{aligned} \tag{346d}$$

where (in this subsection)

$$\begin{aligned} x^{(n)} &\equiv \frac{d^n x}{dt^n} \\ F^{(n)} &\equiv \frac{d^n F}{dt^n} \end{aligned} \tag{347}$$

Clearly, this process of successive differentiation can be continued indefinitely. However, working only to the sixth order, and substituting Eq(s). (346) into Eq. (343), one obtains

$$\begin{aligned} x(t) &= x_0 + \dot{x}_0 (t - t_0) + \frac{1}{2!} F_0 x_0 (t - t_0)^2 + \frac{1}{3!} (\dot{F}_0 x_0 + F_0 \dot{x}_0)(t - t_0)^3 + \frac{1}{4!} [(\ddot{F}_0 + F_0^2)\, x_0 + 2\dot{F}_0 \dot{x}_0] (t - t_0)^4 \\ &\quad + \frac{1}{5!} [(\dddot{F}_0 + 4F_0\dot{F}_0)\, x_0 + (3\ddot{F}_0 + F_0^2)\, \dot{x}_0] (t - t_0)^5 \\ &\quad + \frac{1}{6!} [(F_0^{(4)} + 7F_0\ddot{F}_0 + 4\dot{F}_0^2 + F_0^3)\, x_0 + (4\dddot{F}_0 + 6F_0\dot{F}_0)\, \dot{x}_0] (t - t_0)^6 \end{aligned} \tag{348}$$

From collecting coefficients of x_0 and $\dot{x}_0$, it follows that

$$\begin{aligned} x(t) &= \Big[1 + \frac{1}{2!} F_0 (t - t_0)^2 + \frac{1}{3!} \dot{F}_0 (t - t_0)^3 + \frac{1}{4!} (\ddot{F}_0 + F_0^2)(t - t_0)^4 \\ &\quad + \frac{1}{5!} (\dddot{F}_0 + 4F_0\dot{F}_0)(t - t_0)^5 + \frac{1}{6!} (F_0^{(4)} + 7F_0\ddot{F}_0 + 4\dot{F}_0^2 + F_0^3)(t - t_0)^6 \Big] x_0 \\ &\quad + \Big[(t - t_0) + \frac{1}{3!} F_0 (t - t_0)^3 + \frac{2}{4!} \dot{F}_0 (t - t_0)^4 + \frac{1}{5!} (3\ddot{F}_0 + F_0^2)(t - t_0)^5 + \frac{1}{6!} (4\dddot{F}_0 + 6F_0\dot{F}_0)(t - t_0)^6 \Big] \dot{x}_0 \\ &\equiv f(t)\, x_0 + g(t)\, \dot{x}_0 \end{aligned} \tag{349}$$

where

$$\begin{aligned} f(t) &\equiv \sum_{j=0}^{6} a_j (t - t_0)^j \\ g(t) &\equiv \sum_{j=0}^{6} b_j (t - t_0)^j \end{aligned} \tag{350}$$

The a_j and b_j terms may be expressed in terms of F and its derivatives, evaluated at $t = t_0$, by comparison of Eq. (349) with Eq(s). (350); thus, it remains to compute the derivatives of F. Because

$$F = -\mu r^{-3}, \qquad r = (x^2 + y^2 + z^2)^{1/2} \tag{351}$$

it follows that

$$\dot{F} = 3\mu r^{-4}\dot{r} = 3\mu r^{-5}(r\dot{r}) \tag{352}$$

where $r\dot{r} = x\dot{x} + y\dot{y} + z\dot{z}$ is obtained by differentiating $r^2 = x^2 + y^2 + z^2$ with respect to time. Then

$$\ddot{F} = 3\mu\left[-5r^{-6}\dot{r}(r\dot{r}) + r^{-5}\frac{d}{dt}(r\dot{r})\right]$$

$$= 3\mu\left[-5r^{-7}(r\dot{r})^2 + r^{-5}\frac{d}{dt}(r\dot{r})\right] \tag{353}$$

From Fig. 7, one finds that

$$\dot{\mathbf{r}} \cdot \mathbf{r} = \mathbf{v} \cdot \mathbf{r} = |\mathbf{v}| \cdot |\mathbf{r}| \cos(v,r) = vr\cos(v,r) = r\dot{r} \tag{354}$$

so that

$$\frac{d}{dt}(r\dot{r}) = \frac{d}{dt}(\mathbf{r} \cdot \dot{\mathbf{r}}) = v^2 + \mathbf{r} \cdot \ddot{\mathbf{r}}$$

$$= v^2 - \frac{\mu}{r} \tag{355}$$

The vis-viva integral gives

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a} \tag{356}$$

where a is the semimajor axis of the conic on which the spacecraft moves.

Defining

$$2C \equiv -\frac{\mu}{a} \tag{357}$$

and substituting this quantity into Eq. (356) yields

$$v^2 = \frac{2\mu}{r} + 2C \tag{358}$$

Thus,

$$\frac{d}{dt}(r\dot{r}) = \frac{\mu}{r} + 2C \tag{359}$$

and Eq. (353) may be written as

$$\ddot{F} = 3\mu\left[-5r^{-7}(r\dot{r})^2 + \mu r^{-6} + 2Cr^{-5}\right] \tag{360}$$

Then

$$\dddot{F} = 3\mu\left[35r^{-9}(r\dot{r})^3 - 10r^{-7}(r\dot{r})\left(\frac{\mu}{r} + 2C\right) - 6\mu r^{-8}(r\dot{r}) - 10Cr^{-7}(r\dot{r})\right]$$

$$= 3\mu\left[35r^{-9}(r\dot{r})^3 - 16\mu r^{-8}(r\dot{r}) - 30Cr^{-7}(r\dot{r})\right] \tag{361}$$

Differentiating Eq. (361) once more yields

$$F^{(4)} = 3\mu\left[-315r^{-11}(r\dot{r})^4 + 233\mu r^{-10}(r\dot{r})^2 + 420Cr^{-9}(r\dot{r})^2 - 13\mu^2r^{-9} - 62C\mu r^{-8} - 60C^2r^{-7}\right] \tag{362}$$

The vis-viva integral (Eq. 358) may now be used to eliminate $2C$ from Eqs. (353), (361), and (362). This gives

$$\ddot{F} = 3\mu(-5r^{-5}\dot{r}^2 - \mu r^{-6} + v^2r^{-5}) \tag{363}$$

$$\dddot{F} = 3\mu(35r^{-6}\dot{r}^3 + 14\mu r^{-7}\dot{r} - 15v^2r^{-6}\dot{r}) \tag{364}$$

$$F^{(4)} = 3\mu(-315r^{-7}\dot{r}^4 - 187\mu r^{-8}\dot{r}^2 + 210v^2r^{-7}\dot{r}^2 - 14\mu^2r^{-9} + 29\mu v^2r^{-8} - 15v^4r^{-7}) \tag{365}$$

If Eqs. (345), (352), and (363) through (365) are now used in Eq. (349), and compared with Eq(s). (350), one finds that

$$a_0 = 1 \tag{366a}$$

$$a_1 = 0 \tag{366b}$$

$$a_2 = \frac{1}{2!}F_0 = -\frac{1}{2}\mu r_0^{-3} \tag{366c}$$

$$a_3 = \frac{1}{3!}\dot{F}_0 = \frac{1}{2}\mu r_0^{-4}\dot{r}_0 \tag{366d}$$

$$a_4 = \frac{1}{4!}(\ddot{F}_0 + F_0^2)$$

$$= \frac{1}{24}(-15\mu r_0^{-5}\dot{r}_0^2 - 2\mu^2r_0^{-6} + 3\mu v_0^2 r_0^{-5}) \tag{366e}$$

$$a_5 = \frac{1}{5!}(\dddot{F}_0 + 4F_0\dot{F}_0)$$

$$= \frac{1}{8}(7\mu r_0^{-6}\dot{r}_0^3 + 2\mu^2 r_0^{-7}\dot{r}_0 - 3\mu v_0^2 r_0^{-6}\dot{r}_0) \tag{366f}$$

$$a_6 = \frac{1}{6!}(F_0^{(4)} + 7F_0\ddot{F}_0 + 4\dot{F}_0^2 + F_0^3)$$

$$= \frac{1}{720}(-945\mu r_0^{-7}\dot{r}_0^4 - 420\mu^2\dot{r}_0^{-8} r_0^2 + 630\mu v_0^2 r_0^{-7}\dot{r}_0^2$$

$$-22\mu^3 r_0^{-9} + 66\mu^2 v_0^2 r_0^{-8} - 45\mu v_0^4 r_0^{-7}) \tag{366g}$$

and, for the coefficients b_i,

$$b_0 = 0 \tag{367a}$$

$$b_1 = 1 \tag{367b}$$

$$b_2 = 0 \tag{367c}$$

$$b_3 = \frac{1}{3!}F_0$$

$$= -\frac{1}{6}\mu r_0^{-3} \tag{367d}$$

$$b_4 = \frac{2}{4!}\dot{F}_0$$

$$= \frac{1}{4}\mu r_0^{-4}\dot{r}_0 \tag{367e}$$

$$b_5 = \frac{1}{5!}(3\ddot{F}_0 + F_0^2)$$

$$= \frac{1}{120}(-45\mu r_0^{-5}\dot{r}_0^2 - 8\mu^2 r_0^{-6} + 9\mu v_0^2 r_0^{-5}) \tag{367f}$$

$$b_6 = \frac{1}{6!}(4\dddot{F}_0 + 6F_0\dot{F}_0)$$

$$= \frac{1}{24}(14\mu r_0^{-6}\dot{r}_0^3 + 5\mu^2 r_0^{-7}\dot{r}_0 - 6\mu v_0^2 r_0^{-6}\dot{r}_0) \tag{367g}$$

Thus, Eqs. (366) and (367) represent the desired coefficients in Eq(s). (350).

Finally, comparing Eqs. (349) and (350) with Eq. (343), one finds that

$$\frac{x_0^{(i)}}{i!} = a_i x_0 + b_i \dot{x}_0, \qquad i = 0, 1, \cdots, 6 \tag{368}$$

where the coefficients a_i, b_i are given by Eqs. (366) and (367).

3. Extrapolative start. The Taylor series method for starting a multistep integration procedure is a "true" starting method because it can, in general, always be used, although certain difficulties may occur if this approach is used in some instances (see Section VIII-E-1). In contrast, an extrapolative start depends upon a previously computed difference line.

Let it be assumed that one is in the process of integrating from t_{n-1} to t_n, with step size h, and that an instantaneous maneuver occurs at time t in this time interval. The present integration step is then completed; i.e., the position x_n, velocity $\dot{x}_n$, acceleration $\ddot{x}_n$, and difference line $Dt_{n,h} = [\nabla^i \ddot{x}_n]$ are available. After a new step size h' (Fig. 37) has been determined (see Section VIII-F), position and velocity values at $t'_0 = t$ and at the m points to the left of t'_0 must be computed; i.e., at t'_{-1}, t'_{-2}, $\cdots$, t'_{-m}, where $t'_{-i} - t'_{-i-1} = h'$. This is achieved by using Eqs. (288) and (289) as described below. Let $\bar{t}$ = value of time for which interpolated values are desired, $t_n - mh \le \bar{t} \le t_n$ (clearly, $\bar{t}$ will be the points t'_0, t'_{-1}, $\cdots$, t'_{-m}), and

$$s' = \frac{t_n - \bar{t}}{h} \tag{369}$$

(s' is, in general, not an integer); then

$$x' = x(\bar{t}) = h^2 \sum_{i=-2}^{m} a_i(s')\,\nabla^i \ddot{x}_n \tag{370}$$

and

$$\dot{x}' = \dot{x}(\bar{t}) = h \sum_{i=-1}^{m} b_i(s')\,\nabla^i \ddot{x}_n \tag{371}$$

where the coefficients $a_i(s')$ and $b_i(s')$ are computed according to Eqs. (338) and (340).

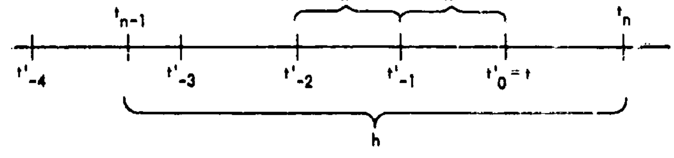

Fig. 37. Step sizes h and h'

Let $x'_0, x'_{-1}, \cdots, x'_{-m}$, and $\dot{x}'_0, \dot{x}'_{-1}, \cdots, \dot{x}'_{-m}$ denote the positions and velocities (corresponding to time points $t'_0, t'_{-1}, \cdots, t'_{-m}$) obtained from Eqs. (370) and (371). These values must be corrected for changes in position and velocity occurring during an instantaneous maneuver (see Section VII-D); hence,

$$x_i \equiv x'_i + \Delta x + \Delta t_i \, \Delta \dot{x}, \qquad i = 0, -1, \cdots, -m \tag{372}$$

$$\Delta t_i = t'_0 - t'_i$$

$$\dot{x}_i \equiv \dot{x}'_i + \Delta \dot{x}, \qquad i = 0, -1, \cdots, -m \tag{373}$$

where Δx, $\Delta \dot{x}$ must be supplied (see Section VII-D). The values obtained from Eqs. (372) and (373) may then be used to compute a new backward difference line at $t'_0 = t$ by using the algorithm (steps 4 through 9) described in Section VIII-E-1. Once a difference line has been found to the desired accuracy, the actual integration process is started to advance the solution from t'_0 to t'_1.

F. Control of Step Size h

When a numerical integration procedure is applied to solve a differential equation, it is most important to select the optimal step size h. Two principal factors determine the choice of h:

(1) Accuracy.

(2) Integration time (and cost).

In general, the accuracy of the solution increases as h decreases (to a certain limit, that is; if h becomes too small, round-off error grows). However, the smaller the h, the more integration steps are required to integrate over a given interval; hence, the total integration time increases as h decreases.

Two methods are used, as described in the two subsections that follow, to select a step size h: (1) finding h from a range list and (2) computing h by an automatic step-size control procedure. For stability reasons, a maximum and a minimum step size h_{max} and h_{min} must be given; these two quantities limit the largest and smallest h that may ever be attained by either of the two methods for controlling step sizes.

1. Step-size control by range list. Associated with each celestial body is a range table and a step size (in seconds). A range table consists of a series of up to 12 increasing positive numbers, which are used to define a pattern of concentric annuli centered at the given body. The step size h associated with such a pattern is used to generate a series of step sizes, each of which is internal to a particular annulus. The step size h itself is used within the central annulus. In the nth annulus, the step size to be used is $h \cdot 2^{n-1}$. Thus, in passing from inner annuli to adjacent outer ones, the step size doubles.

A step size is obtained from the range list at the very beginning of a trajectory, and also at physical central body changes. If the spacecraft is entering the sun phase, the step size is left unchanged. This is because the step size, as determined from the range table for the sun, is usually many times larger than the step size at the same point in space as determined by the range table of the celestial body from whose sphere of influence the spacecraft is emerging. A sudden increase in step size of more than two or three times would introduce errors of such magnitude that the starting procedure would fail to converge. If the spacecraft is entering the sphere of influence of a celestial body other than the sun, the step size is determined from the range tables in the manner described above.

Because the position vector of the spacecraft is referred to the integration central body, and because the integration center may be different from the physical center, a translation is made to the physical central body (PCB). If the PCB has not changed, and the spacecraft is still in the same annulus, the step size need not be recomputed because the original computation would still be available.

When the solar pressure, attitude control, or motor burn models are on, a maximum step size (specified by input) will be passed to the integrator. If the step size, as determined from the range tables, exceeds this maximum, the maximum itself is then returned as the step size to be used.[35]

In other words, the step size associated with a discontinuity always overrides the step size computed from the range list. This is an important point because the integration process has to be restarted whenever a discontinuity occurs in the acceleration of the spacecraft, and, for the starting procedure to converge, the step size must be sufficiently small.

To compute a new difference line for $h' = 2h$, every second acceleration of the last 10 solution points is saved, and another 10 integration steps are taken (with the old h), every second one of which is saved. Hence, at $t'' = t' + 10h$, 10 accelerations are available with a spacing of $h' = 2h$. A new backward difference line is then formed at t'' by use

[35]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

of the 10 points that are separated by a spacing h', and the integration process proceeds with step size h'. Similarly, the current step size h is halved when the spacecraft is moving towards the integration central body; i.e., moving from an outer annulus to an inner annulus. In this case, a new difference line with the new step size $h' = h/2$ (where h is the old step size) is computed according to the following expressions:[36]

$$\nabla^{i}_{h'} \ddot{x}(t) = \sum_{j=i}^{m} H_{i,j} \nabla^{j}_{h} \ddot{x}(t), \qquad i = 1, \cdots, m \tag{374}$$

$$\nabla^{0}_{h'} \ddot{x}(t) = \nabla^{0}_{h} \ddot{x}(t) \tag{375}$$

$$\nabla^{-1}_{h'} \ddot{x}(t) = (h')^{-1} \dot{x}(t) - \sum_{i=0}^{m} b_{i+1}(0) \nabla^{i}_{h'} \ddot{x}(t) \tag{376}$$

$$\nabla^{-2}_{h'} \ddot{x}(t) = (h')^{-2} x(t) + \nabla^{-1}_{h'} \ddot{x}(t) - \sum_{i=0}^{m} a_{i+2}(0) \nabla^{i}_{h'} \ddot{\mathbf{x}}_n \tag{377}$$

Equations (376) and (377) follow immediately from Eqs. (289) and (290), with $s = 0$.

The coefficients $H_{i,j}$ in Eq. (374) are given[37] by

$$H_{1,1} = \frac{1}{2} \tag{378a}$$

$$H_{i,j} = \left(1 - \frac{3}{2j}\right) H_{i,j-1}, \qquad j = 2, \cdots, m \tag{378b}$$

$$H_{i,j} = \sum_{k=1}^{j-i+1} H_{1,k} H_{i-1,j-k}, \qquad i = 2, \cdots, m; \qquad j = i, \cdots, m \tag{378c}$$

The first time step in the inner annulus is denoted by $t = t'$. The integration process is then continued at time t' using the backward differences computed in Eqs. (374–377).

2. ***Automatic step-size control.*** When the integrator is operating under automatic step-size control (ASC), it is checked at each integration step to see whether h should be reduced, doubled, or remain unchanged. The quantity that determines the change (if any) to be made in h is the *local truncation error*, which is defined as the difference between the exact solution of the difference equation and that of the differential equation. Automatic control over the truncation error during the step-by-step integration ensures that the difference equations will represent a good approximation to the differential equations; step-size control is a good method to limit truncation error in an Adams-type numerical integration method.

The truncation error E in a difference line is approximated by the relative amount that the first neglected difference would have contributed to the position vector[38]

$$E = \frac{h^2 \,|a_{m+1}|\, |\nabla^{m+1} \mathbf{r}|}{|\mathbf{r}|} \tag{379}$$

where

$$\ddot{\mathbf{r}} = (\ddot{x}, \ddot{y}, \ddot{z})$$

$$\mathbf{r} = (x, y, z)$$

$$h = \text{current step size}$$

$$a_{m+1} = \begin{cases} a^{(0)}_{m+1} & \left\{ \begin{array}{l} (m+1)\text{st coefficient of Eq. (288)} \\ \text{computed according to Eq. (306),} \\ \text{with } s = 0\text{, if predictor and corrector} \\ \text{are used in integration procedure} \end{array} \right. \\ a^{(1)}_{m+1} & \left\{ \begin{array}{l} (m+1)\text{st coefficient of Eq. (288)} \\ \text{computed according to Eq. (306),} \\ \text{with } s = 1\text{, if only predictor formulas} \\ \text{are applied; or predictor formulas} \\ \text{are applied and corrector formulas} \\ \text{are applied to solution } x(t) \text{ and } \dot{x}(t)\text{,} \\ \text{but not to difference line} \end{array} \right. \end{cases}$$

Two given quantities E_{max} and E_{min} are the upper and lower bounds on truncation error; throughout the integration of a trajectory, the truncation error is kept between E_{min} and E_{max} by appropriately changing h. After each corrector cycle, the truncation error is computed according to Eq. (379).

The current step size h will be reduced to a new h' if

$$E_{max} \leq E \tag{380a}$$

and

$$h_{min} \leq h' \tag{380b}$$

[36]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.
[37]*Ibid.*

[38]Talbot, T., JPL internal document, Feb. 3, 1969.

Step size h will be doubled if

$$E_{min} \geq E \tag{381a}$$

and

$$2h \leq h_{max} \tag{381b}$$

where E is the truncation error.

At time t_n, if $E \leq E_{min}$, then 10 more steps are taken at the current step size h. Provided that the condition $E < E_{min}$ remains satisfied for each of these 10 steps, a new difference line is formed for a spacing $2h$ by differencing every second acceleration of the last 20 accelerations (which were computed by use of h). On the other hand, if $E > E_{max}$ at time t_n, the current step size h_{old} must be reduced to, say, h' to keep E between E_{min} and E_{max}. The change to a smaller step size requires a complete restart of the integration procedure, as described in Section VIII-E-1 (see Eq. 383, below, for the computation of a reduced step size h').

When restarting the integration procedure, one must perform a special truncation error evaluation after the convergence of the starting procedure. This evaluation is necessary because convergence of the starting procedure is not dependent upon truncation error.

The convergence of the starting procedure represents the iterative solution of m algebraic equations in m unknowns (see Section VIII-E-1):

$$\ddot{x}(t - sh) = \sum_{i=0}^{m} c_i(s) \nabla^i x(t), \qquad 1 \leq s \leq m \tag{382}$$

(this is a repetition of Eq. 290).

The convergence of this system of equations will be a good approximation of the differential equations only if the truncation error is within the limits of E_{min} and E_{max}.

The first neglected difference, which is normally used to approximate the truncation error, is not part of the iteration computation. The *last retained* difference is used in place of the first neglected difference when computing the truncation error. This truncation error E is then compared with E_{max}. If $E \leq E_{max}$, the integration process is resumed with h'; otherwise, a new h'' is computed, using h' as the old step size, and the entire starting procedure is repeated.

The integrator will try a maximum of four reduced step sizes when doing a restart. If the converged difference line with the fourth reduced h does not satisfy $E < E_{max}$, the program will terminate with an appropriate error message.

The reduced h' is computed[39] according to

$$h' = h_{old} \left[\frac{0.25\, E_{min}}{E} \right]^{1/10} \tag{383}$$

where E is the truncation error at step size h_{old} based upon the last retained difference in Eq. (379).

G. Reverse Integration

The program has the capability to integrate a trajectory backward in time. All the formulas for forward integration apply, with the step size h now being negative for reverse integration.

IX. Differential Correction Process

The single most important tool in the definitive determination of a spacecraft trajectory from observation is that of differential correction; such procedures have been employed for the improvement of orbits (e.g., of comets) since Gauss.

Given *a priori* estimates of the parameter vector $\mathbf{q}$, the components of which are the so-called "solve-for parameters," the spacecraft acceleration is integrated (see Section VIII) using second-sum numerical integration methods to give position and velocity at any desired time. Use of the spacecraft ephemeris, along with the ephemerides for the other bodies within the solar system and the parameter vector $\mathbf{q}$, allows computation of values for each observed quantity (normally doppler, range, or angles). The residuals Δz are then computed as the difference between the actual observations and the observables computed from the ephemeris of the spacecraft. In addition to integrating the acceleration of the spacecraft to obtain its ephemeris, one integrates the partial derivative of the spacecraft acceleration with respect to the parameter vector $\mathbf{q}$, using the second-sum numerical integration procedure to give the partial derivative of the spacecraft state vector $\mathbf{X}$ (position and velocity components) with respect to $\mathbf{q}$; i.e., $\partial \mathbf{X}/\partial \mathbf{q}$. Using $\partial \mathbf{X}/\partial \mathbf{q}$, one computes the partial derivative of each computed observable quantity z with respect to $\mathbf{q}$; i.e., $\partial z/\partial \mathbf{q}$. Given the residuals Δz

[39]Talbot, T., JPL internal document, Feb. 3, 1969.

$\partial z/\partial \mathbf{q}$ and the weights applied to each residual, along with the *a priori* parameter vector and its covariance matrix (Ref. 17, p. 32), one computes the differential correction $\Delta\mathbf{q}$ to the parameter vector.

Starting with $\mathbf{q}$ and $\Delta\mathbf{q}$, one computes a new spacecraft ephemeris, residuals, and partial derivatives, and a second differential correction is obtained. This process is repeated until convergence occurs and the weighted sum of squares of residual errors between observed and computed quantities is minimized (Ref. 18, p. 24).

Differential correction is applied for two conceptually different purposes. One application considers the effect of errors in the observational data themselves. For the other application, let it be supposed that a preliminary trajectory has been computed on the basis of some simplified physical model (e.g., assume only two-body forces) or on some simplified trajectory (e.g., assume a hyperbolic or elliptic trajectory). Even if perfect observational data are given, subsequent observations would not necessarily agree with what they were computed to be on the basis of the preliminary trajectory.

The quantities that are usually differentially corrected are *a priori* estimates made to minimize the sum of weighted squares of residual errors between observed and computed quantities. These estimates include injection parameters, physical constants (implying that they are not actually constants; that is, their numerical values are subject to improvement—e.g., the astronomical unit or the speed of light), maneuver parameters, and station locations.

A. Interpolation and Differential Correction of Basic Planetary Ephemerides

Predictions of the motion of celestial bodies can be presented in either of two forms: (1) as general but complicated formulas, with time as argument, from which position at any epoch can be computed; or (2) as tables listing discrete, prespecified epochs, from which positions at other than tabular epochs can be obtained. These tables are called ephemerides.

It has become customary to rely exclusively on ephemerides for astronomical work involving lunar and planetary motion because the labor required for the preparation of an ephemeris can be allocated to the solution of many different problems.

*1. **Interpolation.*** The basic planetary ephemeris data consist of predictions of lunar and planetary positions and of the corresponding velocities. The ephemeris data are usually given in heliocentric coordinates for the planets and the earth–moon barycenter and in geocentric coordinates for the moon. However, coordinates referred to any of the bodies as center may be obtained by a translation of centers (see Section VI). As the planetary position ephemerides are tabulated at 4-day intervals (an exception is Mercury, whose data are given in 2-day steps) and the lunar ephemeris at ½-day intervals on a standard ephemeris tape used at JPL, it is necessary to use an interpolation scheme to obtain intermediate values of positions and velocities. An Everett's formula that uses second and fourth differences is usually employed for the positions and velocities; the formula used (Ref. 19, p. 273) is given by

$$x(T_j) = ux_0 + tx_1 + \frac{u(u^2-1)}{3!}\Delta^2_{m0}x_0 + \frac{t(t^2-1)}{3!}\Delta^2_{m1}x_1 + \frac{u(u^2-1)(u^2-4)}{5!}\Delta^4_{m0}x_0 + \frac{t(t^2-1)(t^2-4)}{5!}\Delta^4_{m1}x_1 \tag{384}$$

where

T_j = desired Julian date, $T_i \le T_j < T_i + h$

h = step size of data

T_i = point in time at which data are tabulated

$t = \dfrac{(T_j - T_i)}{h}, \quad 0 \le t \le 1$

$u = 1 - t$

$x_0 = x(T_i)$

$x_1 = x(T_i + h)$

Δ^n_{mi} = nth modified difference

The modified differences are intended to facilitate the use of Everett's fifth-order interpolation formula by "throwing back" sixth- and eighth-order differences on the second and fourth differences:

$$\Delta^2_{mi} = \delta^2_i + a_{26}\delta^6_i + a_{28}\delta^8_i \tag{385}$$

$$\Delta^4_{mi} = \delta^4_i + a_{46}\delta^6_i + a_{48}\delta^8_i \tag{386}$$

where

δ_i^k = ordinary central difference of order k

$a_{26} = -0.013120$

$a_{28} = 0.004299$

$a_{46} = -0.278269$

$a_{48} = 0.068489$

(Ref. 20, p. 6).

Planetary coordinates for centers other than the sun are obtained by the vector subtraction

$$\mathbf{P} = \mathbf{P}_0 - \mathbf{C} \tag{387}$$

where

$\mathbf{P}$ = planetary coordinates referred to desired center

$\mathbf{P}_0$ = planetary coordinates referred to sun

$\mathbf{C}$ = heliocentric coordinates of desired center

A corresponding vector subtraction is performed for velocity vectors.

Calculation of the heliocentric coordinates of the earth or the moon—or the geocentric or selenocentric coordinates of the sun and the planets—requires additional manipulation. Heliocentric lunar and terrestrial coordinates are obtained as

$$\mathbf{M} = \mathbf{B} + \mu_m \mathbf{L} \tag{388}$$

$$\mathbf{E} = \mathbf{B} + \mu_e \mathbf{L} \tag{389}$$

where

$\mathbf{M}$ = heliocentric coordinates of moon

$\mathbf{E}$ = heliocentric coordinates of earth

$\mathbf{B}$ = heliocentric coordinates of earth–moon barycenter

$\mathbf{L}$ = geocentric coordinates of moon

$$\mu_m = \frac{\mu_E}{\mu_E + \mu_M}$$

$$\mu_e = \frac{\mu_M}{\mu_E + \mu_M}$$

where

μ_E = gravitational constant of earth

μ_M = gravitational constant of moon

(Ref. 21, p. 19).

2. *Differential correction of basic ephemerides.* Basic ephemeris information for each of the planets, the earth–moon barycenter, and the moon can be obtained from standard ephemerides by interpolation, as described above. Because the differential correction process makes it possible to solve for corrections to the orbital elements of the ephemeris bodies, the interpolated ephemerides are also subject to correction.

This is accomplished by solving for corrections to an osculating orbit (i.e., a two-body orbit that yields the given ephemerides at a chosen epoch), then projecting these corrections to the desired epoch, assuming that this deviation from the osculating orbit may be added to the ephemerides. Thus, the corrected heliocentric position and velocity vectors of a planet or the earth–moon barycenter (see Ref. 7, p. 24) are given by

$$\mathbf{r}_{corr} = \mathbf{r}_{ephem} A_E + \frac{\partial \mathbf{r}}{\partial E} \Delta E, \quad \text{km} \tag{390}$$

$$\dot{\mathbf{r}}_{corr} = \dot{\mathbf{r}}_{ephem} A_E + \frac{\partial \dot{\mathbf{r}}}{\partial E} \Delta E, \quad \text{km/s} \tag{391}$$

Similarly, the corrected geocentric position and velocity vectors of the moon are given by

$$\mathbf{r}_{corr} = \mathbf{r}_{ephem} R_E + \frac{\partial \mathbf{r}}{\partial E} \Delta E, \quad \text{km} \tag{392}$$

$$\dot{\mathbf{r}}_{corr} = \dot{\mathbf{r}}_{ephem} R_E + \frac{\partial \dot{\mathbf{r}}}{\partial E} \Delta E, \quad \text{km/s} \tag{393}$$

where

$\mathbf{r}_{ephem}, \dot{\mathbf{r}}_{ephem}$ = interpolated ephemeris position and velocity vectors of planet or earth–moon barycenter relative to sun in AU (Eqs. 390 and 391); interpolated ephemeris position and velocity vectors of moon relative to earth in dimensionless lunar units (LU) (Eqs. 392 and 393). The lunar unit differs only slightly from the equatorial radius of the earth.

A_E = conversion factor from AU to km

R_E = conversion factor from dimensionless LU to km

E = osculating two-body orbital elements for heliocentric ephemeris of planet or earth–moon barycenter (Eqs. 390 and 391) or geocentric lunar ephemeris (Eqs. 392 and 393)

$\partial \mathbf{r}/\partial E, \partial \dot{\mathbf{r}}/\partial E$ = partial derivatives of position and velocity, respectively, of osculating conic with respect to orbital elements (see Ref. 8, p. 241, Set III)

ΔE = solve-for corrections to osculating orbital elements

Each of these vectors has rectangular components referred to the mean earth equator and equinox of 1950.0; i.e., the x-axis is along the mean equinox of 1950.0, the z-axis is normal to the mean earth equatorial plane of 1950.0 directed north, and the y-axis completes the right-handed coordinate system.

The partial derivatives of ephemeris position and velocity components at time t with respect to orbital elements $\partial \mathbf{r}(t)/\partial E$, $\partial \dot{\mathbf{r}}(t)/\partial E$ are computed from orbital elements of an osculating conic to the ephemeris at input epoch time t_0, and from position and velocity $\mathbf{r}(t)$, $\dot{\mathbf{r}}(t)$ obtained from the uncorrected ephemeris at time t. The formulation for computing the partial derivatives is given below, along with the method of computing osculating orbital elements from position and velocity components $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$ obtained from the ephemeris at time t_0 and from the parameter μ.

B. Partial Derivatives Formulation

The six parameters that are solved for using Set III (see Ref. 8, p. 241) are $\Delta a/a$, Δe, $(\Delta M_0 + \Delta w)$, Δp, Δq, and $(e\Delta w)$, where

a = semimajor axis

e = eccentricity

M_0 = value of mean anomaly at epoch t

$\Delta p, \Delta q, \Delta w$ = right-handed rotations of $\mathbf{P},\mathbf{Q},\mathbf{W}$ coordinate system about $\mathbf{P},\mathbf{Q},\mathbf{W}$ axes, respectively, where

$\mathbf{P}$ = vector directed from focus to perifocus

$\mathbf{Q}$ = vector 90 deg in advance of $\mathbf{P}$ in osculating plane

$\mathbf{W} = \mathbf{P} \times \mathbf{Q}$ (see Fig. 7)

The partial derivatives of ephemeris position $\mathbf{r}$ with respect to the orbital elements E, $\partial \mathbf{n}/\partial E$ are defined[40] by

$$\Delta \mathbf{r} = \left[\mathbf{r} - \frac{3}{2}(t - t_0)\dot{\mathbf{r}}\right]\frac{\Delta a}{a} + [H_1\mathbf{r} + K_1\dot{\mathbf{r}}]\,\Delta e + \frac{\dot{\mathbf{r}}}{n}(\Delta M_0 + \Delta w) + (\mathbf{P}\times\mathbf{r})\,\Delta p + (\mathbf{Q}\times\mathbf{r})\,\Delta q + \frac{1}{e}\left[\mathbf{W}\times\mathbf{r} - \frac{\dot{\mathbf{r}}}{n}\right](e\Delta w) \tag{394}$$

where n is the mean motion (see Eq. 16). The auxiliary quantities H_1 and K_1 are given by

$$H_1 = \frac{r - a(1 + e^2)}{ae(1 - e^2)} \tag{395}$$

$$K_1 = \frac{r\dot{r}}{a^2n^2e}\left[1 + \frac{r}{a(1 - e^2)}\right] \tag{396}$$

The coefficients of $\Delta a/a$, Δe, $\cdots$, $(e\Delta w)$, H_1,K_1, are developed in Ref. 8, pp. 233–241.

Differentiating Eq. (394) with respect to time defines the partial derivatives as $\dot{\mathbf{r}}$ with respect to the orbital elements $\partial \dot{\mathbf{r}}/\partial E$:

$$\Delta \dot{\mathbf{r}} = \left[-\frac{1}{2}\dot{\mathbf{r}} - \frac{3}{2}(t - t_0)\ddot{\mathbf{r}}\right]\frac{\Delta a}{a} + [H_2\mathbf{r} \pm K_2\dot{\mathbf{r}}]\,\Delta e + \frac{\ddot{\mathbf{r}}}{n}(\Delta M_0 + \Delta w) + (\mathbf{P}\times\dot{\mathbf{r}})\,\Delta p + (\mathbf{Q}\times\dot{\mathbf{r}})\,\Delta q + \frac{1}{e}\left[\mathbf{W}\times\dot{\mathbf{r}} - \frac{\ddot{\mathbf{r}}}{n}\right](e\Delta w) \tag{397}$$

where

$$H_2 = \dot{H}_1 - \frac{\mu}{r^3}K_1 = \frac{\dot{r}}{ae(1 - e^2)}\left\{1 - \frac{a}{r}\left[1 + \frac{a}{r}(1 - e^2)\right]\right\} \tag{398}$$

[40] Peabody, P. R., JPL internal document, Sept. 3, 1963.

$$K_2 = \dot{K}_1 + H_1$$

$$= \frac{1}{e(1-e^2)}\left(1 - \frac{r}{a}\right) \qquad (399)$$

The partial derivatives of ephemeris position and velocity with respect to orbital elements $\partial \mathbf{r}/\partial E$ and $\partial \dot{\mathbf{r}}/\partial E$, defined in Eqs. (394) and (397), are computed at time t from osculating orbital elements a, e, n, $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{W}$, computed once at epoch t_0; from $\mathbf{r}$, $\dot{\mathbf{r}}$, interpolated from the uncorrected ephemeris at time t; and from $\ddot{\mathbf{r}}$, r, $\dot{r}$, computed from

$$r = (\mathbf{r} \cdot \mathbf{r})^{1/2} \qquad (400)$$

$$r\dot{r} = \mathbf{r} \cdot \dot{\mathbf{r}} \qquad (401)$$

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{r^3} \qquad (402)$$

It should be noted that the auxiliary quantities H_1, K_1, H_2, and K_2 must be computed at each time t. The constants a, e, n, $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{W}$ are computed from position and velocity components $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$ obtained from the ephemeris at input epoch t_0, and from μ(planet) or μ(moon) given by

$$\mu(\text{planet}) = \mu_S + \mu_p, \qquad \text{km}^3/\text{s}^2 \qquad (403)$$

where

μ_p = gravitational constant of planet

μ_S = gravitational constant of sun

and

$$\mu(\text{moon}) = \mu_E + \mu_M, \qquad \text{km}^3/\text{s}^2 \qquad (404)$$

where

μ_E = gravitational constant of earth

μ_M = gravitational constant of moon

Given $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$ from the ephemeris at t_0 in AU and AU/s, or LU and LU/s, let us compute

$$r_0 = (\mathbf{r}_0 \cdot \mathbf{r}_0)^{1/2} \qquad (405)$$

$$\dot{s}_0^2 = \dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_0 \qquad (406)$$

$$r_0\dot{r}_0 = \mathbf{r}_0 \cdot \dot{\mathbf{r}}_0 \qquad (407)$$

The semimajor axis a in AU or LU is computed from the vis-viva integral as

$$\frac{1}{a} = \frac{2}{r_0} - \frac{\dot{s}_0^2}{\mu} \qquad (408)$$

The mean motion (see Eq. 16) is given by

$$n = \frac{(\mu)^{1/2}}{a^{3/2}} \qquad (409)$$

Compute

$$e \cos E_0 = 1 - \frac{r_0}{a} \qquad (410)$$

where E_0 is the eccentric anomaly at t_0.

Equation (410) follows directly from the equation of an ellipse in polar coordinates

$$r = a(1 - e\cos E) \qquad (411)$$

Differentiation of Eq. (410) yields

$$e \sin E_0 = \frac{\dot{r}_0}{a\dot{E}_0} \qquad (412)$$

To compute $\dot{E}_0$, one should note that the mean anomaly M (see Eq. 17) is given by

$$M = M_0 + n(t - t_0)$$

$$= M_0 + a^{-3/2}(\mu)^{1/2}(t - t_0) = E - e\sin E$$

Differentiation with respect to t yields

$$\dot{M} = a^{-3/2}(\mu)^{1/2} = (1 - e\cos E)\dot{E}$$

or

$$\left(\frac{\mu}{a}\right)^{1/2} = a(1 - e\cos E)\dot{E} \qquad (413)$$

Using Eq. (411), one may rewrite Eq. (413) as

$$\dot{E} = \frac{1}{r}\left(\frac{\mu}{a}\right)^{1/2} \qquad (414)$$

By evaluating $\dot{E}$ at t_0, one obtains

$$\dot{E}_0 = \frac{1}{r_0}\left(\frac{\mu}{a}\right)^{1/2} \tag{415}$$

Substitution of Eq. (415) into Eq. (412) yields

$$e \sin E_0 = \frac{r_0 \dot{r}_0}{(\mu a)^{1/2}} \tag{416}$$

Then

$$e = [(e \cos E_0)^2 + (e \sin E_0)^2]^{1/2} \tag{417}$$

$$\cos E_0 = \frac{e \cos E_0}{e} \tag{418}$$

$$\sin E_0 = \frac{e \sin E_0}{e} \tag{419}$$

where $e \cos E_0$ and $e \sin E_0$ are given by Eqs. (410) and (416). Furthermore,

$$\mathbf{P} = \frac{\cos E_0}{r_0}\mathbf{r}_0 - \left(\frac{a}{\mu}\right)^{1/2} \sin E_0 \dot{\mathbf{r}}_0 \tag{420}$$

(see Ref. 4, p. 119), and

$$\mathbf{W} = \frac{\mathbf{r}_0 \times \dot{\mathbf{r}}_0}{[\mu a(1 - e^2)]^{1/2}} \tag{421}$$

$$\mathbf{Q} = \mathbf{W} \times \mathbf{P} \tag{422}$$

C. Correction to Ephemeris

At epoch t_1, at which the corrected ephemeris is desired, the partial derivatives of ephemeris position and velocity with respect to the orbital elements $\partial\mathbf{r}(t)/\partial E$ and $\partial\dot{\mathbf{r}}(t)/\partial E$ are computed as described in the preceding section.

The accumulated corrections to osculating orbital elements ΔE given by

$$\Delta E = \sum_{i=1}^{n-1} \begin{bmatrix} \left(\frac{\Delta a}{a}\right)_i \\ (\Delta e)_i \\ (\Delta M_0 + \Delta w)_i \\ (\Delta p)_i \\ (\Delta g)_i \\ (e\Delta w)_i \end{bmatrix} \tag{423}$$

are obtained from the previous $(n - 1)$ iterations that were performed (as described in the beginning of this section) to differentially correct the parameter vector $\mathbf{q}$. It should be recalled that the corrections to the osculating orbit are solve-for quantities. Because the ephemeris is corrected between iterations, the sum of these corrections to orbital elements should converge.[41]

X. Summary of Results Obtained From Integrating the Equations of Motion

Upon integration of the equations of motion, the following information is available:

(1) A spacecraft ephemeris (a sequential file of sum and difference arrays for position and velocity of a spacecraft).

(2) Planetary and lunar ephemerides.

(3) A list of events involving the spacecraft (such as closest approach to a certain body, attainment of a given distance from some body, etc.).

(4) A list of epochs.

It is frequently desirable (e.g., for a mission analysis) to obtain pertinent information regarding the spacecraft and the solar system at the time of occurrence of these events and epochs. This information may be conveniently subdivided into three groups:

(1) The body group, which contains items associated with a given body, such as distances from the body, position and velocity vectors in the body-centered coordinate system, etc.

(2) The conic group, which contains items associated with a conic section, such as semimajor axis, eccentricity, etc.

(3) The angle group; this group contains the angles subtended at the spacecraft by various pairs of celestial bodies.

A. Body Group[42]

In this section, and in the two sections that follow, body C is the physical central body (PCB).

Let

$$\mathbf{R}_p = (X,Y,Z) \tag{424}$$

[41]Moyer, T. D., JPL internal document, Dec. 18, 1964.
[42]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

and

$$\dot{\mathbf{R}}_p = (\dot{X},\dot{Y},\dot{Z}) \tag{425}$$

denote position and velocity vectors of the spacecraft with respect to body C. The distance R from the spacecraft to the center of body C is then given by

$$R = (X^2 + Y^2 + Z^2)^{1/2} \tag{426}$$

The declination angle Φ of the spacecraft (Fig. 38)—i.e., the angle between the spacecraft and the equator measured in a plane normal to the equator that contains the spacecraft and the center of body C—is measured in degrees, and is computed according to

$$\Phi = \sin^{-1}\left(\frac{Z}{R}\right), \qquad -90 \leq \Phi \leq 90 \tag{427}$$

The right ascension of the spacecraft Θ (see Fig. 38)—i.e., the angle measured in the plane of the equator from a fixed inertial axis in space (vernal equinox) to a plane normal to the equator (meridian) that contains the spacecraft—is also measured in degrees, and is given by

$$\Theta = \tan^{-1}\left(\frac{Y}{X}\right), \qquad 0\,\text{deg} \leq \Theta \leq 360\,\text{deg} \tag{428}$$

The velocity V of the spacecraft relative to body C is given by

$$V = (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)^{1/2} \tag{429}$$

The (X,Y,Z) coordinate system is now rotated to the up–east–north coordinate system (X',Y',Z'). This rotation is accomplished by a rotation Θ about the Z-axis, followed by a rotation Φ about the Y' axis. The (X',Y',Z') coordinate system is shown in Fig. 39. The velocity vector $\dot{R}_p^*$ of the spacecraft in the (X',Y',Z') coordinate system is given by

$$\dot{\mathbf{R}}_p^* = \begin{pmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{pmatrix} \begin{pmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{\mathbf{R}}_p$$

with

$$\dot{\mathbf{R}}_p^* = (\dot{X}_p^*,\dot{Y}_p^*,\dot{Z}_p^*)$$

The path angle of the spacecraft Γ is defined in Fig. 40 (see also Fig. 11). Since

$$V = (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)^{1/2} = [(\dot{X}^*)^2 + (\dot{Y}^*)^2 + (\dot{Z}^*)^2]^{1/2} \tag{430}$$

it follows that

$$\Gamma = \sin^{-1}\left(\frac{\dot{X}^*}{V}\right), \qquad -90\,\text{deg} \leq \Gamma \leq 90\,\text{deg} \tag{431}$$

The azimuth angle of the spacecraft Σ, also defined in Fig. 40, is then equal to

$$\Sigma = \tan^{-1}\left(\frac{\dot{Y}^*}{\dot{Z}^*}\right), \qquad 0\,\text{deg} \leq \Sigma \leq 360\,\text{deg} \tag{432}$$

If one lets

$$\mathbf{r}_p = (x,y,z) \tag{433}$$

and

$$\dot{\mathbf{r}}_p = (\dot{x},\dot{y},\dot{z}) \tag{434}$$

denote the body-fixed position and velocity vectors, respectively, of the spacecraft with respect to body C, then

$$R = (x^2 + y^2 + z^2)^{1/2} \tag{435}$$

is the distance of the spacecraft from the center of body C.

If

$$\phi = \text{latitude of spacecraft, deg} \tag{436}$$

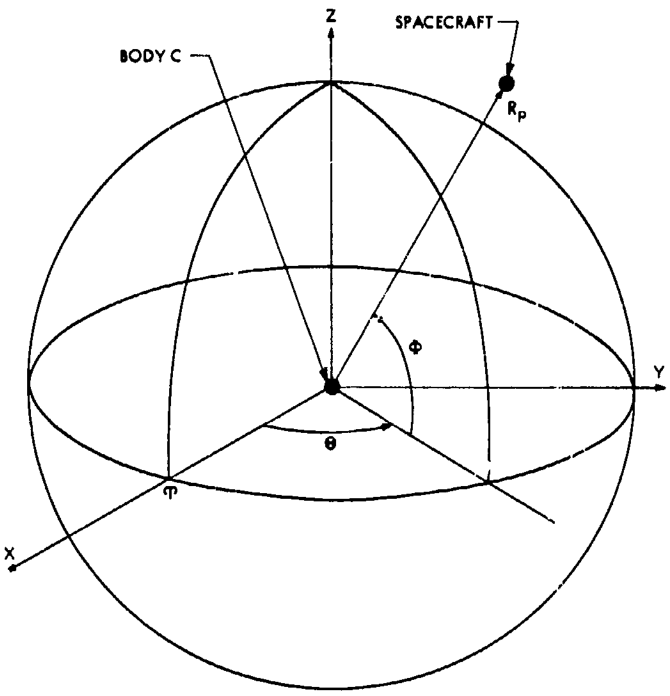

Fig. 38. Right ascension and declination of a spacecraft

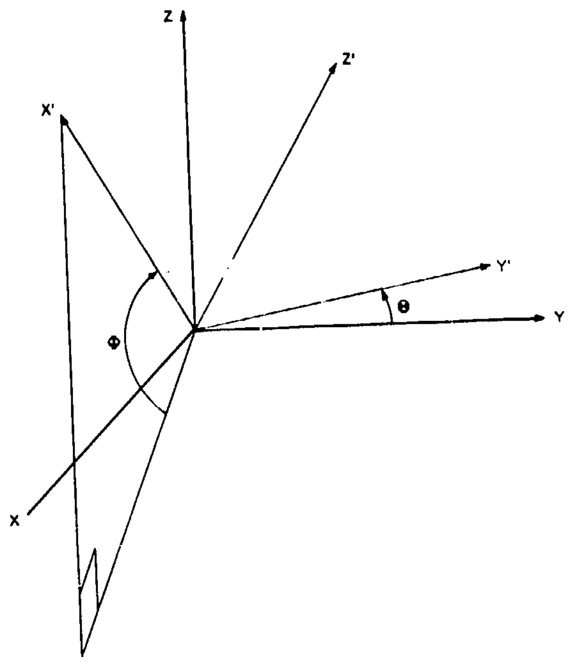

Fig. 39. The vector $\dot{\mathbf{R}}_p^*$

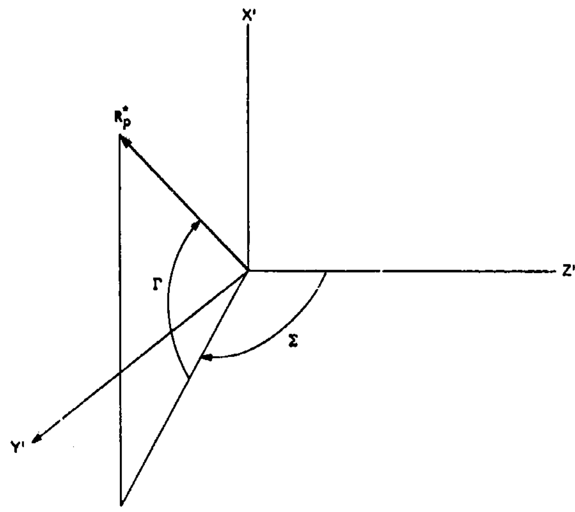

Fig. 40. Angles Γ and Σ

and

$$\theta = \text{longitude of spacecraft, deg} \tag{437}$$

it is then easily seen that

$$\phi = \sin^{-1}\left(\frac{z}{R}\right) \tag{438}$$

and

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{439}$$

The velocity v of the spacecraft relative to body C is then given by

$$v = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \tag{440}$$

Now rotate the (x,y,z) coordinate system into the up–north–east coordinate system (x',y',z'). This rotation is accomplished by rotating first about the z-axis by the angle θ and then about the y'-axis by the angle ϕ. The velocity vector $\dot{\mathbf{r}}_p^*$ in the (x',y',z') coordinate system is given by

$$\dot{\mathbf{r}}_p^* = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{\mathbf{r}}_p$$

where

$$\dot{\mathbf{r}}_p^* = (\dot{x}_p^*, \dot{y}_p^*, \dot{z}_p^*) \tag{441}$$

(Fig. 41).

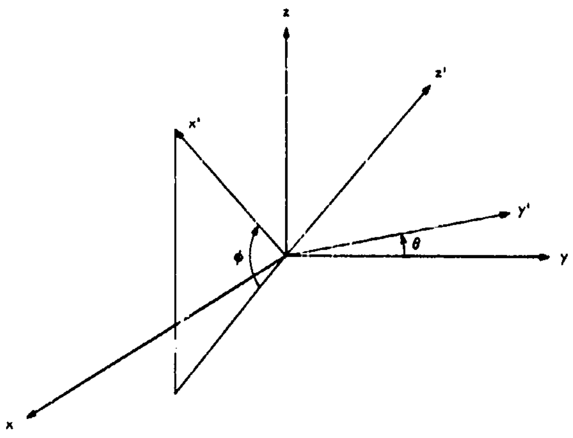

Fig. 41. The vector $\dot{\mathbf{r}}^*$

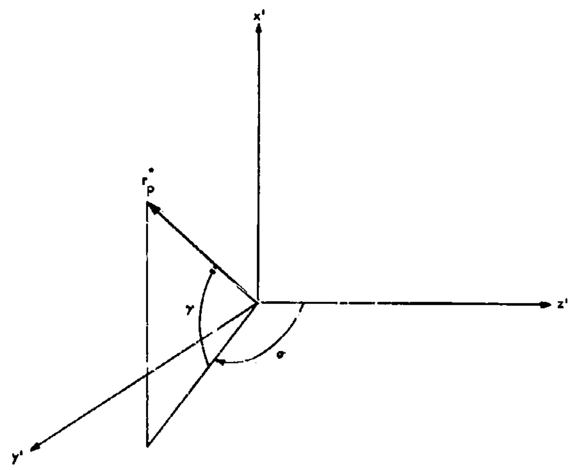

Fig. 42. Angles γ and σ

The body-fixed path angle of the spacecraft γ is shown in Fig. 42.

Clearly,

$$\gamma = \sin^{-1}\left(\frac{\dot{x}^*}{v}\right), \qquad -90 \text{ deg} \leq \gamma \leq 90 \text{ deg} \tag{442}$$

where

$$v = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} = [(\dot{x}^*)^2 + (\dot{y}^*)^2 + (\dot{z}^*)^2]^{1/2} \tag{433}$$

The body-fixed azimuth angle of the spacecraft σ, shown in Fig. 42, is computed according to

$$\sigma = \tan^{-1}\left(\frac{\dot{y}^*}{\dot{z}^*}\right), \qquad 0 \text{ deg} \leq \sigma \leq 360 \text{ deg} \tag{444}$$

It should be noted that the X-axis in Fig. 39 points in the direction of the equinox; i.e., X is *space-fixed,* whereas, in Fig. 41, the x-axis points in the direction of the intersection of prime meridian and equator of body C; i.e., x is *body-fixed.* The Greenwich hour angle H is defined by

$$H = \Theta - \theta \tag{445}$$

It is convenient to number the sun, the planets, and the moon in some definite order; a standard way of doing this is the following:

1 = Mercury
2 = Venus
3 = Earth (E)
4 = Mars
5 = Jupiter
6 = Saturn
7 = Uranus
8 = Neptune
9 = Pluto
10 = Sun (S)
11 = Moon (M)
(12 = Earth–moon barycenter)

The position vector of the spacecraft relative to body i ($i = 1,2,\cdots,11$) has components

$$\mathbf{R}_{ip} = (X_{ip}, Y_{ip}, Z_{ip}) \tag{446}$$

Hence, the distance from the center of body i to the spacecraft R_{ip} is computed according to

$$R_{ip} = (X_{ip}^2 + Y_{ip}^2 + Z_{ip}^2)^{1/2}, \qquad \text{km} \tag{447}$$

The spacecraft velocity vector $\dot{\mathbf{R}}_{ip}$ relative to body i has components

$$\dot{\mathbf{R}}_{ip} = (\dot{X}_{ip}, \dot{Y}_{ip}, \dot{Z}_{ip}) \tag{448}$$

thus, the spacecraft velocity relative to body i is given:

$$\dot{R}_{ip} = (\dot{X}_{ip}^2 + \dot{Y}_{ip}^2 + \dot{Z}_{ip}^2)^{1/2}, \qquad \text{km/s} \qquad (449)$$

Let

$$\mathbf{R}_i = (X_i, Y_i, Z_i) \qquad (450)$$

and

$$\dot{\mathbf{R}}_i = (\dot{X}_i, \dot{Y}_i, \dot{Z}_i) \qquad (451)$$

denote position and velocity vectors, respectively, of body i relative to body C. The distance from body C to body i is then given by

$$R_i = (X_i^2 + Y_i^2 + Z_i^2)^{1/2}, \qquad \text{km} \qquad (452)$$

and the declination angle Φ_i of body i relative to body C is computed from

$$\Phi_i = \sin^{-1}\left(\frac{Z_i}{R_i}\right), \qquad -90\,\text{deg} \leq \Phi_i \leq 90\,\text{deg} \qquad (453)$$

If Θ_i denotes the right ascension of body i relative to body C, then

$$\Theta_i = \tan^{-1}\left(\frac{Y_i}{X_i}\right), \qquad 0\,\text{deg} \leq \Theta_i \leq 360\,\text{deg} \qquad (454)$$

The velocity of body i relative to body C is given by

$$V_i = (\dot{X}_i^2 + \dot{Y}_i^2 + \dot{Z}_i^2)^{1/2}, \qquad \text{km/s} \qquad (455)$$

The body C fixed position and velocity vectors of body i are, respectively,

$$\mathbf{r}_i = (x_i, y_i, z_i), \qquad \text{km} \qquad (456)$$

$$\dot{\mathbf{r}}_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i), \qquad \text{km/s} \qquad (457)$$

Therefore, the body C fixed velocity of body i may be computed from

$$v_i = (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)^{1/2}, \qquad \text{km/s} \qquad (458)$$

and the longitude θ_i of body i is given by

$$\theta_i = \tan^{-1}\left(\frac{y_i}{x_i}\right), \qquad 0\,\text{deg} \leq \theta_i \leq 360\,\text{deg} \qquad (459)$$

The rate of change of the position vector in the radial direction $\dot{R}$ is the projection of $\dot{\mathbf{R}}_p$ onto the unit vector pointing in the radial direction; i.e.,

$$\dot{R} = \dot{\mathbf{R}}_p \cdot \left(\frac{\mathbf{R}_p}{|\mathbf{R}_p|}\right) \qquad (460)$$

as indicated in Fig. 43 (see also Fig. 11).

The angular-momentum vector is equal to

$$\mathbf{R}_p \times \dot{\mathbf{R}}_p \qquad (461)$$

and, since

$$(\mathbf{R}_p \cdot \mathbf{R}_p)\,\dot{\nu} = |\mathbf{R}_p \times \dot{\mathbf{R}}_p|$$

(see Eq. 93), where $\dot{\nu}$ is the rate at which the true anomaly ν is changing, the following relation is obtained:

$$\dot{\nu} = \frac{|\mathbf{R}_p \times \dot{\mathbf{R}}_p|}{|\mathbf{R}_p|^2} \cdot \frac{180}{\pi}, \qquad \text{deg/s} \qquad (462)$$

Let $\mathbf{R}_S$ be the position vector of the sun with respect to body C. The sun-shadow parameter for body C, denoted by d, indicates whether or not the spacecraft is in the shadow of body C (compare with Section VII-B, where the quantities D and D' were computed to determine if the probe was entering or leaving the shadow of body C). Parameter d is given by

$$d = \frac{-|\mathbf{R}_p \times \mathbf{R}_S|}{|\mathbf{R}_S|}\,\text{sign}\,(\mathbf{R}_p \cdot \mathbf{R}_S), \qquad \text{km} \qquad (463)$$

Thus, if d is such that

$$0 < d < RAD(C)$$

where

$$RAD(C) = \text{radius of body } C \qquad (464)$$

then the spacecraft is in the shadow of body C (Fig. 44). In all other cases, the spacecraft is not in the shadow.

During the integration process, the value of d must be computed at each integration step. If d should indicate at some time t_n that the spacecraft is in the sunlight, and that at t_{n+1} the spacecraft is in the shadow of body C, then time t' (when the spacecraft was entering the shadow) is computed, and the integration process must be restarted at t'. This restart is necessary because the solar radiation pressure beginning to act on the acceleration of the spacecraft. The sun shadow parameter d_i for body i is, correspondingly, given by the expression

$$d_i = \frac{-\,|\mathbf{R}_{ip} \times (\mathbf{R}_s - \mathbf{R}_i)|}{|\mathbf{R}_s - \mathbf{R}_i|}\,[\operatorname{sign} \mathbf{R}_{ip} \cdot (\mathbf{R}_s - \mathbf{R}_i)], \quad \text{km} \tag{465}$$

(Fig. 45).

The altitude of the spacecraft above body C is computed according to

$$h = R - RAD(C), \qquad \text{km} \tag{466}$$

where R is the distance of the spacecraft from body C (see Eq. 435).

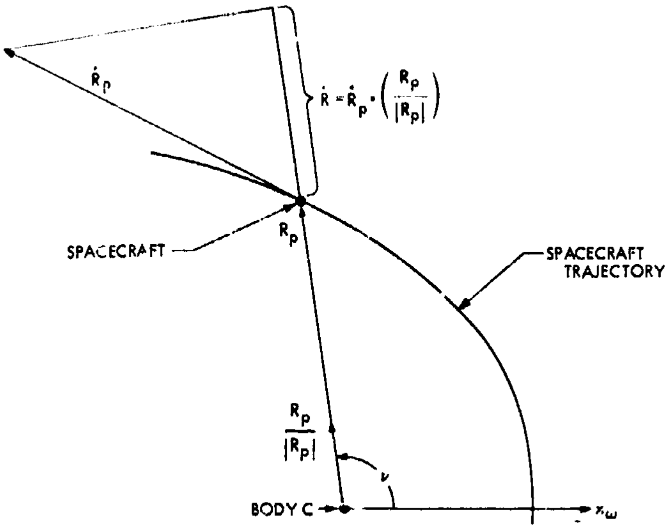

Fig. 43. The quantity $\dot{R}$

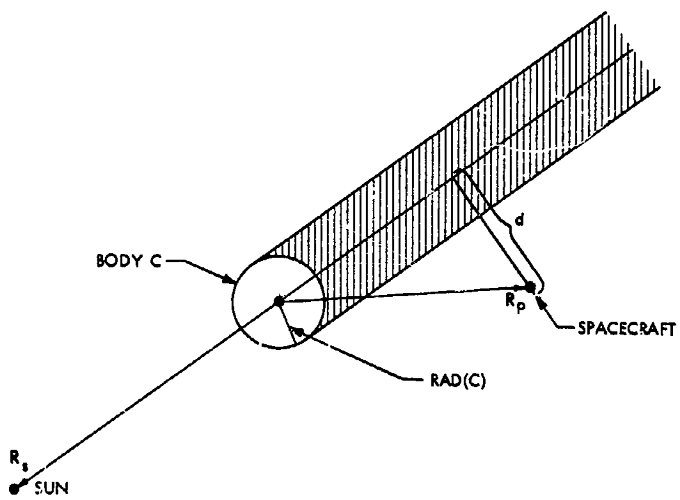

Fig. 44. Sun-shadow parameter d

spacecraft entering the shadow of body C means that, beginning at time t', solar radiation pressure ceases to contribute to the acceleration of the spacecraft; i.e., the acceleration of the spacecraft has a discontinuity at time t'.

When the spacecraft is in the shadow, the value of d must be constantly monitored to determine whether the spacecraft is still in the shadow or has already left it. If the spacecraft has left the shadow, its time of exit (say, t'') is computed, and the integration process is restarted at t''. This restart is necessary because the acceleration of the spacecraft has a discontinuity at t'' caused by

B. Conic Group[43]

From Newton's law of gravitation, it follows that a spacecraft in the force field of another body moves on a conic section. In this section, pertinent information regarding this conic is computed from three quantities:

$$\mu_c = \text{gravitational constant of body } C \tag{467}$$

$$\mathbf{R}_{50} = (X_{50}, Y_{50}, Z_{50}) \tag{468}$$

$$\dot{\mathbf{R}}_{50} = (\dot{X}_{50}, \dot{Y}_{50}, \dot{Z}_{50}) \tag{469}$$

[43]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

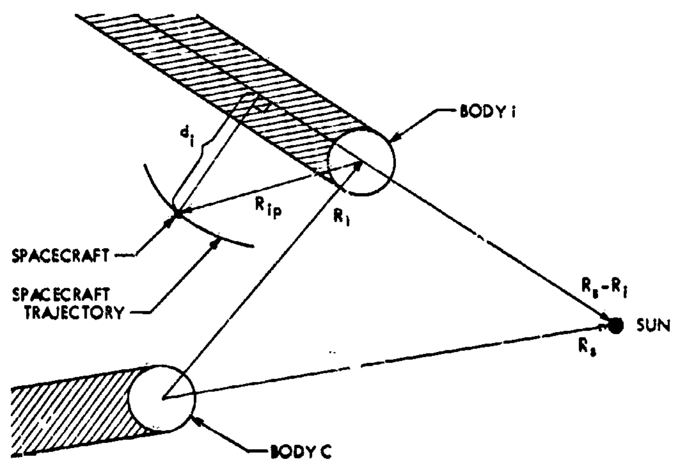

Fig. 45. Sun-shadow parameter d_i

where $\mathbf{R}_{50}$ and $\dot{\mathbf{R}}_{50}$ denote position and velocity vectors, respectively, of the spacecraft with respect to body C in the space-fixed mean earth equator and equinox coordinate system of 1950.0 (see Section V).

Define

$$R_{50} = |\mathbf{R}_{50}|, \qquad \text{km} \tag{470}$$

$$\dot{R}_{50} = |\dot{\mathbf{R}}_{50}|, \qquad \text{km/s} \tag{471}$$

The semimajor axis a of the conic on which the spacecraft is moving is then computed from the vis-viva integral

$$V^2 = \mu\left(\frac{2}{R} - \frac{1}{a}\right) \tag{472}$$

Hence, in the present case,

$$\dot{R}^2_{50} = \mu_c\left(\frac{2}{R_{50}} - \frac{1}{a}\right) \tag{473}$$

so that

$$a = \frac{\mu_c R_{50}}{2\mu_c - R_{50}\dot{R}^2_{50}} \tag{474}$$

Equation (48) develops the equation for the semilatus rectum p; with the notation in this section, one obtains from Eq. (48)

$$p = R_{50}\left(2 - \frac{R_{50}}{a}\right) - \frac{(\mathbf{R}_{50} \cdot \dot{\mathbf{R}}_{50})^2}{\mu_c} \tag{475}$$

The eccentricity e of the conic is given by the standard equation

$$e = \left(1 - \frac{p}{a}\right)^{1/2} \tag{476}$$

Equation (476) may be derived from the basic formula of a conic in polar coordinates

$$r = \frac{p}{1 + e\cos\nu} \tag{477}$$

Setting $\nu = 0$ and $\nu = \pi$, one obtains r_{min} and r_{max}; adding yields

$$r_{min} + r_{max} = 2a = \frac{2p}{1 - e^2}$$

or

$$p = a(1 - e^2) \tag{478}$$

from which Eq. (476) follows.

Clearly, the spacecraft makes its closest approach to body C when $\nu = 0$ in Eq. (477). Thus, the closest approach (or pericenter distance) is given by

$$r_p = q = \frac{p}{1 + e} \tag{479}$$

The apocenter distance is computed from

$$r_a = q_2 = a(1 + e) \tag{480}$$

The quantity C_3 (also called the vis-viva integral) is defined as

$$C_3 = -\frac{\mu_c}{a}, \qquad \text{km}^3/\text{s}^2 \tag{481}$$

The quantity C_3 constitutes twice the total energy E (per unit mass) of the spacecraft,

$$E = \frac{1}{2}v^2 - \frac{\mu_c}{R}, \qquad \text{km}^2/\text{s}^2 \tag{482}$$

where R is the distance from the spacecraft to body C. The first term on the right of Eq. (482) is the kinetic energy and the second is the potential energy of the spacecraft.

By the above remarks,

$$\begin{aligned} C_3 &= 2E \\ &= v^2 - \frac{2\mu_c}{R} \\ &= -\frac{\mu_c}{a} \end{aligned} \tag{483}$$

For orbit determination, the sign of C_3 is important. If $C_3 > 0$, then a must be less than zero; hence, the spacecraft is moving on a hyperbolic orbit. If $C_3 < 0$, then $a > 0$; therefore, the orbit is elliptic. If $C_3 = 0$, then $a = \infty$; thus, the spacecraft is moving on a parabolic orbit (i.e., the velocity of the spacecraft is decreasing as it moves away from body C, and, as its distance from C approaches infinity, its velocity approaches zero). The hyperbolic excess velocity V_∞ is obtained by allowing R

to tend toward infinity. From Eq. (483), one obtains, in this case,

$$V_\infty^2 = C_3 \tag{484}$$

Therefore,

$$V_\infty = (C_3)^{1/2}, \qquad \text{km/s} \tag{485}$$

For actual computations, it is necessary to let

$$\begin{aligned} C_3 &= -\frac{\mu_c}{a} && \text{if } |a| < N \\ &= 0 && \text{if } |a| > N \end{aligned} \tag{486}$$

where N is some large number; e.g., 10^{14} km.

The angular momentum per unit mass, here denoted by C_1, is defined by

$$C_1 = |\mathbf{R}_{50} \times \dot{\mathbf{R}}_{50}| \tag{487}$$

If one computes the cross product in Eq. (487) explicitly, it follows that

$$C_1 = [R_{50}^2 \dot{R}_{50}^2 - (\mathbf{R}_{50} \cdot \dot{\mathbf{R}}_{50})^2]^{1/2} = (\mu_c p)^{1/2} \tag{488}$$

where the last equality stems from the fact that, in the derivation of the standard form of a conic, the quantity p was set equal to C_1^2/μ.

The eccentric anomaly E (in the case of an ellipse) is the angle measured in the orbital plane from the x_ω axis to a line containing the center and another point defined by the projection of the spacecraft in the y_ω direction upon an auxiliary circle that circumscribes the actual ellipse of motion. Geometrically, the eccentric anomaly can be interpreted by means of Fig. 46 as a function of the area of sector OPB, as follows:

$$E = \frac{2 \times \text{area (sector } OPB)}{a^2}, \qquad \text{rad} \tag{489}$$

In direct analogy, it is possible to define a new variable for hyperbolic motion as

$$F = \frac{2 \times \text{area (sector } PBC)}{a^2}, \qquad \text{rad} \tag{490}$$

where the area PBC is defined by means of Fig. 47. When $C_3 \neq 0$, the eccentric anomaly is given by

$$E = \tan^{-1} \frac{a(\mathbf{R}_{50} \cdot \dot{\mathbf{R}}_{50})}{(a - R_{50})(\mu_c a)^{1/2}}, \qquad \text{deg} \tag{491}$$

(see Eq. 66) and, for the hyperbolic case,

$$F = \log [D_1 + (D_1^2 + 1)^{1/2}], \qquad \text{deg} \tag{492}$$

(see Eq. 68), where

$$D_1 = \frac{\mathbf{R}_{50} \cdot \dot{\mathbf{R}}_{50}}{e(|a\mu_c|)^{1/2}} \tag{493}$$

The mean anomaly M is defined by

$$M = E - e \sin E, \qquad \text{deg} \tag{494}$$

and, in the case of a hyperbolic orbit,

$$M_H = e \sinh F - F, \qquad \text{deg} \tag{495}$$

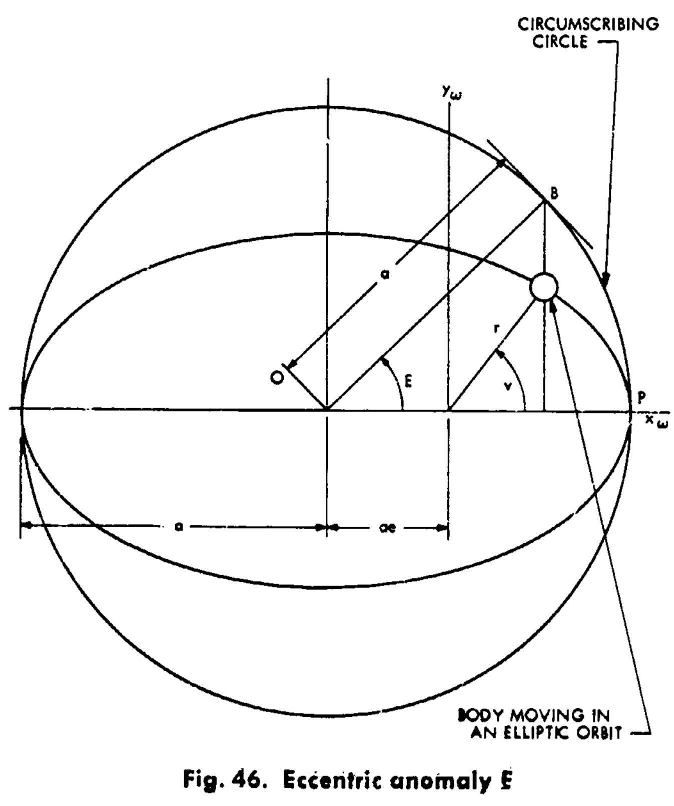

Fig. 46. Eccentric anomaly E

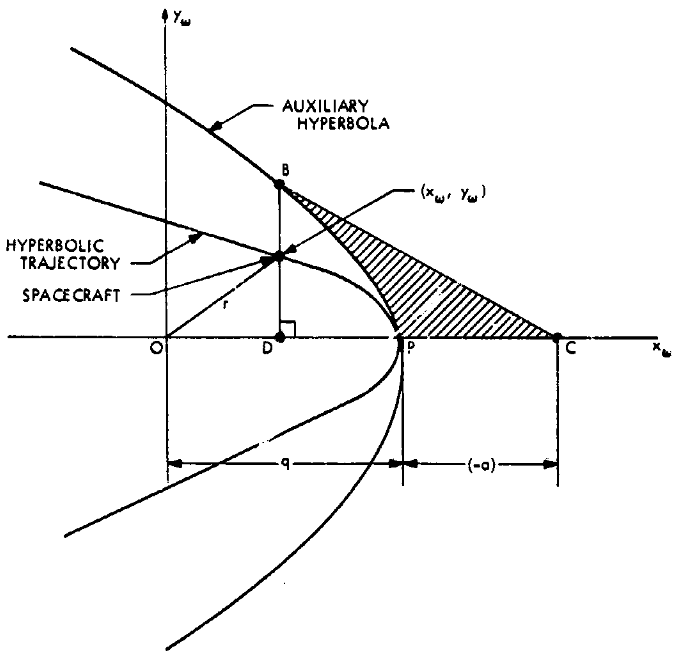

Fig. 47. The quantity F

If $C_3 = 0$—i.e., in the case of a parabolic orbit the mean anomaly is computed from

$$M = qD_2 + \frac{D_2^3}{6}, \quad \text{deg} \tag{496}$$

(Barker's equation; see Eq. 70), where

$$\left.\begin{aligned} M &= (\mu_c)^{1/2}(T - T_p) \\ D_2 &= \frac{\mathbf{R}_{50} \bullet \dot{\mathbf{R}}_{50}}{(\mu_c)^{1/2}} \end{aligned}\right\} \tag{497}$$

In this case, one sets

$$E = M \tag{498}$$

The time from pericenter passage $T - T_p$ is defined by the relation

$$T - T_p = \frac{M}{n}, \quad \text{s} \tag{499}$$

where

$$\begin{aligned} n &= \left(\frac{\mu_c}{|a^3|}\right)^{1/2} && \text{if } C_3 \neq 0 \\ &= (\mu_c)^{1/2} && \text{if } C_3 = 0 \end{aligned} \tag{500}$$

The time from injection to pericenter passage T_f is given by the expression

$$T_f = T - (T - T_p) \tag{501}$$

The maximum true anomaly ν_{max} is defined in Fig. 9. In Section IV-E, the relation

$$\nu_{max} = \cos^{-1}\left(-\frac{1}{e}\right), \qquad 90 \text{ deg} \leq \nu_{max} \leq 180 \text{ deg} \tag{502}$$

is derived, where e = eccentricity.

The angle between the incoming and outgoing asymptote is equal to 2ρ (Fig. 48). It should be noted that the vector $\mathbf{R}$ is parallel to an asymptote when $R \rightarrow \infty$. Thus, from the equation of a conic in polar coordinates,

$$R = \frac{p}{1 + e \cos \nu} \tag{503}$$

it follows that

$$1 + e \cos \nu_{max} = 0 \tag{504}$$

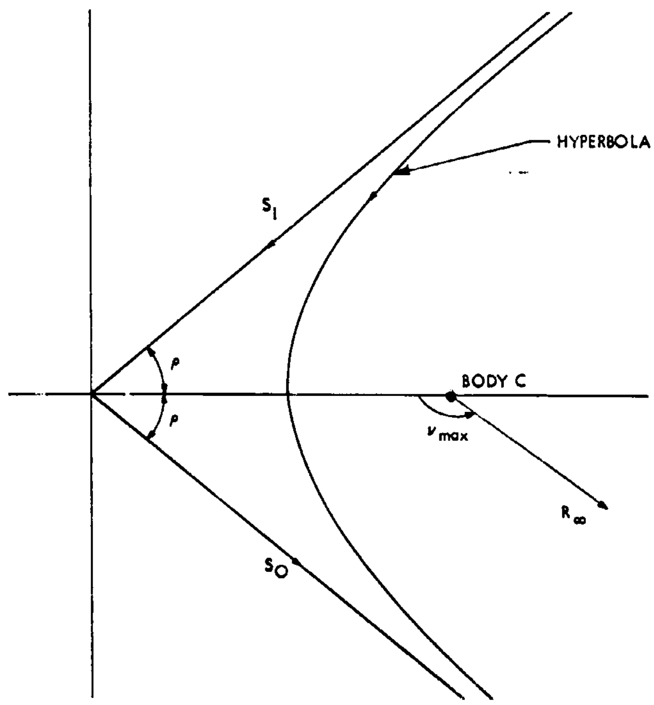

Fig. 48. Angles ρ and ν_{max}

as R→ ∞. Because

$$\nu_{max} = 180\,\text{deg} - \rho \tag{505}$$

one obtains from Eq. (504)

$$1 - e\cos\rho = 0 \tag{506}$$

or

$$\cos\rho = \frac{1}{e} \tag{507}$$

From the half-angle formulas, it follows that

$$\cos 2\rho = \frac{2}{e^2} - 1$$

or

$$2\rho = \cos^{-1}\left(\frac{2}{e^2} - 1\right), \quad \text{deg} \tag{508}$$

The deflection angle D_{EF} between the two asymptote vectors, defined by Fig. 49, is then given by

$$D_{EF} = 180\,\text{deg} - 2\rho$$
$$= \pi - \cos^{-1}\left(\frac{2}{e^2} - 1\right)$$

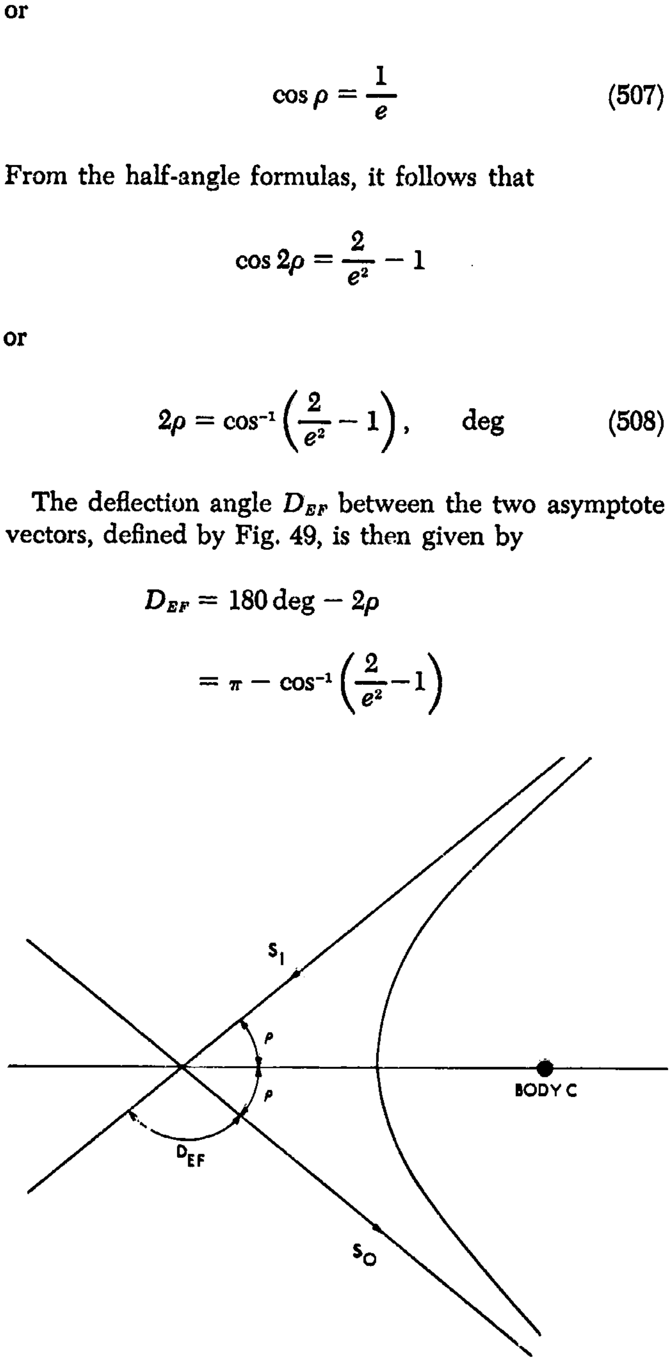

Fig. 49. Deflection angle between asymptote vectors

or

$$D_{EF} = \cos^{-1}\left(1 - \frac{2}{e^2}\right), \quad \text{deg} \tag{509}$$

The velocity at apocenter (of an elliptic orbit) V_A is computed as described below. From the vis-viva integral

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

it follows (for the velocity at apogee)

$$V_A^2 = \mu_c\left(\frac{2}{r_a} - \frac{1}{a}\right) \tag{510}$$

However,

$$a = \frac{1}{2}(r_a + r_p)$$

where r_a and r_p are the apocenter and pericenter distances, respectively, from the focus; i.e., from body C, so that Eq. (510) becomes

$$V_A^2 = \mu_c\left(\frac{2}{r_a} - \frac{2}{r_a + r_p}\right)$$
$$= \frac{2\mu_c r_p}{(r_a + r_p)\,r_a}$$
$$= \frac{\mu_c r_p}{a\,r_a} \tag{511}$$

Since

$$r_p = a(1 - e) \tag{512}$$

$$r_a = a(1 + e) \tag{513}$$

one obtains

$$V_A^2 = \mu_c\frac{(1-e)}{a(1+e)}$$
$$= \mu_c\frac{(1-e)^2}{a(1-e^2)}$$
$$= \mu_c\frac{(1-e^2)}{p}$$

Hence,

$$V_A = \frac{\mu_c(1-e)}{C_1} \tag{514}$$

by use of Eq. (488).

Linearized flight time, denoted by T_L, is defined as the time-to-go on a rectilinear path to the center of the target, and is given by the expression

$$T_L = T_F - \Delta T_f \tag{515}$$

where[44]

$$T_F = \text{time-to-go to closest approach} \tag{516}$$

$$\Delta T_f = \frac{\mu_c}{V_\infty^3} \log e \tag{517}$$

or

$$\Delta T_f = \left(\frac{|a|^3}{\mu_c}\right)^{1/2} \sinh^{-1}\left(\frac{e^2-1}{2e}\right) \tag{518}$$

A derivation of Eqs. (517) and (518) is given in Appendix E.

Called the linearized time of flight correction for the target conic, ΔT_f is used instead of the period P, which is given by

$$P = 2\pi\left(\frac{a^3}{\mu}\right)^{1/2} \bullet \frac{1}{86400} \text{ days} \tag{519}$$

The so-called restricted three-body problem requires the description of the motion of a body P of infinitesimal mass moving under the influence of two bodies P_1 and P_2 of finite mass, which revolve around each other in circular orbits. This situation is approximately realized in many instances in the solar system (e.g., a spacecraft moving in the vicinity of the earth and the moon).

Upon making some simplifying assumptions, Jacobi was able to integrate the equations of motion of the restricted three-body problem. This integral is an expression of the conservation of relative energy of body P with respect to the barycenter of P_1 and P_2. In the special case when P_1 and P_2 are the earth (E) and the moon (M), respectively, and P is a spacecraft moving in the field of E and M, the Jacobi constant is given by the expression[45]

$$C_{3J} = \dot{R}_b^2 - 2\left[\frac{(\mathbf{R}_{EM} \times \dot{\mathbf{R}}_{EM}) \bullet (\mathbf{R}_b \times \dot{\mathbf{R}}_b)}{R_{EM}^2} + \frac{\mu_E}{R_E} + \frac{\mu_M}{R_M}\right] \tag{520}$$

(see also Ref. 23, p. 430), where

$$\mathbf{R}_E = \text{earth–spacecraft position vector}$$
$$= (X_E, Y_E, Z_E)$$

$$\dot{\mathbf{R}}_E = \text{earth–spacecraft velocity vector}$$
$$= (\dot{X}_E, \dot{Y}_E, \dot{Z}_E)$$

$$\mathbf{R}_{EM} = \text{earth–moon position vector}$$
$$= (X_{EM}, Y_{EM}, Z_{EM})$$

$$\dot{\mathbf{R}}_{EM} = \text{earth–moon velocity vector}$$
$$= (\dot{X}_{EM}, \dot{Y}_{EM}, \dot{Z}_{EM})$$

$$\mathbf{R}_b = \text{barycenter–spacecraft position vector}$$
$$= \mathbf{R}_E - \mu_b \mathbf{R}_{EM}$$

where

$$\mu_b = \frac{1}{1 + \frac{\mu_E}{\mu_M}}$$

$$\dot{\mathbf{R}}_b = \text{barycenter–spacecraft velocity vector}$$
$$= \dot{\mathbf{R}}_E - \mu_b \dot{\mathbf{R}}_{EM}$$

$$\mathbf{R}_M = \text{moon–spacecraft position vector}$$
$$= (X_M, Y_M, Z_M)$$

and

$$R_i = |\mathbf{R}_i|, \qquad i = b, E, M, EM$$

[44]Thornton, T. H., JPL internal document, Mar. 1, 1962.

[45]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

The unit vector,

$$\hat{\mathbf{W}}_{50} = \frac{\mathbf{R}_{50} \times \dot{\mathbf{R}}_{50}}{C_1}$$

$$= (W_{x50}, W_{y50}, W_{z50}) \qquad (521)$$

where C_1 is defined by Eq. (487) as the angular momentum per unit mass, is normal to the orbital plane of the spacecraft. The unit vector

$$\hat{\mathbf{N}}_{50} = \frac{(-W_{y50}, W_{x50}, 0)}{(W_{x50}^2 + W_{y50}^2)^{1/2}}$$

$$= (N_{x50}, N_{y50}, N_{z50}) \qquad (522)$$

points in the direction of the ascending node, and the unit vector

$$\hat{\mathbf{M}}_{50} = \hat{\mathbf{W}}_{50} \times \hat{\mathbf{N}}_{50} \qquad (523)$$

completes the right-handed system (Fig. 50).

If $e = 0$, the argument of pericenter is not defined; in this case, the true anomaly ν is computed as

$$\nu = \tan^{-1}\left(\frac{\mathbf{R}_{50} \bullet \hat{\mathbf{M}}_{50}}{\mathbf{R}_{50} \bullet \hat{\mathbf{N}}_{50}}\right), \qquad \text{deg} \qquad (524)$$

i.e., the angle between $\mathbf{R}_{50}$ and $\hat{\mathbf{N}}_{50}$.

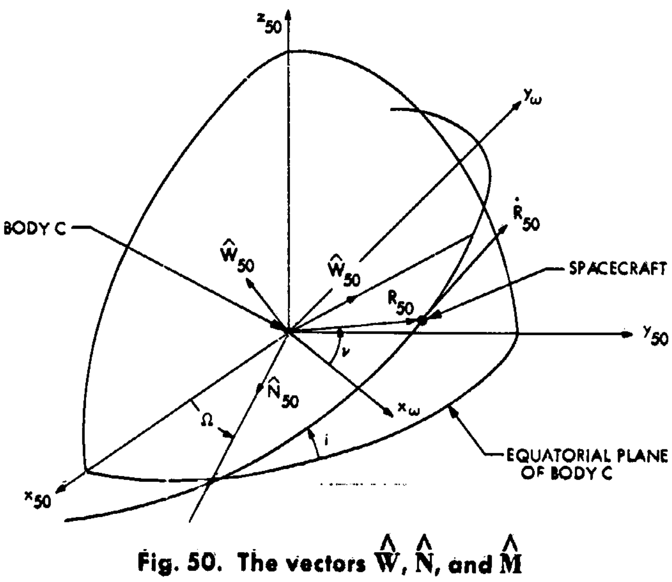

Fig. 50. The vectors $\hat{\mathbf{W}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{M}}$

If $e \neq 0$, then

$$\sin \nu = \frac{\mathbf{R}_{50} \bullet \dot{\mathbf{R}}_{50}}{|\mathbf{R}_{50}|} \frac{1}{e} \left(\frac{p}{\mu_c}\right)^{1/2} \qquad (525)$$

$$\cos \nu = \frac{1}{e}\left(\frac{p}{|\mathbf{R}_{50}|} - 1\right) \qquad (526)$$

For a derivation of the last two equations, see Eqs. (54) and (55).

Thus,

$$\nu = \tan^{-1}\left(\frac{\sin \nu}{\cos \nu}\right), \qquad \text{deg} \qquad (527)$$

If a coordinate system other than the 1950.0 system is requested, one denotes

$$\hat{\mathbf{W}} = \frac{\mathbf{R}_{(I)} \times \dot{\mathbf{R}}_{(I)}}{C_1'}$$

$$= (W_x, W_y, W_z) \qquad (528)$$

where

$\mathbf{R}_{(I)}, \dot{\mathbf{R}}_{(I)}$ = spacecraft position and velocity vectors in requested coordinate system

$C_1' = |\mathbf{R}_{(I)} \times \dot{\mathbf{R}}_{(I)}|$

The inclination of the orbital plane to the coordinate plane (Fig. 51) is given by

$$i = \cos^{-1}(W_z), \qquad \text{deg}$$

and Ω, the longitude of the ascending node, is given by

$$\Omega = \tan^{-1}\left(-\frac{W_x}{W_y}\right)$$

Define

$$\hat{\mathbf{U}}_1 = \frac{\mathbf{R}_{(I)}}{|\mathbf{R}_{(I)}|}$$

$$= (U_{1x}, U_{1y}, U_{1z}) \qquad (529)$$

and

$$\hat{\mathbf{V}}_1 = \left(\frac{R_{(I)}}{C_1'}\right)\dot{\mathbf{R}}_{(I)} - \left(\frac{\hat{\mathbf{U}}_1 \bullet \dot{\mathbf{R}}_{(I)}}{C_1'}\right)\mathbf{R}_{(I)}$$

$$= (V_{1x}, V_{1y}, V_{1z}) \qquad (530)$$

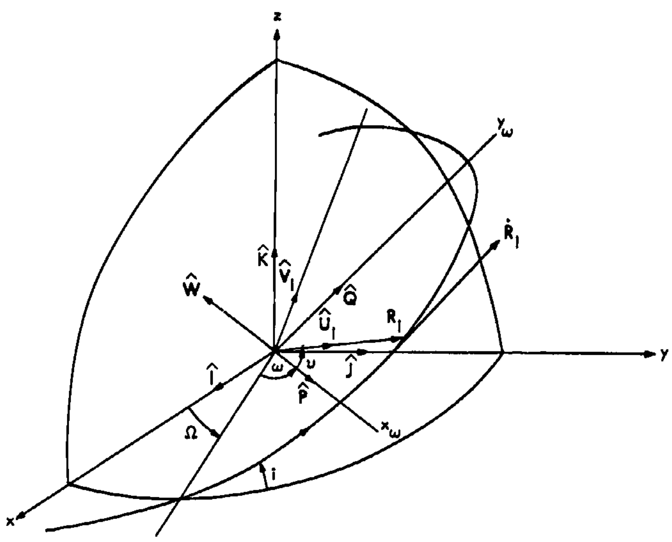

Fig. 51. The vectors U, V, P, and Q

where

$$R_{(I)} = |\mathbf{R}_{(I)}|$$

Thus, $\hat{\mathbf{U}}_1$ and $\hat{\mathbf{V}}_1$ are perpendicular to each other.

The unit vector $\hat{\mathbf{P}}$ points in the direction of perifocus; i.e., $\hat{\mathbf{P}}$ is along the x_ω-axis. Clearly,

$$\begin{aligned}\hat{\mathbf{P}} &= \cos\nu\, \hat{\mathbf{U}}_1 - \sin\nu\, \hat{\mathbf{V}}_1 \\ &= (P_x, P_y, P_z) \qquad (531)\end{aligned}$$

The unit vector $\hat{\mathbf{Q}}$ is normal to $\hat{\mathbf{P}}$; thus,

$$\begin{aligned}\hat{\mathbf{Q}} &= \sin\nu\, \hat{\mathbf{U}}_1 + \cos\nu\, \hat{\mathbf{V}}_1 \\ &= (Q_x, Q_y, Q_z) \qquad (532)\end{aligned}$$

Hence, in matrix notation,

$$\begin{pmatrix}\hat{\mathbf{P}} \\ \hat{\mathbf{Q}}\end{pmatrix} = \begin{pmatrix}\cos\nu & -\sin\nu \\ \sin\nu & \cos\nu\end{pmatrix} \begin{pmatrix}\hat{\mathbf{U}}_1 \\ \hat{\mathbf{V}}_1\end{pmatrix}$$

The argument of pericenter ω is then given by

$$\omega = \tan^{-1}\left(\frac{P_z}{Q_z}\right) \qquad (533)$$

To align the $\hat{\mathbf{P}},\hat{\mathbf{Q}},\hat{\mathbf{W}}$ triad with the $\hat{\mathbf{I}},\hat{\mathbf{J}},\hat{\mathbf{K}}$ triad by performing rotations through the angles Ω, i, ω, it follows that

$$\begin{bmatrix}\hat{\mathbf{P}} \\ \hat{\mathbf{Q}} \\ \hat{\mathbf{W}}\end{bmatrix} = \begin{bmatrix}\cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i\end{bmatrix} \begin{bmatrix}\cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1\end{bmatrix} \begin{bmatrix}\hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}}\end{bmatrix}$$

or

$$\begin{bmatrix}\hat{\mathbf{P}} \\ \hat{\mathbf{Q}} \\ \hat{\mathbf{W}}\end{bmatrix} = \begin{bmatrix}\begin{pmatrix}\cos\omega\cos\Omega \\ -\sin\omega\cos i\sin\Omega\end{pmatrix} & \begin{pmatrix}\cos\omega\sin\Omega \\ +\sin\omega\cos i\cos\Omega\end{pmatrix} & \begin{pmatrix}\sin\omega\sin i\end{pmatrix} \\ \begin{pmatrix}-\sin\omega\cos\Omega \\ -\cos\omega\cos i\sin\Omega\end{pmatrix} & \begin{pmatrix}-\sin\omega\sin\Omega \\ +\cos\Omega\cos\omega\cos i\end{pmatrix} & \begin{pmatrix}\cos\omega\sin i\end{pmatrix} \\ \begin{pmatrix}\sin\Omega\sin i\end{pmatrix} & \begin{pmatrix}-\sin i\cos\Omega\end{pmatrix} & \begin{pmatrix}\cos i\end{pmatrix}\end{bmatrix} \begin{bmatrix}\hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}}\end{bmatrix}$$

so that

$$P_z = \sin\omega\sin i \qquad (534)$$

$$Q_z = \cos\omega\sin i \qquad (535)$$

and Eq. (533) follows.

The impact parameter vector $\mathbf{B}$ is defined as a vector originating at the center of the target planet and directed perpendicular to the incoming asymptote of the target-centered approach hyperbola (Fig. 52). The magnitude of

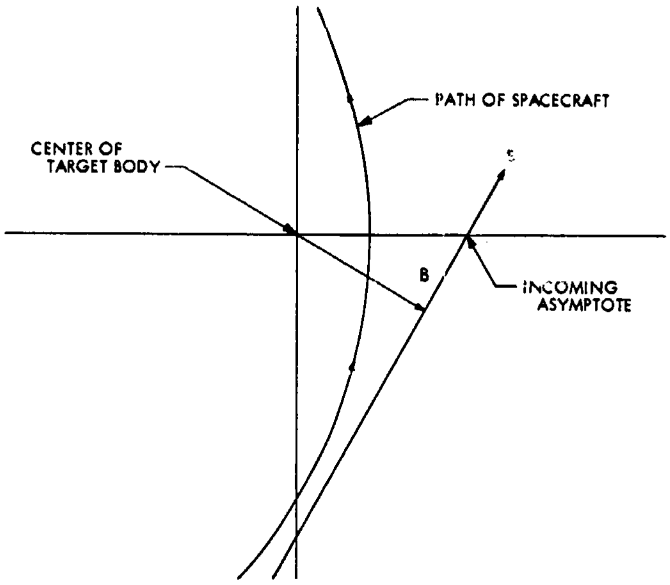

Fig. 52. Impact parameter vector B

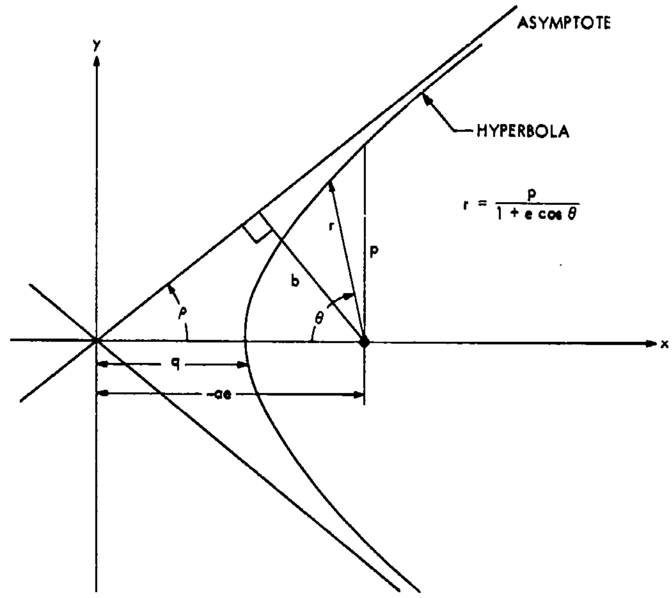

Fig. 53. Magnitude of vector B

B (Fig. 53), which is denoted by b, may be computed as follows:

$$-\frac{b}{ae} = \sin\rho = (1 - \cos^2\rho)^{1/2} \tag{536}$$

But Eq. (507) states that

$$\cos\rho = \frac{1}{e} \tag{537}$$

Substituting Eq. (537) into Eq. (536) yields

$$-\frac{b}{ae} = \left[1 - \left(\frac{1}{e}\right)^2\right]^{1/2} = \frac{(e^2-1)^{1/2}}{e}$$

so that

$$b = a(e^2 - 1)^{1/2} \tag{538}$$

For a hyperbola, $a < 0$; hence, one must take

$$b = -a(e^2 - 1)^{1/2} \tag{539}$$

or

$$b = [a^2(e^2 - 1)]^{1/2} \tag{540}$$

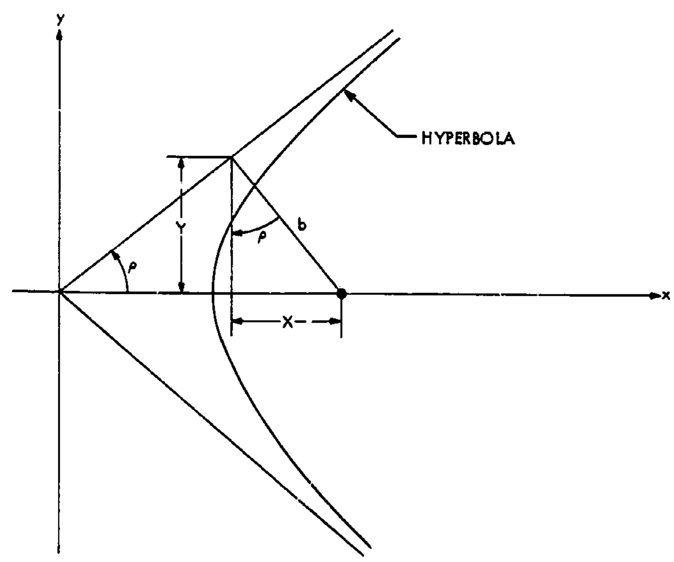

Fig. 54. Coordinates of vector B

From Fig. 54 it can easily be seen that

$$\begin{aligned} X &= b\sin\rho \\ Y &= b\cos\rho \end{aligned} \tag{541}$$

Thus,

$$X = -a(e^2 - 1)^{1/2}\left(1 - \frac{1}{e^2}\right)^{1/2}$$

or

$$X = -a\frac{(e^2-1)}{e} \tag{542}$$

and

$$Y = \frac{-a\,(e^2-1)^{1/2}}{e} \tag{543}$$

by use of Eqs. (537) and (539).

Thus, vector **B** may be written as

$$\mathbf{B} = X\hat{\mathbf{P}} - Y\hat{\mathbf{Q}}, \qquad e > 1 \tag{544}$$

where X and Y are given by Eqs. (542) and (543).

When $e \leq 1$, no asymptote exists, and **B** is not properly defined. However, one can still formally write

$$\mathbf{B} = Z\hat{\mathbf{Q}}, \qquad e < 1 \tag{545}$$

where now

$$Z = a\,(1-e^2)^{1/2}, \qquad e < 1 \tag{546}$$

Clearly,

$$\hat{\mathbf{B}} = \frac{\mathbf{B}}{|\mathbf{B}|}$$

$$= (B_x, B_y, B_z) \tag{547}$$

As a spacecraft approaches a target planet, it becomes necessary to consider the attraction of the planet on the spacecraft, as this effect tends to increase the probability of making a landing. The radius of the effective cross section of the predominant gravitational attraction of the target planet is called the collision parameter, or **B** vector impact radius (Fig. 55), which is denoted by B_{IR} (see Ref. 3, p. 273). It is evident that a spacecraft approaching the planet with an offset distance less than B_{IR} will strike the planet.

Equating the angular momentum $B_{IR} V_\infty$ at a point a great distance from the planet to that at grazing encounter with its surface, one finds that

$$B_{IR} V_\infty = r_0 v_0 \tag{548}$$

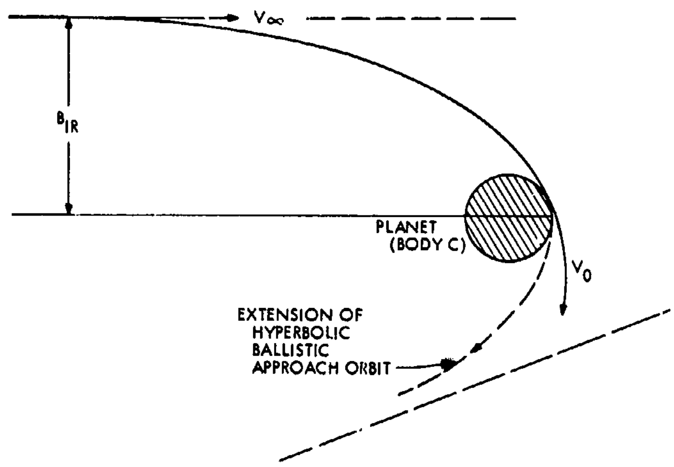

Fig. 55. B vector impact radius

where

V_∞ = hyperbolic excess velocity of spacecraft as it approaches body C

$r_0 = RAD(C)$

v_0 = speed spacecraft would have at a grazing encounter with surface of planet

From the vis-viva integral of Eq. (472), applied to the planetocentric orbit,

$$v_0^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{549}$$

it follows that

$$V_\infty^2 = -\frac{\mu}{a}, \qquad r \to \infty$$

Thus,

$$v_0^2 = \frac{2\mu}{r_0} + V_\infty^2 \tag{550}$$

and solving for B_{IR} yields

$$B_{IR} = r_0\left(\frac{2\mu}{V_\infty^2 r_0} + 1\right)^{1/2} \tag{551}$$

Since $r_p = a\,(1-e)$,

$$V_\infty = \left[\frac{\mu}{r_p}(e-1)\right]^{1/2} \tag{552}$$

where r_p is the pericenter distance from focus (here body C) and, for a grazing encounter, $r_0 = r_p$, it follows that

$$B_{IR} = r_0\left(\frac{e+1}{e-1}\right)^{1/2} \tag{553}$$

or

$$B_{IR} = RAD(C)\left(\frac{e+1}{e-1}\right)^{1/2} \cdot \tag{554}$$

This result implies that, if the magnitude of the vector **B** is equal to B_{IR}, then the spacecraft grazes the planet. It should be noted that, as $e \rightarrow \infty$, $B_{IR} \rightarrow RAD(C)$.

The direction cosines of the incoming asymptote (in the orbital plane) can easily be seen to be equal to

$$\left[\frac{1}{e}, \frac{(e^2-1)^{1/2}}{e}\right], \qquad e \geq 1 \tag{555}$$

and those of the outgoing asymptote

$$\left[\frac{-1}{e}, \frac{(e^2-1)^{1/2}}{e}\right], \qquad e \geq 1 \tag{556}$$

Thus, if

$$\hat{\mathbf{S}}_I = (S_{Ix}, S_{Iy}, S_{Iz}) \tag{557}$$

and

$$\hat{\mathbf{S}}_O = (S_{Ox}, S_{Oy}, S_{Oz}) \tag{558}$$

denote the unit incoming and outgoing asymptote vectors (Fig. 56), respectively, then clearly

$$\hat{\mathbf{S}}_I = \left(\frac{1}{e}\right)\hat{\mathbf{P}} + \frac{(e^2-1)^{1/2}}{e}\hat{\mathbf{Q}} \tag{559}$$

and

$$\hat{\mathbf{S}}_O = \left(\frac{-1}{e}\right)\hat{\mathbf{P}} + \frac{(e^2-1)^{1/2}}{e}\hat{\mathbf{Q}} \tag{560}$$

The impact parameter vector **B** is resolved into two components, which lie in the **B**-plane normal to the incoming asymptote $\mathbf{S}_I$. The orientation of the reference axes in this plane is arbitrary, but one of the axes is selected to lie in a fixed plane. Thus, one defines a unit vector $\hat{\mathbf{T}}$, lying in both the **B**-plane and a specified reference plane. The reference plane is usually either the

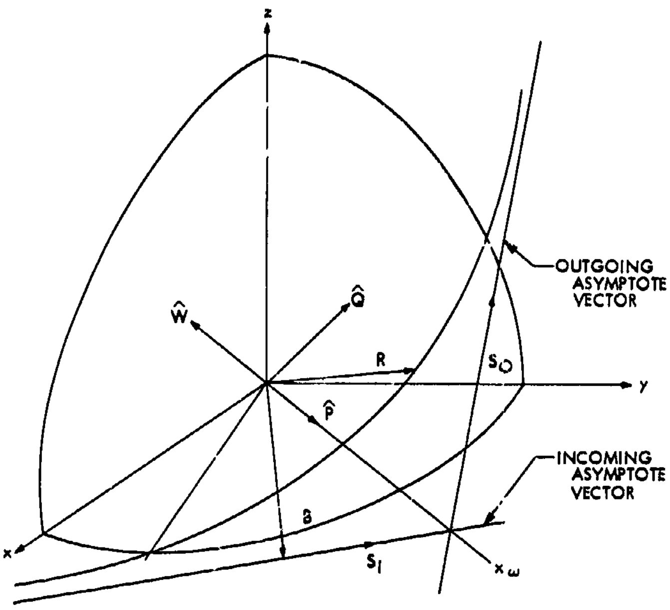

Fig. 56. Incoming and outgoing asymptote vectors

ecliptic or target equatorial plane.[46] The vector $\hat{\mathbf{T}}$ is given by

$$\hat{\mathbf{T}} = \left[\frac{S_{Iy}}{(S_{Ix}^2 + S_{Iy}^2)^{1/2}}, \frac{-S_{Ix}}{(S_{Ix}^2 + S_{Iy}^2)^{1/2}}, 0\right]$$
$$= (T_x, T_y, T_z) \tag{561}$$

The remaining axis is then given by a unit vector $\hat{\mathbf{R}}$ (Fig. 57), defined by

$$\hat{\mathbf{R}} = \hat{\mathbf{S}}_I \times \hat{\mathbf{T}}$$
$$= (R_x, R_y, R_z) \tag{562}$$

The vector **B** lies in the $\hat{\mathbf{R}}\hat{\mathbf{T}}$ plane, and has *miss components* $\mathbf{B} \cdot \hat{\mathbf{T}}$ and $\mathbf{B} \cdot \hat{\mathbf{R}}$ (the T and R components of **B**). The condition

$$\mathbf{B} \cdot \hat{\mathbf{T}} = \mathbf{B} \cdot \hat{\mathbf{R}} = 0 \tag{563}$$

denotes vertical impact on the target (see Ref. 5, p. 4). The angle θ between **B** and $\hat{\mathbf{T}}$ is given by

$$\theta = \tan^{-1}\left(\frac{\mathbf{B} \cdot \hat{\mathbf{R}}}{\mathbf{B} \cdot \hat{\mathbf{T}}}\right) \tag{564}$$

[46]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

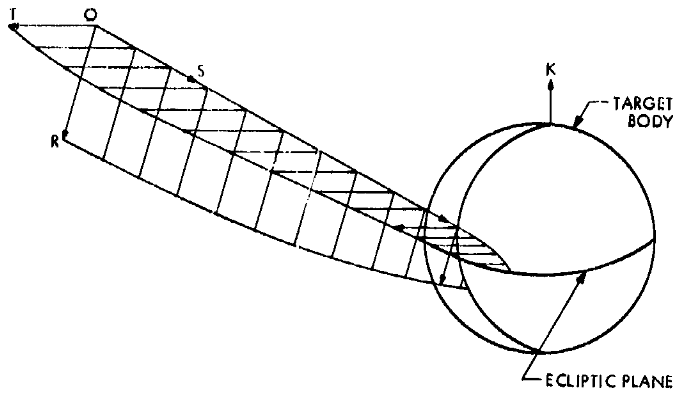

Fig. 57. The R,S,T target coordinate system

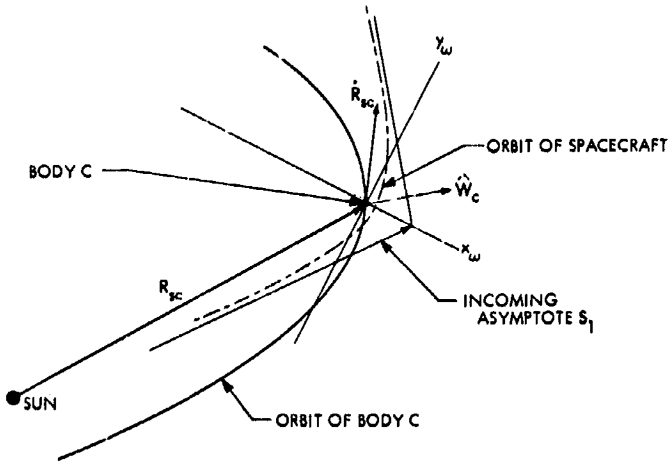

Fig. 58. The vectors $\mathbf{R}_{SC}$ and $\dot{\mathbf{R}}_{SC}$

Let

$$\mathbf{R}_{SC} = (X_{SC}, Y_{SC}, Z_{SC})$$

and

$$\dot{\mathbf{R}}_{SC} = (\dot{X}_{SC}, \dot{Y}_{SC}, \dot{Z}_{SC})$$

denote sun–body C position and velocity vectors, respectively. The unit vector $\hat{\mathbf{W}}_C$, defined by

$$\hat{\mathbf{W}}_C = \frac{\mathbf{R}_{SC} \times \dot{\mathbf{R}}_{SC}}{|\mathbf{R}_{SC} \times \dot{\mathbf{R}}_{SC}|}, \qquad C \neq S \text{ or } M \tag{565}$$

is normal to the plane determined by $\mathbf{R}_{SC}$ and $\dot{\mathbf{R}}_{SC}$; i.e., $\hat{\mathbf{W}}_C$ is normal to the orbital plane of body C (Fig. 58).

To compute the angle G_p between the incoming asymptote $\mathbf{S}_I$ and its projection onto the orbital plane of body C,[47] one forms

$$G_p = \sin^{-1}(\hat{\mathbf{W}}_C \cdot \hat{\mathbf{S}}_I), \qquad \text{deg} \tag{566}$$

In case $C = S$ or $C = M$, one defines

$$\hat{\mathbf{W}}_C = \frac{\mathbf{R}_{EC} \times \dot{\mathbf{R}}_{EC}}{|\mathbf{R}_{EC} \times \dot{\mathbf{R}}_{EC}|} \tag{567}$$

and can then use $\hat{\mathbf{W}}_C$ in Eq. (566), where, for $i = E,S$,

$$\mathbf{R}_{iC} = (X_{iC}, Y_{iC}, Z_{iC}) \tag{568}$$

$$\dot{\mathbf{R}}_{iC} = (\dot{X}_{iC}, \dot{Y}_{iC}, \dot{Z}_{iC}) \tag{569}$$

for the body i–body C position and velocity vectors, respectively.

The latitude of the incoming asymptote ϕ_I is computed according to

$$\phi_I = \sin^{-1}(S_{Iz}), \qquad \text{deg} \tag{570}$$

and the longitude of the incoming asymptote θ_I is given by the expression

$$\theta_I = \tan^{-1}\left(\frac{S_{Iy}}{S_{Ix}}\right), \qquad \text{deg} \tag{571}$$

Similarly, latitude ϕ_O and longitude θ_O of the outgoing asymptote are given by

$$\phi_O = \sin^{-1}(S_{Oz}), \qquad \text{deg} \tag{572}$$

$$\theta_O = \tan^{-1}\left(\frac{S_{Oy}}{S_{Ox}}\right), \qquad \text{deg} \tag{573}$$

Let

$$\hat{\mathbf{R}}_{iC} = (X_{iC}, Y_{iC}, Z_{iC}) \tag{574}$$

denote the body i–body C unit position vector, and let

$$\hat{\mathbf{R}}_{Ci} = (X_{Ci}, Y_{Ci}, Z_{Ci}) \tag{575}$$

denote the body C–body i unit position vector, where $i = E$ (earth), S (sun), or C (Canopus). Then the angle E_{Ti} (Fig. 59), which lies between the vector $\hat{\mathbf{T}}$ (defined by

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[47] Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

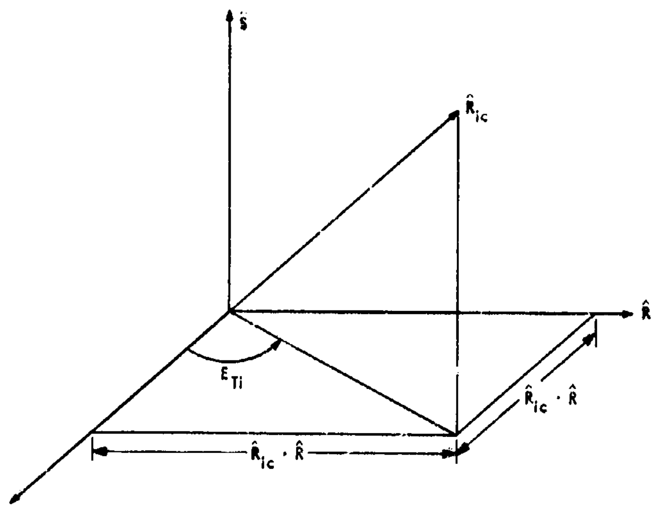

Fig. 59. The angle E_{Ti}

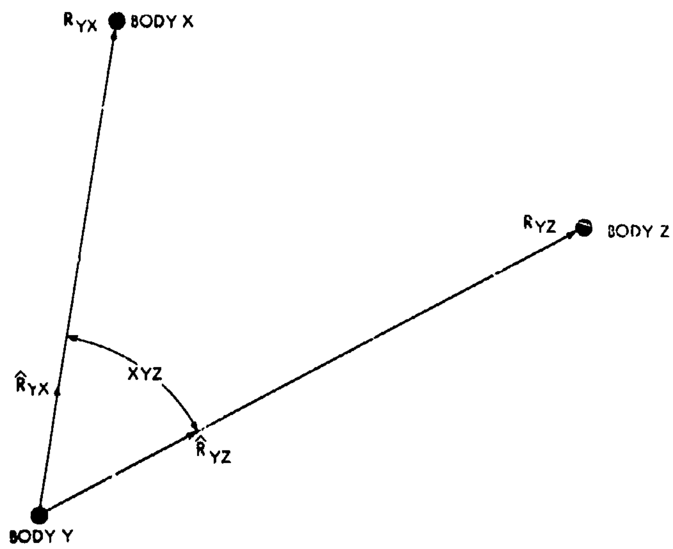

Fig. 60. The angle XYZ

Eq. 561) and the projection $\hat{\mathbf{R}}_{iC}$ onto the $\hat{\mathbf{R}}\hat{\mathbf{T}}$ plane (where $i = E$, S, or C), is given by

$$E_{Ti} = \tan^{-1}\left(\frac{\hat{\mathbf{R}} \cdot \hat{\mathbf{R}}_{iC}}{\hat{\mathbf{T}} \cdot \hat{\mathbf{R}}_{iC}}\right), \quad \text{deg} \qquad (576)$$

The angle between the incoming asymptote $\mathbf{S}_I$ and $\hat{\mathbf{R}}_{Ci}$, denoted by Z_{Ai}, is obviously given by

$$Z_{Ai} = \cos^{-1}(\hat{\mathbf{S}}_I \cdot \hat{\mathbf{R}}_{Ci}) \qquad (577)$$

C. Angle Group[48]

The vectors $\mathbf{R}_{YX}$ and $\mathbf{R}_{YZ}$ will denote the position vectors of bodies X and Z with respect to body Y, and XYZ will be the angle between $\mathbf{R}_{YX}$ and $\mathbf{R}_{YZ}$ (Fig. 60).

Clearly,

$$XYZ = \cos^{-1}(\hat{\mathbf{R}}_{YX} \cdot \hat{\mathbf{R}}_{YZ}), \quad \text{deg} \qquad (578)$$

where

$X = 1, \cdots, 11, C$ (C = Canopus)

$Y = 1, \cdots, 11, C, P$ (P = spacecraft)

$Z = 1, \cdots, 11, P$

The meaning of the numbers is explained above, following Eq. (445). The vectors $\mathbf{R}_{YX}$, $\mathbf{R}_{YZ}$ can easily be obtained from the ephemerides of the planets and the spacecraft.

[48]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

Some of the angles in the angle group are not of the type given by Eq. (578); these are described in the subsections that follow.

1. Clock and cone angles. Assume $\hat{\mathbf{R}}_{pS}$ to be a unit vector from the sun to the spacecraft (this vector can always be determined because the positions of the sun and the spacecraft are known), and $\hat{\mathbf{R}}_{px}$ to be a unit vector from the spacecraft to a reference body (earth or Canopus; i.e., $x = E$ or C). A unit vector normal to the plane determined by the spacecraft, sun, and reference body (Fig. 61) is given by

$$\hat{\mathbf{A}} = \frac{\hat{\mathbf{R}}_{pS} \times \hat{\mathbf{R}}_{px}}{\hat{\mathbf{R}}_{pS} \times \hat{\mathbf{R}}_{px}} \qquad (579)$$

A unit vector $\hat{\mathbf{B}}$ normal to $\hat{\mathbf{R}}_{pS}$ and $\hat{\mathbf{A}}$ may then be computed[49] according to

$$\hat{\mathbf{B}} = \frac{\hat{\mathbf{A}} \times \hat{\mathbf{R}}_{pS}}{|\hat{\mathbf{A}} \times \hat{\mathbf{R}}_{pS}} \qquad (580)$$

The clock angle of body i with respect to the earth or Canopus is defined by Fig. 62; thus,

$CLEi, i = 1, \cdots, 11, C$ = clock angle of body i with respect to earth

$CLCi, i = 1, \cdots, 11$ = clock angle of body i with respect to Canopus

[49]In some references, the vectors $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are interchanged.

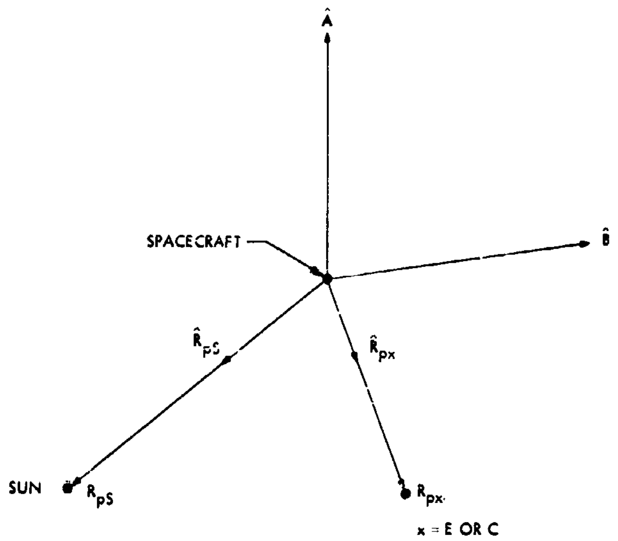

Fig. 61. Vectors $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$

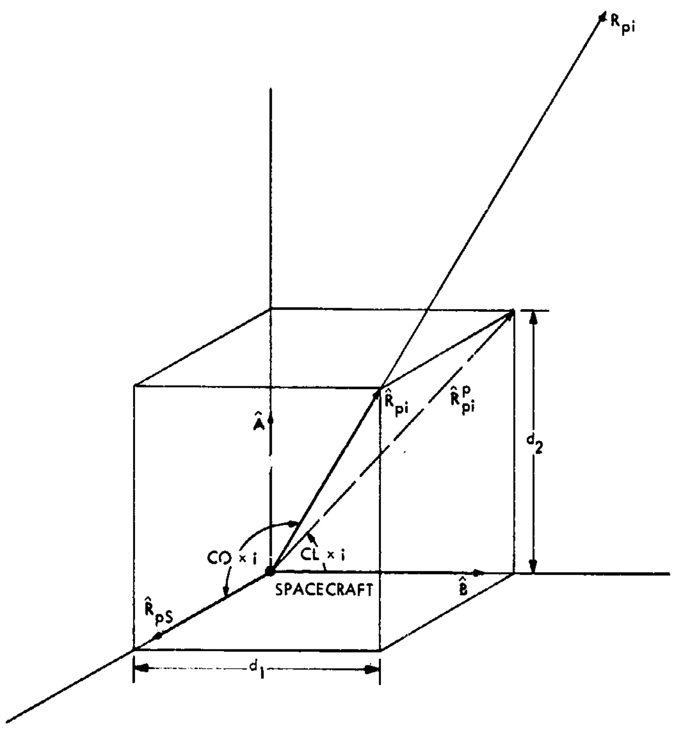

Fig. 62. Clock and cone angles

Since

$$\mathbf{R}^P_{pi} \cdot \hat{\mathbf{B}} = |\mathbf{R}^P_{pi}| \cos CLxi$$

$$\mathbf{R}^P_{pi} \cdot \hat{\mathbf{A}} = |\mathbf{R}^P_{pi}| \cos (90 - CLxi)$$

$$= |\mathbf{R}^P_{pi}| \sin CLxi$$

it follows that

$$\tan CLxi = \frac{\mathbf{R}^P_{pi} \cdot \hat{\mathbf{A}}}{\mathbf{R}^P_{pi} \cdot \hat{\mathbf{B}}} \tag{581}$$

But the coordinates of $\mathbf{R}^P_{pi}$ in the $(\hat{\mathbf{R}}_{pS}, \hat{\mathbf{A}}, \hat{\mathbf{B}})$ coordinate system are

$$(0, \hat{\mathbf{R}}_{pi} \cdot \hat{\mathbf{A}}, \hat{\mathbf{R}}_{pi} \cdot \hat{\mathbf{B}})$$

Therefore,

$$\tan CLxi = \frac{\hat{\mathbf{R}}_{pi} \cdot \hat{\mathbf{A}}}{\hat{\mathbf{R}}_{pi} \cdot \hat{\mathbf{B}}}$$

or

$$CLxi = \tan^{-1} \left(\frac{\hat{\mathbf{R}}_{pi} \cdot \hat{\mathbf{A}}}{\hat{\mathbf{R}}_{pi} \cdot \hat{\mathbf{B}}} \right) \tag{582}$$

The cone angle of body i with respect to the earth or Canopus is also defined in Fig. 62; it is the angle $COxi$, where

$COEi$, $i = 1, \cdots, 11, C$ = cone angle of body i with respect to earth

$COCi$, $i = 1, \cdots, 11$ = cone angle of body i with respect to Canopus

Clearly,

$$COxi = \cos^{-1} (\hat{\mathbf{R}}_{pi} \cdot \hat{\mathbf{R}}_{pS}) \tag{583}$$

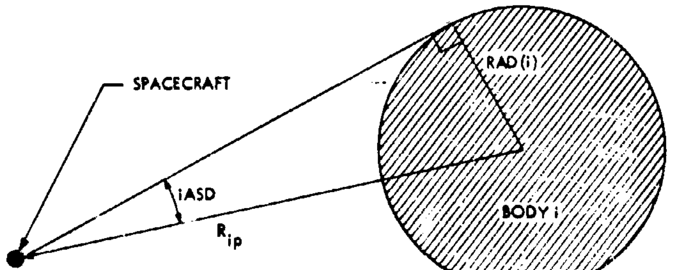

Fig. 63. The angle *iASD*

2. ***Angle iASD and limb angles.*** The angular semidiameter of body i as seen from a spacecraft is denoted by $iASD$ and defined by Fig. 63, where $i = 1,\cdots,11$.[50]

Clearly,

$$iASD = \sin^{-1}\left[\frac{RAD(i)}{|\mathbf{R}_{pi}|}\right] \tag{584}$$

where $RAD(i)$ is the radius of body i.

The angle $xPNi$ is defined by Fig. 64. Here x is either S (sun), E (earth), or C (Canopus), and $i = 1,\cdots, 11$.[51] Thus,

(1) $SPNi$ (the sun–spacecraft near-limb angle of body i).

(2) $EPNi$ (the earth–spacecraft near-limb angle of body i).

(3) $CPNi$ (the Canopus–spacecraft near-limb angle of body i).

Clearly,

$$xPNi = xPi - iASD, \qquad \text{deg} \tag{585}$$

[50]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

[51]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

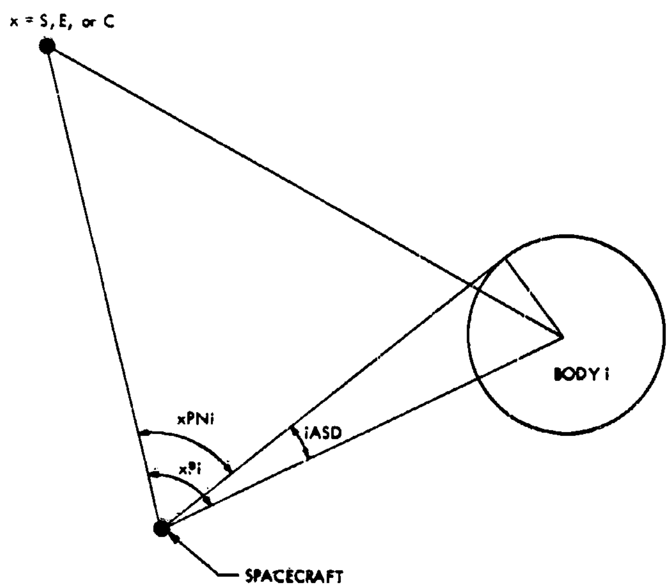

Fig. 64. The angle xPNi

where xPi and $iASD$ are given by Eqs. (578) and (584). This angle is computed to determine occultation; i.e., when the spacecraft cannot be "seen" any longer from body x.

3. ***Hinge and swivel angles.*** To define the hinge and swivel angles of body i, one needs the auxiliary unit vector $\hat{\mathbf{S}}$ (Fig. 65).

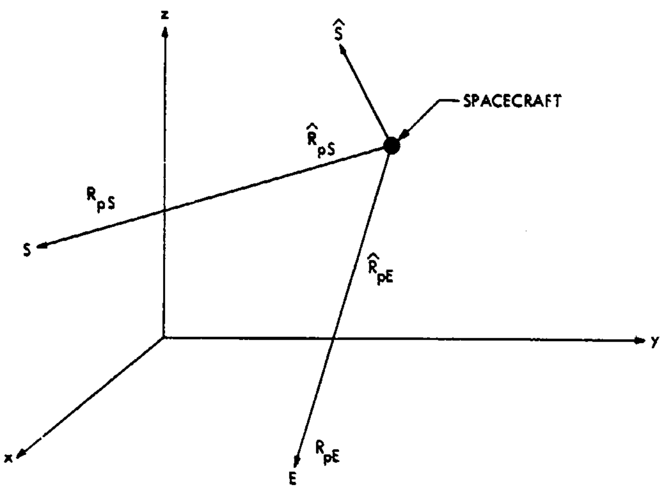

Fig. 65. The vector $\hat{\mathbf{S}}$

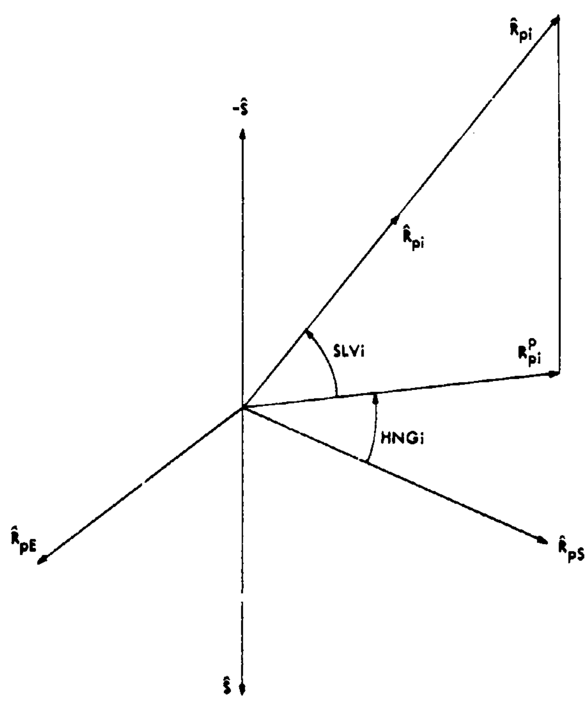

Fig. 66. Hinge and swivel angles

Vector $\hat{\mathbf{S}}$ is given[52] by the expression

$$\hat{\mathbf{S}} = \frac{\mathbf{R}_{pS} \times \mathbf{R}_{pE}}{|\mathbf{R}_{pS} \times \mathbf{R}_{pE}|} \tag{586}$$

(see Ref. 22, p. 53). The hinge angle $HNGi$ of body i is defined in Fig. 66. This angle is computed according to the formula[52]

$$HNGi = \tan^{-1}\left[\frac{\hat{\mathbf{R}}_{pi} \cdot (\hat{\mathbf{R}}_{pS} \times \hat{\mathbf{S}})}{\hat{\mathbf{R}}_{pi} \cdot \hat{\mathbf{R}}_{pS}}\right], \quad \text{deg} \tag{587}$$

[52]Warner, M. R., et al., JPL internal document, Oct. 30, 1968.

This formula may be derived in the same way as was Eq. (582).

The swivel angle $SVLi$ of body i ($i = 1,\cdots,11$) is also defined in Fig. 66. Clearly,

$$-\hat{\mathbf{S}} \cdot \hat{\mathbf{R}}_{pi} = \cos(90 - SVLi)$$

$$= \sin SVLi$$

Hence,

$$SVLi = -\sin^{-1}(\hat{\mathbf{S}} \cdot \hat{\mathbf{R}}_{pi}), \quad \text{deg} \tag{588}$$

Appendix A

Proof of Kepler's Laws

Early in the seventeenth century, Kepler empirically obtained the following three laws:

(1) Within the domain of the solar system, all planets describe elliptical paths, with the sun at one focus.

(2) The radius vector from the sun to a planet generates equal areas in equal times ("law of areas").

(3) The squares of the periods of revolution of the planets about the sun are proportional to the cubes of their mean distances from the sun.

The concept of Newtonian gravitational theory and Newton's second and third laws will be used to prove Kepler's laws in a rigorous manner.

Let a point P be fixed in space and let two point masses M_1 and M_2 move around one another subject only to each other's gravitational attraction (Fig. A-1). Newton's law of gravitation states that the force acting on body M_1 caused by body M_2 is given by the expression

$$\mathbf{F}_{12} = -G\frac{M_1M_2}{r_{12}^2}\frac{\mathbf{r}_{12}}{r_{12}} \tag{A-1}$$

where

G = universal gravitational constant; its value in the mks[53] system is

$$G = 6.673 \times 10^{-11}\ \text{N-m}^2/\text{kg}^2$$

and in the British engineering system is

$$G = 3.436 \times 10^{-8}\ \text{lb-ft}^2/\text{slug}^2$$

(Ref. 24, p. 322)

$\mathbf{r}_{12}$ = vector from M_1 to M_2

$r_{12} = |\mathbf{r}_{12}|$

From Newton's third law (every action has an opposite and equal reaction), it follows that

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \tag{A-2}$$

[53]mks = meter-kilogram-second (system of units).

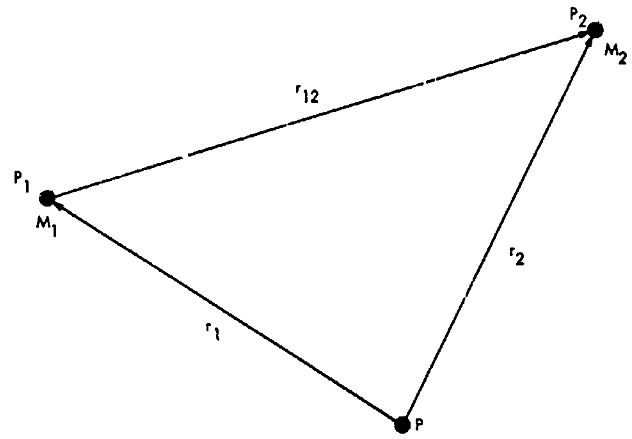

Fig. A-1. Two-body gravitational attraction

Newton's second law ($\mathbf{F} = m\mathbf{a}$) allows one to write

$$\begin{aligned}\mathbf{F}_1 &= M_1\ddot{\mathbf{r}}_1\\ \mathbf{F}_2 &= M_2\ddot{\mathbf{r}}_2\end{aligned} \tag{A-3}$$

However, again making use of Newton's law of gravitation,

$$\begin{aligned}\mathbf{F}_1 &= G\frac{M_1M_2}{|\mathbf{r}_2-\mathbf{r}_1|^2}\frac{(\mathbf{r}_2-\mathbf{r}_1)}{|\mathbf{r}_2-\mathbf{r}_1|}\\ &= G\frac{M_1M_2}{r_{12}^3}\mathbf{r}_{12}\end{aligned} \tag{A-4}$$

and, similarly,

$$\begin{aligned}\mathbf{F}_2 &= G\frac{M_1M_2}{|\mathbf{r}_1-\mathbf{r}_2|^2}\frac{(\mathbf{r}_1-\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|}\\ &= -G\frac{M_1M_2}{r_{12}^3}\mathbf{r}_{12}\end{aligned} \tag{A-5}$$

because

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i = -\mathbf{r}_{ji} \tag{A-6}$$

Combining Eqs. (A-3) through (A-5), one obtains

$$\ddot{\mathbf{r}}_1 = \frac{GM_2}{r_{12}^3}\mathbf{r}_{12} \tag{A-7}$$

$$\ddot{\mathbf{r}}_2 = -\frac{GM_1}{r_{12}^3}\mathbf{r}_{12} \tag{A-8}$$

It is obvious from Eq. (A-6) that

$$\ddot{\mathbf{r}}_{12} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1$$

thus, subtraction of Eq. (A-7) from Eq. (A-8) yields

$$\ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = -G(M_1 + M_2)\frac{\mathbf{r}_{12}}{r_{12}^3}$$

or

$$\ddot{\mathbf{r}}_{12} = -G(M_1 + M_2)\frac{\mathbf{r}_{12}}{r_{12}^3} \tag{A-9}$$

To simplify the notation, define

$$\mathbf{r} \equiv \mathbf{r}_{12}$$

hence,

$$r \equiv r_{12}$$

$$\ddot{\mathbf{r}} \equiv \ddot{\mathbf{r}}_{12}$$

also, let

$$G(M_1 + M_2) \equiv \mu$$

Then

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} \tag{A-10}$$

Equation (A-10) is the fundamental equation of relative motion for a two-body system. If one takes the vector product of Eq. (A-10) with $\mathbf{r}$, it results in

$$\mathbf{r} \times \ddot{\mathbf{r}} = -\frac{\mu}{r^3}(\mathbf{r} \times \mathbf{r})$$

or

$$\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} \tag{A-11}$$

because

$$\mathbf{r} \times \mathbf{r} = \mathbf{0}$$

But

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{r} \times \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \dot{\mathbf{r}}$$

$$= \mathbf{r} \times \ddot{\mathbf{r}} \tag{A-12}$$

thus, Eq. (A-11) implies

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{0} \tag{A-13}$$

or

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h} \tag{A-14}$$

where $\mathbf{h}$ is a constant vector. From the definition of a cross product, $\mathbf{h}$ is perpendicular to $\mathbf{r}$ and $\dot{\mathbf{r}}$; i.e., the bodies are moving in a plane, the equation of which is clearly

$$\mathbf{r} \cdot \mathbf{h} = 0 \tag{A-15}$$

The quantity $\mathbf{h}$ is called the angular-momentum vector of the system, and Eq. (A-14) shows that the angular momentum of the system is constant. If Eq. (A-10) is crossed with $\mathbf{h}$,

$$\mathbf{h} \times \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{h} \times \mathbf{r}$$

$$= -\frac{\mu}{r^3}(\mathbf{r} \times \dot{\mathbf{r}}) \times \mathbf{r} \tag{A-16}$$

Because

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$$

Eq. (A-16) may be written as follows:

$$\mathbf{h} \times \ddot{\mathbf{r}} = -\frac{\mu}{r^3}[(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\dot{\mathbf{r}} \cdot \mathbf{r})\mathbf{r}] \tag{A-17}$$

Now, $\dot{r}$ is the component of $\dot{\mathbf{r}}$ in the radial direction (Fig. A-2); i.e., $\dot{r} \neq |\dot{\mathbf{r}}|$, but rather

$$\dot{r} = v\cos\phi \tag{A-18}$$

where

$$v = |\dot{\mathbf{r}}|$$

ϕ = angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$

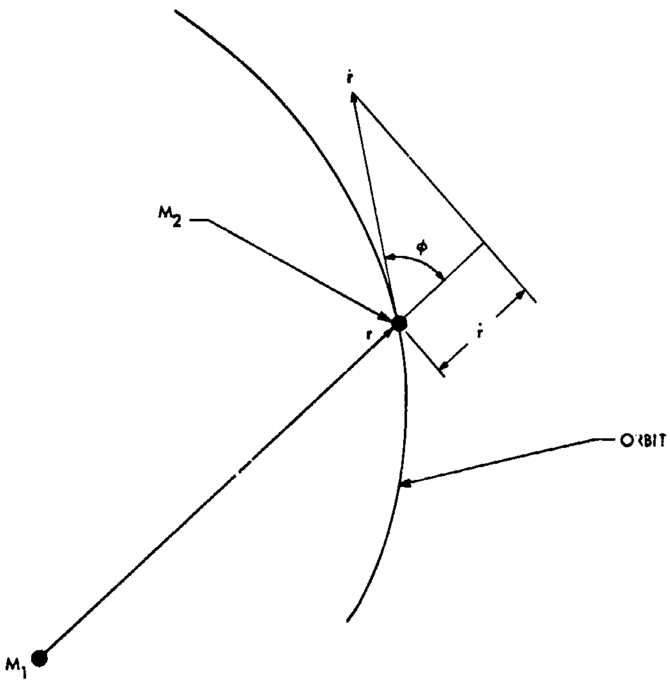

Fig. A-2. Radial component of $\dot{\mathbf{r}}$

Therefore,

$$\dot{\mathbf{r}} \cdot \mathbf{r} = r v \cos\phi = r\dot{r}$$

and the following expression for $\mathbf{h} \times \ddot{\mathbf{r}}$ is obtained:

$$\mathbf{h} \times \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\left[r^2\dot{\mathbf{r}} - (r\dot{r})\mathbf{r}\right]$$

$$= -\mu\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\dot{r}}{r^2}\right)$$

or

$$\mathbf{h} \times \ddot{\mathbf{r}} = -\mu \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \qquad \text{(A-19)}$$

Upon integrating,

$$\mathbf{h} \times \dot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{r} - \mathbf{P} \qquad \text{(A-20)}$$

where $\mathbf{P}$ is a constant vector of integration. Dotting this last result with the vector $\mathbf{h}$, one obtains

$$(\mathbf{h} \times \dot{\mathbf{r}}) \cdot \mathbf{h} = -\frac{\mu}{r}\mathbf{r} \cdot \mathbf{h} - \mathbf{P} \cdot \mathbf{h} \qquad \text{(A-21)}$$

But

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$$

so one may rewrite Eq. (A-21) as

$$\dot{\mathbf{r}} \cdot (\mathbf{h} \times \mathbf{h}) = -\frac{\mu}{r}\mathbf{r} \cdot \mathbf{h} - \mathbf{P} \cdot \mathbf{h} \qquad \text{(A-22)}$$

But

$$\mathbf{h} \times \mathbf{h} = 0$$

and

$$\mathbf{r} \cdot \mathbf{h} = 0$$

because $\mathbf{h}$ is normal to the plane of motion. Therefore,

$$\mathbf{P} \cdot \mathbf{h} = 0 \qquad \text{(A-23)}$$

Equation (A-23) indicates that $\mathbf{P}$ is normal to $\mathbf{h}$; that is, $\mathbf{P}$ is a fixed vector in the orbital plane. Upon dotting Eq. (A-20) with $\mathbf{r}$, one obtains

$$(\mathbf{h} \times \dot{\mathbf{r}}) \cdot \mathbf{r} = -\frac{\mu}{r}\mathbf{r} \cdot \mathbf{r} - \mathbf{P} \cdot \mathbf{r} \qquad \text{(A-24)}$$

From the identity

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = -\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B})$$

it follows that

$$\mathbf{h} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \mu r + \mathbf{P} \cdot \mathbf{r} \qquad \text{(A-25)}$$

and, by Eq. (A-14),

$$\mathbf{h} \cdot \mathbf{h} = h^2 = \mu r + \mathbf{P} \cdot \mathbf{r} \qquad \text{(A-26)}$$

or

$$\frac{h^2}{\mu} = r\left(1 + \frac{\mathbf{P} \cdot \mathbf{r}}{\mu r}\right) \qquad \text{(A-27)}$$

Let the unit vector

$$\hat{\mathbf{U}} = \frac{\mathbf{r}}{r}$$

make an angle ν with the constant vector $\mathbf{P}$; then

$$\frac{\mathbf{P}\cdot\mathbf{r}}{r} = \mathbf{P}\cdot\hat{\mathbf{U}}$$

$$= P\cos\nu$$

where

$$P = |\mathbf{P}|$$

and Eq. (A-27) may be rewritten in the form

$$\frac{h^2}{\mu} = r\left(1 + \frac{P}{\mu}\cos\nu\right) \qquad \text{(A-28)}$$

Letting

$$p = \frac{h^2}{\mu} \qquad \text{(A-29)}$$

$$e = \frac{P}{\mu} \qquad \text{(A-30)}$$

one obtains

$$p = r(1 + e\cos\nu) \qquad \text{(A-31)}$$

which is the general equation of a conic in polar coordinates, where p is the semilatus rectum and e is the eccentricity of the conic. This proves Kepler's first law—the orbit of an object about its primary is a conic with the primary at one focus. It may now be seen that the vector $\mathbf{P}$, because it lies in the orbital plane and makes an angle ν with $\mathbf{r}$, points toward the perifocus (the point of closest approach) of the conic (Fig. A-3).

In Fig. A-3, the path velocity

$$v = |\dot{\mathbf{r}}| \qquad \text{(A-32)}$$

has a radial component $\dot{r}$ and a transverse component $r\,\dot{\nu}$. The angular momentum is the moment of the transverse component, which is equal to

$$r(r\,\dot{\nu}) = r^2\dot{\nu}$$

$$= h \qquad \text{(A-33)}$$

An element of area dA in polar coordinates is

$$dA = \frac{1}{2}r^2\,d\nu \qquad \text{(A-34)}$$

Thus,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\nu}{dt}$$

$$= \frac{1}{2}r^2\dot{\nu} \qquad \text{(A-35)}$$

or

$$dA = \frac{1}{2}h\,dt \qquad \text{(A-36)}$$

Integrating between times $t = t_1$ and $t = t_2$ yields

$$A = \frac{1}{2}h(t_2 - t_1) \qquad \text{(A-37)}$$

Equation (A-37) proves Kepler's second law—the radius vector of the object moving on a conic section sweeps over equal areas in equal times. It remains to prove Kepler's third law.

In the case of a closed conic (i.e., an ellipse), the area over one complete orbit is

$$A = \frac{1}{2}h\,P$$

where P is the period of revolution. The area of an ellipse is given by

$$A = \pi a b$$

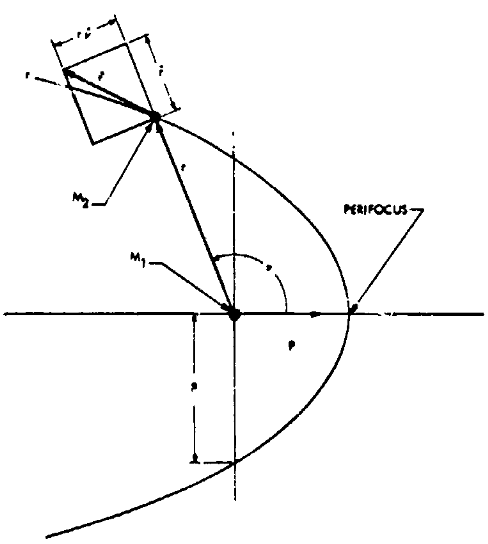

Fig. A-3. Velocities in polar reference frame

so that

$$\pi a b = \frac{1}{2} h P$$

or

$$P = \frac{2\pi a b}{h} \qquad \text{(A-38)}$$

From Eq. (A-29) is obtained

$$h = (p\mu)^{1/2} \qquad \text{(A-39)}$$

In Eq. (A-31), if one sets $\nu = 0$ and $\nu = \pi$, one obtains r_{min} and r_{max}, respectively; and, because

$$r_{min} + r_{max} = 2a \qquad \text{(A-40)}$$

where

$$a = \text{semimajor axis} \qquad \text{(A-41)}$$

it follows that

$$p = r_{min}(1 + e)$$

$$p = r_{max}(1 - e)$$

and

$$r_{min} + r_{max} = 2a$$

$$= \frac{2p}{1 - e^2}$$

or

$$p = a(1 - e^2) \qquad \text{(A-42)}$$

One may then write

$$h = [\mu a (1 - e^2)]^{1/2} \qquad \text{(A-43)}$$

For an ellipse, the relationship is

$$a^2 + b^2 = c^2 \qquad \text{(A-44)}$$

where $c = ae$, which is the distance of the focus from the center of the ellipse. Thus,

$$b = a(1 - e^2)^{1/2} \qquad \text{(A-45)}$$

If Eqs. (A-43) and (A-45) are substituted into Eq. (A-38), it follows that

$$P = \frac{2\pi a^2 (1 - e^2)^{1/2}}{(\mu a)^{1/2} (1 - e^2)^{1/2}}$$

or

$$P = \frac{2\pi a^{3/2}}{\mu^{1/2}} \qquad \text{(A-46)}$$

Hence,

$$P^2 = \frac{(2\pi)^2}{\mu} a^3$$

and Kepler's third law has been proved—the squares of the periods of revolution are proportional to the cubes of the semimajor axes.

Appendix B

Kepler's Equation

An important equation relating the position of a body to the time in orbit is Kepler's equation. It is not directly related to Kepler's laws, but is a separate and independent equation. Kepler's first law states that the planets move on conic sections, and it was shown in Appendix A that the conic may be represented in polar coordinate formulation as

$$r = \frac{p}{1 + e\cos\nu} \tag{B-1}$$

From the theory of conic sections, it follows that, provided $p \neq 0$, if $e = 0$, the conic is a circle; if $0 < e < 1$, the conic is an ellipse; if $e = 1$, the conic is a parabola; and if $1 < e < \infty$, the conic is a hyperbola.

The semimajor axis a of the orbit is $a = \infty$ for parabolic motion, $0 < a < \infty$ for elliptic motion, and $-\infty < a < 0$ for hyperbolic motion.

For future convenience, a set of axes x_ω and y_ω is introduced with the origin at the focus. The positive x_ω-axis points in the direction of perifocus, and the positive y_ω-axis is advanced by a right angle to x_ω in the orbit plane.

I. Elliptic Formulation

To derive the elliptic formulation of Kepler's equation, it is useful to relate the x_ω and y_ω parameters to the angles ν and E. The angle ν is called the *true anomaly*, and was defined in Appendix A. The auxiliary angle E is defined by Fig. B-1, and is called the *eccentric anomaly*.

For elliptic (including circular) motion, it is clear from Fig. B-1 that the coordinates of the spacecraft are given by

$$x_\omega = r\cos\nu \tag{B-2}$$

$$y_\omega = r\sin\nu \tag{B-3}$$

and, in terms of E,

$$x_\omega = a\cos E - ae \tag{B-4}$$

or

$$x_\omega = a(\cos E - e) \tag{B-5}$$

From Eq. (B-2),

$$\cos\nu = \frac{x_\omega}{r} \tag{B-6}$$

Substitution of this result into Eq. (B-1) yields

$$r + ex_\omega = p = a(1 - e^2) \tag{B-7}$$

or, making use of Eq. (B-5),

$$r = a(1 - e\cos E) \tag{B-8}$$

Because

$$r^2 = x_\omega^2 + y_\omega^2$$

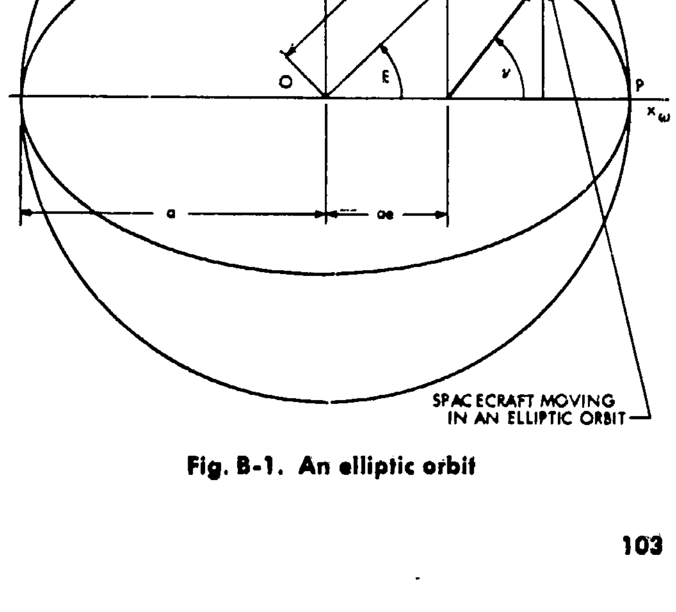

Fig. B-1. An elliptic orbit

the relation follows

$$y_\omega = r \sin \nu = a(1-e^2)^{1/2} \sin E \tag{B-9}$$

so that, in summary,

$$\left.\begin{aligned} r &= a(1 - e\cos E) \\ x_\omega &= r\cos\nu = a(\cos E - e) \\ y_\omega &= r\sin\nu = a(1-e^2)^{1/2}\sin E \end{aligned}\right\} \tag{B-10}$$

Differentiating this set of equations with respect to time, one obtains

$$\begin{aligned} \dot{r} &= a\,e\,\dot{E}\sin E \\ \dot{x}_\omega &= \dot{r}\cos\nu - r\,\dot{\nu}\sin\nu = -a\,\dot{E}\sin E \\ \dot{y}_\omega &= \dot{r}\sin\nu + r\,\dot{\nu}\cos\nu = a\,\dot{E}\,(1-e^2)^{1/2}\cos E \end{aligned} \tag{B-11}$$

Let $\hat{\mathbf{W}}$ be a unit vector normal to the plane of the orbit. Then

$$\begin{aligned} h\hat{\mathbf{W}} &= \mathbf{r} \times \dot{\mathbf{r}} \\ &= \mathbf{r} \times (\dot{\mathbf{r}}_R + \dot{\mathbf{r}}_P) \end{aligned} \tag{B-12}$$

where $\dot{\mathbf{r}}_R$, $\dot{\mathbf{r}}_P$ are velocity components of $\dot{\mathbf{r}}$ at right angles and parallel to $\mathbf{r}$, so that (by Fig. A-3)

$$|\dot{\mathbf{r}}_R| = r\dot{\nu}$$

$$|\dot{\mathbf{r}}_P| = \dot{r}$$

Clearly,

$$\begin{aligned} \mathbf{r} \times (\dot{\mathbf{r}}_R + \dot{\mathbf{r}}_P) &= \mathbf{r} \times \dot{\mathbf{r}}_R + \mathbf{r} \times \dot{\mathbf{r}}_P \\ &= \mathbf{r} \times \dot{\mathbf{r}}_R \\ &= r\,r\,\dot{\nu}\sin\frac{\pi}{2}\,\hat{\mathbf{W}} \\ &= r^2\dot{\nu}\mathbf{W} \end{aligned}$$

because

$$\mathbf{r} \times \dot{\mathbf{r}}_P = 0$$

Thus,

$$h\hat{\mathbf{W}} = r^2\dot{\nu}\hat{\mathbf{W}} \tag{B-13}$$

or

$$h = r^2\dot{\nu} \tag{B-14}$$

Therefore, the following relationships exist:

$$\begin{aligned} p &= a(1-e^2) && \text{(by Eq. A-42)} \\ &= \frac{h^2}{\mu} && \text{(by definition)} \\ &= \frac{r^4\dot{\nu}^2}{\mu} && \text{(by Eq. B-14)} \\ &= \frac{(\mathbf{r}\times\dot{\mathbf{r}})\cdot(\mathbf{r}\times\dot{\mathbf{r}})}{\mu} && \text{(by Eq. B-12)} \end{aligned} \tag{B-15}$$

The cross product $\mathbf{r} \times \dot{\mathbf{r}}$, referred to the orbital axes x_ω, y_ω, can now be formulated as

$$\begin{aligned} |\mathbf{r}\times\dot{\mathbf{r}}| &= \begin{vmatrix} 1 & 1 & 1 \\ x_\omega & y_\omega & 0 \\ \dot{x}_\omega & \dot{y}_\omega & 0 \end{vmatrix} \\ &= x_\omega\dot{y}_\omega - \dot{x}_\omega y_\omega \end{aligned} \tag{B-16}$$

or, since

$$\begin{aligned} h &= |\mathbf{r}\times\dot{\mathbf{r}}| \\ &= (\mu\,p)^{1/2} \end{aligned} \tag{B-17}$$

it follows that

$$(\mu p)^{1/2} = x_\omega\dot{y}_\omega - \dot{x}_\omega y_\omega \tag{B-18}$$

and, using Eqs. (B-10) and (B-11), one obtains

$$\begin{aligned} (\mu p)^{1/2} &= a(\cos E - e)\,a\,\dot{E}(1-e^2)^{1/2}\cos E \\ &\quad - a\,\dot{E}\sin E\,a(1-e^2)^{1/2}\sin E \\ &= a^{3/2}(p)^{1/2}(\sin^2 E + \cos^2 E - e\cos E)\,\dot{E} \end{aligned} \tag{B-19}$$

or

$$\frac{(\mu)^{1/2}}{a^{3/2}} = (1 - e\cos E)\frac{dE}{dt} \tag{B-20}$$

Upon integration,

$$\frac{(\mu)^{1/2}}{a^{3/2}}(t - t_0) = E - e\sin E \tag{B-21}$$

The epoch time t_0 corresponds to the point on the orbit where $E = 0$. This time is called the *time of perifocal passage,* and is denoted by T. If one defines

$$n = \frac{(\mu)^{1/2}}{a^{3/2}} \quad \text{(B-22)}$$

where n is called the *mean motion,* Eq. (B-21) may be rewritten in the form

$$n(t - T) = E - e \sin E \quad \text{(B-23)}$$

Equation (B-23) is Kepler's equation, and relates position in an elliptic orbit to time. The product $n(t - T)$ is called the *mean anomaly,* denoted by M; that is,

$$M = n(t - T) \quad \text{(B-24)}$$

thus,

$$M = E - e \sin E \quad \text{(B-25)}$$

II. Hyperbolic Formulation

When a spacecraft flies by a planet, the planet-centered conic section on which the spacecraft is moving is a hyperbola. To relate position and time in this kind of orbit (Fig. B-2), recourse must be taken to hyperbolic functions.

If the spacecraft P has coordinates (x_ω, y_ω), hyperbolic functions are defined a

$$\overline{DC} = a \cosh F \quad \text{(B-26)}$$

so that

$$\begin{aligned} x_\omega &= OD \\ &= q - (-a \cosh F + a) \\ &= a(e - 1) + a \cosh F - a \\ &= a(\cosh F - e) \end{aligned} \quad \text{(B-27)}$$

Repeating Eq. (B-7), i.e.,

$$r + e x_\omega = a(1 - e^2)$$

and using Eq. (B-27), one obtains

$$r = a(1 - e \cosh F) \quad \text{(B-28)}$$

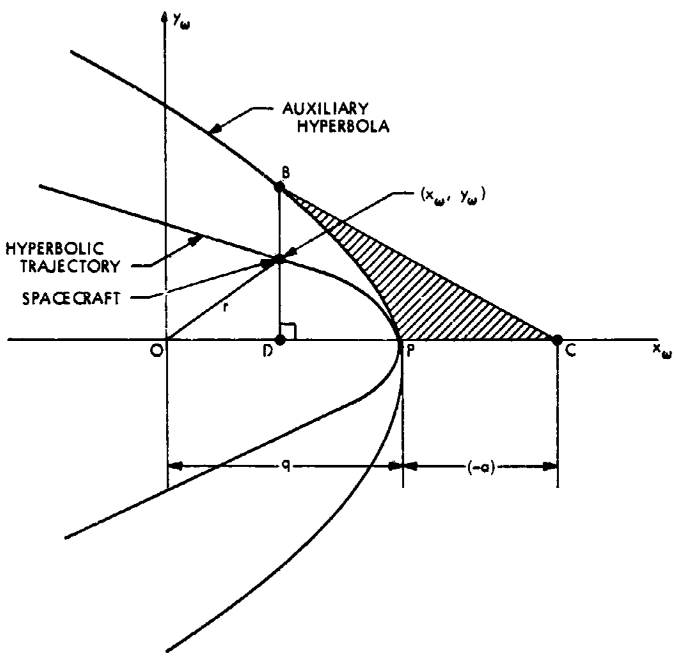

Fig. B-2. A hyperbolic orbit

Because

$$r^2 = x_\omega^2 + y_\omega^2 \quad \text{(B-29)}$$

it follows that

$$y_\omega = -a(e^2 - 1)^{1/2} \sinh F \quad \text{(B-30)}$$

and

$$\dot{x}_\omega = a\dot{F} \sinh F \quad \text{(B-31)}$$

$$\dot{y}_\omega = -a(e^2 - 1)^{1/2} \dot{F} \cosh F \quad \text{(B-32)}$$

Substituting Eqs. (B-27), (B-30), (B-31), and (B-32) into Eq. (B-18), one obtains

$$\begin{aligned} (\mu p)^{1/2} &= a(\cosh F - e)[-a(e^2 - 1)^{1/2} \dot{F} \cosh F] \\ &\quad - a\dot{F} \sinh F [-a(e^2 - 1)^{1/2} \sinh F] \\ &= a^2 (e^2 - 1)^{1/2} (-\cosh^2 F + \sinh^2 F + e \cosh F) \dot{F} \\ &= (-a)^{3/2} (p)^{1/2} (e \cosh F - 1) \dot{F} \end{aligned} \quad \text{(B-33)}$$

or

$$\frac{(\mu)^{1/2}}{(-a)^{3/2}} = (e\cosh F - 1)\frac{dF}{dt} \qquad \text{(B-34)}$$

and, upon integration of Eq. (B-34),

$$\frac{(\mu)^{1/2}}{(-a)^{3/2}}(t - t_0) = e\sinh F - F \qquad \text{(B-35)}$$

As in the case of the elliptic formulation,

$$T = t_0 \qquad \text{(B-36)}$$

denotes the time of perifocal passage; the mean hyperbolic motion is defined by

$$n = \frac{(\mu)^{1/2}}{(-a)^{3/2}} \qquad \text{(B-37)}$$

and the mean anomaly by

$$\begin{aligned} M_H &= n(t - T) \\ &= \frac{(\mu)^{1/2}}{(-a)^{3/2}}(t - t_0) \end{aligned} \qquad \text{(B-38)}$$

With this notation, Eq. (B-35) takes the form

$$M_H = e\sinh F - F \qquad \text{(B-39)}$$

This is Kepler's equation for hyperbolic motion.

III. Parabolic Formulation

In the case of a parabolic orbit, $a = \infty$; therefore, a new relationship must be established between the position and time of a body moving on a parabolic path (Fig. B-3).

The general equation of a conic in polar coordinates, in terms of the true anomaly ν, is given by Eq. (A-31) as

$$r = \frac{p}{1 - e\cos\nu} \qquad \text{(B-40)}$$

For a parabola, because $e = 1$,

$$\begin{aligned} p &= a(1 - e)(1 + e) \\ &= q(1 + e) \end{aligned}$$

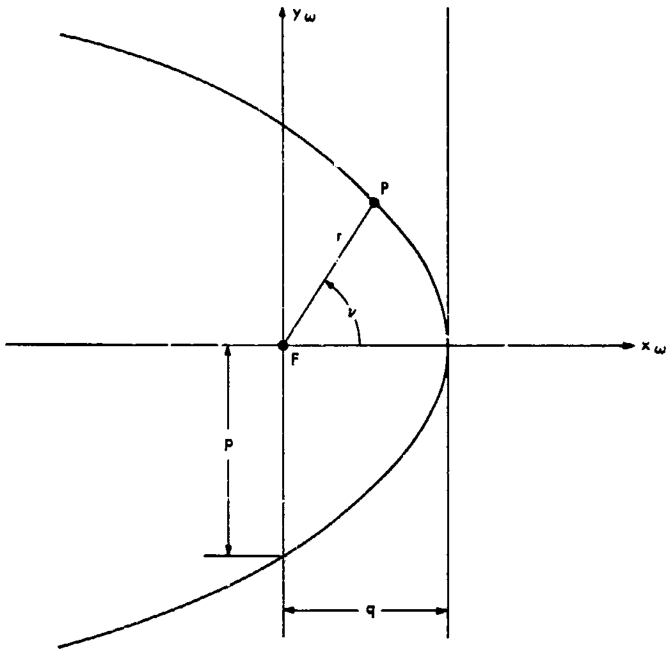

Fig. B-3. A parabolic orbit

or

$$p = 2q \qquad \text{(B-41)}$$

A trigonometric identity states that

$$1 + \cos\nu = 2\cos^2\frac{1}{2}\nu$$

Thus, Eq. (B-40) may be rewritten as

$$r = \left(\sec^2\frac{1}{2}\nu\right)q \qquad \text{(B-42)}$$

Upon squaring Eq. (B-42), and multiplying both sides by $\dot\nu$, one obtains

$$r^2\dot\nu = \left(\sec^2\frac{1}{2}\nu\right)\left(\sec^2\frac{1}{2}\nu\right)q^2\dot\nu \qquad \text{(B-43)}$$

and, from Eq. (B-15),

$$(2q\mu)^{1/2} = r^2\dot\nu \qquad \text{(B-44)}$$

Therefore,

$$(2q\mu)^{1/2}dt = q^2\left(1 + \tan^2\frac{1}{2}\nu\right)\left(\sec^2\frac{1}{2}\nu\right)d\nu \qquad \text{(B-45)}$$

and

$$\frac{(2\mu)^{1/2}}{q^{3/2}}\int_{t_0}^{t} dt = \int_0^{\nu} \sec^2\left(\frac{\nu}{2}\right)d\nu + \int_0^{\nu} \tan^2\left(\frac{\nu}{2}\right)\sec^2\left(\frac{\nu}{2}\right)d\nu \tag{B-46}$$

Equation (B-46) can be integrated at once to yield

$$\frac{(2\mu)^{1/2}}{q^{3/2}}(t - t_0) = 2\left(\tan\frac{\nu}{2} + \tan^3\frac{\nu}{2}\right) \tag{B-47}$$

or

$$\frac{(\mu)^{1/2}}{(2)^{1/2}q^{3/2}}(t - t_0) = \tan\frac{\nu}{2} + \frac{1}{3}\tan^3\frac{\nu}{2} \tag{B-48}$$

Equation (B-48) is known as Barker's equation; it is the parabolic form of Kepler's equation. As in the other two cases, $T = t_0$ denotes the time of perifocal passage.

By introduction of the parameter D, defined by

$$D \equiv \frac{r\dot{r}}{(\mu)^{1/2}} = (2q)^{1/2}\tan\frac{\nu}{2} \tag{B-49}$$

Eq. (B-48) may be transformed into a cubic:

$$M_P = qD + \frac{D^3}{6} \tag{B-50}$$

with

$$M_P = n(t - T) \tag{B-51}$$

and

$$\begin{aligned} n &= \text{parabolic mean motion} \\ &\equiv (\mu)^{1/2} \end{aligned} \tag{B-52}$$

In summary, it has been shown that, for various types of motion, there exist different forms of Kepler's equation; that is,

$$M = E - e\sin E \quad \text{(for elliptic motion)} \tag{B-53a}$$

$$M_H = e\sinh F - F \quad \text{(for hyperbolic motion)} \tag{B-53b}$$

$$M_P = qD + \frac{D^3}{6} \quad \text{(for parabolic motion)} \tag{B-53c}$$

Appendix C

Solution of Kepler's Equation

For convenience, the three forms of Kepler's equation will be repeated here:

$$M = E - e \sin E \qquad \text{(elliptic motion)} \tag{C-1a}$$

$$M_H = e \sinh F - F \qquad \text{(hyperbolic motion)} \tag{C-1b}$$

$$M_P = qD + \frac{D^3}{6} \qquad \text{(parabolic motion)} \tag{C-1c}$$

The mean anomaly M or M_H can be found at once when $(t - T)$ is given, after which Eq. (C-1a) or (C-1b) must be solved for E or F. The solution of Kepler's equation in the case of a parabolic orbit will not be considered here, principally because parabolic orbits do not occur in practice, although some comets have nearly parabolic paths ($e = 0.96$). When E or F is known, r and ν can be found from Eqs. (B-10), (B-28), and (B-1). Because r and ν are important quantities for orbit determination, much attention has been devoted to the solution of Kepler's equation, and many methods of solving it have been discovered.

Obviously, Kepler's equation is a transcendental equation in E (or F, or ν), and the solution for this quantity cannot be expressed in a finite number of terms. Some iterative technique is usually employed for solving transcendental equations—e.g., Newton's method or the method of false position (regula falsi)—because they can easily be programmed for use on electronic computers.

For the elliptic case (that is, $0 \le e < 1$), it will be shown that Kepler's equation has one (and only one) real solution for every value of M and e in the range stated above.

If Eq. (C-1a) is rewritten in the form

$$f(E) \equiv E - e \sin E - M = 0 \tag{C-2}$$

and M is assigned some given (fixed) value between $n\pi$ and $(n + 1)\pi$, where n is an integer, then exactly one real value of E exists that satisfies this equation, and it lies between $n\pi$ and $(n + 1)\pi$. The function $f(E)$, when $E = n\pi$, is

$$f(n\pi) = n\pi - M < 0 \tag{C-3}$$

because, by hypothesis, $n\pi < M < (n + 1)\pi$.

When $E = (n + 1)\pi$, then

$$f[(n + 1)\pi] = (n + 1)\pi - M > 0 \tag{C-4}$$

Consequently, there is an odd number of real solutions for E that lie between $n\pi$ and $(n + 1)\pi$. However, the derivative

$$f'(E) = 1 - e \cos E \tag{C-5}$$

is always positive because

$$|e \sin E| \le e < 1 \tag{C-6}$$

Therefore, $f(E)$ is a monotonic function of E and takes the value of zero only once on the interval $[n\pi, (n + 1)\pi]$.

From Eqs. (C-3) and (C-4), it follows that the root of the function given by Eq. (C-2) can be bracketed; hence, one can find values both to the left and to the right of the true solution E^* of Eq. (C-2). If E is replaced by $M + e$ in Eq. (C-2), then

$$f(M + e) = (M + e) - e \sin(M + e) - M$$

$$= e[1 - \sin(M + e)] \ge 0 \tag{C-7}$$

and if the argument of the function f given by Eq. (C-2) is chosen to be

$$\frac{M + e}{2^n}$$

30

then

$$f\left(\frac{M+e}{2^n}\right) = \frac{M+e}{2^n} - e\sin\left(\frac{M+e}{2^n}\right) - M$$

$$= e\left[\frac{1}{2^n} - \sin\left(\frac{M+e}{2^n}\right)\right] - M\left(1 - \frac{1}{2^n}\right) \tag{C-8}$$

and, for some n (say, $n = k$),

$$f\left(\frac{M+e}{2^k}\right) < 0$$

The method of false position (Fig. C-1) is described by

$$x_{\nu+1} = x_\nu - f(x_\nu)\frac{x_\nu - x_{\nu-1}}{f(x_\nu) - f(x_{\nu-1})}, \qquad \nu = 1, 2, \cdots \tag{C-9}$$

As starting points x_0 and x_1, one may choose, according to the above discussion,

$$x_0 = M + e \tag{C-10}$$

$$x_1 = \frac{M+e}{2^k} \tag{C-11}$$

where k is such an integer that

$$f\left(\frac{M+e}{2^k}\right) < 0$$

(f is the function defined by Eq. C-2).

The method of false position is said to have converged when

$$|f(x_\nu)| < \epsilon$$

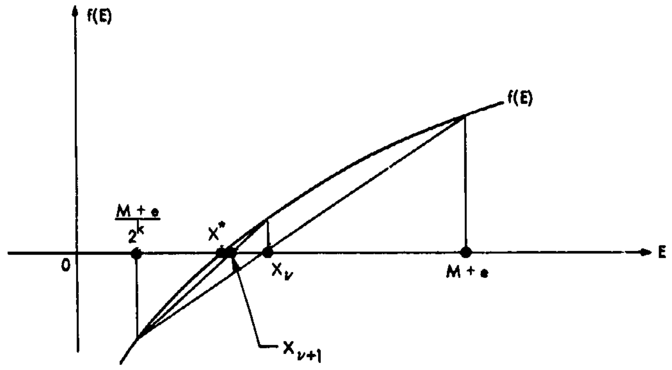

Fig. C-1. The method of false position

where ϵ is a given small number. When this method is employed on an automatic computing machine, an iteration counter should limit the maximum number of iterations. Figure C-1 is not meant to imply that one of the x_ν is always fixed; for the function sketched, the point x_1 happened to be stationary. Equation (C-9) indicates that, in general, not all lines pass through one of the original estimates.

The method of false position has the disadvantage that two successive iterates, x_0 and x_1, must be estimated before the recursion formula can be used. However, only one function evaluation $f(x_\nu)$ is required at each step because the previous value $f(x_{\nu-1})$ may be retained. It can be shown (see Ref. 19, p. 101) that the method of false position is of the order of ≈ 1.618; that is, the convergence is not quite as good as that obtained by second-order methods (e.g., Newton's method).

It remains to be demonstrated that, for the hyperbolic case ($e > 1$), the root of the equation

$$f(F) = e\sinh F - F - M_H = 0 \tag{C-12}$$

can be bracketed; the method of false position may then be used to solve Eq. (C-12) for F.

In Eq. (C-12), let

$$\bar{F} = (6M_H)^{1/3} \tag{C-13}$$

Then

$$f\left[(6M_H)^{1/3}\right] = e(6M_H)^{1/3}\left\{1 + \frac{[(6M_H)^{1/3}]^2}{3!} + \frac{[(6M_H)^{1/3}]^4}{5!} + \cdots\right\} - (6M_H)^{1/3} - M_H \tag{C-14}$$

But

$$e\,(6M_H)^{1/3} > (6M_H)^{1/3} \tag{C-15}$$

because $e > 1$. Also,

$$M_H = \frac{[(6M_H)^{1/3}]^3}{6} \tag{C-16}$$

thus,

$$e\,(6M_H)^{1/3}\left\{1 + \frac{[(6M_H)^{1/3}]^2}{3!} + \frac{[(6M_H)^{1/3}]^4}{5!} + \cdots\right\}$$

$$- (6M_H)^{1/3} - M_H > \frac{[(6M_H)^{1/3}]^5}{5!} + \cdots > 0 \tag{C-17}$$

that is,

$$f\,[(6M_H)^{1/3}] > 0 \tag{C-18}$$

From Eq. (C-14), it follows that, for some (positive) integer n,

$$f\left[\frac{(6M_H)^{1/3}}{2^n}\right] < 0 \tag{C-19}$$

Therefore, the method of false position may be employed to find the root of Eq. (C-12). Two initial successive iterates are given by

$$x_0 = (6M_H)^{1/3} \tag{C-20}$$

and

$$x_1 = \frac{(6M_H)^{1/3}}{2^k} \tag{C-21}$$

where k is such an integer that

$$f\left[\frac{(6M_H)^{1/3}}{2^k}\right] < 0 \tag{C-22}$$

(f is the function defined by Eq. C-12).

Appendix D

The Vis-Viva Integral

A body b, subject to no frictional forces, moving in the gravitational field of another body B, moves faster as it approaches the attracting mass because some of its potential energy has been converted into kinetic energy. It is shown in physics that the loss in potential energy equals the increase in kinetic energy (if there are no frictional forces); in other words, the sum of kinetic and potential energy is constant. The kinetic energy (KE) of body b having mass m and moving with velocity v is equal to $1/2\ mv^2$; the potential energy (PE) of this body is given by the expression

$$\mathrm{PE} = -\frac{\mu m}{r} \tag{D-1}$$

where

r = distance of b from B

$\mu = G(M + m)$

where

G = universal gravitational constant

M = mass of body B

The expression given for the PE is negative because, by convention, it is taken to be zero at $r = \infty$; hence, PE grows more negative as r increases, approaching negative infinity as r approaches zero. For simplicity, and without loss of generality, one may assume that body b has unit mass; its total energy is then given by the expression

$$\frac{1}{2}v^2 - \frac{\mu}{r} \tag{D-2}$$

and, as mentioned above, this sum is constant. The constant to which this expression is equal is evaluated below.

A unit mass is assumed to move on an elliptic path about a mass M. At any point in its orbit, its velocity v has a radial component v_r and a transverse component v_T; that is,

$$v^2 = v_r^2 + v_T^2 \tag{D-3}$$

where

$$v = |\mathbf{v}|$$

$$v_r = |\mathbf{v}_r|$$

$$v_T = |\mathbf{v}_T|$$

The longest and shortest distances of the unit mass from mass M (which, by Kepler's second law, is located at one of the foci of the ellipse) occur at opposite ends of the semimajor axis. These points are called apogee and perigee, respectively, and their distances from the mass M are labeled r_{max} and r_{min} (Fig. D-1). It should be noted that, at apogee and perigee, the velocity is normal to the radius vector.

Let r_i denote either r_{min} or r_{max}, and let v_i denote the corresponding velocities at r_i. By Kepler's second law, the angular momentum is constant; hence,

$$r v_T = r_i v_i \tag{D-4}$$

or

$$v_i = \left(\frac{r}{r_i}\right) v_T \tag{D-5}$$

where

$$r = |\mathbf{r}|$$

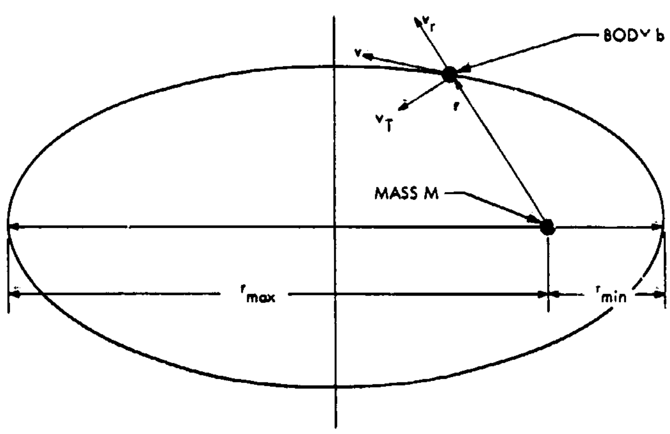

Fig. D-1. Velocity components of body *b*

Also,

$$\frac{1}{2}v^2 - \frac{\mu}{r} = \frac{1}{2}v_i^2 - \frac{\mu}{r_i} \tag{D-6}$$

because the sum of KE and PE is constant.

Using Eq. (D-5), one may write

$$\frac{1}{2}v^2 = \frac{\mu}{r} = \frac{1}{2}\left(\frac{r}{r_i}\right)^2 v_T^2 - \frac{\mu}{r_i} \tag{D-7}$$

or

$$\left(v^2 - \frac{2\mu}{r}\right) r_i^2 + 2\mu r_i - r^2 v_T^2 = 0 \tag{D-8}$$

This quadratic equation in r_i has two solutions. Because there are two values of i, each solution gives one of the r_i, but

$$r_{\max} + r_{\min} = 2a \tag{D-9}$$

where a is the semimajor axis.

From the theory of quadratic equations, the sum of the two solutions of

$$Ax^2 + Bx + C = 0 \tag{D-10}$$

is equal to $-B/A$. It may be deduced, therefore, from Eq. (D-8), that

$$2a = r_{\min} + r_{\max} = -\frac{2\mu}{v^2 - \dfrac{2\mu}{r}} \tag{D-11}$$

or

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{D-12}$$

Equation (D-12) is the so-called *vis-viva integral* or *vis-viva equation.* It follows that the sum of KE and PE of body b is equal to

$$\frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \tag{D-13}$$

It is remarkable that the velocity of a body in orbit depends upon only the semimajor axis of the conic on which it is moving.

Although the vis-viva integral has been derived herein only for the case in which the conic is an ellipse, the vis-viva integral is generally valid. Hence, in the case of a hyperbola,

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{D-14}$$

A parabola is the limiting case of an ellipse in which the semimajor axis a becomes infinite. In this case, Eq. (D-12) takes the form

$$v^2 = \frac{2\mu}{r} \tag{D-15}$$

Appendix E

Linearized Flight Time

In this appendix, the two trajectories shown in Fig. E-1 will be considered. The spacecraft enters the sphere of influence on the standard trajectory at time t_1 with hyperbolic excess velocity $V_{\infty S}$, descends vertically to the target, and arrives at time t_2. The spacecraft enters the sphere of influence on a perturbed trajectory at time $(t_1 + \Delta r/V_\infty)$ with hyperbolic excess velocity V_∞, and descends to the point of closest approach at time t_3. From Fig. E-1, the following definition applies:

$$\Delta T_f = t_3 - t_2 \tag{E-1}$$

From Kepler's equation for hyperbolic motion,

$$\Delta t = t_3 - t_1 = \frac{e \sinh F - F}{n} \tag{E-2}$$

where n is the mean motion and

$$n = \left(\frac{\mu_c}{|a|^3}\right)^{1/2} \tag{E-3}$$

Because

$$V^2 = \mu_c\left(\frac{2}{r} - \frac{1}{a}\right) \qquad \text{(vis-viva integral)} \tag{E-4}$$

for the hyperbolic excess velocity, it follows that

$$V_\infty^2 = -\frac{\mu_c}{a} \tag{E-5}$$

or

$$\frac{1}{V_\infty^3} = \frac{-a^{3/2}}{\mu_c^{3/2}}$$

Thus,

$$\frac{\mu_c}{V_\infty^3} = \left(\frac{|a|^3}{\mu_c}\right)^{1/2} \tag{E-6}$$

and Eq. (E-3) becomes

$$n = \frac{V_\infty^3}{\mu_c}$$

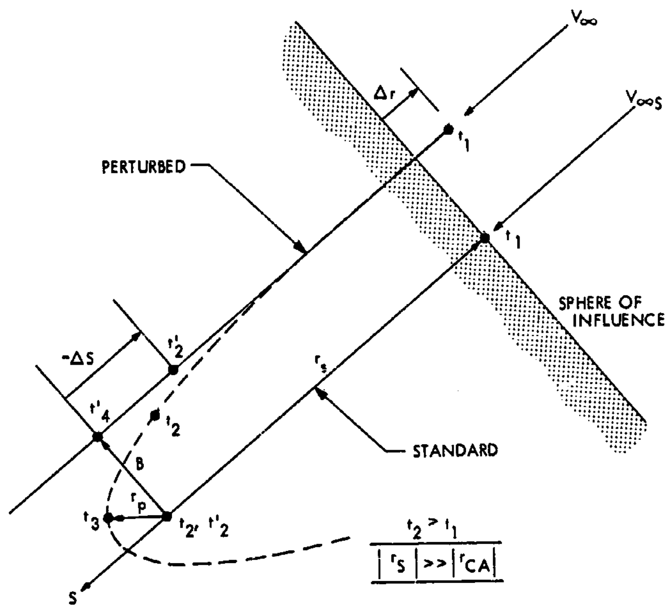

Fig. E-1. Trajectory approach geometry

Substitution of this expression for n into Eq. (E-2) yields

$$\Delta t = \frac{\mu_c}{V_\infty^3}(e \sinh F - F) \tag{E-7}$$

For a hyperbolic orbit,

$$r = a(1 - e \cosh F) \tag{E-8}$$

hence,

$$1 - e \cosh F = \frac{r}{a}$$

or

$$\cosh F = \frac{a - r}{ae} \tag{E-9}$$

If the well-known identity

$$\cosh^{-1} x = \log\left[x + (x^2 - 1)^{1/2}\right]$$

is used, it follows from Eq. (E-8) that

$$F = \log\left\{\frac{a-r}{ae} + \left[\left(\frac{a-r}{ae}\right)^2 - 1\right]^{1/2}\right\} \tag{E-10}$$

Now

$$\begin{aligned}\sinh F &= (\cosh^2 F - 1)^{1/2} \\ &= \left(\frac{a^2 - 2ar + r^2 - a^2e^2}{a^2e^2}\right)^{1/2}\end{aligned}$$

so that

$$e\sinh F = \left[1 - 2\frac{r}{a} + \left(\frac{r}{a}\right)^2 - e^2\right]^{1/2}$$

or

$$e\sinh F = \left[\left(1 - \frac{r}{a}\right)^2 - e^2\right]^{1/2} \tag{E-11}$$

Substitution of Eqs. (E-10) and (E-11) into Eq. (E-7) yields

$$\begin{aligned}\Delta t = \frac{\mu_c}{V_\infty^3}\Bigg\{&\left[\left(1 - \frac{r}{a}\right)^2 - e^2\right]^{1/2} \\ &- \log\left[\frac{a-r}{ae} + \sqrt{\left(\frac{a-r}{ae}\right)^2 - 1}\right]\Bigg\}\end{aligned} \tag{E-12}$$

Let us now consider the expression

$$\frac{\mu_c}{V_\infty^3}\left[\left(1 - \frac{r}{a}\right)^2 - e^2\right]^{1/2} \tag{E-13}$$

which is the first term of Eq. (E-12). This term may be rewritten as

$$\frac{\mu_c}{V_\infty^3}\left[\left(1 - \frac{r}{a}\right)^2 - e^2\right]^{1/2} = \frac{\mu_c}{V_\infty^3}\left(1 - \frac{2r}{a} + \frac{r^2}{a^2} - e^2\right)^{1/2} = \frac{\mu_c}{V_\infty^3}\left(1 + \frac{2rV_\infty^2}{\mu_c} + \frac{r^2V_\infty^4}{\mu_c^2} - e^2\right)^{1/2} \tag{E-14}$$

as, according to Eq. (E-5),

$$-\frac{1}{a} = \frac{V_\infty^2}{\mu_c}$$

Equation (E-14) may be rewritten as

$$\begin{aligned}\frac{\mu_c}{V_\infty^3}\left(1 + \frac{2rV_\infty^2}{\mu_c} + \frac{r^2V_\infty^4}{\mu_c^2} - e^2\right)^{1/2} &= \left(\frac{\mu_c^2}{V_\infty^6} + \frac{2\mu_c r}{V_\infty^4} + \frac{r^2}{V_\infty^2} - \frac{e^2\mu_c^2}{V_\infty^6}\right)^{1/2} \\ &= \frac{r}{V_\infty}\left(\frac{\mu_c^2}{V_\infty^4 r^2} + \frac{2\mu_c}{rV_\infty^2} + 1 - \frac{e^2\mu_c^2}{r^2V_\infty^4}\right)^{1/2}\end{aligned} \tag{E-15}$$

This last expression is approximately equal to

$$\frac{r}{V_\infty} \tag{E-16}$$

when $|\mathbf{r}| \gg |\mathbf{r}_p|$, that is,

$$\frac{\mu_c}{V_\infty^3}\left[\left(1 - \frac{r}{a}\right)^2 - e^2\right]^{1/2} = \frac{r}{V_\infty} \text{ for } |\mathbf{r}| \gg |\mathbf{r}_p| \tag{E-17}$$

The second term on the right side of Eq. (E-12) is

$$-\frac{\mu_c}{V_\infty^3}\log\left[\frac{a-r}{ae}+\sqrt{\left(\frac{a-r}{ae}\right)^2-1}\right] \quad \text{(E-18)}$$

which may be rewritten as

$$-\frac{\mu_c}{V_\infty^3}\log\left[\frac{a-r}{ae}+\sqrt{\left(\frac{a-r}{ae}\right)^2-1}\right]=-\frac{\mu_c}{V_\infty^3}\log\left[\frac{a-r+(a^2-2ar+r^2-a^2e^2)^{1/2}}{ae}\right]$$

$$=-\frac{\mu_c}{V_\infty^3}\log\left[\frac{1-\frac{r}{a}+\left(1-\frac{2r}{a}+\frac{r^2}{a^2}-e^2\right)^{1/2}}{e}\right]$$

$$=-\frac{\mu_c}{V_\infty^3}\log\left[1-\frac{r}{a}+\left(1-\frac{2r}{a}+\frac{r^2}{a^2}-e^2\right)^{1/2}\right]+\frac{\mu_c}{V_\infty^3}\cdot\log e \quad \text{(E-19)}$$

Now,

$$1-\frac{r}{a}+\left(1-\frac{2r}{a}+\frac{r^2}{a^2}-e^2\right)^{1/2}=r\left[\frac{1}{r}-\frac{1}{a}+\left(\frac{1}{r^2}-\frac{2}{ar}+\frac{1}{a^2}-\frac{e^2}{r^2}\right)^{1/2}\right] \quad \text{(E-20)}$$

which, for $|\mathbf{r}| \gg |\mathbf{r}_p|$, is approximately

$$r\left[-\frac{1}{a}+\left(\frac{1}{a^2}\right)^{1/2}\right]=r\left[-\frac{1}{a}+\left(-\frac{1}{a}\right)\right]=-\frac{2r}{a}=\frac{2rV_\infty^2}{\mu_c} \quad \text{(E-21)}$$

From the last term of Eq. (E-19) and the second term on the right side of Eq. (E-21), it follows that, for $|\mathbf{r}| \gg |\mathbf{r}_p|$,

$$-\frac{\mu_c}{V_\infty^3}\log\left\{\frac{a-r}{ae}+\left[\left(\frac{a-r}{ae}\right)^2-1\right]^{1/2}\right\}=-\frac{\mu_c}{V_\infty^3}\log\frac{2rV_\infty^2}{\mu_c}+\frac{\mu_c}{V_\infty^3}\log e \quad \text{(E-22)}$$

Substituting Eqs. (E-17) and (E-22) into Eq. (E-12), one obtains the following expression for Δt:

$$\Delta t = t_3 - t_1 = \frac{r}{V_\infty}-\frac{\mu_c}{V_\infty^3}\log\frac{2rV_\infty^2}{\mu_c}+\frac{\mu_c}{V_\infty^3}\log e$$

$$|\mathbf{r}| \gg |\mathbf{r}_p| \quad \text{(E-23)}$$

where

$$r = |\mathbf{r}_S| + \Delta r$$

From Eq. (E-23) and Fig. E-1, it follows that

$$t_2 - t_1 = \frac{r_S}{V_{\infty S}}-\frac{\mu_c}{V_{\infty S}^3}\log\frac{2r_S V_{\infty S}^2}{\mu_c} \quad \text{(E-24)}$$

In this case, because there is no rectilinear motion, no correction term (as in Eq. E-23) is required. From Eqs. (E-23) and (E-24) is obtained

$$\Delta T_f = t_3 - t_2 = \frac{r}{V_\infty}-\frac{r_S}{V_{\infty S}}+\frac{\mu_c}{V_\infty^3}\log e \quad \text{(E-25)}$$

Now

$$\frac{r_S}{r}\approx 1$$

and[54]

$$V_{\infty S} = V_\infty$$

[54]Thornton, T. H., JPL internal document, Mar. 1, 1962.

Hence, Eq. (E-25) takes the form

$$\Delta T_f = \frac{\mu_c}{V_\infty^3} \log e \tag{E-26}$$

This establishes Eq. (517) in Section X-B.

From the well-known identity

$$\sinh^{-1} x = \log [x + (x^2 + 1)^{1/2}]$$

it follows that

$$e = x + (x^2 + 1)^{1/2}$$

or

$$e - x = (x^2 + 1)^{1/2}$$

Thus,

$$x = \frac{e^2 - 1}{2e}$$

and

$$\log e = \sinh^{-1}\left(\frac{e^2 - 1}{2e}\right) \tag{E-27}$$

By use of Eqs. (E-6) and (E-27), Eq. (E-26) may be rewritten as

$$\Delta T_f = \left(\frac{|a|^3}{\mu_c}\right)^{1/2} \sinh^{-1}\left(\frac{e^2 - 1}{2e}\right)$$

which shows that Eqs. (517) and (518) in Section X-B are indeed equivalent.

Glossary

Acceleration: 1. Rate of change of velocity. 2. Act or process of accelerating, or the state of being accelerated. Negative acceleration is called deceleration.

Angular momentum vector: Cross product (vector product) of the position vector of a body and its velocity vector relative to its primary; usually angular momentum vector per unit mass.

Anomalistic period: Interval between two successive perigee passes of a satellite in orbit about its primary. Also called perigee-to-perigee period.

Apo-: In the orbit of one body about another, this prefix indicates the greatest separation.

Apofocus: The apsis on an elliptic orbit farthest from the principal focus or center of force.

Apogee: 1. Point on an ellipse farthest from the principal focus or the center of force. 2. Point on a geocentric elliptic orbit farthest from the center of the earth.

Apsis (pl. *apsides*)*:* Point on a conic where $dr/dt = 0$; i.e., where the radius vector is a minimum or a maximum.

Areo-: Combining form of Mars (Ares), as in *areocentric.*

Argument of latitude: Angle in the orbit from the ascending node to the object; the sum of the argument of perifocus and the true anomaly.

Argument of perifocus: Angular distance measured in the orbit plane from the line of nodes to the line of apsides.

Ascending node: Point at which the orbital plane of a body crosses a fixed plane (e.g., the ecliptic) with a positive component of velocity in the z direction.

Astronomical unit: A unit of length; the mean distance or semimajor axis of the orbit of a fictitious unperturbed planet having the mass 0.000002819 $m_{\odot}$ (mass of sun) and sidereal period (365.2563835 mean solar days) that Gauss adopted for the earth in his original determination of the gravitational constant k_s ($= 0.01720209895$). According to the IAU system of constants (see Ref. 10, p. 34), 1 astronomical unit = 1 AU = 149,600,000 km.

Atomic clock: A timekeeping device controlled by the frequency of the natural vibrations of certain atoms (e.g., cesium atoms).

Atomic time: The unit of atomic time is derived from the atomic resonance corresponding to transition between the two hyperfine levels of the ground state of cesium 133. The frequency of this resonance is 9,192,631,770 Hz (ephemeris time). The epoch for atomic time (A.1) is supplied by definition as follows: At the epoch 1958 January 1, at $0^h0^m0^s$ UT2, A.1 was precisely $0^h0^m0^s$ (see Ref. 10, p. 16).

Autumnal equinox: See *Equinoxes.*

Azimuth: Coordinate in the horizontal coordinate system that is measured westward in the plane of the horizon from the prime vertical to the intersection of the vertical circle through the object with the horizon.

Glossary (contd)

Azimuth angle: An angle measured at the center of the celestial sphere in the plane of the elevated pole (north or south to agree with the latitude) east or west through 180 deg.

Backward difference (∇)*:* Defined as

$$\nabla f(x) = f(x) - f(x - h)$$

where h is spacing of tabulated abscissas; used for interpolation and related processes near a tabular point at the end of the tabulated range.

Barker's equation: An equation that relates position to time for an object traveling on a parabolic orbit:

$$(\mu)^{1/2}(t - T) = q\,D + \frac{D^3}{6}$$

where

$$D = \frac{r\dot{r}}{(\mu)^{1/2}} = (2q)^{1/2}\tan\frac{\nu}{2}$$

Barycenter: Center of mass of a system of masses.

Besselian year: Period of one complete orbit of the fictitious mean sun in right ascension, beginning at the instant when the right ascension is 18 h 40 min. This instant, designated by the notation ".0" after the year (e.g., 1950.0), always falls near the beginning of the Gregorian calendar year. The Besselian year is shorter than the tropical year by the amount of $0^{s}\!.148\ T$, where T is measured in tropical centuries from 1900.0.

Body-fixed coordinate system: The xy-plane is the true of-date equatorial plane of the body; the x-axis points toward the prime meridian of the body, the z-axis is normal to the xy-plane, and the y-axis completes the right-handed coordinate system.

Celestial sphere: A hypothetical sphere of infinite dimensions centered at the observer (or center of the earth, sun, etc.).

Collision parameter: Offset distance between the extension of a velocity vector of an object at a great distance from the center of attraction or repulsion and this center.

Conic (section): 1. A curve formed by the intersection of a plane and a right circular cone. 2. In reference to satellite orbital parameters, such a curve formed without consideration of the perturbing effects of the actual shape or distribution of mass of the primary.

Coordinate time: Identified with the ideal of uniform time on which the definition of ephemeris time is based. For a transformation from proper time to coordinate time, see Ref. 10, p. 16.

Glossary (contd)

Covariance matrix: A symmetric matrix that has diagonal elements consisting of the square of the standard deviations σ_i of the parameters q_i of the parameter vector $\mathbf{q}$; the off-diagonal elements are of the form $\sigma_i \sigma_j \rho_{ij}$, where ρ_{ij} is the correlation coefficient between parameters q_i and q_j.

Cowell's method: Direct step-by-step integration of the total acceleration, central as well as perturbative, of a body or a spacecraft. It is a simple, straightforward method, but has the disadvantage that a large number of significant figures must be carried because of the large central force term.

Cross product (or *vector product*): An operation on two vectors; say, **A** and **B**, denoted by $\mathbf{A} \times \mathbf{B}$, which is defined as a vector normal to both **A** and **B** with magnitude $|\mathbf{A}||\mathbf{B}| \sin (\measuredangle \mathbf{A},\mathbf{B})$. The resultant vector **C** is computed according to

$$\mathbf{C} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\hat{\mathbf{I}} + (A_z B_x - A_x B_z)\hat{\mathbf{J}} + (A_x B_y - A_y B_x)\hat{\mathbf{K}}$$

where the subscripts denote the components of the vectors on the three orthogonal axes denoted by the unit vectors $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$.

Declination: Arc of an hour circle (great circles passing through the poles) intercepted between the celestial equator and the object.

Definitive orbit: An orbit defined in a highly precise manner, with due regard taken for accurate constants and observational data, and precision computational techniques, including perturbations.

Differential correction: A method of finding small corrections from the observed minus-computed residuals, which, when applied to the elements or constants, will reduce the deviations from the observed motion to a minimum.

Discrete integration method: A method of finding approximate values of the solution of a differential equation on a set of discrete points.

Diurnal: Daily.

Doppler shift: 1. A shift in observed frequency when the source of the frequency is receding from or approaching relative to the observer. If v_r is the radial velocity of the moving transmitter, the doppler shift f_d (i.e., the difference between the true frequency at the transmitter and that observed at the receiver) is given by

$$f_d = f_0 \frac{v_r}{c}$$

or

$$v_r = \frac{f_d}{f_0} c$$

where f_0 is the true frequency and c is the speed of light. 2. The magnitude of the doppler effect, measured in cycles per second (Hz).

Glossary (contd)

Dot product (or *scalar product*): An operation on two vectors; say, **A** and **B**, denoted by **A** • **B**, which can be defined by $|\mathbf{A}||\mathbf{B}| \cos (\measuredangle \mathbf{A},\mathbf{B})$ or equivalently by $A_xB_x + A_yB_y + A_zB_z$, where the subscripts denote the components of the vectors on three orthogonal axes.

Dynamical center of a body: Point of mass concentration of the body.

Eccentric anomaly: An angle at the center of an ellipse between the line of apsides and the radius of the auxiliary circle through a point that has the same x-coordinate as a given point on the ellipse.

Eccentricity: Ratio of the radius vector through a point on a conic to the distance from the point to the directrix.

Ecliptic: A great circle on the celestial sphere cut by the plane of the orbit of the earth; the apparent annual path of the sun.

Ecliptic coordinate system: Rectangular axes with the ecliptic as the fundamental plane and spherical coordinates: celestial longitude and latitude.

Elements of orbit: Constants defining the orbit: (1) Orientation elements: Ω, longitude of ascending node; i, inclination of the orbit plane; ω, longitude of perifocus. (2) Dimensional elements: e, eccentricity; a, semimajor axis; M_0, mean anomaly; t_0, epoch.

Elevation, angle of: Angle between the inertial velocity vector $\dot{\mathbf{r}}$ and the local horizontal; i.e., the plane normal to $\mathbf{r}$ and passing through the spacecraft.

Ellipse: A plane curve constituting the locus of all points, the sum of whose distances from two fixed points (called foci) is constant.

Ephemeris (pl. *ephemerides*): A table of calculated position (and velocity) coordinates of an object with equidistant dates as arguments.

Ephemeris second: Tropical second at 1900 January 0.5 E.T.

Ephemeris time (E.T.): Uniform measure of time that is the independent variable for the equations of motion and, hence, the argument for the ephemerides of the planets, the moon, and a spacecraft.

Epoch: Arbitrary instant of time for which the elements of an orbit are valid (e.g., initial, injection, or correction time).

Equator system: Rectangular axes referred to the equator as the fundamental plane and having spherical coordinates (right ascension and declination).

Equinoxes: Intersections of the equator and the ecliptic, the vernal equinox being the point at which the sun crosses the equator going from south to north in the spring. The autumnal equinox is the point at which the sun crosses the equator going from north to south in the autumn.

Escape velocity: Radial speed that a body must attain to escape from the gravitational field of a planet or a star. When friction is neglected, the escape velocity is $(2Gm/r)^{1/2}$, where G is the universal gravitational constant, m is the mass of the planet or star, and r is the radial distance from the center of the planet or star to the body.

Glossary (contd)

Field: A region of space within which each point has a definite value of a given physical or mathematical quantity.

Free fall: Free and unhampered motion of a body along a Keplerian trajectory, in which the force of gravity is counterbalanced by the force of inertia.

Gaussian gravitational constant k_s: Factor of proportionality in Kepler's third law:

$$k_s = \frac{(2\pi a^{3/2})}{P(m_1 + m_2)^{1/2}}$$

The numerical value depends upon the units employed.

General equation of a conic: In polar coordinates, the equation of a conic is given by

$$r = \frac{p}{1 + e \cos \nu}$$

where p is the semilatus rectum, e is the eccentricity, and ν is the true anomaly.

General precession: Combined effect of lunisolar and planetary precession.

Geo-: Combining form of earth (geos), as in *geocentric.*

Gravitation: Acceleration produced by the mutual attraction of two masses, directed along the line joining their centers of mass, and of magnitude inversely proportional to the square of the distance between the two centers of mass.

Gravitational potential: At a point, the work needed to remove an object from that point to infinity.

Gravity: Viewed from a frame of reference fixed in the earth, force imparted by the earth to a mass that is at rest relative to the earth. Because the earth is rotating, the force observed as gravity is the resultant of the force of gravitation and the centrifugal force arising from this rotation and the use of an earthbound rotating frame of reference.

Greenwich hour angle: Angle between the vernal equinox of the earth and the Greenwich meridian (measured in the equatorial plane of the earth).

Greenwich meridian: Zero meridian from which the geographical longitude is measured, passing through the Greenwich Observatory at Greenwich, England.

Harmonics of the gravitational field of the earth: A series, representing the gravitational potential of the earth, whose terms form a harmonic progression (i.e., include powers of the reciprocal of r).

Heliocentric: Referred to the center of the sun as origin.

Hour angle: Angle at the celestial pole between the observer's meridian and the hour circle passing through the object; a coordinate in the equatorial system.

Hour circle: A great circle that passes through the celestial poles and is, therefore, at right angles to the equator.

Glossary (contd)

Inclination i: Angle between orbit plane and reference plane; e.g., the ecliptic (for heliocentric orbits).

Inertial axes: Axes that are not in accelerated or rotational motion.

Injection: 1. The time following launch when nongravitational forces (thrust, lift, and drag) become negligible in their effect on the trajectory of a spacecraft. 2. The process of accelerating a spacecraft to escape velocity.

Integration central body (ICB): A celestial body relative to which the equations of motion of another body or of a spacecraft are integrated.

Interpolation: Approximation from tabulated values of a function (and possibly its derivatives) of values not included in a table.

Jacobi integral: An integral of the equations of motion in a rotating coordinate system that relates the square of the velocity and the coordinates of an infinitesimal body referred to the rotating coordinate system.

Jerk: A vector that specifies the time rate of change of acceleration; the third derivative of displacement with respect to time.

Julian century: 36,525 Julian days.

Julian date: Number of mean solar days that have elapsed, beginning at Greenwich noon, since January 1, 4713 B.C. For example, the Julian date of April 3, 1986, is 2446253.

Julian year: Average of the calendar years in the Julian calendar. It has the advantage of an exact decimal fraction representation: Julian year = 365.25 Julian days.

Kepler's equation: Relates position to time for an object traveling on a conic section:

$$n_E(t - T) = E - e \sin E \text{ (elliptic orbit)}$$

$$n_H(t - T) = e \sinh F - F \text{ (hyperbolic orbit)}$$

$$n_P(t - T) = qD + D^3/6 \text{ (parabolic orbit; see \textit{Barker's equation})}$$

Kepler's laws: 1. Every planet moves in an ellipse about the sun with the sun at one focus. 2. Every planet moves in such a way that its radius vector sweeps over equal areas in equal intervals of time. 3. The squares of the periods of revolution of two planets are to each other as the cubes of their mean distances from the sun.

Latitude (celestial): Angular distance of an object north (positive) or south (negative) of the ecliptic plane, a coordinate in the ecliptic system.

Latitude (geocentric): Angle at the center of the earth between the radius through a given point and the equatorial plane.

Launch window: Postulated opening in the continuum of time or of space through which a spacecraft must be launched to achieve a desired encounter, rendezvous, impact, etc.

Glossary (contd)

Legendre polynomials: Coefficients of the expansion

$$(1 - 2q\alpha + \alpha^2)^{-1/2} = \sum_{i=0}^{\infty} P_i(q)\,\alpha^i$$

where $P_0(q) = 1$, $P_2(q) = q$, $P_2(q) = 1/2\,(3q^2 - 1)$; the nth Legendre polynomial is given by the recursion formula

$$P_n = \frac{2n-1}{n}\,q\,P_{n-1} - \frac{n-1}{n}\,P_{n-2}$$

Libration: 1. Apparent tilting and side-to-side movements of the moon that render approximately 18% of its surface alternately visible and invisible. 2. Periodic perturbative oscillations in orbital elements.

Limb: Edge of the visible disk of the sun, the moon, a planet, etc.

Linearized flight time: Time-to-go on a rectilinear path to the center of the target.

Line of apsides: A line connecting the near to the far apsis; i.e., it defines the semimajor or transverse axis.

Line of nodes: Intersection of a reference plane and an orbit plane.

Longitude (celestial): Angular distance measured along the ecliptic from the vernal equinox eastward to the foot of a great circle passing through the object and through the poles of the celestial sphere.

Longitude (terrestrial): Angular distance from the foot of the Greenwich meridian, measured along the equator (east or west) to the foot of the meridian through the location of interest.

Longitude of ascending node: Angular distance from the vernal equinox measured eastward in the fundamental plane (ecliptic or equator) to the point of intersection of the orbit plane where the object crosses from south to north.

Longitude of perifocus: Sum of the angle in the fundamental plane between the vernal equinox and the line of nodes and the angle in the orbit plane, between the line of nodes and the line of apsides, measured in the direction of motion.

Lunicentric: Referred to the center of the moon as origin; selenocentric.

Lunisolar precession: See *precession of the equinoxes.*

Mass: A quantity characteristic of a body that relates the attraction of this body toward another body. All masses are referred to a standard kilogram.

Mass–energy equivalence: Equivalence of a quantity of mass m and a quantity of energy E, the two quantities being related by the mass–energy relation $E = mc^2$, where c is the speed of light.

Maximum true anomaly: In the case of a hyperbola, the angle measured from a vector directed toward perifocus to the outgoing asymptote vector; given by $\nu_{max} = \cos^{-1}(-1/e)$. For an ellipse, the angle measured from a vector directed

Glossary (contd)

toward perifocus to a given radius vector (denoted by R_{max} and called the pseudo-asymptote); in this case,

$$\nu_{max} = \cos^{-1}\left(\frac{p - R_{max}}{eR_{max}}\right)$$

Mean anomaly: Angle through which an object would move at the uniform average angular speed, measured from perifocus.

Mean distance: Semimajor axis. Because the term does not represent the time-average distance from the focus of a body traveling on an ellipse, which would be $a(1 + e^2/2)$, it can be considered only as an historical term.

Mean ecliptic, mean equator, mean equinox: Fictitious ecliptic, equator, and equinox represented by the precessional motions only; i.e., the effect of nutation is removed.

Mean solar day: Elapsed time between successive passages of the meridian of an observer past the mean sun; 86,400 s.

Mean sun: A fictitious sun that moves along the celestial equator with the mean speed with which the true sun apparently moves along the ecliptic throughout the year.

Meridian: 1. Terrestrial meridian—a great circle passing through the north and south poles; e.g., the observer's local meridian passes through his local zenith and the north and south poles. 2. Celestial meridian—a great circle on the celestial sphere in the plane of the observer's terrestrial meridian.

Moon's celestial equator: A great circle on the celestial sphere in the plane of the lunar equator; i.e., in a plane perpendicular to the axis of lunar rotation.

n-body problem: Concerned with the gravitational interactions of masses m_i, m_j, which are assumed to be homogeneous in spherical layers, under the Newtonian law.

Newton's laws: Law of gravitation: Every particle of matter in the universe attracts every other particle with a force proportional to the product of their masses and inversely as the square of the distance between them. Laws of motion: A set of three fundamental postulates forming the basis of the mechanics of rigid bodies. (1) Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by a force impressed upon it. (2) The rate of change of momentum is proportional to the force impressed, and takes place in the direction of the straight line in which the force acts. (3) To every action there is an equal and opposite reaction.

Nodal passage, time of: Time when an object passes through the node from the southern hemisphere to the northern hemisphere.

Node: Points of intersection of the great circle on the celestial sphere cut by the orbital plane and a reference plane, e.g., the ecliptic or equator reference plane.

Numerical differentiation: A process that allows for the numerical evaluation of the derivative of a quantity when tabular values of the quantity are given.

Glossary (contd)

Numerical integration: A process that allows for the numerical evaluation of a definite integral.

Nutation: Short-period terms in the precession arising from the obliquity, the eccentricity, and the inclination of the lunar orbit and the regression of its nodes (approximately a 19-yr period).

Obliquity of the ecliptic: Inclination of the ecliptic to the celestial equator; the angle of 23 deg, 27 min between the orbital plane of the earth and its equator.

Occultation: Disappearance of a body behind another body of larger apparent size.

Orbit: Path of a body or particle under the influence of a gravitational or other force; e.g., the orbit of a celestial body in its path relative to another body around which it revolves.

Orbital elements: A set of seven parameters defining the orbit of a body attracted by a central, inverse-square force.

Orbital period: Interval between successive passages of a satellite through the same point of its orbit.

Orientation angles: Classical orientation elements; i.e., inclination, longitude of ascending node, and longitude of perifocus.

Osculating orbit: An orbit, tangent to the actual or disturbed trajectory, having the same coordinates and velocity.

Parameter: Semilatus rectum $= a(1 - e^2)$; not to be confused with the generic term "parameters."

Peri-: In the orbit of one body about another, this prefix indicates the least separation.

Perifocal passage, time of: Time corresponding to the point on the orbit when the eccentric anomaly $E = 0$ (in the case of a hyperbolic orbit, when $F = 0$).

Perifocus: Point of an orbit nearest to the dynamical center.

Perigee: 1. Point on an ellipse nearest the principal focus or center of force. 2. Point on a geocentric orbit nearest the center of the earth.

Perihelion: Point on a heliocentric orbit nearest to the sun.

Period: See *orbital period.*

Perturbations: Deviations from exact reference motion caused by the gravitational attractions of other bodies or forces. (1) General perturbations—a method of calculating the perturbative effects by expanding and integrating in series. (2) Special perturbations—methods of deriving the disturbed orbit by numerically integrating the rectangular coordinates or the elements.

Phase: A portion of a trajectory influenced by a single physical central body.

Photon: According to the quantum theory of radiation, the elementary quantity, or quantum, of radiant energy. It is regarded as a discrete quantity, having

Glossary (contd)

momentum equal to $h\nu/c$, where h is Planck's constant, ν is the frequency of the radiation, and c is the speed of light.

Physical central body (PCB): Celestial body that dominates the acceleration of the spacecraft.

Planet: A celestial body of the solar system, revolving around the sun in a nearly circular orbit; a similar body revolving around a star.

Planetary precession: Because of perturbations from the other planets on the orbit of the earth, the ecliptic is not fixed in space, but is gradually changing.

Planetocentric: Referred to the center of a planet as the dynamical center or origin of coordinates.

Precession of the equinoxes: Slow, 26,000-yr period, westward motion of the equinoxes along the ecliptic arising from solar and lunar perturbations on the equatorial bulge of the earth, which cause the terrestrial axis to precess.

Predictor: An integration formula used in the numerical solution of ordinary differential equations in which the integral is expressed in terms of equally spaced ordinates to the left of (but not including) the end point. This provides an initial approximation to the new ordinate. Thereafter, the approximation is used in a corrector to improve or check the approximation. A predictor formula is open; a corrector formula is closed.

Primary (= primary body): Celestial body or central force field about which a satellite or other body orbits, or from which it is escaping, or towards which it is falling.

Prime meridian (of the earth): Meridian that passes through Greenwich, England. For other planets and the moon, the meridian that passes through a distinct mark on the surface of the planet or the moon.

Principal axes: Axes of a body about which the products of inertia vanish.

Products of inertia: Products of inertia of a body about the x,y,z axes are defined as

$$I_{xy} = \int xydm, I_{xz} = \int xzdm, I_{yz} = \int yzdm$$

Proper time: In a general relativistic framework, atomic time kept by an observer is interpreted as the observer's proper time.

Radiation: 1. Process by which electromagnetic energy is propagated through free space by virtue of joint undulatory variations in the electric and magnetic fields in space. 2. Radiant energy.

Radiation pressure: Pressure exerted upon any material body by electromagnetic radiation incident upon it. It is caused by the momentum transferred to the surface by the absorption and reflection of the radiation.

Rectilinear orbit: A trajectory for which $q = 0, e = 1$, where q is the perifocal distance from the principal focus to the nearer apsis $[= a(1 - e)]$.

Red shift, gravitational: An effect predicted by the general theory of relativity in which the frequency of light emitted by atoms in stellar atmospheres is decreased

Glossary (contd)

by a factor proportional to the (mass/radius) quotient of the star; confirmed observationally by the spectra of white stars.

Reference orbit: Usually (but not exclusively) the best two-body orbit available on the basis of which the perturbations are computed.

Relativistic: In general, pertaining to material (e.g., a particle) moving at speeds that are an appreciable fraction of the speed of light, thus increasing the mass.

Relativity: A principle that postulates the equivalence of the description of the universe, in terms of physical laws, by various observers or for various frames of reference.

Residuals: 1. $(O - C)$: small differences between the observed and computed coordinates in the sense observed minus computed. 2. $(O - I)$: small differences between the precomputed ideal observational data and the actual observed data on, e.g., an interplanetary mission.

Retrograde motion: Westward or clockwise motion as seen from the north pole; i.e., motion in an orbit in which $i > 90$ deg.

Revolution: 1. Motion of a celestial body in its orbit; circular motion about an axis usually external to the body. 2. One complete cycle of the movement of a celestial body in its orbit or of a body about an external body, as "a revolution of the earth about the sun."

Right ascension: Angular distance from the vernal equinox measured counterclockwise along the equator to the foot of the hour circle through the object.

Rotation: Turning of a body about an axis within the body, as "the daily rotation of the earth."

Secular terms: Expressions for perturbations that are proportional to time; usually terms of extremely long period.

Selenocentric: Referred to the center of the moon; lunicentric.

Selenocentric equatorial coordinates: A right-handed coordinate system, centered at the moon, with its three axes defined by the vernal equinox, north celestial pole (of the earth), and a direction perpendicular to these two; i.e., an equatorial coordinate system translated to the moon.

Selenographic coordinates: Coordinates that are rigidly attached to the moon, defined by the lunar equator and prime meridian.

Semilatus rectum: See *parameter*.

Semimajor axis: Distance from the center of an ellipse to an apsis; one half of the longest diameter; one of the orbital elements.

Semiminor axis: One half of the shortest diameter of an ellipse.

Sidereal time: Hour angle of the vernal equinox.

Sidereal year: Time required by the earth to complete one revolution of its orbit; equal to 365.25636 mean solar days.

Glossary (contd)

Slug: A unit of mass; the mass of a free body that, if acted upon by a force of 1 lbf, would experience an acceleration of 1 ft/s/s; thus, approximately 32.17 lb.

Solar constant: Rate at which solar radiation is received outside the atmosphere of the earth on a surface normal to the incident radiation and at the mean distance of the earth from the sun.

Solar radiation: Total electromagnetic radiation emitted by the sun.

Space: 1. Specifically, the part of the universe lying outside the limits of the atmosphere of the earth. 2. More generally, the volume in which all celestial bodies (including the earth) move.

Spacecraft: An instrumented vehicle, manned or unmanned, designed to be placed into an orbit about the earth or into a trajectory to another celestial body, to orbit or land thereon, to obtain information about an environment; previously used synonymously with *probe.*

Spacecraft state vector: An ordered sextuple of parameters $x,y,z,\dot{x},\dot{y},\dot{z}$, whose numerical values are the position and velocity components of a spacecraft.

Space-fixed coordinate system: A system in which the x-axis is in the direction of the vernal equinox, the z-axis is normal to the specified plane (equatorial or ecliptic) in the direction of the angular momentum vector, and the y-axis completes the right-handed coordinate system.

Spheroid: An oblate ellipsoid that closely approximates the mean-sea-level figure of the earth or geoid.

Station time (ST): Time obtained at each tracking station around the world by counting cycles of a rubidium atomic frequency standard (see Ref. 2, p. 37).

Surface harmonic: Of degree n, an expression of the type

$$a_n P_n(\cos\theta) + \sum_{m=1}^{n} [a_n^m \cos m\phi + b_n^m \sin m\phi] P_n^m \cos\theta$$

where P_n is a Legendre polynomial and P_n^m is an associated Legendre function. A surface harmonic of type $(\cos m\phi)\ P_n^m(\cos\theta)$ or $(\sin m\phi)\ P_n^m(\cos\theta)$ is a tesseral harmonic if $m < n$ and a sectoral harmonic if $m = n$; it is a solution of the differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 y}{\partial\theta^2} + n(n+1)y = 0$$

A tesseral harmonic is zero on n-m parallels of latitude and $2m$ meridians (on a sphere with its center at the origin of spherical coordinates); a sectoral harmonic is zero along $2n$ meridians (which divide the surface of the sphere into sectors).

Synodic period of a planet: Interval of time between two successive oppositions or conjunctions with the sun, as observed from the earth.

Tesseral harmonic: A surface harmonic of type $(\cos m\phi)\ P_n^m(\cos\theta)$ or $(\sin m\phi)$ $P_n^m(\cos\theta)$ is a tesseral harmonic if $m < n$ (and a sectoral harmonic if $m = n$); see also *surface harmonic.*

Glossary (contd)

Three-body problem: Problem of integrating the equations of motion of three bodies (e.g., sun–moon–earth) moving under their mutual gravitational attractions; directly solvable only in particular cases.

Time block: Period of time for which the coefficients of certain polynomials used for transformation between time scales (e.g., from A.1 to UT1 time) are valid.

Time of perifocal passage: Time when a spacecraft traveling upon an orbit passes by the nearer apsis or perifocal point.

Topocentric: Referred to the position of an observer on the earth as origin.

Topocentric equatorial coordinates: A right-handed coordinate system, centered at the observer, with its axes defined by the vernal equinox, north celestial pole, and a direction perpendicular to these two; i.e., an equatorial coordinate system translated to the topos.

Trajectory: In general, the path traced by any body moving as a result of an externally applied force considered in three dimensions.

Transfer orbit: In interplanetary travel, an elliptic trajectory tangent to the orbits of both the departure planet and the target planet.

Tropical year: Time required (365.2422 mean solar days) by the sun to make an apparent revolution of the ecliptic from vernal equinox to vernal equinox; shorter than the solar year because of the precession of the equinoxes. It is the civil year of the seasons.

True anomaly: Angle at the focus between the line of apsides and the radius vector measured from perifocus in the direction of motion.

True ecliptic, true equator, true equinox: Actual positions of the ecliptic, equator, and equinox, taking into account both precession and nutation.

Two-body orbit: Motion of a body of negligible mass around a center of attraction.

Unit vector: A vector whose magnitude or length is unity; used to define directions in space.

Universal time (UT): Mean solar time referred to the meridian of Greenwich; nonuniform because of the irregular rotation of the earth.

UTC: A time scale, which is Greenwich civil time; an approximation of UT2, UTC is derived from the U.S. Frequency Standard at Boulder, Colo., and deviates from UT2 by a known amount having a maximum of about 0.200 s (see Ref. 2, p. 36).

Vector **B**: A vector originating at the center of the target planet and directed perpendicular to the incoming asymptote of the target-centered approach hyperbola; also called the impact-parameter vector.

Vector component: Projection of a vector on a given axis in space; e.g., if it is the x-axis, the component of the vector $\mathbf{A}$ on this axis is denoted by A_x.

Glossary (contd)

Vector equation: An equation whose terms include vectors that can be resolved into three component equations; e.g., $\ddot{\mathbf{r}} = -\mu\mathbf{r}/r^3$ actually represents three component equations:

$$\ddot{x} = -\frac{\mu x}{r^3}$$

$$\ddot{y} = -\frac{\mu y}{r^3}$$

$$\ddot{z} = -\frac{\mu z}{r^3}$$

where $\mathbf{r}$ has been replaced by its three components x, y, and z and $\mathbf{r}$ by its three components x, y, and z.

Vernal equinox: Point of intersection of the ecliptic and celestial equator at which the sun crosses the equator from south to north in its apparent annual motion along the ecliptic.

Vis-viva integral: An important integral of the two-body problem, giving the orbital velocity

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

Year: Defined as the period of revolution of the earth in its motion about the sun.

Zenith: Point at which the upward extension of the plumb-bob direction intersects the celestial sphere.

Zonal harmonic: A function P_n (cos θ), where P_n is the Legendre polynomial of degree n. The function P_n (cos θ) is zero along n great circles on a sphere with its center at the origin of a system of spherical coordinates; these circles pass through the poles and divide the sphere into n zones (see also *surface harmonic*).

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