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16. Abstract

Nonlinear panel flutter is examined by using the multiple-time-scaling method and the harmonic-balance method. The two methods are described in detail. The solution of transient response and the stability of the limit-cycle solution are included in the discussion.

The two methods are applied to the analysis of (1) simply-supported two-dimensional plates and (2) clamped-clamped three-dimensional plates. The effect of mass-ratio and inplane loadings are analyzed in both cases. The influence of the aspect ratio is considered also for the three-dimensional plate. The two methods yield very similar results.

A comparison with existing methods is made. It is shown that the results are in very good agreement with those obtained by Dowell in which the more straightforward direct numerical integration method was used; the present approaches require much less computing time.

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FOREWORD

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SECTION 1

INTRODUCTION

Panel flutter is the self-excited oscillation of a plate or shell when exposed to an airflow along its surface. Linear theory indicates there is a critical dynamic pressure above which the plate motion becomes unstable and grows exponentially with time. At large deflections, however, nonlinear effects (mainly midplane stretching forces) come into play and restrain the motion to a bounded limit cycle. For realistic assessments of the significance of panel flutter, the nonlinear theory must be investigated. An excellent, up-to-date summary of panel flutter is given by Dowell[1].

The physical nature of the problem is described in Appendix A. Various methods can be used to investigate nonlinear panel flutter. First of all, there is the direct numerical integration of the equations of motion from given initial conditions. This has been used with considerable success by Dowell[2,3,4]; see also Evensen and Olson[5]. Secondly, the equations of motion can be placed on an analogue computer with nonlinear components, and the response observed. Kobayashi[6] and Bolotin[7] have used this method. Thirdly, the harmonic balance method can be used to determine limit cycles; see for example, Kobayashi[6], Bolotin[7], Evensen and Olson[5], and Eastep and McIntosh[8]. Fourthly, perturbation methods can be used to obtain neighboring solutions to the linear problem: Olson and Fung[9] used the Kryloff and Bogoliuboff method; more recently Morino[10] has used the multiple time scaling technique. Finally, the Lyapunov stability criteria can be used to study qualitatively the nonlinear behavior of the response. Such a technique has been employed by Librescu[11].

The present report will concentrate on perturbation methods and harmonic balance methods for nonlinear panel flutter. The previously published work with these two methods has been limited specifically to two modes. This will be extended here in a systematic fashion to an arbitrary number of modes, and results will be given for as many as twenty modes. It is believed that the present perturbation and harmonic balance methods developed here may provide useful and efficient alternatives to the longer, more straightforward direct numerical integration method in studying nonlinear panel flutter problems. Also, these methods developed here may be of interest in the study of related vibration problems.

SECTION 2

PERTURBATION METHODS

2.1 Introduction to the Multiple-Time-Scaling Technique

In order to introduce the multiple time-scaling technique in a simple way, it is convenient to discuss first an application of this method to an elementary nonlinear problem. Consider the vibrations of a system of a mass with a nonlinear spring, described by the equation

$$\ddot{X} + X + \varepsilon X^3 = 0 \qquad 0 < \varepsilon << 1 \tag{2.1}$$

If $\varepsilon = 0$, the problem reduces to the linear equation

$$\ddot{X}_0 + X_0 = 0 \tag{2.2}$$

whose solution is given by

$$X_0 = A e^{it} + A^* e^{-it} = 2\,Real\,[A e^{it}] \tag{2.3}$$

where $i = \sqrt{-1}$, A is an arbitrary constant and A* is its complex conjugate. For $\varepsilon \neq 0$, one can think of solving the problem by "disturbing" the solution of the linear system; that is by setting

$$X(t,\varepsilon) = X_0(t) + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + \cdots\cdots \tag{2.4}$$

Combining Eqs. 1.3 and 1.4, and equating terms of the same order of magnitude (i.e., terms multiplying like powers of ε) yields the following system of recurrence equations:

$$\ddot{X}_0 + X_0 = 0 \qquad \text{(order 1)} \tag{2.5}$$

$$\ddot{X}_1 + X_1 = -X_0^3 \qquad \text{(order } \varepsilon\text{)} \tag{2.6}$$

and so on. Equation 2.5 is equal to Eq. 2.2 and its solution is given by Eq. 2.3. Combining Eqs. 2.3 and 2.6 yields

$$\ddot{X}_1 + X_1 = -(A^3 e^{3it} + 3A^2 A^* e^{it} + 3A A^{*2} e^{-it} + A^{*3} e^{-3it}) \tag{2.7}$$

whose solution is

$$X_1 = (B e^{it} + B^* e^{-it}) + \frac{1}{8}(A^3 e^{-i3t} + A^{*3} e^{-i3t}) + \frac{3}{2}(i A^2 A^* t e^{it} - i A^{*2} A t e^{-it}) \tag{2.8}$$

where B is another arbitrary constant, with B* being its complex conjugate.

Finally, combining Eqs. 2.3 and 2.8 with 2.4 yields

$$X = 2\,\mathrm{Real}\left[(A+\varepsilon B)e^{it} + \varepsilon \frac{1}{8}A^3 e^{i3t} + \varepsilon \frac{3}{2} i A^2 A^* t e^{it}\right] + O(\varepsilon^2) \tag{2.9}$$

The term which contains te^{it} is called a "secular term". Terms of this kind play a very important role in the singular perturbation techniques, and the nature of secular terms is discussed in the following. First, it should be noted that the solution obtained by neglecting the term of order ε^2 is not bounded. In fact, the amplitude of the term te^{it} becomes infinitely large as t goes to infinity. This is in contradiction to the fact that Eq. 2.1 has a first integral (energy integral)

$$\frac{\dot{X}^2}{2} + \frac{X^2}{2} + \varepsilon \frac{X^4}{4} = \text{constant} \tag{2.10}$$

which implies that the solution remains finite. In addition, the solution of Eq. 2.1 can be obtained in terms of Jacobian elliptic functions which are periodic functions, whereas the function obtained from Eq. 2.9 disregarding terms of order ε^2 does not have this property of periodicity. Thus, this method seems to destroy the finiteness and the periodicity present in the true solution.

The nature of the secular terms can be understood better by examining the solution obtained by using the multiple-time-scaling technique which allows one to obtain the "correct" solution. This solution is given (later) in Eq. 2.26 as

$$X = 2\,\mathrm{Real}\left[(A_1+\varepsilon B_1)e^{i(1+\varepsilon\frac{3}{2}A_1A_1^*)t} + \varepsilon\frac{1}{8}A_1^3 e^{i3(1+\varepsilon\frac{3}{2}A_1A_1^*)t}\right] + O(\varepsilon^2) \tag{2.11}$$

The Taylor series representation of this equation is given by

$$X = 2\,\mathrm{Real}\left[A_1 e^{it} + \varepsilon\left(B_1 e^{it} + \frac{1}{8}A_1^3 e^{3it} + \frac{3}{2}iA_1^2 A_1^* t e^{it}\right)\right] + O(\varepsilon^2) \tag{2.12}$$

which is identical to Eq. 2.9. It is apparent from Eqs. 2.11 and 2.12 that since the frequency changes with ε, any attempt to obtain a power series representation (as given by Eq. 2.4) for the solution, will always yield secular terms. In other words, the solution given by Eq. 2.9 is correct but has the defect that in order to maintain the properties of finiteness and periodicity, the complete series must be considered. This makes the method useless from the practical point of view.

In order to circumvent this difficulty several methods have been introduced, as for instance the slowly varying amplitude method (described in Section 3), the Kryloff-Bogoliuboff method (Ref. 12), the two-variable expansion procedure (Refs. 13 and 14), and the multiple time-scaling technique. In the following, the multiple time-scaling technique is described in detail.

It should be emphasized that other methods (Refs. 13 and 14, for example) yield exactly the same results (to order of ε^5) in this case, but they cannot be applied in a more systematic way to creat more general cases. In particular, the two-variable expansion procedure introduced by Cole and Kevorkian (see Refs. 13 and 14) is often simpler to use than the multiple-time-scaling technique, but it is not always applicable. A comparison of the two methods is made in Appendix B of Ref. 19.

The multiple time-scaling technique was introduced by different researchers independently (see e.g., Refs. 15, 16, and 17). The basic concept is the following. Note that ε disappears from the arguments of the functions on the right-hand side of the equation, if Eq. 2.11 is written in the form

$$X = 2\,\mathrm{Real}\left[\left(A_1 + \varepsilon B_1\right) e^{i\left(t_0 + \frac{3}{2} A_1 A_1^{*} t_1\right)} + \varepsilon \frac{1}{8} A_1^{3} e^{i3\left(t_0 + \frac{3}{2} A_1 A_1^{*} t_1\right)}\right] + O(\varepsilon^2) \tag{2.13}$$

where $t_0 = t$, $t_1 = \varepsilon t$. This suggests that the solution can be found as follows: it is assumed that the solution of Eq. 2.1 is of the form

$$X = X_0(t_0, t_1, \cdots) + \varepsilon X_1(t_0, t_1, \cdots) + \cdots + O(\varepsilon^{N+1}) \tag{2.14}$$

where

$$t_0 = t\ ,\ t_1 = \varepsilon t\ ,\ \cdots\ ,\ t_n = \varepsilon^n t\ , \cdots \tag{2.15}$$

are considered to be independent (and are called "independent multiple time scales") so that

$$\frac{d}{dt} = \sum_{n=0}^{N} \frac{\partial}{\partial t_n} \frac{dt_n}{dt} = \sum_{n=0}^{N} \varepsilon^n \frac{\partial}{\partial t_n} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \cdots \tag{2.16}$$

where N may be finite.

As is shown in the following, the use of the multiple-time-scaling introduces enough flexibility in the scheme so that the secular terms can be avoided. It should be noted that this is not possible in all cases. Cases in which the method fails are discussed in Sections 3, 4, and 5 of Ref. 19.

Next, the concepts described above are applied to solve Eq. 2.1. Combining Eqs. 2.1, 2.15, and 2.16 yields

$$\left(\frac{\partial^2}{\partial t_0^2} + 2\varepsilon \frac{\partial^2}{\partial t_0 \partial t_1} + \cdots\right)\left(X_0 + \varepsilon X_1 + \cdots\right) = -\left(X_0 + \varepsilon X_1 + \cdots\right) + \varepsilon\left(X_0 + \varepsilon X_1 + \cdots\right)^3 \tag{2.17}$$

Equating terms of the same order of magnitude (i.e., terms which are multiplied by the same power of ε) yields

$$\frac{\partial^2 X_0}{\partial t_0^2} + X_0 = 0 \qquad \text{(order 1)} \tag{2.18}$$

$$\frac{\partial^2 X_1}{\partial t_0^2} + X_1 = -\left(2\frac{\partial^2 X_0}{\partial t_0 \partial t_1} + X_0^3 \right) \quad \text{(order } \varepsilon\text{)} \tag{2.19}$$

and so on. The solution of Eq. 2.18 is given by

$$X_0 = A_0(t_1, \cdots)\, e^{it_0} + A_0^*(t_1, \cdots)\, e^{-it_0} \tag{2.20}$$

where now A_0 is not an arbitrary constant but rather a function of t_1, t_2, and so on. Combining Eqs. 2.19 and 2.20 yields

$$\frac{\partial^2 X_1}{\partial t_0^2} + X_1 = -2\left(i\frac{\partial A_0}{\partial t_1} e^{it_0} - i\frac{\partial A_0^*}{\partial t_1} e^{-it_0} \right)$$
$$- \left(A_0^3 e^{i3t_0} + 3A_0^2 A_0^* e^{it_0} + 3A_0 A_0^{*2} e^{-it_0} + A_0^{*3} e^{-i3t_0} \right) \tag{2.21}$$

The solution of this equation is given by

$$X_1 = 2\,\mathrm{Real}\left[B_0 e^{it_0} + \frac{1}{8} A_0^3 e^{i3t_0} + \left(\frac{3}{2} i A_0^2 A_0^* - \frac{\partial A_0}{\partial t_1} \right) t_0 e^{it_0} \right] \tag{2.22}$$

which contains a secular term unless

$$\frac{\partial A_0}{\partial t_1} - \frac{3}{2} i A_0^2 A_0^* = 0 \tag{2.23}$$

This is a differential equation for A as a function of t_1 and its solution is*

$$A_0 = A_1 e^{i\frac{3}{2} A_1 A_1^* t_1} \tag{2.24}$$

---

* In fact, setting $A_0 = \rho e^{i\theta}$ in Eq. 1.22 yields $\partial\rho/\partial t_1 = 0$, $\partial\theta/\partial t_1 = 3/2(\rho^2)$ which yield $\rho = \rho_1 = \text{const.}$ and $\theta = 3/2(\rho^2 t_1 + \theta_1)$; that is Eq. 2.24 where $A_1 = \rho_1 e^{i\theta_1}$ is a complex function of t_2, t_3 and so on.

Combining Eqs. 2.22 and 2.23 yields

$$X_1 = 2\ Real\left\{B_0 e^{it_0} + \frac{1}{8} A_0^3 e^{i3t_0}\right\} \tag{2.25}$$

which does not contain secular terms. Finally, combining Eqs. 2.14, 2.20, and 2.25 yields

$$X = 2\,Real\left\{(A_0 + \varepsilon B_0) e^{it_0} + \varepsilon \frac{1}{8} A_0^3 e^{i3t_0}\right\} + O(\varepsilon^2) \tag{2.26}$$

or using Eqs. 2.15 and 2.24

$$X = 2\,Real\left\{(A_1 + \varepsilon B_1)\, e^{i(1 + \varepsilon \frac{3}{2} A_1 A_1^*)t} + \varepsilon \frac{1}{8} A_1^3 e^{i3(1 + \varepsilon \frac{3}{2} A_1 A_1^*)t}\right\} + O(\varepsilon^2) \tag{2.27}$$

where

$$B_1 = B_0 e^{-i\frac{3}{2} A_1 A_1^* t_1} = B_0 + O(\varepsilon) \tag{2.28}$$

Finally, Eq. 2.27 can be written as

$$X = 2\,Real\left\{(A_1 + \varepsilon B_1) e^{i\omega t} + \varepsilon \frac{1}{8} A_1^3 e^{i3\omega t}\right\} + O(\varepsilon^2) \tag{2.29}$$

with

$$\omega = 1 + \varepsilon \frac{3}{2} A_1 A_1^* \tag{2.30}$$

which shows that the solution is a periodic function with frequency depending upon ε.

2.2 Analysis of Two Modes

In Subsection 2.1, the multiple-time-scaling technique is used to study a very simple equation. In order to study nonlinear panel flutter problems, the method should be generalized to treat systems of differential equations. For the sake of simplicity, the generalization of the method is described first for the case of a system of two differential equations with two unknowns. This is done in this subsection. The more general case of N equations with N unknowns is described in Subsection 2.3.

2.2.1 The Method of Solution

Consider a two-mode system* (see Appendix A for the expression of the coefficients)..

$$\ddot{W}_1 + g_1 \dot{W}_1 + \Omega_1^2 W_1 - \Lambda W_2 + (C_{11} W_1^2 + C_{12} W_2^2) W_1 = 0$$

$$\ddot{W}_2 + g_2 \dot{W}_2 + \Omega_2^2 W_2 + \Lambda W_1 + (C_{21} W_1^2 + C_{22} W_2^2) W_2 = 0 \qquad (2.31)$$

Consider the corresponding linear system

$$\ddot{W}_1 + g_1 \dot{W}_1 + \Omega_1^2 W_1 - \Lambda W_2 = 0$$

$$\ddot{W}_2 + g_2 \dot{W}_2 + \Omega_2^2 W_2 + \Lambda W_1 = 0 \qquad (2.32)$$

For $\Lambda = 0$, the solution is stable (vibrations of exponentially decreasing amplitude), but there exists a critical (flutter) value, Λ_F, of the dynamic pressure parameter such that for $\Lambda > \Lambda_F$, the solution is unstable (vibrations of exponentially growing amplitude occur). For $\Lambda = \Lambda_F$, the solution is at the limit of stability; that is, the solution contains a component which represents harmonic vibrations (constant amplitude) of frequency ω_F (flutter frequency). The values of Λ_F and ω_F are obtained as follows. Consider the algebraic system corresponding to Eq. 2.32, given by

$$\begin{bmatrix} p^2 + g_1 p + \Omega_1^2 & -\Lambda_F \\ \Lambda_F & p^2 + g_2 p + \Omega_2^2 \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = 0 \qquad (2.33)$$

The values of Λ_F and ω_F are obtained by setting

$$p = i\omega_F \qquad (2.34)$$

and setting the determinant (its real and imaginary part, independently) of the coefficients of Eq. 2.33 equal to zero. This yields

$$(-\omega_F^2 + \Omega_1^2)(-\omega_F^2 + \Omega_2^2) - g_1 g_2 \omega_F^2 + \Lambda_F^2 = 0 \qquad (2.35)$$

$$g_1 \omega_F (-\omega_F^2 + \Omega_2^2) + g_2 \omega_F (-\omega_F^2 + \Omega_1^2) = 0 \qquad (2.36)$$

---

*Note that $\Lambda = e_{21}\lambda$ and that for simplicity g_1 and g_2 are assumed to be independent of ε (see also Subsection 2.4.3).

From Eq. 2.36 one obtains*

$$\omega_F^2 = \frac{g_1\Omega_2^2 + g_2\Omega_1^2}{g_1 + g_2} \tag{2.37}$$

Combining Eqs. 2.35 and 2.37 and solving for Λ_F yields

$$\Lambda_F^2 = \frac{g_1 g_2}{(g_1+g_2)^2}(\Omega_2^2 - \Omega_1^2)^2 + g_1 g_2 \frac{g_1\Omega_2^2 + g_2\Omega_1^2}{g_1 + g_2} \tag{2.38}$$

Thus the undamped** part of the solution of the linear problem for $\Lambda = \Lambda_F$ (that is at the stability boundary) is given by

$$\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = A \begin{Bmatrix} 1 \\ u \end{Bmatrix} e^{i\omega_F t} + A^* \begin{Bmatrix} 1 \\ u^* \end{Bmatrix} e^{-i\omega_F t}$$

$$= 2\,\mathrm{Real}\left(A \begin{Bmatrix} 1 \\ u \end{Bmatrix} e^{i\omega_F t} \right) \tag{2.39}$$

where, combining Eqs. 2.32 and 2.39

$$u = \frac{-\omega_F^2 + i g_1 \omega_F + \Omega_1^2}{\Lambda_F} = \frac{-\Lambda_F}{-\omega_F^2 + i g_2 \omega_F + \Omega_2^2} \tag{2.40}$$

or***

$$u = -\sqrt{\frac{g_1}{g_2}}\, e^{i\chi} \tag{2.41}$$

with

$$\chi = -\tan^{-1}\frac{g_1 + g_2}{\Omega_2^2 - \Omega_1^2}\,\omega_F < 0 \tag{2.42}$$

---

*It is assumed $\omega_F \neq 0$. The case $\omega_F = 0$ is discussed in Ref. 19.

**It is worth noting that the full solution of the problem contains also a term which decreases exponentially to zero.

***In fact, with the use of Eqs. 2.37 and 2.38:

$$u = \frac{1}{\Lambda_F}\left[(\Omega_1^2 - \omega_F^2) + i g_1 \omega_F\right] = \frac{1}{\Lambda_F}\left[-\frac{g_1}{g_1+g_2}(\Omega_1^2 - \Omega_2^2) + i g_1 \omega_F\right]$$

$$= -\frac{1}{\Lambda_F}\left[\frac{g_1^2}{(g_1+g_2)^2}(\Omega_2^2 - \Omega_1^2)^2 + g_1^2 \frac{g_1\Omega_2^2 + g_2\Omega_1^2}{g_1 + g_2}\right]^{1/2} e^{i\tan^{-1}\frac{g_1+g_2}{\Omega_1^2 - \Omega_2^2}\omega_F}$$

$$= -\sqrt{\frac{g_1}{g_2}}\, e^{-i\tan^{-1}\frac{g_1+g_2}{\Omega_1^2 - \Omega_2^2}\omega_F}$$

As mentioned above, the solution of the linear problem becomes unstable, that is the amplitude becomes infinitely large for $\Lambda > \Lambda_F$. On the other hand, the solution of the nonlinear problem grows only to a finite value. The limiting solution (as $t \to \infty$) is a periodic solution with constant amplitude. This is due to the fact that the increase in Λ is "compensated for" by the effect of the nonlinear terms.

Thus, the increase in Λ and the effect of the nonlinear terms must be (mathematically speaking) of the same order of magnitude. Assume that ε is the order of magnitude of w_1 and w_2; then a "compensation" is obtained by setting

$$\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \varepsilon \begin{Bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{Bmatrix} \tag{2.43}$$

and*

$$\Lambda = \Lambda_F \pm \varepsilon^2 \tag{2.44}$$

Combining Eqs. 2.31, 2.43, and 2.44 yields

$$\begin{aligned} \ddot{\tilde{w}}_1 + g_1 \dot{\tilde{w}}_1 + \Omega_1^2 \tilde{w}_1 - \Lambda_F \tilde{w}_2 &= \varepsilon^2 \left[\pm \tilde{w}_2 - (c_{11}\tilde{w}_1^2 + c_{12}\tilde{w}_2^2)\tilde{w}_1 \right] \\ \ddot{\tilde{w}}_2 + g_2 \dot{\tilde{w}}_2 + \Omega_2^2 \tilde{w}_2 + \Lambda_F \tilde{w}_1 &= \varepsilon^2 \left[\mp \tilde{w}_1 - (c_{21}\tilde{w}_1^2 + c_{22}\tilde{w}_2^2)\tilde{w}_2 \right] \end{aligned} \tag{2.45}$$

which shows that Eqs. 2.43 and 2.44 produce the desired "compensation" between the variation of Λ and the effect of the nonlinear terms.

Next, note that only ε^2 appears in Eq. 2.45 so that only even powers of ε are involved in the solution. This suggests that the solution can be obtained in the form

$$\begin{aligned} w_1 = \varepsilon \tilde{w}_1 &= \varepsilon w_{11}(t_0, t_2, \cdots) + \varepsilon^3 w_{13}(t_0, t_2, \cdots) + \cdots \\ w_2 = \varepsilon \tilde{w}_2 &= \varepsilon w_{21}(t_0, t_2, \cdots) + \varepsilon^3 w_{23}(t_0, t_2, \cdots) + \cdots \end{aligned} \tag{2.46}$$

where (just as in the preceding subsection)

---

*The plus sign is used to study the behavior for $\Lambda > \Lambda_F$ and the minus sign is used to study the behavior of $\Lambda < \Lambda_F$.

$$t_0 = t\,, \quad t_2 = \varepsilon^2 t\,, \ldots . \tag{2.47}$$

are considered to be independent variables so that

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon^2 \frac{\partial}{\partial t_2} + \ldots . \tag{2.48}$$

2.2.2 The Solution of the Problem and the Equation for $A(t_2)$

Combining Eqs. 2.32, 2.46, and 2.48 yields

$$\left(\frac{\partial^2}{\partial t_0^2} + \varepsilon^2 2\frac{\partial^2}{\partial t_0 \partial t_2} + \cdots\right)(\varepsilon W_{11} + \varepsilon^3 W_{13} + \cdots) + g_1\left(\frac{\partial}{\partial t_0} + \varepsilon^2 \frac{\partial}{\partial t_2} + \cdots\right)$$

$$\times (\varepsilon W_{11} + \varepsilon^3 W_{13} + \cdots) + \Omega_1^2 (\varepsilon W_{11} + \varepsilon^3 W_{13} + \cdots) - (\Lambda_F \pm \varepsilon^2)$$

$$\times (\varepsilon W_{21} + \varepsilon^3 W_{23} + \cdots) + C_{11}(\varepsilon W_{11} + \varepsilon^3 W_{13} + \cdots)^3$$

$$+ C_{12}(\varepsilon W_{11} + \varepsilon^3 W_{13} + \cdots)(\varepsilon W_{21} + \varepsilon^3 W_{23} + \cdots)^2 = 0$$

$$\left(\frac{\partial^2}{\partial t_0^2} + \varepsilon^2 2\frac{\partial^2}{\partial t_0 \partial t_2} + \cdots\right)(\varepsilon W_{21} + \varepsilon^3 W_{23} + \cdots) + g_2\left(\frac{\partial}{\partial t_0} + \varepsilon^2 \frac{\partial}{\partial t_2} + \cdots\right)$$

$$\times (\varepsilon W_{21} + \varepsilon^3 W_{23} + \cdots) + \Omega_2^2 (\varepsilon W_{21} + \varepsilon^3 W_{23} + \cdots) + (\Lambda_F \pm \varepsilon^2)$$

$$\times (\varepsilon W_{11} + \varepsilon^3 W_{13} + \cdots) + C_{21}(\varepsilon W_{11} + \varepsilon^3 W_{13} + \cdots)^2 (\varepsilon W_{21} + \varepsilon^3 W_{23} + \cdots)$$

$$+ C_{22}(\varepsilon W_{21} + \varepsilon^3 W_{23} + \cdots)^3 = 0 \tag{2.49}$$

Equating terms of the same order of magnitude yields the following set of recurrent systems of differential equations:

System of order ε

$$\frac{\partial^2 W_{11}}{\partial t_0^2} + g_1 \frac{\partial W_{11}}{\partial t_0} + \Omega_1^2 W_{11} - \Lambda_F W_{21} = 0$$

$$\frac{\partial^2 W_{21}}{\partial t_0^2} + g_2 \frac{\partial W_{21}}{\partial t_0} + \Omega_2^2 W_{21} + \Lambda_F W_{11} = 0 \tag{2.50}$$

System of order ε^3

$$\frac{\partial^2 W_{13}}{\partial t_0^2} + g_1 \frac{\partial W_{13}}{\partial t_0} + \Omega_1^2 W_{13} - \Lambda_F W_{23} =$$
$$- \left[2 \frac{\partial^2 W_{11}}{\partial t_0 \partial t_2} + g_1 \frac{\partial W_{11}}{\partial t_2} \mp W_{21} + C_{11} W_{11}^3 + C_{12} W_{11} W_{21}^2 \right]$$

$$\frac{\partial^2 W_{23}}{\partial t_0^2} + g_2 \frac{\partial W_{23}}{\partial t_0} + \Omega_2^2 W_{23} + \Lambda_F W_{13} =$$
$$- \left[2 \frac{\partial^2 W_{21}}{\partial t_0 \partial t_2} + g_2 \frac{\partial W_{21}}{\partial t_2} \pm W_{11} + C_{21} W_{11}^2 W_{21} + C_{22} W_{21}^3 \right] \quad (2.51)$$

and so on.

Equation 2.50 is equivalent to the linear system given by Eq. 2.32. The undamped part of its solution is given by (see Eq. 2.39)

$$\begin{Bmatrix} W_{11} \\ W_{21} \end{Bmatrix} = A \begin{Bmatrix} 1 \\ u \end{Bmatrix} e^{i\omega_F t_0} + A^* \begin{Bmatrix} 1 \\ u^* \end{Bmatrix} e^{-i\omega_F t_0}$$
$$= 2\, Real \left[A \begin{Bmatrix} 1 \\ u \end{Bmatrix} e^{i\omega_F t_0} \right] \quad (2.52)$$

where A is a function of t_2, . . . and u is given by Eq. 2.41. As mentioned earlier, the full solution of Eq. 2.18 contains also a part which is damped. This part is not of interest for the solution of the nonlinear problem. In fact, only the asymptotic behavior (as $t \to \infty$) of the solution is of interest here and as is shown in Ref. 10, all of the terms originating from the damped part of the solution of the linear system go exponentially to zero. Thus, for the sake of simplicity, the damped part of the solution is disregarded in the following.

Next, consider the solution of Eq. 2.51 Combining Eqs. 2.51 and 2.52 yields

$$\frac{\partial^2 W_{13}}{\partial t_0^2} + g_1 \frac{\partial W_{13}}{\partial t_0} + \Omega_1^2 W_{13} - \Lambda_F W_{23} + Z_1 = 0$$

$$\frac{\partial^2 W_{23}}{\partial t_0^2} + g_2 \frac{\partial W_{23}}{\partial t_0} + \Omega_2^2 W_{23} + \Lambda_F W_{13} + Z_2 = 0 \quad (2.53)$$

where

$$
\begin{aligned}
Z_1 = 2\,\mathrm{Real}\Big\{ & (2i\omega_F + g_1)\frac{\partial A}{\partial t_2} e^{i\omega_F t_0} \mp u A e^{i\omega_F t_0} \\
& + C_{11}\left[A^3 e^{i3\omega_F t_0} + 3A^2A^* e^{i\omega_F t_0}\right] \\
& + C_{12}\left[A^3 u^2 e^{i3\omega_F t_0} + A^2A^*(u^2 + 2uu^*) e^{i\omega_F t_0}\right]\Big\} \\
Z_2 = 2\,\mathrm{Real}\Big\{ & (2i\omega_F + g_2) u \frac{\partial A}{\partial t_2} e^{i\omega_F t_0} \pm A e^{i\omega_F t_0} \\
& + C_{21}\left[A^3 u e^{i3\omega_F t_0} + A^2A^*(2u + u^*) e^{i\omega_F t_0}\right] \\
& + C_{22}\left[A^3 u^3 e^{i3\omega_F t_0} + A^2A^* 3u^2u^* e^{i\omega_F t_0}\right]\Big\}
\end{aligned}
\tag{2.54}
$$

or

$$
\begin{aligned}
Z_1 &= 2\,\mathrm{Real}\left[\, Z_1^{(1)} e^{i\omega_F t_0} + Z_1^{(3)} e^{3i\omega_F t_0} \,\right] \\
Z_2 &= 2\,\mathrm{Real}\left[\, Z_2^{(1)} e^{i\omega_F t_0} + Z_2^{(3)} e^{3i\omega_F t_0} \,\right]
\end{aligned}
\tag{2.55}
$$

with

$$
\begin{aligned}
Z_1^{(1)} &= (2i\omega_F + g_1)\frac{\partial A}{\partial t_2} \mp uA + \left[3C_{11} + (u^2 + 2uu^*)C_{12}\right]A^2A^* \\
Z_2^{(1)} &= (2i\omega_F + g_2)\frac{\partial A}{\partial t_2} u \pm A + \left[(2u + u^*)C_{21} + 3u^2u^*C_{22}\right]A^2A^*
\end{aligned}
\tag{2.56}
$$

and

$$Z_1^{(3)} = (C_{11} + C_{12} U^2) A^3$$

$$Z_2^{(3)} = (C_{21} U + C_{22} U^2) U A^3 \qquad (2.57)$$

Next, note that Z_1 and Z_2 contain terms proportional to $e^{i\omega_F t}$. These yield secular terms unless $Z_1^{(1)}$ and $Z_2^{(1)}$ satisfy a special condition (Eq. 2.62).

In order to derive this condition, it is convenient to consider the system

$$\frac{\partial^2 W_{13}}{\partial t_0^2} + g_1 \frac{\partial W_{13}}{\partial t_0} + \Omega_1^2 W_{13} - \Lambda_F W_{23} + Z_1^{(1)} e^{i\omega_F t_0} = 0$$

$$\frac{\partial^2 W_{23}}{\partial t_0^2} + g_2 \frac{\partial W_{23}}{\partial t_0} + \Omega_2^2 W_{23} + \Lambda_F W_{13} + Z_2^{(1)} e^{i\omega_F t_0} = 0 \qquad (2.58)$$

Assume that

$$W_{13} = P_1^{(1)} e^{i\omega_F t_0}$$

$$W_{23} = P_2^{(1)} e^{i\omega_F t_0} \qquad (2.59)$$

Combining Eqs. 2.58 and 2.59 yields

$$\begin{bmatrix} -\omega_F^2 + i g_1 \omega_F + \Omega_1^2 & -\Lambda_F \\ \Lambda_F & -\omega_F^2 + i g_2 \omega_F + \Omega_2^2 \end{bmatrix} \begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} \end{Bmatrix} = - \begin{Bmatrix} Z_1^{(1)} \\ Z_2^{(1)} \end{Bmatrix} \qquad (2.60)$$

The determinant of the matrix is zero (by definition of ω_F and Λ_F), which means that the first row is proportional to the second row, the constant of proportionality, C_p, being equal to u; ir fact Eq. 2.40 yields

$$C_p = \frac{-\omega_F^2 + i g_1 \omega_F + \Omega_1^2}{\Lambda_F} = \frac{-\Lambda_F}{-\omega_F^2 + i g_2 \omega_F + \Omega_2^2} = U \qquad (2.61)$$

with u given by Eq. 2.41 Thus, Eq. 2.60 cannot be solved*, unless

$$Z_1^{(1)} = U\, \dot{z}_2^{(1)} \tag{2.62}$$

If this condition is satisfied, the two equations are not independent. Thus, it is sufficient to satisfy only one of them; for instance, the second one. Furthermore, there are more unknowns than equations. Hence a particular solution of Eq. 2.60 can be obtained by setting $p_1^{(1)} = 0$ and obtaining $p_2^{(1)}$ from the second equation; this yields

$$\begin{Bmatrix} p_1^{(1)} \\ p_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{-Z_2^{(1)}}{-\omega_F^2 + i g_2 \omega_F + \Omega_2^2} \end{Bmatrix} \tag{2.63}$$

Thus, the solution of Eq. 2.51, disregarding the damped part, is given by

$$\begin{Bmatrix} W_1^{(3)} \\ W_2^{(3)} \end{Bmatrix} = 2\,\mathrm{Real}\left[\left(B\begin{Bmatrix} 1 \\ u \end{Bmatrix} + \begin{Bmatrix} p_1^{(1)} \\ p_2^{(1)} \end{Bmatrix}\right) e^{i\omega_F t} + \begin{Bmatrix} p_1^{(3)} \\ p_2^{(3)} \end{Bmatrix} e^{i3\omega_F t}\right] \tag{2.64}$$

where $p_1^{(1)}$ and $p_2^{(1)}$ are given by Eq. 2.63, B is a complex function of t_2, t_4, . . . , and

$$\begin{Bmatrix} p_1^{(3)} \\ p_2^{(3)} \end{Bmatrix} = \begin{bmatrix} -9\omega_F^2 + i g_1 3\omega_F + \Omega_1^2 & -\Lambda_F \\ \Lambda_F & -9\omega_F^2 + i g_2 3\omega_F + \Omega_2^2 \end{bmatrix}^{-1} \begin{Bmatrix} Z_1^{(3)} \\ Z_2^{(3)} \end{Bmatrix} \tag{2.65}$$

Finally, combining Eqs. 2.46, 2.52, and 2.64 yields

$$\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = 2\,\mathrm{Real}\left[\left((\varepsilon A + \varepsilon^3 B)\begin{Bmatrix} 1 \\ u \end{Bmatrix} + \varepsilon^3 \begin{Bmatrix} p_1^{(1)} \\ p_2^{(1)} \end{Bmatrix}\right) e^{i\omega_F t_0} + \varepsilon^3 \begin{Bmatrix} p_1^{(3)} \\ p_2^{(3)} \end{Bmatrix} e^{i3\omega_F t_0}\right] + O(\varepsilon^5) \tag{2.66}$$

*In other words, the assumption made with Eq. 2.59 is not correct. In fact, the solution contains, in general, terms like $t\, e^{i\omega_F t}$ (secular term).

where ω_F is given by Eq. 2.37 A and B are complex functions of t_2, . . . , $p_1^{(1)}$ and $p_2^{(1)}$ are given by Eq. 2.63, and finally $p_1^{(3)}$ and $p_2^{(3)}$ are given by Eq. 2.65.

2.2.3 The Function $A(t_2)$

Thus far, the original problem has been solved to the order ε^3 under the condition that Eq. 2.62 is satisfied. This condition yields a differential equation for A as a function of t_2. It should be noted that Eq. 2.62 is necessary in order to avoid secular terms; hence (as in the problem considered in Section 1), avoiding secular terms defines the functional dependence of A on t_2. The reminder of this subsection is devoted to the discussion of this function dependence.

Combining Eqs. 2.54 and 2.62 yields

$$z_1^{(1)} - u z_2^{(1)} = \left[2 i \omega_F (1-u^2) + (g_1 - g_2 u^2)\right] \frac{\partial A}{\partial t_2} \mp 2uA + \left[3C_{11} + (u^2 + 2uu^*) C_{12} - (2u^2 + uu^*) C_{21} - 3u^3 u^* C_{22}\right] A^2 A^* = 0 \tag{2.67}$$

or

$$\frac{\partial A}{\partial t_2} + \beta A + \gamma A^2 A^* = 0 \tag{2.68}$$

with

$$\beta = \mp \frac{1}{\alpha} 2u \tag{2.69}$$

$$\gamma = \frac{1}{\alpha}\left[3C_{11} + (u^2 + 2uu^*) C_{12} - (2u^2 + uu^*) C_{21} - 3u^3 u^* C_{22}\right] \tag{2.70}$$

where

$$\alpha = 2 i \omega_F (1-u^2) + (g_1 - g_2 u^2) \tag{2.71}$$

The solution of Eq. 2.68 is given by*

$$A = |A| e^{i\varphi} \tag{2.72}$$

---

*In fact, combining Eqs. 2.68, 2.72, and 2.75 yields

$$\left(\frac{\partial |A|}{\partial t_2} + i|A| \frac{\partial \varphi}{\partial t_2}\right) e^{i\varphi} = -(\beta_R + i\beta_I)|A| e^{i\varphi} - (\gamma_R + i\gamma_I)|A|^3 e^{i\varphi}$$

or by separating real and imaginary parts

$$\frac{\partial |A|}{\partial t_2} = -\beta_R |A| - \gamma_R |A|^3 \tag{a}$$

$$\frac{\partial \varphi}{\partial t_2} = -\beta_I - \gamma_I |A|^2 \tag{b}$$

Equation (a) can be written as

$$\frac{\partial}{\partial t_2}\left(\frac{1}{|A|^2}\right) = 2\beta_R \left(\frac{1}{|A|^2} + \frac{\gamma_R}{\beta_R}\right)$$

whose solution is

$$\frac{1}{|A|^2} + \frac{\gamma_R}{\beta_R} = K e^{2\beta_R t_2}$$

which is equivalent to Eq. 2.73. Then Eq. (b) yields

$$\varphi = -\beta_I t_2 - \gamma_I \int |A|^2 dt_2 + \varphi_0 = -\beta_I t_2 - \gamma_I \int \frac{dt_2}{-\frac{\gamma_R}{\beta_R} + K e^{2\beta_R t_2}} + \varphi_0$$

which is equivalent to Eq. 2.74.

with

$$|A| = \left(-\frac{\gamma_R}{\beta_R} + K e^{2\beta_R t_2}\right)^{-1/2} \tag{2.73}$$

$$\begin{aligned}\varphi &= -\beta_I t_2 - \frac{\gamma_I}{2\gamma_R} \ln\left(-\frac{\gamma_R}{\beta_R} e^{-2\beta_R t_2} + K\right) + \varphi_0 \\ &= -\left(\beta_I - \frac{\beta_R}{\gamma_R}\gamma_I\right) t_2 - \frac{\gamma_I}{2\gamma_R} \ln\left(-\frac{\gamma_R}{\beta_R} + K e^{2\beta_R t_2}\right) + \varphi_0 \\ &= -\left(\beta_I - \frac{\beta_R}{\gamma_R}\gamma_I\right) t_2 + \frac{\gamma_I}{\gamma_R} \ln|A| + \varphi_0\end{aligned} \tag{2.74}$$

where k and ϕ_o are two arbitrary functions of t_4, t_6, . . . , and the following relations have been used:

$$\begin{aligned}\beta &= \beta_R + i\beta_I \\ \gamma &= \gamma_R + i\gamma_I\end{aligned} \tag{2.75}$$

To summarize, the solution of Eq. 2.31 is given by Eq. 2.66 where A is obtained by combining Eqs. 2.72, 2.73, and 2.74. All the quantities are known, except B, k, and ϕ_o. It is worth noting that B is a function of t_2, whereas k and ϕ_o are a function of t_4; these functional dependences are obtained by avoiding the secular term in the expressions of w_{15} and w_{25} (see Section 2 of Ref. 15).

Next, the asymptotic behavior of the solution as $t \to \infty$ is examined. For the sake of simplicity, only the terms of order ε are considered here; combining Eqs. 2.66 and 2.72 and disregarding terms of order ε^3 yields

$$\begin{aligned}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} &= 2\,\mathrm{Real}\left[\varepsilon A \left\{u'\right\} e^{i\omega_f t_0}\right] \\ &= 2\varepsilon|A|\,\mathrm{Real}\left[\left\{u'\right\} e^{i(\omega_f t_0 + \varphi)}\right]\end{aligned} \tag{2.76}$$

where ϕ is given by Eq. 2.74 and according to Eq. 2.73, $|A|$ is given by

$$|A| = \left[-\frac{\gamma_R}{\beta_R} + K e^{2\beta_R t_2} \right]^{-1/2}$$

$$= \left[-\frac{\gamma_R}{\beta_R} + S_K e^{2\beta_R (t_2 - t_{2\rho})} \right]^{-1/2} \qquad (2.77)$$

with

$$S_K = \text{sign}(K)$$

$$t_{2\rho} = -\frac{1}{2\beta_R} \ln |K| \qquad (2.78)$$

The function $|A|$ has a different behavior, depending upon the signs of β_R and γ_R (the four possible behaviors are plotted in Figs. 1, 2, 3, and 4). Note that:

$$\beta_R \gtrless 0 \qquad \text{for } \Lambda \lesseqgtr \Lambda_F \qquad (2.79)$$

since the linear case is unstable ($\beta_R < 0$) for $\Lambda > \Lambda_*$.* The four cases are examined in the following. Assume first that (Cases 1 and 2)

$$\gamma_R > 0 \qquad (2.80)$$

which corresponds to "hard-spring" nonlinear terms (stabilizing nonlinear terms).

Case 1:

If $\Lambda > \Lambda_F$ ($\beta_R < 0$) as $t_2 \to \infty$, one obtains (see Eq. 2.73 and Fig. 1):

$$|A|_\infty = \left| \frac{\beta_R}{\gamma_R} \right|^{1/2} \qquad (2.81)$$

and (see Eq. 2.74)

*The solution of the linear case behaves like $e^{-\varepsilon^2 \beta_R t} \cos(\omega t + \phi)$ as is seen by setting $\gamma_R = 0$ in Eq. 2.77.

$$\varphi_\infty = \omega_2 t_2 + \varphi_{\infty,0} \tag{2.82}$$

with

$$\omega_2 = -\beta_I + \frac{\beta_R}{\gamma_R}\gamma_I \tag{2.83}$$

$$\varphi_{\infty,0} = \frac{\gamma_I}{2\gamma_R} \ln\left|\frac{\beta_R}{\gamma_R}\right| + \varphi_0 \tag{2.84}$$

Thus for $\gamma_R > 0$, and $\Lambda > \Lambda_F (\beta_R < 0)$, the asymptotic behavior of the solution as $t \to \infty$ is given by

$$\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = 2\left|\frac{\beta_R}{\gamma_R}\right|^{1/2} \text{Real}\left\{\begin{matrix} 1 \\ u \end{matrix}\right\} e^{i(\omega_F t_0 + \omega_2 t_2 + \varphi_{\infty,0})} \tag{2.85}$$

or, using the definition $t_o = t$, $t_2 = \varepsilon^2 t$ (Eq. 2.47) and the definition of u (Eq. 2.41)

$$W_{1\infty} = 2\left|\frac{\beta_R}{\gamma_R}\right|^{1/2} \cos(\omega t + \varphi_{\infty,0})$$

$$W_{2\infty} = -2\left|\frac{g_1}{g_2}\cdot\frac{\beta_R}{\gamma_R}\right|^{1/2} \cos(\omega t + \varphi_{\infty,0} + \chi) \tag{2.86}$$

with (see Eq. 2.83)

$$\omega = \omega_F + \varepsilon^2\omega_2 = \omega_F - \varepsilon^2\left(\beta_I - \frac{\beta_R}{\gamma_R}\gamma_I\right) \tag{2.87}$$

Equation 2.86 shows that the steady solution represents oscillations of given amplitude (steady limit cycle oscillations). The transient solution given by Eq. 2.76 shows that the amplitude $|A|$ of the solution grows (decrease) to the value $|A|_\infty$ (see Fig. 1), if the amplitude at time $t = 0$ is lower (greater) than $|A|_\infty$.

Case 2:

On the other hand, for $\Lambda < \Lambda_F (\beta_R > 0)$, Eq. 2.77 shows that

$$|A| \approx e^{-\beta_R (t_2 - t_{2_0})}$$

as $t_2 \to \infty$. Thus, the solution is fully stable and its amplitude goes to zero as $t \to \infty$ (Fig. 2).

Next, assume (Cases 3 and 4) that

$$\gamma_R < 0$$

which corresponds to "soft-spring" nonlinear terms (destabilizing nonlinear terms). Note that Eq. 2.77 is still valid.

Case 3:

For $\Lambda < \Lambda_F(\beta_R > 0)$, the diagram of $|A|$ (see Fig. 3), shows that the solution grows to infinity (or goes to zero) if the initial condition is such that, at

$$t=0, \; |A| > \left|\frac{\beta_R}{\gamma_R}\right|^{1/2} \quad \left(\text{or } |A| < \left|\frac{\beta_R}{\gamma_R}\right|\right)$$

respectively. This case corresponds to unstable limit cycle oscillations.*

Case 4:

On the other hand, for $\Lambda > \Lambda_F$, the solution is strongly unstable, and goes to infinity in finite time ($t_2 = t_{2\rho}$) as is shown in Fig. 4.

Summarizing, consider first, $\Lambda > \Lambda_F$. Then, if there exists a limit cycle, it is stable (Case 1, Fig. 1); otherwise the solution is strongly unstable (Case 4, Fig. 4). On the other hand, for $\Lambda < \Lambda_F$, if there exists a limit cycle, this is unstable (Case 3, Fig. 3); otherwise the solution is completely stable (Case 2, Fig. 2).**

---

*This case is discussed in more detail in Ref. 10.

**It should be noted that these conclusions (limit cycle stable for $\Lambda > \Lambda_F$; unstable for $\Lambda < \Lambda_F$) are true only if $\beta_R > 0$ (linear solution unstable) for $\Lambda > \Lambda_F$.

2.3 The N Mode Case

In the preceding section, the nonlinear flutter problem was studied for the two-mode case. The results can be generalized to the N mode case. This generalization is described in the following.

2.3.1 The Formulation of the Problem

Consider the N mode Galerkin system for the nonlinear panel-flutter problem. For a plate without imperfections, the differential equations are given by the following (see Appendix A):

$$\sum_{p=1}^{N} M_{np}\frac{d^2W_p}{dt^2} + \sum_{p=1}^{N} G_{np}\frac{dW_p}{dt} + \sum_{p=1}^{N} K_{np}W_p + \lambda\sum_{p} E_{np}W_p + \sum_{p,q,r=1}^{N} C_{npqr}W_pW_qW_r + O(\varepsilon^5) = 0 \tag{2.88}$$

where M_{np}, G_{np}, K_{np}, E_{np} and C_{npqr} are known coefficients and ε is the order of magnitude on w_n. This equation generalizes Eq. 2.31 of Subsection 2.2.

In order to study Eq. 2.88, it is convenient to write it in matrix notation as

$$\left([M]\frac{d^2}{dt^2} + [G]\frac{d}{dt} + [K] + \lambda[E]\right)\{W\} + \{C\} + O(\varepsilon^5) = 0 \tag{2.89}$$

where*

$$\{W\} = \{W_n\} \tag{2.90}$$

$$[M] = [M_{np}] \tag{2.91}$$

$$[G] = [G_{np}] \tag{2.92}$$

$$[K] = [K_{np}] \tag{2.93}$$

---

*Note that here G_{np} are assumed to be independent of ε. A generalization is considered in Subsection 2.4.3.

$$[E] = [E_{np}] \tag{2.94}$$

$$\{C\} = \left\{ \sum_{p,q,r} C_{npqr} W_p W_q W_r \right\} \tag{2.95}$$

The corresponding linear system is given by

$$\left([M]\frac{d^2}{dt^2} + [G]\frac{d}{dt} + [K] + \lambda [E] \right)\{W\} = 0 \tag{2.96}$$

Just as for the two-mode case, there is a value, λ_F, of λ such that for

$$\lambda = \lambda_F \tag{2.97}$$

Equation 2.96 has a particular solution which is purely harmonic:

$$\{W\} = \{U\} e^{i\omega_F t} + \{U^*\} e^{-i\omega_F t} \tag{2.98}$$

In order to find the values of λ_F and ω_F, combine Eqs. 2.96 and 2.98. This yields the corresponding algebraic system

$$[F]\{U\} = 0 \tag{2.99}$$

with

$$[F] = \left(-\omega_F^2 [M] + i\omega_F [G] + [K] + \lambda_F [E] \right) \tag{2.100}$$

This system has a nontrivial solution, if

$$\text{Det.}\,[F] = 0 \tag{2.101}$$

This is a complex equation; by setting the real and the imaginary parts equal to zero, independently, one obtains a set of two equations with two unknowns: λ_F and ω_F. If Eq. 2.101 is satisfied, the eigenvector {U} can be obtained by solving Eq. 2.99.

Consider next, the nonlinear problem. In order to solve it, the same concepts used in Subsection 2.2 are used here in a slightly more general form. Set

$$\lambda = \lambda_F \pm \varepsilon^2 \tag{2.102}$$

and

$$\{W\} = \varepsilon\{W_1\} + \varepsilon^3\{W_3\} + \cdots \tag{2.103}$$

where

$$\{W_k\} = \{w_{nk}\} \tag{2.104}$$

are functions of t_o, t_2, and so on, with

$$t_n = \varepsilon^n t \tag{2.105}$$

so that

$$\frac{d}{dt} = \frac{\partial}{\partial t_o} + \varepsilon^2 \frac{\partial}{\partial t_2} + \cdots \tag{2.106}$$

Combining Eqs. 2.88, 2.102, 2.103, and 2.106 yields

$$\left([M]\left(\frac{\partial^2}{\partial t_o^2} + 2\varepsilon^2 \frac{\partial^2}{\partial t_o \partial t_2} + \cdots\right) + [G]\left(\frac{\partial}{\partial t_o} + \varepsilon^2 \frac{\partial}{\partial t_2} + \cdots\right) + [K] + (\lambda_F \pm \varepsilon^2)[E]\right)\left(\varepsilon\{W_1\} + \varepsilon^3\{W_3\} + \cdots\right) + \left(\varepsilon^3\{C_3\} + \cdots\right) + O(\varepsilon^5) = 0 \tag{2.107}$$

with

$$\{C_3\} = \left\{\sum_{p,q,r=1}^{N} C_{npqr} W_{p1} W_{q1} W_{r1}\right\} \tag{2.108}$$

Equating terms of the same order of magnitude yields the following set of recurrent systems:

<u>System of Order ε</u>

$$\left([M]\frac{\partial^2}{\partial t_o^2} + [G]\frac{\partial}{\partial t_o} + [K] + \lambda_F[E]\right)\{W_1\} = 0 \tag{2.109}$$

System of Order ε^3

$$\left([M]\frac{\partial^2}{\partial t_0^2}+[G]\frac{\partial}{\partial t_0}+[K]+\lambda_F[E]\right)\{W_3\}$$
$$=-2[M]\frac{\partial^2}{\partial t_0\partial t_2}\{W_1\}-[G]\frac{\partial}{\partial t_2}\{W_1\}\mp[E]\{W_1\}-\{C_3\} \qquad (2.110)$$

and so on. Note that the differential operator on the left-hand side is the same in all the systems.

2.3.2 The Solution of the Problem

Equation 2.109 is equivalent to the linearized system, Eq. 2.96; the steady-state solution* of Eq. 2.109 is given by (see Eq. 2.98)

$$\{W_1\}=A\{U\}e^{i\omega_F t_0}+A^*\{U^*\}e^{-i\omega_F t_0}$$
$$=2\,Real\left(A\{U\}e^{i\omega_F t_0}\right) \qquad (2.111)$$

where A is a function of t_2, t_4, . . .; ω_F is a root of Eq. 2.101 and

$$\{U\}=\{U_i\} \qquad (2.112)$$

is the complex eigenvector of the linear problem and is found by solving Eq. 2.99.

Next consider the system of order ε^3 (Eq.2.110). First, the explicit expression for $\{C_3\}$ is derived. Combining Eqs.2.108, 2.111, and 2.112 yields

$$\{C_3\}=\sum_{p,q,r=1}^{N}C_{npqr}\left(Au_p e^{i\omega_F t}+A^*u_p^* e^{-i\omega_F t_0}\right)$$
$$\times\left(Au_q e^{i\omega_F t_0}+A^*u_q^* e^{-i\omega_F t_0}\right)\times\left(Au_r e^{i\omega_F t_0}+A^*u_r^* e^{-i\omega_F t_0}\right)$$
$$=2\,Real\left(A^3\{H_0\}e^{i3\omega_F t_0}+A^2A^*\{H_1\}e^{i\omega_F t_0}\right) \qquad (2.113)$$

---

*As mentioned in the preceding section, the exponentially damped part does not give any relevant contribution to the limit solution. For a complete solution of the problem see Ref. 10.

with

$$\{H_0\} = \left\{ \sum_{p,q,r=1}^{N} C_{npqr}\, u_p\, u_q\, u_r \right\}$$

$$\{H_1\} = \left\{ \sum_{p,q,r=1}^{N} C_{npqr} \left(u_p\, u_q\, u_r^* + u_p\, u_q^*\, u_r + u_p^*\, u_q\, u_r \right) \right\} \qquad (2.114)$$

Finally, combining Eqs. 2.110 and 2.113 yields

$$\left([M] \frac{\partial^2}{\partial t_0^2} + [G] \frac{\partial}{\partial t_0} + [K] + \lambda_F [E] \right) \{W_s\} + \{Z\} = 0 \qquad (2.115)$$

where

$$\{Z\} = 2\, Real \left[\{Z^{(1)}\} e^{is\omega_F t} + \{Z^{(11)}\} e^{i\omega_F t} \right] \qquad (2.116)$$

with

$$\{Z^{(s)}\} = \{H_0\}\, A^s \qquad (2.117)$$

$$\{Z^{(11)}\} = (2i\omega [M] + [G]) \{U\} \frac{\partial A}{\partial t_2}$$

$$\pm [E] \{U\} A + \{H_1\} A^2 A^* \qquad (2.118)$$

Next, it should be noted that the term which contains $e^{i\omega_F t}$ yields secular terms unless the vector $\{z_1\}$ satisfies a special condition. This condition generalizes the condition for the two-mode case (Eq. 2.62), which can be written as

$$\begin{bmatrix} I, -U \end{bmatrix} \begin{Bmatrix} z_1^{(11)} \\ z_2^{(11)} \end{Bmatrix} = 0 \qquad (2.119)$$

The condition for N modes is discussed in the following. Consider a particular integral of the equation

$$\left([M] \frac{d^2}{dt_0^2} + [G] \frac{d}{dt_0} + [K] + \lambda_F [E] \right) \{W_s\}$$

$$+ \{Z^{(1)}\} e^{i\omega_F t_0} + \{Z^{(1)*}\} e^{-i\omega_F t_0} = 0 \qquad (2.120)$$

Assume a solution of the form

$$\{W_3\} = \{P^{(1)}\} e^{i\omega_p t_0} + \{P^{(1)*}\} e^{-i\omega_p t_0} \tag{2.121}$$

Then $\{P^{(1)}\}$ must satisfy the equation

$$[F]\{P^{(1)}\} = -\{Z^{(1)}\} \tag{2.122}$$

with [F] given by Eq. 2.100. The determinant of this equation is zero (Eq. 2.101), which implies that it is possible to obtain a linear combination of the coefficient of the equation which is identically equal to zero. In other words, there exists a vector $\{U^L\}$ such that

$$\lfloor U^L \rfloor [F] = 0 \tag{2.123}$$

The vector $\lfloor U^L \rfloor$ will be mentioned as the left eigenvector (i.e., the transpose of the eigenvector of the transpose of [F]).

Thus, Eq. 2.122 cannot be solved unless the linear combinations (with the same coefficients) of the components of $Z^{(1)}$ are also zero; in other words, Eq. 2.122 can be solved (that is the solution of Eq. 2.120 has the form given in Eq. 2.122), only if the condition

$$\lfloor U^L \rfloor \{Z^{(1)}\} = 0 \tag{2.124}$$

is satisfied. This is the desired condition for avoiding secular terms. Equation 2.124 is a generalization of Eq. 2.119 which is valid only for the two-mode case.

This condition for "no-secular terms" yields a differential equation for A as a function of t_2. In fact, combining Eqs. 2.118 and 2.123 one obtains

$$\frac{\partial A}{\partial t_2} + \beta A + \gamma A^2 A^* = 0 \tag{2.125}$$

where

$$\beta = \pm \frac{1}{\alpha} \lfloor U^L \rfloor [E] \{U\} \tag{2.126}$$

$$\gamma = \frac{1}{\alpha} \lfloor U^L \rfloor \{H_1\} \tag{2.127}$$

with

$$\alpha = \langle U^* \rangle \left(2i\omega_t [M] + [G] \right) \{U\} \tag{2.128}$$

Equation 2.125 is identical to Eq. 2.68. Thus its solution is obtained by combining Eqs. 2.72, 2.73, and 2.74, which yields

$$A = \left(-\frac{\gamma_R}{\beta_R} + K e^{2\beta_R t_2} \right)^{-\gamma_2} e^{i\left[-\beta_2 t_2 - \frac{\gamma_2}{2\gamma_R} \ln\left(-\frac{\gamma_R}{\beta_R} e^{-2\beta_R t_2} + K\right) + \varphi_0\right]} \tag{2.129}$$

where according to Eq. 2.75, β_R and β_I (or γ_R and γ_I) are the real and the imaginary part of β (or γ, respectively). Note that the definitions of β and γ given in Eqs. 2.126 and 2.127 are similar to those given in Eqs. 2.69 and 2.70 for the two-mode case.* Finally, Eq. 2.122 can be solved by setting the first component of $\{P^{(1)}\}$ equal to zero and disregarding the first equation. In matrix notation this can be written as

$$\{P^{(1)}\} = [L^{(1)}] \{Z^{(1)}\} \tag{2.130}$$

---

*In this case

$$\{U\} = \begin{Bmatrix} 1 \\ u \end{Bmatrix} \quad \{U^*\} = \begin{Bmatrix} 1 \\ -u \end{Bmatrix} \quad [E] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad [M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[G] = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \quad \{H_1\} = \begin{Bmatrix} 3C_{11} + (u^2 + 2uu^*) C_{12} \\ (2u + u^*) C_{21} + 3u^2 u^* C_{22} \end{Bmatrix}$$

which yields, by use of Eqs. 2.126, 2.127, and 2.128

$$\beta = \pm \frac{1}{\alpha} \langle U^* \rangle [E] \{U\} = \mp \frac{1}{\alpha} 2u$$

$$\gamma = \frac{1}{\alpha} \langle U^* \rangle \{H_1\} = \frac{1}{\alpha} \left[3C_{11} + (u^2 + 2uu^*) C_{12} - (2u^2 + uu^*) C_{21} - 3u^2 u^{*2} C_{22} \right]$$

with

$$\alpha = \langle U^* \rangle \left(2i\omega_t [M] + [G] \right) \{U\} = 2i\omega_t (1 - u^2) + g_1 - g_2 u^2$$

in agreement with Eqs. 2.69, 2.70, and 2.71.

where

$$[L^{(1)}] = -\begin{bmatrix} 0 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & L \end{bmatrix} \tag{2.131}$$

and where [L] is the inverse of the (N-1) x (N-1) matrix obtained from [F] by eliminating the first row and the first column.

Thus the solution for $\{W_3\}$ can now be obtained and is given by

$$\{W_3\} = 2\,Real\left[\left(B\{U\} + \{P^{(1)}\}\right) e^{i\omega_F t_0} + \{P^{(3)}\}\, e^{i3\omega_F t_0}\right] \tag{2.132}$$

where $\{P^{(1)}\}$ is given by Eq. 2.130, B is a function of $t_2, \ldots,$ and

$$\{P^{(3)}\} = [L^{(3)}]\{Z^{(3)}\} \tag{2.133}$$

where

$$[L^{(3)}] = -\left[-9\omega_F^2[M] + i3\omega_F[G] + [K] + \lambda_F[E]\right]^{-1} \tag{2.134}$$

Finally, combining Eqs. 2.103, 2.111, and 2.132, the solution of Eq. 2.88 is given by

$$\{W\} = 2\,Real\left[(\varepsilon A + \varepsilon^3 B)\{U\} e^{i\omega_F t_0} + \varepsilon^3\{P^{(1)}\}e^{i\omega_F t_0} + \varepsilon^3\{P^{(3)}\}e^{i3\omega_F t_0}\right] \tag{2.135}$$

with A given by Eq. 2.129.

2.3.3 Discussion of Results

In order to describe the behavior of the solution, it is convenient to consider only the terms of order ε; combining Eqs. 2.129 and 2.135 and neglecting terms of order ε^3 yields

$$\begin{aligned}[W] &= 2\,Real\left[\varepsilon A\{U\} e^{i\omega_F t_0}\right] \\ &= 2\varepsilon|A|\,Real\left[\{U\} e^{i\omega_F t_0 + \varphi}\right]\end{aligned} \tag{2.136}$$

with ϕ given by Eq. 2.131 and

$$|A| = \left[-\frac{\gamma_R}{\beta_R} + K e^{2\beta_R t_2} \right]^{-1/2} = \left[-\frac{\gamma_R}{\beta_R} + S_K e^{2\beta_R (t_2 - t_{2,0})} \right]^{-1/2} \quad (2.137)$$

with S_k and $t_{2,0}$ defined by Eq. 2.46. Equation 2.136 is similar to Eq. 2.76. The behavior is exactly the same. The conclusions made in Subsection 2.2.3 are repeated here. Note first that (just as for the two-mode case)

$$\beta_R \lesseqgtr 0 \qquad for \quad \lambda \gtrless \lambda_F \quad (2.138$$

since the linear solution is unstable for $\lambda > \lambda_F$. Thus, for $\gamma_R > 0$ (stabilizing or hard-spring nonlinear terms), the solution is stable for $\lambda < \lambda_F$ (Fig. 2), whereas for $\lambda > \lambda_F$, there is a stable limit cycle (Fig. 1), and the steady-state oscillations are given by

$$\{W\}_{L.C.} = 2\varepsilon \sqrt{\frac{\beta_R}{\gamma_R}} \, Real\left(\{U\} e^{i(\omega_F t + \phi_{\infty,0})} \right) + O(\varepsilon^3) \quad (2.139)$$

with ω given by Eq. 2.87 and $\phi_{\infty,0}$ is an arbitrary constant (see Eq. 2.84) or

$$W_{n_{L.C.}} = 2\varepsilon \sqrt{\frac{\beta_R}{\gamma_R}} \, Real\left(u_n e^{i(\omega_F t + \phi_{\infty,0})} \right) \quad (2.140$$

Note that ε is defined by Eq.2.102. On the other hand, for $\gamma_R < 0$ (destabilizing or soft-spring nonlinear terms) the solution is strongly unstable (goes to infinity in finite time) for $\lambda > \lambda_F$, whereas for $\lambda < \lambda_F$ there exists a limit cycle (with oscillation given by Eq.2.140), but this is unstable (see Figs. 3 and 4).

It is worth noting that, from the above conclusions, one can say, in general, that limit cycles with $\lambda > \lambda_F$ are stable, whereas those with $\lambda < \lambda_F$ are unstable.

2.4 The Numerical Procedure

In this subsection the procedure used for the numerical application is described.

2.4.1 Summary of the Method

In the preceding section it is shown that the limit cycle solution is obtained as follows. First, solve the linear problem

$$[F]\{U\}=0 \tag{2.141}$$

with

$$[F]=-\omega_F^2[M]+i\omega_F[G]+[K]+\lambda_F[E] \tag{2.142}$$

which yields the two eigenvalues ω_F and λ_F and the eigenvector $\{U\}$. Evaluate also the left eigenvector $\{U^L\}$ defined by

$$\lfloor U^L \rfloor [F]=0 \tag{2.143}$$

or, by taking the transpose equation

$$[F]^T\{U^L\}=0 \tag{2.144}$$

where the superscript T means "transpose". Evaluate the following complex numbers (Eqs. 2.126 and 2.127)

$$\beta=\pm\frac{1}{\alpha}\lfloor U^L \rfloor [E]\{U\} \qquad (\lambda \gtrless \lambda_F) \tag{2.145}$$

$$\gamma=\frac{1}{\alpha}\lfloor U^L \rfloor \{H_1\} \tag{2.146}$$

where (Eq. 2.128)

$$\alpha=\lfloor U^L \rfloor (2i\omega_F[M]+[G])\{U\} \tag{2.147}$$

In Eq. 2.146 the vector $\{H_1\}$ is given by (see Eq. 2.114)

$$\{H_1\}=\{h_n\} \tag{2.148}$$

with

$$h_n=\sum_{p,q,r} C_{npqr}\left(u_p u_q u_r^* + u_p u_q^* u_r + u_p^* u_q u_r\right) \tag{2.149}$$

where u_k is the k^{th} component of $\{U\}$ and u^*_k is the complex conjugate of u_k. Next, consider the real parts and the imaginary parts of β and γ

$$\beta = \beta_R + i\beta_I \qquad (2.150)$$
$$\gamma = \gamma_R + i\gamma_I$$

Note that according to Eq. 2.138

$$\beta_R \lesseqgtr 0 \qquad for \qquad \lambda \gtrless \lambda_F \qquad (2.151)$$

Next, consider the sign of γ_R. If $\gamma_R > 0$ then the limit cycle is stable and exists only for $\lambda > \lambda_F$; on the other hand, for $\gamma_R < 0$ the limit cycle is unstable and exists only for $\lambda < \lambda_F$. Thus, in both cases the solution is given as follows. For

$$\lambda = \lambda_F \pm \varepsilon^2 \qquad (for \ \lambda \gtrless \lambda_F) \qquad (2.152)$$

there exists a limit cycle (stable for $\gamma_R > 0$, unstable for $\gamma_R < 0$). The limit cycle oscillations are given by

$$\{W\}_{t\to\infty} = \varepsilon \psi \, Real\left(\{U\} e^{i(\omega t + \varphi)} \right) \qquad (2.153)$$

where ϕ is an arbitrary phase angle,

$$\varepsilon = \sqrt{|\lambda - \lambda_F|} \qquad (2.154)$$

and finally

$$\psi = 2 \left| \frac{\beta_R}{\gamma_R} \right|^{1/2}$$

is called the "limit cycle amplitude parameter". The frequency ω, is given by (see Eq. 2.87)

$$\omega = \omega_F \mp \varepsilon^2 \left(\beta_I - \frac{\beta_R}{\gamma_R} \gamma_I \right) \qquad (for \ \lambda \gtrless \lambda_F) \qquad (2.155)$$

2.4.2 The Left Eigenvector

Next, it should be noted that the left-eigenvector $[U^L]$ can be obtained from the vector $\{U\}$ in a very simple way, if the matrices [M], [G], [K], and [E] are such that

$$[M_{np}]=[M_{pn}]\,,\ [G_{np}]=[G_{pn}],\ [K_{np}]=[K_{pn}] \quad \text{for } n+p=\text{even}$$

$$[M_{np}]=0\ ,\ [G_{np}]=0,\ [K_{np}]=0 \quad \text{for } n+p=\text{odd} \qquad (2.156)$$

and

$$[E_{np}]=-[E_{pn}] \quad \text{for } n+p=\text{odd}$$

$$[E_{np}]=0 \quad \text{for } n+p=\text{even} \qquad (2.157)$$

Then it is immediately verified that

$$[J][M]^T[J]=[M]$$

$$[J][G]^T[J]=[G]$$

$$[J][K]^T[J]=[K]$$

$$[J][E]^T[J]=[E] \qquad (2.158)$$

where

$$[J]=\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}=-\left[(-1)^n\right] \qquad (2.159)$$

Thus, combining Eqs. 2.142 and 2.158 yields

$$[J][F]^T[J]=[F] \qquad (2.160)$$

where the superscript T indicates the "transpose". Thus, Eq.2.141 is equivalent to

$$[J][F]^T[J]\{u\}=0 \qquad (2.161)$$

which shows that the solution of Eq. 2.143 is given by

$$\{u^*\}=[J]\{u\} \qquad (2.162)$$

or

$$\llcorner U^{t} \lrcorner = \llcorner U_1, -U_2, U_3, -U_4, \cdots \lrcorner \tag{2.163}$$

2.4.3 Damping Coefficients Depending Upon λ

As shown in Ref. 10 (see also Appendix A), the matrix of the damping coefficients is of the form

$$[G] = g_E [G'] + \sqrt{\lambda}\,[\tilde{G}] \tag{2.164}$$

or, by expanding in series of ε (see Eq. 2.152)

$$[G] = g_E [G'] + \sqrt{\lambda_F}\,[\tilde{G}] \pm \frac{1}{2\sqrt{\lambda_F}}\,\varepsilon^2 [\tilde{G}] + O(\varepsilon^4)$$

$$= [G_0] \pm \varepsilon^2 [G_2] + O(\varepsilon^4) \tag{2.165}$$

with

$$[G_0] = g_E [G'] + \sqrt{\lambda_F}\,[\tilde{G}]$$

$$[G_2] = \frac{1}{2\sqrt{\lambda_F}}\,[\tilde{G}] \tag{2.166}$$

It is easily shown that the results summarized in Subsection 2.4.1 are still valid, if [G] is replaced by $[G_0]$ and the definition of β is modified as follows*:

$$\beta = \pm \frac{1}{\alpha}\, \llcorner U^{t} \lrcorner \left([E] + i\omega_F [G_2] \right) \{U\} \tag{2.167}$$

This is the formulation used in the numerical application, with $g_E = 0$

---

*Because Eq.2.110 contains, in this case, an additional term, on the right-hand side, of the form

$$\mp [G_2] \frac{\partial}{\partial t_0} \{W_1\} = \mp 2\,\mathrm{Real}\left[i\omega_F [G_2] \{U\} A e^{i\omega_F t} \right]$$

2.5 Extension of the Theoretical Analysis

The analysis presented here is limited in several respects:

(a) It is limited to w^3 order terms

(b) It assumes implicitly that the flutter frequency ω_F is greater than zero.

(c) The terms g_n are often greater than $\varepsilon = \sqrt{\lambda - \lambda_F}$ which is the asymptotic expansion parameter.

It is of interest to investigate further these limitations. In order to obtain simple results, these three problems have been studied, using the simple model of two modes

$$\ddot{W}_1 + g_1 \dot{W}_1 + \Omega_1^2 W_1 - \Lambda W_2 + C_{11} W_1^3 + C_{12} W_1 W_2^2 + d_{11} W_1^5 + d_{12} W_1^3 W_2^2 + d_{13} W_1 W_2^4 = 0$$

$$\ddot{W}_2 + g_2 \dot{W}_2 + \Omega_2^2 W_2 + \Lambda W_1 + C_{21} W_1^2 W_2 + C_{22} W_2^3 + d_{21} W_2^5 + d_{22} W_1^2 W_2^3 + d_{23} W_2^5 = 0 \qquad (2.168)$$

The details of the analysis are given in Ref. 19. The results summarized here extend the analysis to include the items (a), (b), and (c) mentioned above.

2.5.1 Effect of the Fifth Order Nonlinear Terms

Depending upon the sign of a coefficient which represents the effect of the fifth order nonlinear terms, the parabola which gives the amplitude of the limit cycle versus Λ can bend back. The limit cycle oscillations disappear for a value

$$\Lambda > \Lambda_{critical} \qquad (2.169)$$

2.5.2 Flutter-Buckling Interaction

The region of stability of the linear system is obtained by studing the stability of the equation

$$\text{Det.} \begin{bmatrix} p^2 + g_1 p + \Omega_1^2 & -\Lambda \\ \Lambda & p^2 + g_2 p + \Omega_2^2 \end{bmatrix} = 0 \qquad (2.170)$$

This is a fourth order algebraic equation of the type

$$p^4 + a p^3 + b p^2 + c p + d = 0 \qquad (2.171)$$

The stability conditions for this equation are

$$a>0 \quad , \; b>0 \; , \; c>0 \, , d>0 \, , \; c^2-abc+a^2d<0 \qquad (2.172)$$

The curves c = 0, d = 0, and $c^2 - abc + a^2d = 0$ define the region of stability.

It can be noted that the condition c = 0 corresponds to the case where $\omega_F = 0$. Thus, the analysis given before fails in the neighborhood of the point of intersection of the flutter curve with the buckling curve (where the flutter frequency ω_F can be of the order ε).

To study the nonlinear behavior of the system, the multiple time-scaling technique was used. The zero order system is the one with $\Lambda = \Lambda_*$ and membrane force $N = N_*$. These are (by definition) the values of Λ and N such that

$$c^2 - abc + a^2d = 0 \qquad \text{(Flutter condition)}$$

$$d = 0 \qquad \text{(Buckling condition)} \qquad (2.173)$$

which also imply that c = 0. Since c = 0 and d = 0, the algebraic equation given by Eq. 5.4 has a double root p = 0. The other two roots are complex conjugates with negative real parts. Thus, the "nondamped" part of the solution does not depend upon the time t_o. The solution can be written as

$$\{W_1\} = \{U\} A(t_1, t_2, \cdots) \qquad (2.174)$$

The first order system does not yield any secular terms. In order to avoid the secular terms in the second order system, the following equation must be satisfied

$$\frac{\partial^2 A}{\partial t_1^2} + \beta A + \gamma A^3 = 0 \qquad (2.175)$$

where α, β, and γ are known coefficients which depend upon the coefficients of Eq. 2.168. Equation 2.175 can be integrated to yield

$$\frac{1}{2}\left(\frac{\partial A}{\partial t_1}\right)^{1/2} + \frac{1}{2}\beta A^2 + \frac{1}{4}\gamma A^4 = \mathcal{E} \qquad (2.176)$$

where $\mathcal{E}$ is the energy of the system. The solution of this equation can be obtained in terms of Jacobian elliptic functions cn(T,k) and dn(T,k) as follows:

1. For $\beta < 0$ and $\xi > 0$, or $\beta > 0$

$$A = A_1 \, cn(\omega t, k) \tag{2.177}$$

2. For $\beta < 0$ and $\xi < 0$

$$A = A_1 \, dn(\omega t, k) \tag{2.178}$$

In Eqs. 2.177 and 2.178, ξ, ω and k are expressed in terms of A, β and γ.

Finally, avoiding secular terms in the third order system yields a differential equation which gives the variation of ξ with t_2.

2.5.3 Small Damping Coefficients

In order to verify the validity of the analysis for very small values of the damping coefficient, the system obtained from Eq. 2.31 by setting

$$g_n = \varepsilon \bar{g}_n \tag{2.179}$$

has been studied. The results show that in this case the multiple-time-scaling technique fails. More precisely, it can predict the variation of the amplitude during the transient, but cannot predict the change in frequency. It is believed that the failure is due to the fact that the function $e^{-1/\varepsilon g_n}$ (which replaces the function e^{-1/g_n}) does not have any asymptotic expansion as ε tends to zero. Nevertheless, the limit cycle oscillation obtained by assuming $g_n = 0(\varepsilon)$ appears similar to that obtained by assuming $g_n = 0(1)$. Further analysis is needed.

SECTION 3

HARMONIC BALANCE METHODS

3.1 Analysis for Two Modes

The governing differential equations can be written in the form*

$$\ddot{W}_1 + \tilde{g}_1 \lambda^{1/2} \dot{W}_1 + \Omega_1^2 W_1 + e_{12} \lambda W_2 + C_{11} W_1^3 + C_{12} W_1 W_2^2 = 0 \qquad (3.1)$$

$$\ddot{W}_2 + \tilde{g}_2 \lambda^{1/2} \dot{W}_2 + \Omega_2^2 W_2 + e_{21} \lambda W_1 + C_{21} W_1^2 W_2 + C_{22} W_2^3 = 0 \qquad (3.2)$$

where the various coefficients are defined in Appendix A for panels with 2-dimensional simply-supported, 3-dimensional simply-supported, and 3-dimensional clamped-clamped edge conditions.

Assume the limit cycle solutions to be of the form

$$W_n(t) = a_n \sin \omega t + b_n \cos \omega t \qquad n = 1, 2 \qquad (3.3)$$

where a_n and b_n are constant coefficients.

Combining Eqs. 3.1, 3.2, and 3.3 and balancing the first harmonic, yields

$$\begin{aligned}
f_1 &= [\Omega_1^2 - \omega^2 + \tfrac{3}{4} C_{11}(a_1^2 + b_1^2) + \tfrac{1}{4} C_{12}(3a_2^2 + b_2^2)]a_1 \\
&\quad + [-\tilde{g}_1 \lambda^{1/2} \omega + \tfrac{1}{2} C_{12} a_2 b_2]b_1 + \lambda e_{12} a_2 = 0 \\
f_2 &= [\Omega_1^2 - \omega^2 + \tfrac{3}{4} C_{11}(a_1^2 + b_1^2) + \tfrac{1}{4} C_{12}(a_2^2 + 3b_2^2)]b_1 \\
&\quad + [\tilde{g}_1 \lambda^{1/2} \omega + \tfrac{1}{2} C_{12} a_2 b_2]a_1 + \lambda e_{12} b_2 = 0 \\
f_3 &= [\Omega_2^2 - \omega^2 + \tfrac{3}{4} C_{22}(a_2^2 + b_2^2) + \tfrac{1}{4} C_{21}(3a_1^2 + b_1^2)]a_2 \\
&\quad + [-\tilde{g}_2 \lambda^{1/2} \omega + \tfrac{1}{2} C_{21} a_1 b_1]b_2 + \lambda e_{21} a_1 = 0 \\
f_4 &= [\Omega_2^2 - \omega^2 + \tfrac{3}{4} C_{22}(a_2^2 + b_2^2) + \tfrac{1}{4} C_{21}(a_1^2 + 3b_1^2)]b_2 \\
&\quad + [\tilde{g}_2 \lambda^{1/2} \omega + \tfrac{1}{2} C_{21} a_1 b_1]a_2 + \lambda e_{21} b_1 = 0
\end{aligned} \qquad (3.4)$$

*For $g_E = 0$, one can write $g_i = \sqrt{\lambda}\tilde{g}_i$ (see Appendix A and Eq. 2.31)

These four equations are solved by the Newton-Raphson method (see Appendix B). The λ, which has to be greater than $\lambda_{\text{critical}}$ in order to obtain any nontrivial solution, is treated as an unknown. Also ω, which is not known beforehand, is treated as an unknown in the iteration process. There are thus four equations for four unknowns, $\underset{\sim}{x} = (\lambda, \omega, a_2, b_2)$. Values for the quantities a_1 and b_1 are assumed, and remain unchanged in the iteration process. If one allows $a_1 = 0$, then the magnitude of b_1 will determine the level of the limit cycle amplitude. The Newton-Raphson iteration formula is as follows:

$$\begin{pmatrix} \lambda^{i+1} \\ \omega^{i+1} \\ a_2^{i+1} \\ b_2^{i+1} \end{pmatrix} = \begin{pmatrix} \lambda^{i} \\ \omega^{i} \\ a_2^{i} \\ b_2^{i} \end{pmatrix} - \left[\begin{matrix} \frac{\partial f_1}{\partial \lambda} & \frac{\partial f_1}{\partial \omega} & \frac{\partial f_1}{\partial a_2} & \frac{\partial f_1}{\partial b_2} \\ \frac{\partial f_2}{\partial \lambda} & \frac{\partial f_2}{\partial \omega} & \frac{\partial f_2}{\partial a_2} & \frac{\partial f_2}{\partial b_2} \\ \frac{\partial f_3}{\partial \lambda} & \frac{\partial f_3}{\partial \omega} & \frac{\partial f_3}{\partial a_2} & \frac{\partial f_3}{\partial b_2} \\ \frac{\partial f_4}{\partial \lambda} & \frac{\partial f_4}{\partial \omega} & \frac{\partial f_4}{\partial a_2} & \frac{\partial f_4}{\partial b_2} \end{matrix} \right]^{-1}_{\underset{\sim}{x}=\underset{\sim}{x}^i} \begin{pmatrix} f_1(\underset{\sim}{x}^i) \\ f_2(\underset{\sim}{x}^i) \\ f_3(\underset{\sim}{x}^i) \\ f_4(\underset{\sim}{x}^i) \end{pmatrix} \tag{3.5}$$

The criterion of convergence employed is that the absolute value of the ratio of correction to the previous value of each $\underset{\sim}{x}$ element is smaller than 1×10^{-5}.

3.2 Analysis for N Modes

The governing differential equations can be written in the form*

*For $g_E = 0$, one can write $G_{mn} = \lambda^{1/2} \tilde{G}_{mn}$ (see Appendix A and Eq. 2.88).

$$\sum_{n=1}^{N} M_{mn}\ddot{W}_n + \lambda^{1/2}\sum_{n=1}^{N}\tilde{G}_{mn}\dot{W}_n + \sum_{n=1}^{N} K_{mn} W_n + \lambda\sum_{n=1}^{N} E_{mn}W_n$$

$$+ \sum_{n,p,q=1}^{N} C_{mnpq} W W_p W_q = 0 \qquad (3.6)$$

for m = 1, 2, ... N

where M_{mn}, G_{mn}, K_{mn}, C_{mnpq} are all defined in Appendix A for 2-dim. simply-supported, 3-dim. simply-supported, and 3-dim. clamped-clamped edge boundary conditions.

Again, assume that

$$W_n(t) = a_n \sin\omega t + b_n \cos\omega t \qquad \text{for } n = 1, 2 \ldots N \qquad (3.7)$$

where a_n and b_n are constant coefficients. Substituting Eq. 3.6 in Eq. 3.5 and balancing the first harmonic, one obtains 2-N simultaneous algebraic equations:

$$f_{2m-1}(\underset{\sim}{x}) = \left[-\sum_{n=1}^{N} M_{mn}\omega^2 + \sum_{n=1}^{N} K_{mn} + \lambda\sum_{n=1}^{N} E_{mn}\right] a_n$$
$$- \lambda^{1/2}\omega\sum_{n=1}^{N}\tilde{G}_{mn} b_n + \frac{1}{4}\sum_{n,p,q=1}^{N} C_{mnpq}\left[3a_n a_p a_q\right.$$
$$\left. + b_n b_p a_q + b_n a_p b_q + a_n b_p b_q\right] = 0 \qquad (3.8)$$

m = 1, 2, . . . N

$$f_{2m}(\underset{\sim}{x}) = \left[-\sum_{n=1}^{N} M_{mn}\omega^2 + \sum_{n=1}^{N} K_{mn} + \lambda\sum_{n=1}^{N} E_{mn}\right] b_n -$$
$$+ \lambda^{1/2}\omega\sum_{n=1}^{N}\tilde{G}_{mn} a_n + \frac{1}{4}\sum_{n,p,q=1}^{N} C_{mnpq}\left[3b_n b_p b_q +\right.$$
$$\left. + a_n a_p b_q + a_n b_p a_q + b_n a_p a_q\right] = 0 \qquad (3.9)$$

m = 1, 2, . . . N

Equations 3.8 and 3.9 represent 2N equations which may be written in vector form as

$$\underset{\sim}{F}(\underset{\sim}{x}) = \begin{pmatrix} f_1(\underset{\sim}{x}) \\ f_2(\underset{\sim}{x}) \\ \vdots \\ \vdots \\ f_{2N}(\underset{\sim}{x}) \end{pmatrix} = 0 \quad , \qquad \underset{\sim}{x} = \begin{pmatrix} \lambda \\ \omega \\ a_2 \\ b_2 \\ a_3 \\ \vdots \\ b_N \end{pmatrix} \tag{3.10}$$

Again, solving by the Newton-Raphson iteration technique

$$(\underset{\sim}{x}^{i+1}) = (\underset{\sim}{x}^{i}) - J^{-1}(\underset{\sim}{x}^{i})\, F(\underset{\sim}{x}^{i}) \tag{3.11}$$

where $J(\underset{\sim}{x}^{i})$ is the Jacobian of $F(\underset{\sim}{x}^{i})$ and is defined in Appendix B.

The convergence criterion is the same as that in the two-mode solution.

3.3 Stability of Solutions

The present method gives both stable and unstable solutions; hence each limit-cycle solution must be checked for stability by giving a small perturbation to the limit cycle solution.

Let

$$W_n(t) = [a_n + \rho_n(t)] \sin \omega t + [b_n + \beta_n(t)] \cos \omega t \tag{3.12}$$

where

$$|\rho_n(t)| \ll |a_n| \quad , \quad |\beta_n(t)| \ll |b_n|$$

and a_n, b_n are limit cycle solutions; the quantities $\zeta_n(t)$ and $\xi_n(t)$ represent perturbations to a_n and b_n, respectively.

Upon employing Eq. 3.12 in Eq. 3.6, subtracting the steady-state limit cycle solution, and retaining only the linear terms in ζ_n and ξ_n, one obtains a set of linear differential equations in ζ_n and ξ_n. Assuming that $\zeta_n(t)$ and $\xi_n(t)$ are slowly varying functions of time, one can neglect $\dot{\zeta}_n(t)$, $\dot{\xi}_n(t)$ in comparison with $\zeta_n(t)$ and $\xi_n(t)$, respectively, since

$$\left|\frac{\ddot{\varphi}_n(t)}{\dot{\varphi}_n(t)}\right| \frac{2\pi}{\omega} \ll 1 \quad , \quad \left|\frac{\ddot{\zeta}_n(t)}{\dot{\zeta}_n(t)}\right| \frac{2\pi}{\omega} \ll 1 \tag{3.13}$$

where ω is the frequency of the steady-state oscillations. Finally, one obtains a set of 2N first-order differential equations,

$$\begin{aligned}
&2\omega \sum_{n=1}^{N} M_{mn}\dot{\varphi}_n - \omega^2 \sum_{n=1}^{N} M_{mn}\zeta_n + \lambda^{1/2} \sum_{n=1}^{N} \tilde{G}_{mn}(\omega\varphi_n + \dot{\zeta}_n) \\
&+ \sum_{n=1}^{N} K_{mn}\zeta_n + \lambda \sum_{n=1}^{N} E_{mn}\zeta_n + \frac{1}{4}\sum_{n,p,q=1}^{N} C_{mnpq}[3(b_n b_p \zeta_q + \\
&+ b_n b_q \zeta_p + b_p b_q \zeta_n) + b_n a_p \varphi_q + b_n a_q \varphi_p \\
&+ a_p a_q \zeta_n + b_p a_n \varphi_q + b_q a_n \varphi_p + a_n a_q \zeta_p \\
&+ b_q a_p \varphi_n + a_n a_p \zeta_q + b_p a_q \varphi_n] = 0
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&-2\omega \sum_{n=1}^{N} M_{mn}\dot{\zeta}_n - \omega^2 \sum_{n=1}^{N} M_{mn}\varphi_n - \lambda^{1/2} \sum_{n=1}^{N} \tilde{G}_{mn}(\omega\zeta_n - \dot{\varphi}_n) \\
&+ \sum_{n=1}^{N} K_{mn}\varphi_n + \lambda \sum_{n=1}^{N} E_{mn}\varphi_n + \frac{1}{4}\sum_{n,p,q=1}^{N} C_{mnpq}[3(a_n a_p \varphi_q \\
&+ a_n a_q \varphi_p + a_p a_q \varphi_n) + a_n b_p \zeta_q + a_n b_q \zeta_p \\
&+ b_p b_q \varphi_n + a_p b_n \zeta_q + a_p b_q \zeta_n + b_n b_q \varphi_p \\
&+ a_q b_n \zeta_p + a_q b_p \zeta_n + b_n b_p \varphi_q] = 0
\end{aligned} \tag{3.15}$$

Let

$$y_n = z_{2n-1}\, e^{\alpha t} \quad , \quad z_n = z_{2n}\, e^{\alpha t}$$

and placing these in Eqs. 3.14 and 3.15, dividing Eq. 3.14 by -2ω, Eq. 3.15 by 2ω, and writing in matrix form, one obtains

$$\left\{ \underset{\sim}{Q} - \alpha \underset{\sim}{M}_1 \right\} \underset{\sim}{z} = 0 \tag{3.16}$$

where M_1 and Q are defined in Appendix C. This is an eigenvalue problem; α can be found by the power method. If the real parts of all α's are negative, one has a stable limit cycle. If one of the real parts is positive, one has an unstable limit-cycle solution.

3.4 Static Buckling Solution

Setting dw_n/dt and d^2w_n/dt^2 in Eq. 3.5 equal to zero, one gets the equation for static buckling

$$\sum_{n=1}^{N} K_{mn} W_n + \lambda \sum_{n=1}^{N} E_{mn} W_n + \sum_{n,p,q=1}^{N} C_{mnpq} W_n W_p W_q = 0 \tag{3.17}$$

This can be programmed for solution by the Newton-Raphson iteration technique in the same way as the previous flutter equations, Eqs. 3.8 and 3.9, only now, the unknown vector is taken as $\underset{\sim}{x} = (W_1, W_2, \ldots W_N)$ and λ is assumed known. The corresponding Jacobian, $J(\underset{\sim}{x})$, is defined in Appendix B. There is, however, a limit on λ beyond which no static buckling exists. This point can be visualied by solving the two-mode solution of a 2-dim. simply-supported panel analytically. The governing equations are

$$\Omega_1^2 W_1 + \lambda e_{12} W_2 + (C_{11} W_1^2 + 4 C_{11} W_2^2) W_1 = 0 \tag{3.18}$$

$$\Omega_2^2 W_2 - \lambda e_{12} W_1 + (4 C_{11} W_1^2 + 16 C_{11} W_2^2) W_2 = 0 \tag{3.19}$$

Taking 4 x W_2 x Eq. 3.18 - W_1 x Eq. 3.19, and dividing by λe_{12} gives

$$W_1^2 + 4 W_2^2 = -\frac{1}{\lambda e_{12}} (4\Omega_1^2 - \Omega_2^2) W_1 W_2 \tag{3.20}$$

Employing Eq. 3.20 in Eq. 3.13 gives

$$W_2 = - \frac{\Omega_1^2 W_1 \lambda e_{12}}{\lambda^2 e_{12}^2 + (4\Omega_1^2 - \Omega_2^2) C_{11} W_1^2} \tag{3.21}$$

Substituting the above two relations in either Eq.3.19, and using the relation $4\Omega_1^2 - \Omega_2^2 = -12\pi^4$ gives

$$W_1^2 = \frac{1}{12\pi^4 C_{11}} \left\{ \Omega_1^2 \left[-6\pi^4 \pm \sqrt{36\pi^8 - 4\lambda^2 e_{12}^2} \right] - \lambda^2 e_{12}^2 \right\}$$

$$W_2^2 = \frac{-\lambda^2 e_{12}^2 \left\{ \Omega_1^2 \left[6\pi^4 \mp \sqrt{36\pi^8 - 4\lambda^2 e_{12}^2} \right] + \lambda^2 e_{12}^2 \right\}}{12\pi^4 C_{11} \left[-6\pi^4 \pm \sqrt{36\pi^8 - 4\lambda^2 e_{12}^2} \right]^2} \tag{3.22}$$

W_1 is real, and $\sqrt{36\pi^8 - 4\lambda^2 e_{12}^2} < 6\pi^4$ so the first requirement is

$$\Omega_1^2 < 1 \qquad \text{or} \qquad R_x < -\pi^2 \tag{3.23}$$

The second requirement is that

$$36\pi^8 - 4\lambda^2 e_{12}^2 > 0 \quad \text{or} \quad \lambda^2 \leq 9\pi^8 / e_{12}^2 \tag{3.24}$$

This is the upper bound of λ.

Note:

(a) If $\Omega_1^2 \left[-6\pi^4 + \sqrt{36\pi^8 - 4\lambda^2 e_{12}^2} \right] - \lambda^2 e_{12}^2 > 0$

there are two solutions.

(b) If $\Omega_1^2 \left[-6\pi^4 + \sqrt{36\pi^8 - 4\lambda^2 e_{12}^2} \right] < \lambda^2 e_{12}^2 < \Omega_1^2 \left[-6\pi^4 - \sqrt{36\pi^8 - 4\lambda^2 e_{12}^2} \right]$

there is one solution.

(c) If $\Omega_1^2 \left[-6\pi^4 - \sqrt{36\pi^8 - 4\lambda^2 e_{12}^2} \right] < \lambda^2 e_{12}^2$

there is no solution.

This may give another bound to λ, but it will depend upon R_x.

The general N-mode case (Eq. 3.17) is solved by the Newton-Raphson technique. The Jacobian of Eq. 3.17 is defined in Appendix B.

3.5 Transient Solutions

The time needed for the panel flutter to reach its steady state is of interest. Of immediate concern is the transient response of the envelope of the panel flutter. For simplicity, the fluttering frequency is assumed to be constant at its final steady-state value, throughout the process of reaching the steady state. Assume

$$W_n = a_n(t) \sin \omega t + b_n(t) \cos \omega t \tag{3.25}$$

where $a_n(t)$ and $b_n(t)$ are slowly varying functions of time. Again, using the properties of a slowly varying function, one finds that

$$\dot{W}_n = (\dot{a}_n - \omega b_n) \sin \omega t + (\dot{b}_n + \omega a_n) \cos \omega t$$

$$\ddot{W}_n \doteq (-2\omega \dot{b}_n - \omega^2 a_n) \sin \omega t + (2\omega \dot{a}_n - \omega^2 b_n) \cos \omega t \tag{3.26}$$

Substituting these relations into Eq. 3.6, the first harmonic gives

$$2\omega \sum_{n=1}^{N} M_{mn} \dot{a}_n + \lambda^{1/2} \sum_{n=1}^{N} \tilde{G}_{mn} \dot{b}_n + \lambda^{1/2} \omega \sum_{n=1}^{N} \tilde{G}_{mn} a_n + \sum_{n=1}^{N} (K_{mn} + \lambda E_{mn} - \omega^2 M_{mn}) b_n + \frac{1}{4} \sum_{n,p,q=1}^{N} C_{mnpq} [3 b_n b_p b_q + b_n a_p a_q + b_p a_n a_q + b_q a_n a_p] = 0 \tag{3.27}$$

$$-2\omega \sum_{n=1}^{N} M_{mn} \dot{b}_n + \lambda^{1/2} \sum_{n=1}^{N} \tilde{G}_{mn} \dot{a}_n - \lambda^{1/2} \omega \sum_{n=1}^{N} \tilde{G}_{mn} b_n + \sum_{n=1}^{N} (K_{mn} + \lambda E_{mn} - \omega^2 M_{mn}) a_n + \frac{1}{4} \sum_{n,p,q=1}^{N} C_{mnpq} [3 a_n a_p a_q + a_n b_p b_q + a_p b_n b_q + a_q b_n b_p] = 0 \tag{3.28}$$

Solving Eqs. 3.27 and 3.28 by the Runge-Kutta method, one can get the transient response of the panel-flutter envelope.

SECTION 4

RESULTS FOR THE TWO-DIMENSIONAL PLATE

A computer program has been written for solving general panel-flutter problems subject to different boundary conditions.

For the two-dimensional plate, the harmonic balance method takes less than 1/10 of a minute of computing time on the IBM 360/65 computer to get one point in a 6-mode analysis and less than 3/10 of a minute for a 20-mode analysis. The perturbation method takes even less time, since after finding the parameter ψ and the right-hand vector, the entire amplitude versus dynamic pressure can be obtained almost immediately.

Compared with Dowell's direct integration method (Ref. 4), which takes from 2 to 10 minutes, the harmonic balance method and the perturbation method save much time, yet the results agree very well as can be seen in Fig. 5.

4.1 The Harmonic Balance Method

Consider, first, the question of convergence. Several observations can be made by looking at the results shown in Fig. 6:

(a) for the range of aerodynamic force and amplitude of interest here, six modes are required for good convergence.

(b) convergence is better for a limit cycle of low amplitude than for high amplitude because a high aerodynamic pressure associated with a high amplitude tends to blow the peak of deflection to the rear; hence the deflection shape deviates more from the shape of the first few modes.

It has been found that four modes are required for good convergence for $R_x = -4\pi^2$, $\mu/M = .01$ while six modes are needed for $R_x = 0$, $\mu/M = .01$. The reason for this observation is that the peak moves toward the center either by the midplane force or by the lessening of aerodynamic force needed for the same amplitude. A comparison of deflection shapes for these two cases is shown in Fig. 7. Since the limit cycle solution is a traveling wave, the deflection shapes shown in Fig. 7 represent the maximum deflection of every point of the plate in the airflow direction.

4.1.1 Limit Cycle Solution

The effects of two important parameters (the mass ratio, μ/M, and the in-plane loading, R_x) on the limit cycle amplitude are described in the following. Consider first, the effect of mass ratio: the limit cycle amplitude, W/h, vs. dynamic pressure for a different mass ratio μ/M is shown in Fig. 8. This mass ratio μ/M is a measure of the amount of aerodynamic damping present. It is obvious that damping has a stabilizing effect. The results obtained here agree with Dowell's[4].

Next, consider the effect of the inplane loading. As shown in Fig. 9, the inplane compressive loading aggravates the peak deflection and also causes the panel to flutter at a lower aerodynamic pressure. Note that the classical Euler buckling load is $R_x = -\pi^2$. For $R_x < -\pi^2$, there is both a flutter solution and static buckling solution. Some static buckling solutions are shown in Fig. 10 for the two-mode case. From Fig. 10 one sees that λ has an upper limit for static buckling to exist. For $R_x > -3\pi^2$, the limit is well below the $9\pi^2/e_{12}^2$ limit set in Subsection 3.4. For $R_x = -4\pi^2$, there are two solutions, and the stability analysis shows that the lower branch is unstable.

4.1.2 Transient Response

A program has been written for a general N-mode solution. The Runge-Kutta-Gill integration method has been used. For the two-mode case it works out quite well. The time increment can be set very large (about 1/4 of the period of real vibration). But as one increases the mode number to six, one has to decrease the time increment to such a small size (about 1/120 of the period of limit cycle oscillation) that this envelope analysis has little advantage over the direct integration method. One explanation is that during the transient period, amplitudes of damped modes are not necessarily small and they vibrate at their own frequencies, which are quite far away from the limit cycle frequency; hence the assumption that a slowly varying character of a_n and b_n is no longer valid.* When a large time increment is used, higher

*If $W_6(t) = \bar{A}_6 e^{(-\alpha + i\omega_6)t} = A_6(t) e^{i\omega t}$ where ω is the limit cycle frequency, $\bar{A}_6$ is a constant, and $A_6(t)$ is a damped oscillating function. Then $A_6(t) = \bar{A}_6 e^{[-\alpha + i(\omega_6 - \omega)]t}$, and when $\alpha \ll 1$, $\omega_6 \gg \omega$, $A_6(t)$ is no longer a slowly-varying function.

modes could cause numerical instability. A typical transient response curve is shown in Fig. 11.

4.2 The Perturbation Method

As can be seen in Fig. 5, the results of the perturbation method agree very well with those of the harmonic balance method. Since the perturbation method gives the solution in analytical form, additional information can be obtained by considering the parameters upon which the solution depends.

4.2.1 The Limit Cycle Solution

It should be noted that the solution (Eqs. 2.153 and 2.154) obtained by the perturbation method shows that the curves A vs. λ correspond to a parabola with a horizontal axis. The parabola is fully described by the two parameters ψ and λ_p. Thus it is convenient to consider the continuous variation of these two parameters by changing the inplane loading.

Consider, first, the effect of inplane loading on a critical dynamic pressure. From Fig. 12, it is clear that the critical dynamic pressure decreases as inplane compressive loading increases. For the range of inplane loading investigated here, a linear relation seems to hold.

Next, consider the effect of inplane loading on $|\psi|$: $|\psi|$ is a measurement of the opening of the W/h vs. λ curve and hence is a measurement of the "rigidity" of the panel. Figure 13 shows $|\psi|$ as a function of the inplane loading. It is clear that $|\psi|$ grows with inplane compressive loading. In the range of R_x investigated here, a linear relation seems to hold fairly well.

4.2.2 The Transient Response

The transient response (Eq. 2.129) is described by the parameter β_R: β_R is a scale to measure the time required for reaching a steady-state vibration. The effect of inplane loading on β_R is given in Fig. 14. It is clear from Fig. 14 that the time required to reach a steady-state solution decreases with increasing inplane compressive stress.

SECTION 5

RESULTS FOR A THREE-DIMENSIONAL CLAMPED-CLAMPED PLATE

A complete edge restraint, as typical of a clamped-clamped panel, is used in this analysis. Other edge restraints could be similarly incorporated in the analysis if desired. All of the results presented here are for complete edge restraints.

For the three-dimensional clamped-clamped plate (see Fig. A.2 of Appendix A), the harmonic balance method takes approximately 6 seconds of computing time on the IBM 360/65 computer for one point in the 6-mode solution. On the other hand, the perturbation method takes 9 seconds to obtain all the necessary information for the whole curve. Again, both methods are highly effective when compared with three minutes for one point in the direct integration method of Ref. 18.

5.1 Harmonic Balance

The present investigation concerning limit cycle behavior shows that both the inplane stresses and the length-to-width ratio (a/b) affect the convergence of the solution. From Figs. 15 and 16, one can see that for a/b = 1, six modes are required for good convergence if $R_x = -4\pi^2$, whereas an eight-mode analysis is needed if $R_x = 0$. Also, it has been found that (not shown in a plot) that for a/b = 3 and $R_x = 0$, fourteen modes are required for good convergence. Thus, length-to-width ratio is another important factor which affects convergence.

The coefficients of the nondimensional panel-flutter equation depend upon the parameters μ/M, R_x, R_y, and a/b. The effect of these parameters on the limit-cycle solution is described in the following.

The effect of mass ratio for the 2-dim. and 3-dim. plate is essentially of the same character. As can be seen in Fig. 17, when the mass ratio decreases, the plate becomes more and more flexible.

Concerning the effect of the inplane force, it is noted that the effect of R_y at low a/b is small as can be expected, since $(a/b)^2$ is associated with every R_y in the equation. The effect of R_y at a/b = 1, as can be seen in

Table 1, is still negligible. Accordingly, the effect of R_x was studied with $R_y = 0$. As can be seen in Figs. 18 and 19, the effect of the inplane loading is of the same pattern for both a/b = .01 and a/b = 1.0. An interesting observation has been made: that is, if one transfers these curves for different R_x (on Fig. 18, for example) horizontally, so that the W/h = 0 of each of these curves coincides, then for W/h < 1.0, all of these curves almost coincide. Therefore, the effect of inplane loading is mainly on the critical aerodynamic pressure, whereas the inplane loading effect on the opening of the curve is negligible.

Finally, consider the effect of the length-to-width ratio, a/b. In Fig. 20, the flutter amplitude vs dynamic pressure is shown for a/b = .01, a/b = 1.0, and a/b = 2.0. It is clear from the figure that an increase of a/b will affect not only the critical aerodynamic pressure, but also the "opening" of the curve. The large a/b is, the "stiffer" is the plate.

5.2 The Perturbation Method

The perturbation solution for a 3-dim. plate also shows good agreement with the harmonic balance solution for W/h < 1.0. As can be seen in Figs. 18 and 20, some curves are in good agreement even up to W/h = 1.4. As already mentioned, the solution obtained by the perturbation method is given by a parabola with a horizontal axis. The behavior of the parameters λ_p and ψ (which determine the parabola) as a function of the length-to-width ratio a/b is described in the following.

5.2.1 The Limit-Cycle Solution

In Fig. 21, the critical aerodynamic pressure λ_p at which flutter occurs is given as a function of the aspect ratio for $R_x = 0, -\pi^2$, and $-2\pi^2$, and $R_y = 0$. It is clear from the figure that λ_p is greater for greater a/b, and smaller for larger inplane compressive stress. Also the characteristics are similar for different inplane stress. The parameter $|\psi|$, which is a measure of the "opening" in the dynamic pressure vs amplitude curve is given in Fig.22. It is clear from Fig. 22 that $|\psi|$ decreases as a/b increases; for the same a/b, $|\psi|$ does not change much as the inplane stress varies. This is consistent with the observation made in Subsection 5.1.1 about the deflection shape.

5.2.2 The Transient Response

The transient response is determined by the parameter β_R, which is a measurement for the time required to reach the steady state. Figure 23 shows that the larger the length-to-width ratio, the shorter the time required to reach a steady state. It is also clear that the influence of inplane stress on β_R is greater for a smaller length-to-width ratio.