NASA TECHNICAL NOTE

NASA TN D-6279

c. 1

NASA TN D-6279

THE INFLUENCE OF THE GREAT INEQUALITY ON THE SECULAR DISTURBING FUNCTION OF THE PLANETARY SYSTEM

by Peter Musen

Goddard Space Flight Center
Greenbelt, Md. 20771

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1971

1. Report No. NASA TN D-6279	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle The Influence of the Great Inequality on the Secular Disturbing Function of the Planetary System		5. Report Date May 1971
		6. Performing Organization Code
7. Author(s) Peter Musen		8. Performing Organization Report No. G-1022
9. Performing Organization Name and Address Goddard Space Flight Center Greenbelt, Maryland 20771		10. Work Unit No.
		11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Technical Note
		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

This paper derives the contribution F_2^* by the great inequality to the secular disturbing function of the principal planets. Andoyer's expansion of the planetary disturbing function and von Zeipel's method of eliminating the periodic terms is employed; thereby, the corrected secular disturbing function for the planetary system is derived. An earlier solution suggested by Hill is based on Leverrier's equations for the variation of elements of Jupiter and Saturn and on the semi-empirical adjustment of the coefficients in the secular disturbing function. Nowadays there are several modern methods of eliminating periodic terms from the Hamiltonian and deriving a purely secular disturbing function. Von Zeipel's method is especially suitable. The conclusion is drawn that the canonicity of the equations for the secular variation of the heliocentric elements can be preserved if there be retained, in the secular disturbing function, terms only of the second and fourth order relative to the eccentricity and inclinations.

The Krylov-Bogolubov method is suggested for eliminating periodic terms, if it is desired to include the secular perturbations of the sixth and higher order in the heliocentric elements. The additional part of the secular disturbing function F_2^* derived in this paper can be included in existing theories of the secular effects of principal planets.

17. Key Words Suggested by Author Great Inequality Secular Disturbing Function Celestial Mechanics	18. Distribution Statement Unclassified—Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price *
Unclassified	Unclassified	21	3.00

*For sale by the National Technical Information Service, Springfield, Virginia 22151

CONTENTS

BASIC NOTATIONS

a = semi-major axis of the orbit of Jupiter

a' = semi-major axis of the orbit of Saturn

D $= a\dfrac{d}{da}$

f_s = secular part of f

f_p = periodic part of f

i = orbital inclination of Jupiter

i' = orbital inclination of Saturn

k^2 = Gaussian constant

m = mass of Jupiter

m' = mass of Saturn

n = mean motion of Jupiter

n' = mean motion of Saturn

$N^{\sigma}_{p_1,p_2}$ = modified Newcomb's differential operator, associated with the expansion in powers of ϵ

$N'^{\sigma}_{p_1,p_2}$ = modified Newcomb's differential operator associated with the expansion in powers of ϵ'

r = radius vector of Jupiter

r' = radius vector of Saturn

R $= \dfrac{1}{\rho} - \dfrac{r \cos S}{r'^2}$ = disturbing function for Jupiter

R' $= \dfrac{1}{\rho} - \dfrac{r' \cos S}{r^2}$ = disturbing function for Saturn

R_0 $= \dfrac{1}{\rho}$ = direct part of the disturbing function for Jupiter and Saturn

$R_1 \quad = -\dfrac{r \cos S}{r'^2}$ = indirect part of the disturbing function for Jupiter

$R_1' \quad = -\dfrac{r' \cos S}{r^2}$ = indirect part of the disturbing function for Saturn

S = angle between r and r'

$x \quad = (x_1, x_2, x_3)$

$x' \quad = (x_1', x_2', x_3')$

$x_1 \quad = \epsilon_1 \lambda$

$x_1' \quad = \epsilon_1' \lambda'$

$x_2 \quad = \epsilon_2 \lambda^{-1}$

$x_2' \quad = \epsilon_2' \lambda'^{-1}$

$X_{p,q}^{i,j}$ = Hansen coefficients

$y \quad = (y_1, y_2, y_3)$

$y' \quad = (y_1', y_2', y_3')$

y_1 = mean longitude of Jupiter

y_1' = mean longitude of Saturn

(In the later part of the exposition, these notations designate the canonical variables.)

$a \quad = \dfrac{a}{a'}$

$\beta \quad = \dfrac{1}{a} + a$

$\gamma \quad = \dfrac{1}{a} - a$

(Beginning with Section 3, γ designates $2 \sin i/2$.)

$\gamma \quad = 2 \sin \dfrac{i}{2}$

$\gamma' \quad = 2 \sin \dfrac{i'}{2}$

$\gamma_1 \quad = \frac{1}{2}\gamma e^{-i\theta}$

$\gamma_1' \quad = \frac{1}{2}\gamma' e^{-i\theta'}$

$\gamma_2 \quad = \frac{1}{2}\gamma e^{+i\theta}$

$\gamma_2' \quad = \frac{1}{2}\gamma' e^{+i\theta'}$

ϵ = orbital eccentricity of Jupiter

ϵ' = orbital eccentricity of Saturn

$\epsilon_1 \quad = \frac{\epsilon}{2} e^{-i\pi}$

$\epsilon_1' \quad = \frac{\epsilon'}{2} e^{-i\pi'}$

$\epsilon_2 \quad = \frac{\epsilon}{2} e^{+i\pi}$

$\epsilon_2' \quad = \frac{\epsilon'}{2} e^{+i\pi'}$

θ = longitude of the ascending node of the orbital plane of Jupiter

θ' = longitude of the ascending node of the orbital plane of Saturn

$\lambda \quad = e^{+iy_1}$

$\lambda' \quad = e^{+iy_1'}$

$\mu \quad = k^2(1 + m)$

$\mu' \quad = k^2(1 + m')$

ν = $2n - 5n'$ = mean motion of the critical argument

π = longitude of the perigee of Jupiter

π' = longitude of the perigee of Saturn

ρ = distance between Jupiter and Saturn

χ = $2y_1 - 5y_1'$ = critical argument of the great inequality

ω = argument of the perihelion of Jupiter

ω' = argument of the perihelion of Saturn

$\frac{\partial}{\partial x}$ = gradient operator relative to x

$\frac{\partial}{\partial x'}$ = gradient operator relative to x'

$\frac{\partial}{\partial y}$ = gradient operator relative to y

$\frac{\partial}{\partial y'}$ = gradient operator relative to y'

THE INFLUENCE OF THE GREAT INEQUALITY ON THE SECULAR DISTURBING FUNCTION OF THE PLANETARY SYSTEM

by
Peter Musen
Goddard Space Flight Center

1. INTRODUCTION

This paper discusses the influence of the great inequality, between Jupiter and Saturn, on the secular (long-period) disturbing function of the principal planets. Earlier results concerning this influence were obtained by Hill more than 70 years ago.

The great inequality is a near-resonance effect produced by commensurability close to 2:5 between the mean motions of Jupiter and Saturn.

The corresponding critical argument $2y_1 - 5y_1'$ in the trigonometrical expansion of the disturbing function has a period of approximately 900 yr. So long a period and close a commensurability produce large perturbative effects in the elements, especially in the mean longitude of Jupiter, with amplitude nearly 1200″, and Saturn, with amplitude nearly 2900″. Such periodic perturbations produce, in higher approximations, an appreciable effect on the secular disturbing function and on the secular behavior of the elements of the principal planets.

Secular perturbations are the source of the long-period effects, with periods ranging from 5.7×10^4 to 2×10^6 yr.

These perturbations are chiefly responsible for the behavior of the elements of the principal planets over millions of years. The amplitudes and mean motions of the arguments of these periodic terms determine the secular change in the longitudes of the perihelia and nodes. The amplitudes also provide information about the range of oscillation of the eccentricities and inclinations. Knowledge of the long-period perturbations of the motion of the principal planets is essential if we are to understand the secular behavior of the asteroidal ring, because these perturbations induce the forced long-period oscillations in asteroid elements.

Since Lagrange, the classical way of treating secular changes in the motion of the principal planets has been based on the linearization of Poincaré's canonical elements or some of their approximations.

Only terms quadratic with respect to the eccentricities and inclinations are retained in the expansion of the secular disturbing functions.

Modern computational technique permits terms of higher order relative to the eccentricities and inclinations to be included in the secular disturbing function and in the perturbations. Harzer (1895) in his classical work on secular perturbations solved the linearized problem and made an attempt to include terms of higher order, but unfortunately this last part of his work was never completed. Only recently Anolik, Krasinsky, and Pius (1969) succeeded in developing the trigonometrical theory of direct secular perturbations (of rank zero) including terms of the fourth order with respect to the eccentricities and inclinations.

Secular perturbations of rank 1 produced by the great inequality were omitted from their work.

Hill (1897) tried to derive these perturbations from the Leverrier (1874) differential equations for the eccentricities and perihelia of Jupiter and Saturn.

Brouwer and van Woerkom (1950) and Sharaf and Budnikova (1967) incorporated Hill's contribution to the secular disturbing function in their linearized theories of secular perturbations of major planets.

Brouwer and van Woerkom, however, assert the following "[Hill's computations] cannot be considered as definitive, because many of the coefficients were not derived in a rigorous manner. They were obtained as the means of the coefficients arising in four equations .... While in some cases the agreement was satisfactory, there were contradictions in other cases. These contradictions were resolved by Hill in an empirical manner."

In other words, Hill's results need revision. We can fully understand the difficulties he encountered, because at that time the theory of elimination of periodic terms from a Hamiltonian was not fully developed.

Owing to the publications of von Zeipel (1916), Brouwer (1959), Hori (1966), and Deprit (1969) on the modernization of the method of Delaunay, we now possess an easy and exact algorithm for the elimination of short-period terms from a Hamiltonian. Even when the equations of motion do not have a canonical form, short-period terms can be eliminated directly from the differential equations by the method of Krylov and Bogolubov (Bogolubov and Mitropolsky, 1961). This method is described by Musen (1965) in a form convenient for application to astronomical problems.

We may consider the great inequality as a short-period term compared with the secular effects. We can eliminate the great inequality either from the Hamiltonian or from the differential equations for the variation of elements. This elimination results in the introduction of secular effects of rank 1 and class 1/2 into the expression for the perturbations of the elements.

Hill's addition to the secular Hamiltonian contains terms of the fourth and sixth order with respect to the mean orbital eccentricities of Jupiter and Saturn. We can add those terms of the fourth order that depend on the inclinations (they are omitted in Hill's exposition). The use of heliocentric elements of motion is the most convenient from an astronomical standpoint.

In our problem we can write the equations for variation of heliocentric elements in a canonical form for the whole system only if we retain the direct parts of the disturbing functions and neglect the indirect parts. This is because the indirect parts contain great-inequality terms of the fifth and higher orders and because they are different for Jupiter and Saturn.

In order to retain the heliocentric elements and the canonicity we must neglect such terms. However, we retain those terms of the third order that depend on the great-inequality argument, because they appear only in the direct parts of the disturbing function. Thus, in the secular disturbing function we can retain terms of the second, fourth, sixth ... order and of rank zero, also terms of the fourth order of rank 1 and class 1/2. These last terms are produced by the elimination of the great-inequality terms of the third order mentioned above.

If we decide to include in the disturbing functions terms of the sixth and higher orders in ϵ, ϵ', γ, and γ' and to retain the canonicity of the equations of motion, then we shall use Jacobi's reduction of the differential equations of planetary motion to the canonical form and use some modern version of Delaunay to eliminate the periodic terms.

If the motion is referred to the Sun and we wish to retain terms of the sixth and higher orders, we can eliminate the periodic terms by the method of Krylov and Bogolubov. This choice is more convenient from an astronomical standpoint.

Hill's empirical adjustment of the coefficients stands between these two methods and evidently does not solve the problem. In our exposition we limit ourselves to terms of class 1/2 and, therefore, neglect the secular effects produced by short-period terms in higher approximations.

We found that Andoyer's (1923) expansion of the planetary disturbing function suits our purpose very well, because he expands the function in powers of the inclinations relative to a fixed ecliptic.

The transition from Andoyer's expansion to Harzer's expansion in the canonical elements is very easy.

We found, in general, that many theories developed by Andoyer now sound quite modern; we believe that interest in his remarkable works should be revived.

2. ANDOYER EXPANSION OF DISTURBING FUNCTION

The Andoyer expansion of the direct part of the disturbing function is given by

$$R_0\sqrt{aa'} = \sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} \lambda^{+s-j}\lambda'^{-s+j} A_s^j B_s^j ,$$

where

$$A_s^j = \sum x_1^{p_1} x_2^{p_2} x_1'^{p_1'} x_2'^{p_2'} N_{p_1,p_2}^{+s+j} N'^{-s+j}_{p_1',p_2'} ,$$

$$B_s^j = \sum \frac{(2q-1)!!}{q_1!\,q_2!\,q_1'!\,q_2'!\,2^q} \sigma_1^{q_1} \sigma_2^{q_2} \sigma_1'^{q_1'} \sigma_2'^{q_2'} b_{s-q_1+q_2}^{q+1/2} ,$$

$$q_1 + q_2 + q_1' + q_2' = q ,$$

$$q_1' - q_2' = j ,$$

$$\sigma_1 = (\gamma_1 - \gamma_1')\gamma_2' - (\gamma_2 - \gamma_2')\gamma_1 - \gamma_1\gamma_2'(\gamma_1 - \gamma_1')(\gamma_2 - \gamma_2') + \cdots ,$$

$$\sigma_2 = (\gamma_2 - \gamma_2')\gamma_1' - (\gamma_1 - \gamma_1')\gamma_2 - \gamma_2\gamma_1'(\gamma_1 - \gamma_1')(\gamma_2 - \gamma_2') + \cdots ,$$

$$\sigma_1' = (\gamma_1 - \gamma_1')^2 + \gamma_1\gamma_1'(\gamma_1 - \gamma_1')(\gamma_2 - \gamma_2') + \cdots ,$$

and

$$\sigma_2' = (\gamma_2 - \gamma_2')^2 + \gamma_2\gamma_2'(\gamma_1 - \gamma_1')(\gamma_2 - \gamma_2') + \cdots .$$

The definitions of Laplace coefficients and Newcomb operators given by Andoyer differ slightly from the classical ones.

The term

$$x_1^{p_1} x_2^{p_2} x_1'^{p_1'} x_2'^{p_2'}$$

is of the order

$$p_1 + p_2 + p_1' + p_2' \text{ in } \epsilon \text{ and } \epsilon' ,$$

whereas B_s^j is of the order $2j$ in γ and γ'.

The Laplace coefficients and their derivatives in the Andoyer expansion are successively computed from the following equations:

$$b_n^{1/2} = a^{n+1/2} \int_0^{\pi/2} (1 - a^2 \cos^2 \psi)^{-1/2} \cos^{2n} \psi \, d\psi ,$$

$$b_n'^{1/2} = a b_n^{1/2} - b_{n+1}^{1/2} ,$$

$$D b_n^{1/2} = \left(n + \frac{1}{2}\right)\left(b_n^{1/2} + \frac{2}{\gamma} b_n'^{1/2}\right) ,$$

$$b_n^{p+1} = \frac{1}{p\gamma} D b_n^p ,$$

$$D^2 b_n^p = (n \pm p)^2 b_n^p + 4p^2 b_{n\pm1}^{p+1} ,$$

$$Db_n^{p+1} = \frac{1}{p\gamma}\left(\frac{\beta}{\gamma} Db_n^p + D^2 b_n^p\right),$$

and

$$D^{k+2} b_n^p = (n \pm p)^2 D^k b_n^p + 4p^2 D^k b_{n\pm 1}^{p+1}.$$

The differential operators $N^{\sigma}_{p_1,p_2}$ and $N'^{\sigma}_{p_1,p_2}$ are computed from the following recursive equations (Andoyer, 1923):

$$\begin{aligned} p_1 N^{\sigma}_{p_1,p_2} &= \left(2\sigma + \frac{1}{2} - D\right) N^{\sigma+1}_{p_1-1,p_2} \\ &+ \left(\sigma + \frac{1}{2} - D\right) N^{\sigma+2}_{p_1-2,p_2} \\ &+ \left(4\sigma + 5p_1 - p_2 - \frac{7}{2} - D\right) N^{\sigma}_{p_1-1,p_2-1} \\ &- 3(\sigma + p_1 - p_2) \sum_{p=2}^{m} \frac{1}{p(2p-3)} \binom{2p-2}{p-1} N^{\sigma}_{p_1-p,p_2-p}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} p_2 N^{\sigma}_{p_1,p_2} &= \left(-2\sigma + \frac{1}{2} - D\right) N^{\sigma-1}_{p_1,p_2-1} \\ &+ \left(-\sigma + \frac{1}{2} - D\right) N^{\sigma-2}_{p_1,p_2-2} \\ &+ \left(-4\sigma - p_1 + 5p_2 - \frac{7}{2} - D\right) N^{\sigma}_{p_1-1,p_2-1} \\ &+ 3(\sigma + p_1 - p_2) \sum_{p=2}^{m} \frac{1}{p(2p-3)} \binom{2p-2}{p-1} N^{\sigma}_{p_1-p,p_2-p}, \end{aligned} \tag{2}$$

where

$$m = \min(p_1, p_2).$$

We have also

$$N^{\sigma}_{p_1,p_2} = N^{-\sigma}_{p_2,p_1},$$

$$N^{\sigma}_{0,0} = 1,$$

and

$$N'^{\sigma}_{p_1,p_2}(+D) = N^{\sigma}_{p_1,p_2}(-D).$$

C. Hipkins of the Analytical Mechanics Associates prepared a program, using an analytical language, for the computation of operators N and N' in rational-fractions arithmetic. In his program, N and N' can be obtained up to any desired order in the form of polynomials in σ and D. The use of both equations, (1) and (2), provides a sharp check of the whole calculation. Hipkins found a misprint in (2) as it is given by Andoyer. Instead of

$$N^{-\sigma}_{p_1-p,\,p_2-p}$$

under the summation sign in the book, one should read

$$N^{+\sigma}_{p_1-p,\,p_2-p} \;.$$

Such a program paves the way for the production of analytical planetary theory on electronic machines.

3. TERMS ASSOCIATED WITH THE GREAT INEQUALITY

If we want to extract from the disturbing function those terms having arguments that are multiples of the given argument $iy_1 + i'y_1'$, or, what is the same thing, to extract those terms having the factors

$$\lambda^{+i}\,, \quad \lambda'^{+i'}\,, \quad \lambda^{-i}\,, \quad \text{and} \quad \lambda'^{-i'}\,,$$

from Andoyer's expansion up to the order P in the eccentricities and inclinations, then p_1, p_2, p_1', and p_2' must be selected in such a manner as to satisfy the following conditions:

$$p_1 + p_2 + p_1' + p_2' + j \leqslant P\,,$$

$$p_1 - p_2 + s + j = ik\,,$$

$$p_1' - p_2' - s + j = i'k\,,$$

and

$$s, j, k = 0, \pm 1, \pm 2, \ldots$$

Evidently the inequality

$$|p_1 - p_2| + |p_1' - p_2'| \leqslant P$$

or the equivalent inequality

$$|ik - s - j| + |i'k + s - j| \leqslant P$$

must also be satisfied. The use of the last inequality facilitates the search for the prospective candidates for k, s, j, p_1, p_2, p_1', and p_2'. In our case,

$$i = 2\,, \quad i' = -5\,,$$

and Table 1 shows the admissible values of the indices and exponents. To the required accuracy we obtain

$$R_0\sqrt{aa'} = (A^0_{-2}B^0_{-2} + A^0_{+2}B^0_{+2}) + (A^0_{-3}B^0_{-3} + A^0_{+3}B^0_{+3})$$

$$+ (A^0_{-4}B^0_{-4} + A^0_{+4}B^0_{+4}) + (A^0_{-5}B^0_{-5} + A^0_{+5}B^0_{+5})$$

$$+ (A^{+1}_{-3}B^{+1}_{-3} + A^{-1}_{+3}B^{-1}_{+3}) + (A^{+1}_{-4}B^{+1}_{-4} + A^{-1}_{+4}B^{-1}_{+4}) ,$$

or, if we substitute

$$x_1 = \lambda^{+1}\epsilon_1 , \quad x_2 = \lambda^{-1}\epsilon_2 , \quad x'_1 = \lambda'^{+1}\epsilon'_1 , \quad x'_2 = \lambda'^{-1}\epsilon'_2$$

and take, with the necessary accuracy,

$$B^0_s = b^{1/2}_s ,$$

$$B^{+1}_s = \tfrac{1}{2}\sigma'_1 b^{3/2}_s ,$$

and

$$B^{-1}_s = \tfrac{1}{2}\sigma'_2 b^{3/2}_s ,$$

we obtain for the direct part of the disturbing function associated with the great inequality the following expression:

$$R_0\sqrt{aa'} = (\lambda^{-2}\lambda'^{+5}\epsilon_1^3 + \lambda^{+2}\lambda'^{-5}\epsilon_2^3)P_0$$

$$+ (\lambda^{-2}\lambda'^{+5}\epsilon_1^2\epsilon'_1 + \lambda^{+2}\lambda'^{-5}\epsilon_2^2\epsilon'_2)P_1$$

Table 1—Admissible values of exponents and subscripts.

j	s	k	p_1	p_2	p'_1	p'_2
0	−2	−1	0	0	3	0
0	+2	+1	0	0	0	3
0	−3	−1	1	0	2	0
0	+3	+1	0	1	0	2
0	−4	−1	2	0	1	0
0	+4	+1	0	2	0	1
0	−5	−1	3	0	0	0
0	+5	+1	0	3	0	0
+1	−3	−1	0	0	1	0
−1	+3	+1	0	0	0	1
+1	−4	−1	1	0	0	0
−1	+4	+1	0	1	0	0

$$+ (\lambda^{-2}\lambda'^{+5}\epsilon_1\epsilon_1'^2 + \lambda^{+2}\lambda'^{-5}\epsilon_2\epsilon_2'^2)P_2$$

$$+ (\lambda^{-2}\lambda'^{+5}\epsilon_1'^3 + \lambda^{+2}\lambda'^{-5}\epsilon_2'^3)P_3$$

$$+ (\lambda^{-2}\lambda'^{+5}\epsilon_1\sigma_1' + \lambda^{+2}\lambda'^{-5}\epsilon_2\sigma_2')Q_0$$

$$+ (\lambda^{-2}\lambda'^{+5}\epsilon_1'\sigma_1' + \lambda^{+2}\lambda'^{-5}\epsilon_2'\sigma_2')Q_1 , \tag{3}$$

where

$$P_0 = N_{3,0}^{-5}N_{0,0}'^{+5}b_5^{1/2}$$

$$= \left(-\frac{2473}{48} - \frac{527}{24}D - \frac{13}{4}D^2 - \frac{1}{6}D^3\right)b_5^{1/2} , \tag{4}$$

$$P_1 = N_{2,0}^{-4}N_{1,0}'^{+4}b_4^{1/2}$$

$$= \left(+\frac{2567}{16} + \frac{559}{8}D + \frac{41}{4}D^2 + \frac{1}{2}D^3\right)b_4^{1/2} , \tag{5}$$

$$P_2 = N_{1,0}^{-3}N_{2,0}'^{+3}b_3^{1/2}$$

$$= \left(-\frac{2585}{16} - \frac{587}{8}D - \frac{43}{4}D^2 - \frac{1}{2}D^3\right)b_3^{1/2} , \tag{6}$$

$$P_3 = N_{0,0}^{+2}N_{3,0}'^{+2}b_2^{1/2}$$

$$= \left(+\frac{2455}{48} + \frac{611}{24}D + \frac{15}{4}D^2 + \frac{1}{6}D^3\right)b_2^{1/2} , \tag{7}$$

$$Q_0 = +\frac{1}{2}N_{1,0}^{-3}N_{0,0}'^{+5}b_4^{3/2}$$

$$= \left(-\frac{11}{2} - D\right)b_4^{3/2} , \tag{8}$$

and

$$Q_1 = +\frac{1}{2}N_{0,0}^{+2}N_{1,0}'^{+4}b_3^{3/2}$$

$$= \left(+\frac{17}{2} + D\right)b_3^{3/2} . \tag{9}$$

The first four terms in (3) and the coefficients in (4)–(7) are given by Andoyer.

For the indirect parts of the disturbing functions we have (Andoyer, 1923)

$$R_1 = -\frac{a}{2a'^2}\sum x_1^{p_1}x_2^{p_2}x_1'^{p_1'}x_2'^{p_2'}\Big[(1+\sigma_1)\lambda\lambda'^{-1}X_{p_1,p_2}^{+1,+1}X_{p_1',p_2'}^{-2,-1}$$

$$+\sigma_1'\lambda\lambda' X_{p_1,p_2}^{+1,+1}X_{p_1',p_2'}^{-2,+1} + (1+\sigma_2)\lambda^{-1}\lambda' X_{p_1,p_2}^{+1,-1}X_{p_1',p_2'}^{-2,+1}$$

$$+\sigma_2'\lambda^{-1}\lambda'^{-1}X_{p_1,p_2}^{+1,-1}X_{p_1',p_2'}^{-2,-1}\Big],$$

and similarly

$$R_1' = -\frac{1}{2}\frac{a'}{a^2}\sum x_1^{p_1}x_2^{p_2}x_1'^{p_1'}x_2'^{p_2'}\Big[(1+\sigma_1)\lambda\lambda'^{-1}X_{p_1,p_2}^{-2,+1}X_{p_1',p_2'}^{+1,-1}$$

$$+\sigma_1'\lambda\lambda' X_{p_1,p_2}^{-2,+1}X_{p_1',p_2'}^{+1,+1} + (1+\sigma_2)\lambda^{-1}\lambda' X_{p_1,p_2}^{-2,-1}X_{p_1',p_2'}^{+1,+1}$$

$$+\sigma_2'\lambda^{-1}\lambda'^{-1}X_{p_1,p_2}^{-2,-1}X_{p_1',p_2'}^{+1,-1}\Big].$$

The great-inequality term of the lowest order in R_1 is

$$-\frac{a}{2a'^2}\Big[(1+\sigma_1)\lambda^{+1}\lambda'^{-1}x_1x_2'^4X_{1,0}^{+1,+1}X_{0,4}^{-2,-1}$$

$$+(1+\sigma_2)\lambda^{-1}\lambda'^{+1}x_2x_1'^4X_{0,1}^{+1,-1}X_{4,0}^{-2,+1}\Big],$$

and in R_1' the lowest term is

$$-\frac{1}{2}\frac{a'}{a^2}\Big[(1+\sigma_1)\lambda^{+1}\lambda'^{-1}x_1x_2'^4X_{1,0}^{-2,+1}X_{0,4}^{+1,-1}$$

$$+(1+\sigma_2)\lambda^{-1}\lambda'^{+1}x_2x_1'^4X_{0,1}^{-2,-1}X_{4,0}^{+1,+1}\Big].$$

All these terms produce terms of the sixth order in the secular disturbing function; and they can be omitted if we decide to retain the canonicity of the differential equations for the heliocentric elements.

There will be a large number of sixth-order terms. This usually means that they have a small weight in the combined secular effects.

4. ELIMINATION OF THE GREAT INEQUALITY FROM THE HAMILTONIAN

In performing the elimination of the great inequality we can consider only Jupiter and Saturn. For the Hamiltonian we can take

$$F = \frac{m\mu}{2a} + \frac{m'\mu'}{2a'} + \frac{fmm'}{\rho},$$

and neglect terms above the third order in the expansion of F.

The canonical elements of Delaunay in our case are

$$\begin{aligned}
&L = m\sqrt{\mu a} = mna^2 && l = \text{mean anomaly of Jupiter}\\
&G = L\sqrt{1-e^2} && g = \omega\\
&H = G\cos i && h = \theta\\
&L' = m'\sqrt{\mu' a'} = m'n'a'^2 && l' = \text{mean anomaly of Saturn}\\
&G' = L'\sqrt{1-e'^2} && g' = \omega'\\
&H' = G'\cos i' && h' = \theta'
\end{aligned}$$

Dividing F into purely secular and purely periodic parts, we have

$$F = F_0 + F_{1s} + F_{1p},$$

where

$$F_0 = \frac{m^3\mu^2}{2L^2} + \frac{m'^3\mu'^2}{2L'^3}, \tag{10}$$

$$\begin{aligned}
F_{1s} = \frac{fmm'}{\sqrt{aa'}}\Bigg\{ & b_0^{1/2} + \left(\epsilon_1\epsilon_2 + \epsilon_1'\epsilon_2' + \frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2\right)b_1^{3/2} - (\epsilon_1\epsilon_2' + \epsilon_2\epsilon_1')b_2^{3/2}\\
&+ (\epsilon_1^2\epsilon_2^2 + \sigma_1'\epsilon_2'^2 + \sigma_2'\epsilon_1'^2)\left(b_1^{3/2} + \frac{9}{4}b_0^{5/2} - Db_1^{3/2}\right)\\
&+ (\epsilon_1'^2\epsilon_2'^2 + \sigma_1'\epsilon_2^2 + \sigma_2'\epsilon_1^2)\left(b_1^{3/2} + \frac{9}{4}b_0^{5/2} + Db_1^{3/2}\right)\\
&- \epsilon_1\epsilon_2(\epsilon_1\epsilon_2' + \epsilon_2\epsilon_1')\left(\frac{1}{2}b_2^{3/2} + \frac{9}{2}b_1^{5/2} - Db_2^{3/2}\right)\\
&- \epsilon_1'\epsilon_2'(\epsilon_1\epsilon_2' + \epsilon_2\epsilon_1')\left(\frac{1}{2}b_2^{3/2} + \frac{9}{2}b_1^{5/2} + Db_2^{3/2}\right)\\
&+ 9\left[\epsilon_1\epsilon_2\epsilon_1'\epsilon_2' + \frac{1}{2}(\sigma_1 + \sigma_2)(\epsilon_1\epsilon_2 + \epsilon_1'\epsilon_2') + \frac{1}{12}(\sigma_1\sigma_2 + \sigma_1'\sigma_2')\right]b_0^{5/2}
\end{aligned}$$

$$+\frac{9}{4}\left[\epsilon_1^2\epsilon_2'^2+\epsilon_2^2\epsilon_1'^2+\frac{1}{6}(\sigma_1^2+\sigma_2^2)\right]b_2^{5/2}$$

$$-\frac{9}{2}[\sigma_1\epsilon_2\epsilon_1'+\sigma_2\epsilon_1\epsilon_2'+\sigma_1'\epsilon_2\epsilon_2'+\sigma_2'\epsilon_1\epsilon_1']b_1^{5/2}$$

$$\left.+(\sigma_1\epsilon_1\epsilon_2'+\sigma_2\epsilon_2\epsilon_1')\left(b_2^{3/2}-\frac{9}{2}b_1^{5/2}\right)\right\}, \tag{11}$$

and

$$F_{1p}=\frac{fmm'}{\sqrt{aa'}}(\tau\lambda^{+2}\lambda'^{-5}+\sigma\lambda^{-2}\lambda'^{+5}), \tag{12}$$

where

$$\tau=P_0\epsilon_2^3+P_1\epsilon_2^2\epsilon_2'+P_2\epsilon_2\epsilon_2'^2+P_3\epsilon_2'^3+(Q_0\epsilon_2+Q_1\epsilon_2')\sigma_2',$$

and

$$\sigma=P_0\epsilon_1^3+P_1\epsilon_1^2\epsilon_1'+P_2\epsilon_1\epsilon_1'^2+P_3\epsilon_1'^3+(Q_0\epsilon_1+Q_1\epsilon_1')\sigma_1'. \tag{12'}$$

At the beginning we set up elimination of the great inequality, using the following canonical elements:

$$\begin{array}{ll}
x_1=L & x_1'=L' \\
x_2=L-G & x_2'=L'-G' \\
x_3=G-H & x_3'=G'-H' \\
y_1=l+g+h & y_1'=l'+g'+h' \\
y_2=-g-h=-\pi & y_2'=-g'-h'=-\pi' \\
y_3=-h=-\theta & y_3'=-h=-\theta'
\end{array}$$

At a later stage we switch to the canonical elements of Harzer and Poincaré, because of their intimate connections with Andoyer's expansion. Harzer's elements are

$$\begin{array}{ll}
x_1=L=g^2 & x_1'=L'=g'^2 \\
y_1=y_1 & y_1'=y_1' \\
\xi_1=2g\sin\frac{\psi}{2}e^{-i\pi} & \xi_1'=2g'\sin\frac{\psi'}{2}e^{-i\pi'} \\
\xi_2=2g\sin\frac{\psi}{2}e^{+i\pi} & \xi_2'=2g'\sin\frac{\psi'}{2}e^{+i\pi'}
\end{array}$$

$$p_1 = 2g\sqrt{\cos\psi}\,\sin\frac{i}{2}\,e^{-i\theta} \qquad p_1' = 2g'\sqrt{\cos\psi'}\,\sin\frac{i'}{2}\,e^{-i\theta'}$$

$$p_2 = 2g\sqrt{\cos\psi}\,\sin\frac{i}{2}\,e^{+i\theta} \qquad p_2' = 2g'\sqrt{\cos\psi'}\,\sin\frac{i'}{2}\,e^{+i\theta'}$$

where we set

$$g = a\sqrt{mn} \qquad g' = a'\sqrt{m'n'}$$

$$e = \sin\psi \qquad e' = \sin\psi'$$

In eliminating the great inequality we do not go beyond terms quadratic in fmm' and thus must use very simple differential operators associated with the process of elimination. (We are not particularly interested in the inversion process of expressing the old elements in terms of the final ones.)

For all these reasons we found it convenient to use von Zeipel's method in its classical form to eliminate the great-inequality terms. In addition, when using this method we need compute at each step only one half of each Poissonian bracket to obtain the unknown secular terms. When using other methods we must compute full Poissonian brackets. The author is indebted to Dr. Garfinkel* for pointing out both these facts.

We make the canonical transformation

$$x_i = x_i^* + \frac{\partial S(x^*, y; x'^*, y')}{\partial y_i}, \quad x_i' = x_i'^* + \frac{\partial S(x^*, y; x'^*, y')}{\partial y_i'}$$

and

$$y_i^* = y_i + \frac{\partial S(x^*, y; x'^*, y')}{\partial x_i^*}, \quad y_i'^* = y_i' + \frac{\partial S(x^*, y; x'^*, y')}{\partial x_i'^*}, \tag{13}$$

where

$$i = 1, 2, 3,$$

in such a manner that the new Hamiltonian F^* does not contain the angles y_1^* and $y_1'^*$.

The essence of von Zeipel's method can be described in a contracted form as follows: changing notations in (13) from x^*, x'^* to x, x', we can write the relation

$$F(x, y; x', y') = F^*(x^*, y^*; x'^*, y'^*)$$

in the form (von Zeipel, 1916)

$$F\left(x + \frac{\partial S(x, y; x', y')}{\partial y}, y; x' + \frac{\partial S(x, y; x', y')}{\partial y'}\right)$$

$$= F^*\left(x, y + \frac{\partial S(x, y; x', y')}{\partial x}; x', y' + \frac{\partial S(x, y; x', y')}{\partial x'}\right). \tag{14}$$

*Garfinkel, B.: 1969, Private Communication.

Introducing the Taylor operators

$$T = \exp\left(\frac{\partial S}{\partial y}\frac{\partial}{\partial x} + \frac{\partial S}{\partial y'}\frac{\partial}{\partial x'}\right)$$

and

$$T^* = \exp\left(\frac{\partial S}{\partial x}\frac{\partial}{\partial y} + \frac{\partial S}{\partial x'}\frac{\partial}{\partial y'}\right),$$

we can rewrite (14) as

$$TF(x, y; x', y') = T^*F^*(x, y; x', y') . \tag{15}$$

Assuming the expansions

$$S = S_1 + S_2 + \cdots ,$$

$$F = F_0 + F_1 + F_2 + \cdots ,$$

and

$$F^* = F_0^* + F_1^* + F_2^* + \cdots ,$$

in powers of fmm', we can obtain the expansions

$$T = T_0 + T_1 + T_2 + \cdots$$

and

$$T^* = T_0^* + T_1^* + T_2^* + \cdots ,$$

in terms of Faa-de-Bruno operators (1855) T_j and T_j^*. The operators T_j and T_j^* are polynomials in the operators

$$\delta_j = \frac{\partial S_j}{\partial y}\frac{\partial}{\partial x} + \frac{\partial S}{\partial y'}\frac{\partial}{\partial x'} \tag{16}$$

and

$$\delta_j^* = \frac{\partial S_j}{\partial x}\frac{\partial}{\partial y} + \frac{\partial S_j}{\partial x'}\frac{\partial}{\partial y'}, \tag{17}$$

respectively. From (15) we derive the system of partial differential equations

$$\sum_{k=0}^{n} T_{n-k}F_k = \sum_{k=0}^{n} T_{n-k}^*F_k^* \quad (n = 0, 1, 2, \ldots)$$

for the determination of $S_1, S_2, \ldots$. The values of $F_0^*, F_1^*, F_2^* \ldots$ must be determined in such a manner that $S_1, S_2, \ldots$ do not contain any secular terms. Further details on this approach to von Zeipel's method and the general equations for operators T_i, T_i^* $(i = 0, 1, 2, \ldots)$ are given by Musen (1965).

In particular, we have

$$F_0 = F_0^* , \tag{18}$$

$$\delta_1 F_0 + F_1 = \delta_1^* F_0^* + F_1^* ,$$

and

$$\left(\delta_2 + \frac{1}{2}\delta_1^2\right) F_0 + \delta_1 F_1 + F_2 = \left(\delta_2^* + \frac{1}{2}\delta_1^{*2}\right) F_0^* + \delta_1^* F_1^* + F_2^* .$$

Substituting (10) for F_0 and substituting $F_{1s} + F_{1p}$ for F_1 [where F_{1s} and F_{1p} are given by (11) and (12), respectively] and taking (16) and (17) into consideration, we obtain

$$-\nu \frac{\partial S_1}{\partial \chi} + F_{1s} + F_{1p} = F_1^*$$

and

$$-\nu \frac{\partial S_2}{\partial \chi} + \frac{1}{2}\left(4 \frac{\partial^2 F_0}{\partial x_1^2} + 25 \frac{\partial^2 F_0}{\partial x_1'^2}\right)\left(\frac{\partial S_1}{\partial \chi}\right)^2 + \left(\frac{\partial S_1}{\partial y_1}\frac{\partial}{\partial x_1} + \frac{\partial S_1}{\partial y_2}\frac{\partial}{\partial x_2}\right.$$

$$\left. + \frac{\partial S_1}{\partial y_3}\frac{\partial}{\partial x_3} + \frac{\partial S_1}{\partial y_1'}\frac{\partial}{\partial x_1'} + \frac{\partial S_1}{\partial y_2'}\frac{\partial}{\partial x_2'} + \frac{\partial S_1}{\partial y_3'}\frac{\partial}{\partial x_3'}\right)(F_{1p} + F_{1s})$$

$$= \left(\frac{\partial S_1}{\partial x_2}\frac{\partial}{\partial y_2} + \frac{\partial S_1}{\partial x_3}\frac{\partial}{\partial y_3} + \frac{\partial S_1}{\partial x_2'}\frac{\partial}{\partial y_2'} + \frac{\partial S_1}{\partial x_3'}\frac{\partial}{\partial y_3'}\right) F_1^* + F_2^* . \tag{19}$$

From (18) we have

$$F_1^* = F_{1s} \tag{20}$$

and, making use of (12), we have

$$\nu \frac{\partial S_1}{\partial \chi} = \frac{fmm'}{\sqrt{aa'}} (\tau \lambda^{+2} \lambda'^{-5} + \sigma \lambda^{-2} \lambda'^{+5}) .$$

Consequently, after the integration, we have

$$S_1 = \frac{fmm'}{i\nu\sqrt{aa'}} (\tau \lambda^{+2} \lambda'^{-5} - \sigma \lambda^{-2} \lambda'^{+5}) . \tag{21}$$

Because F_{1s} and F_1^* are purely secular and S_1 is purely periodic, the only contribution to F_2^* in (19) comes from the terms and

$$\frac{\partial S_1}{\partial y_1}\frac{\partial F_{1p}}{\partial x_1}+\frac{\partial S_1}{\partial y_2}\frac{\partial F_{1p}}{\partial x_2}+\frac{\partial S_1}{\partial y_3}\frac{\partial F_{1p}}{\partial x_3}+\frac{\partial S_1}{\partial y_1'}\frac{\partial F_{1p}}{\partial x_1'}+\frac{\partial S_1}{\partial y_2'}\frac{\partial F_{1p}}{\partial x_2'}+\frac{\partial S_1}{\partial y_3'}\frac{\partial F_{1p}}{\partial x_3'}$$

$$+\frac{1}{2}\left(4\frac{\partial^2 F_0}{\partial x_1^2}+25\frac{\partial^2 F_0}{\partial x_1'^2}\right)\left(\frac{\partial S_1}{\partial \chi}\right)^2 .$$

Thus,

$$F_2^* = \left\{\frac{1}{2}\left(4\frac{\partial^2 F_0}{\partial x_1^2}+25\frac{\partial^2 F_0}{\partial x_1'^2}\right)\left(\frac{\partial S_1}{\partial \chi}\right)^2\right\}_s + \left\{\left(\frac{\partial S_1}{\partial y_1}\frac{\partial F_{1p}}{\partial x_1}+\frac{\partial S_1}{\partial y_2}\frac{\partial F_{1p}}{\partial x_2}\right.\right.$$

$$\left.\left.+\frac{\partial S_1}{\partial y_3}\frac{\partial F_{1p}}{\partial x_3}+\frac{\partial S_1}{\partial y_1'}\frac{\partial F_{1p}}{\partial x_1'}+\frac{\partial S_1}{\partial y_2'}\frac{\partial F_{1p}}{\partial x_2'}+\frac{\partial S_1}{\partial y_3'}\frac{\partial F_{1p}}{\partial x_3'}\right)\right\}_s .$$

The term

$$\left\{\frac{1}{2}\left(4\frac{\partial^2 F_0}{\partial x_1^2}+25\frac{\partial^2 F_0}{\partial x_1'^2}\right)\left(\frac{\partial S_1}{\partial \chi}\right)^2\right\}_s$$

is of the sixth order in ϵ, ϵ', γ, γ', and thus can be neglected. Substituting (12) and (21) in the last equation, we obtain

$$F_2^* = \frac{1}{i\nu}\left(\frac{fmm'}{\sqrt{aa'}}\right)^2\left[\frac{\partial(\tau,\sigma)}{\partial(y_2,x_2)}+\frac{\partial(\tau,\sigma)}{\partial(y_3,x_3)}+\frac{\partial(\tau,\sigma)}{\partial(y_2',x_2')}+\frac{\partial(\tau,\sigma)}{\partial(y_3',x_3')}\right] .$$

Transforming the last equation to Harzer canonical variables, we obtain

$$F_2^* = \frac{2}{\nu}\left(\frac{fmm'}{\sqrt{aa'}}\right)^2\left[\frac{\partial(\tau,\sigma)}{\partial(\xi_1,\xi_2)}+\frac{\partial(\tau,\sigma)}{\partial(p_1,p_2)}+\frac{\partial(\tau,\sigma)}{\partial(\xi_1',\xi_2')}+\frac{\partial(\tau,\sigma)}{\partial(p_1',p_2')}\right] . \tag{22}$$

With sufficient accuracy we can set

$$\xi_1 = 2g\epsilon_1 \qquad \xi_1' = 2g'\epsilon_1'$$

$$\xi_2 = 2g\epsilon_2 \qquad \xi_2' = 2g'\epsilon_2'$$

$$p_1 = 2g\gamma_1 \qquad p_1' = 2g'\gamma_1'$$

$$p_2 = 2g\gamma_2 \qquad p_2' = 2g'\gamma_2'$$

Substituting these values in (22) and taking

$$\frac{\partial\tau}{\partial\epsilon_1} = \frac{\partial\sigma}{\partial\epsilon_2} = \frac{\partial\tau}{\partial\gamma_1} = \frac{\partial\sigma}{\partial\gamma_2} = \frac{\partial\tau}{\partial\epsilon_1'} = \frac{\partial\sigma}{\partial\epsilon_2'} = \frac{\partial\tau}{\partial\gamma_1'} = \frac{\partial\sigma}{\partial\gamma_2'} = 0$$

into consideration, we obtain

$$F_2^* = -\frac{1}{2\nu}\left(\frac{fmm'}{\sqrt{aa'}}\right)^2 \left[\frac{1}{mna^2}\left(\frac{\partial\tau}{\partial\epsilon_2}\frac{\partial\sigma}{\partial\epsilon_1} + \frac{\partial\tau}{\partial\gamma_2}\frac{\partial\sigma}{\partial\gamma_1}\right)\right.$$

$$\left. + \frac{1}{m'n'a'^2}\left(\frac{\partial\tau}{\partial\epsilon_2'}\frac{\partial\sigma}{\partial\epsilon_1'} + \frac{\partial\tau}{\partial\gamma_2'}\frac{\partial\sigma}{\partial\gamma_1'}\right)\right]. \tag{23}$$

We set

$$\kappa = -\frac{fmm'}{2\nu aa'}\frac{1}{mna^2}$$

and

$$\kappa' = -\frac{fmm'}{2\nu aa'}\frac{1}{m'n'a'^2}.$$

Taking

$$f(1+m) = n^2a^3$$

and

$$f(1+m') = n'^2a'^3$$

into consideration, we obtain

$$\kappa = -\frac{m'n}{2\nu(1+m)a'}$$

and

$$\kappa' = -\frac{mn'}{2\nu(1+m')a}.$$

Substituting the values of τ and σ given by (12′) into (23), we obtain

$$F_2^* = fmm'\Big\{(9P_0^2\kappa + P_1^2\kappa')\epsilon_1^2\epsilon_2^2 + (P_2Q_0\kappa + 3P_3Q_1\kappa')(\epsilon_2'^2\sigma_1' + \epsilon_1'^2\sigma_2')$$

$$+ (P_2^2\kappa + 9P_3^2\kappa')\epsilon_1'^2\epsilon_2'^2 + (3P_0Q_0\kappa + Q_1P_1\kappa')(\epsilon_2^2\sigma_1' + \epsilon_1^2\sigma_2')$$

$$+ 3P_1(3P_0\kappa + P_2\kappa')(\epsilon_1^2\epsilon_2\epsilon_2' + \epsilon_1\epsilon_2^2\epsilon_1')$$

$$+ 2P_2(P_1\kappa + 3P_3\kappa')(\epsilon_2\epsilon_1'^2\epsilon_2' + \epsilon_1\epsilon_1'\epsilon_2'^2)$$

$$+ 4(P_1^2\kappa + P_2^2\kappa')\epsilon_1\epsilon_2\epsilon_1'\epsilon_2' + 3(P_0P_2\kappa + P_1P_3\kappa')(\epsilon_1^2\epsilon_2'^2 + \epsilon_2^2\epsilon_1'^2)$$

$$+ 2(P_1Q_0\kappa + P_2Q_1\kappa')(\epsilon_2\epsilon_2'\sigma_1' + \epsilon_1\epsilon_1'\sigma_2') + (Q_0^2\kappa + Q_1^2\kappa')\sigma_1'\sigma_2'$$

$$- 4(\kappa + \kappa')Q_0^2\epsilon_1\epsilon_2 \cdot \tfrac{1}{2}(\sigma_1 + \sigma_2)$$

$$- 4(\kappa + \kappa')Q_0Q_1(\epsilon_1\epsilon_2' + \epsilon_2\epsilon_1') \cdot \tfrac{1}{2}(\sigma_1 + \sigma_2)$$

$$- 4(\kappa + \kappa')Q_1^2\epsilon_1'\epsilon_2' \cdot \tfrac{1}{2}(\sigma_1 + \sigma_2)\Big\} . \tag{24}$$

Taking into account (11), (20), and (24), and omitting useless terms, we obtain the transformed secular Hamiltonian associated with Jupiter and Saturn:

$$\begin{aligned}
F^* &= F_1^* + F_2^* \\
&= fmm'\Big[B_1(\epsilon_1\epsilon_2 + \epsilon_1'\epsilon_2' + \tfrac{1}{2}\sigma_1 + \tfrac{1}{2}\sigma_2) + B_2(\epsilon_1\epsilon_2' + \epsilon_2\epsilon_1') \\
&\quad + B_3\epsilon_1^2\epsilon_2^2 + B_4(\epsilon_2'^2\sigma_1' + \epsilon_1'^2\sigma_2') + B_5\epsilon_1'^2\epsilon_2'^2 + B_6(\sigma_1'\epsilon_2^2 + \sigma_2'\epsilon_1^2) \\
&\quad + B_7(\epsilon_1^2\epsilon_2\epsilon_2' + \epsilon_1\epsilon_2^2\epsilon_1') + B_8(\epsilon_1\epsilon_1'\epsilon_2'^2 + \epsilon_2\epsilon_1'^2\epsilon_2') + B_9\epsilon_1\epsilon_2\epsilon_1'\epsilon_2' \\
&\quad + B_{10}\epsilon_1\epsilon_2 \cdot \tfrac{1}{2}(\sigma_1 + \sigma_2) + B_{11}\epsilon_1'\epsilon_2' \cdot \tfrac{1}{2}(\sigma_1 + \sigma_2) + B_{12}\sigma_1\sigma_2 + B_{13}\sigma_1'\sigma_2' \\
&\quad + B_{14}(\epsilon_1^2\epsilon_2'^2 + \epsilon_2^2\epsilon_1'^2) + B_{15}(\epsilon_2\epsilon_1'\sigma_1 + \epsilon_1\epsilon_2'\sigma_2 + \epsilon_2\epsilon_2'\sigma_1' + \epsilon_1\epsilon_1'\sigma_2') \\
&\quad + B_{16}(\sigma_1\epsilon_1\epsilon_2' + \sigma_2\epsilon_2\epsilon_1') + B_{17}\epsilon_1'\epsilon_2' \cdot \tfrac{1}{2}(\sigma_1 + \sigma_2) + B_{18}(\sigma_1^2 + \sigma_2^2)\Big] ,
\end{aligned}$$

where

$$B_1 = +\frac{1}{a'\sqrt{a}}b_1^{3/2} ,$$

$$B_2 = -\frac{1}{a'\sqrt{a}}b_2^{3/2} ,$$

$$B_3 = +\frac{1}{a'\sqrt{a}}\left(b_1^{3/2} + \frac{9}{4}b_0^{5/2} - Db_1^{3/2}\right) + (9P_0^2\kappa + P_1^2\kappa') ,$$

$$B_4 = +\frac{1}{a'\sqrt{a}}\left(b_1^{3/2} + \frac{9}{4}b_0^{5/2} - Db_1^{3/2}\right) + (P_2Q_0\kappa + 3P_3Q_1\kappa') ,$$

$$B_5 = +\frac{1}{a'\sqrt{a}}\left(b_1^{3/2} + \frac{9}{4}b_0^{5/2} + Db_1^{3/2}\right) + (P_2^2\kappa + 9P_3^2\kappa') ,$$

$$B_6 = +\frac{1}{a'\sqrt{a}}\left(b_1^{3/2} + \frac{9}{4}b_0^{5/2} + Db_1^{3/2}\right) + (3P_0Q_0\kappa + Q_1P_1\kappa') ,$$

$$B_7 = -\frac{1}{a'\sqrt{a}}\left(\frac{1}{2}b_2^{3/2} + \frac{9}{2}b_1^{5/2} - Db_2^{3/2}\right) + (9P_1P_0\kappa + 3P_1P_2\kappa') ,$$

$$B_8 = -\frac{1}{a'\sqrt{a}}\left(\frac{1}{2}b_2^{3/2} + \frac{9}{2}b_1^{5/2} + Db_2^{3/2}\right) + (2P_2P_1\kappa + 6P_2P_3\kappa') ,$$

$$B_9 = +\frac{9}{a'\sqrt{a}}b_0^{5/2} + (4P_1^2\kappa + 4P_2^2\kappa') ,$$

$$B_{10} = +\frac{9}{a'\sqrt{a}}b_0^{5/2} - 4(\kappa + \kappa')Q_0^2 ,$$

$$B_{11} = +\frac{9}{a'\sqrt{a}}b_0^{5/2} - 4(\kappa + \kappa')Q_1^2 ,$$

$$B_{12} = +\frac{3}{4}\frac{1}{a'\sqrt{a}}b_0^{5/2} ,$$

$$B_{13} = +\frac{3}{4}\frac{1}{a'\sqrt{a}}b_0^{5/2} + (Q_0^2\kappa + Q_1^2\kappa') ,$$

$$B_{14} = +\frac{9}{4a'\sqrt{a}}b_2^{5/2} + (3P_0P_2\kappa + 3P_1P_3\kappa') ,$$

$$B_{15} = -\frac{9}{2a'\sqrt{a}}b_1^{5/2} ,$$

$$B_{16} = +\frac{1}{a'\sqrt{a}}\left(b_2^{3/2} - \frac{9}{2}b_1^{5/2}\right) ,$$

$$B_{17} = +\frac{9}{a'\sqrt{a}}b_0^{5/2} - 4(\kappa + \kappa')Q_1^2 ,$$

and

$$B_{18} = +\frac{3}{8}\frac{1}{a'\sqrt{a}}b_2^{5/2} .$$

All elements in F^* and F_2^* are in fact the "asterisk elements." We change the notations and omit the asterisks.

With the elements given in the work of Anolik et al. (1969), we obtain (using C. Hipkins's program)

$$P_0 = -3.42811 ,$$

$$P_1 = +17.1336 ,$$

$$P_2 = -28.3531 ,$$

$$P_3 = +15.4786 ,$$

$$Q_0 = -1.95863\ ,$$

$$Q_1 = +3.76733\ ,$$

and

$$\frac{F_2^*}{fmm'} = +0.924899\epsilon_1^2\epsilon_2^2$$

$$+ 0.542825(\epsilon_2'^2\sigma_1' + \epsilon_1'^2\sigma_2')$$

$$+ 6.82370\epsilon_1'^2\epsilon_2'^2$$

$$+ 0.199901(\sigma_1'\epsilon_2^2 + \sigma_2'\epsilon_1^2)$$

$$- 4.59557(\epsilon_1^2\epsilon_2\epsilon_2' + \epsilon_1\epsilon_2^2\epsilon_1')$$

$$- 8.32165(\epsilon_1\epsilon_1'\epsilon_2'^2 + \epsilon_2\epsilon_1'^2\epsilon_2')$$

$$+ 10.1485\epsilon_1\epsilon_2\epsilon_1'\epsilon_2'$$

$$+ 2.51217(\epsilon_1^2\epsilon_2'^2 + \epsilon_2^2\epsilon_1'^2)$$

$$- 0.662104(\epsilon_2\epsilon_2'\sigma_1' + \epsilon_1\epsilon_1'\sigma_2')$$

$$+ 0.0432945\sigma_1'\sigma_2'$$

$$- 0.0592693 \cdot \tfrac{1}{2}(\sigma_1 + \sigma_2)\epsilon_1\epsilon_2$$

$$+ 0.114001 \cdot \tfrac{1}{2}(\sigma_1 + \sigma_2)(\epsilon_1\epsilon_2' + \epsilon_2\epsilon_1')$$

$$- 0.219276 \cdot \tfrac{1}{2}(\sigma_1 + \sigma_2)\epsilon_1'\epsilon_2'\ .$$

5. CONCLUSION

In order to meet the requirements of modern Celestial Mechanics in improving the theory of secular perturbations of the principal planets, it is necessary to include the influence of the great inequality.

The contribution of the great inequality to the secular disturbing function is considerable. A comparison between F_1^* and F_2^* shows that the great inequality contributes the largest portion of the coefficients of many fourth-order terms in ϵ, ϵ', γ, and γ'. The part F_2^* produced by the great inequality can be added to the disturbing function given by Anolik et al. (1969). Better still, we can preserve the homogeneity of the theory and recompute the secular perturbations of principal planets from the very start, using the Andoyer's symbolism.

We have already pointed out that the canonicity of the differential equations and the heliocentric elements can be jointly preserved only if we do not go beyond the fourth order in ϵ, ϵ', γ, and γ' in the secular disturbing function. The use of heliocentric elements is important from an astronomical standpoint; if we wish to retain them and at the same time include the terms of sixth and possibly higher orders, we can use the method of Krylov and Bogolubov for eliminating periodic terms.

We may expect, in general, that the Krylov-Bogolubov method will occupy an important place in developing the analytical theories of heliocentric planetary motions.

ACKNOWLEDGMENT

The author wishes to thank his colleague, C. Hipkins, who programmed the differential operators of the Andoyer expansion, using symbolic programming language, and computed the numerical results.

Goddard Space Flight Center
National Aeronautics and Space Administration
Greenbelt, Maryland, January 6, 1971
188-41-51-06-51

REFERENCES

Andoyer, H.: 1923, *Cours de Mécanique Celeste* Gauthier-Villars, Paris.

Anolik, M. V., Krasinsky, B. A., and Pius, L. J.: 1969, "The Trigonometrical Theory of Secular Perturbations of Major Planets," *Trudy Instituta Teoreticheskoy Astronomii* XIV, 3-47.

Bogolubov, N. N. and Mitropolsky, Y. A.: 1961, *Asymptotic Method in the Theory of Nonlinear Oscillations* Gordon and Breach, New York.

Brouwer, D.: 1959, "Solution of the Problem of Artificial Satellite Theory Without Drag," *Astron. J.* 64, No. 9, 378-397.

Brouwer, D. and van Woerkom, A. J. J.: 1950, "The Secular Variations of the Orbital Elements of the Principal Planets," *Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac* 13, part 2, U.S. Printing Office, Washington.

Deprit, A.: 1969, "Canonical Transformations Depending on a Small Parameter," *Celestial Mechanics* 1, No. 1, 12-30.

Faa di Bruno: 1855, "Note sur une Nouvelle Formule de Calcul Differentiel," *Quarterly J. of Math.* 1, 359.

Harzer, P.: 1895, *Die Säcularen Veränderungen der Bahnen der Grossen Planeten* S. Hirzel, Leipzig.

Hill, G. W.: 1897, "The Values of the Eccentricities and Longitudes of the Perihelia of Jupiter and Saturn for Distant Epochs," *Astron. J.* 17, 81-87. (*Also published in 1907, The Collected Mathematical Works of George William Hill* IV, 123, The Carnegie Institution of Washington, Washington, D.C.)

Hori, G.: 1966, "Theory of General Perturbations with Unspecified Canonical Variables," *Publ. Astr. Soc. Japan* 18, 287-296.

Leverrier, U. J.: 1874, *Annales de L' Observatoire de Paris* Memoirs X, 239-290.

Musen, P.: 1965, "On the High Order Effects in the Methods of Krylov-Bogoliubov and Poincaré," *J. Astronautical Sciences* 12, 129-134.

Sharaf, S. G. and Boudnikova, N. A.: 1967, "On the Secular Perturbations in the Elements of the Earth's Orbit and their Influence on the Climates in the Geological Past," *Bulletin Instituta Teoreticheskoy Astronomii* No. 4(127), 231-235.

von Zeipel, H.: 1916, "Recherches sur le Mouvement des Petites Planètes," *Arkiv foer Matematik, Astronomi, och Fysik* 11, No. 1, 1-58.

NASA-Langley, 1971 —— 30

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546

OFFICIAL BUSINESS
PENALTY FOR PRIVATE USE $300

FIRST CLASS MAIL

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS AND
SPACE ADMINISTRATION

02U 001 55 51 3DS 71118 00903
AIR FORCE WEAPONS LABORATORY /WLOL/
KIRTLAND AFB, NEW MEXICO 87117

ATT E. LOU BOWMAN, CHIEF, TECH. LIBRARY

POSTMASTER: If Undeliverable (Section 158 Postal Manual) Do Not Return

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

— NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION OFFICE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Washington, D.C. 20546