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PROPAGATORS IN STRONG PLASMA TURBULENCE

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ABSTRACT

Straightforward relationships among Weinstock's propagator U_A, the Vlasov propagator U, and the ensemble average Vlasov propagator $\langle U \rangle$ are derived. U and $\langle U \rangle$ are related to the characteristic trajectories of the Vlasov Equation.

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In Dupree's[1] theory of strong plasma turbulence, the fundamental role is played by the operator $\langle U t, t_0) \rangle$, the average of the Vlasov propagator $U(t, t_0)$ over an ensemble of plasma realizations. $\langle U(t, t_0) \rangle$, as we shall see, can be related to various statistical correlations of the turbulent fields.

Weinstock[2] amplified Dupree's ideas and obtained several formally exact results, all involving yet another propagator $U_A(t, t_0)$. A pair of complicated non-linear integro-differential equations implicitly relate U_A, U, and $\langle U \rangle$ in the general case.[2] Specific application of the theory[3] is however, limited to the weak coupling approximation, where U_A is expressed explicitly in terms of $\langle U \rangle$ (and hence in terms of fluctuation correlations): $U_A = \langle U \rangle$ to lowest order in a perturbation series in δF, the amplitude of the fluctuations.

In this note we first show how in the weak coupling limit U_A can be expressed straightforwardly in terms of $\langle U \rangle$ to arbitrary order in δF. Calculation of higher order corrections to Dupree's[1] plasma kinetic equation and dispersion relation is thus facilitated. We relate U_A to $\langle U \rangle$ in two steps, first relating U_A to U and then relating U to $\langle U \rangle$. Our results are a significant simplification of Weinstock's equations.

We next show the relationship between U (and $\langle U \rangle$) and the characteristic trajectories of the Vlasov Equation. The value of these latter relationships is that the propagators can be visualized in terms of the Newtonian orbits of Vlasov fluid elements and properties of the plasma turbulence.

Consider an ensemble of Vlasov plasmas. For each realization, δf, the deviation of the one particle distribution function from its ensemble average $\langle f \rangle$, obeys the equation

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} + (\langle \underset{\sim}{F} \rangle + \delta \underset{\sim}{F}) \cdot \frac{\partial}{\partial \underset{\sim}{v}}\right] \delta f - \langle \delta \underset{\sim}{F} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \delta f \rangle = - \delta \underset{\sim}{F} \cdot \frac{\partial \langle f \rangle}{\partial \underset{\sim}{v}} \tag{1}$$

Here $\langle\ \rangle$ is the ensemble average and δ indicates the fluctuation of a quantity from its average value. $\underset{\sim}{F}$ is the total force per unit mass on the plasma element at the phase space point $\underset{\sim}{r}$, $\underset{\sim}{v}$.

Weinstock's formal solution to Eq. (1) is

$$\delta f(\underset{\sim}{x}, \underset{\sim}{v}, t) = U_A(t, t_0)\, \delta f(\underset{\sim}{x}, \underset{\sim}{v}, t_0) - \int_{t_0}^{t} d\tau\, U_A(t, \tau)\, L(\tau)\, \langle f(\underset{\sim}{x}, \underset{\sim}{v}, \tau) \rangle \tag{2}$$

where $\delta f(\underset{\sim}{x}, \underset{\sim}{v}, t_0)$ is the initial value of δf, $L(\tau) = \delta \underset{\sim}{F}(\tau) \cdot \partial / \partial \underset{\sim}{v}$, and $U_A(t, t_0)$ is defined by Weinstock's[2] Eq. (8).

Alternatively we can place the $\langle \delta \underset{\sim}{F} \cdot \partial / \partial \underset{\sim}{v}\, \delta f \rangle$ term on the right hand side of Eq. (1), recognize $\partial / \partial t + \underset{\sim}{v} \cdot \nabla + (\langle \underset{\sim}{F} \rangle + \delta \underset{\sim}{F}) \cdot \partial / \partial \underset{\sim}{v}$ as the Vlasov operator, and iterate with respect to the $\langle \delta \underset{\sim}{F} \cdot \partial \delta f / \partial \underset{\sim}{v} \rangle$ term. The solution obtained in this way is

$$\delta f(\underset{\sim}{x},\underset{\sim}{v},t) = U(t,t_0)\,\delta f(\underset{\sim}{x},\underset{\sim}{v},t_0) - \int_{t_0}^{t} d\tau\, U(t,\tau)\, L(\tau) \langle f \rangle (\tau)$$
$$- \sum_{m=1}^{\infty} \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \ldots \int_{t_0}^{\tau_{m-1}} d\tau_m\, U(t,\tau_1)\, A\, L(\tau_1)\, U(\tau_1,\tau_2)\, A\, L(\tau_2) \ldots \tag{3}$$
$$\left[\int_{t_0}^{\tau_m} d\tau_{m+1}\, U(\tau_m,\tau_{m+1})\, L(\tau_{m+1}) \langle f \rangle (\tau_{m+1}) - U(\tau_m,t_0)\,\delta f(\underset{\sim}{x},\underset{\sim}{v},t_0) \right]$$

The Vlasov propagator $U(t, t_0)$ satisfies the equation

$$\frac{\partial U}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} U + \left(\langle \underset{\sim}{E} \rangle + \delta \underset{\sim}{E} \right) \cdot \frac{\partial U}{\partial \underset{\sim}{v}} = 0 \qquad U(t_0,t_0) = 1 \tag{4}$$

The averaging operator A in Eq. (3) averages <u>everything</u> <u>to</u> <u>its</u> <u>right</u> over the ensemble of plasmas. If the term $\langle \delta \underset{\sim}{F} \cdot \partial \delta f / \partial \underset{\sim}{v} \rangle$ were neglected in Eq. (1), the solution, Eq. (3), would just be

$$\delta f(\underset{\sim}{x},\underset{\sim}{v},t) = U(t,t_0)\,\delta f(\underset{\sim}{x},\underset{\sim}{v},t_0) - \int_{t_0}^{t} d\tau\, U(t,\tau)\, \delta \underset{\sim}{E}(\tau) \cdot \frac{\partial \langle f \rangle (\tau)}{\partial \underset{\sim}{v}} \tag{5}$$

After reversing the order of integrations in Eq. (3), changing the variable τ_{n+1} to τ, and comparing the resulting form with Eq. (2), we conclude

$$U_A(t,t_0) = U(t,t_0) + \sum_{m=1}^{\infty} \int_{t_0}^{t} d\tau_m \int_{\tau_m}^{t} d\tau_{m-1} \ldots \int_{\tau_2}^{t} d\tau_1$$
$$\left\{ U(t,\tau_1)\, A L(\tau_1)\, U(\tau_1,\tau_2)\, A L(\tau_2) \ldots U(\tau_m,t_0) \right\} \tag{6}$$

Equation 6 is an exact relationship between U_A and U. It becomes approximate only when the series is truncated. Here A also operates on whatever function U_A is propagating.

An equation relating U and ⟨U⟩ is obtained by averaging Eq. (4) and subtracting the resulting equation from Eq. (4) itself:

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} + \left(\langle \underset{\sim}{E}\rangle + \delta \underset{\sim}{E}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}}\right]\left(u - \langle u\rangle\right) = \left\langle \delta \underset{\sim}{E}\cdot\frac{\partial u}{\partial \underset{\sim}{v}}\right\rangle - \delta \underset{\sim}{E}\cdot\frac{\partial\langle u\rangle}{\partial \underset{\sim}{v}} \tag{7}$$

Equation (7) formally integrates to

$$U(t,t_0) = \langle U(t,t_0)\rangle + \int_{t_0}^{t} d\tau\, U(t,\tau)\left[\langle L(\tau)U(\tau,t_0)\rangle - L(\tau)\langle U(\tau,t_0)\rangle\right] \tag{8}$$

It is straightforward to iterate Eq. (8) to determine U in terms of ⟨U⟩ to any desired order in δF. The term quadratic in U on the right side prohibits us, however, from writing U succinctly in terms of ⟨U⟩ to all orders in δF (i.e., writing an equation analogous to Eq. 6 between U_A and U). Iterated through $\mathcal{O}(\delta F)^2$, Eq. 8 is equivalent to Dupree's' Eq. (4.1).

By combining Eqs. (6) and (8), it is possible to obtain directly a relationship between U_A and ⟨U⟩ . Through $(\delta F)^2$,

$$\begin{aligned} U_A(t,t_0) = {} & \langle U(t,t_0)\rangle - \int_{t_0}^{t} d\tau_1 \langle U(t,\tau_1)\rangle (1-A) L(\tau_1)\langle U(\tau_1,t_0)\rangle \\ & + \int_{t_0}^{t} d\tau_1 \int_{\tau_1}^{t} d\tau_2 \langle U(t,\tau_2)\rangle \Big[(1-A)L(\tau_2)\langle U(\tau_2,\tau_1)\rangle (1-A)L(\tau_1)\langle U(\tau_1,t_0)\rangle \\ & - \big\langle L(\tau_2)\langle U(\tau_2,\tau_1)\rangle L(\tau_1)\langle U(\tau_1,t_0)\rangle\big\rangle\Big] \end{aligned} \tag{9}$$

As a check, it is readily shown that with this form of U_A inserted into Eq. 2, $A\,\delta f = 0 + \mathcal{O}(\delta F)^3$. This must, of course, be the case, since δf has by definition zero ensemble average.

We finally indicate the relationship between U and ⟨U⟩ and the characteristic trajectories of the Vlasov Equation. If $\langle \delta \underset{\sim}{F} \cdot \partial \delta f / \partial \underset{\sim}{v} \rangle$ is neglected in Eq. (1), the solution to this equation is

$$\delta f(\underset{\sim}{x}, \underset{\sim}{v}, t) = \delta f\left[\underset{\sim}{x}^*(t_0), \underset{\sim}{v}^*(t_0), t_0\right] - \int_{t_0}^{t} d\tau\, \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(\tau), \underset{\sim}{v}^*(\tau), \tau\right] \cdot \frac{\partial \langle f \rangle \left[\underset{\sim}{x}^*(\tau), \underset{\sim}{v}^*(\tau), \tau\right]}{\partial \underset{\sim}{v}^*(\tau)} \tag{10}$$

Here $\underset{\sim}{x}^*$ and $\underset{\sim}{v}^*$ are solutions to the characteristic equations

$$\frac{d\underset{\sim}{x}^*}{d\tau} = \underset{\sim}{v}^* \qquad\qquad \frac{d\underset{\sim}{v}^*}{d\tau} = \underset{\sim}{F}\left(\underset{\sim}{x}^*, \underset{\sim}{v}^*, \tau\right) \tag{11}$$

(The boundary conditions to be applied are $\underset{\sim}{x}^*(\tau = t) = \underset{\sim}{x}$, $\underset{\sim}{v}^*(\tau = t) = \underset{\sim}{v}$).

Next we Taylor expand the $\underset{\sim}{x}^*$, $\underset{\sim}{v}^*$ dependence in Eq. (10) about $\underset{\sim}{x}$, $\underset{\sim}{v}$. Comparing the result with Eq. (5), we conclude

$$U(t, t_0) = \exp\left\{\left[\underset{\sim}{x}^*(t_0) - \underset{\sim}{x}\right] \cdot \underset{\sim}{\nabla} + \left[\underset{\sim}{v}^*(t_0) - \underset{\sim}{v}\right] \cdot \frac{\partial}{\partial \underset{\sim}{v}}\right\} \tag{12}$$

Operating on an arbitrary function $\psi(\underset{\sim}{x}, \underset{\sim}{v})$, $U(t, t_0)$ translates the point at which ψ is evaluated to $\underset{\sim}{x}^*(t_0)$, $\underset{\sim}{v}^*(t_0)$, the $t = t_0$ phase space coordinates of the plasma element located at the point $\underset{\sim}{x}$, $\underset{\sim}{v}$ at time t. The trajectory from $\underset{\sim}{x}^*(t_0)$, $\underset{\sim}{v}^*(t_0)$ to $\underset{\sim}{x}$, $\underset{\sim}{v}$ (Eqs. 11) is the exact, fluctuating Vlasov orbit for the element.

For conciseness of notation we introduce $\boldsymbol{\pi}$, the six component phase space vector. Thus

$$\langle U(t,t_0)\rangle = \left\langle \exp\left\{\left[\boldsymbol{\pi}^*(t_0)-\boldsymbol{\pi}\right]\cdot\frac{\partial}{\partial\boldsymbol{\pi}}\right\}\right\rangle = \left\langle \exp\left[\Delta\boldsymbol{\pi}^*(t_0)\cdot\frac{\partial}{\partial\boldsymbol{\pi}}\right]\right\rangle$$

$$= \exp\left\{\langle\Delta\boldsymbol{\pi}^*(t_0)\rangle\cdot\frac{\partial}{\partial\boldsymbol{\pi}} + \frac{1}{2}\left[\langle\Delta\boldsymbol{\pi}^*(t_0)\Delta\boldsymbol{\pi}^*(t_0)\rangle - \langle\Delta\boldsymbol{\pi}^*(t_0)\rangle\langle\Delta\boldsymbol{\pi}^*(t_0)\rangle\right] : \frac{\partial}{\partial\boldsymbol{\pi}}\frac{\partial}{\partial\boldsymbol{\pi}} + \sum_{n=3}^{\infty}\frac{C_n}{n!}\right\} \tag{13}$$

In the last form of Eq. (13), we have, following Weinstock[2], made a cumulant expansion[5]. C_n is the cumulant of $\left\langle\left[\Delta\boldsymbol{\pi}^*(t_0)\cdot\frac{\partial}{\partial\boldsymbol{\pi}}\right]^n\right\rangle$.

We have explicitly written out C_2 in Eq. (13). By integrating the characteristic equations, Eqs. (11), $\Delta\boldsymbol{\pi}^*(t_0)$ can be expressed in terms of the fluctuating fields along a particle trajectory. $\langle U(t, t_0)\rangle$ can thus be represented in terms of statistical correlations of the fluctuating field [2,6]. One can then further make reasonable estimates about the strength of these correlations.

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