General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.

- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

Produced by the NASA Center for Aerospace Information (CASI)

X-551-71-122
PREPRINT

NASA TM X- 65533

CLOSED FORM SATELLITE TRACKING DATA CORRECTIONS FOR AN ARBITRARY TROPOSPHERIC PROFILE

JOHN W. MARINI

MARCH 1971

GSFC — GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

N71-25689
(ACCESSION NUMBER) (THRU)
56 G3
(PAGES) (CODE)
TMX-65533 30
(NASA CR OR TMX OR AD NUMBER) (CATEGORY)
FACILITY FORM 602

JUN 1971
RECEIVED
NASA STI FACILITY
INPUT BRANCH

X-551-71-122

CLOSED FORM SATELLITE TRACKING DATA CORRECTIONS FOR AN ARBITRARY TROPOSPHERIC PROFILE

John W. Marini

March 1971

Goddard Space Flight Center

Greenbelt, Maryland

CLOSED FORM SATELLITE TRACKING DATA CORRECTIONS FOR AN ARBITRARY TROPOSPHERIC PROFILE

John W. Marini

ABSTRACT

The formulas commonly used to correct satellite tracking data for the effects of atmospheric refraction are based, for the sake of mathematical convenience, on simple model atmospheric profiles that are not physically realistic. A method is given here for deriving correction formulas that do not suffer from this limitation, and can be used in precision orbit calculations. The refractivity of the atmosphere is assumed to have spherical symmetry, but may have any given vertical profile. The method was tested for numerical accuracy by application to the simple exponential profile, and the corrections calculated agreed closely at all elevation angles from 0 to 90 degrees with those obtained by double-precision ray-tracing. The error was in all cases less than 1%, and less than 1/3% above 1 degree elevation.

The method has been used to obtain improved correction formulas, and these will be published in a separate report.

Page intentionally left blank

CONTENTS

CONTENTS—(continued)

CORRECTING SATELLITE TRACKING DATA FOR A SPHERICALLY-SYMMETRIC ATMOSPHERE

1. INTRODUCTION

1.1 Background

In the radio tracking of a satellite by a ground station, measurements are made of the satellite elevation, range, and range-rate. The elevation can be determined by a measurement of the angle-of-arrival of a radio wave from the satellite. The range is obtained from a measurement of the delay of signals propagated between the ground station and the satellite, while the range-rate is obtained by counting cycles of the Doppler-shifted received signals over a prescribed period of time, or, alternatively, by noting the time interval required to count a prescribed number of cycles.

The passage of the radio waves through the atmosphere of the earth introduces into these measurements errors that may require correction. Both tropospheric and ionospheric effects are present. However, if the radio frequency used is sufficiently high, the ionospheric correction may either be neglected or superposed on the tropospheric correction [1].

1.2 Related Work

A formula for the bending of a radio ray in an exponential atmosphere as a function of the angle-of-arrival has been given by Thayer [2]. Freeman [3] has obtained, for an exponential atmosphere, a range-error correction as a function of the elevation angle of the satellite. The correction is of the first order in the surface refractivity. Reichley [4] has obtained both second order elevation-error and range-error correction formulas as functions of the elevation angle. His formulas can be applied to profiles other than exponential. Hopfield [5], by neglecting path curvature, has obtained range and range-rate corrections for the two-quartic tropospheric refractivity profile. Rowlandson and Moldt [6] have derived closed-form corrections for an exponential atmosphere.

1.3 Satellite Tracking Data and Correction Formulas

The satellites tracked are ordinarly above the region, which extends from the ground to about 70 km in altitude, where most of the radio-ray bending caused by the nonionized atmosphere of the earth takes place.

In a typical pass of satellite over a ground station, the satellite might be under observation for a number of minutes. During this period of time, angle-of-arrival range and range-rate measurements are taken periodically at a rate, perhaps, of one set of data per second [7]. Consequently, a large volume of data is generated, which must be processed automatically by electronic computer.

To be usable in a practical sense, the equations employed to correct this data for the effects of the atmosphere should require a minimum of computer time.

It is not easy, however, to derive correction equations that, on the one hand, accurately represent the atmospheric model assumed and, on the other hand, are suitable for use in the processing of large quantities of data. The need to obtain a mathematically tractable formula was a consideration in the choice of the fourth (quartic) power in the two-quartic profile [5]. The same mathematical difficulty has also resulted in the existence and use of a multitude of approximate correction formulas [8]. Most of these are in agreement at high elevation angles, but disagree at low angles, apparently because of differing approximations used, even when the same profile was initially assumed. Fortunately this discrepancy is not too serious since most satellite tracking measurements are taken at the higher elevation angles. Nevertheless, it is possible to obtain, for any assumed profile, corrections that are both accurate and yet suitable for use on tracking data. To do so it is necessary to take advantage of the circumstance that the corrections ordinarily are exercised repeatedly using varying values of elevation angle and range but with a fixed atmospheric profile. Consequently, initial or "pre-pass" calculations which involve only the atmospheric conditions at the time of the satellite pass and which are independent of satellite position may be lengthy without causing a significant percentage increase in the total computer time per satellite pass. For efficiency in calculation, therefore, the mathematical formulation of the corrections should be such that quantities functionally dependent on the parameters of the atmospheric profile are separated from those dependent on elevation and range.

Such a separation can be effected by expansions in rational functions of the sine of the elevation angle and negative powers of the range. The coefficients in these expansions will depend on the atmospheric profile alone, and can be calculated in advance of the satellite pass.

2. GEOMETRY AND NOTATION

The geometry involved is shown in Figure 1. The earth is taken to be spherical with a radius 'a' nominally equal to 6369.95 kilometers. The tracking station is located at a distance r_0 from the center of the earth, and at a height h_0 above sea level. The satellite is at a distance r_1 from the center of the earth, and at a height h_1. The radio ray path between the satellite and the station is shown as a dotted line. The distance between the center of the earth and a given point on the ray path is 'r'. The height of the point above sea level is 'h', and the elevation of the ray path at the point is θ.

The angle-of-arrival is the angle θ_0 above the local horizontal at the ground station. The angle ΔE is the elevation error, or difference between the angle-of-arrival and the true elevation angle E of the satellite with respect to the ground station.

$$\Delta E = \theta_0 - E \tag{1}$$

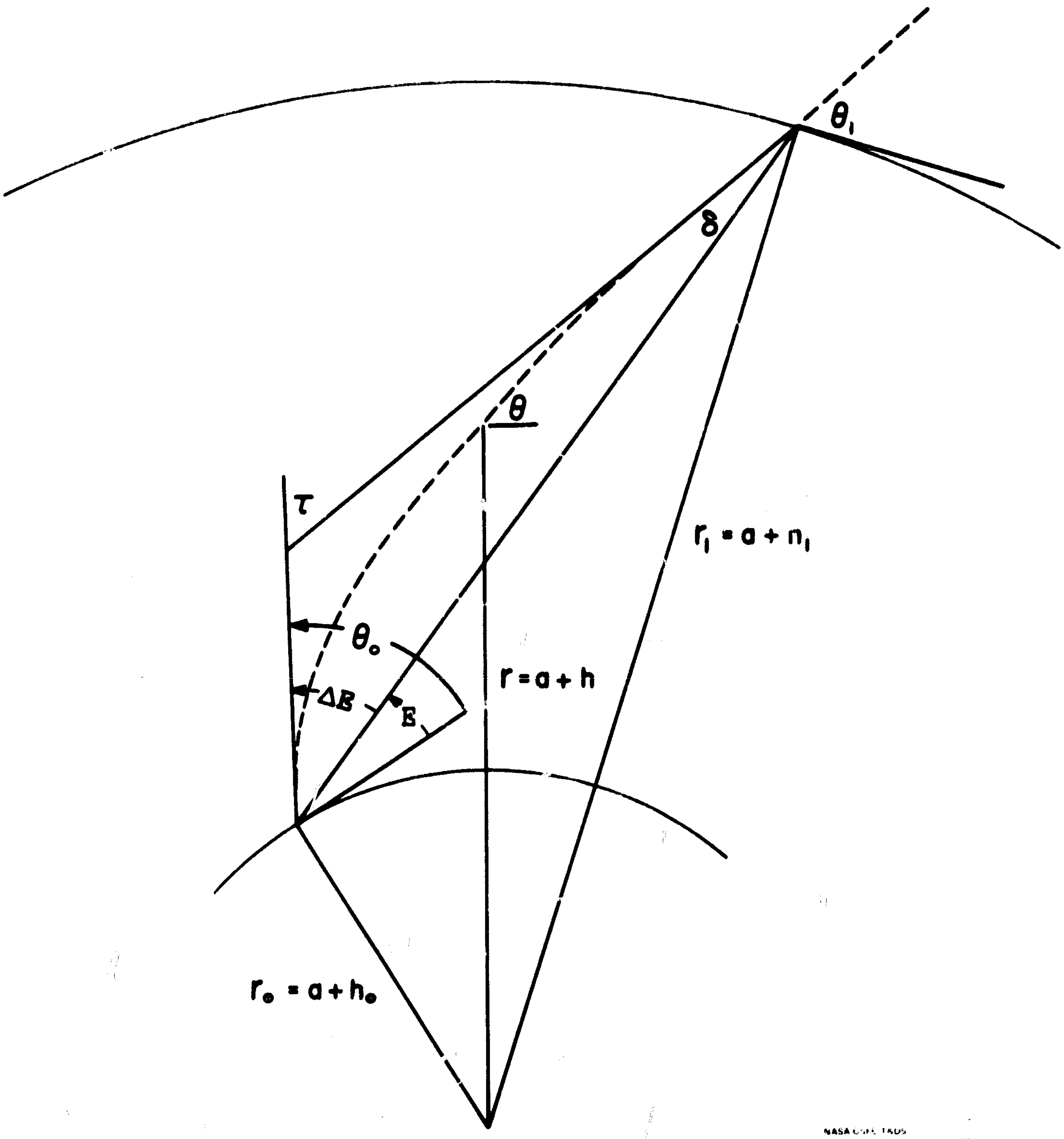


Figure 1. Satellite and Tracking Station Geometry

and τ is the total bending of the radio path. The refractive index n is assumed to depend only on the height h above the surface of the earth. The radio refractivity

$$N(h) = 10^6 \, [n(h) - 1] \tag{2}$$

will be used in normalized form, and a normalized height above the tracking station will be employed. Taking the refractive index and the refractivity at the tracking station to be

$$n_0 = n(h_0) \tag{3}$$

$$N_0 = N(h_0) \tag{4}$$

respectively, and taking the effective height of the troposphere above the tracking station to be

$$H = (1/N_0) \int_{h_0}^{\infty} N(h) \, dh \tag{5}$$

the normalized height above the tracking station

$$x = (h - h_0)/H \tag{6}$$

is defined. In terms of this variable, the refractivity may be written as

$$N(h) = N_0 \, f(x) \tag{7}$$

where the normalized profile $f(x)$, which will be abbreviated as f, is

$$f = N(h_0 + H x)/N_0 \tag{8}$$

which is equal to unity at $x = 0$

$$f(0) = 1 \tag{9}$$

and integrates to unity

$$\int_0^\infty f\, dx = 1 \tag{10}$$

from the tracking station upward. In the particular case of an exponential profile, f is equal to exp (-x).

The radio range, or electrical distance along the ray path is designated as R_e,

$$R_e = \int_{r_0}^{r_1} (n/\sin\theta)\, dr \tag{11}$$

the geometrical distance along the ray path is

$$R_g = \int_{r_0}^{r_1} (1/\sin\theta)\, dr \tag{12}$$

and the straight-line distance or slant range is R. The range error is then the difference

$$\Delta R = R_e - R \tag{13}$$

The parameters

$$p = \sqrt{2H/r_0} \tag{14}$$

$$q = 10^{-6} N_0 r_0/H = 2 \times 10^{-6} N_0/P^2 \tag{15}$$

$$Q = q \cos^2 \theta_0 \tag{16}$$

the normalized sine of the angle of arrival

$$\alpha = (1/p) \sin \theta \tag{17}$$

the normalized sine of the elevation angle,

$$\beta = (1/p) \sin E \tag{18}$$

and the normalized inverse range

$$\rho = p\, r_0/R \tag{19}$$

will be used. Typical values are p = 0.05 and q = 0.25. The value of α range from 0 on the horizon to about 20 for a wave arriving vertically downward.

3. INPUTS TO THE CORRECTION FORMULAS

The desired form of the correction formulas for the elevation error ΔE and the range error ΔR depend on the intended use. In the first case to be considered, which applies, for example, to a tracking radar, it is assumed that measured values of the angle-of-arrival θ_0 and the radio-range R_e are available. The quantities ΔE and ΔR should then be given as functions of the variables θ_0 and R_e.

In the second case, the situation is somewhat different. It is assumed that the satellite ephemeris is known from previous tracking and orbit determination. In this case the true slant range R and the true elevation angle E will be known quite accurately as functions of time, and one or both of the corrections ΔE and ΔR may be needed to provide accurate predictions of the angle-of-arrival and/or radio-range and/or range rate either for acquisition or for comparison with values to be measured. The latter comparisons are used to improve the satellite ephemeris by an iterative process that minimizes some weighted function of the differences observed. Here ΔE and ΔR need to be expressed as functions of E and R.

In both cases above, the percentage difference between R and R_e is small. R_e is at worst about 200 meters larger than R, while R has been taken to be at least 70 km. Consequently the distinction between these quantities may be neglected in their use as inputs to the error formulas. The distinction between E and θ_0, however, must be retained at lower elevation angles.

4. CORRECTIONS USING KNOWN ANGLE OF ARRIVAL

4.1 Formula for Elevation Error

A procedure for determining the elevation error has been given by Bean and Thayer [9]. The bending τ is first calculated by integration, and the angle δ is next calculated from the geometry of Figure 1 using the known value of τ. The elevation error is then given by

$$\Delta E = \tau - \delta \tag{20}$$

Using the notation of Section 2, the usual expression for the bending is [2]

$$\tau = 10^{-6} N_0 \cos \theta_0 I(\alpha)/p \tag{21}$$

where the bending integral is defined as

$$I(\alpha) = \int_0^{\infty} \frac{-f'}{\sqrt{x + \alpha^2 - Q(1 - f)}} dx \tag{22}$$

in which it has been assumed that the satellite height h_1 is great enough to permit the upper limit to be extended to infinity.

The equation giving δ in terms of τ is derived in Appe lix 1. Using (21) and (1-4) in (20), the formula for the elevation error is

$$\Delta E = 10^{-3} N_0 (1/p) \cos \theta_0 [I(\alpha) - \rho L(\alpha)] \quad \text{mrad} \tag{24}$$

with

$$L(\alpha) = 1 - \alpha I(\alpha) + \frac{1}{4} q I^2(\alpha) \tag{25}$$

4.2 A Digression on the Approximations Used

The quantities $10^{-6}\ N_0 \sim 300 \times 10^{-6}$ and $p^2 \sim 0.0025$ are neglected compared to 1. However the quantity $q \sim 0.25$ cannot be so neglected, even though it contains the surface refractivity as a factor.

The quantity Q that appears in the radical in Equation (22) is not independent of satellite position because of its dependence on θ_0. This would lead to complication of the formulas to be derived. Fortunately Q may be replaced by q with negligible error, since the term neglected thereby is small compared to the square of α

$$q \sin^2 \theta_0 \,(1 - f) \ll \alpha^2 \doteq (1 - p^2) \sin^2 \theta_0$$

It will also be found that the same approximation can[1], and in some cases should[2], be made elsewhere in the formulas to be derived.

4.3 Expansions of the Bending Integral

In order to use the formula (24) for the elevation error a rapid method for calculating the bending integral, Equation (22), is needed. Since α is as large as 20 at high elevation angles, it is natural to expand (22) asymptotically[3] in powers

1. In the asymptotic expansions that follows, when $Q = q - q p^2 \alpha^2$ is set equal to q, the term neglected is small compared to the preceeding term in the series.
2. If the approximation is used in calculating $I(\alpha)$ then it should be used in calculating polynomials involving $I(\alpha)$ (for example, in eq 25) in order to obtain the correct asymptotic expansion of the polynomial.
3. The expansion is asymptotic if $q = 0$ and $f = \exp(-x)$. The nature of the expansion was not investigated for other cases, and the development proceeds on a formal basis.

of $1/\alpha$. This can be accomplished either by a formal binomial expansion of the radical in Equation (22) after the square of α has been factored out, or, alternatively, by repeated integration by parts of the numerator of the integrand. In either case there results (see Appendix 2)

$$I(\alpha) \sim (1/\alpha) - I_1 (1/\alpha)^3 + I_2 (1/\alpha)^5 - \ldots \tag{26}$$

where

$$I_1 = \frac{1}{2}\left(1 - \frac{1}{2} q\right) \tag{27}$$

and

$$I_2 = (3/4)\left[\int_0^{\alpha} x\, f\, dx - q\left(1 - \frac{1}{2}\int_0^{\infty} f^2\, dx\right) + (1/6)\, q^2\right] \tag{28}$$

Use of the first term of (26) in (21) results in the familiar

$$\tau \sim 10^{-3} N_0 \cot \theta_0 \qquad \text{mrad} \tag{29}$$

which holds at high elevation angles.

Equation (26) suffers from the usual property of an asymptotic expansion – it is not useful at small values of the argument α. There is, it happens, a procedure [10] for converting a divergent series such as Equation (26) into a continued fraction expansion that converges, in this case, for all $\alpha > 0$. The expansion diverges at $\alpha = 0$, however, and converges only slowly when α is near zero. Rather than apply the procedure directly, therefore, the integral $I(\alpha)$ is expanded, with small, as

$$I(\alpha) = i_0 + i_1 \alpha + \ldots \tag{30}$$

where (Appendix 3)

$$i_0 = I(0) = 2 \int_0^{\infty} \sqrt{x + q(1 - f)} \; \frac{f''}{(1 + q f')^2} \, dx \tag{31}$$

and

$$i_1 = -I'(0) = -2 f'(0) / [1 + q f'(0)] \tag{32}$$

Noting that Equation (26) approximates $I(\alpha)$ when α is large, and that Equation (30) approximates $I(\alpha)$ when α is small, the approximation of $I(\alpha)$ over the entire range of α is accomplished by means of a ratio of polynomials in α. The coefficients of the polynomials are chosen in such a way that the expansion of the ratio in inverse powers of α agrees with the leading terms of Equation (26) on the one hand, and its expansion in ascending powers of α agrees with the leading terms of Equation (30) on the other hand. This method of approximation insures accuracy if α is either large or small. Accuracy with intermediate values is obtained by the inclusion of a sufficient number of terms from each series expansion. The number of terms used here — three from Equation (26) and two from Equation (30) — is not necessarily optimum, but worked out well when the method was applied to an exponential profile. It is evident that the method requires a certain degree of smoothness in the profile if an accurate approximation is to be obtained with this number of terms.

4.4 Form of the Approximation

Consider a rational function of α, $F(\alpha;\ F_1,\ F_2,\ f_0,\ f_1)$, which depends on four parameters F_1, F_2, f_0 and f_1, and which is expressed in the form of a continued fraction

$$F(\alpha;\ F_1,\ F_2;\ f_0,\ f_1) = \cfrac{1}{\alpha + \cfrac{\underline{f}_1}{\alpha + \cfrac{\underline{f}_2}{\alpha + \cfrac{\underline{f}_3}{\alpha + \underline{f}_4}}}} \tag{33}$$

where the intermediate constants $\underline{f}_1$, $\underline{f}_2$, $\underline{f}_3$, and $\underline{f}_4$ are calculated from the set of parameters F_1, F_2, f_0, and f_1 using in sequence

$$\underline{f}_1 = F_1 \tag{34}$$

$$\underline{f}_2 = (F_2 / \underline{f}_1) - \underline{f}_1 \tag{35}$$

$$\underline{f}_3 = \underline{f}_2 \Big/ \left[f_0^2\, \underline{f}_1 \left(1 + \frac{\underline{f}_1}{\underline{f}_2}\right) - (1 + f_1\, \underline{f}_1) \right] \tag{36}$$

$$\underline{f}_4 = f_0\, \underline{f}_1\, \underline{f}_3 / \underline{f}_2 \tag{37}$$

On clearing the denominator of Equation (33) of fractions, and expanding the resulting fractional form by long division in descending powers of α

$$F(\alpha;\ F_1,\ F_2;\ f_0,\ f_1) = (1/\alpha) - F_1\ (1/\alpha)^3 + F_2\ (1/\alpha)^5 - \text{constant}\cdot(1/\alpha)^7\cdots \tag{38}$$

If the long division is carried out using ascending powers of α

$$F(\alpha;\ F_1,\ F_2;\ f_0,\ f_1) = f_0 - f_1\ \alpha + \text{constant}\ \alpha^2 \ldots \tag{39}$$

Thus the function $F(\alpha;\ F_1,\ F_2,\ f_0,\ f_1)$ is well-suited to approximate $I(\alpha)$, Equation (21), provided that the parameters F_1, F_2, f_0, and f_1 are chosen as I_1, I_2, i_0, and i_1 respectively, i.e., $I(\alpha)$ is approximately equal to $F(\alpha;\ I_1,\ I_2;\ i_0,\ i_1)$. Note that these latter parameters, Equations (27), (28), (31), and (32), depend only on the refractivity through the parameter q and the profile f(x). Their calculation is independent of satellite position, and numerical integration prior to each satellite pass need not be ruled out. If f is a given model profile, moreover, the integrals can be evaluated analytically if the functional form of f(x) permits, or, otherwise, may be evaluated numerically and curve or surface fitted empirically.

4.5 Formula for Range Error

The range error Equation (13) may be written as the sum of the difference between the electrical and geometric distances along the ray path and of the difference between the geometric distance along the ray path and the slant range

$$\Delta R = (R_e - R_g) + (R_g - R) \tag{40}$$

The first term in Equation (40), the difference along the ray path, is, after division by r_0

$$(R_e - R_g)/r_0 = (1/r_0) \int_0^{\infty} [10^{-6} N(h)/\sin \theta] \, dr \doteq \frac{1}{2} 10^{-6} N_0 p J(\alpha) \qquad (41)$$

where

$$J(\alpha) = \int_0^{\infty} \frac{f}{\sqrt{x + \alpha^2 - q(1 - f)}} \, dx \qquad (42)$$

While the geometrical difference $(R_g - R)$ between the ray and straight-line paths is smaller than the electrical difference $(R_e - R_g)$ along the ray path, it is not entirely negligible. In Appendix 4 an expression for this difference is derived. Substituting Equations (4-8) and (41) into (40), the expression for the range error is

$$\Delta R/r_0 = \frac{1}{2} 10^{-6} N_0 p \left[M(\alpha) - \frac{1}{2} \rho Q L^2(\alpha)\right] \qquad (43)$$

where

$$M(\alpha) = J(\alpha) + q \left[I(\alpha) - \frac{1}{2} K(\alpha) - \frac{1}{2} \alpha I^2(\alpha) + \frac{1}{12} q I^3(\alpha)\right] \qquad (44)$$

with

$$K(\alpha) = \int_0^{\infty} \frac{-2 f f'}{\sqrt{x + \alpha^2 - q(1 - f)}} \, dx \qquad (45)$$

The expansions of $J(\alpha)$ and $K(\alpha)$ for large and for small values of α have been given in Appendices 2 and 3. Thus it is possible to calculate these integrals in the same way as was done for $I(\alpha)$ by using Equation (33) with suitable choices for the parameters F_1, F_2, f_0, and f_1 and then to compute $M(\alpha)$ by substituting the calculated values of $I(\alpha)$, $J(\alpha)$ and $K(\alpha)$ into Equation (44). It is more efficient to calculate $M(\alpha)$ directly however. Thus, substituting Equations (26), (2-3), and (2-6) into (44)

$$M(\alpha) \sim (1/\alpha) - M_1 (1/\alpha)^3 + M_2 (1/\alpha)^5 - \ldots \tag{46}$$

where

$$M_1 = \frac{1}{2}\left[\int_0^\infty x\, f\, d\, x - q\left(1 - \frac{1}{2}\int_0^\infty f^2\, d\, x\right)\right] \tag{47}$$

$$M_2 = (3/4)\left[\frac{1}{2}\int_0^\infty x^2\, f\, d\, x - q\left(\frac{1}{6} + \int_0^\infty x\, f\, d\, x - \frac{1}{2}\int_0^\infty x\, f^2\, d\, x\right) + q^2\left(\frac{1}{2} - \frac{1}{2}\int_0^\infty f^2\, d\, x + \frac{1}{6}\int_0^\infty f^3\, d\, x\right)\right] \tag{48}$$

Substituting Equations (30), (3-7) and (3-11) into (44)

$$M(\alpha) = m_0 - m_1\, \alpha + \ldots \tag{49}$$

$$m_0 = j_0 + q i_0 + (1/12)\, q^2\, i_0^3 - \frac{1}{2} q\, k_0 \tag{50}$$

$$m_1 = j_1 + \frac{1}{2} q\, i_0^2 \left(1 + \frac{1}{2} q\, i_1\right) \tag{51}$$

$M(\alpha)$ is now calculated using Equations (33-37); i.e., is given by $F(\alpha;\ M_1,\ M_2,\ m_0,\ m_1)$.

A formula for the range-rate correction can be derived (Appendix 5) and, presumably, modelled as above, but this has not been done. In practice (with the Goddard Range and Range Rate System) the range-rate correction can be obtained by dividing the difference between successive range corrections by the corresponding time interval.

5. CORRECTIONS USING KNOWN ELEVATION ANGLE

The formulas for the corrections ΔE and ΔR as functions of β and R rather than as functions of α are more difficult to obtain Reichly [4] in an excellent treatment of the problem, has used a perturbation method to obtain the first two terms of the expansion of these corrections in powers of the surface refractivity. This, in the notation and with the approximations used here, is equivalent to an expansion in powers of the parameter q which may be as large as 0.64 at $N_0 = 450$. Such an expansion is accurate at large values of the elevation angles as may be determined by an examination of the asymptotic expansions, in which the higher powers of q appear only in the higher order terms. More terms are needed, however, if formulas numerically accurate at small values of the elevation angle are to be obtained by this method. Such formulas can be obtained by expansion, not in powers of q, but in powers of ρ, β and $1/\beta$, and by the use of the method of the preceding paragraphs.

5.1 Elevation Correction

It is assumed that R is large enough to permit ΔE to be calculated from the first two terms of its expansion in powers of the normalized inverse range ρ. Setting

$$\Delta E = 10^{-3} N_0 \, (1/p) \, [U(\beta) - \rho V(\beta)] \cos E \qquad \text{mrad} \tag{52}$$

where U and V are functions of β that must be determined, noting that[4]

$$\cos \theta \overset{o}{=} \cos E \tag{53}$$

and that

$$\alpha \overset{o}{=} \beta + (1/p) \, \Delta E \cos E = \beta + \frac{1}{2} q \, (U - \rho V) \tag{54}$$

there results, equating Equations (24) and (52)

$$U - \rho V = I \left(\beta + \frac{1}{2} q U - \frac{1}{2} q \rho V \right) - \rho L \left(\beta + \frac{1}{2} q U - \frac{1}{2} q \rho V \right) \tag{55}$$

If I and L in Equation (55) above are expanded in powers of ρ, and if coefficients of like powers of ρ are equated

$$U = I \left(\beta + \frac{1}{2} q U \right) \tag{56}$$

$$V(\beta) = L \left(\beta + \frac{1}{2} q U \right) \Big/ \left[1 - \frac{1}{2} q I' \left(\beta + \frac{1}{2} q U \right) \right] \tag{57}$$

Equation (56) gives U implicitly as a function of β. Taking

$$U(\beta) \sim (1/\beta) - U_1 \, (1/\beta)^3 + U_2 \, (1/\beta)^5 - \ldots \tag{58}$$

4. $\cos E = \cos(\theta_0 - \Delta E) = \cos \theta_0 (\cos \Delta E - \sin \Delta E \sin \theta_0 / \cos \theta_0)$
$\overset{o}{=} \cos \theta_0 \, [1 - 10^{-6} N_0 (I - \rho L) \, \alpha] \overset{o}{=} \cos \theta_0$ since $I > \rho L$ and $\alpha I < 1$.

the parameters U_1 and U_2 are determined by substituting Equation (58) into Equation (56) with I in (56) given by Equation (26), expanding in powers of $1/\beta$, and equating coefficients of like powers. The results are

$$U_1 = \frac{1}{2}\left(1 + \frac{1}{2} q\right) \tag{59}$$

$$U_2 = \left(\frac{3}{4}\right)\left[\int_0^x x\, f\, dx + q\left(\frac{1}{3} + \frac{1}{2}\int_0^x f^2\, dx\right) + (1/6)\, q^2\right] \tag{60}$$

At small values of

$$U(\beta) = u_0 - u_1\,\beta + u_2\,\beta^2 - \ldots \tag{61}$$

To obtain u_0, β is set equal to zero in Equation (56) to obtain

$$u_0 = I\left(\frac{1}{2} q\, u_0\right) \tag{62}$$

which can be solved iteratively. The parameters u_1 and u_2 are obtained by differentiating Equation (56)

$$U'(\beta) = I'\left(\beta + \frac{1}{2} q\, u_0\right) \Big/ \left[1 - \frac{1}{2} q\, I'\left(\beta + \frac{1}{2} q\, u_0\right)\right] \tag{63}$$

whence

$$u_1 = -\, I'\left(\frac{1}{2} q\, u_0\right) \Big/ \left[1 - \frac{1}{2} q\, I'\left(\frac{1}{2} q\, u_0\right)\right] \tag{64}$$

Similarly, a second differentiation results in

$$u_2 = I''\left(\frac{1}{2} q\, u_0\right) \Big/ \left[1 - \frac{1}{2} q\, I'\left(\frac{1}{2} q\, u_0\right)\right]^3 \tag{65}$$

Using U_1, U_2, u_0, and u_1, $U(\beta)$ in Equation (52) is approximated by

$$F(\beta;\ U_1,\ U_2;\ u_0,\ u_1).$$

To calculate $V(\beta)$ Equation (51), Equation (63) is solved for $I'(\beta + 1/2\, q\, u_0)$ and substituted into Equation (57):

$$V = \left[1 - \beta U(\beta) - \frac{1}{4} q U^2(\beta)\right]\left[1 + \frac{1}{2} q U'(\beta)\right] \tag{66}$$

Here $U'(\beta)$ may be calculated using U_1, U_2, u_0, and u_1 as shown in Appendix 6.

5.2 Range Correction

Substitution of Equation (54) into (43) and expansion in powers of ρ gives (using Equation 5-1)

$$\Delta R/r_0 = \frac{1}{2} 10^{-6} N_0 \rho \left[M\left(\beta + \frac{1}{2} q U\right) + \frac{1}{2} Q \rho L^2 \left(\beta + \frac{1}{2} q U\right)\right] \tag{67}$$

which becomes

$$\Delta R/r_0 = \frac{1}{2} 10^{-6} N_0 p \left\{W(\beta) + \frac{1}{2} Q \rho \left[1 - \beta U(\beta) - \frac{1}{4} q U^2(\beta)\right]^2\right\} \tag{68}$$

where, using the asymptotic expansion of U in $M(\beta + 1/2\, q\, U)$ as given by Equation (46)

$$W(\beta) \sim (1/\beta) - W_1 (1/\beta)^3 + W_2 (1/\beta)^5 - \ldots \tag{69}$$

$$W_1 = \frac{1}{2}\left(\int_0^\infty x\, f\, dx + \frac{1}{2} q \int_0^\infty f^2\, dx\right) \tag{70}$$

$$W_2 = (3/4)\left[\frac{1}{2}\int_0^{\infty} x^2 f\, d x + (q/6)\left(1 + 3\int_0^{\infty} x f^2 d x\right) + \frac{1}{6} q^2 \int_0^{\infty} f^3 d x\right] \tag{71}$$

and where, setting $\beta = 0$

$$W(\beta) = w_0 - w_1 \beta + \ldots \tag{72}$$

$$w_0 = J\left(\frac{1}{2} q u_0\right) + q\left[u_0 - \frac{1}{2} K\left(\frac{1}{2} q u_0\right) - \frac{1}{6} q u_0^3\right] \tag{73}$$

$$w_1 = 2\left(1 - \frac{1}{4} q u_0^2\right) \tag{74}$$

6. EXAMPLE USING AN EXPONENTIAL PROFILE

If an exponential refractivity profile is assumed, the normalized profile f becomes

$$f(x) = e^{-x} \tag{75}$$

The effective height H is equal to the reciprocal of the decay constant of the profile and will be estimated from the refractivity at the station using the empirical formula [10]

$$1/H = \ln \frac{N_0}{N_0 - 7.32\, e^{0.005577 N_0}} \tag{76}$$

6.1 Known Arrival-Angle

Most of the required integration, that in Equations (28), (47) and (48), may be performed directly. Equation (3-5) or (31) and Equation (3-12) however, must be handled numerically. The simplest procedure is to evaluate these integrals numerically [12] at, perhaps, 20 different values of q spread over the expected range $0.1 < q < 0.7$. The values obtained may then be approximated by a polynomial using, since it is simple to apply, a least squares fit. One finds, for example, that a cubic polynomial in q approximates i_0 to better than 1/4 percent over the range 0 to 0.7. As it happened, however, an exponential expression gave a better approximation with fewer empirical constants:

$$i_0 = \sqrt{\pi}\,(1 - 0.9206\, q)^{-0.4468} \pm 0.04\% \quad 0 \le q \le 0.7 \tag{77}$$

The integral k_0 was also closely approximated by a similar exponential expression.

The equations for the corrections are collected in Appendix 7.

6.2 Known Elevation Angle

Here, also, the coefficients of the asymptotic expansions, Equations (60), (70), and (71), may be integrated. Equation (62) was solved numerically for u_0 at a number of values of q by Wegstein's method [13], using repeated numerical integration. The calculated values of u_0 were fitted by

$$u_0 = \sqrt{\pi}\,(1 + 1.4844\, q)^{-0.39144} \pm 0.02\% \; 0 \le q \le 0.7 \tag{78}$$

The coefficients u_1 and u_2 were also obtained numerically using the same calculated values of u_0 in Equations (64) and (65) with I' and I'' given by Equations (3-3) and (3-4).

The equations are collected in Appendix 8.

6.3 Numerical Examples and Comparisons

Let the refractivity at the tracking station be $N_0 = 313$. If the tracking station is at sea level, $r_0 = 6369.95$ kilometers. Performing the pre-pass calculations of Appendix 7, $H = 6.951$ km, $p = 0.04672$, $q = 0.2868$, and the equation for the elevation error is

$$\Delta E = 0.313 \cos \theta_0 \; [i - (6369.95/R)\, L] \qquad \text{mrad} \tag{79}$$

with

$$i = \cfrac{1}{\sin \theta_0 + \cfrac{0.0009348}{\sin \theta_0 + \cfrac{0.002117}{\sin \theta_0 + \cfrac{0.006054}{\sin \theta_0 + 0.1163}}}} \tag{80}$$

and

$$L = 1 - i \sin \theta_0 + 0.0001565\, i^2 \tag{81}$$

The equation for the range correction becomes

$$\Delta R = 0.002176 \; [m - (913.5/R)\, L^2 \cos^2 \theta_0] \qquad \text{km} \tag{82}$$

$$m = \cfrac{1}{\sin\theta_0 + \cfrac{0.0008565}{\sin\theta_0 + \cfrac{0.002173}{\sin\theta_0 + \cfrac{0.006082}{\sin\theta_0 + 0.1157}}}} \tag{83}$$

Using twelve values of θ_0 from 0 to 900 milliradians and two different values of the range R (in kilometers) at each value of θ_0 (these ranges are for satellites at heights with respect to the tracking station of 70 km and 475 km), the corrections in Table 1 were calculated. The corrections calculated using a double-precision ray trace program [14] are also listed for comparison.[6]

The largest difference is 0.3 percent. If desired, a final empirical adjustment of the coefficients can be made to reduce this error (Appendix 9). In Table 2 the corrections calculated from Appendix 8, using the values of elevation angle and range listed, are compared with those obtained from the ray-trace program. Since in this case the corrections do not hold for negative elevation angles, all but the smallest of these have been omitted from the table. The maximum error in Table 2, 0.9 percent, is larger than that found in Table 1, apparently because only two terms were retained in Equation (52).

6. These values will be found to differ somewhat from those in the CRPL ray-trace tables [15] which used single precision and slightly different values of r_0.

Table 1
Calculated Corrections, Arrival Angle Known

```
        PREPASSCALCULATIONS1
        TIME
0   0   2    (MINUTES:SECONDS:60THS)

        CORRECTIONS1
        TIME
0   0   2    (MINUTES:SECONDS:60THS)

        TABLE1
```

NO	THETA0 (MRAD)	R (KM)	ΔE (MRAD)	ΔE (MRAD) RAY TRACE	ΔR (KM)	ΔR (KM) RAY TRACE	PERCENT-ERROR ΔE	PERCENT-ERROR ΔR
313	0	1020.2	1.109E01	1.108E01	1.018E¯01	1.018E¯01	0.00	¯0.06
313	0	2587.1	1.262E01	1.262E01	1.038E¯01	1.038E¯01	0.00	¯0.06
313	1	1011.3	1.079E01	1.079E01	9.853E¯02	9.859E¯02	0.00	¯0.06
313	1	2578.2	1.227E01	1.227E01	1.004E¯01	1.004E¯01	0.00	¯0.06
313	2	1002.3	1.050E01	1.050E01	9.545E¯02	9.552E¯02	¯0.02	¯0.07
313	2	2569.5	1.193E01	1.194E01	9.714E¯02	9.721E¯02	¯0.01	¯0.07
313	4	985.7	9.968E00	9.972E00	8.977E¯02	8.986E¯02	¯0.04	¯0.10
313	4	2552.4	1.131E01	1.131E01	9.122E¯02	9.131E¯02	¯0.04	¯0.10
313	8	953.6	9.031E00	9.041E00	8.003E¯02	8.016E¯02	¯0.11	¯0.16
313	8	2519.6	1.022E01	1.023E01	8.111E¯02	8.124E¯02	¯0.10	¯0.16
313	15	901.8	7.719E00	7.736E00	6.689E¯02	6.705E¯02	¯0.22	¯0.24
313	15	2465.6	8.691E00	8.708E00	6.756E¯02	6.773E¯02	¯0.19	¯0.24
313	30	805.4	5.817E00	5.833E00	4.879E¯02	4.892E¯02	¯0.28	¯0.27
313	30	2360.3	6.498E00	6.513E00	4.908E¯02	4.921E¯02	¯0.22	¯0.27
313	65	633.5	3.589E00	3.594E00	2.900E¯02	2.904E¯02	¯0.13	¯0.15
313	65	2146.8	3.965E00	3.968E00	2.906E¯02	2.911E¯02	¯0.09	¯0.15
313	100	511.9	2.547E00	2.548E00	2.027E¯02	2.029E¯02	¯0.04	¯0.10
313	100	1962.4	2.798E00	2.799E00	2.029E¯02	2.032E¯02	¯0.02	¯0.10
313	200	316.8	1.350E00	1.350E00	1.073E¯02	1.073E¯02	0.02	¯0.06
313	200	1546.4	1.477E00	1.477E00	1.073E¯02	1.074E¯02	0.02	¯0.09
313	400	174.9	6.616E¯01	6.615E¯01	5.556E¯03	5.560E¯03	0.03	¯0.08
313	400	1046.4	7.234E¯01	7.233E¯01	5.556E¯03	5.561E¯03	0.02	¯0.09
313	900	89.1	2.234E¯01	2.233E¯01	2.774E¯03	2.776E¯03	0.03	¯0.08
313	900	593.8	2.443E¯01	2.443E¯01	2.774E¯03	2.776E¯03	0.02	¯0.09

RO USED WAS 6369.95

Table 2
Calculated Corrections, Elevation Angle Known

```
      PREPASSCALCULATIONS2
      TIME
0  0  4   (MINUTES:SECONDS:60THS)

      CORRECTIONS2
      TIME
0  0  1   (MINUTES:SECONDS:60THS)

      TABLE2
```

NO	E (MRAD)	R (KM)	ΔE (MRAD)	ΔE (MRAD) RAY TRACE	ΔR (KM)	ΔR (KM) RAY TRACE	PERCENT-ERROR ΔE	PERCENT-ERROR ΔR
313	¯1.04	953.6	9.124E00	9.041E00	8.003E¯02	8.016E¯02	0.92	¯0.17
313	¯2.23	2519.6	1.024E01	1.023E01	8.117E¯02	8.124E¯02	0.13	¯0.09
313	7.26	901.8	7.780E00	7.736E00	6.691E¯02	6.705E¯02	0.56	¯0.21
313	6.29	2465.6	8.710E00	8.708E00	6.763E¯02	6.773E¯02	0.02	¯0.14
313	24.17	805.4	5.835E00	5.833E00	4.876E¯02	4.892E¯02	0.04	¯0.33
313	23.49	2360.3	6.498E00	6.513E00	4.905E¯02	4.921E¯02	¯0.22	¯0.31
313	61.41	633.5	3.592E00	3.594E00	2.898E¯02	2.904E¯02	¯0.05	¯0.21
313	61.03	2146.8	3.964E00	3.968E00	2.904E¯02	2.911E¯02	¯0.11	¯0.21
313	97.45	511.9	2.548E00	2.548E00	2.027E¯02	2.029E¯02	0.00	¯0.14
313	97.20	1962.4	2.798E00	2.799E00	2.029E¯02	2.032E¯02	¯0.02	¯0.14
313	198.65	316.8	1.351E00	1.350E00	1.072E¯02	1.073E¯02	0.03	¯0.11
313	198.52	1546.4	1.477E00	1.477E00	1.073E¯02	1.074E¯02	0.02	¯0.12
313	399.34	174.9	6.617E¯01	6.615E¯01	5.554E¯03	5.560E¯03	0.03	¯0.11
313	399.28	1046.4	7.234E¯01	7.233E¯01	5.554E¯03	5.561E¯03	0.02	¯0.12
313	899.78	89.1	2.234E¯01	2.233E¯01	2.773E¯03	2.776E¯03	0.03	¯0.11
313	899.76	593.8	2.443E¯01	2.443E¯01	2.773E¯03	2.776E¯03	0.02	¯0.12

RO USED WAS 6369.95

Similar checks were made at $N_0 = 200$ and $N_0 = 450$ for both kinds of corrections with equivalent results.

The computer time required to perform the calculations is also shown in the tables. The pre-pass calculations in Table 1 took two sixtieths of a second (IBM 360/95). The two sixtieths of a second used to compute the twenty four pairs of corrections in Table 1 represents a rate of about 700 per second. Programs designed for operational use should show improvements over these times.

7. SUMMARY

A method is given for deriving atmospheric elevation-error and range-error correction equations in a form suitable for use in the processing of satellite tracking data. The method can be applied to any sufficiently-smooth spherically-symmetric model of the atmospheric refractivity. The method was tested by application to an exponential profile and comparison of the resulting corrections with those given by a double-precision ray trace program. The results were in agreement to better than one percent over the entire range of elevation angle (0 - 90°) and better than 0.3 percent over most of the range.

An inspection of the coefficients in the correction equations shows that the corrections at high elevations do not depend on the final detail of the refractivity profile N(h), but rather on the value of the surface refractivity and, in apparent order of importance, the height integrals $\int$ Ndh, $\int$ hNdh, and $\int N^2$dh, etc. The derivative N'(0) of the refractivity at the tracking station is also significant because it weighs heavily in the determination of both coefficients of the small-angle expansion of the bending integral. Where improved accuracy is required, emphasis should be placed on studies leading to better estimates for these significant quantities.

8. ACKNOWLEDGEMENT

The author wishes to acknowledge the help and co-operation of Dr. S. Rangaswamy, who supplied the ray-trace data used in Tables 1 and 2.

REFERENCES

1. Weisbrod, S. and Anderson, L., "Simple Methods for Computing Tropospheric and Ionospheric Refraction Effects on Radio Waves," Proc. IRE vol. 47, pp. 1770-1777; Oct. 1959.
2. Thayer, G., "A Formula for Radio Ray Refraction in an Exponential Atmosphere," Jour. of Res. of the N.B.S. - D. Radio Propagation Vol. 65, No. 2, March-April 1961, pp. 181-182.
3. Freeman, J., "Range-Error Compensation for a Troposphere With Exponentially Varying Refractivity," Jour. of Res. of the N.B.S. - D. Radio Propagation Vol. 66D, No. 6, Nov.-Dec. 1962, pp. 695-697.

4. Reichly, P., Jet Propulsion Laboratory Space Programs Summary 37-43 vol. IV, Feb. 28, 1967, pp. 314-321.

5. Hopfield, H., "Two-Quartic Tropospheric R ractivity Profile for Correcting Satellite Data," Jou . of Geophysical Research, Vol. 74, No. 18, August 20, 1969, pp. 4487-4499.

6. Rowlandson, L. and Moldt, R., "Derivation of Closed Functions to Compensate Range and Angle Errors in an Exponential Atmosphere," Radio Science, Vol. 4, No. 10, pp. 927-933, October 1969.

7. "Design Evaluation Report, Goddard Range and Range Rate System," General Dynamics Electronics Division Report R-67-042, December 13, 1967. Prepared for Goddard Space Flight Center under contract NAS 5-10555.

8. Berbert, J. and Parker, H., "GEOS Satellite Tracking Corrections for Refraction in the Troposphere," Goddard Space Flight Center Report X-514-70-55, Feb. 1970.

9. Bean, B. and Thayer, G., "Models of the Atmospheric Radio Refractive Index," Proc. IRE vol. 47, May 1959, pp. 740-755. See also Bean, B., and Dutton E., "Radio Meteorology," NBS Monograph 92, March 1, 1966.

10. Wall, H. S., "Analytic Theory of Continued Fractions," D. Van Nostrand Company, Inc., New York, 1948; p. 362.

11. Saxton, J. (ed.), "Advances in Radio Research," Vol. I, Academic Press, N.Y. 1964, p. 65.

12 Krylov, V. I., "Approximate Calculation of Integrals," McMillan, N.Y., 1962.

13. Lance, G. C., "Numerical Methods for High Speed Computers," Iliffe, London, 1960, pp. 134-138.

14. Developed by S. Rangaswamy, National Academy of Science Research Associate, Goddard Space Flight Center.

15. Bean, B. and Thayer, G., "CRPL Exponential Reference Atmosphere," NBS Monograph 4, October 29, 1959.

16. APL/360 Users Manual, International Business Machines Corporation, 1968.

APPENDIX 1

EXPRESSION FOR δ IN TERMS OF τ

The equation for the angle δ follows from the geometry of Figure 2, and from Snell's law for a spherically stratified medium. In Figure 2 r_1, r_0, and R are all projected on a line perpendicular to the tangent to the ray path at the satellite. It follows that

$$r_1 \cos \theta_1 = r_0 \cos (\theta_0 - \tau) + R \sin \delta \tag{1-1}$$

Snell's law for a spherically stratified medium is

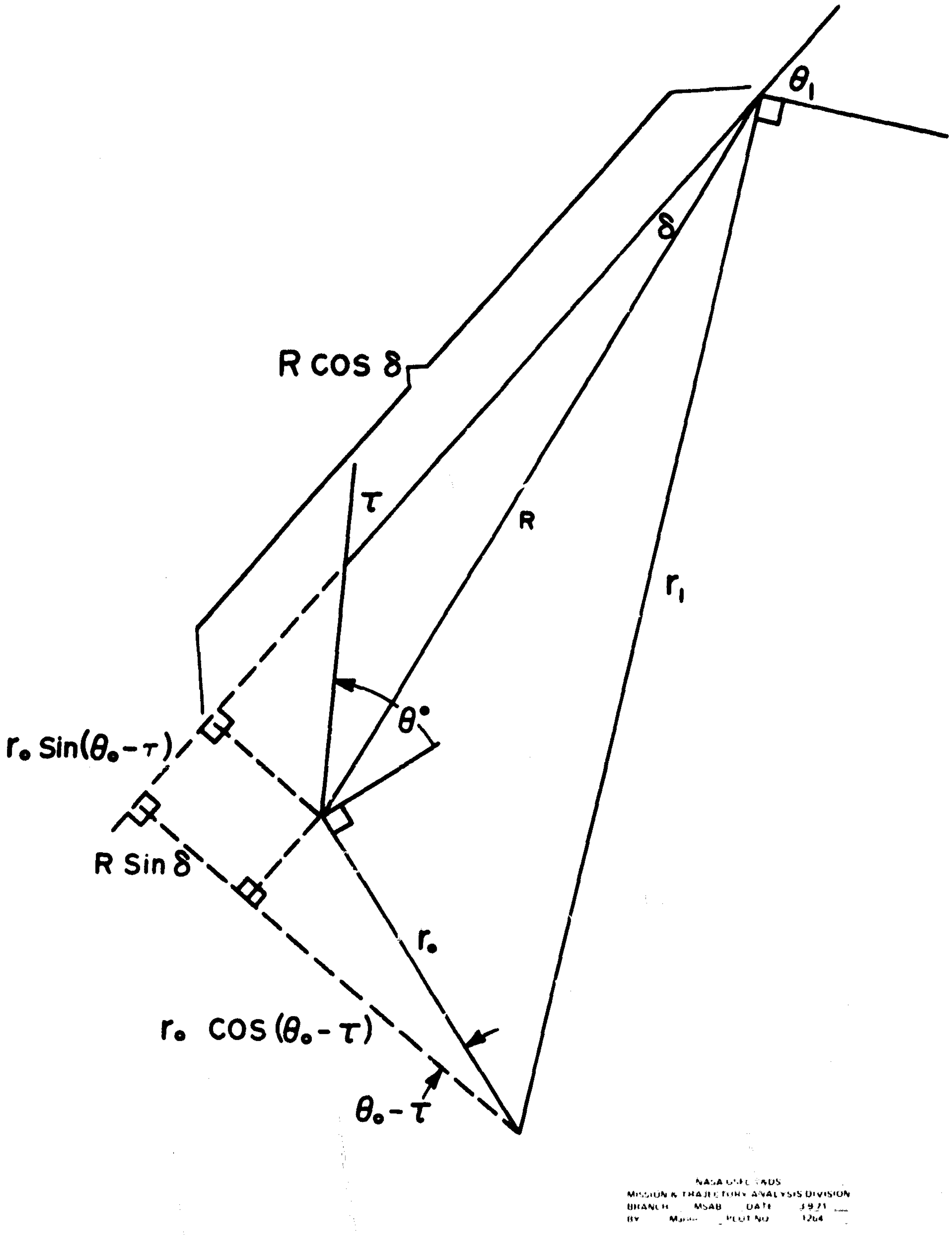
$$n\ r \cos \theta = n_0\ r_0 \cos \theta_0 \tag{1-2}$$

At the high altitude of the satellite n is equal to unity, $r = r_1$, and $\theta = \theta_1$. Combining Equations (1-1) and (1-2) gives

$$\sin \delta = (r_0/R)\ [n_0 \cos \theta_0 - \cos (\theta_0 - \tau)] \tag{1-3}$$

Approximating $\sin \delta$ by δ, $\sin \tau$ by τ, and $\cos \tau$ by $1 - 1/2\ \tau^2$,

$$\delta \stackrel{\circ}{=} (r_0/R) \left[\left(10^{-6}\ N_0 + \frac{1}{2}\tau^2\right) \cos \theta_0 - \tau \sin \theta_0\right] \tag{1-4}$$



MISSION & TRAJECTORY ANALYSIS DIVISION
BRANCH MSAB DATE
BY PLOT NO

Figure 2. Projection of R, r_0, and r_1

APPENDIX 2

ASYMPTOTIC EXPANSION OF I, J, AND K

Setting Q = q in Equation (22), factoring α out of the radical, and expanding the latter by the binomial theorem

$$I(\alpha) \sim (1/\alpha) \int_0^\infty - f' \Big\{ 1 - \frac{1}{2} [x - q(1 - f)] (1/\alpha)^2$$

$$+ (3/8) [x - q(1 - f)]^2 (1/\alpha)^4 - \ldots \Big\} dx \qquad (2\text{-}1)$$

The terms in (2-1) that do not contain x as a factor may be integrated directly. Those that do contain x can be integrated by parts on the factor containing f' and powers of f. The result of these integrations is given in Equations (26), (27), and (28). (The same result can be obtained by repeatedly integrating the numerators starting with Equation (22), using the definition

$$f^{(-1)} = - \int_x^\infty f \, dx$$

and assuming that $f^{(-1)}$ vanishes with sufficient rapidity at large values of x).

The integral $J(\alpha)$ Equation (42) may be expanded in the same way with the result

$$J(\alpha) \sim (1/\alpha) - J_1 (1/\alpha)^3 + J_2 (1/\alpha)^5 - \ldots \qquad (2\text{-}3)$$

$$J_1 = \frac{1}{2}\left[\int_0^\infty x\, f\, d\, x - q\left(1 - \int_0^\infty f^2\, d\, x\right)\right] \tag{2-4}$$

$$J_2 = (3/4)\left[\frac{1}{2}\int_0^\infty x^2\, f\, d\, x - q\left(\int_0^\infty x\, f\, d\, x - \int_0^\infty x\, f^2\, d\, x\right)\right.$$

$$\left. + q^2\left(\frac{1}{2} - \int_0^\infty f^2\, d\, x + \frac{1}{2}\int_0^\infty f^3\, d\, x\right)\right] \tag{2-5}$$

Similarly Equation (45) yields

$$K(\alpha) \sim (1/\alpha) - K_1\,(1/\alpha)^3 + K_2\,(1/\alpha)^5 - \cdots \tag{2-6}$$

$$K_1 = \frac{1}{2}\int_0^\infty f^2\, d\, x - (1/6)\, q \tag{2-7}$$

$$K_2 = (3/4)\left\{\int_0^\infty x\, f^2\, d\, x - q\left[\int_0^\infty f^2\, d\, x - (2/3)\int_0^\infty f^3\, d\, x\right] + (1/12)\, q^2\right\} \tag{2-8}$$

APPENDIX 3

INTEGRALS FOR I, J AND K, AND EXPANSIONS FOR SMALL VALUES OF α

The integral Equation (22) can be integrated by parts. Multiplying the numerator and denominator of the integrand by $1 + qf'$ and integrating the radical by parts

$$I(\alpha) = \frac{2 f'(0)}{1 + q f'(0)} \alpha + 2 \int_0^\infty \sqrt{x + \alpha^2 - q(1 - f)} \frac{f''}{(1 + q f')^2} dx \tag{3-1}$$

The first term may be absorbed into the integral

$$I(\alpha) = 2 \int_0^\infty [\sqrt{x + \alpha^2 - q(1 - f)} - \alpha] \frac{f''}{(1 + q f')^2} dx \tag{3-2}$$

Differentiating Equation (3-2)

$$I'(\alpha) = -2 \int_0^\infty \left[1 - \frac{\alpha}{\sqrt{x + \alpha^2 - q(1 - f)}}\right] \frac{f''}{(1 + q f')^2} dx \tag{3-3}$$

Differentiating Equation (3-3)

$$I''(\alpha) = 2 \int_0^\infty \frac{x - q(1 - f)}{[x + \alpha^2 - q(1 - f)]^{3/2}} \frac{f''}{(1 + q f')^2} dx \tag{3-4}$$

Equation (31) for I(0) is obtained by sitting α equal to zero in Equation (3-2). The integral

$$I(0) = \int_0^\infty \frac{-f'}{\sqrt{x - q(1-f)}} \, dx \tag{3-5}$$

obtained directly from Equation (22) can also be used, but more care must then be exercised since the integrand diverges at x = 0.

Equation (32) follows from Equation (3-3) by setting $\alpha = 0$ and integrating. In the same way, from Equation (42)

$$J(\alpha) = -2 \int_0^\infty [\sqrt{x + \alpha^2 - q(1-f)} - \alpha] \frac{f' + q(f'^2 - f f'')}{(1 + q f')^2} \, dx \tag{3-6}$$

$$J(\alpha) = j_0 - j_1 \alpha + \cdots \tag{3-7}$$

with

$$j_0 = J(0) = -2 \int_0^\infty \sqrt{x - q(1-f)} \frac{f' + q(f'^2 - f f'')}{(1 + q f')^2} \, dx \tag{3-8}$$

$$j_1 = -2/[1 + q f'(0)] = i_1/-f'(0) \tag{3-9}$$

Similarly,

$$K(\alpha) = 4 \int_0^\infty [\sqrt{x + \alpha^2 - q(1-f)} - \alpha] \frac{f f'' + f'^2 + q f'^3}{(1 + qf')^2} \, dx \tag{3-10}$$

and setting

$$K(\alpha) = k_0 - k_1 \alpha + \ldots \tag{3-11}$$

$$k_0 = 4 \int_0^\infty \sqrt{x - q(1 - f)}\, \frac{f f'' + f'^2 + q f'^3}{(1 + q f')^2}\, dx \tag{3-12}$$

$$k_1 = -4 f'(0)/[1 + q f'(0)] = 2 i_1 \tag{3-13}$$

APPENDIX 4

APPROXIMATION FOR THE GEOMETRICAL DIFFERENCE

Using Equation (1-2), R_g equation (12) may be written as

$$R_g = \int_{r_0}^{r_1} \frac{(r + n_0^2 r_0^2 \cos^2 \theta_0 n'/n^3) - n_0^2 r_0^2 \cos^2 \theta_0 n'/n^3}{\sqrt{r^2 - (n_0^2 r_0^2 \cos^2 \theta_0)/n^2}} d r \tag{4-1}$$

The left-hand term in the integrand can be integrated. On setting $n(r_1) = 1$ and making use of the geometrical relationship

$$r_1 \sin \theta_1 = R \cos \delta + r_0 \sin (\theta_0 - \tau) \tag{4-2}$$

which follows from Figure 2, and using the exact integral for τ [9]

$$\tau = - r_0 \cos \theta_0 \int_0^{\infty} \frac{n_0 n'}{n^2 \sqrt{r^2 - n_0^2 r_0^2 \cos^2 \theta_0 / n^2}} d r \tag{4-3}$$

there results

$$(R_g - R)/r_0 = - (R/r_0) (1 - \cos \delta) - (1 - \cos \tau) \sin \theta_0$$

$$+ (\tau - \sin \tau) \cos \theta_0 + \cos^2 \theta_0 \int_0^{\infty} \frac{n_0 (n - n_0) n'}{n^3 \sqrt{r^2 - \dfrac{n_0^2 r_0^2 \cos^2 \theta_0}{n^2}}} d r \tag{4-4}$$

On setting

$$\cos \delta \doteq 1 - \frac{1}{2}\delta^2 \tag{4-5}$$

$$\sin \tau \doteq \tau - \tau^3/6 \tag{4-6}$$

$$\cos \tau \doteq 1 - \frac{1}{2}\tau^2 \tag{4-7}$$

and making the usual approximations in the integrals, there results

$$\begin{aligned}(R_g - R)/r_0 = \frac{1}{2}\,10^{-6}\,N_0\,p\,O\Bigg[&\left(I - \frac{1}{2}K - \frac{1}{2}\alpha\,I^2 + q\,I^3/12\right) \\ &- (r_0\,p/2\,R)\left(1 - \alpha\,I + \frac{1}{4}q\,I^2\right)^2\Bigg]\end{aligned} \tag{4-8}$$

APPENDIX 5

RANGE-RATE CORRECTION

The range-rate correction is the difference between $n(r_1)$ times the projection of the satellite velocity on the ray path at the satellite and its projection on the straight line joining the satellite and the ground station.

It may be found by differentiating Equation (44) with respect to time. Using primes to denote differentiation with respect to α, and the dot notation to denote differentiation with respect to time, and making use of the relation (obtained by performing the indicated differentiation and then integrating by parts)

$$J'(\alpha) - \frac{1}{2} q K'(\alpha) = -2 \left[1 - \alpha I(\alpha)\right] \tag{5-1}$$

one obtains

$$\begin{aligned} \Delta \dot{R}/r_0 = (\dot{R}_e - \dot{R})/r_0 = -\dot{\theta}_0 10^{-6} N_0 \cos \theta_0 L(\alpha) \Big\{ 1 - \frac{1}{2} q I'(\alpha) \\ - \rho \frac{1}{2} q \left[I(\alpha) + \alpha I'(\alpha) - \frac{1}{2} q I(\alpha) I'(\alpha) + \frac{1}{2} \delta^2 \dot{R}/r_0 \right] \Big\} \end{aligned} \tag{5-2}$$

APPENDIX 6

CALCULATION OF $U'(\beta)$

From Equations (59) and (60)

$$U' \sim -\left[(1/\beta)^2 - 3\,U_1\,(1/\beta)^4 + 5\,U_2\,(1/\beta)^6 - \dots\right] \tag{6-1}$$

$$U' \sim -\left[u_1 - 2\,u_2\,\beta + \dots\right] \tag{6-2}$$

The calculation of U' is accomplished by means of the form

$$G(\beta) = \cfrac{1}{\beta^2 + \cfrac{g_1}{1 + \cfrac{g_2}{\beta^2 + g_4\,\beta + g_3}}} \tag{6-3}$$

where

$$g_1 = G_1 \tag{6-4}$$

$$g_2 = (G_2/g_1) - g_1 \tag{6-5}$$

$$g_3 = g_2/(g_1\,g_0 - 1) \tag{6-6}$$

$$g_4 = g_3^2\,g_1\,g_1/g_2 \tag{6-7}$$

whence

$$G = (1/\beta)^2 - G_1\,(1/\beta)^4 + G_2\,(1/\beta)^6 - \text{const.}/\beta^7 \dots \tag{6-8}$$

and

$$G = g_0 - g_1 \beta + \text{constant} \ \beta^2 + \ldots \tag{6-9}$$

Setting $G_1 = 3U_1$, $G_2 = 5U_2$, $g_0 = u_1$, and $g_1 = 2u_2$, U' will be approximately equal to minus G.

APPENDIX 7

CORRECTION EQUATIONS FOR AN EXPONENTIAL PROFILE, ARRIVAL ANGLE KNOWN

$$H = 1/\ln \frac{N_0}{N_0 - 7.32\, e^{0.005577\, N_0}} \qquad \text{km}$$

$$p = \sqrt{2H/r_0}$$

$$q = 10^{-6}\, N_0\, r_0/H$$

$$i_0 = \sqrt{\pi}\,(1 - 0.9206\, q)^{-0.4468}$$

$$i_1 = 2/(1 - q)$$

$$I_1 = \frac{1}{2}\left(1 - \frac{1}{2} q\right)$$

$$I_2 = 0.75\,[1 - 0.75\, q + (1/6) q^2]$$

$$k_0 = \sqrt{2\pi}\,(1 - 0.9408\, q)^{-0.4759}$$

$$m_0 = i_0 \left(1 + q + \frac{1}{12} q^2\, i_0^2\right) - \frac{1}{2} q\, k_0$$

$$m_1 = 2\left(1 + \frac{1}{4} q\, i_0^2\right) \Big/ (1 - q)$$

$$M_1 = \frac{1}{2}\left(1 - \frac{3}{4} q\right)$$

$$M_2 = 0.75\left(1 - \frac{25}{24} q + \frac{11}{36} q^2\right)$$

$$i = F(\sin\theta_0;\ p^2 I_1,\ p^4 I_2;\ i_0/p,\ i_1/p^2$$

$$L = 1 - i \sin\theta_0 + \frac{1}{2} 10^{-6}\, N_0\, i^2$$

$$\Delta E = 0.001\, N_0 \cos \theta_0 \,(i - r_0 L/R) \qquad \text{mrad}$$

$$m = F(\sin \theta_0 ;\ p^2 M_1,\ p^4 M_2;\ m_0/p,\ m_1/p^2)$$

$$\Delta R = 10^{-6} N_0\, H \left(m - \frac{1}{2}\, 10^{-6} N_0\, r_0^2 \cos^2 \theta_0\, L^2/RH \right) \qquad \text{km}$$

The function i above is equal to $I(\alpha)/p$, and the intermediate constant, Equations (34-37), associated with i can be obtained from those of $I(\alpha)$ by multiplying the first three by p^2 and the last by p.

The programs used to perform the calculations were written in APL language [16], and are shown in Figure 3.

```
      ∇PREPASSCALCULATIONS1[⎕]∇
    ∇ PREPASSCALCULATIONS1
[1]   HH←÷⍟NN0÷NN0-7.32×*0.005577×NN0
[2]   PSQUARED←2×HH÷R0
[3]   P←PSQUARED*0.5
[4]   Q←1E¯6×NN0×R0÷HH
[5]   F0←I0←((○1)*0.5)×(1-0.9206×Q)*¯0.4468
[6]   F1←I1←2÷1-Q
[7]   FF1←II1←0.5×1-0.5×Q
[8]   FF2←II2←0.75×1+Q×¯0.75+Q÷6
[9]   CALCULATEF
[10]  I1←PSQUARED×F1
[11]  I2←PSQUARED×F2
[12]  I3←PSQUARED×F3
[13]  I4←P×F4
[14]  K0←((○2)*0.5)×(1-0.9408×Q)*¯0.4759
[15]  F0←M0←(I0×1+Q×1+Q×I0×I0÷12)-Q×K0÷2
[16]  F1←M1←2×(1+Q×I0×I0÷4)÷1-Q
[17]  FF1←MM1←0.5×1-0.75×Q
[18]  FF2←MM2←0.75×1+Q×(¯25÷24)+11×Q÷36
[19]  CALCULATEF
[20]  M1←PSQUARED×F1
[21]  M2←PSQUARED×F2
[22]  M3←PSQUARED×F3
[23]  M4←P×F4
    ∇

      ∇CORRECTIONS1[⎕]∇
    ∇ CORRECTIONS1
[1]   SIN←1○THETA0×0.001
[2]   COS←2○THETA0×0.001
[3]   I←÷SIN+I1÷SIN+I2÷SIN+I3÷SIN+I4
[4]   LL←1-I×SIN-0.5×1E¯6×NN0×I
[5]   DEE←0.001×NN0×COS×I-R0×LL÷RR
[6]   M←÷SIN+M1÷SIN+M2÷SIN+M3÷SIN+M4
[7]   DRR←1E¯6×NN0×HH×M-0.5×1E¯6×NN0×R0×R0×COS×COS×LL×LL÷RR×HH
    ∇

      ∇CALCULATEF[⎕]∇
    ∇ CALCULATEF
[1]   F1←FF1
[2]   F2←(FF2÷F1)-F1
[3]   F3←F2÷(F0×F0×F1×1+F1÷F2)-1+F1×F1
[4]   F4←F0×F1×F3÷F2
    ∇
```

Figure 3. APL Programs for Corrections, Arrival Angle Known.

APPENDIX 8

CORRECTION EQUATIONS FOR AN EXPONENTIAL PROFILE, ELEVATION ANGLE KNOWN

H, p, and q are calculated as in Appendix 7.

$$I'\left(\frac{1}{2} q u_0\right) = -2(1 + 1.482 q)^{-0.3826}$$

$$I''\left(\frac{1}{2} q u_0\right) = 2\sqrt{\pi}(1 + 1.71 q)^{0.1}$$

$$u_0 = \sqrt{\pi}(1 + 1.4844 q)^{-0.39144}$$

$$u_1 = -I'\left(\frac{1}{2} q u_0\right) \Big/ \left[1 - \frac{1}{2} q I'\left(\frac{1}{2} q u_0\right)\right]$$

$$u_2 = I''\left(\frac{1}{2} q u_0\right) \Big/ \left[1 - \frac{1}{2} q I'\left(\frac{1}{2} q u_0\right)\right]^3$$

$$U_1 = \frac{1}{2}\left(1 + \frac{1}{2} q\right)$$

$$U_2 = \frac{3}{4}\left(1 + \frac{7}{12} q + \frac{1}{6} q^2\right)$$

$$K\left(\frac{1}{2} q u_0\right) = \sqrt{2\pi}(1 + 1.6454 q)^{-0.583}$$

$$w_0 = u_0\left(1 + q - \frac{1}{6} q u_0^2\right) - \frac{1}{2} q K\left(\frac{1}{2} q u_0\right)$$

$$w_1 = 2\left(1 - \frac{1}{4} q u_0^2\right)$$

$$W_1 = \frac{1}{2}\left(1 + \frac{1}{4} q\right)$$

$$W_2 = \frac{3}{4}\left(1 + \frac{7}{24} q + \frac{1}{18} q^2\right)$$

$$u = F(\sin E;\ p^2 U_1,\ p^4 U_2;\ u_0/p,\ u_1/p^2)$$

$$u' = -G(\sin E;\ 3p^2 U_1,\ 5p^4 U_2;\ u_1/p^2,\ 2u_2/p^3)$$

$$v = \left(1 - u \sin E - \frac{1}{2}\,10^{-6} N_0 u^2\right)(1 + 10^{-6} N_0 u')$$

$$\Delta E = 0.001\, N_0 \cos E\,(u - v\, r_0/R) \qquad \text{mrad}$$

$$w = \mathbb{F}(\sin e;\ p^2 W_1,\ p^4 W_2;\ w_0/p,\ w_1/p^2)$$

$$\Delta R = 10^{-6} N_0 H \left[w - \frac{1}{2}\,10^{-6} N_0 \frac{r_0^2}{RH} \cos^2 E \left(1 - u \sin E - \frac{1}{2}\,10^{-6} N_0 u^2\right)^2\right] \qquad \text{km}$$

The corresponding APL program is shown in Figure 4.

```
      ∇PREPASSCALCULATIONS2[⎕]∇
    ∇ PREPASSCALCULATIONS2
[1]   HH←÷⍟NN0÷NN0-7.32××0.005577×PP0
[2]   PSQUARED←2×HH÷R0
[3]   P←PSQUARED*0.5
[4]   Q←1E¯6×NN0×R0÷HH
[5]   IPU←¯2×(1+1.482×Q)*¯0.3826
[6]   DEN←1-0.5×Q×IPU
[7]   IPPU←2×((○1)*0.5)×(1+1.71×Q)*0.1
[8]   F0←U0←((○1)*0.5)×(1+1.4844×Q)*¯0.39144
[9]   F1←U1←-IPU÷DEN
[10]  FF1←UU1←0.5×1+0.5×Q
[11]  FF2←UU2←0.75×1+Q×(7÷12)+Q÷6
[12]  CALCULATEE
[13]  U1←PSQUARED×E1
[14]  U2←PSQUARED×E2
[15]  U3←PSQUARED×E3
[16]  U4←P×E4
[17]  G0←UP0←U1
[18]  G1←UP1←U2←IPPU÷DEN*3
[19]  GG1←UUP1←3×UU1
[20]  GG2←UUP2←5×UU2
[21]  CALCULATEG
[22]  UP1←PSQUARED×G1
[23]  UP2←PSQUARED×G2
[24]  UP3←PSQUARED×G3
[25]  UP4←P×G4
[26]  KU←((○2)*0.5)×(1+1.6454×Q)*¯0.583
[27]  F0←W0←(U0×1+Q×1-Q×U0×U0÷6)-0.5×Q×KU
[28]  F1←W1←2×1-0.25×Q×U0×U0
[29]  FF1←WW1←0.5×1+0.25×Q
[30]  FF2←WW2←0.75×1+Q×(7÷24)+Q÷18
[31]  CALCULATEE
[32]  W1←PSQUARED×E1
[33]  W2←PSQUARED×E2
[34]  W3←PSQUARED×E3
[35]  W4←P×E4
    ∇

      ∇CORRECTIONS2[⎕]∇
    ∇ CORRECTIONS2
[1]   SINE←1○EE×0.001
[2]   COSE←2○EE×0.001
[3]   U←÷SINE+U1÷SINE+U2÷SINE+U3÷SINE+U4
[4]   UP←-÷(SINE×SINE)+UP1÷1+UP2÷UP3+SINE×UP4+SINE
[5]   PAREN←1-U×SINE+0.5×1E¯6×NN0×U
[6]   DEE←0.001×NN0×COSE×U-PAREN×(1+1E¯6×NN0×UP)×R0÷RP
[7]   W←÷SINE+W1÷SINE+W2÷SINE+W3÷SINE+W4
[8]   DRR←1E¯6×NN0×HH×W+0.5×1E¯6×NN0×(÷HH×RP)×(R0×COSE×PAREN)*2
    ∇

      ∇CALCULATEG[⎕]∇
    ∇ CALCULATEG
[1]   G1←GG1
[2]   G2←(GG2÷G1)-G1
[3]   G3←G2÷¯1+G1×G0
[4]   G4←G3×G3×G1×G1÷G2
    ∇
```

Figure 4. APL Programs for Corrections, Elevation Angle Known

APPENDIX 9

IMPROVED FIT

The correction equations given in Appendix 7 can easily be modified to give a closer fit to ray-trace calculations. The method chosen was to modify the coefficients $\underline{f}_2$, $\underline{f}_3$, and $\underline{f}_4$ using a least squares adjustment that brought $F(\alpha; 1/2, 3/4; \sqrt{\pi}, 2)$, (which is the approximation for $I(\alpha)$ where $q = 0$) into better agreement with its theoretical value $\sqrt{\pi}\ e^{\alpha^2}$ erfc α. The factors by which $\underline{f}_2$, $\underline{f}_3$, and $\underline{f}_4$ are multiplied are 1.08885, 1.320903, and 1.213133 respectively. The resulting corrections are shown in Table 3.

Table 3
Calculated Corrections, F Modified

```
      ∇CALCULATEE[⎕]∇
    ∇ CALCULATEE
[1]   E1←FF1
[2]   E2←(FF2÷E1)-E1
[3]   E3←E2÷(F0×F0×E1×1+E1÷E2)-1+F1×E1
[4]   E4←F0×E1×E3÷E2
[5]   E2←E2×1.08685
[6]   E3←E3×1.32093
[7]   E4←E4×1.21313
    ∇

      PREPASSCALCULATIONS1
      CORRECTIONS1

      TABLE1
```

NO	THETA0 (MRAD)	R (KM)	ΔE (MRAD)	ΔE (MRAD) RAY TRACE	ΔR (KM)	ΔR (KM) RAY TRACE	PERCENT-ERROR ΔE	PERCENT-ERROR ΔR
313	0	1020.2	1.109E01	1.108E01	1.018E¯01	1.018E¯01	0.06	¯0.06
313	0	2587.1	1.262E01	1.262E01	1.038E¯01	1.038E¯01	0.00	¯0.06
313	1	1011.3	1.079E01	1.079E01	9.853E¯02	9.859E¯02	0.01	¯0.06
313	1	2578.2	1.227E01	1.227E01	1.004E¯01	1.004E¯01	0.01	¯0.06
313	2	1002.1	1.050E01	1.050E01	9.546E¯02	9.552E¯02	0.00	¯0.06
313	2	2569.5	1.194E01	1.194E01	9.715E¯02	9.721E¯02	0.01	¯0.06
313	4	985.7	9.973E00	9.972E00	8.981E¯02	8.986E¯02	0.01	¯0.06
313	4	2552.4	1.132E01	1.131E01	9.125E¯02	9.131E¯02	0.01	¯0.06
313	8	953.6	9.042E00	9.041E00	8.011E¯02	8.016E¯02	0.00	¯0.07
313	8	2519.6	1.023E01	1.023E01	8.118E¯02	8.124E¯02	0.00	¯0.07
313	15	901.8	7.736E00	7.736E00	6.700E¯02	6.705E¯02	0.00	¯0.08
313	15	2465.6	8.708E00	8.708E00	6.768E¯02	6.773E¯02	0.00	¯0.08
313	30	805.4	5.834E00	5.833E00	4.889E¯02	4.892E¯02	0.02	¯0.06
313	30	2360.3	6.514E00	6.513E00	4.918E¯02	4.921E¯02	0.02	¯0.06
313	65	633.5	3.596E00	3.594E00	2.903E¯02	2.904E¯02	0.07	¯0.05
313	65	2146.8	3.971E00	3.968E00	2.909E¯02	2.911E¯02	0.05	¯0.05
313	100	511.9	2.550E00	2.548E00	2.028E¯02	2.029E¯02	0.08	¯0.06
313	100	1962.4	2.800E00	2.799E00	2.030E¯02	2.032E¯02	0.05	¯0.06
313	200	316.8	1.351E00	1.350E00	1.073E¯02	1.073E¯02	0.08	¯0.08
313	200	1546.4	1.477E00	1.477E00	1.073E¯02	1.074E¯02	0.03	¯0.08
313	400	174.9	6.617E¯01	6.615E¯01	5.556E¯03	5.560E¯03	0.04	¯0.08
313	400	1046.4	7.235E¯01	7.233E¯01	5.556E¯03	5.561E¯03	0.02	¯0.09
313	900	89.1	2.234E¯01	2.233E¯01	2.774E¯03	2.776E¯03	0.03	¯0.08
313	900	593.8	2.443E¯01	2.443E¯01	2.774E¯03	2.776E¯03	0.02	¯0.09

RO USED WAS 6369.95