NASA TECHNICAL NOTE

NASA TN D-6326

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NASA TN D-6326

FINITE-ELEMENT MODELING STUDIES IN THE NORMAL-MODE METHOD AND NORMAL-MODE SYNTHESIS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1971

1. Report No. NASA TN D-6326	2. Government Accession No.
3. Recipient's Catalog No.	
4. Title and Subtitle: Finite-Element Modeling Studies in the Normal-Mode Method and Normal-Mode Synthesis	5. Report Date: May 1971
	6. Performing Organization Code
7. Author(s): Roy Leon Courtney	8. Performing Organization Report No. G-1009
9. Performing Organization Name and Address: Goddard Space Flight Center, Greenbelt, Maryland 20771	10. Work Unit No.
	11. Contract or Grant No.
12. Sponsoring Agency Name and Address: National Aeronautics and Space Administration, Washington, D.C. 20546	13. Type of Report and Period Covered: Technical Note
	14. Sponsoring Agency Code

15. Supplementary Notes

The information presented herein was submitted as a thesis in partial fulfillment of the requirements for the degree of Master of Science in Engineering, The George Washington University, Washington, D. C., August 1970.

16. Abstract

This report compares the results of finite-element and analytic solutions of the normal modes of a uniform beam under various end conditions. The normal modes of a single component are synthesized to represent different single-component and multi-component structures. The results are compared for accuracy with the analytic structural solutions. Guidelines for finite-element structural modeling are developed which determine the minimum number of elements required to obtain specified levels of accuracy in normal-mode analysis and synthesis. The finite-element modeling studies are implemented on NASA's NASTRAN structural analysis computer program. Modeling techniques that represent a part of a structure by its normal modes in a computer structural analysis are presented.

17. Key Words Suggested by Author: Finite-Element Modeling; Normal-Mode Analysis; Structural Analysis	18. Distribution Statement: Unclassified–Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	46	3.00

*For sale by the National Technical Information Service, Springfield, Virginia 22151

FOREWORD

It is the policy of the National Aeronautics and Space Administration to employ, in all formal publications, the international metric units known collectively as the Systême Internationale d'Unités and designated SI in all languages. In certain cases, however, utility requires that other systems of units be retained in addition to the SI units.

This document contains data so expressed because the use of the SI equivalents alone would impair communication. The non-SI units, given in parentheses following their computed SI equivalents, are the basis of the measurements and calculations reported here.

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FINITE-ELEMENT MODELING STUDIES IN THE NORMAL-MODE METHOD AND NORMAL-MODE SYNTHESIS*

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CHAPTER 1

INTRODUCTION

1.1 Problem Statement

The normal-mode method of structural analysis has been developed into a powerful tool in the analysis of large and complex aerospace structures. The value of this method lies in the fact that the simultaneous differential equations of motion which describe the linear dynamic characteristic of a structure are decoupled into independent differential equations when the displacements are expressed in terms of the normal modes. A normal or natural mode of vibration occurs when each point in the structure executes harmonic motion about a point of static equilibrium, with every point passing through its equilibrium position at the same instant and reaching its maximum displacement at the same instant. The form of the displacement of a structure is known as the normal-mode shape, and the frequency of the harmonic motion is known as the normal-mode frequency. Additionally, normal-mode shapes and frequencies are used to compute the modal properties of generalized mass, stiffness, and structural damping associated with the modes.

There are basically two numerical methods of approach to the problem of determining the normal modes of structural systems: An exact problem formulation solved approximately and an approximate formulation solved exactly. For certain simple systems the exact solution for normal modes can be obtained by analytical methods. When it is impossible or very difficult to obtain an exact solution of the partial differential equations governing a vibrating system, the system may be redefined in a discrete form. The structure is approximated by an assembly of discrete structural elements having an assumed distribution of stress and displacement. The complete solution is obtained by combining these individual, approximate stress and displacement distributions in a manner that satisfies the force

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equilibrium and displacement compatibility at the connection between elements. Methods based on the discrete or finite-element approximations involve appreciable quantities of linear algebra that must be organized into a systematic sequence of operations, which are handled most conveniently by the use of matrix algebra. The formulation of a specific method of analysis in matrix algebra is ideally suited for solution on the digital computer, which can be programmed for systematic compilation of data and for execution of required operations.

As mentioned above, normal-mode analysis of structural systems has been the subject of a great deal of investigation, especially in the aerospace field, where the dynamic response of efficient, lightweight structures is of paramount interest. The National Aeronautics and Space Administration (NASA) prepared a general review and useful bibliography of normal-mode analysis.[1] Application of this method of analysis to space vehicles and a bibliography of both analytical and applied references are presented by NASA.[2,3,4]

Finite-element analysis of complex structures requires large numbers of coordinates in the model, resulting in equations so large as to overwhelm the best of analysts and computers. The technique of normal-mode synthesis is a process whereby the dynamic characteristics of the several components of the system are calculated separately and then brought together to evaluate the dynamic characteristics of the entire system, thereby increasing computer efficiency.

Another major advantage of the normal-mode-synthesis technique is that the modal data may come from such diverse sources as analytic solutions, finite-element analyses, and vibration tests designed to determine normal modes. Thus, one structural analysis using modal synthesis can incorporate the representation of individual substructures analyzed by completely different techniques by separate engineering groups that may never have conferred about the complete structure.

The concept of modal synthesis is founded on the principle that a substructure is completely (or adequately) represented by its primary modes; if this is true, then the connections of these substructures at their interfaces can be described in terms of the modal quantities of the components. A set of equations can be written with coordinates in terms of normal-mode shapes and modal factors of the components. Solving these equations for the amplitude factors and multiplying by the component modes yields system modes.

Several investigators have developed complete methods of analysis by substructures; these methods incorporate the concept of normal-mode synthesis. Gladwell[5] developed the branch mode method of vibration analysis, which involves the imposition of a sequence of constraints on the system so that in each constrained system or branch only a few adjoining components vibrate in modes, called branch modes. These branch modes and appropriate rigid-body modes are used in a Rayleigh-Ritz analysis of the complete system. Craig and Bampton[6] extended the branch mode method to systems having highly redundant substructure boundaries.

Hurty[7,8] presented the method of component-mode synthesis, in which the vibration modes of the composite structure are synthesized from generalized coordinates that are defined by a finite number of displacement modes for each component structure. These displacement modes are generated in three categories: Rigid-body, "constraint" modes, and "normal" modes.

MacNeal[9] devised a general solution to the problem of representing a part of a structure by its vibration modes when some or all of its connection points to the rest of the structure are restrained during measurement or calculation of substructure modes. This procedure is incorporated in the present study and presented in detail in the following chapter.

Many computer programs have been written to solve the myriad structural problems, both specific and general, to which finite-element theory can be applied. For example, NASA has committed a large amount of time, funds, and manpower to create a general-purpose computer program for structural analysis called NASTRAN[10] (an acronym for NAsa STRuctural ANalysis). Although demonstration problems involving normal-mode modeling have been solved, no extensive study of the normal-mode-synthesis technique has been performed with this computer program to date.

In this report the basis of normal-mode analysis and normal-mode synthesis is presented and modeling studies using finite elements and modal synthesis for structural problems are evolved, performed on NASTRAN, and compared to known analytic solutions for accuracy. The purpose of these studies is threefold:

(1) To compare the results of the normal-mode solutions of structural problems analyzed by finite-element and other methods for accuracy.

(2) To develop guidelines for efficient finite-element modeling so that required accuracies will result from the analysis.

(3) To investigate modeling techniques in representing a part of a structure by its normal modes as implemented on a structural-analysis computer program.

1.2 Notation

The letter symbols used in the present study are defined in the text where they first appear. For convenience they are listed here in alphabetical order. Throughout the text square brackets, [ ], denote two-dimensional arrays and braces, { }, indicate column vectors.* The transpose symbol, T, appended to braces identifies a row vector. Subscripts are used with matrix notation to designate subsets of displacement components.

a = Subscript denoting degrees of freedom.

A = Cross-sectional area; also accuracy.

b = Subscript denoting restrained degrees of freedom; also generalized damping factor.

c = Subscript denoting connection points; also angular constant.

E = Modulus of elasticity.

$\{f\}$ = Force.

$\{F\}$ = Force.

*This is the matrix notation adopted by NASTRAN.[10]

g = Damping factor.

i = Subscript.

I = Moment of inertia.

J = Torsional constant.

k = Generalized stiffness.

$[k]$ = Diagonal matrix of modal coefficient.

$[K]$ = Structural stiffness matrix.

l = Length.

m = Subscript; also generalized mass.

$[m]$ = Diagonal matrix of modal coefficient.

$[M]$ = Structural mass matrix.

n = Subscript denoting degrees of freedom.

N = Number of elements in model.

p = Subscript denoting physical points; also differential operator.

t = Time.

T = Superscript denoting transpose of matrix.

$\{u\}$ = Displacement vector.

x = Coordinate.

y = Coordinate.

z = Coordinate.

β = Characteristic number.

ζ = Auxiliary modal coordinates.

λ = Eigenvalue.

ν = Poisson's ratio.

ξ = Modal coordinates.

ρ = Mass per unit length.

ϕ = Characteristic function.

$\{\phi\}$ = Eigenvector.

$[\phi]$ = Transformation matrix of eigenvectors.

$[\psi]$ = Transformation matrix.

ω = Natural frequency of a vibration mode.

CHAPTER 2

METHOD OF ANALYSIS

2.1 Normal-Mode Analysis

In the normal-mode or modal method of dynamic problem formulation, the vibration modes of the structure in a selected frequency range are used as the degrees of freedom. Thus the number of degrees of freedom is reduced while the accuracy in the selected frequency range is maintained. In the direct method, the degrees of freedom are simply the displacements at connection points between substructures.

The advantage of the normal-mode method lies in the fact that the differential equations of motion of the structure are decoupled when the displacements are expressed in terms of the normal modes. Thus, a structure with n degrees of freedom may be expressed by n independent differential equations rather than by a system of n simultaneous differential equations.

The modal method of dynamic problem formulation is important in maximization of computational efficiency in certain types of problems. This method will usually be more efficient in problems where a small fraction of the modes are sufficient to produce the desired accuracy in the range of interest and where the stiffness matrix used in the direct method is not well banded. For problems without dynamic coupling, i.e., for problems in which the matrices of the modal formulation are diagonal, the modal method will frequently be more efficient, even though a large fraction of the modes are needed.[10]

The results of a normal-mode analysis of a part of a structural system modeled by finite elements may be combined directly with modal data obtained from other sources to analyze the entire system.

2.2 Eigenvalue Analysis

Eigenvalue analysis yields structural vibration modes from the symmetric mass and stiffness matrices, $[M_{aa}]$ and $[K_{aa}]$, generated by static analysis. The eigenvectors and eigenvalues produced by this analysis may be used to generate modal coordinates for further dynamic analysis.

The general form of the eigenvalue problem for vibration modes is

$$[K_{aa} - \lambda M_{aa}]\{u_a\} = 0\,, \tag{1}$$

where $\{u_a\}$ is the displacement vector and the eigenvalues $\lambda_i = \omega_i^2$ are the squares of the natural frequencies. The results of the calculation are the eigenvalues and corresponding eigenvectors, $\{\phi_{ai}\}$, normalized so that the largest element of each eigenvector is unity.

The eigenvalue extraction method used in the present analysis is called the inverse power method with shifts, a particularly effective method of analysis for problems formulated by the displacement approach when only a fraction of all of the eigenvalues are required. Section 10.4 of Reference 10 gives a complete description of the theory and application of this method as implemented in NASTRAN.

2.3 Normal-Mode Synthesis

The normal-mode synthesis technique in structural analysis permits part of a structure to be described by its orthogonal vibration modes. In some instances structural information may not be available in other forms. Thus, normal-mode information derived from diverse sources, including vibration tests and other analyses such as energy methods and finite elements, may be combined in one structural analysis. Normal-mode synthesis has been found useful in many practical situations.[2]

Section 14.1 of Reference 10 develops the modeling technique utilized in the present study, whereby a part of a structure is represented by its vibrational modes. This development is presented below for completeness.

Description of part of a structure by vibration modes requires knowledge of how the connection points between parts of the structure were supported when the vibration modes were measured or computed. Three cases are distinguished:

(1) All connection coordinates free.

(2) All connection coordinates restrained.

(3) Some connection coordinates free and some restrained.

The first condition is usually employed in vibration tests or analyses of large parts. Often it is not possible to achieve effectively unrestrained test conditions; however, unrestrained conditions can be obtained from calculated modes.

2.3.1 Case 1

For Case 1, in which the substructure modes are free at all connection coordinates, the required data are the vibration mode frequencies, ω_i, the mode shapes or eigenvectors, $\{\phi_i\}$, and the mass distribution of the part, expressed by the mass matrix $[M_p]$. The eigenvectors need not be normalized in any particular manner. Let the degrees of freedom at the points of connection to the remainder of the structure be designated by the vector $\{u_c\}$. Then the motions of these points are related to the modal coordinates, $\{\xi_i\}$, of the part by

$$\{u_c\} = [\phi_{ci}]\{\xi_i\} . \tag{2}$$

The columns of $[\phi_{ci}]$ are the eigenvectors, $\{\phi_i\}$, abbreviated to include only the degrees of freedom at the connection points, $\{u_c\}$. The usual approximation of including only a finite number of eigenvectors in $[\phi_{ci}]$ produces an idealized model for the part that is too stiff. Specification of the part is completed by calculation of the generalized mass, m_i, stiffness, k_i, and damping, b_i, associated with each modal coordinate, ξ_i, as follows:

$$m_i = \{\phi_i\}^{\mathsf{T}}[M_p]\{\phi_i\} , \tag{3}$$

$$k_i = \omega_i^2 m_i , \tag{4}$$

and

$$b_i = g_i m_i \omega_i , \tag{5}$$

where g_i is a damping factor for the ith mode. Frequently g_i will not be accurately known.

The equation of motion for the generalized coordinate, ξ_i, is

$$(m_i p^2 + b_i p + k_i)\xi_i = \{\phi_{ci}\}^{\mathsf{T}}\{f_c\} , \tag{6}$$

where p is the differential operator, $\{f_c\}$ is the vector of forces applied to the substructure at the connection points, and $\{\phi_{ci}\}$ is the eigenvector $\{\phi_i\}$ abbreviated to include only the degrees of freedom at connection points.

Equations (2) through (6) contain all of the information required to describe the part. In the construction of the idealized model, each of the rows of Equation (2) is regarded as an equation of constraint between a constrained degree of freedom, u_c, and the generalized coordinates, $\{\xi_i\}$. The generalized mass, stiffness, and damping elements connected to ξ_i are m_i, k_i, and b_i, respectively. Figure 1 illustrates the interconnection of the elements in diagrammatic form.

2.3.2 Cases 2 and 3

The derivation of an idealized model for Cases 2 and 3, in which some or all of the connection points are restrained during measurement or calculation of the substructure modes, is considerably more involved. A general solution devised by MacNeal[9,10] is developed below.

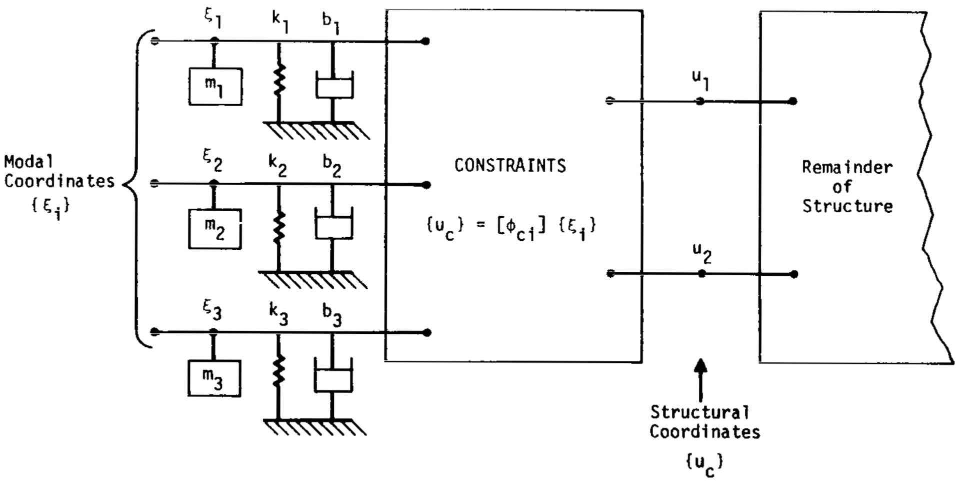


Figure 1—Representation of a part of a structure by its vibration modes, Case 1: All connection points are free while the modes are calculated.

The objective of this analysis is to derive a set of relationships that can be treated as equations of constraint between the modal coordinates and the degrees of freedom at connection points (both free and restrained). The modal mass, damping, and stiffness properties will be simulated by scalar structural elements, as in Case 1.

Let the degrees of freedom of the substructure be partitioned into $\{u_a\}$, degrees of freedom that are free in the substructure modes and $\{u_b\}$, degrees of freedom that are restrained in the substructure modes, i.e., the connection points. The equations of motion for the substructure (without damping) can then be written as

$$\left\{\begin{array}{c} f_a \\ \hline f_b \end{array}\right\} = \left[\begin{array}{c|c} K_{aa} + M_{aa}p^2 & K_{ab} \\ \hline K_{ab}^{\mathrm{T}} & K_{bb} \end{array}\right] \left\{\begin{array}{c} u_a \\ \hline u_b \end{array}\right\}, \tag{7}$$

where $\{f_a\}$ and $\{f_b\}$ are forces applied to the substructure. The mass of the substructure is assumed to be concentrated at the free coordinates, $\{u_a\}$, which include all coordinates not restrained in the substructure modes. Any substructure mass on the restrained coordinates, $\{u_b\}$, should be lumped into the remainder of the structure because the masses on the restrained coordinates produce no effect during the vibration modes of the substructure. They are, therefore, ignored in the modal representation of the substructure. The stiffness matrix is partitioned in Equation (7) according to free and restrained coordinates. Note that $\{u_a\}$ contains the free connection coordinates as a subset.

The substructure modal shapes are described by a modal transformation between the free coordinates, $\{u_a\}$, and modal coordinates, $\{\xi_i\}$, by

$$\{u_a\} = [\phi_{ai}]\{\xi_i\} . \tag{8}$$

The corresponding generalized forces on the modal coordinates are

$$\{f_i\} = [\phi_{ai}]^{\mathrm{T}}\{f_a\} . \tag{9}$$

By virtue of the orthogonality property of vibration modes,

$$[\phi_{ai}]^{\mathrm{T}}[K_{aa} + p^2 M_{aa}][\phi_{ai}] = [k_i + m_i p^2] , \tag{10}$$

where $[k_i]$ and $[m_i]$ are diagonal matrices of the modal coefficients computed by Equations (3) and (4). Now, if we use Equations (8), (9), and (10) to transform Equation (7),

$$\left\{\begin{array}{c} f_i \\ \hline f_b \end{array}\right\} = \left[\begin{array}{c|c} k_i + m_i p^2 & \phi_{ai}^{\mathrm{T}} K_{ab} \\ \hline K_{ab}^{\mathrm{T}}\phi_{ai} & K_{bb} \end{array}\right] \left\{\begin{array}{c} \xi_i \\ \hline u_b \end{array}\right\}. \tag{11}$$

It is convenient to separate the inertia forces from Equation (11), so that if we define

$$\{\bar{f}_i\} = \{f_i\} - [m_i]\{p^2 \xi_i\} , \tag{12}$$

$$\begin{Bmatrix} \bar{f}_i \\ \hdashline f_b \end{Bmatrix} = \left[\begin{array}{c:c} k_i & \phi_{ai}^{\mathsf{T}} K_{ab} \\ \hdashline K_{ab}^{\mathsf{T}} \phi_{ai} & K_{bb} \end{array}\right] \begin{Bmatrix} \xi_i \\ \hdashline u_b \end{Bmatrix} . \tag{13}$$

Equation (13) is a stiffness equation in standard form. Placing ξ_i on the left-hand side leads more directly to a useful physical model:

$$\begin{Bmatrix} \xi_i \\ \hdashline f_b \end{Bmatrix} = \left[\begin{array}{c:c} k_i^{-1} & \psi_{ib} \\ \hdashline -\psi_{ib}^{\mathsf{T}} & \bar{K}_{bb} \end{array}\right] \begin{Bmatrix} \bar{f}_i \\ \hdashline u_b \end{Bmatrix} , \tag{14}$$

where

$$[\psi_{ib}] = -[k_i]^{-1}[\phi_{ai}]^{\mathsf{T}}[K_{ab}] , \tag{15}$$

and

$$[\bar{K}_{bb}] = [K_{bb}] - [\psi_{ib}]^{\mathsf{T}}[k_i][\psi_{ib}] . \tag{16}$$

If the set of restrained points $\{u_b\}$ is nonredundant, the matrix $[\bar{K}_{bb}]$ is null; this condition will be assumed. The matrix $[\psi_{ib}]$ is calculated from properties of the vibration modes as follows: During a vibration mode, $\{u_b\} = 0$, and the vector of forces acting on the constraints is, from Equation (14),

$$\{F_b\} = -\{f_b\} = [\psi_{ib}]^{\mathsf{T}}\{\bar{f}_i\} = [\psi_{ib}]^{\mathsf{T}}[k_i]\{\xi_i\} . \tag{17}$$

We define $[K_{bi}]$ to be the matrix of forces on the constraints due to unit values of the modal coordinates while the substructure is vibrating in its normal modes:

$$\{F_b\} = [K_{bi}]\{\xi_i\} . \tag{18}$$

Then, if we compare Equations (17) and (18),

$$[\psi_{ib}] = [k_i]^{-1}[K_{bi}]^{\mathsf{T}} , \tag{19}$$

or, in other words, $[\psi_{ib}]$ is equal to $[K_{bi}]^{\mathsf{T}}$, with each row divided by the appropriate element of $[k_i]$. We may also use $[\psi_{ib}]$ to define an auxiliary set of modal coordinates

$$\{\zeta_i\} = [\psi_{ib}]\{u_b\} . \tag{20}$$

Then, from the top half of Equation (14),

$$\{\bar{f}_i\} = [k_i]\{\xi_i - \zeta_i\} . \tag{21}$$

The free connection coordinates $\{u_c\}$ are a subset of $\{u_a\}$. The relation between $\{u_c\}$ and the modal coordinates $\{\xi_i\}$ is

$$\{u_c\} = [\phi_{ci}]\{\xi_i\} , \tag{22}$$

where $[\phi_{ci}]$ is the appropriate partition of $[\phi_{ai}]$.

Equations (12), (20), (21), and (22) provide a complete description of the substructure. They are also used to construct the idealized model of the substructure, shown in Figure 2. The modal dampers, b_i, are placed across the modal springs, k_i, if they simulate structural damping. If they simulate damping due to the viscosity of a surrounding fluid environment, they should be placed between the modal coordinates and ground. Equation (20) expresses a new set of constraint equations between the auxiliary modal coordinates and the degrees of freedom that are restrained in substructure modes.

The techniques discussed above provide the capability for the complete dynamic partitioning of a structure, since all of the parts, rather than a few, may be represented by their respective vibration modes.[8] The general case diagrammed in Figure 2 is particularly useful for this purpose. Consider, for example, the missile structure shown in Figure 3. The missile is physically partitioned with support conditions for the calculation of uncoupled vibration modes, as shown in the figure. The first partition, (a), is unsupported, while the others are cantilevered. The lumped-element model for the composite system consists of parts with the form of Figure 2 connected in tandem. It is evident from the form of the lumped-element model that the independent degrees of freedom consist of the modal coordinates $\{\xi_a\}$, $\{\xi_b\}$, $\{\xi_c\}$, etc. The displacement sets $\{u_{a,b}\}$, $\{\xi_b\}$, $\{u_{b,c}\}$, etc., are all constrained. The dynamic equations, when written by the stiffness method, are banded with bandwidths equal to the number of modal coordinates in three successive partitions.

The analyst should be cautioned against an uncritical use of dynamic partitioning techniques. Use of a smaller number of modes as degrees of freedom to represent a dynamical system always results in a loss of mass, a loss of flexibility, or both. Procedures have been developed[9,11] for incorporating the "residual mass" or the "residual flexibility" into the analysis with substantial increase in accuracy. In general, however, established techniques for truncating the modes of a complete system do not automatically give good results when applied to substructures.

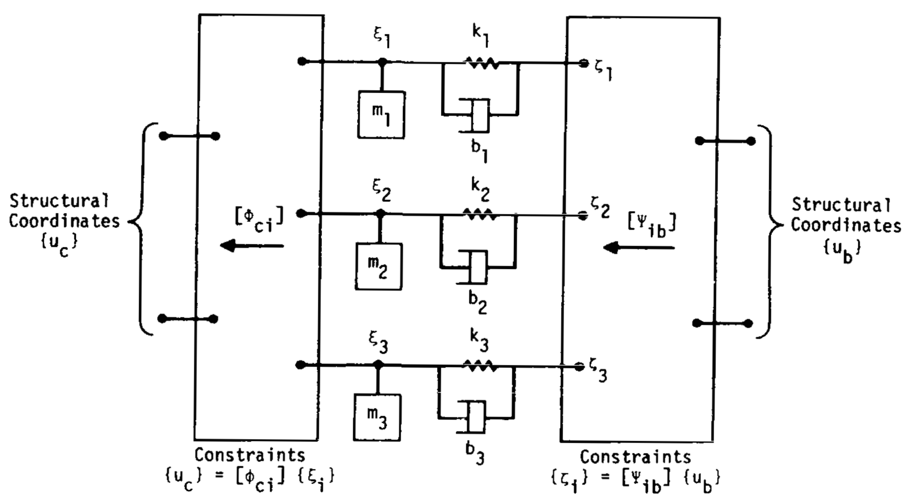
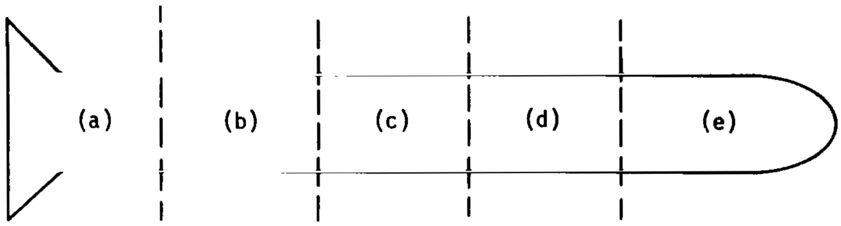
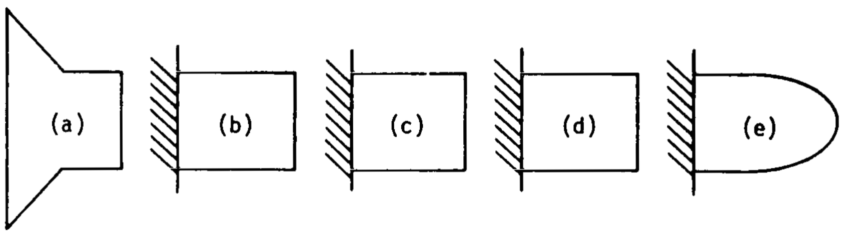


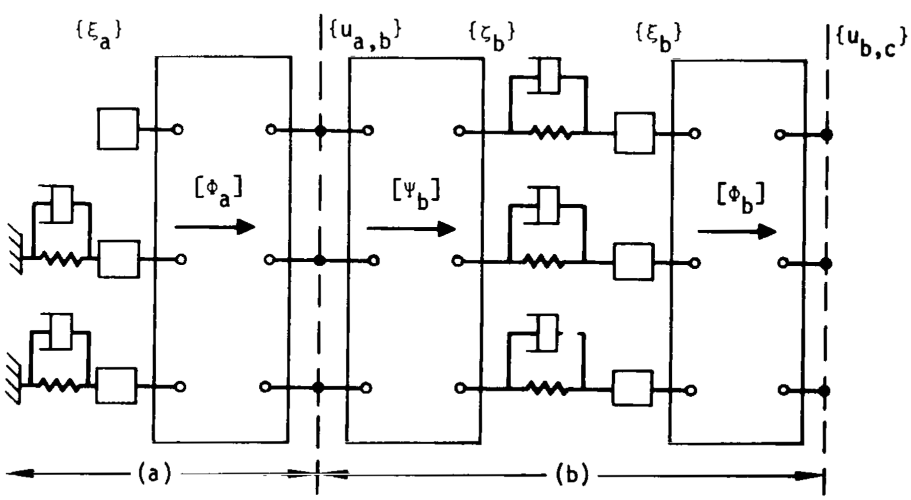
Figure 2—Representation of a part of a structure by its vibration modes, Cases 2 and 3: Some connection points are free and some are rigidly constrained.



Missile structure, unpartitioned



Support conditions for partitions while calculating substructure vibration modes.



Portion of composite model

Figure 3—Dynamic partitioning of missile structure.

CHAPTER 3

MODELING STUDIES

3.1 General

The theory of normal-mode analysis and normal-mode synthesis presented in the previous chapter is applied in this chapter to studies in problem formulation, i.e., modeling technique, for these two methods of structural analysis. Structural problems for which known analytic solutions exist are defined and modeled by finite elements. The results of the normal-mode analysis are directly incorporated in the modeling studies of the normal-mode-synthesis technique. The analysis of these results is presented in the next chapter.

3.2 Normal-Mode Modeling Study

3.2.1 Description

The structural problem chosen to compare the accuracy of solutions obtained by different methods is the determination of the normal modes of vibration of a constant-property beam with various end conditions. Vibration transverse to the axis of the beam is considered in only one plane. Damping and axial vibration are neglected.

The analytic method for the determination of the natural frequencies and modes of vibration of a beam and a detailed derivation of the corresponding characteristic functions are given in standard texts[12,13] on structural dynamics. Young and Felgar[14] tabulated the solution of the characteristic functions of this problem, with vibrations governed by the well known differential equation

$$EI\frac{\partial^4 y}{\partial x^4} - \rho\frac{\partial^2 y}{\partial t^2} = 0 \,, \tag{23}$$

where x and y are the coordinates parallel and perpendicular, respectively, to the longitudinal axis of the beam; t is time; E is the modulus of elasticity; I is the moment of inertia; and ρ is the mass per unit length of the beam. Each of the functions for a given beam satisfies the differential equation

$$\frac{d^4\phi_n}{dx^4} = \beta_n^4\phi_n \tag{24}$$

and also satisfies the boundary conditions corresponding to the end conditions of the beam. In Equation (24) β_n is the characteristic number and $\phi_n(x)$ is the characteristic function for each type of beam. The natural circular frequency of the nth mode of vibration of the beam, ω_n, is given by

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho}} = (\beta_n l)^2 \sqrt{\frac{EI}{\rho l^4}}, \tag{25}$$

where l is the length of the beam. These results are assumed to be the "known" solution to which the finite-element results are compared for accuracy.

The general type of finite element used to represent the beam in this study is a bar element that undergoes extension, torsion, bending in two perpendicular planes, and the associated shears. The restrictive assumptions for this element are that it is straight and unloaded except at its end and that its properties are uniform from end to end. The stiffness matrix of the bar element is a 12 X 12 matrix of coefficients that express the forces and moments acting on the six degrees of freedom (three translations and three rotations) at each of its two ends. Figure 4 shows a representation of the bar element, its coordinate system, and its degrees of freedom. A lumped-mass distribution is applied in the development of the mass matrix of the element. The specific element used in the present modeling study is identified in the NASTRAN computer program by the mnemonic, CBAR, denoting the "connection, bar," element. Section 5.2 of Reference 10 presents the detailed analytic description of this finite element.

The uniform beam is represented by four different finite-element models consisting of 5, 10, 20, and 40 elements, respectively, to ascertain the effect of the number of elements upon the accuracy of results. For ease in correlation of finite-element and differential equation results, beam properties were chosen to reduce to 1 the value of the terms within the radical in the second expression for ω in Equation (25). Figure 5 shows the four beam models. Thus, the natural frequency in radians per second of the nth mode resulting from the eigenvalue analysis of the finite-element formulation is directly comparable to the term $(\beta_n l)^2$ of Equation (25) as tabulated in Reference 14 for various beam types.

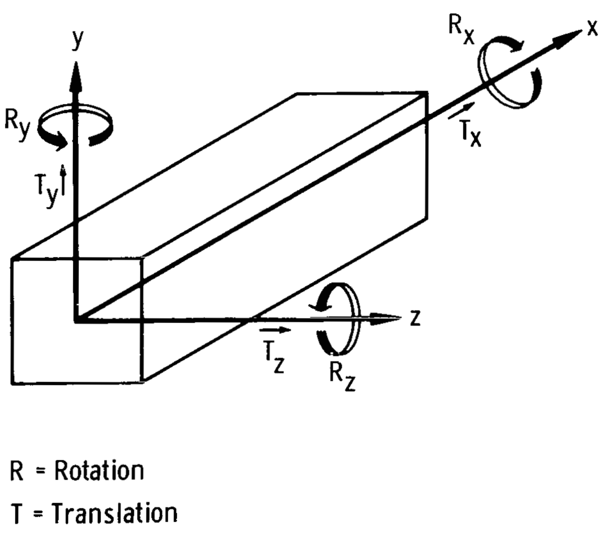


Figure 4—Bar element.

Free, supported, and clamped beam end conditions are combined to obtain six different types of beams: (1) Free-free, (2) free-supported, (3) free-clamped, (4) supported-supported, (5) supported-clamped, and (6) clamped-clamped.

3.2.2 Implementation

The normal-mode modeling study is implemented on the NASTRAN computer program. NASTRAN embodies a lumped-element approach, whereby the distributed physical properties of a structure are represented by a model consisting of a finite number of idealized substructures or elements that are interconnected at a finite number

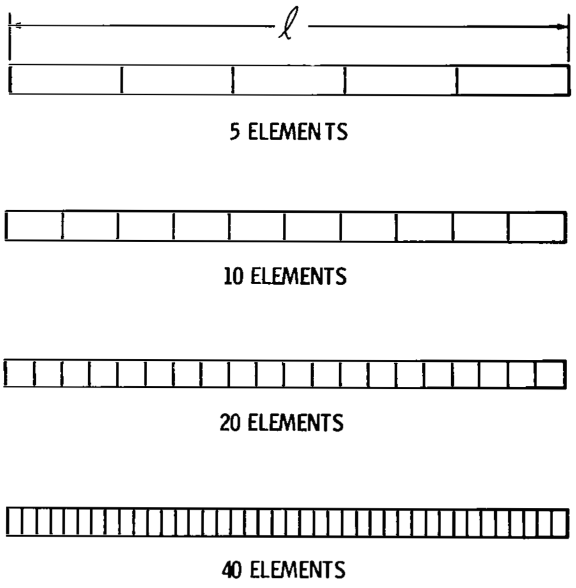


Figure 5—Four finite-element models of a beam.

of grid points, to which loads are applied. All input and output pertain to the idealized structural model.

The structural problem is defined by encoding the required information on punch cards according to prescribed formats given in Reference 15. For the present study, the following information is required: Coordinate-system definition, grid-point definition, elements connected between grid points, cross-sectional and material properties, constraints on ends of the beam, and method of eigenvalue extraction.

The problem thus defined undergoes eigenvalue analysis, which computes the following for each of the modes analyzed: Eigenvalue, eigenvector normalized to the maximum displacement, natural frequency in radians per second and cycles per second, generalized stiffness, and generalized mass.

In Chapter 4, the natural frequency resulting from the finite-element, normal-mode analysis is compared for accuracy to the known solution. The eigenvector and generalized properties of each mode are used in the following modeling studies involving normal-mode synthesis.

3.3 Normal-Mode-Synthesis Modeling Study

3.3.1 General

The results of the normal-mode analysis provides a basis for solution of the following problems by synthesis of the normal modes of the constituent parts, i.e., by representing a part of a structure by its vibration modes. The eigenvectors and generalized properties of the first i modes of the beam with free-free end conditions are used in the application of the theory developed for normal-mode synthesis in Section 2.3.1.

3.3.2 Single-Component Structure

The structural problem chosen to demonstrate the synthesis of normal modes and to investigate the accuracy of the technique is the determination of the normal modes of a uniform cantilevered beam, i.e., a beam with fixed-free end conditions. The beam consists of a single component modeled by the modal properties of a beam with free-free end conditions. The first i free-free normal modes are selected to model the beam, where i must include the first two modes that are rigid-body modes. The generalized stiffness given by Equation (4) is zero for rigid-body modes since the natural frequency is zero. The various values of i selected include the two rigid-body modes plus arbitrarily selected flexible modes. The source of the modal information is the four finite-element models used in the normal-mode analysis of Section 3.2, which consisted of 5, 10, 20, and 40 elements, respectively. Thus, for example, in the 10-element model the first i modes were used, where $i = 3, 5, 7, 9$, and 11. The following chapter presents the numerical results and compares them to known solutions for the modes of the cantilevered beam.

Implementation of the normal-mode synthesis on the NASTRAN program requires the declaration of the following for each mode used to model the structural part: (1) Scalar point, which is a point in vector space at which one degree of freedom is defined, (2) scalar mass and stiffness connection elements whose values are the generalized mass and stiffness, and (3) the coefficients of the matric $[\phi_{ci}]$ of Equation (2), the columns of which are the eigenvectors of the ith mode, truncated to include only the degrees of freedom at the connection points. The scalar points become the generalized coordinates and degrees of freedom for the analysis. The relationship given by Equation (2) is declared in NASTRAN by a multipoint constraint equation.

Simulation of restrained end conditions of the beam is most easily accomplished by the connection of a spring of very large spring constant, e.g., 10^{12} N/m (10^{10} lb/in.),* at the connection points between ground and the restrained degrees of freedom. Figure 6 shows a representation of the model used for normal-mode synthesis of a single cantilevered beam.

3.3.3 Multiple-Component Structures

The modeling techniques described for a single-component structure are extended to structures with multiple components. In each problem formulation, the required modal information for each substructure is identical to that required in the preceding structure. Boundary conditions of the multiple-substructure problem are handled in the same manner as the end conditions of a single-element cantilever beam. Connections between adjoining substructures are made by declaring a scalar

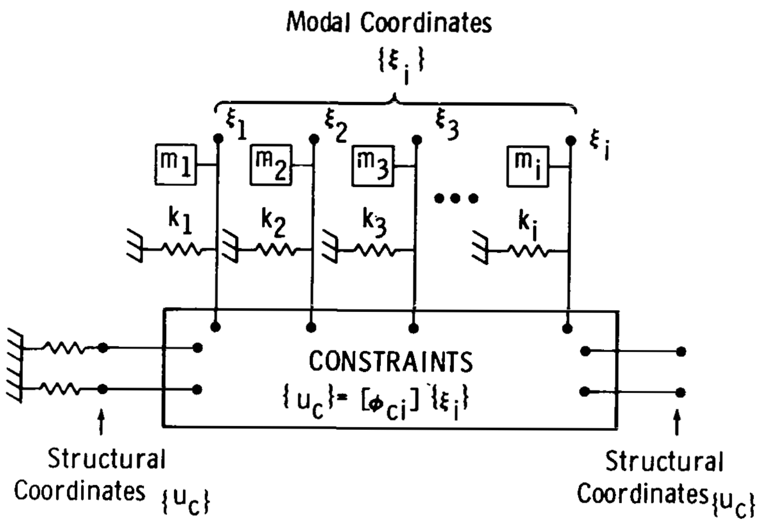
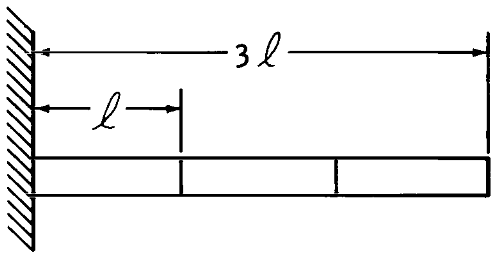


Figure 6—Representation of the model for normal-mode synthesis of a single cantilevered beam.

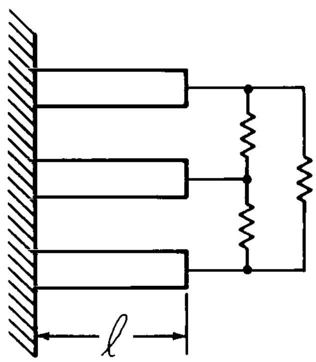
*The quantity that appears in parentheses in the text is the value that was used in the modeling study.

spring element with a very large spring constant between appropriate translational and rotational degrees of freedom.

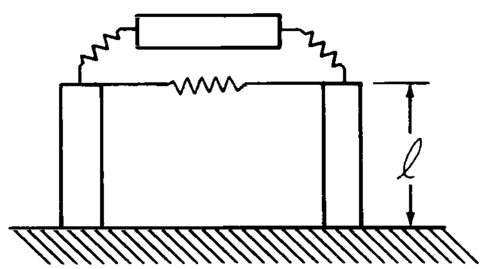
Structural normal modes are obtained by the normal-mode-synthesis technique as applied to the following structures, each divided into more than one substructure: (1) Three beams in series, cantilevered, (2) three beams in parallel, cantilevered, and (3) three beams in a portal arch. Figure 7 shows sketches of these structures. The next chapter presents the comparison of the results of the modeling study and the known solution as given by Equation (25) for the first and second structures and by a 120-finite-element model for the third structure.



(a) Three beams in series, cantilevered.



(b) Three beams in parallel, cantilevered.



(c) Three beams in a portal arch.

Figure 7—Multiple-component structures modeled by normal-mode synthesis.

CHAPTER 4

ANALYSIS OF MODELING STUDIES

4.1 General

This chapter presents the analysis and numerical results of the modeling studies described in the preceding chapter. These studies are implemented on the NASTRAN computer program, Release 11.1, which is operational on the IBM System 360/95 at Goddard Space Flight Center.

4.2 Normal-Mode Analysis

Values and units for the parameters required for the finite-element models described in Section 3.2 are chosen as follows:*

Modulus of elasticity,	$E = 6.9 \times 10^{11}$ N/m^2 (10^8 lb/in.^{2}).
Poisson's ratio,	$\nu = 0.3$.
Mass per unit length,	$\rho = 6.9 \times 10^3$ N-sec^2/m^2 (1.0 lb-sec^2/in.^{2}).
Area moments of inertia,	$I_{1,2} = 41.6$ cm^4 (1.0 in.^{4}).
Cross-sectional area,	$A = 6.5$ cm^2 (1.0 in.^{2}).
Torsional constant,	$J = 41.6$ cm^4 (1.0 in.^{4}).
Length,	$l = 254$ cm (100 in.).

Thus, the four models of the uniform beam consist of 5, 10, 20, and 40 bar elements, whose elemental lengths are 50.8, 25.4, 12.7, and 6.4 cm (20, 10, 5, and 2.5 in.),* respectively (Figure 5). As previously stated, this choice of parameter values permits direct comparison of the results of the eigenvalue analysis with those tabulated in Reference 14.

The complete matrix of problems studied consists of the four finite-element models solved for each of the six end conditions: (1) Free-free, (2) free-supported, (3) free-clamped, (4) supported-supported, (5) supported-clamped, and (6) clamped-clamped. Appendix A records the numerical results of the eigenvalue analyses for these six end conditions. Recorded data include—

(1) Mode number, $i = 1, 2, 3, \ldots 20$.

(2) Natural circular frequency of the ith mode of vibration of the beam in radians per second for the known solution given in Reference 14.

*The quantities that appear in parentheses in the text, and the value of Poisson's ratio, are the exact values that were used in the modeling study.

(3) Accuracy, A, defined as the ratio of the finite-element solution to the known solution. This definition of accuracy shall be maintained throughout the analysis of results.

In general, if n is the number of unrestrained, independent degrees of freedom of the problem, then n normal modes exist. For example, six grid points are required to model the 5-element beam. For the free-free type beam, $n = 6$. These consist of two rigid-body modes and four flexible modes. Supporting one end of the beam reduces n by one and results in one rigid-body mode and four flexible modes. Clamping one end prohibits rigid-body motion. In Appendix A, only flexible modes are recorded.

Figure 8, a representative of the data recorded for the six end conditions, shows the normal-mode results for modes 1, 2, 3, 4, 5, 9, and 15 of the free-free type of beam. The coordinates of the figure are the number of finite elements in the model, N, versus the accuracy, A, of modal frequency, defined above. In general, accuracy approaches 1.0 asymptotically from below as the number of elements increases. The plot passes through the origin, since zero finite elements result in zero accuracy. This relationship is expressed by

$$N = \tan\left[Af(i,c)\right] , \tag{26}$$

where N is the number of elements in the model, $f(i,c)$ is a function of the mode number, i, and c is an angular constant.

Equation (26) can be used to estimate the number of finite elements required to model the beam for selected values of mode number, i, desired minimum accuracy, A, and beam end conditions. For example, for the beam with free-free end conditions, when

$$N > i + 2 , \tag{27}$$

these parameters determined by curve-fitting the empirical data are defined as

$$\begin{aligned} &f(i,c) = c + i \\ \text{and} \quad &c = 87.7 \text{ deg} . \end{aligned} \tag{28}$$

For $A = 0.90$ and $i = 2$, Equation (26) yields $N = 6.1$; when rounded to the next highest integer, $N = 7$. This estimate is verified by Figure 8.

A comparison of accuracies for both mode numbers and finite-element models as recorded in Appendix A reveals a general trend in the accuracy of the finite-element solution. For a specific mode number or for a specific number of finite elements in the beam model, a reduction in the degrees of freedom at the ends of the beam results in increased accuracy of the solution. This observation is explained by the fact that a decrease in freedom of the structure at the boundary requires a given number of elements to describe less complex displacements in the eigenvector, i.e., less difference in the end displacements of individual elements. Therefore, Equations (26), (27), and (28), when applied to beams with increased end fixity, represent overestimates of the minimum number of elements that will yield specified accuracies in the eigenvalue solution.

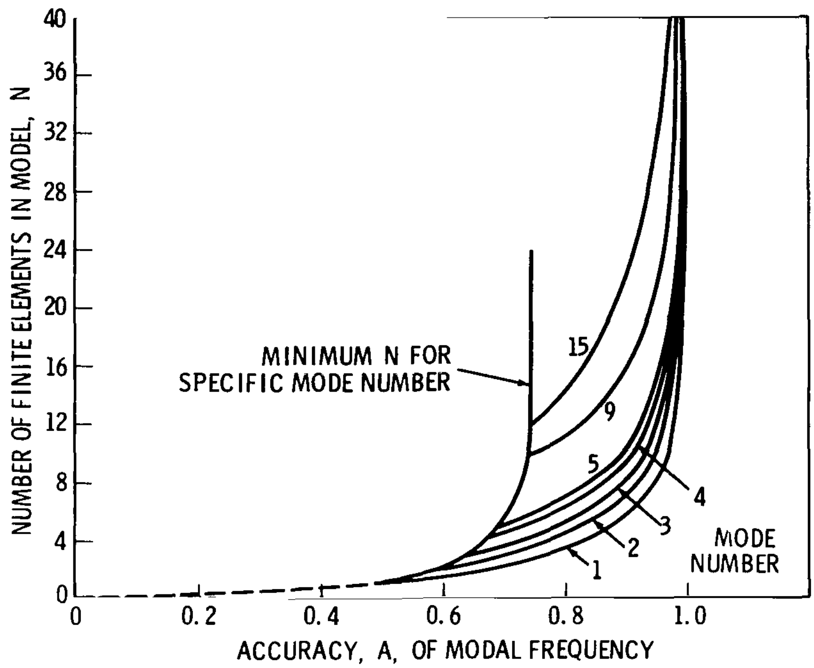


Figure 8—Results of normal-mode analysis of a beam with free-free end conditions.

4.3 Normal-Mode Synthesis

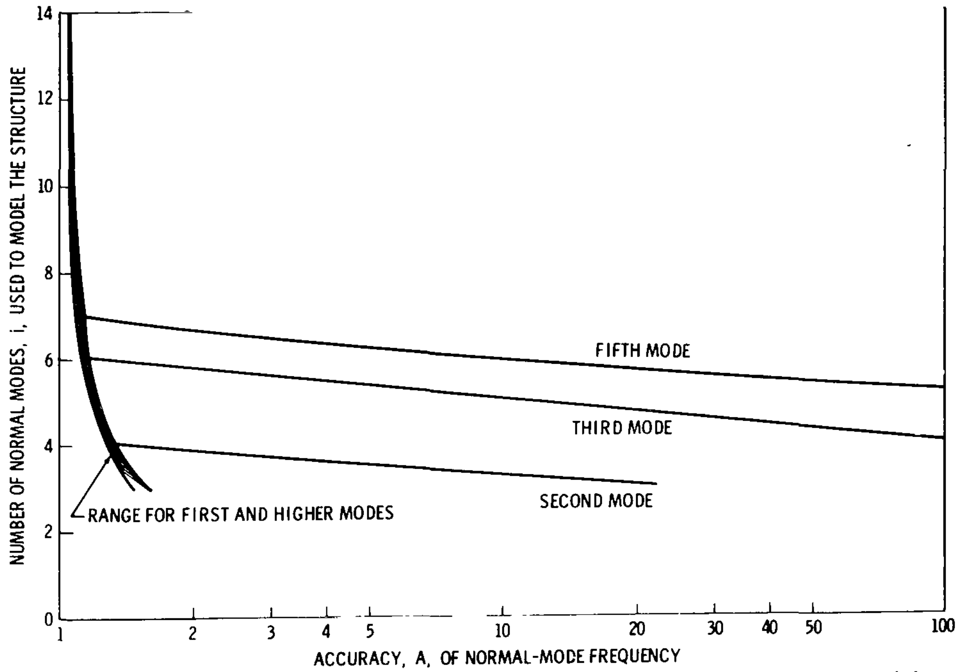
4.3.1 Single-Component Structure

Numerical results of the single-component problem discussed in Section 3.3.2 are tabulated in Appendix B, Tables B1 through B4. They correspond to the 5-, 10-, 20-, and 40-element beam models, respectively, used as sources of the substructure normal modes. The normal modes are computed for free-free end conditions. The first i modes are used to model the substructure. Data recorded in Tables B1 through B4 include the number of normal modes, i, used to synthesize the model, the known solution, and the accuracy, A, of the modal frequency defined as the ratio of the synthesized result to the known solution, as in the normal-mode analysis.

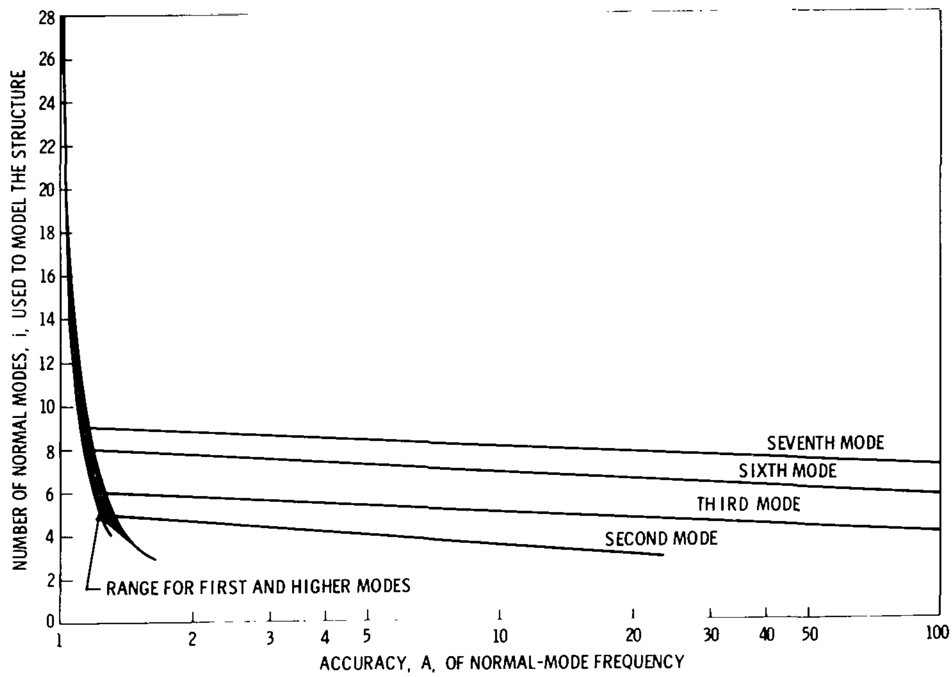
Representative data are shown in Figure 9a, which plots i versus A for the first, second, third, and fifth model frequencies of the uniform cantilevered beam. Curves are included for the normal-mode source beams consisting of 5, 10, 20, and 40 elements.

Figure 9b shows the effect of varying i to represent a substructure in a plot of i versus A for modal data recorded in Table B3, i.e., only for the normal-mode source model consisting of 20 finite elements.

The following generalizations are drawn from the recorded and plotted data for this structure modeled by component modes:



(a) Cantilevered beam modeled by free-free modes computed from source beams containing 5, 10, 20, and 40 elements (Tables B1 through B4).



(b) Cantilevered beam modeled by free-free modes computed from a 20-element source beam (Table B3).

Figure 9—Results of normal-mode synthesis of a single-component structure.

(1) There is nearly constant accuracy in model frequency for the $i - 2$ modes where the first i normal modes are used to represent the substructure.

(2) Accuracy of modes i and $i - 1$ are extremely poor, i.e., inaccurate by more than an order of magnitude.

(3) For the minimum number of synthesized normal modes used to model the substructure, best accuracy for a specific normal mode, i', of the structure is obtained when $i = i' + 2$. Thus, for a specific i', a nontruncated set of normal modes represents a substructure more accurately than does a truncated set of modes.

4.3.2 Multiple-Component Structures

The several problems described in Section 3.3.3 exercise the normal-mode-synthesis technique for structures with more than one separable substructure.

Each structure is composed of three uniform beams previously discussed and analyzed by the normal-mode method. Each component beam is represented by the first 12 normal modes, i.e., two rigid-body modes and 10 flexible modes evaluated by the 40-element beam analysis for free-free end conditions. The 40-element source beam produces the most accurate results, therefore it is used to minimize error in the component modes. Stiffness connection elements, i.e., springs with very large spring constants, are used to couple the degrees of freedom of the substructures that are unrestrained when their normal modes are computed. Similarly, these fictitious springs provide the means to fix boundary conditions by connecting the appropriate degrees of freedom to ground.

The first structure studied is a cantilevered beam whose length is $3l$, composed of three uniform segments, each l in length, in series (see Figure 7). Substitution of $3l$ for l in Equation (25) yields the known solution for modal frequency. Table B5 presents the known solutions and the accuracy, A, of normal-mode-synthesis results as a function of the number of modes, i, used to model each component. Figure 10 shows these data plotted as i versus A for representative normal modes of the complete structure. The curves approach an accuracy of 1.0 asymptotically from above as the number of normal modes synthesized for the substructures increases. The accuracy of the synthesized-structure modal frequency varies over a narrow range of values for modes numbered from 1 to nearly $2i$. For example, when seven modes are used to model each substructure, accuracy of the model frequency varies between 1.07 and 1.16 for all the structural modes numbered from 1 to 15. The model frequencies are inaccurate by an order of magnitude for structural modes greater than $2i$. As i increases, the range of values for A narrows and approaches 1.0. Reasonable engineering accuracy, e.g., within 10 percent, is obtained by use of a moderate number of component modes in the model. For example, 10-percent accuracy is obtained in the first mode when $i = 6$ and in the second mode when $i = 7$. When $i = 12$, all of the first 20 modes of the structure are computed with less than 10-percent error.

For the second structure, three parallel beams are fixed at one end and connected at the other. Table B6 gives data similar to that for the preceding structure. The known solution is the same as that for the single cantilevered beam, since the 3's cancel in the numerator and denominator of the term within the radical of Equation (25). Connections between components are made by the technique

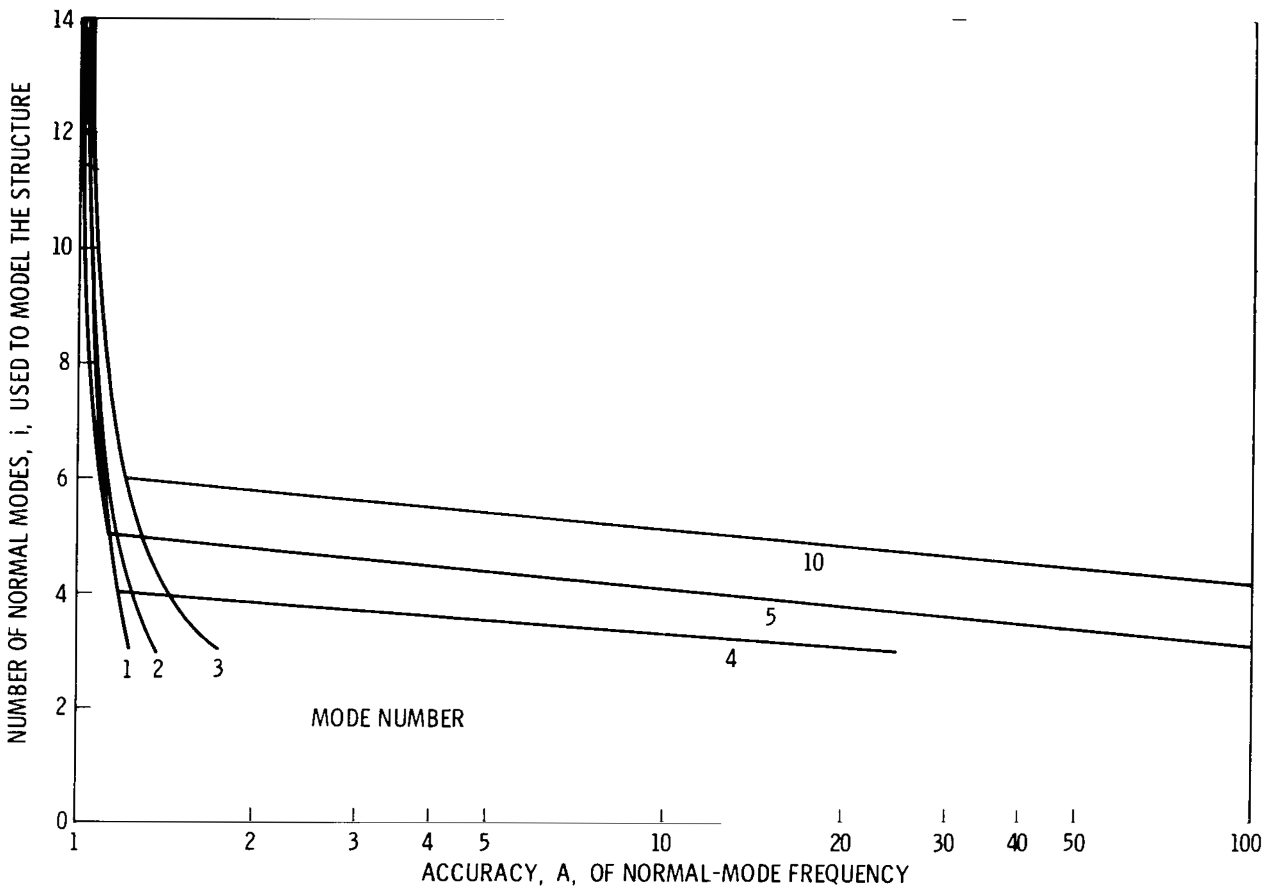


Figure 10—Results of normal-mode synthesis of a multiple-component structure, three uniform beams in parallel, cantilevered.

discussed for the previous structure. Normal-mode synthesis yields three computed modes corresponding to each mode of a single cantilevered beam. In the first synthesized mode all three parallel beams move in the same characteristic mode. In each of the other two synthesized modes, two beams move in the same cantilevered mode, while the third parallel beam takes a different mode shape. Good accuracy in modal frequency is obtained for structural modes in which each of the three components are characterized by the same substructure mode. This problem formulation demonstrates the ability of the synthesis technique to determine modes of the complete structure, which cannot be obtained from a simple combination of modes of the substructures.

The third structure is a three-member portal frame chosen to demonstrate the interconnection between substructures. Here the fictitious springs connect the appropriate degrees of freedom (see Figure 7). Rotations at the ends of adjoining beams are connected to maintain continuity of displacement between substructures. Transverse displacements at the top of the two vertical-support beams are connected, since longitudinal extension of the horizontal beam is neglected. Table B7 tabulates the following data for this structure: The known solution for normal modes, the mode number and type with respect to symmetry, and the accuracy, A, of the normal-mode-synthesis technique. The results of a finite-element, normal-mode analysis are assumed to be the known solutions. This model consists of 40 bar elements for each of the three constituent beams. Accuracy is poor for the first symmetric and antisymmetric modes, i.e., approximately 55 percent error. However, accuracy improves with higher mode numbers.

CHAPTER 5

CONCLUSIONS

In this report, the bases in finite-element analysis of the normal-mode method and the normal-mode-synthesis technique were presented and their features and applications were discussed. A normal-mode modeling study of a uniform beam resulted in guidelines for prediction of accuracy in analysis compared to a known solution for the problem. Modal data from this modeling study were used to represent substructures in a modeling study of the normal-mode synthesis. One single-component structure and three multiple-component structures demonstrated normal-mode synthesis and its modeling techniques as implemented on NASTRAN, a structural analysis computer program.

The limited examples treated in the present study are intended to investigate accuracy obtained by the analyses and to demonstrate the application of these techniques on a generally available computer program. Accuracy guidelines can be applied judiciously to similar structures, e.g., tapered beams. Though normal modes may be synthesized from many diverse sources, the analyst should be cautioned to use extreme care in dynamic partitioning techniques, especially when truncating the modes used to represent a substructure.

The selection of an adequate number of finite elements needed to model a structure yields good accuracy in the solution obtained by normal-mode analysis. The synthesis of components represented by their normal modes yields more accurate results for the entire structure when the component modes are characteristic of the modal contribution of the substructure than when these modes are not characteristic of the modal contributions. Further study of the reason for the less accurate results may lead to a better understanding of the general applications and limitations of normal-mode synthesis.

ACKNOWLEDGMENT

The author wishes to express his gratitude to Dr. Theodore G. Toridis for his insight and assistance during the research and preparation of this report.

Goddard Space Flight Center
National Aeronautics and Space Administration
Greenbelt, Maryland, July 31, 1970
861-51-75-01-51

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Appendix A

Numerical Results of Normal-Mode Analysis

Appendix A presents the numerical results of the normal-mode analysis of a uniform beam for the following end conditions:

(1) Free-free (Table A1)

(2) Free-supported (Table A2)

(3) Free-clamped (Table A3)

(4) Supported-supported (Table A4)

(5) Supported-clamped (Table A5)

(6) Clamped-clamped (Table A6)

In Appendices A and B the accuracy, A, is defined as the ratio of the finite-element solution to the known solution. In Appendix A, N denotes the number of bar elements used to model the uniform beam.

Table A1—Uniform beam with free-free end conditions.

Mode number	Frequency of known solution (rad/sec)	Accuracy, A			
		$N = 5$	$N = 10$	$N = 20$	$N = 40$
1	22.373	0.893	0.970	0.992	0.998
2	61.673	.838	.951	.987	.997
3	120.90	.796	.933	.982	.995
4	199.86	.734	.916	.977	.994
5	298.56		.900	.972	.993
6	416.99		.883	.967	.992
7	555.16		.857	.962	.990
8	713.08		.815	.957	.989
9	890.73		.743	.951	.988
10	1088.1			.945	.986
11	1305.3			.939	.985
12	1542.1			.930	.983
13	1798.7			.919	.982
14	2075.1			.905	.980
15	2371.2			.886	.979
16	2687.0			.860	.977
17	3022.6			.826	.975
18	3377.9			.783	.973
19	3752.9			.730	.971
20	4147.7				.969

Table A2—Uniform beam with free-supported end conditions.

Mode number	Frequency of known solution (rad/sec)	Accuracy, A			
		$N = 5$	$N = 10$	$N = 20$	$N = 40$
1	15.418	0.951	0.987	0.997	0.999
2	49.965	.916	.977	.994	.999
3	104.25	.882	.967	.992	.998
4	178.27	.815	.957	.989	.997
5	272.03		.945	.986	.997
6	385.53		.930	.983	.996
7	518.77		.905	.980	.995
8	671.75		.860	.977	.995
9	844.47		.783	.973	.994
10	1036.9			.969	.993
11	1249.1			.963	.992
12	1481.1			.955	.991
13	1732.7			.945	.991
14	2004.1			.931	.990
15	2295.3			.911	.989
16	2606.2			.884	.987
17	2936.8			.849	.986
18	3287.2			.803	.985
19	3657.3			.749	.984
20	4047.1				.983

Table A3—Uniform beam with free-clamped end conditions.

Mode number	Frequency of known solution (rad/sec)	Accuracy, A			
		$N = 5$	$N = 10$	$N = 20$	$N = 40$
1	3.5160	0.982	0.995	0.999	1.000
2	22.034	.941	.984	.996	0.999
3	61.697	.907	.974	.993	0.998
4	120.90	.863	.964	.991	0.998
5	199.86	.766	.954	.988	0.997
6	298.56		.941	.986	0.996
7	416.99		.922	.982	0.996
8	555.16		.890	.980	0.995
9	713.08		.835	.976	0.994
10	890.73		.749	.972	0.994
11	1088.1			.967	0.993
12	1305.2			.960	0.992
13	1542.1			.952	0.991
14	1798.7			.940	0.990
15	2075.1			.923	0.989
16	2371.2			.900	0.988
17	2687.0			.871	0.987
18	3022.6			.833	0.986
19	3377.9			.786	0.984
20	3752.9			.731	0.982

Table A4—Uniform beam with supported-supported end conditions.

Mode number	Frequency of known solution (rad/sec)	Accuracy, A			
		$N = 5$	$N = 10$	$N = 20$	$N = 40$
1	9.8696	0.999	0.999	1.000	1.000
2	39.478	.997	.999	0.999	1.000
3	88.826	.981	.999	0.999	1.000
4	157.91	.909	.997	0.999	0.999
5	246.74		.993	0.999	0.999
6	355.30		.981	0.999	0.999
7	483.61		.957	0.999	0.999
8	631.65		.909	0.998	0.999
9	799.44		.825	0.996	0.999
10	986.96			0.993	0.999
11	1194.2			0.988	0.999
12	1421.2			0.981	0.999
13	1668.0			0.971	0.999
14	1934.4			0.971	0.999
15	2220.7			0.937	0.998
16	2526.6			0.909	0.998
17	2852.3			0.872	0.997
18	3197.7			0.825	0.995
19	3562.9			0.768	0.994
20	3947.8				0.993

Table A5—Uniform beam with supported-clamped end conditions.

Mode number	Frequency of known solution (rad/sec)	Accuracy, A			
		$N = 5$	$N = 10$	$N = 20$	$N = 40$
1	15.418	0.999	1.000	1.000	1.000
2	49.965	.994	1.000	1.000	1.000
3	104.25	.963	0.999	1.000	1.000
4	178.27	.852	0.996	1.000	1.000
5	272.03		0.989	0.999	1.000
6	385.53		0.973	0.998	0.999
7	518.77		0.941	0.998	0.999
8	671.75		0.882	0.997	0.999
9	844.47		0.789	0.995	0.999
10	1036.9			0.991	0.999
11	1249.1			0.986	0.999
12	1481.1			0.978	0.999
13	1732.7			0.966	0.999
14	2004.1			0.949	0.998
15	2295.3			0.927	0.998
16	2606.2			0.896	0.997
17	2936.8			0.856	0.996
18	3287.2			0.807	0.995
19	3657.3			0.749	0.994
20	4047.1				0.992

Table A6—Uniform beam with clamped-clamped end conditions.

Mode number	Frequency of known solution (rad/sec)	Accuracy, A			
		$N = 5$	$N = 10$	$N = 20$	$N = 40$
1	22.373	0.999	1.000	1.000	1.000
2	61.673	.988	0.999	1.000	1.000
3	120.90	.936	0.998	1.000	1.000
4	199.86	.790	0.994	1.000	1.000
5	298.56		0.984	0.999	1.000
6	416.99		0.963	0.999	1.000
7	555.16		0.922	0.998	1.000
8	713.08		0.852	0.996	1.000
9	890.73		0.754	0.993	1.000
10	1088.1			0.989	1.000
11	1305.3			0.983	0.999
12	1542.1			0.974	0.999
13	1798.7			0.960	0.999
14	2075.1			0.941	0.998
15	2371.2			0.915	0.998
16	2687.0			0.882	0.997
17	3022.6			0.840	0.996
18	3377.9			0.789	0.995
19	3752.9			0.731	0.993
20	4147.7				0.991

Appendix B

Numerical Results of Normal-Mode Synthesis

Appendix B presents the numerical results of the normal-mode-synthesis modeling studies. Each table includes a problem description for the tabulated data.

Table B1—Normal-mode synthesis of a uniform cantilevered beam modeled by the first i free-free normal modes computed from a source beam consisting of 5 finite elements.

Mode number	Frequency of known solution (rad/sec)	Accuracy, A		
		$i = 3$	$i = 5$	$i = 6$
1	3.5160	1.485	1.221	1.115
2	22.034	22.10	1.236	1.095
3	61.697	424.3	16.02	1.088
4	120.90		246.2	1.072
5	199.86			9.329
6	298.56			106.1

Table B2—Normal-mode synthesis of a uniform cantilevered beam modeled by the first i free-free normal modes computed from a source beam consisting of 10 finite elements.

Mode number	Frequency of known solution (rad/sec)	Accuracy, A				
		$i = 3$	$i = 5$	$i = 7$	$i = 9$	$i = 11$
1	3.5160	1.585	1.192	1.096	1.064	1.058
2	22.034	21.83	1.206	1.095	1.059	1.052
3	61.697	449.4	1.236	1.095	1.056	1.048
4	120.90		12.80	1.100	1.053	1.044
5	199.86		187.6	1.112	1.051	1.040
6	298.56			10.26	1.067	1.035
7	416.99			102.3	1.040	1.023
8	555.16				8.299	0.9895
9	713.08				62.67	0.9135
10	890.73					5.894
11	1088.1					41.43

Table B3—Normal-mode synthesis of a uniform cantilevered beam modeled by the first i free-free normal modes computed from a source beam consisting of 20 finite elements.

Mode number	Frequency of known solution (rad/sec)	Accuracy, A				
		$i = 3$	$i = 7$	$i = 12$	$i = 17$	$i = 21$
1	3.5160	1.628	1.147	1.069	1.041	1.027
2	22.034	21.69	1.157	1.071	1.041	1.027
3	61.697	458.0	1.167	1.073	1.042	1.027
4	120.90		1.183	1.075	1.042	1.027
5	199.86		1.210	1.077	1.043	1.027
6	298.56		9.018	1.080	1.043	1.027
7	416.99		117.0	1.083	1.044	1.027
8	555.16			1.088	1.045	1.027
9	713.08			1.095	1.046	1.027
10	890.73			1.107	1.047	1.027
11	1088.1			6.419	1.048	1.027
12	1305.3				1.050	1.027
13	1542.1				1.053	1.028
14	1798.7				1.057	1.028
15	2075.1				1.063	1.028
16	2371.2				5.416	1.029
17	2687.0				29.13	1.029
18	3022.6					1.030
19	3377.9					1.032
20	3752.9					3.872

Table B4—Normal-mode synthesis of a uniform cantilevered beam modeled by the first i free-free normal modes computed from a source beam consisting of 40 finite elements.

Mode number	Frequency of known solution (rad/sec)	Accuracy, A				
		$i = 3$	$i = 7$	$i = 12$	$i = 17$	$i = 22$
1	3.5160	1.619	1.132	1.052	1.031	1.029
2	22.034	21.73	1.139	1.052	1.030	1.027
3	61.697	456.3	1.147	1.051	1.029	1.026
4	120.90		1.162	1.054	1.030	1.028
5	199.86		1.183	1.052	1.027	1.024
6	298.56		9.455	1.053	1.026	1.023
7	416.99		114.1	1.054	1.025	1.021
8	555.16			1.056	1.024	1.020
9	713.08			1.059	1.022	1.018
10	890.73			1.066	1.020	1.016
11	1088.1			7.140	1.018	1.013
12	1305.3				1.013	1.008
13	1542.1				1.007	1.001
14	1798.7				0.996	0.990
15	2075.1				0.981	0.973
16	2371.2				5.530	0.949
17	2687.0				24.11	0.914
18	3022.6					0.868
19	3377.9					0.809
20	3752.9					1.034

Table B5—Normal-mode synthesis of a uniform cantilevered beam composed of three identical substructures, each of which is modeled by the first i free-free modes computed from a 40-finite-element source beam.

Mode number	Frequency of known solution (rad/sec)	Accuracy, A			
		$i = 3$	$i = 5$	$i = 7$	$i = 12$
1	0.3906	1.231	1.113	1.073	1.040
2	2.4482	1.365	1.159	1.099	1.047
3	6.8552	1.754	1.277	1.159	1.065
4	13.433	25.09	1.122	1.063	1.026
5	22.206	114.7	1.189	1.105	1.047
6	33.173	150.7	1.240	1.146	1.075
7	46.332		1.134	1.066	1.030
8	61.684		1.195	1.110	1.049
9	79.231		1.262	1.173	1.070
10	98.970		13.77	1.127	1.030
11	120.90		94.62	1.140	1.051
12	145.02			1.154	1.079
13	171.34			1.123	1.039
14	199.85			1.138	1.055
15	230.56			1.160	1.074
16	263.46			9.990	1.038
17	298.56			74.22	1.055
18	335.84				1.080
19	375.32				1.056
20	416.99				1.063

Table B6–Normal-mode synthesis of three parallel beams fixed at one end and mutually connected at their other ends. Each component beam is modeled by the first 12 free-free modes computed from a 40-finite-element source beam.

Mode number of single cantilever beam	Frequency of known solution (rad/sec)	Mode number of three parallel beams	Computed natural Frequency (rad/sec)	Accuracy, A
1	3.5160	1	3.760	1.069
2	22.034	2	23.60	1.070
	none	2a	26.01	
	none	2b	26.01	
3	61.697	3	66.18	1.073
	none	3a	70.82	
	none	3b	70.82	
4	120.90	4	129.9	1.075
	none	4a	142.3	
	none	4b	142.3	
5	199.86	5	215.2	1.077
	none	5a	231.9	
	none	5b	231.9	
6	298.56	6	322.3	1.080
	none	6a	357.1	
	none	6b	357.1	

Table B7–Normal-mode synthesis of a three-beam portal frame, each component of which is modeled by the first 12 free-free modes computed from a 40-finite-element source beam. Known solutions are the eigenvalue results of a finite-element model of the portal frame in which 40 bar elements represent each beam.

Portal frame mode number	Mode	Frequency of known solution (rad/sec)	Accuracy, A
1	antisymmetric	3.2035	1.532
1	symmetric	12.620	0.573
2	antisymmetric	20.621	0.989
2	symmetric	22.275	1.037
3	antisymmetric	44.794	0.875
3	symmetric	54.953	1.004

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