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AN ALGORITHM FOR SOLVING THE POPOV CRITERION APPLIED TO SAMPLED DATA SYSTEMS

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SUMMARY

A systematic means is provided for analytically solving the Popov stability criterion that was developed by Tsypkin [1] for application to sampled data systems. It is developed in a manner that is amenable to solution by a digital computer, regardless of the order of the polynomials defining the numerator and denominator of the z-transform of the transfer function representing the linear part of the plant.

INTRODUCTION

Lindorff [2] has presented a derivation of Popov's stability criterion applied to sampled data systems which follows the development prepared by Tsypkin [1]. It provides a criterion for establishing sufficient conditions for absolute stability of a class of nonlinear sampled data systems described in Figure 1. The system must be capable of being separated into a single-valued nonlinearity represented by $\phi(\cdot)$ and a linear plant represented by $G(s)$. If this system meets the conditions,

$$\phi(0) = 0 \quad , \tag{1}$$

$$K > \frac{\phi(\sigma)}{\sigma} > 0 \text{ for } \sigma \neq 0 \quad , \tag{2}$$

and $G(s)$ represents a stable plant with at most one free integrator, then the Tsypkin version of the Popov stability criterion is met if

$$\operatorname{Re}\left\{ G(z) \Big|_{z = e^{i\omega T}} \right\} > -\frac{1}{K} \quad , \tag{3}$$

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1. A portion of this report was presented as Paper No. G5 at the Third Annual Southeastern Symposium on System Theory at the Georgia Institute of Technology, Atlanta, Georgia, on April 5-6, 1971.

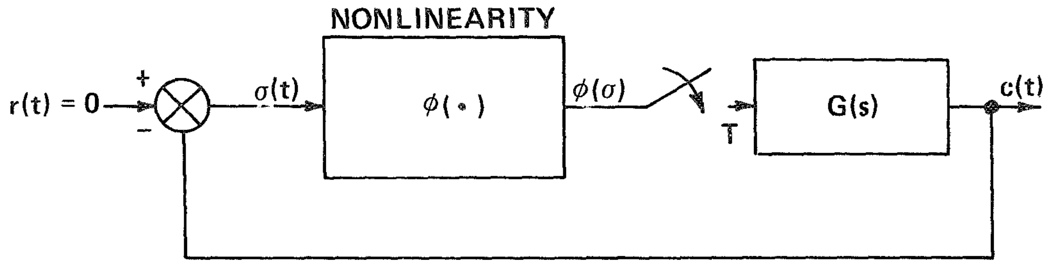

Figure 1. Model of class of nonlinear sampled data systems.

where $z \equiv e^{Ts}$, T is the sampling period, and G(z) represents the z-transform of G(s) which is evaluated at $z = e^{i\omega T}$, i.e., on the unit circle in the z-plane.

As with continuous systems, the stability requirement on the linear part of the plant can be weakened to include open-loop unstable plants [3]. This is shown through a coordinate transformation,

$$G_{tr}(s) = \frac{G(s)}{1 + \epsilon G(s)} \quad , \qquad (4)$$

where gain ϵ is chosen so that the poles of the transformed system open-loop transfer $G_{tr}(s)$ lie within the unit circle in the z-plane [2]. This is equivalent to requiring the nonlinear function $\phi(\sigma)$ to lie within the sector $[\epsilon, k]$ for the original system to be absolutely stable.

In determining the upper bound K of the sector, the real part of G(z) must be determined in some manner. This may be done by plotting G(z) in a complex plane or by analytically determining the real part of G(z) and then finding its minimum value. Either procedure becomes tedious as the order of a system under investigation increases.

DESCRIPTION OF PROPOSED ALGORITHM

A systematic means for analytically solving the criterion of equation (3) is provided, as a subsequent development to one suggested earlier by the author in a more primitive form [4]. It is developed in a manner that is particularly amenable to solution by a digital computer or electronic calculator, regardless of the order of the polynomials describing the numerator and denominator of $G(z)$. It requires only that $G(z)$ be expressed in terms of two polynomials in z; i.e.,

$$G(z) = \frac{\sum_{j=0}^{n} \gamma_j z^j}{\sum_{j=0}^{d} \eta_j z^j} \quad . \tag{5}$$

The criterion of equation (3) may be recast in the form

$$\frac{N_1 D_1 + (1 - \beta^2) N_2 D_2}{D_1^2 + (1 - \beta^2) D_2^2} > -\frac{1}{K} \quad , \tag{6}$$

where

$$N_1 = \sum_{j=0}^{n} \gamma_j X_j \quad ,$$

$$N_2 = \sum_{j=0}^{n} \gamma_j Y_j \quad ,$$

$$D_1 = \sum_{j=0}^{d} \eta_j X_j \quad ,$$

$$D_2 = \sum_{j=0}^{d} \eta_j Y_j \quad , \tag{7}$$

and

$$\beta = \cos \omega T \quad . \tag{8}$$

The terms X_j and Y_j are developed by means of the recursive relationships,

$$X_{j+1} - 2\beta X_j + X_{j-1} = 0 \quad ,$$

$$Y_{j+1} - 2\beta Y_j + Y_{j-1} = 0 \quad , \tag{9a}$$

where

$$X_0 = Y_1 = 1 \quad ,$$

$$Y_0 = 0 \quad ,$$

and

$$X_1 = \beta \quad ,$$

or by the alternate recursive relationships,

$$X_{j+1} = \beta X_j - (1 - \beta^2) Y_j \quad ,$$

$$Y_{j+1} = X_j + \beta Y_j \quad . \tag{9b}$$

See Appendix A for the development of these relationships.

Figure 2 represents a logic diagram demonstrating the ease with which a general program can be established to solve equation (6) for any plant meeting the conditions specified in the introduction. The linear portion of the plant may be of any order as long as it can be expressed in terms of a z-domain open-loop transfer function. See Appendix B for computer program.

DERIVATION

The derivation of equation (6) from equation (3) is brief. Defining $\bar{z}$ as

$$\bar{z} \equiv e^{i\omega T} \tag{10}$$

it follows that $\bar{z}^j$ may be expressed as

$$\bar{z}^j = X_j + i Y_j \sqrt{1 - \beta^2} \quad . \quad \text{(See Appendix A.)} \tag{11}$$

It is apparent from the definition of equation (10) that

$$X_0 = 1 \quad , \qquad Y_0 = 0 \quad ,$$

$$X_1 = \beta \quad , \qquad Y_1 = 1 \quad . \tag{12}$$

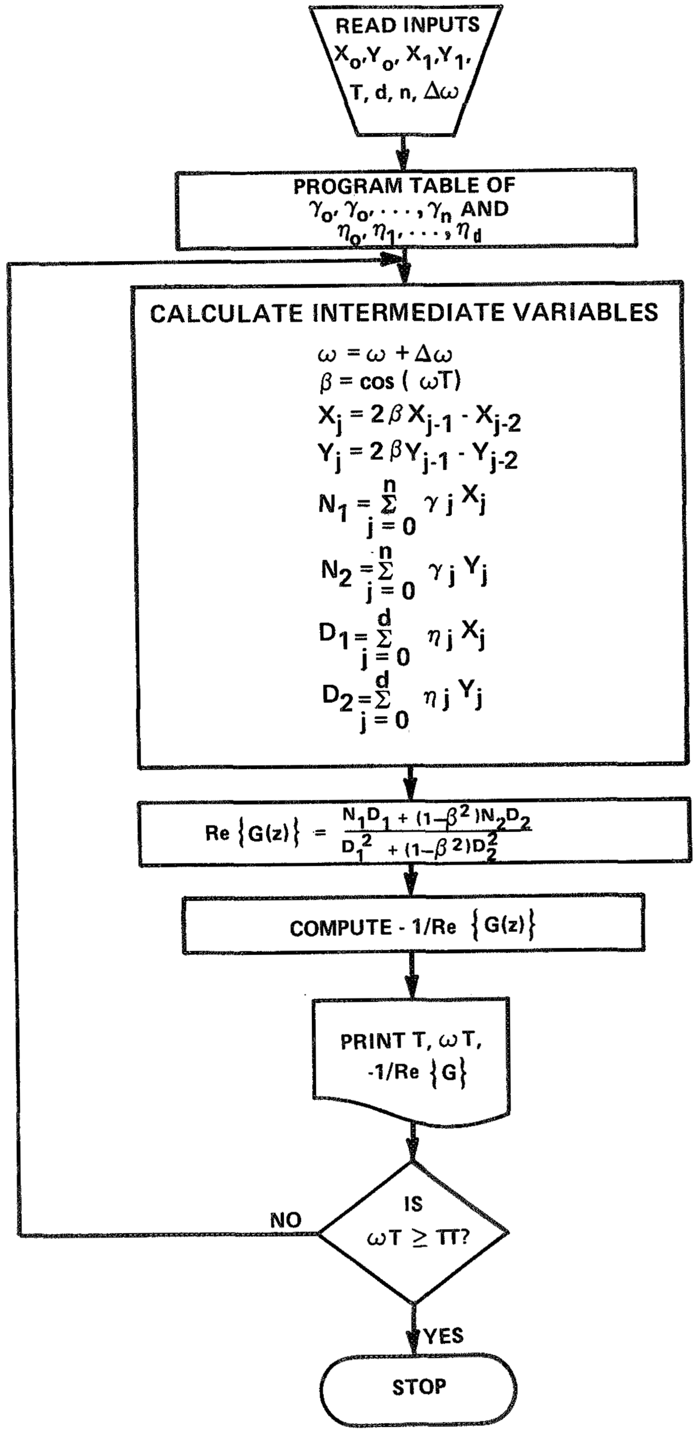

Figure 2. Digital computer logic diagram.

One may use equations (12) and the recursion relations of equations (9) to generate values of X_j and Y_j for $j > 1$. Then, equation (11) may be substituted into equation (5), letting $z = \bar{z}$,

$$G(z)\Big|_{z=e^{i\omega T}} = \frac{\sum_{j=0}^{n} \gamma_j X_j + i \sum_{j=0}^{n} \gamma_j Y_j \sqrt{1 - \beta^2}}{\sum_{j=0}^{d} \eta_j X_j + i \sum_{j=0}^{d} \eta_j Y_j \sqrt{1 - \beta^2}} . \tag{13}$$

Taking the real part of equation (13) yields the left half side of equation (6).

EXAMPLE

Several examples show how the proposed algorithm is implemented. Consider an integral control servo system [5] shown in Figure 3. The corresponding z-transform of the linear portion of the plant may be put in the form of equation (5) and is described in Table 1. By using Table 1 as an input to the digital program, the left side of inequality (6) may be solved as ωT varies in value from zero to π for selected values of T. The results are printed out, and the minimum value of

$$- 1/\mathrm{Re} \left\{ G(z)\Big|_{z=e^{i\omega T}} \right\}$$

is selected as the maximum value which K may possess. The results, reproduced in Table 2, agree with those of Reference 5. The same algorithm has been used to find stability limits on K for two other examples of Reference 5 (Bertram's example and the integral-control example with digital compensation added). The results have been identical.

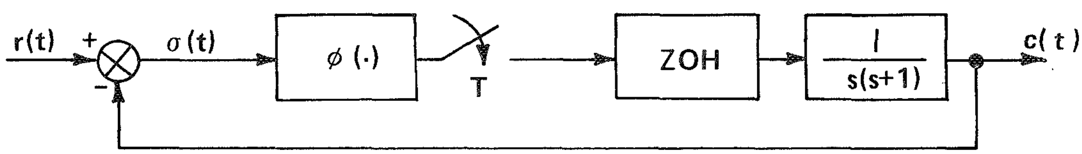

Figure 3. Model of integral-control servo system.

TABLE 1. TRANSFER FUNCTION COEFFICIENTS FOR INTEGRAL-CONTROL SERVO SYSTEM

j	γ_j	η_j
0	$1 - e^{-T} - Te^{-T}$	e^{-T}
1	$T - 1 + e^{-T}$	$-(1 + e^{-T})$
2	0	1

TABLE 2. STABILITY LIMIT ON K

	Sampling Period T				
	1	2	3	4	5
K<	0.666	0.5	0.4	0.333	0.286

CONCLUSION

A general computer program can be developed to determine the sufficient condition for stability of a class of nonlinear sampled data systems in terms of a sector within which the nonlinearity must lie. The criterion used in developing the computer algorithm is Tsypkin's development of the Popov stability criterion. Work is now in progress to investigate a similar algorithm for Jury's and Lee's [5] less restrictive stability criterion,

$$\mathrm{Re}\{G(z)[1 + q(z - 1)]\} + \frac{1}{K} - \frac{K'|q|}{2}\left|(z - 1)G(z)\right|^2 \leq 0 , \tag{14}$$

where

$$-\infty \leq \frac{d\phi(\sigma)}{d\sigma} < K' ,$$

and q is a real number.

APPENDIX A[2]

DERIVATION OF RECURSION RELATIONSHIP

The two sets of recursive relations, equations (9a) and equations (9b), are developed from the definition of $\bar{z}$,

$$\bar{z} \equiv e^{i\omega T} = \cos \omega T + i \sin \omega T \quad . \tag{10}$$

Letting

$$\beta = \cos \omega T \quad , \tag{8}$$

$\bar{z}$ may be rewritten as

$$\bar{z} = \beta + i\sqrt{1 - \beta^2} \quad . \tag{A-1}$$

If $\bar{z}$ is raised to the j^{th} power, it may be written as

$$\bar{z}^j = e^{ij\omega T} = \cos j\omega T + i \sin j\omega T \quad . \tag{A-2}$$

Two identities [6] for $\cos j\omega t$ and $\sin j\omega t$ may be rewritten in terms of β:

$$\cos j\omega T = 2^{j-1}\beta^j - \sum_{p=0}^{\frac{1}{2}(j-3)} (-1)^p \binom{j-p-2}{p} \left(\frac{j}{p+1}\right) 2^{j-2p-3} \beta^{j-2p-2} \tag{A-3}$$

and

$$\sin j\omega T = \sqrt{1 - \beta^2} \sum_{p=0}^{\frac{1}{2}(j-1)} (-1)^p \binom{j-p-1}{p} 2^{j-2p-1} \beta^{j-2p-1} \quad , \tag{A-4}$$

where p is an integer. If $\sin j\omega T$ and $\cos j\omega T$ are defined as

2. I am indebted to Mr. Hans Hosenthien, Chief, R&D Analysis Office, Astrionics Laboratory, NASA, Marshall Space Flight Center, for the development of the alternate set of recursive relations of equations (9b).

$$\cos j\omega T \equiv X_j$$

and

$$\sin j\omega T \equiv Y_j\sqrt{1-\beta^2} \quad , \tag{A-5}$$

examination of equations (A-3) and (A-4) shows that X_j and Y_j are polynomials in β. If equations (A-5) are substituted into equation (A-2), the resulting expression becomes of the form

$$\bar{z}^j = X_j + iY_j\sqrt{1-\beta^2} \quad . \tag{11}$$

The recursive relations of equations (9a) or (9b) may be derived by writing $\bar{z}^{j+1}$ as

$$\begin{aligned} \bar{z}^{j+1} &= \bar{z}^j\bar{z} \\ &= \left(X_j + iY_j\sqrt{1-\beta^2}\right)\left(\beta + i\sqrt{1-\beta^2}\right) \\ &= \left[X_j\beta - Y_j(1-\beta^2)\right] + i\left[\left(X_j + Y_j\beta\right)\sqrt{1-\beta^2}\right] \\ &= X_{j+1} + iY_{j+1} \quad , \end{aligned}$$

yielding the recursive tions

$$X_{j+1} = X_j\beta - Y_j(1-\beta^2)$$

and

$$Y_{j+1} = X_j + Y_j\beta \quad . \tag{9b}$$

From the definition of $\bar{z}$,

$$X_0 = Y_1 = 1 \quad , \qquad Y_0 = 0 \quad , \qquad X_1 = \beta \quad . \tag{A-7}$$

Using equations (9b), one may readily derive equations (9a).

APPENDIX B

DIGITAL COMPUTER PROGRAM DESCRIPTION

The Popov-Tsypkin digital program was written in Fortran IV, and computer runs were made on an SEL 840 digital computer. Using the Tsypkin version of Popov's criterion, the program calculates and prints the negative reciprocal of the real part of the z-domain transfer function representing the linear portion of the plant. Values of the negative reciprocal are calculated for incremental values of the independent argument ωT and scanned; the smallest value is selected as the upper bound of the nonlinear (or Popov) sector. The following mathematical model is described by equations (3) and (6) through (9a) and is the basis for all program calculations (see program symbol table for relations of mathematical symbols and program symbols).

Program Symbol Table

Math Symbol	Program Symbol	Description
X_0	X(1)	Initial value for X.
Y_0	Y(1)	Initial value for Y.
X_1	X(2)	Initial condition for X.
Y_1	Y(2)	Initial condition for Y.
	AIN	Increment denominator which is divided into RA to form the omega increment value.
	RA	Range value of omega.
	WI	Initial omega.
T	T	T parameter which should be set to 1.0 and associated with ωT limit values.
$\Delta\omega$	DW	Delta omega which value is used if omega equals 0.

Math Symbol	Program Symbol	Description
d	ID	Upper limit on D_1 and D_2 summation and has value d + 1.
n	N	Upper limit on N_1 and N_2 summation and has value n + 1.
K'	KP	Program flag which when set to 0 causes a Popov-Tsypkin run to be made. This relates to a program modification now under development to solve equation (14).

See program inputs for all other values.

Program Inputs

Card 1 — Reads X(1), Y(1), and Y(2) under a 4F15.8 format. These are the initial X and Y values.

Card 2 — Reads AIN, RA, and WI which are the increment value of omega, the range of omega, and the initial omega. Reads under a 4F15.8 format.

Card 3 — Reads T under an F15.8 format.

Card 4 — Reads DW, the delta omega value, under an F15.8 format.

Card 5 — Reads ID, the value for d, under an I5 format.

Card 6 — Reads N, the value for n, under an I5 format.

Card 7 — Reads KP, the value which when set to 0 causes the program to make a Popov-Tsypkin run using the initial equations to compute Re $\{G(Z)\}$ and its negative reciprocal. When KP = 1, the program omits this phase and computes a K' equation set; where KP = 2, the above options are omitted and the program solves a quadratic K equation. For the Popov-Tsypkin option always set KP = 0.

Cards 8 through 11 contain the inputs involving options other than the Popov-Tsypkin equation solutions that were part of the program study and are included in the present version of the program. These inputs are included here only for the programmers information.

Card 8 — Reads IQ, the number of Q elements, under an I5 format.

Card 9 — Reads IK, the number of K elements, under an I5 format.

Card 10 — Reads array Q, containing all values of q, under a 4F15.8 format.

Card 11 — Reads array AK, containing all K elements, under a 4F15.8 format.

Sample Run Case

The following sample case was run successfully on an SEL 840 digital computer. The appropriate gamma and eta functions listed in Table 1 of the computer printout were used in the Popov-Tsypkin program with the input data listed here and described under the card input data. The input data will always be followed by the printed variable values for T, ωT, Re $\{G(Z)\}$, and $K < -1/\text{Re}\,\{G(Z)\}$.

```
C* * * * * * * * * * * * *  PØPØV - TSYPKIN PRØGRAM    * * * * * * * * * *
      DIMENSIØN Q(20),AK(20)
      DIMENSIØN X(20),Y(20),G(20),E(20)
      DIMENSIØN IALA(20)
      TRACE 9999
      READ (4,10) X(1),Y(1),Y(2)
      READ (4,10) AIN,RA,WI
      READ (4,11) T
      READ (4,11) DW
      READ (4,12) ID
      READ (4,12) N
      READ (4,12) KP
      READ (4,12) IQ
      READ (4,12) IK
      READ (4,10) (Q(I),I=1,IQ)
      READ (4,10) (AK(I),I=1,IK)
      READ (4,120) IALA
 10   FØRMAT (4F15.8)
 11   FØRMAT (F15.8)
 12   FØRMAT (I5)
 120  FØRMAT (20A4)
      WRITE (5,120) IALA
      WRITE (5,20)
 20   FØRMAT(5X,10HINPUT DATA//)
      WRITE (5,21) X(1),Y(1),Y(2)
 21   FØRMAT (10X,7HX(1) = ,F10.5,3X,7HY(1) = ,F10.5,3X,7HY(2) = ,F10.5)
      WRITE (5,26) AIN,RA,WI
 26   FØRMAT (10X,18HØMEGA INCREMENT = ,F10.5,3X,14HØMEGA RANGE = ,F10.5
     1,3X,16HØMEGA INITIAL = ,F10.5)
      WRITE (5,22) ID
 22   FØRMAT (10X,4HD = ,I5)
      WRITE (5,23) N
 23   FØRMAT (10X,4HN = ,I5)
      WRITE (5,24) T
 24   FØRMAT (10X,4HT = ,F15.8)
      WRITE (5,27) DW
 27   FØRMAT (10X,14HDELTA ØMEGA = ,F15.8)
      WRITE (5,500) KP,IQ,IK
 500  FØRMAT (10X,5HKP = ,I5,5X,5HIQ = ,I5,5X,5HIK = ,I5)
      WRITE (5,501) (I,Q(I),I=1,IQ)
 501  FØRMAT (10X,2HQ(,I2,4H) = ,F10.5)
      WRITE (5,502) (I,AK(I),I=1,IK)
 502  FØRMAT (10X,2HK(,I2,4H) = ,F10.5)
      W=0.0
C***FØRM GAMMA AND ETA FUNCTIØN
      E1=EXP(-T)
      G(1)=(1.0-E1)-T*E1
      G(2)=T-(1.0-E1)
      E(1)=E1
      E(2)=-(1.0+E1)
      E(3)=1.0
C***FØRM REAL RØØT FUNCTIØN
      IN=AIN
      IC=IN+1
      VA=RA/AIN
      IF (KP-1) 4010,401,401
 4010 CØNTINUE
```

```
      WRITE (5,41)
41    FØRMAT (11X,1HT,18X,2HWT,14X,8HRE(G(Z)),4X,23HK LESS THAN -1/RE(G(
     1Z)))
401   CØNTINUE
      IF (KP-1) 39,301,390
39    CØNTINUE
      DØ 40 I=1,IC
      AI=I
      W=WI+VA*AI
      IF (W.EQ.0.0) W=DW
      WT=W*T
      B=CØS(WT)
      X(2)=B
      IF (ID.LT.3) GØ TØ 410
      DØ 30 J=3,ID
      X(J)=2.0*B*X(J-1)-X(J-2)
      Y(J)=2.0*B*Y(J-1)-Y(J-2)
30    CØNTINUE
      CALL SUM (N,ID,IP,X,Y,E,G,AN1,AN2,D1,D2)
      GØ TØ 300
410   CØNTINUE
      CALL SUM (N,ID,IP,X,Y,E,G,AN1,AN2,D1,D2)
300   CØNTINUE
      ANUM=AN1*D1+(1.0-B**2)*AN2*D2
      DEN=D1*D1+(1.0-B**2)*D2*D2
      RE=ANUM/DEN
      REC=-1.0/RE
      WRITE (5,31) T,WT,RE,REC
31    FØRMAT (3X,E15.8,5X,E15.8,5X,E15.8,5X,E15.8)
40    CØNTINUE
      GØ TØ 4200
301   CØNTINUE
      DØ 380 IJ=1,IK
      DØ 380 II=1,IQ
      DØ 380 I=1,IC
      AI=I
      W=WI+VA*AI
      IF (W.EQ.0.0) W=DW
      WT=W*T
      B=CØS(WT)
      X(2)=B
      BM1=1.0-B
      BSQ=1.0-B*B
      IF (ID.LT.3) GØ TØ 4100
      DØ 3000 J=3,ID
      X(J)=2.0*B*X(J-1)-X(J-2)
      Y(J)=2.0*B*Y(J-1)-Y(J-2)
3000  CØNTINUE
      CALL SUM (N,ID,IP,X,Y,E,G,AN1,AN2,D1,D2)
      GØ TØ 3111
4100  CØNTINUE
      CALL SUM (N,ID,IP,X,Y,E,G,AN1,AN2,D1,D2)
3111  CØNTINUE
      RK=1.0/AK(IJ)
      DBN=AN1*D1+BSQ*AN2*D2
      QB=1.0-Q(II)*BM1
      BQ=Q(II)*BSQ
      DN=AN1*D2-AN2*D1
```

```
      DB=D1*D1+BSQ*D2*D2
      TØP=DBN*QB+BQ*DN+RK*DB
      AQB=Q(II)*BM1
      BN=AN1*AN1+BSQ*AN2*AN2
      BØT=AQB*BN
      AKP=TØP/BØT
      WRITE (5,383) IJ,AK(IJ),II,Q(II),WT,AKP
 383  FØRMAT (3X,2HK(,I2,4H) = ,E15.8,2X,2HQ(,I2,4H) = ,E15.8,2X,5HWT =
     1,E15.8,2X,7HK PR = ,E15.8)
 380  CØNTINUE
      GØ TØ 4200
 390  CØNTINUE
      DØ 399 II=1,IQ
      DØ 399 I=1,IC
      AI=I
      W=WI+VA*AI
      IF (W.EQ.0.0) W=DW
      WT=W*T
      B=CØS(WT)
      X(2)=B
      BM1=1.0-B
      BSQ=1.0-B*B
      IF (ID.LT.3) GØ TØ 4101
      DØ 3001 J=3,ID
      X(J)=2.0*B*X(J-1)-X(J-2)
      Y(J)=2.0*B*Y(J-1)-Y(J-2)
 3001 CØNTINUE
      CALL SUM (N,ID,IP,X,Y,E,G,AN1,AN2,D1,D2)
      GØ TØ 3112
 4101 CØNTINUE
      CALL SUM (N,ID,IP,X,Y,E,G,AN1,AN2,D1,D2)
 3112 CØNTINUE
      AA=Q(II)*BM1*(AN1*AN1+BSQ*AN2*AN2)
      BB=-(AN1*D1+BSQ*AN2*D2)*(1.0-Q(II)*BM1)
      CC=-(D1*D1+BSQ*D2*D2)
      DISC=SQRT(BB*BB-4.0*AA*CC)
      AK1=(-BB+DISC)/(2.0*AA)
      AK2=(-BB-DISC)/(2.0*AA)
      WRITE (5,3990) (II,Q(II),AK1,AK2,WT)
 3990 FØRMAT (3X,2HQ(,I2,4H) = ,E15.8,5X,5HK1 = ,E15.8,5X,5HK2 = ,E15.8,
     15X,5HWT = ,E15.8)
 399  CØNTINUE
 4200 CØNTINUE
      WRITE (5,42)
 42   FØRMAT (25X,14HCASE CØMPLETED)
      WRITE (5,43)
 43   FØRMAT (1H1)
      GØ TØ 1
      END
 PATH
00000       Q
00050       AK
00120       X
00170       Y
00240       G
00310       E
00360       IALA
 UNRF       $9999
```

00405	$1
00576	$10
03251	AIN
03253	RA
03255	WI
00601	$11
03257	T
03261	DW
00604	$12
03263	ID
03264	N
03265	KP
03266	IQ
03267	IK
03270	I
03271	T$1000
00606	$120
00624	$20
00647	$21
00704	$26
00743	$22
00757	$23
00773	$24
01010	$27
01036	$500
01100	$501
01136	$502
03272	W
03276	E1
SUBR	EXP
03300	T$2000
03304	IN
03305	IC
03306	VA
01266	$4010
01316	$401
01275	$41
01330	$39
01705	$301
02502	$390
01674	$40
03310	AI
03312	WT
03314	B
SUBR	CØS
01537	$410
01512	$30
03316	J
03321	T$1001
03322	T$1002
03323	T$1003
SUBR	SUM
03324	IP
03325	AN1
03327	AN2
03331	D1
03333	D2
01556	$300
03335	ANUM
03337	DEN
03341	RE
03343	REC
01661	$31
03217	$4200
02457	$380
03347	IJ
03350	II
03351	BM1
03353	BSQ
02137	$4100
02112	$3000
02156	$3111
03355	RK
03357	DBN
03361	QB
03363	BQ
03365	DN
03367	DB
03371	TØP
03373	T$2001
03375	AQB
03377	BN
03401	BØT
03403	AKP
02430	$383
03202	$399
02732	$4101
02705	$3001
02751	$3112
03405	AA
03407	BB
03411	T$2002
03413	CC
03415	DISC
SUBR	SQRT
03421	AK1
03423	AK2
03156	$3990
03226	$42
03241	$43

```
      SUBRØUTINE SUM (N,ID,IP,X,Y,E,G,AN1,AN2,D1,D2)
      DIMENSIØN X(20),Y(20),G(20),E(20)
      TRACE 9999
      AN1=0.0
      AN2=0.0
      D1=0.0
      D2=0.0
      DØ 10 J=1,N
      AN1=G(J)*X(J)+AN1
      AN2=G(J)*Y(J)+AN2
 10   CØNTINUE
      DØ 11 J=1,ID
      D1=E(J)*X(J)+D1
      D2=E(J)*Y(J)+D2
 11   CØNTINUE
      RETURN
      END
00000     N        (DUMMY)
00001     ID       (DUMMY)
00003     X        (DUMMY)
00004     Y        (DUMMY)
00005     E        (DUMMY)
00006     G        (DUMMY)
00007     AN1      (DUMMY)
00010     AN2      (DUMMY)
00011     D1       (DUMMY)
00012     D2       (DUMMY)
 UNRF     $9999
00115     $10
00212     J
00213     T$1000
00214     T$1001
00215     T$1002
00176     $11
```

TABLE 1 04/19/71

INPUT DATA

```
X(1) =     1.00000   Y(1) =      .00000   Y(2) =     1.00000
ØMEGA INCREMENT =  314.00000   ØMEGA RANGE =     3.14000   ØMEGA INITIAL =      .00000
D =       3
N =       2
T =      1.00000000
DELTA ØMEGA =          .00100000
KP =      0      IQ =      9     IK =      4
Q( 1) =      .00010
Q( 2) =      .20000
Q( 3) =      .40000
Q( 4) =      .60000
Q( 5) =      .80000
Q( 6) =     1.00000
Q( 7) =     2.00000
Q( 8) =    10.00000
Q( 9) =   100.00000
K( 1) =      .48990
K( 2) =      .49000
K( 3) =     1.99000
K( 4) =     2.00000
```

T	WT	RE(G(Z))	K LESS THAN -1/RE(G(Z))
.10000000E 01	.10000000E-01	-.14998289E 01	.66674273E 00
.10000000E 01	.20000000E-01	-.14993156E 01	.66697097E 00
.10000000E 01	.30000000E-01	-.14984609E 01	.66735141E 00
.10000000E 01	.40000000E-01	-.14972657E 01	.66788411E 00
.10000000E 01	.50000000E-01	-.14957316E 01	.66856916E 00
.10000000E 01	.60000000E-01	-.14938602E 01	.66940668E 00
.10000000E 01	.70000000E-01	-.14916539E 01	.67039679E 00
.10000000E 01	.80000000E-01	-.14891153E 01	.67153967E 00
.10000000E 01	.90000000E-01	-.14862474E 01	.67283550E 00
.10000000E 01	.10000000E 00	-.14830535E 01	.67428449E 00
.10000000E 01	.11000000E 00	-.14795375E 01	.67588689E 00
.10000000E 01	.12000000E 00	-.14757034E 01	.67764296E 00
.10000000E 01	.13000000E 00	-.14715556E 01	.67955299E 00
.10000000E 01	.14000000E 00	-.14670989E 01	.68161730E 00
.10000000E 01	.15000000E 00	-.14623385E 01	.68383621E 00
.10000000E 01	.16000000E 00	-.14572796E 01	.68621012E 00
.10000000E 01	.17000000E 00	-.14519280E 01	.68873939E 00
.10000000E 01	.18000000E 00	-.14462896E 01	.69142447E 00
.10000000E 01	.19000000E 00	-.14403706E 01	.69426578E 00
.10000000E 01	.20000000E 00	-.14341774E 01	.69726380E 00
.10000000E 01	.21000000E 00	-.14277168E 01	.70041904E 00
.10000000E 01	.22000000E 00	-.14209955E 01	.70373201E 00
.10000000E 01	.23000000E 00	-.14140206E 01	.70720326E 00
.10000000E 01	.24000000E 00	-.14067994E 01	.71083339E 00
.10000000E 01	.25000000E 00	-.13993393E 01	.71462298E 00
.10000000E 01	.26000000E 00	-.13916477E 01	.71857267E 00
.10000000E 01	.27000000E 00	-.13837323E 01	.72268314E 00
.10000000E 01	.28000000E 00	-.13756009E 01	.72695505E 00
.10000000E 01	.29000000E 00	-.13672612E 01	.73138914E 00
.10000000E 01	.30000000E 00	-.13587212E 01	.73598614E 00
.10000000E 01	.31000000E 00	-.13499889E 01	.74074683E 00
.10000000E 01	.32000000E 00	-.13410722E 01	.74567201E 00
.10000000E 01	.33000000E 00	-.13319791E 01	.75076251E 00

.10000000E 01	.34000000E 00	-.13227177E 01	.75601919E 00
.10000000E 01	.35000000E 00	-.13132960E 01	.76144295E 00
.10000000E 01	.36000000E 00	-.13037220E 01	.76703470E 00
.10000000E 01	.37000000E 00	-.12940036E 01	.77279540E 00
.10000000E 01	.38000000E 00	-.12841487E 01	.77872602E 00
.10000000E 01	.39000000E 00	-.12741652E 01	.78482757E 00
.10000000E 01	.40000000E 00	-.12640609E 01	.79110111E 00
.10000000E 01	.41000000E 00	-.12538435E 01	.79754769E 00
.10000000E 01	.42000000E 00	-.12435206E 01	.80416844E 00
.10000000E 01	.43000000E 00	-.12330996E 01	.81096448E 00
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.10000000E 01	.45000000E 00	-.12119931E 01	.82508718E 00
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.10000000E 01	.47000000E 00	-.11905817E 01	.83992555E 00
.10000000E 01	.48000000E 00	-.11797791E 01	.84761631E 00
.10000000E 01	.49000000E 00	-.11689209E 01	.85548990E 00
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.10000000E 01	.51000000E 00	-.11470638E 01	.87179110E 00
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.10000000E 01	.54000000E 00	-.11140204E 01	.89764965E 00
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.10000000E 01	.56000000E 00	-.10918887E 01	.91584428E 00
.10000000E 01	.57000000E 00	-.10808088E 01	.92523306E 00
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.10000000E 01	.59000000E 00	-.10586470E 01	.94460192E 00
.10000000E 01	.60000000E 00	-.10475751E 01	.95458548E 00
.10000000E 01	.61000000E 00	-.10365156E 01	.96477084E 00
.10000000E 01	.62000000E 00	-.10254729E 01	.97515983E 00
.10000000E 01	.63000000E 00	-.10144515E 01	.98575433E 00
.10000000E 01	.64000000E 00	-.10034556E 01	.99655628E 00
.10000000E 01	.65000000E 00	-.99248921E 00	.10075676E 01
.10000000E 01	.66000000E 00	-.98155617E 00	.10187904E 01
.10000000E 01	.67000000E 00	-.97066021E 00	.10302266E 01
.10000000E 01	.68000000E 00	-.95980486E 00	.10418785E 01
.10000000E 01	.69000000E 00	-.94899350E 00	.10537480E 01
.10000000E 01	.70000000E 00	-.93822936E 00	.10658375E 01
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.10000000E 01	.72000000E 00	-.91685485E 00	.10906852E 01
.10000000E 01	.73000000E 00	-.90625016E 00	.11034481E 01
.10000000E 01	.74000000E 00	-.89570407E 00	.11164402E 01
.10000000E 01	.75000000E 00	-.88521905E 00	.11296639E 01
.10000000E 01	.76000000E 00	-.87479744E 00	.11431218E 01
.10000000E 01	.77000000E 00	-.86444146E 00	.11568163E 01
.10000000E 01	.78000000E 00	-.85415318E 00	.11707502E 01
.10000000E 01	.79000000E 00	-.84393456E 00	.11849260E 01
.10000000E 01	.80000000E 00	-.83378741E 00	.11993465E 01
.10000000E 01	.81000000E 00	-.82371344E 00	.12140144E 01
.10000000E 01	.82000000E 00	-.81371425E 00	.12289326E 01
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.10000000E 01	.85000000E 00	-.78417955E 00	.12752182E 01
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.10000000E 01	.88000000E 00	-.75536462E 00	.13238640E 01
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.10000000E 01	.91000000E 00	-.72729568E 00	.13749566E 01

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.10000000E 01	.94000000E 00	-.69999184E 00	.14285881E 01
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.10000000E 01	.96000000E 00	-.68222089E 00	.14658009E 01
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.10000000E 01	.10000000E 01	-.64772511E 00	.15438648E 01
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.10000000E 01	.10300000E 01	-.62277190E 00	.16057243E 01
.10000000E 01	.10400000E 01	-.61462899E 00	.16269978E 01
.10000000E 01	.10500000E 01	-.60657326E 00	.16486055E 01
.10000000E 01	.10600000E 01	-.59860452E 00	.16705520E 01
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.10000000E 01	.15000000E 01	-.32661175E 00	.30617392E 01
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.10000000E 01	.15200000E 01	-.31743837E 00	.31502178E 01
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.10000000E 01	.17400000E 01	-.23079561E 00	.43328381E 01
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.10000000E 01	.18000000E 01	-.21119930E 00	.47348642E 01
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.10000000E 01	.18200000E 01	-.20500734E 00	.48778742E 01
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.10000000E 01	.19000000E 01	-.18183613E 00	.54994572E 01
.10000000E 01	.19100000E 01	-.17911175E 00	.55831067E 01
.10000000E 01	.19200000E 01	-.17642412E 00	.56681591E 01
.10000000E 01	.19300000E 01	-.17377283E 00	.57546396E 01
.10000000E 01	.19400000E 01	-.17115745E 00	.58425737E 01
.10000000E 01	.19500000E 01	-.16857756E 00	.59319875E 01
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.10000000E 01	.19700000E 01	-.16352267E 00	.61153599E 01
.10000000E 01	.19800000E 01	-.16104687E 00	.62093724E 01
.10000000E 01	.19900000E 01	-.15860498E 00	.63049724E 01
.10000000E 01	.20000000E 01	-.15619661E 00	.64021877E 01
.10000000E 01	.20100000E 01	-.15382136E 00	.65010467E 01
.10000000E 01	.20200000E 01	-.15147894E 00	.66015779E 01
.10000000E 01	.20300000E 01	-.14916890E 00	.67038104E 01
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.10000000E 01	.20600000E 01	-.14242965E 00	.70210100E 01
.10000000E 01	.20700000E 01	-.14024569E 00	.71303437E 01

.10000000E 01	.20800000E 01	-.13809239E 00	.72415284E 01
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.10000000E 01	.22600000E 01	-.10421926E 00	.95951551E 01
.10000000E 01	.22700000E 01	-.10259068E 00	.97474739E 01
.10000000E 01	.22800000E 01	-.10098700E 00	.99022643E 01
.10000000E 01	.22900000E 01	-.99407988E-01	.10059554E 02
.10000000E 01	.23000000E 01	-.97853399E-01	.10219369E 02
.10000000E 01	.23100000E 01	-.96323004E-01	.10381736E 02
.10000000E 01	.23200000E 01	-.94816575E-01	.10546679E 02
.10000000E 01	.23300000E 01	-.93333887E-01	.10714222E 02
.10000000E 01	.23400000E 01	-.91874721E-01	.10884387E 02
.10000000E 01	.23500000E 01	-.90438859E-01	.11057194E 02
.10000000E 01	.23600000E 01	-.89026090E-01	.11232662E 02
.10000000E 01	.23700000E 01	-.87636203E-01	.11410809E 02
.10000000E 01	.23800000E 01	-.86268995E-01	.11591650E 02
.10000000E 01	.23900000E 01	-.84924263E-01	.11775198E 02
.10000000E 01	.24000000E 01	-.83601811E-01	.11961463E 02
.10000000E 01	.24100000E 01	-.82301443E-01	.12150455E 02
.10000000E 01	.24200000E 01	-.81022970E-01	.12342179E 02
.10000000E 01	.24300000E 01	-.79766204E-01	.12536638E 02
.10000000E 01	.24400000E 01	-.78530961E-01	.12733831E 02
.10000000E 01	.24500000E 01	-.77317063E-01	.12933756E 02
.10000000E 01	.24600000E 01	-.76124332E-01	.13136404E 02
.10000000E 01	.24700000E 01	-.74952595E-01	.13341766E 02
.10000000E 01	.24800000E 01	-.73801681E-01	.13549827E 02
.10000000E 01	.24900000E 01	-.72671426E-01	.13760567E 02
.10000000E 01	.25000000E 01	-.71561664E-01	.13973962E 02
.10000000E 01	.25100000E 01	-.70472236E-01	.14189985E 02
.10000000E 01	.25200000E 01	-.69402985E-01	.14408602E 02
.10000000E 01	.25300000E 01	-.68353757E-01	.14629774E 02
.10000000E 01	.25400000E 01	-.67324402E-01	.14853455E 02
.10000000E 01	.25500000E 01	-.66314772E-01	.15079596E 02
.10000000E 01	.25600000E 01	-.65324723E-01	.15308140E 02
.10000000E 01	.25700000E 01	-.64354113E-01	.15539022E 02
.10000000E 01	.25800000E 01	-.63402803E-01	.15772173E 02
.10000000E 01	.25900000E 01	-.62470659E-01	.16007515E 02
.10000000E 01	.26000000E 01	-.61557547E-01	.16244962E 02
.10000000E 01	.26100000E 01	-.60663338E-01	.16484421E 02
.10000000E 01	.26200000E 01	-.59787905E-01	.16725791E 02
.10000000E 01	.26300000E 01	-.58931124E-01	.16968962E 02
.10000000E 01	.26400000E 01	-.58092874E-01	.17213815E 02
.10000000E 01	.26500000E 01	-.57273036E-01	.17460223E 02

.10000000E 01	.26600000E 01	-.56471496E-01	.17708049E 02
.10000000E 01	.26700000E 01	-.55688140E-01	.17957145E 02
.10000000E 01	.26800000E 01	-.54922857E-01	.18207356E 02
.10000000E 01	.26900000E 01	-.54175542E-01	.18458514E 02
.10000000E 01	.27000000E 01	-.53446088E-01	.18710443E 02
.10000000E 01	.27100000E 01	-.52734394E-01	.18962956E 02
.10000000E 01	.27200000E 01	-.52040359E-01	.19215855E 02
.10000000E 01	.27300000E 01	-.51363888E-01	.19468931E 02
.10000000E 01	.27400000E 01	-.50704885E-01	.19721965E 02
.10000000E 01	.27500000E 01	-.50063260E-01	.19974728E 02
.10000000E 01	.27600000E 01	-.49438922E-01	.20226978E 02
.10000000E 01	.27700000E 01	-.48831784E-01	.20478465E 02
.10000000E 01	.27800000E 01	-.48241763E-01	.20728927E 02
.10000000E 01	.27900000E 01	-.47668776E-01	.20978092E 02
.10000000E 01	.28000000E 01	-.47112745E-01	.21225679E 02
.10000000E 01	.28100000E 01	-.46573592E-01	.21471395E 02
.10000000E 01	.28200000E 01	-.46051242E-01	.21714941E 02
.10000000E 01	.28300000E 01	-.45545623E-01	.21956006E 02
.10000000E 01	.28400000E 01	-.45056667E-01	.22194274E 02
.10000000E 01	.28500000E 01	-.44584304E-01	.22429418E 02
.10000000E 01	.28600000E 01	-.44128471E-01	.22661107E 02
.10000000E 01	.28700000E 01	-.43689103E-01	.22889003E 02
.10000000E 01	.28800000E 01	-.43266142E-01	.23112761E 02
.10000000E 01	.28900000E 01	-.42859529E-01	.23332034E 02
.10000000E 01	.29000000E 01	-.42469207E-01	.23546472E 02
.10000000E 01	.29100000E 01	-.42095124E-01	.23755720E 02
.10000000E 01	.29200000E 01	-.41737228E-01	.23959425E 02
.10000000E 01	.29300000E 01	-.41395470E-01	.24157233E 02
.10000000E 01	.29400000E 01	-.41069803E-01	.24348790E 02
.10000000E 01	.29500000E 01	-.40760183E-01	.24533746E 02
.10000000E 01	.29600000E 01	-.40466568E-01	.24711757E 02
.10000000E 01	.29700000E 01	-.40188916E-01	.24882482E 02
.10000000E 01	.29800000E 01	-.39927191E-01	.25045589E 02
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.10000000E 01	.30000000E 01	-.39451379E-01	.25347657E 02
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.10000000E 01	.30900000E 01	-.38090685E-01	.26253138E 02
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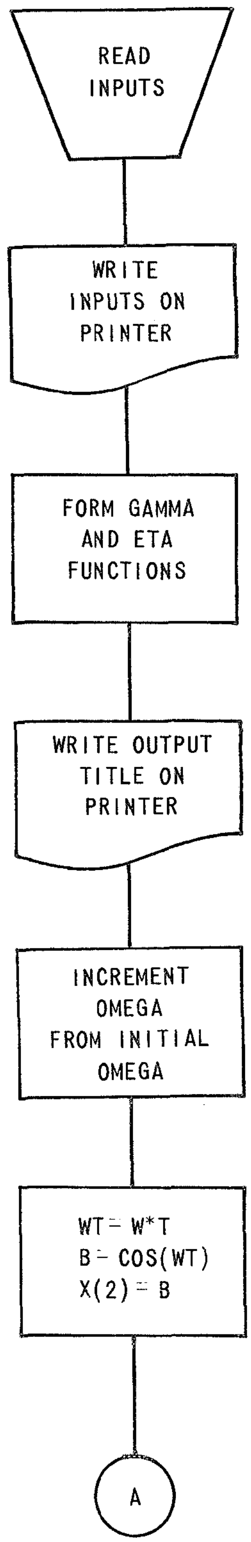
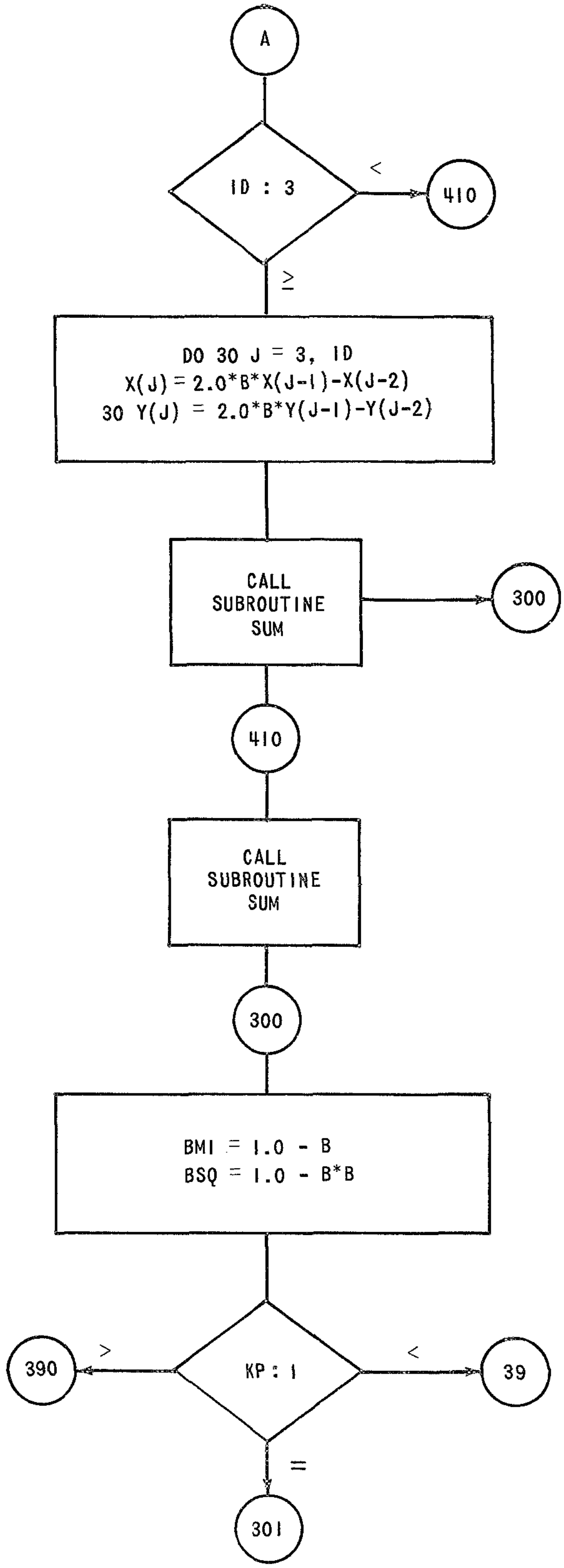
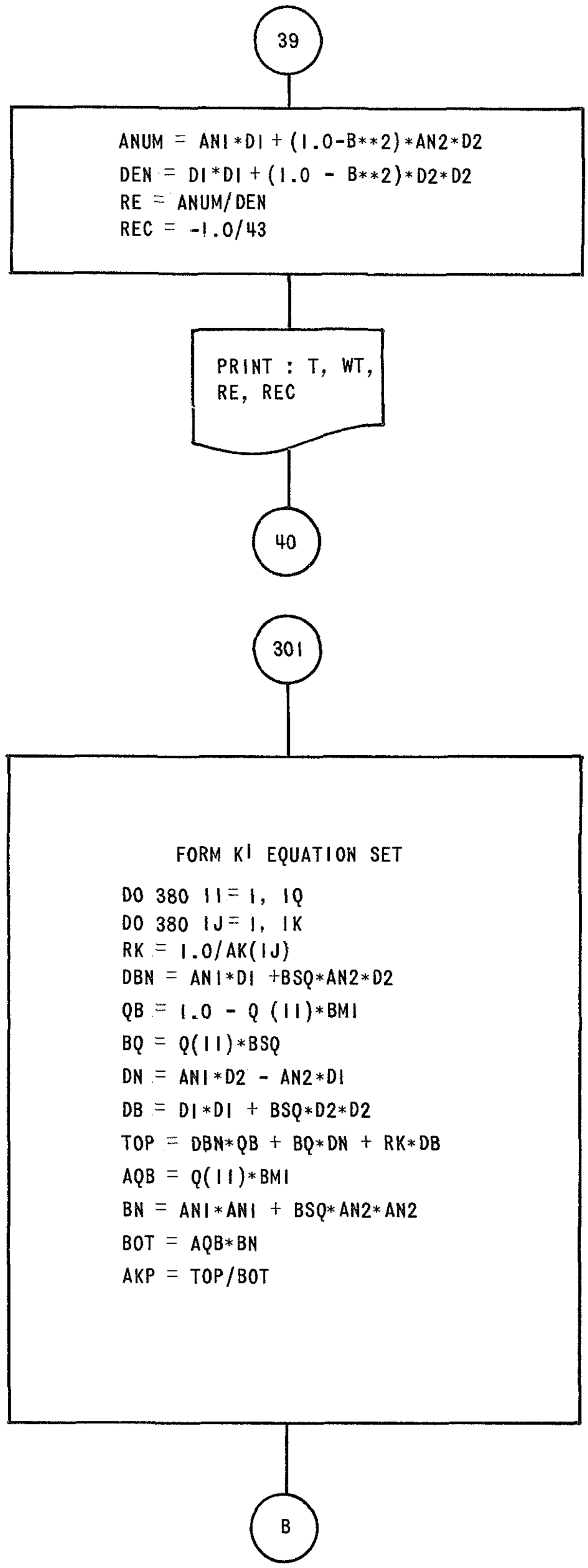
.10000000E 01	.31300000E 01	-.37893333E-01	.26389866E 02
.10000000E 01	.31400000E 01	-.37883041E-01	.26397036E 02
.10000000E 01	.31500000E 01	-.37888360E-01	.26393330E 02

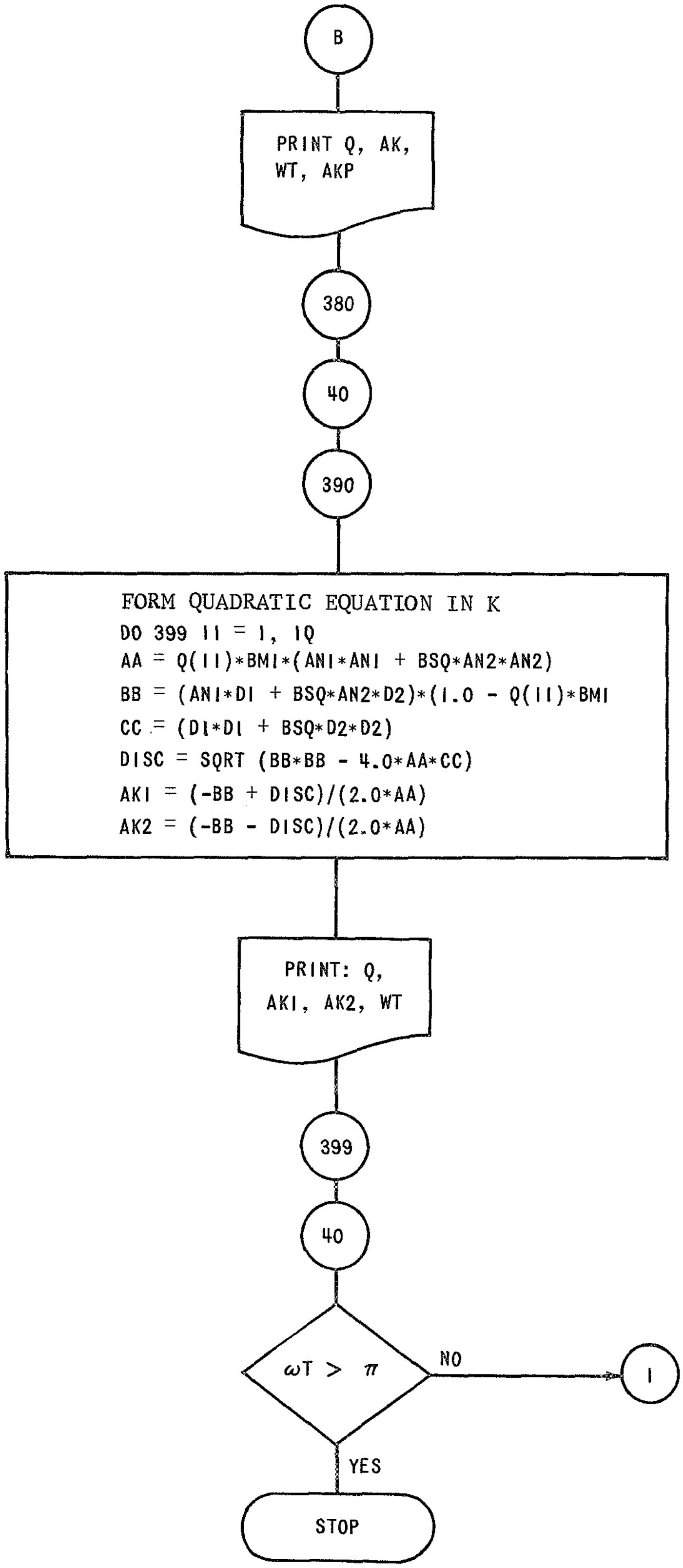
CASE CØMPLETED

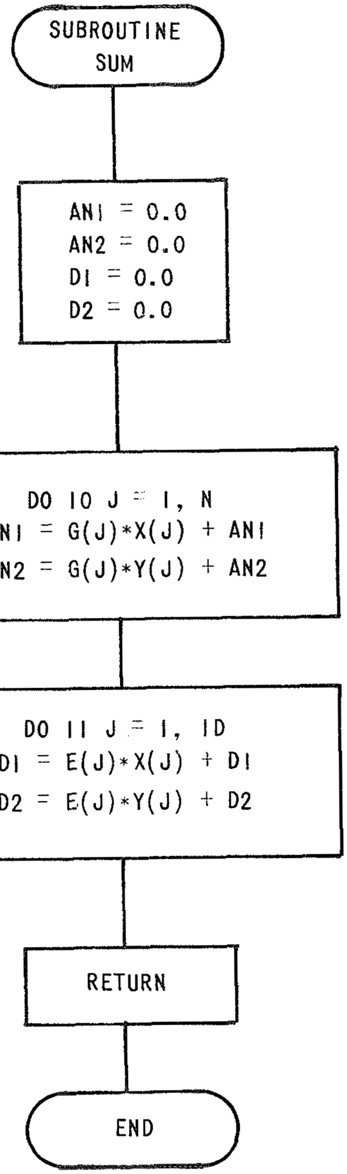
POPOV - TSYPKIN MAIN PROGRAM
FLOW DIAGRAM




A
ID : 3
<
410
≥
DO 30 J = 3, ID
X(J) = 2.0*B*X(J-1)-X(J-2)
30 Y(J) = 2.0*B*Y(J-1)-Y(J-2)
CALL SUBROUTINE SUM
300
410
CALL SUBROUTINE SUM
300
BMI = 1.0 - B
BSQ = 1.0 - B*B
KP : 1
>
390
<
39
=
301


39
ANUM = AN1*D1 + (1.0-B**2)*AN2*D2
DEN = D1*D1 + (1.0 - B**2)*D2*D2
RE = ANUM/DEN
REC = -1.0/43
PRINT : T, WT, RE, REC
40
301
FORM K1 EQUATION SET
DO 380 I1= 1, IQ
DO 380 IJ= 1, IK
RK = 1.0/AK(IJ)
DBN = AN1*D1 +BSQ*AN2*D2
QB = 1.0 - Q (I1)*BM1
BQ = Q(I1)*BSQ
DN = AN1*D2 - AN2*D1
DB = D1*D1 + BSQ*D2*D2
TOP = DBN*QB + BQ*DN + RK*DB
AQB = Q(I1)*BM1
BN = AN1*AN1 + BSQ*AN2*AN2
BOT = AQB*BN
AKP = TOP/BOT
B


B
PRINT Q, AK,
WT, AKP
380
40
390
FORM QUADRATIC EQUATION IN K
DO 399 I1 = 1, IQ
AA = Q(I1)*BM1*(AN1*AN1 + BSQ*AN2*AN2)
BB = (AN1*D1 + BSQ*AN2*D2)*(1.0 - Q(I1)*BM1
CC = (D1*D1 + BSQ*D2*D2)
DISC = SQRT (BB*BB - 4.0*AA*CC)
AK1 = (-BB + DISC)/(2.0*AA)
AK2 = (-BB - DISC)/(2.0*AA)
PRINT: Q,
AK1, AK2, WT
399
40
ωT > π
NO
1
YES
STOP


SUBROUTINE
SUM
AN1 = 0.0
AN2 = 0.0
D1 = 0.0
D2 = 0.0
DO 10 J = 1, N
AN1 = G(J)*X(J) + AN1
AN2 = G(J)*Y(J) + AN2
DO 11 J = 1, ID
D1 = E(J)*X(J) + D1
D2 = E(J)*Y(J) + D2
RETURN
END

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APPROVAL

NASA TM X-64600

AN ALGORITHM FOR SOLVING THE POPOV CRITERION APPLIED TO SAMPLED DATA SYSTEMS

By S. M. Seltzer

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This document has also been reviewed and approved for technical accuracy.

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