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ANALYSIS OF A CONTROL SYSTEM CONTAINING TWO NONLINEARITIES

By S. M. Seltzer
Astrionics Laboratory

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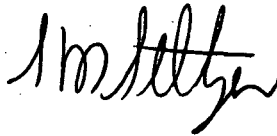| 1. REPORT NO. NASA TM X-64601	2. GOVERNMENT ACCESSION NO.	3. RECIPIENT'S CATALOG NO.
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16. ABSTRACT

A control system containing two nonlinear elements may be analyzed in the parameter plane to determine the existence, stability, and values of amplitude and frequency of limit cycles. By using describing functions to represent the nonlinearities, the system characteristic equation may be obtained. Two adjustable parameters are selected, each containing one of the describing functions and possibly several control system gains. A correlation between these parameters and the roots of the characteristic equation is determined by mapping stability contours from the complex s-plane onto the chosen parameter plane. A relationship between the inputs to the two nonlinearities is determined next, and a locus representing the variation of the describing functions also is plotted on the parameter plane. If this locus and the stability contour associated with a pair of pure imaginary roots intersect, the existence and characteristics of a limit cycle are indicated. Particular emphasis is placed on organizing the equations so that they may be readily solved with a digital computer.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ANALYSIS OF A CONTROL SYSTEM
CONTAINING TWO NONLINEARITIES

SUMMARY

A control system containing two nonlinear elements may be analyzed in the parameter plane to determine the existence, stability, and values of amplitude and frequency of limit cycles. By using describing functions to represent the nonlinearities, the system characteristic equation may be obtained. Two adjustable parameters are selected, each containing one of the describing functions and possibly several control system gains. A correlation between these parameters and the roots of the characteristic equation is determined by mapping stability contours from the complex s-plane onto the chosen parameter plane. A relationship between the inputs to the two nonlinearities is determined next, and a locus representing the variation of the describing functions also is plotted on the parameter plane. If this locus and the stability contour associated with a pair of pure imaginary roots intersect, the existence and characteristics of a limit cycle are indicated. Particular emphasis is placed on organizing the equations so that they may be readily solved with a digital computer.

INTRODUCTION

A technique has been developed by Siljak [1] to analyze systems containing two nonlinear elements if they have a common input or if their inputs are related by a linear differential equation. Viswanadham and Deekshatulu [2] and Gelb and Vander Velde [3] present techniques for analyzing a system whose inputs to the nonlinear elements are related by a nonlinear differential equation. A technique similar to that developed by Siljak is presented here to enable one to analyze this latter system.

By using describing functions to represent the nonlinearities, the system characteristic equation may be obtained. Two adjustable parameters are selected, each containing one of the describing functions and possibly several control system gains. A correlation between these parameters and the roots of the characteristic equation is determined by mapping stability contours from the complex s-plan onto the chosen parameter plane.

A relationship between the inputs to the two nonlinearities is determined next, and a locus representing the variation of the describing functions also is plotted on the parameter plane. If this locus and the stability contour associated with a pair of pure imaginary roots intersect, the existence and characteristics of a limit cycle are indicated. From this point of intersection, the frequency and magnitude of the indicated limit cycle may be determined as a function of the characteristics of the nonlinearities and of the adjustable control system gains. The stability of the limit cycle is investigated by determining if all characteristic equation roots, other than the pair of pure imaginary roots, have negative real parts. This condition is indicated readily on the parameter plane. The behavior of the limit cycle when a small perturbation is applied to its amplitude also is apparent on the parameter plane.

To demonstrate the application of this technique, a typical attitude control system for a large space vehicle is analyzed. The results of the analysis are confirmed by analog simulation.

ANALYSIS

The system under consideration is portrayed in Figure 1. It is assumed that the inputs x_1 and x_2 to the nonlinearities are sinusoidal and that the applicability conditions permitting the use of the describing functions N_1 $(A_1,\ \omega)$ and N_2 $(A_2,\ \omega)$ in lieu of the actual nonlinearities n_1 and n_2 are satisfied, where

$$x_1 = A_1 \sin \omega t$$

and $\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}$ (1)

$$x_2 = A_2 \sin (\omega t - \psi) \quad .$$

It also is assumed that only the first harmonic of the signal x_1 passes through the first nonlinearity n_1. Finally, it is assumed that the system parameters are time invariant.

The describing functions are used to linearize the system characteristic equations; i.e.

$$1 + N_1 G_1 (s) N_2 G_2 (s) = 0,$$

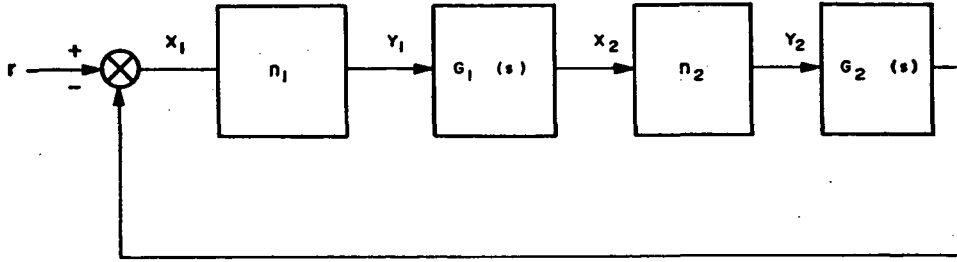which leads to

$$\Delta(s) = \sum_{k=0}^{n} f_k s^k = 0 \quad . \tag{2}$$

Figure 1. Control system containing two nonlinearities.

where

$$f_k = b_k \, P_1 + c_k \, P_2 + d_k \quad . \tag{3}$$

Parameters P_1 and P_2 are defined as

$$P_1 \equiv l_1 \, N_1 \text{ and } P_2 \equiv l_1 \, N_2 \quad , \tag{4}$$

where b_k, c_k, d_k, l_1, and l_2 are constants or system parameters of the linear portion of the system.

A correlation between parameters P_1 and P_2 and the roots of the characteristic equation is determined by mapping certain stability contours from the complex s-plane onto the P_1-P_2 parameter plane. The stability contour chosen for this portion of the analysis is the imaginary axis of the s-plane.

$$s = i\Omega \quad , \tag{5}$$

which will be referred to as the $\zeta = 0$ contour. To facilitate computation and to provide flexibility for use in later analysis, s^k of equation (2) may be replaced by

$$s^k = X_k + iY_k \quad , \tag{6}$$

where X_k and Y_k may be obtained by using the recurrence formulas [1]:

3

$$X_{k+1} + 2\omega_n \zeta X_k + \omega_n^2 X_{k-1} = 0$$

and

$$Y_{k+1} + 2\omega_n \zeta Y_k + \omega_n^2 Y_{k-1} = 0. \qquad (7)$$

Equation (2) may be separated into its real and imaginary parts and regrouped:

$$\text{Re}\{\Delta(s)\} = \sum_{k=0}^{n} f_k X_k = B_1 P_1 + C_1 P_2 + D_1 = 0 \qquad (8.a)$$

and

$$\text{Im}\{\Delta(s)\} = \sum_{k=0}^{n} f_k Y_k = B_2 P_1 + C_2 P_2 + D_2 = 0 \quad , \qquad (8.b)$$

where

$$B_1 = \sum_{k=0}^{n} b_k X_k \quad , \quad C_1 = \sum_{k=0}^{n} c_k X_k \quad , \quad D_1 = \sum_{k=0}^{n} d_k X_k \quad ,$$

$$B_2 = \sum_{k=0}^{n} b_k Y_k \quad , \quad C_2 = \sum_{k=0}^{n} c_k Y_k \quad , \quad \text{and} \quad D_2 = \sum_{k=0}^{n} d_k Y_k \quad .$$

Equations (8.a) and (8.b) may be solved for P_1 and P_2 which may be plotted on the P_1-P_2 parameter plane as functions of frequency Ω. This curve is the $\zeta = 0$ stability contour associated with complex conjugate roots. The boundary separating stable real roots from unstable ones is determined by setting s = 0 in equation (2) and solving for P_2 as a function of P_1. The resulting curve also is plotted on the P_1-P_2 parameter plane. The stable region (corresponding to the left half of the s-plane) in the parameter plane is determined by observing the sign of the Jacobian (J) [1].

In determining regions of stability and instability on the parameter plane, one must be alert for the possible existence of singular cases where the Jacobian $J \equiv 0^2$. This may happen for certain combinations of N_1 and N_2 —a point not mentioned by Viswanadham and Deekshatulu [2]. (Analysis of the Jacobian associated with the example presented in Viswanadham's and Deekshatulu's paper [2] shows the existence of several singular points for certain combinations of values of N_1, N_2, and ω.) In such cases, the mapping of

certain points from the complex s-plane onto the parameter plane may result in loci of points or in distinct points. Hence, additional stability boundaries may occur on the parameter plane. One such singular case corresponds to the mapping of the origin of the s-plane onto the parameter plane as the real root stability boundary.

To obtain some insight into the choice of values for adjustable gains of the linear portion of the system, two such gains (or combinations of gains) may be chosen as parameters (α_1 and α_2). Considering linear operation of the system, the characteristic equation (2) may be written in terms of α_1 and α_2 by setting N_1 and N_2 equal to unity; i.e.,

$$f_k = b_k \; \alpha_1 + c_k \; \alpha_2 + d_k \; . \tag{9}$$

Stability contours associated with chosen values of the damping ratio ζ may be plotted as functions of ω_n on the α_1-α_2 parameter plane by mapping the line

$$s = - \zeta \omega_n + i \omega_n \sqrt{(1 - \zeta^2)} \tag{10}$$

from the s-plane onto the α_1-α_2 parameter plane. Any roots lying on this line will possess an associated damping ratio specified by ζ. If equation (9) is substituted for equation (3) and P_1 and P_2 are replaced by α_1 and α_2 in equations (8.a) and (8.b), they may be solved by α_1 and α_2 for various values of ζ. Then, values of α_1 and α_2 may be chosen for desired values of ζ and ω_n.

The relationship between x_1 and x_2 may be determined, assuming that $G_1(s)$ is a low-pass function and that only the first harmonic passes through n_1. Since $N_1 (A_1, \omega)$ is known, A_2 may be calculated as a function of A_1 and ω:

$$A_2 = |G_1 (i\omega)| N_1 (A_1, \omega) \; A_1 \; . \tag{11}$$

For chosen values of A_1 and ω, N_2 may be found using equation (11). Thus, N_2 may be plotted versus N_1 as a function of A_1 and ω. Using equation (4), one may plot this nonlinear locus on the P_1-P_2 parameter plane, as a function of A_1, for various values of ω. If the locus intersects the $\zeta = 0$ curve, a limit cycle is indicated if the frequencies of the locus (ω) and the $\zeta = 0$ curve (Ω) match; this is the frequency of oscillation of the limit cycle. The amplitude of the limit cycle is determined from the value of A_1 corresponding to the intersection. For a limit cycle to be stable, it is necessary but not sufficient that all roots (other than the pair of complex roots lying on the imaginary axis of the s-plane)

lie to the left of the imaginary axis. Hence, the intersection of the locus and the $\zeta = 0$ curve must lie on a boundary of the stable region (the entire $\zeta = 0$ curve is not necessarily a stability boundary—a point omitted by Siljak [1]). If the locus crosses from the unstable region as A_1 increases, a stable limit cycle is indicated (and vice versa if it crosses from the stable region into the unstable region). If no intersection occurs and the locus remains in the stable region for all values of A_1, linear stable operation is indicated. If the locus remains in the unstable region, unstable operation without a limit cycle is indicated.

Example. Consider a typical pitch-plane attitude control system for a space vehicle (Fig. 2). The rigid body dynamics are represented by the transfer function

$$G_{RB}(s) \equiv \dot{\theta}(s)/\beta_L(s) = c/s \quad , \tag{12}$$

where

$$c = (\partial L/\partial \beta_L)\beta_L 1_g/I \quad .$$

The distance between the center of mass and the point of application of the thrust is denoted by 1_g, the lateral component of thrust by L, and the moment of inertia about the pitch axis by I.

The system characteristic equation may be written in the form of equation (2):

$$\Delta(s) = \tau_s^3 + s^2 + a_1 c N_s N_D s + a_0 c N_s = \sum_{k=0}^{3} f_k s^k = 0 \quad , \tag{13}$$

where N_s and N_D represent describing functions associated with the saturation characteristic and the dead-zone characteristic, respectively. Convenient parameters are

$$P_0 \equiv a_0 N_s \quad \text{and} \quad P_1 \equiv a_1 N_s N_D \quad , \tag{14}$$

where

$$f_k = b_k P_0 + c_k \tilde{P}_1 + d_k \quad . \tag{15}$$

Figure 2. Space vehicle attitude control model.

In the manner indicated by equations (8.a) and (8.b), the $\zeta = 0$ contour may be determined:

$$P_0 = \Omega^2/c \tag{16}$$

and

$$P_1 = \tau \Omega^2/c = \tau P_0 \quad , \tag{17}$$

where

$$J = c^2 \Omega > 0, \quad \forall \Omega > 0 \quad . \tag{18}$$

In a design problem it is desirable to separate the describing functions N_s and N_D rather than have them appear as a product in the parameter P_1. This is achieved by defining P_1:

$$P_1 \equiv a_1 N_D/a_0 \quad . \tag{19}$$

Examination of the Jacobian reveals that it will become identically equal to zero only if ω equals zero (since c is non-zero). This corresponds to the singular case associated with the real root stability boundary which is

$$P_0 = 0. \tag{20}$$

Now the stability boundaries may be plotted, using equations (16), (19), and (20) (Fig. 3). The number of stable roots of the characteristic equation is indicated within square brackets for each region.

To determine suitable values for the adjustable control gains, a_0 and a_1, linear operation is assumed and contours corresponding to various values of ζ may be plotted as functions of ω_n. A typical contour (for $\zeta = 0.5$) also is shown in Figure 3; values of ω_n and Ω are shown in parentheses. Numerical values must be assigned to the constants τ, a_0, a_1, k_s, k_D, S, D, and c. Constants τ, k_D, and D are fixed by the design of the actuator. Values representative of contemporary hardware are $\tau = 0.2$s, $k_s = 1$, and S = 1 deg. A typical rate gyroscope has values of $k_D = 1$ and D = 0.1 deg/s. Parameter c is a function of flight time and, for a typical launch vehicle, varies slowly between 0.5 and 2.5. The value of 0.7 is chosen from the example. Values of a_0 and a_1 are chosen from Figure 3, assuming linear operation and $\zeta \approx 0.7$ and $\omega_n \approx 0.6$ rad/s; $a_0 = 0.5$ and $a_1 = 1$.

Assuming the following forms for x_1 and x_2:

$$x_1 \equiv \beta = A_s \sin \Omega t$$

and
                                                                                    (21)

$$x_2 \equiv \dot{\theta} = A_D \sin (\Omega t + \psi_1)$$

and utilizing the assumptions of the foregoing section, the relationship between A_D and A_s is

$$A_D = cN_s (A_s, \omega) A_s / \omega .$$ (22)

Describing function relationships for $N_s (k_s, A_s, S)$ and $N_D (k_D, A_D, D)$ are well known and readily available. From these relationships and equations (16), (17), (20), and (22), loci representing the variation of N_s and N_D may be plotted on the P_0-P_1 parameter plane as a function of A_s for various values of ω. These loci are shown as dashed lines in Figure 3. Sets of two figures within parentheses indicate corresponding values of A_s/S and A_D/D. Since most loci shown rise vertically at $P_0 = 0.5$, the values shown for A_s/S and A_D/D correspond to the locus associated with $\omega = 0.592$. It is seen that a limit cycle
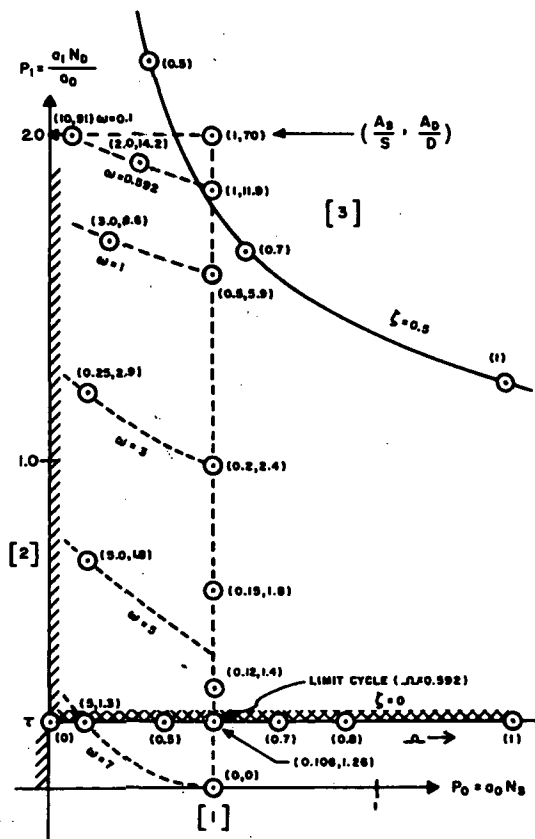
8

Figure 3. P_0-P_1 parameter plane
($a_0 = 0.5$, $a_1 = 1$, $\tau = 0.2$, and
$c = 0.7$).

cannot be avoided if otherwise nonlinear operation is to be precluded. It also is observed that only one limit cycle is possible, and it will be stable. The characteristics of the limit cycle for this example are read from the intersection of the $\zeta = 0$ contour and the nonlinear locus as: $\Omega = 0.592$; $P_0 = 0.5$; $P_1 = 0.2$; $A_s = 0.106$, $S = 1$; and $A_D = 0.126$, $D = 0.1$. The predicted numerical results were confirmed by simulating the system on an analog computer, thereby increasing confidence in the technique and the assumptions. Figures 4(a) and 4(b) show these results for several initial conditions on θ. Also of interest (although not analyzed) is the case where an input signal is applied. Simulation results, shown in Figure 5, indicate that limit cycle operation will not occur when the system is forced by a ramp input, but as soon as the input levels off to a step input, sustained oscillations again occur.
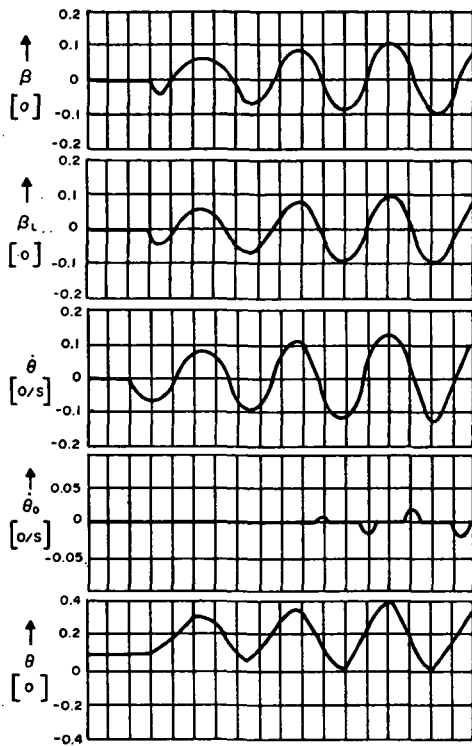
In the actual design of a control system, the values chosen for τ, a_0, a_1, k_s, k_D, S, and D would be varied (within constraints imposed by actual hardware implementation) to achieve other limit cycle characteristics so that the least undesirable characteristics could be obtained. Also, the problem would be analyzed for the entire range of values that c assumes during predicted flight.

CONCLUSIONS

The parameter plane technique of stability analysis can be applied to a model of a system containing two nonlinearities whose inputs are related by a nonlinear differential equation. The constraints imposed on the system are prescribed, and limit cycle operation and characteristics may be predicted. In cases where limit cycle operation can be avoided, relative stability characteristics can be established by choosing the distribution of the roots of the characteristic equation through the variation of two adjustable parameters.

Figure 4(a). Computer simulation:
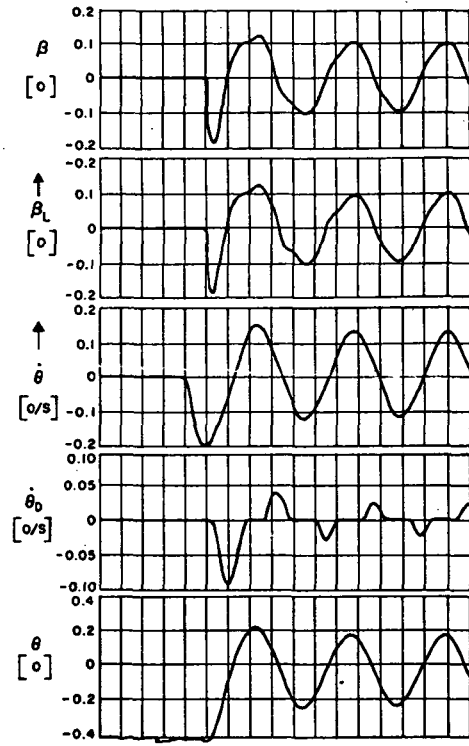θ (0) = 0.4 deg (time scale: 2.5
s/div).

Figure 4(b). Computer simulation:
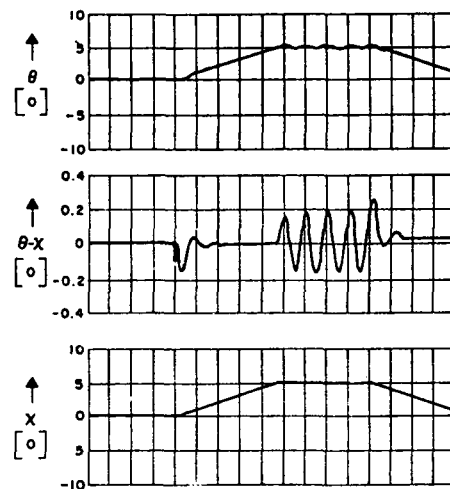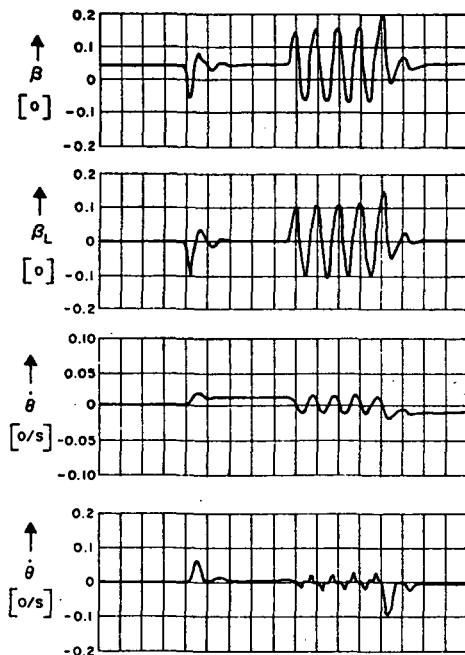θ (0) = 0.1 deg (time scale: 2.5
s/div).

Figure 5. Computer simulation: forced solution (time scale: 2.5 s/div).

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APPROVAL

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CONTAINING TWO NONLINEARITIES

By S. M. Seltzer

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

HANS H. HOSENTHIEN
Chief, Research and Development Analysis Office

F. B. MOORE
Director, Astrionics Laboratory