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P. J. PALMADESSO
G. SCHMIDT

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P. J. Palmadesso
Goddard Space Flight Center
Greenbelt, Maryland 20771

and

G. Schmidt
Stevens Institute of Technology
Hoboken, New Jersey 07030

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ABSTRACT

The behavior of weak electrostatic waves in a collisionless magnetoplasma supporting a steady large amplitude whistler wave has been studied. All waves are assumed to propagate parallel to a uniform background magnetic field, B_0. In the presence of the whistler wave fields each particle executes an oscillatory motion parallel to B_0, in addition to a translation along B_0 and transverse motions. This oscillation causes the Landau resonance to be replaced by a series of new resonances between particles and the electrostatic modes. A distribution function for the perturbed plasma is constructed by solving the Vlasov equation, linearized in the electrostatic wave amplitudes. A dispersion relation is obtained and solved approximately for the growth/damping rate of the perturbations. Growing electrostatic modes are found to be approximately uncoupled. Trapped particles have a strong influence on the stability of the system.

STABILITY OF A STEADY, LARGE

AMPLITUDE WHISTLER WAVE

I INTRODUCTION

The existence of steady, arbitrary amplitude whistler mode solutions to the Maxwell-Vlasov equations was demonstrated in 1966 by Lutomirski and Sudan[1]. These authors considered right circularly polarized waves propagating along a uniform background magnetic field, B_0, in an infinite collisionless plasma. It was shown that no dispersion relation exists for such waves. Undamped, finite amplitude whistlers with arbitrarily chosen frequency ω_0 and wave number k_0 are therefore possible, in principle.

In order to decide which of these modes are likely to be observed in real plasmas, one must be able to answer two questions. First, does a physical mechanism exist which "constructs" the mode in a plasma? Second, is the resulting equilibrium stable enough to persist for a reasonable time? In this paper we consider the question of stability. Specifically, we investigate the stability against the growth of electrostatic perturbing waves of a plasma supporting a large amplitude steady whistler. Of particular interest is an electrostatic instability associated with the distribution of trapped particles.

There is a second motivation for studying the stability of undamped whistlers. A recent theoretical paper[2] describes a process which produces, in the absence of instabilities, an approximate whistler mode equilibrium. In the case considered in Reference 2, a right circularly polarized transverse wave propagates along B_0 in an infinite collisionless plasma initially in thermal equilibrium. The wave damps exponentially for a short time after it is launched at $t = 0$.

The resonance damping mechanism acting on the wave rapidly saturates, due to the phase mixing of resonant particles. The wave amplitude then executes damped oscillations about a finite value, and the system approaches a steady state. The presence of instabilities may modify this process. One can avoid the difficulties inherent in a stability analysis of the complete, time dependent system by considering just the asymptotic steady state. This can be done in a straightforward manner, and yields some insight into the nature of the instability mechanisms involved in the dynamic process.

We begin in Sec. II by linearizing the Vlasov equation for the electrons in the perturbation amplitude. Ion motions are neglected and all waves are assumed to propagate along B_0. The solution to the linearized Vlasov equation is constructed in terms of unperturbed single particle trajectories. (The equations of motion for a single particle in B_0 and the whistler fields have been solved by Roberts and Buchsbaum,[3] and others.[1,2,4]) The perturbed distribution function is used to obtain a dispersion relation for the Fourier components of the perturbation.

It is known[1-4] that the fields of a finite amplitude whistler wave trap particles, and that the motion of both trapped and untrapped charged particles in these fields consists of an oscillation parallel to B_0, a translation along B_0, and transverse motions. One finds that the parallel oscillation part of the unperturbed particle motion, which is greatly enhanced for trapped particles, leads to new resonances with the electrostatic perturbing waves.

One also finds that the perturbing waves are approximately uncoupled, due to the symmetry of the unperturbed wave-plasma system. If we assume that the whistler phase velocity ω_0/k_0 is less than c and work in the wave frame, there is no electric field and the magnetic field lines are helices.

$$\mathbf{B}(z) = B_1(\mathbf{e}_x \sin k_0 z + \mathbf{e}_y \cos k_0 z) + B_0 \mathbf{e}_z$$

The unperturbed electron distribution function also has helical symmetry, i.e., the electron distribution at any point z_1 differs from that at another point z_2 only by a rotation in velocity space through an angle $\psi = k_0(z_2 - z_1)$. The electron density is therefore uniform throughout space. This is in contrast to the situation encountered by Kruer, Dawson, and Sudan[5], and Goldman[6], in their study of the stability of a large amplitude electrostatic wave. In that case there is a strong periodic spatial variation in the unperturbed electron density, and this leads to the coupling of an infinite number of discrete perturbing waves.

Finally, in Sec. III, an approximate expression for the growth/damping rate is derived and studied, with particular emphasis on the effects of trapped particles.

II THE DISPERSION RELATION

The wave frame electron distribution function for a magnetoplasma supporting a large amplitude, parallel propagating steady whistler is a function of the single particle constants of motion[1-4] v and W, where

$$v = (v_\parallel^2 + v_\perp^2)^{1/2} \tag{1a}$$

$$W = \frac{1}{2}\left(v_\parallel + \frac{\Omega_0}{k_0}\right)^2 + \frac{\Omega_1}{k_0} v_\perp \sin(k_0 z + \psi) \tag{1b}$$

The quantities $v_\parallel$, $v_\perp$, and ψ describe the wave frame velocity

$$\mathbf{v} = v_\perp \cos\psi\, \mathbf{e}_x + v_\perp \sin\psi\, \mathbf{e}_y + v_\parallel \mathbf{e}_z$$

and

$$\Omega_0 = |e| B_0/m, \; \Omega_1 = |e| B_1/m.$$

A small electrostatic perturbation of the form $E_{\parallel}(z, t)\mathbf{e}_z$ produces a correction to the distribution function ($\hat{f} \to \tilde{f} + f_1$) which may be computed by solving the linearized Vlasov equation.

$$\left[\frac{\partial}{\partial t} + v_{\parallel} \frac{\partial}{\partial z} - \frac{e}{m} \mathbf{v} \times \mathbf{B}(z) \cdot \nabla_v\right] f_1 = \frac{e}{m} E_{\parallel}(z, t) \frac{\partial \tilde{f}(v, W)}{\partial v_{\parallel}}$$

(2)

$$= \frac{e}{m} E_{\parallel}\left[\frac{v_{\parallel}}{v} \frac{\partial \tilde{f}}{\partial v} + \left(v_{\parallel} + \frac{\Omega_0}{k_0}\right) \frac{\partial \tilde{f}}{\partial W}\right]$$

This equation may be solved formally by introducing the function $z(z_0, v_0, t)$, which describes the zero order motion along the z axis of a particle initially at z_0, v_0. The functions $z_0(z, v, t)$ and $v_0(z, v, t)$ are, like v and W, constants of the unperturbed motion and solutions of the homogeneous equation associated with (2). Thus

$$f_1 = f_{10}(z_0, v_0) + \frac{e}{m_{\parallel}} \int_0^t E_{\parallel}(z(t'), t')$$

(2a)

$$\times \left[\frac{1}{v} \frac{dz(t')}{dt'} \frac{\partial \tilde{f}(v, W)}{\partial v} + \left(\frac{dz(t')}{dt'} + \frac{\Omega_0}{k_0}\right) \frac{\partial \tilde{f}(v, W)}{\partial W}\right] dt'$$

If one considers only growing disturbances,

$$f_{10} = \frac{e}{m} \int_{-\infty}^{0} E_{\parallel}(z(t'), t')[\;\text{''}\;]dt' \qquad (2b)$$

First order corrections to the perpendicular current field and the charge density can be obtained directly from f_1.

$$\mathbf{J}_{1_\perp} = -n_e e \iiint v_\perp f_1 d^3 v \qquad \rho_1 = -n_e e \iiint f_1 d^3 v$$

The current density $\mathbf{J}_{1_\perp}$, which is in general nonzero, leads to time varying magnetic and induced transverse electric fields. The omission of these fields from (2) is justified when the perpendicular thermal velocity V_T is much less than c. To see this, we estimate that $J_{1_\perp} \sim V_T \rho_1 \sim \epsilon_0 V_T k E_\parallel$, where k is a typical wavenumber in the spectrum of $E_\parallel(z, t)$. After using Maxwell's equations to relate the transverse field corrections $E_{1_\perp}$ and $B_{1_\perp}$ to $J_{1_\perp}$, and introducing the characteristic frequency ω_1 and wavenumber k_1 for $E_{1_\perp}$, $B_{1_\perp}$, we find that

$$\left| \frac{v \times \mathbf{B}_{1_\perp}}{E_\parallel} \right| \sim \frac{k}{k_1} \frac{V_T v}{c^2 + \omega_1^2/k_1^2} << 1$$

$$\left| \frac{E_{1_\perp}}{E_\parallel} \right| \sim \frac{k}{k_1} \frac{V_T(\omega_1/k_1)}{c^2 + \omega_1^2/k_1^2} << 1$$

The charge density ρ_1 produces an electric field which must be identical with $E_\parallel(z, t)$, for self consistency. If we express $E_\parallel(z, t)$ in the form

$$E_\parallel(z, t) = \int_{-\infty}^{\infty} E_\parallel(k)\, e^{i(kz-\omega t)}\, dk,$$

construct the charge density by integrating (2) over velocity space, and apply Poisson's equation, we obtain

$$0 = \int_{-\infty}^{\infty} E_{\parallel}(k) \Bigg\{ e^{i(kz - \omega' t)} k \qquad (4)$$

$$- i \omega_p^2 \iiint d^3v \int_{-\infty}^{t} dt' \, e^{i[kz(t') - \omega' t']} \left[\frac{1}{v} \frac{dz}{dt'} \frac{\partial \tilde{f}}{\partial v} + \left(\frac{dz}{dt'} + \frac{\Omega_0}{k_0} \right) \frac{\partial \tilde{f}}{\partial W} \right] \Bigg\} dk$$

where $\omega'(k) = \omega(k) - k\,\omega_0/k_0$ and $\omega_p^2 = n_e e^2/\epsilon_0 m$. $\omega(k)$ is the complex frequency of $E_{\parallel k}$ (t) in the lab frame.

As noted previously, the unperturbed motion of a particle along the z axis consists of a pure translation plus an oscillatory motion. If $t_+(z_0, v_0)$ is any time at which the particle reaches the upper turning point of its oscillatory motion, the function $z(z_0, v_0, t)$ may be written in the form

$$z = z_0 + \overline{v}_{\parallel}(v, W)\,t + \left[S(v, W, t - t_+) - S_0 \right] \qquad (5)$$

where $S(v, W, t - t_+)$ is periodic in time with period $T(v, W)$, S_0 is $S(v, W, -t_+)$, and $\overline{v}_{\parallel}(v, W)$ is dz/dt averaged over a period. The functions $\overline{v}_{\parallel}$, S, T, t_+ are known, in the sense that they can be constructed in a straightforward way from the exact, closed form expressions for dz/dt obtained by Roberts and Buchsbaum (Ref. 3), or from the simpler approximate expressions given in Ref. 2. The time dependence of exp [i k z (t)], which appears in (4), can be conveniently exhibited by expanding exp [i k S (t)] in a Fourier series, so that

$$\exp\left\{ ikz(t) \right\} = \exp\left[ik(z_0 + \overline{v}_{\parallel} t - S_0) \right] \sum_{-\infty}^{\infty} A_n \exp\left[2\pi i n (t - t_+)/T \right] \qquad (6a)$$

where

$$A_n(v, W, k) = \frac{1}{T(v, W)} \int_0^T \exp\left[ikS(t - t_+) - 2\pi i n \frac{(t - t_+)}{T} \right] d(t - t_+) \qquad (6b)$$

After noting that $[\exp(i k z)]\, dz/dt$ is $(i k)^{-1}\, d[\exp(i k z)]/dt$ and applying (6), one is able to write (4) in the form

$$0 = \int_{-\infty}^{\infty} E_{\parallel}(k) \Bigg\{ e^{i(kz-\omega' t)} k$$

$$- i \omega_p^2 \iiint d^3 v\, e^{ik(z_0 - S_0)} \int_{-\infty}^{t} \sum_n A_n I_n \exp\left[i \left(k \bar{v}_{\parallel} t + 2\pi n \frac{t - t_+}{T} \right) \right] dt' \Bigg\} dk$$

where

$$I_n(v, W, k) = \frac{\Omega_0}{k_0} \frac{\partial \tilde{f}}{\partial W} + \left(\bar{v}_{\parallel} + \frac{2\pi n}{kT} \right) \left(\frac{1}{v} \frac{\partial \tilde{f}}{\partial v} + \frac{\partial \tilde{f}}{\partial W} \right) \qquad (7)$$

After integrating over t', using (5) to express $(z_0 - S_0)$ in terms of current dynamical variables, and expanding $\exp(-i k S)$, we obtain

$$0 = \int_{-\infty}^{\infty} E_{\parallel}(k)\, e^{i(kz-\omega' t)} \Bigg\{ k$$

$$- \frac{\omega_p^2}{k} \iiint d^3 v \sum_n \sum_m{}' A_n A_m^* I_n \frac{\exp[2\pi i (n - m)(t - t_+)/T]}{(\bar{v}_{\parallel} + 2\pi n / kT - \omega'/k} \Bigg\} dk$$

The terms in the double summation above for which (n - m) is not zero phase mix away for large t, when integrated over velocities. Note also that the quantity in curly brackets in (8) is independent of z. The implicit z dependence in $W(v_{\parallel}, v_{\perp}, k_0 z + \psi)$ is removed by the integration over ψ $(d^3 v = v_{\perp}\, dv_{\perp}\, dv_{\parallel}\, d\psi)$. It follows that the Fourier components $E_{\parallel}(k)$ are uncoupled, and the dispersion relation is

$$1 - \frac{\omega_p^2}{k^2} \iiint \sum_n |A_n(v, W)|^2 \frac{I_n(v, W, k)\, d^3v}{(\bar{v}_\parallel + 2\pi n/kT - \omega'/k)} = 0 \quad (9)$$

This expression was derived for growing modes, hence it is understood that the integrals are to be taken along the real velocity axes when the frequency ω' has a positive imaginary part. The integration contours are deformed in the usual manner to construct the analytic continuation of (9) in the lower half ω'-plane. It is easily verified that (9) reduces to the dispersion relation obtained by Landau in the limit

$$\Omega_i \to 0 \quad (A_n \to \delta_{n,0}\, ;\ \bar{v}_\parallel \to v_\parallel\, ;\ I_0 \to \partial f/\partial v\ ).$$

III ESTIMATION OF Im $\{\omega\}$

There is an alternative form of (9) which provides a convenient starting point for a study of the resonant amplification or damping of perturbing waves. After a little algebraic manipulation, and after using the fact that $\Sigma|A_n|^2 = 1$ [from (6)], (9) may be written in the form

$$\epsilon(k,\omega) = 1 - \frac{\omega_p^2}{k^2} \iiint \left\{ \frac{1}{v}\frac{\partial \tilde{f}}{\partial v} + \frac{\partial \tilde{f}}{\partial W} + \sum \frac{|A_n|^2\, I(v, W, \omega'/k)}{(\bar{v}_\parallel + 2\pi n/kT - \omega'/k)} \right\} d^3v = 0 \quad (10a)$$

where

$$I(v, W, \omega'/k) = \frac{\omega'}{kv}\frac{\partial \tilde{f}}{\partial v} + \left[\frac{\omega'}{k} + \frac{\Omega_0}{k_0}\right]\frac{\partial \tilde{f}}{\partial W} \quad (10b)$$

When the growth/damping rate is small, the imaginary part of (10a) may be expanded in powers of ω_i/ω_p, where $\omega_i = \mathrm{Im}\,\omega'$. After neglecting terms higher than first order and solving for ω_i, one obtains

$$\omega_i = \omega_r < I(v, W, \omega_r'/k) >_k / Q(k, \omega_r) \qquad (11a)$$

where ω_r is the real part of the frequency of the perturbation in the lab frame, and

$$< \ >_k \equiv \pi \frac{\omega_P^2}{k^2} \iiint (\ ) \sum_n |A_n(v, W)|^2 \, \delta(u_n) \, d^3v \qquad (11b)$$

$$Q(k, \omega_r) = \omega_r \frac{\partial}{\partial \omega_r} \epsilon_r(k, \omega_r) \qquad (11c)$$

$$\epsilon_r(k, \omega_r) = 1 - \frac{\omega_P^2}{k^2} P \iiint \left\{ \frac{1}{v} \frac{\partial \tilde{f}}{\partial v} + \frac{\partial \tilde{f}}{\partial W} + \sum \frac{|A_n|^2 I}{u_n} \right\} d^3v \qquad (11d)$$

$$u_n = \bar{v}_{\parallel}(v, W) + 2\pi n/kT - \omega'/k \qquad (11e)$$

Similarly, the real part of (10a) yields $\epsilon_r(k, \omega_r) \approx 0$ when ω_i is small, and this determines the function $\omega_r(k)$. The quantity $Q(k, \omega_r)$ determines the sign of the energy associated with the wave E_k in the lab frame. In a Maxwellian plasma Q is positive for all values of k, but negative energy waves are possible in media sufficiently far from thermal equilibrium ("anomalous dispersion")[7].

The growth or damping rate of perturbations on a given whistler equilibrium can, in principle, be determined by direct application of (11), but the computations involved are clearly formidable. It is therefore helpful to obtain some qualitative or semiquantitative information about factors influencing the stability of the system. To simplify the problem, we assume that B_1 is small compared to B_0 and study effects of lowest order in ϵ, where

$$\epsilon = B_1/B_0 = \Omega_1/\Omega_0 .$$

The Cyclotron Resonance

We begin by considering the unperturbed particle orbits. The unperturbed path of a particle in the two dimensional phase space with coordinates $k_0 z + \psi$, $v_\parallel$ can be determined by solving (1b) for $k_0 z + \psi$ and plotting $k_0 z + \psi$ versus $v_\parallel$. A typical set of such paths, for particles having the same v and various values of W, is illustrated in Fig. 1. Note that the angle $90° + k_0 z + \psi$, which is the angle between $\mathbf{B}_1$ and $\mathbf{v}_\perp$, oscillates about zero for particles following the closed orbits. These particles are said to be "trapped" by the whistler. Untrapped particles close to the separatrix and trapped particles interact strongly with the wave and are termed "cyclotron resonant." ($\omega_0 - k_0 V_{\parallel\,\mathrm{LAB\ FRAME}} \equiv - k_0 v_\parallel \sim \Omega_0$ for cyclotron resonance.)

The following results are based on the approximate unperturbed particle orbits developed in Ref. 2. For trapped particles,

$$1/\Omega_0 T(v, W) = \epsilon^{1/2} \left[(k_0 v/\Omega_0)^2 - 1\right]^{1/4} / 4K(1/r) + O(\epsilon)$$

$$\overline{v}_\parallel (v, W) = - \Omega_0/k_0 + O(\epsilon) \qquad \text{(12a,b,c)}$$

$$S(v, W, t_+, t) = \frac{8}{k_0} \sum_{m=0}^{\infty} \frac{q_{(1/r)}^{m+1/2} \sin\left[2\pi(2m+1)(t - t_+)/T\right]}{(2m+1)\left(1 + q_{(1/r)}^{2m+1}\right)} + O(\epsilon^{1/2})$$

Here K (1/r) is the complete elliptic integral of the first kind, and

$$q(1/r) = \exp\left\{- \pi K\left[(1 - 1/r^2)^{1/2}\right] / K(1/r)\right\}$$

$$r(v,W) = (2\epsilon)^{1/2} \left[\left(\frac{k_0 v}{\Omega_0}\right)^2 - 1\right]^{1/4} \left\{\left(\frac{k_0}{\Omega_0}\right)^2 W + \epsilon\left[\left(\frac{k_0 v}{\Omega_0}\right)^2 - 1\right]^{1/2}\right\}^{-1/2} \qquad (13)$$

For untrapped cyclotron resonant particles,

$$1/\Omega_0 T(v, W) = \epsilon^{1/2} \left[(k_0 v/\Omega_0)^2 - 1\right]^{1/4} / 2r\, K(r) + O(\epsilon)$$

$$\bar{v}_{\parallel}(v, W) = -\Omega_0/k_0 + 2\pi\sigma/k_0 T(v, W) + O(\epsilon) \qquad \text{(14a,b,c)}$$

$$S(v, W, t_+, t) = \frac{4\sigma}{k_0} \sum_{m=1}^{\infty} \frac{q_{(r)}^m}{m\left(1 + q_{(r)}^{2m}\right)} \sin\left[2\pi m(t - t_+)/T\right] + O(\epsilon^{1/2})$$

where $\sigma = \text{sgn}(v_{\parallel} + \Omega_0/k_0)$. Finally, for non-cyclotron-resonant particles,

$$1/\Omega_0 T(v, W) = \epsilon^{1/2} \left[(k_0 v/\Omega_0)^2 - 1\right]^{1/4} / \pi r + O(\epsilon)$$

$$\bar{v}_{\parallel}(v, W) = -\Omega_0/k_0 + 2\pi\sigma/k_0 T(v, W) + O(\epsilon) \qquad \text{(15a,b,c,d)}$$

$$S(v, W, t_+, t) = O(\epsilon) \qquad |A_n(v, W)|^2 = \delta_{n,0} + O(\epsilon)$$

The parameter r (v, W) "classifies" the unperturbed orbits according to type. r (v, W) is greater than one for trapped particles, with $r \to \infty$ for particles in the center of the nested set of closed orbits in Fig. 1. r (v, W) has the value unity for particles on the separatrix and is less than one for untrapped particles, with $r \lesssim O(\epsilon^{1/2})$ for non-cyclotron resonant particles.

Locus of Electrostatic Resonances

Resonance between particles and a perturbing electrostatic wave occurs when the Doppler shifted frequency of the perturbing wave, as measured in a frame moving with the average velocity $\bar{v}_{\parallel}$ of a particle, matches a harmonic of the particle's oscillatory motion. It follows that

$$Y(v, r, k) \equiv \left[k\,\overline{v}_{\parallel}(v, r) - \omega_r'(k)\right] T(v, r)/2\pi = n \qquad (16)$$

at electrostatic resonance, where $n = 0, \pm 1, \pm 2, \ldots$. With the aid of (12)—(15), we depict in Fig. 2 the variation of $[k\overline{v}_{\parallel} - \omega_r']\, T/2\pi$ with $1/r\,(v, W)$, for a fixed value of v and for particles above and below cyclotron resonance velocity. The intersections of the curve $Y(v, r, k)$ with the horizontal lines $Y = 0, \pm 1, \pm 2, \ldots$ mark the locations in phase space of electrostatic resonances. Note that the higher harmonic resonances tend to cluster in the vicinity of the cyclotron resonance. This is due to the singularity in $T(v, r)$ at $r = 1$. ($T(v, r) \sim r\,K(r) \sim r \ln[(1 - r^2)^{1/2}]$ for untrapped particles, etc.)

It follows from (12a), (12b), and (16) that particles on the flat portion of the curve near $\sigma/r \sim 0$ in Fig. 2 resonate with the perturbing wave when

$$(v^2 - \Omega_0^2/k_0^2)/V_T^2 = |(\omega_r' + k\Omega_0/k_0)/n(k_0\Omega_1 V_T)^{1/2}|^4 .$$

These particles oscillate coherently, which greatly enhances the effect of the resonance. The frequency $(k_0 \Omega_1 V_T)^{1/2}$ is the "bounce frequency" ω_B for typical trapped particles at the bottom of the magnetic wells, i.e., $2\pi/T(v, r) \rightarrow \omega_B$ when $r \rightarrow \infty$, $v^2 - \Omega_0^2/k_0^2 \approx v_\perp^2 \rightarrow V_T^2$.

Trapped Particle Effects

In view of the preceding, one expects cyclotron resonant particles, especially trapped particles, to play an important role in the interaction between the whistler and the perturbing waves. The effects of trapped particles are most prominent when the Doppler shifted frequency of the perturbing wave, as measured in the frame moving with the average velocity $\overline{v}_{\parallel} = -\Omega_0/k_0$ of the trapped particles, is of the order of the bounce frequency $\omega_B = (k_0 \Omega_1 V_T)^{1/2}$.

It is therefore convenient to introduce the frequency ω'', where $\omega''(k) = \omega_r'(k) + k\,\Omega_0/k_0$, and to consider k chosen so that $|\omega''(k)| \sim O(\omega_B)$.

It is instructive to solve (10b) and (11) for $< \partial \tilde{f}/\partial W >_k$. We assume that the perturbing waves have positive energy in the lab frame ($Q > 0$), and that $\omega_0 < \Omega_0$. The latter assumption is reasonable in the small ϵ limit, where one expects the whistler dispersion relation obtained from linear theory to hold approximately, but is not necessary in general. One finds that the system is unstable ($\omega_i > 0$) if

$$\left\langle \frac{\partial \tilde{f}}{\partial W} \right\rangle_k > -\left[\frac{k\,\Omega_0}{k_0\,|\omega''|} + 1\right] \left\langle \frac{\partial \tilde{f}}{v\,\partial v} \right\rangle_k \qquad \frac{\omega''}{k} < 0$$

or (17a,b)

$$\left\langle \frac{\partial \tilde{f}}{\partial W} \right\rangle_k < \left[\frac{k\,\Omega_0}{k_0\,\omega''} - 1\right] \left\langle \frac{\partial \tilde{f}}{v\,\partial v} \right\rangle_k \qquad 0 < \frac{\omega''}{k} < (\Omega_0 - \omega_0)/k_0$$

The quantity in square brackets in each of these inequalities is positive.

We first consider the case in which $< \partial \tilde{f}/v\,\partial v >_k$ is positive for some choice of k. If one considers trapped particles only, it is possible to show (Appendix A) that $< \partial \tilde{f}/\partial W >_k$ and $< \partial \tilde{f}/v\,\partial v >_k$ are even functions of ω'', when $|\omega''| \sim O(\omega_B)$ and to lowest order in ϵ. Given this symmetry it is easy to convince oneself that either (17a) or (17b) is satisfied if $< \partial \tilde{f}/v\,\partial v >_k$ is positive, indicating that trapped particles are destabilizing.

The case in which $< \partial \tilde{f}/v\,\partial v >_k$ is negative may be more important physically, since $(\partial \tilde{f}/v\,\partial v)_W \sim (\partial \tilde{f}/v_\perp \partial v_\perp)_{v_\parallel}$ in the small ϵ limit. In this case (17) implies that the plasma is unstable if both of the following conditions are satisfied:

$$\left| \left\langle \frac{\partial \tilde{f}}{\partial W} \right\rangle_k \Big/ \left\langle \frac{\partial \tilde{f}}{v \, \partial v} \right\rangle_k \right| > \left| \frac{k \Omega_0}{k_0 \omega''} - 1 \right| \qquad \frac{\omega''}{k} < \frac{\Omega_0 - \omega_0}{k_0} \tag{18a}$$

and

$$\operatorname{sgn} \left\langle \frac{\partial \tilde{f}}{\partial W} \right\rangle_k = - \operatorname{sgn} \omega'' \tag{18b}$$

The fact that $< \partial \tilde{f} / \partial W >_k$ is even in ω'' guarantees that (18b) will be satisfied for some choice of k. A semi-quantitative trapped particle instability condition may be obtained from (18a). We estimate $(\partial \tilde{f}/v \partial v)_W \sim - \tilde{f}/V_T^2$ and $\partial \tilde{f}/\partial W \sim \beta f/\Delta W$. ΔW is the change in W in going from the separatrix to the bottom of a well. For typical values of v this is $\sim 2 \, \Omega_1 V_T/k_0$. β scales the fractional variation of f over the trapping region, in the direction normal to the surfaces of constant W in phase space (Fig. 1). After setting $| \omega'' | \sim (k_0 \Omega_1 V_T)^{1/2}$, one finds from (18a) that trapped particles are destabilizing if

$$|\beta| \gtrsim 2 \frac{k}{k_0} \left(\frac{\Omega_0}{k_0 V_T} \right)^{3/2} \epsilon^{1/2}$$

Special Cases

For example, consider a distribution in which the trapped particles are strongly concentrated near the centers of the nested sets of closed orbits (one set for each value of v), and approximate the distribution function in this region of phase space by

$$\tilde{f}_T \approx (2n_{eT}/n_e V_T^2) \, e^{-v_\perp^2/V_T^2} \, \delta(v_\parallel + \Omega_0/k_0) \, \delta(\psi + k_0 z + \pi/2) \tag{19a}$$

where n_{eT} is the density of trapped particles. It is shown in Appendix B that this is equivalent to

$$\tilde{f}_T = \frac{k_0 n_{eT} H\left(v^2 - \Omega_0^2/k_0^2\right)}{2\pi \Omega_0 n_e \eta(v) V_T^2} \exp\left\{-\frac{\left(v^2 - \Omega_0^2/k_0^2\right)}{V_T^2}\right\} \delta\left(\frac{1}{r^2} - 0\right) \quad (19b)$$

where H (x) is the unit step function, and

$$\eta(v) = \epsilon^{1/2}\left[\left(\frac{k_0 v}{\Omega_0}\right)^2 - 1\right]^{1/4} \qquad \delta\left(\frac{1}{r^2} - 0\right) \equiv \lim_{R\to\infty} \delta\left(\frac{1}{r^2} - \frac{1}{R^2}\right)$$

Substitution of (19b) into (10b) yields, to lowest order in ϵ,

$$I \approx \frac{\omega'' n_{eT} k_0^3 H\left(v^2 - \Omega_0^2/k_0^2\right)}{4\pi k n_e V_T^2 \eta^3 \Omega_0^3} \exp\left\{-\frac{\left(v^2 - \Omega_0^2/k_0^2\right)}{V_T^2}\right\} \delta'\left(\frac{1}{r^2} - 0\right) \quad (20)$$

where

$$\delta'(1/r^2 - 0) \equiv \partial \delta / \partial r^{-2}.$$

It follows from (16) and (12b) that the n = 0 resonance condition for trapped particles is $-\omega''/k = 0$. Since $I(v, W, \omega')$ vanishes when $\omega'' = 0$, the n = 0 resonance makes no contribution in this case. For very large r, $q(1/r) \sim (4r)^{-2}$. After using (12c) in (6b) and retaining only terms of lowest order in $1/r$, one has

$$|A_{\pm 1}(v, W)|^2 \sim (1/r)^2 (k/k_0)^2 \qquad r \sim \infty \quad (21)$$

The higher harmonic oscillator strengths go to zero faster than $(1/r)^2$, and these terms do not contribute to <I> in this case.

Equations (20) and (21) may be used in (11) to obtain a growth rate for the perturbations. The integrations are most conveniently performed in terms of the variables[8] r, v, and $\xi = \frac{1}{2}\left(\frac{1}{2}\pi + k_0 z + \psi\right)$. One obtains from (1a,b), (11e), (12a,b) and (13) the results

$$\delta(u_n) = \frac{8k\ K(1/r)\ (v^2 - \Omega_0^2/k_0^2)^{3/4}}{\pi\,|n|\,(k_0\Omega_1)^{1/2}}\ \delta(v^2 - v_n^2)\ \delta_{\sigma(n),\sigma(\omega'')} \tag{22a}$$

where

$$v_n^2 = \frac{\Omega_0^2}{k_0^2} + \left|\frac{4\ K\ (1/r)\ \omega''}{2\pi n\ \omega_B}\right|^4 V_T^2$$

$$\sigma(x) \equiv \text{sgn } x$$

and

$$d^3 v = \left[4\Omega_0\,\eta(v)\ (1/r^2 - \sin^2\xi)^{-1/2}/k_0 r^3\right] v dv\ dr\ d\xi \tag{22b}$$

The variable ξ appears only in the expression for the volume element d^3v, and leads to an integral of the form

$$\int_{-\sin^{-1}(1/r)}^{+\sin^{-1}(1/r)} (1/r^2 - \sin^2\xi)^{-1/2}\, d\xi = 2\ K\ (1/r)$$

in (11b). The remaining double integral contains two δ functions and is easily evaluated. If the approximation

$$\epsilon_r(\omega_r, k) \approx 1 - \frac{\omega_P^2}{\omega_r^2}\left(1 + \frac{3}{2}\frac{k^2 V_T^2}{\omega_P^2}\right)$$

is used to evaluate $Q(\omega_r, k)$, one obtains for ω_i

$$\omega_i \approx -\pi\omega_r\left(\omega_T\omega''/\omega_B^2\right)^2 e^{-(\omega''/\omega_B)^4}\ \text{sgn}(\omega'') \tag{23}$$

where $\omega_T^2 = n_{eT}\, e^2/\epsilon_0\, m$. The factor $\exp\ \{-(\omega''/\omega_B)^4\}$, which strongly suppresses the trapped particle contribution to ω_i when $|\omega''| >> \omega_B$, comes from the factor $\exp\ \{-v_\perp{}^2/V_T^2\}$ in (19a).

To estimate the maximum growth rate, one sets $\omega_r \approx -\omega_P$ and maximizes (23) with respect to ω''. For $\omega'' = 2^{-1/4}\,\omega_B$,

$$(\omega_i)_{max.} \simeq 1.35\,\omega_P\,(\omega_T/\omega_B)^2$$

It is interesting to study the dependence of $(\omega_i)_{max.}$ on the amplitude of the whistler. If ω_T^2 is approximately proportional to the width of the trapping region in velocity space, then $\omega_T^2 \sim \epsilon^{1/2}$ and $(\omega_i)_{max.} \sim \epsilon^{-1/2}$. The growth rate of the instability therefore decreases with increasing $\epsilon = B_1/B_0$. This result is not valid for extremely small ϵ, because of the assumption that $\omega_i << \omega_r$.

Finally, consider the equilibrium established when a finite amplitude Whistler is "turned on" in an initially Maxwellian magnetoplasma and then allowed to evolve. The approximate asymptotic distribution function for trapped particles obtained[2] by the present authors for such a process is

$$\tilde{f}_T \approx \left(\pi^{1/2} V_T^3\right)^{-3} \exp\left\{-\left[v^2 + \frac{\omega_0^2 - 2\omega_0\Omega_0}{k_0^2}\right]\Big/ V_T^2\right\}.$$

This distribution shows no variation with W, and $\partial\tilde{f}_T/\partial v$ is less than zero for all v. In this case trapped particles help stabilize the plasma against perturbations having $|\omega''| \sim \omega_B$. As in the preceeding case, the trapped particle contribution to the imaginary part of ω becomes negligible when $|\omega''| >> \omega_B$.

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APPENDIX A

We consider the effect of the transformation $k \to k + \Delta k$, $\omega''(k) \to \omega''(k + \Delta k) = -\omega''(k)$ on the quantity $\langle g \rangle_k$, when untrapped particles are ignored. Here g is any single valued function of v and W only. It follows from (22) and (11b) that ω'' occurs only in the combination ω''/ω_B in $\langle g \rangle_k$, so that $\langle g \rangle_k = G(k, \omega''/\omega_B)$. When $|\omega''|$ is comparable to ω_B, $\Delta k \approx -2\omega'' (d\omega''/dk)^{-1} \sim O(\epsilon^{1/2})$. Hence $G(k + \Delta k, -\omega''/\omega_B) \approx G(k, -\omega''/\omega_B)$, to lowest order in ϵ. Using this fact, and again referring to (22), one finds that the sole effect of the transformation $\omega'' \to -\omega''$ is to replace n by -n in each of the terms which contribute to the summation in (11b). Thus, to show that the trapped particle contribution to $\langle g \rangle_k$ is an even function of ω'', it is only necessary to show that $|A_n|^2 = |A_{-n}|^2$.

Equation (12c) may be written in the form

$$S(v, W, t_+, t) \approx \frac{8}{k_0} \sum_{m=0}^{\infty} \frac{q_{(1/r)}^{m+1/2} (-1)^{m+1} \cos\left[(2m + 1)\left(2\pi \frac{t - t_+}{T} + \frac{\pi}{2}\right)\right]}{(1 + q^{2m+1})(2m + 1)}$$

The function $\exp[i k S]$ is periodic in $[2\pi(t - t_+)/T + \pi/2]$, and may be expanded in a Fourier series. The coefficients of this expansion are, after using (6b),

$$B_n = e^{-in\pi/2} A_n$$

But $B_n = B_{-n}$, because S is an even function of $[2\pi(t - t_+)/T + \pi/2]$, hence

$$A_n = e^{in\pi} A_{-n}$$

and

$$|A_n|^2 = |A_{-n}|^2$$

for trapped particles.

APPENDIX B

A straightforward change of variables in (19a), using (1), (13), and the definition

$$\xi = \frac{1}{2}\left(\frac{1}{2}\pi + k_0 z + \psi\right),$$

yields

$$\tilde{f}_T = \frac{n_{eT} k_0 H(v^2 - \Omega_0^2/k_0^2)}{2\Omega_0 n_e \eta(v) V_T^2} \exp\left[-\frac{(v^2 - \Omega_0^2/k_0^2)}{V_T^2}\right] \delta(1/r - 0)\ \delta(\xi) \qquad \text{(B1)}$$

$\tilde{f}_T$ must be a function of v, W or v, r only, i.e., the density of particles in phase space must be uniform along the particle orbits. The distribution described by (B1) satisfies this requirement, even though it appears to have a third independent variable. In fact, the orbit with $1/r = 0$ consists of a single point on the line $\xi = 0$.

A more convenient representation of $\tilde{f}_T$ is obtained by first imagining the trapped particles to be evenly distributed along a small but finite radius orbit near the bottom of the well, then allowing the radius of this orbit to go to zero. Thus

$$\tilde{f}_T = \frac{n_{eT} k_0 H(v^2 - \Omega_0^2/k_0^2)}{2\Omega_0 n_e \eta(v) V_T^2} \exp\left[-\frac{(v^2 - \Omega_0^2/k_0^2)}{V_T^2}\right] \delta(1/r - 0) \times C \qquad \text{(B2)}$$

where C is determined by the condition

$$\int_{-\sin^{-1}(1/r)}^{+\sin^{-1}(1/r)} \delta(\xi)\,(1/r^2 - \sin^2\xi)^{-1/2}\,d\xi \;=\; 2C\int_{-\sin^{-1}(1/r)}^{+\sin^{-1}(1/r)} (1/r^2 - \sin^2\xi)^{-1/2}\,d\xi \qquad \text{(B3)}$$

$$= 4\,C\,K\,(1/r).$$

$$C \;=\; r/4\,K\,(1/r) \qquad \text{(B4)}$$

[See (22b)]. The factor of 2 on the right hand side of (B3) is due to the degeneracy[8] of the coordinates r, v, ξ. After noting that K $(1/r) \sim \pi/2$ for large r and inserting (B4) into (B2), one obtains the result quoted in (19b).

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8. It should be noted that the coordinates r, v, and ξ are degenerate in the sense that each point in r, v, ξ space represents two points in the original phase space; e.g., the two small circles in Figure 1 mark points with the same r, v, ξ.

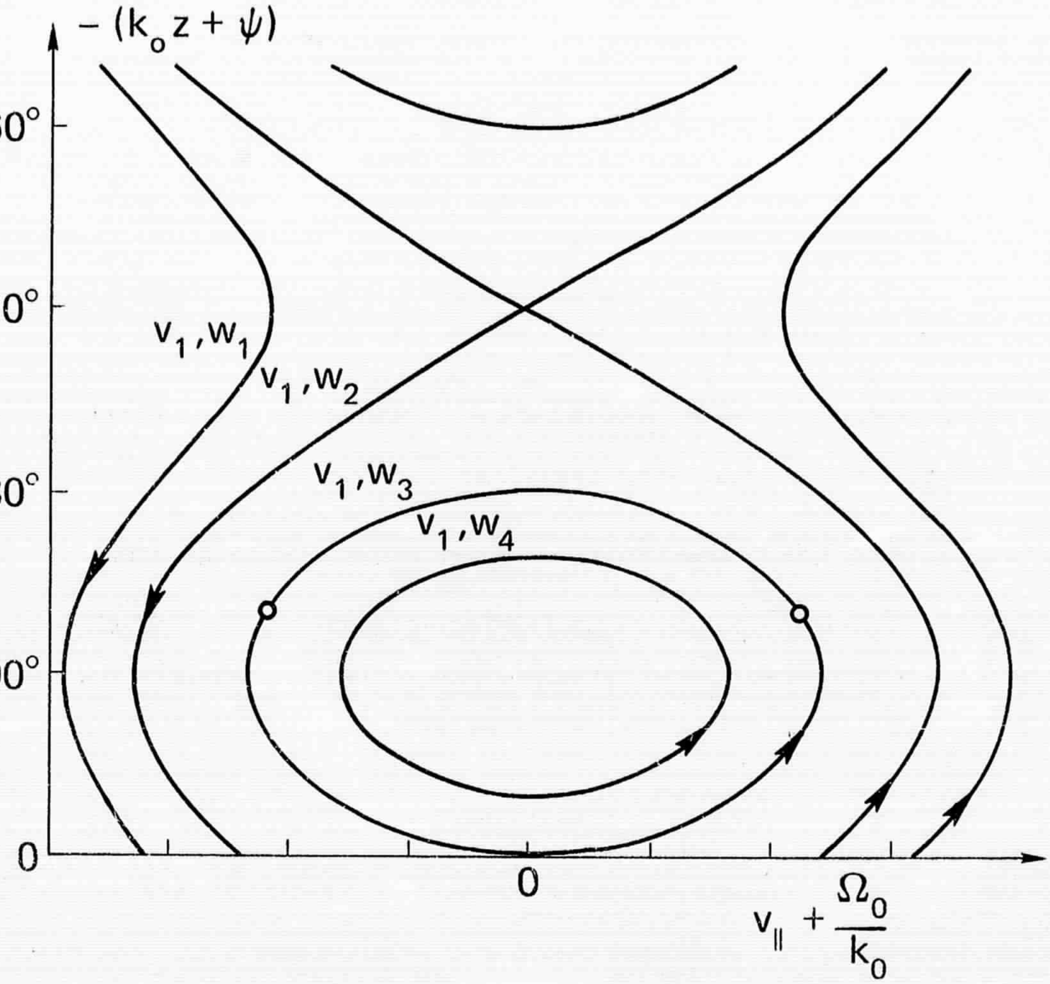

Figure 1. Phase Plane Trajectories for Typical Trapped and Untrapped Cyclotron Resonant Particles

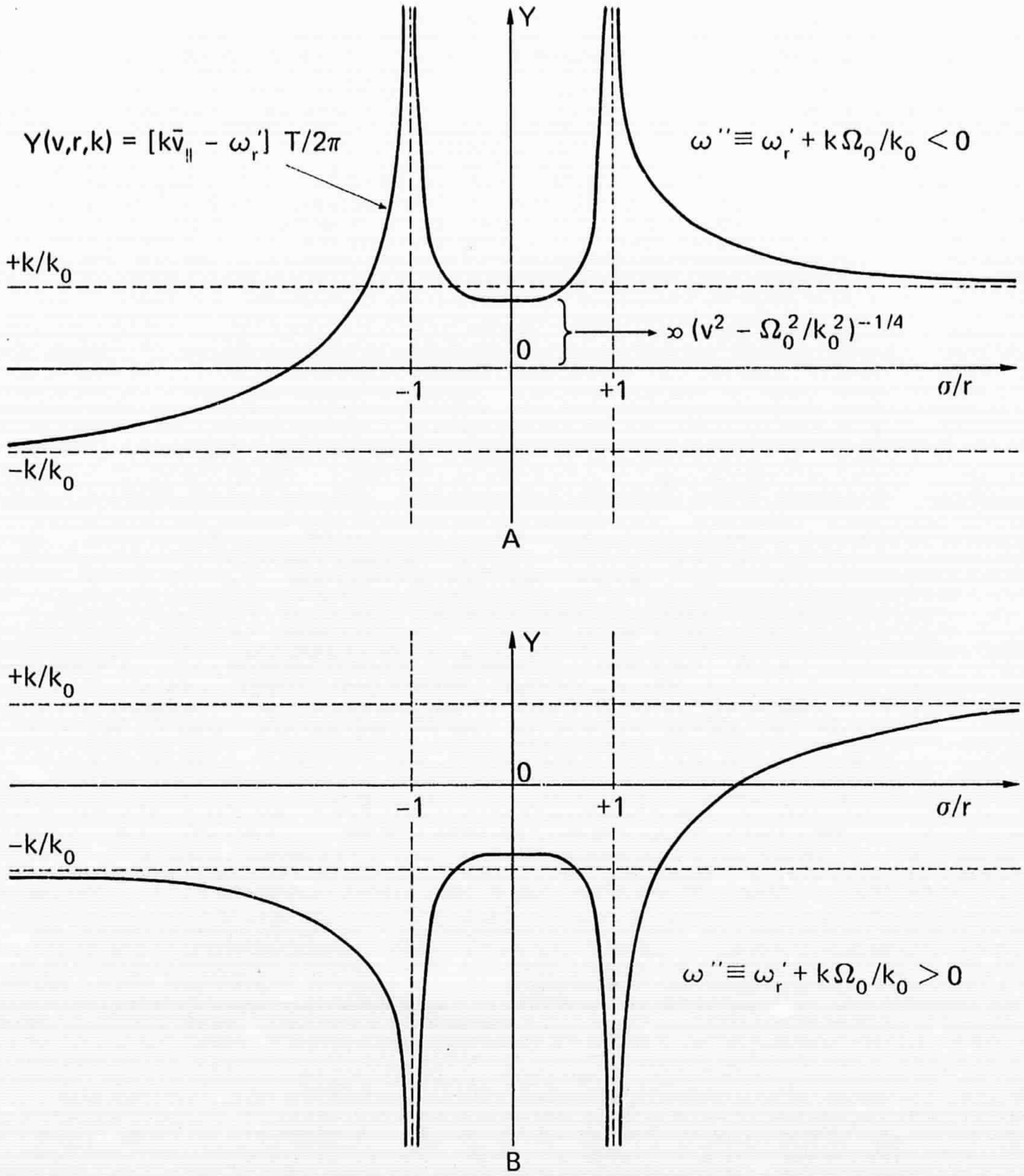

Figure 2. Electrostatic resonances occur at those points in phase space for which $[k\bar{v}_{\parallel} - W_r''] T/2\pi$ is zero or a positive or negative integer. Trapped particles have $r > 1$; untrapped particles have $r < 1$. $\sigma = \mathrm{Sgn}(v_{\parallel} + \Omega_0/k_0)$.