THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

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John E. Walsh and Grace J. Kelleher

Technical Report No. 61
Department of Statistics THEMIS Contract

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ABSTRACT

Considered is discrete two-person game theory where the players choose their strategies independently. Use of mixed strategies introduces probabilistic aspects, so that the payoff to a player has a probability distribution. Determination of optimum strategies is simplified when only some reasonable "representative value" is considered for a distribution. The distribution mean is used for this purpose in expected-value game theory. Another reasonable choice is the distribution median, and this is the basis for median game theory. Median game theory has huge application advantages over expected-value game theory. Payoffs of a very general nature are allowable for median game theory (some payoffs may not even be numbers). Also, optimum solutions are obtainable for virtually all games. These solutions are obtained through orderings of the outcomes of the game (pairs of payoffs, one to each player) according to desirability, with each player doing a separate ordering. This paper first provides an introduction to median game theory and then gives the generally applicable solution, which depends on choices of "relative desirability" functions by the two players (to order the outcomes). Finally, to illustrate the flexibility of median game theory, there is a discussion (including some examples) about considerations in selection of relative desirability functions.

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INTRODUCTION AND DISCUSSION

Considered is the case of two players with finite numbers of strategies. Separately and independently, each player chooses one of his strategies. Every possible combination of strategies determines a pair of payoffs, one to each player. These pairs are the possible outcomes for the game. For a given player, his payoffs can be expressed conveniently in matrix form, where the rows constitute his strategies and the columns the strategies of the other player. Both payoff matrices are known to both players.

A mixed strategy occurs for a player when he assigns probabilities (sum to unity) to his strategies and randomly selects the strategy he uses according to these probabilities. When at least one player uses a randomly chosen strategy, the payoff to each player is a random variable, whose distribution is determined by the probabilities that the players use. These distributions constitute the most information attainable about the outcome for the game.

Determination of an optimum choice for the probabilities of the mixed strategies, with unit probabilities possible, is a basic problem of game theory. This determination encounters many difficulties when all the properties of distributions are taken into account. Great simplification occurs, however, when all that is considered is some reasonable kind of "representative value" for a distribution. The distribution mean (expected payoff to the player) is used to represent a distribution when the well known expected-value method is used. Another reasonable choice is to represent a distribution by its median, and this is the basis for median game theory.

Optimum solutions that are of a "controlling" nature are desirable. That is, an optimum use of mixed strategies controls the game outcome according to some plausible criterion (such as expected payoff). The minimax method used for expected-value game theory yields results of this nature. Also the results developed for median game theory have this property (with respect to a median criterion).

The first several results developed for median game theory are for the situation where the players behave competitively. These results emphasize the ranking of payoffs, separately within each matrix (refs. 1 and 2). This initial method has very desirable features with regard to the effort needed for application (ref. 3). For example, very general kinds of payoffs can occur. Also, an ordering of the payoffs within each matrix, plus accurate evaluation of at most two payoffs in each matrix (whose locations are identified by the orderings), is sufficient for application. Virtually all payoffs need to be accurately evaluated for expected-value game theory.

This initial median method also has strong advantages over expected-value game theory with regard to generality of application. The players behave as competitors in both cases. Also, the games with minimax solutions are a very small subclass of the class of games where optimum median solutions exist when the initial method is used. Also, a median optimum solution can exist for one player but not for the other, which seems to have no analogue in expected-value game theory (ref. 2).

However, the class of games with a median optimum solution (for at least one player) on this basis is a very small subclass of all the discrete two-person games where the players behave competitively, and an exceedingly small subclass of the games where competitive behavior need not occur.

A change of the emphasis to ordering of outcomes (rather than payoffs) results in a median approach that is applicable for virtually all discrete two-person games. Moreover, the players need not behave competitively. The only requirement is that, separately, each player is able to order the possible outcomes according to increasing desirability to him (with equal desirability possible at places in the ordering). The first result of this nature, but with the players using "relative desirability" function of a specialized kind (appropriate for a type of competitive behavior) to order the outcomes, occurs, in ref. 4. This is easily extended to situations where general kinds of relative desirability functions can be used (based on an idea in ref. 5) and is a result of this paper. The way a player behaves is specified by the relative desirability function that he uses for ordering the outcomes.

The availability of suitable relative desirability functions is important with respect to the effort needed for application. Ordering of the outcomes, by each player, does not require very much effort when the functions are available. Otherwise, some general method, such as paired comparisons, might need to be used. If the number of outcomes N is large the number of possible paired comparisons for a player, N(N-1)/2, is huge. It is to be noted that virtually all payoffs ordinarily need to be accurately evaluated if the outcomes are to be ordered. An exception to this requirement occurs when the specialized kind of function introduced in ref. 4 is used.

So much freedom is available in the selection of relative desirability functions that difficulties can arise in making a definite choice. To aid in such selections, several possible types of functions, and how these can reflect a player's desires, receive consideration. Of course, virtually any function (with one-dimensional numerical values) of the two payoffs of an outcome could be used as a relative desirability function

The next section is devoted to the generally applicable median approach that is based on orderings of outcomes. The final section contains some examples of relative desirability functions.

GENERALLY APPLICABLE APPROACH

The same results apply to each player and are given for player i (i=1, 2). These results are stated in terms of a marking of outcome positions in the payoff matrix for player i (that is, the payoffs to player i, in the outcomes considered, are marked). The method of verification is very analogous to that given in refs. 2, 4 and no details are stated here.

First, mark the position(s) in the payoff matrix for player i of the outcome(s) with the highest level of desirability to player i. Next, mark the position(s) of the outcome(s) with the next to highest level of desirablilty. Continue this marking, according to decreasing level of desirability, until the first time that marks in all columns can be obtained from two or fewer rows. Now remove the mark(s) for the least desirable outcome(s) of those that received marks. Then, by the following procedure, determine whether some one of the remaining outcomes can be assured with probability at least 1/2. The procedure is to replace every marked position in the matrix by unity and all other positions with zero. The resulting matrix is considered to be that for player i in a zero-sum game with an expected-value basis, and is solved for the value of the game to player i. Some one of the outcomes corresponding to the marked positions can be assured with probability at least 1/2 if and only if this game value is at least 1/2.

Suppose that the game value is less than 1/2. Then, the largest level of desirability that can be assured with probability at least 1/2 is the level that corresponds to the outcome(s) with marking(s) removed at this step. Otherwise, when the game value is at least 1/2, remove the mark(s) for the least desirable outcome(s) of those still having marks. Then, by another use of the procedure given above, determine whether some one of the remaining marked outcomes can be assured with probability at least 1/2. If not, the maximum level of desirability that can be assured with probability at least 1/2 is the level that corresponds to the outcome(s) with marking(s) removed at this step. If a probability of at least 1/2 can be assured, continue in the same manner until the first time some one of the remaining marked outcomes cannot be assured with probability at least 1/2. Then, the largest desirability level that can be assured with probability at least 1/2 is the level for the outcome(s) with marking(s) removed at this last step.

Now, consider determination of a median optimum strategy for player i. Use the markings in the matrix of player i that, by the method used, ultimately resulted in the smallest set of marked outcomes such that some one of these outcomes can be assured with probability at least 1/2. Replace the marked positions by unity and the unmarked positions by zero. Treat the resulting matrix as that for player i in a zero-sum game with an expected-value basis. An optimum strategy for player i in this zero-sum game is median optimum for him.

The method used here is similar to that of ref. 2, 4. It is a simplification (of the method in ref. 1) that, for a specified minimum desirability level, maximizes the probability that at least this desirability level occurs.

EXAMPLES OF DESIRABILITY FUNCTIONS

Virtually complete freedom is available in expressing the desires of a player (for the outcomes of a game) by use of a relative desirability function. However, this does not imply that any choice that might be made is necessarily satisfactory. On the contrary, great care can be needed to determine a function that is suitable. This great freedom of choice is a valuable property, but only if used wisely. Several examples are given to illustrate considerations in the development of relative desirability functions. In general, an ordering function for player i is denoted by $D_i(p_1,p_2)$, where p_1 and p_2 are the payoffs received by players 1 and 2, respectively. For simplicity, but without much loss of generality, p_1 and p_2 are expressed as numbers which are such that increasing values of p_i represent nondecreasing desirability of the payoffs to player i (i=1, 2).

The first situation is one where player 1 is considered and the players behave competitively. Suppose that an increase of 1 in p_1 has the same desirability, to player 1, as a decrease of 10 in p_2. Then, use of

$$D_1(p_1, p_2) = p_1 - p_2/10$$

would seem appropriate, where it is to be noted that $D_1(p_1+1, p_2)$ equals $D_1(p_1, p_2-10)$ for all possible values for p_1 and p_2. Incidentally, the same ordering would be obtained if $D_1(p_1, p_2)$ were replaced by any strictly monotonic increasing function of $p_1-p_2/10$.

Next, consider player 2 and a situation where an increase in p_1 is desirable to player 2, although not nearly as desirable as the same increase in p_2. Suppose that an increase of 1 in p_2 has the same desirability, to player 2, as an increase of 8 in p_1. Then, use of

$$D_2(p_1, p_2) = p_2 + p_1/8$$

would seem suitable, where the relation

$$D_2(p_1+8, p_2) = D_2(p_1, p_2+1)$$

is seen to hold for all possible p_1 and p_2. Here the same ordering is obtained when $D_2(p_1, p_2)$ is replaced by any strictly monotonic increasing function of $p_2 + p_1/8$.

Now, consider player 1 and a more complicated type of competitive behavior. Here, $p_1 > 0$ and $p_2 > 0$ for the situation that occurs. Suppose that an increase of 10 percent in p_1 has the same desirability, to player 1, as a 40 percent decrease in p_2. Then use of

$$D_1(p_1, p_2) = \log_{10}p_1 + \left[(\log 1.1)/(\log .6)\right]\log_{10}p_2$$

would seem suitable. It is to be noticed that

$$D_1(1.1\, p_1, p_2) = D_1(p_1, .6\, p_2)$$

for all allowable p_1 and p_2. Again, the same ordering would be obtained if a strictly monotonic increasing function of $D_1(p_1, p_2)$ is used in place of $D_1(p_1, p_2)$.

Finally, consider player 2 and another complicated type of competitive behavior. Here, $p_1 > 0$ for the situation that occurs. Suppose that an increase of 1 in p_2 has the same desirability, to player 2, as a 15 percent decrease in p_1. Then, use of

$$D_2(p_1, p_2) = p_2 - (\log_{10}.85)^{-1}\log_{10}p_1$$

would seem appropriate, where

$$D_2(p_1, p_2+1) = D_2(.85p_1, p_2)$$

is seen to hold for all allowable p_1 and p_2.

It is to be observed that a change in the value of p_1 and/or the value of p_2 does not necessarily result in a change in the value of $D_1(p_1, p_2)$ or of $D_2(p_1, p_2)$. This occurs, for example, when the special kind of relative desirability function introduced in ref. 4 is used. For this kind of function, all (p_1, p_2) such that $p_1 \geq P_1$ and also $p_2 \leq P_2'$ have maximum desirability for player 1, and all (p_1, p_2) such that $p_1 \leq P_1'$ and also $p_2 \geq P_2$ have maximum desirability for player 2. Determination of P_1, P_2, P_1', P_2' is considered in refs. 1 and 2 (with P_I, P_{II}, P_I', P_{II}' used as the notation, respectively). However, the (p_1, p_2) that do not satisfy $p_1 \geq P_1$ and $p_2 \leq P_2'$ are ordered by player 1 through use of some function $D_1(p_1, p_2)$, perhaps of a type considered above, for the situation of players behaving competitively. Also, the (p_1, p_2) not satisfying $p_1 \leq P_1'$ and $p_2 \geq P_2$ are ordered through use of some $D_2(p_1, p_2)$ by player 2 for the situation of players who behave as competitors.

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