THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

A NEW APPROACH TO TWO-PERSON GAME THEORY USING MEDIAN CONSIDERATIONS

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John E. Walsh and Grace J. Kelleher

Technical Report No. 58
Department of Statistics THEMIS Contract

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Research sponsored by the Office of Naval Research
Contract N00014-68-A-0515
Project NR 042-260

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ABSTRACT

The concept of mixed strategies greatly extends the opportunity to obtain optimum strategies for discrete two-person game theory but introduces probability considerations. That is, the payoffs to the players can have probability distributions. This complicates the determination of optimum strategies. However, this determination can be greatly simplified by only considering some reasonable kind of "representative value" for a distribution. The distribution mean is used in the expected-value approach. Another reasonable choice is the distribution median, and this is the basis for median game theory. The median approach not only can be applied to the payoffs for each player but also to the possible outcomes (pairs of payoffs, one to each player) for the game. Application of the median approach to the payoffs for each player yields some results that are much more widely applicable than those for expected-value game theory. Median results that are generally usable, for virtually all types of payoffs, are developed by applying the median approach to outcomes. The median approach has strong application advantages, of several kinds, over the expected-value approach. The research results obtained to date are stated, somewhat more generally than they were originally given, and are discussed. Also, anticipated extensions are outlined.

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*Research partially supported by Mobil Research and Development Corporation. Also associated with ONR contract N00014-68-A-0515 and NASA Grant NGR 44-007-028.

INTRODUCTION AND DISCUSSION

The situation considered is that of two players, each of whom has the choice of a finite number of strategies. Separately and independently, each player selects one of his strategies. A known pair of payoffs, one to each player, is associated with every combination of strategy choices for the players. These pairs are called the outcomes of the game. The payoffs to a player for the various strategy combinations can be stated conveniently in matrix form, where the rows correspond to his strategies and the columns to the strategies of the other player. Both payoff matrices are known to the players.

When a player assigns probabilities to his strategies and makes the selection randomly (according to these probabilities), he is said to use a mixed strategy. The concept of mixed strategies introduces probabilistic aspects into game theory. When at least one player selects his strategy randomly, the payoff to each player has a probability distribution (determined by the probabilities used by the players). Knowledge of these distributions is the maximum information that can possibly be obtained about the payoffs to the players.

A fundamental problem of game theory is determination of optimum mixed strategies for the players. More specifically, the problem is to make an optimum choice for the probabilities that determine the mixed strategies, where unit probabilities are possible (and represent definite choice of strategy). Such a determination is, unfortunately, subject to many complications when all the properties of probability distributions are taken into consideration. However, determination of an optimum choice can be greatly simplified by only considering some kind of "representative value" for a distribution. The well known expected-value approach uses the distribution mean (the expected payoff to the player) as the representative value. For zero-sum games, the situation is essentially one-dimensional for outcomes (one payoff of a pair determines the other), so that consideration of expected payoffs is equivalent to consideration of expected outcomes.

Another reasonable way to represent a (univariate) distribution is by its median. This is the basis for median game theory. For the <u>first approach</u>, the payoff matrices are considered separately. Within each matrix, the payoffs are ranked according to increasing desirability, and the situation is such that the resulting rankings are the same for both players. The median of payoffs to a player is considered (with respect to the ordering). For the <u>second approach</u>, the outcomes are fist ordered according to relative desirability separately by each player. These orderings need not be even roughly the same. The median of outcomes is considered (with respect to the ordering for the player). Situations can occur that involve both approaches.

A strong advantage of median game theory is that the payoff "values" can be of a very general nature. The allowable kinds of payoffs are more restricted for the first approach, due to the condition that the rankings are the same for both players. The only condition for the second approach is that the outcomes can be ranked by each player. For both approaches, some or all of the payoffs need not even be numbers (for example, some payoffs may designate categories). A ranking, of payoffs or outcomes, should nearly always be possible on a paired comparison basis (with equal desirability as one possibility).

The first approach has advantages over the second approach with regard to necessity for evaluation of payoffs. All the payoff "values" ordinarily need to be determined for the second approach. Knowledge of the relative ranking within each matrix, combined with "values" for at most two payoffs in each matrix (which can be identified by the ranking), is sufficient for the first approach. The effort needed for evaluating payoffs can be a very important practical consideration (ref.1).

An important advantage of the second approach is its generality of application. Not only is it usable for virtually all kinds of payoffs but also optimum solutions, for both players, are obtainable for all cases where the outcomes can be ranked. Moreover, situations that are only partially competitive or not competitive, in the behavior of the players, can be handled by a suitably chosen method for ranking the outcomes.

A competitive viewpoint is adopted in developing the results for the first approach. An optimum solution occurs for a player if and only if the game situation is median competitive for him (called one player median competitive: OPMC). The class of games that are OPMC for both players is a small subset of the set of all games. Even the class that is OPMC for at least one player is a small subset of all games. Thus, the first approach has strong disadvantages (compared to the second approach) with respect to generality of application. However, its advantage with respect to application effort suggests that the first approach be considered for possible use when a competitive viewpoint occurs for both players.

A combination of the two approaches can occur when the viewpoint is competitive but the situation is not OPMC for both players. Then outcomes are ordered according to increasing desirability, separately for each player. However, the group with maximum desirability is determined from OPMC considerations. Within this group, only ranking of payoffs for each player, and "values" of at most two identified payoffs for each player, need to be determined. Substantially less effort than would be required to evaluate all payoffs is sometimes sufficient when results based on this combination of the approaches are used.

Both approaches have appreciable application advantages over expected-value game theory, with respect to the classes of games where they are usable and in the effort needed for their application. Games that satisfy the zero-sum condition (and slight modifications), which leads to optimum solutions for discrete two-person game theory of an expected-value mature, are included among the median competitive games. Moreover, they are a very small subset of the median competitive games. Also, expected-value game theory requires accurate evaluation of all (or nearly all) payoffs. Median game theory is generally applicable and can often be used without accurate evaluation of more than a small fraction of the payoffs. A discussion of the application advantages of median game theory over expected-value game theory is given in ref. 1 (also see refs. 2, 3, 4, 5).

The results that have been developed for the first approach are outlined in the next section. Then, the results for the second approach are described in the following section (including a combination of the two approaches). The next to last section reports on some results where cooperation is possible. The final section outlines some anticipated extensions of median game theory.

RESULTS FOR FIRST APPROACH

For brevity, the desirability of a payoff and the "value" of a payoff are used interchangeably in the remainder of this paper. The original results were stated in terms of payoff values, with the payoffs being numbers. This paper extends the applicability of these results to situations where relative desirability can be determined among payoffs, and the resulting ordering of payoffs is the same for both players.

When the viewpoint is competitive (as considered for the first approach), the concepts of a player acting protectively, or vindictively, are useful in determining optimum strategies (ref. 2). That is, a protective player tries to maximize his payoff, regardless of the payoff to the other player. A vindictive player attempts to minimize the payoff to the other player, without consideration of his own payoff. When a player has a strategy whereby he can be simultaneously protective and vindictive, this is considered to be an optimum strategy for him.

The player are designated as I and II. It is always true that: A largest value P_I (P_{II}) occurs in the payoff matrix for player I (II) such that, when acting protectively, he can assure at least this payoff with probability at least 1/2. Also, a smallest value P'_I (P'_{II}) occurs in the matrix for player I (II) such that vindictive player II (I) can assure, with probability at least 1/2, that player I (II) receives at most this payoff. Here, $P'_I \le P_I$ and $P'_{II} \le P_{II}$, with equality possible. A method for evaluation of P_I, P_{II}, P'_I, P'_{II} is given in Appendix A. Also given there is a procedure for determining a median optimum strategy for player I (II) when he is acting protectively, and for determining a median optimum strategy for player I (II) when he is acting vindictively.

Payoff matrices exist such that a player can be simultaneously protective and vindictive. This occurs if and only if the game situation is OPMC for him. Specifically, let set I (II) consist of those outcomes such that the payoff to player I (II) is at least P_I (P_{II}) and the payoff to player II (I) is at most P'_{II} (P'_I). A game is OPMC for player I (II) if and only if player I (II) can assure, with probability at least 1/2, that some outcome in set I (II) occurs. A game is median competitive if and only if it is OPMC for both players. These results are taken from ref. 3. A subclass of the median competitive games is identified in ref. 2. A method of determining whether a game is OPMC for a specified player is given in Appendix A. Also given there is method for determining a median optimum strategy for a player when the game is OPMC for him.

When a game is not OPMC for player I (II), an optimum solution, for him, can be obtained by suitably supplementing the outcomes of set I (II) with additional outcomes, until the first time some outcome of the augmented set can be assured with probability at least 1/2. This method is an example of the second approach for the special case of a competitive viewpoint, and is considered in the next section.

RESULTS FOR SECOND APPROACH

The case of suitably supplementing sets I and II is considered first. Specifically, each player chooses a suitable "relative desirability" function for ordering the outcomes. These functions have the special property that all outcomes of set I (II) have maximum desirability for player I (II). Further, where (p_I, p_{II}) denotes a general outcome, a condition on player I is that his relative desirability is a nonincreasing function of $P_I - p_I$ for fixed $p_{II} - P'_{II}$, and a nonincreasing function of $p_{II} - P'_{II}$ for fixed $P_I - p_I$. Likewise, for player II, relative desirability is a nonincreasing function of $P_{II} - p_{II}$ for fixed $p_I - P'_I$, and a nonincreasing function of $p_I - P'_I$ for fixed $P_{II} - p_{II}$. These conditions are motivated by the competitive viewpoint. The material using this kind of relative desirability function is taken from ref. 4.

The preceding method for ordering outcomes according to increasing desirability is a special case of the second approach. Actually, each player could use almost any possible way that results in an ordering of the outcomes. However, if (p_I, p_{II}) is a general outcome, virtually any ordering method should have the property that, for player I (II), relative desirability is a nondecreasing function of p_I (p_{II}) for a fixed value of p_{II} (p_I). Once an ordering according to increasing desirability has been determined for player I (II), he can identify a smallest set of outcomes S_I (S_{II}) such that the other outcomes are less desirable and such that he can assure an outcome of S_I (S_{II}) with probability at least 1/2. A method for determining S_I and S_{II} is outlined in Appendix B. A way of determining median optimum strategies is also given there. This material using general ways of establishing relative desirability is based on a method given in ref. 5.

These results are optimum, according to the median criterion, when the players choose their strategies separately and independently. Cases where the players can cooperate in the selection of their strategies offer additional possibilities and are discussed in the next section.

CASES WITH COOPERATION

Use of median game theory may not be advantageous when the players can cooperate in the choice of strategies. That is, the players may be able to guarantee a game outcome that is more desirable, to both, than use of solutions that are medium optimum for the case of no cooperation.

Some situations where cooperation is definitely preferable are identified in ref. 5. Two types of cooperation are considered. No side payments are made for one type. Cooperation of this type can occur for any situation where median game theory is applicable (first or second approach). Side payments can be made for the other type of cooperation. This type can occur for situations where all payoffs can be expressed in a common unit and satisfy the arithemetical operations.

Suppose that the outcomes have been ordered according to increasing desirability, separately for each player (as for the second approach). Then sets S_I and S_{II} are determined by optimum use of median game theory (as in the preceding section). General Rule: Cooperation is definitely preferable when an acievable outcome exists that is at least as desirable to player I as one or more outcomes in S_I and also is at least as desirable to player II as one or more outcomes in S_{II}. Here, the achievable outcomes are the possible outcomes when side payments are not made. When side payments can be made, any pair of payoffs whose sum equals the sum for a possible outcome is considered to be an achievable outcome.

Some implications of this rule are discussed in ref. 5. Cooperation can be definitely preferable for many situations that are median competitive. However cooperation is seldom useful when the game is of a competitive type. (A game is competitive when the totality of outcomes can be arranged so that the payoffs to player I are nondecreasing and also the payoffs to player II are nonincreasing.). Cooperation tends to be more attractive when side payments can be made. In fact, cooperation often is preferable to median game theory when side payments can be made.

ANTICIPATED EXTENSIONS

Many extensions of median game are being carried out or are planned. Generally applicable two-person percentile game theory, where each player decides on the percentile he uses, is being developed using relative desirability orderings like those of the second approach. This seems to be directly extendable to generally applicable N-person percentile game theory where the players choose their strategies separately and independently. Identification of cases where cooperation is definitely preferable will be considered for these more general results. This identification can be very complicated for N-person games. Also, difficulties in practical application seem to be very important for N-person games.

Percentile (including median) game theory has difficulties in suitably allowing for the sizes of highly desirable and highly undesirable payoffs and outcomes. Ways of doing this more meaningfully are being developed.

Game theory in which a player simultaneously considers two or more percentiles for use, and chooses one according to probabilities he specifies, is under consideration. Optimum choice of the probabilities for the percentiles, combined with use of an optimum strategy for the percentile chosen, may tend to strongly increase the desirability of the outcome that occurs.

Another direction for extensions is cases where one or more players do not know the percentiles being used by one or more of the players. This can be important whether cooperation is allowed or not.

Other possibilities such as extension to the continuous case also exist. This field is just starting its development and a multitude of worthwhile results should be developed over the next few years.

APPENDIX A

Determination of values for P_I, P_{II}, P'_I, P'_{II} is considered first. This can be accomplished by marking of some of the values in the payoff matrices. The method given here is the simplification of the procedure in ref. 2 that is introduced in ref. 3 (and is not based on development and use of preferred sequences).

For player I (II) acting protectively, first mark the position(s) in his matrix of the largest payoff value. Then also mark the position(s) of the next to largest payoff value. Continue this marking, according to decreasing payoff value, until the first time that marks in all the columns can be obtained from two or fewer of the rows (then a marked value can be assured with probability at least 1/2, perhaps greater than 1/2). Now, remove the mark(s) for the smallest of the payoffs used and (by the following method) determine whether some one of the remaining marks can be assured with probability at least 1/2. This cannot occur unless there are still marks in all the columns. When all columns are still marked, replace every marked payoff by the value unity and every unmarked payoff by zero. Consider the resulting matrix of ones and zeroes to be for a zero-sum game with expected-value basis and solve for the value of this game to player I (II). Some one of the remaining marks can be obtained with probability at least 1/2 if and only if this game value is at least 1/2.

When protective player I (II) cannot assure a remaining mark with probability at least 1/2, the value of P_I (P_{II}) is the payoff value in the matrix of player I (II) that had its marking(s) removed last. Otherwise (a game value of at least 1/2), those of the remaining marks that correspond to the smallest of the remaining marked payoffs are removed. Then, as just discussed, determine whether some one of the marks remaining now can be assured with probability at least 1/2. If not, P_I (P_{II}) equals the payoff in the matrix of player I (II) that had its marking(s) removed last. If a probability of at least 1/2 can be assured, continue in the same way (removing the mark(s) for the smallest of the remaining payoffs with marks) until the first time that some one of the remaining marks cannot be assured with probability at least 1/2. Then P_I (P_{II}) is the payoff in the matrix of player I (II) that had its marking(s) removed last. It is to be noted that P_I and P_{II} are often the payoffs which provided the first time that two or fewer rows contained marks in all columns of the respective matrices.

For player I (II) acting vindictively, first mark the position(s) in the matrix for player II (I) of the smallest payoff value. Then also mark the position(s) of the next to smallest payoff value. Continue this marking, according to increasing payoff value, until the first time that marks in all the rows can be obtained from two or fewer of the columns (assures that a marked value can be obtained with probability at least 1/2). Next remove the marks for the largest of the payoffs used and (by the following method) determine whether some one of the remaining marks can be assured with probability at least 1/2. This is not possible unless there are still marks in all the rows. When all rows still contain marks, replace every marked payoff by the value zero and every unmarked payoff by unity. The resulting matrix of ones and zeroes is considered to be for a zero-sum game with an expected value basis, with rows corresponding to strategies for player II (I). Solve this game for its value to player II (I). Some one of the remaining marks can be obtained with probability at least 1/2 by player I (II) if and only if this game value is at most 1/2.

When vindictive player I (II) cannot assure a remaining mark with probability at least 1/2, the value of P'_{II} (P'_I) is the payoff value in the matrix of player II (I) that had its marking removed last. Otherwise (a game value of at most 1/2), those of the remaining marks that correspond to the largest of the remaining marked payoffs are removed.

Then, as just discussed, determine whether some one of the marks still remaining can be assured by player I (II) with probability at least 1/2. If not, P'_{II} (P'_{I}) equals the payoff in the matrix of player II (I) that had its marking(s) removed last. If a probability of at least 1/2 can be assured, continue in the same manner (removing the mark(s) for the largest of the remaining payoffs with marks) until the first time that some one of the remaining marks cannot be assured by player I (II) with probability at least 1/2. Then P'_{II} (P'_{I}) is the payoff in the matrix of player II (I) that had its marking(s) removed last. Often, P'_{II} and P'_{I} are the payoffs which furnished the first time that two or fewer columns contained marks in all the rows of the respective matrices.

Consider determination of a median optimum strategy for player I (II) when he acts protectively. Let all payoffs with value at least P_{I} (P_{II}) in the matrix for player I (II) be replaced by unity and all other payoffs replaced by zero. The resulting matrix of ones and zeroes is considered to be for a zero-sum game with an expected-value basis. An optimum strategy for player I (II) in this game is a protective median optimum strategy for him.

Now consider determination of a median optimum strategy for player I (II) when he acts vindictively. Let all payoffs with values at most P'_{II} (P'_{I}) in the matrix for player II (I) be replaced by zero and all other payoffs replaced by unity. This matrix of ones and zeroes is used for a zero-sum game with an expected-value basis. An optimum strategy is determined for player I (II) in the game where the matrix (ones and zeroes) for player II (I) is used. This is a vindictive median optimum strategy for player I (II).

To determine whether a OPMC situation occurs for player I (II), mark the positions of his matrix for his payoffs in the outcomes of set I (II). If marks in all columns can be obtained from two or fewer rows, the situation is automatically OPMC for player I (II). If at least one column contains no marks, the situation is not OPMC for player I (II). Otherwise, replace every marked payoff by unity and every unmarked payoff by zero. Consider the resulting matrix to be for a zero-sum game with an expected value basis. The situation is OPMC for player I (II) if and only if the value of this game is at least 1/2.

Now, consider determination of an optimum strategy for player I (II) when the situation is OPMC for him. Use the same marking as for the preceding paragraph and replace marked values by unity and unmarked values by zero. Again treat the resulting matrix as a zero-sum game with an expected value basis. An optimum strategy for player I (II) in this zero-sum game is a median optimum OPMC strategy for that player. The probability that player I (II) receives at least P_I (P_{II}) and simultaneously player II (I) receives at most P'_{II} (P'_I) is at least equal to the game value. This method of choosing a median optimum OPMC strategy tends to maximize the game value and also tends to minimize the application effort. Other methods based on choice of a preferred sequence order for the pairs of payoffs (see ref.2) can be developed in a straightforward manner, but only this method is considered here.

APPENDIX B

Determination of S_I (S_{II}) is very similar to determination of P_I (P_{II}). The only difference is that the markings are made according to the ranking of the outcomes by player I (II) instead of according to the payoffs to player I (II). That is, for player I (II), first mark the position(s) in his matrix for the payoff(s) in the outcome(s) with highest desirability. Then also mark the position(s) for the payoff(s) in the outcome(s) with the next to highest desirability. Continue this marking, according to decreasing desirability, and use the method for evaluating P_I (P_{II}) to determine the smallest set of marked positions such that a marked position can be assured with probability at least 1/2 and such that outcomes with unmarked payoffs are less desirable than those with marked payoffs. The resulting outcomes with marked payoffs constitute S_I (S_{II}) for player I (II).

To determine a median optimum strategy for player I (II) in the second approach, let all payoffs in his matrix that belong to outcomes of S_I (S_{II}) be replaced by unity and all other payoffs replaced by zero. The resulting matrix of ones and zeroes is considered to be for a zero-sum game with an expected-value basis. An optimum strategy for player I (II) in this game is median optimum for him.

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UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) SOUTHERN METHODIST UNIVERSITY	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED
	2b. GROUP UNCLASSIFIED

3. REPORT TITLE

"A new approach to two-person game theory using median considerations"

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (First name, middle initial, last name)

John E. Walsh

Grace J. Kelleher

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 13, 1970	17	5

8a. CONTRACT OR GRANT NO. N00014-68-A-0515	9a. ORIGINATOR'S REPORT NUMBER(S) 58
b. PROJECT NO. NR 042-260	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research

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DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

UNCLASSIFIED
Security Classification

A-31408