NASA TECHNICAL NOTE

NASA TN D-6372

C.1

NASA TN D-6372

THEORY OF THOMSON SCATTERING FROM A WEAKLY IONIZED PLASMA

by Richard G. Seasholtz

Lewis Research Center
Cleveland, Ohio 44135

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JUNE 1971

1. Report No.	2. Government Accession No.	3. Recipie...
NASA TN D-6372		

4. Title and Subtitle	5. Report Date
THEORY OF THOMSON SCATTERING FROM A WEAKLY IONIZED PLASMA	June 1971
	6. Performing Organization Code
7. Author(s) Richard G. Seasholtz	8. Performing Organization Report No. E-6051
	10. Work Unit No. 120-26
9. Performing Organization Name and Address Lewis Research Center National Aeronautics and Space Administration Cleveland, Ohio 44135	11. Contract or Grant No.
	13. Type of Report and Period Covered Technical Note
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D. C. 20546	14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

The theory of Thomson scattering from a weakly ionized plasma is extended to include the effects of unequal electron and ion temperatures, a constant magnetic field, and electron drift. The plasma is described by the two fluid continuum equations that are valid if the wavelength of the probing wave is longer than the ion-neutral mean free path. The spectrum and total scattered power are calculated. Collisions and a magnetic field are both shown to influence the scattered power for unequal electron and ion temperatures.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Incoherent scattering, Electromagnetic scattering, Plasma diagnostics, Electron density, Plasma temperature	Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	46	$3.00

*For sale by the National Technical Information Service, Springfield, Virginia 22151

CONTENTS

THEORY OF THOMSON SCATTERING FROM A WEAKLY IONIZED PLASMA*

by Richard G. Seasholtz

Lewis Research Center

SUMMARY

The theory of Thomson scattering from a weakly ionized plasma is extended to include the effects of unequal electron and ion temperatures, a constant magnetic field, and electron drift. The Born scattering formula and the fluctuation-dissipation theorem are used to calculate the differential backscattering cross section. The plasma is described by the hydrodynamic (continuum) equations for the conservation of mass, momentum, and energy of each species. Included are electron-neutral and ion-neutral collisions but not collisions between charged particles. The results given are valid if the wavelength of the probing wave is greater than the ion-neutral mean free path. With no magnetic field the backscattered power is shown to be generally less than that found from a collisionless calculation and to be dependent on the ion-neutral and electron-neutral collision frequencies when the electron to ion temperature ratio T_e/T_i does not equal 1. A magnetic field normal to the incident wave causes the backscattered power to be increased for T_e/T_i in the common experimental range of 1 to 10; it is shown that the scattered power may increase with T_e/T_i rather than falling off in accord with the usual $(1 + T_e/T_i)^{-1}$ relation. The effect of electron drift (less than the critical velocity corresponding to the onset of instability) on the Thomson scattering spectrum and on the total scattered power is calculated. As the electron drift velocity is increased, the spectrum develops a sharp peak at a frequency shift corresponding to the ion acoustic wave, and the total scattered power is enhanced.

INTRODUCTION

An important new technique of plasma diagnostics is based on Thomson, or incoherent, scattering of electromagnetic radiation. Although originally applied to the study of

*Based, in part, on a dissertation submitted to Case Western Reserve University in partial fulfillment of the requirements for the degree of doctor of philosophy in January 1970. The work was partially supported by the National Science Foundation.

the ionosphere (refs. 1 to 6), Thomson scattering measurements have also been used in the diagnosis of laboratory plasmas using high power lasers as the radiation source (e.g., ref. 7). The measurement of the Thomson scattered light of a pulsed ruby laser has been suggested (ref. 8) as a method of studying re-entry plasmas. Other possible applications of Thomson scattering of laser light include measurements in rocket exhaust and explosion generated plasmas.

Thomson scattering measurements offer the important potential advantage of being able to measure many quantities of interest, such as electron and ion densities, temperatures, drift velocities, and even velocity distributions at a specific interior point in the plasma. Techniques that involve the insertion of physical probes (e.g., Langmuir probes) into the plasma offer much less spatial resolution and can disturb the plasma. Other radiation based diagnostic methods either measure what leaks out of the plasma, as with spectroscopic methods, or measure an average value along a path through the plasma as with microwave and laser interferometry techniques.

Thomson scattering measurements were first used in studies of the ionosphere. In 1958, Gordon (ref. 1) suggested that, if a beam of radio waves with a frequency well above the plasma cutoff frequency were sent into the ionosphere, the radiation scattered by the free electrons could be detected. Analysis of the spectrum and scattered power could provide information about the electron density and temperature. He noted that this measurement could be made above, as well as below, the altitude of maximum electron density. In his calculation he assumed that the electrons were in random thermal motion similar to neutral particles. He predicted that the scattered signal should have a spectral width corresponding to the thermal velocity of the electrons since their motion would cause the scattered signal to be Doppler shifted with respect to the incident wave.

Shortly after Gordon's initial work, Bowles (ref. 2) successfully observed this phenomenon using high power radar. The scattered power agreed with that predicted by Gordon; however, Bowles was not able to observe the broadening of the spectrum that had been calculated. He explained this discrepancy by modifying the theory to include the electrostatic interaction between the electrons and ions and concluded that the spectral width should correspond to the ion, rather than to the electron thermal velocity.

With the observations of Bowles as a stimulus many workers attacked the problem theoretically. The early work was done by Dougherty and Farley (ref. 3), Salpeter (refs. 9 and 10), and Fejer (ref. 11), who dealt primarily with plasmas in thermal equilibrium. More recent works by Salpeter (ref. 12), Rosenbluth and Rostoker (ref. 13), Buneman (ref. 14), and Farley (ref. 6) have extended the theory to include unequal electron and ion temperatures. All this work has been based on kinetic theory calculations.

Cohen (ref. 15) took a somewhat different approach and used very simple two-fluid collisionless hydrodynamic (continuum) equations to calculate the Thomson scattering spectrum. His continuum theory disagreed with the kinetic theory calculations in that the spectrum he obtained consisted of only sharp spikes since the broadening effect of

Landau damping was not included. A more complete continuum theory based on both two and three fluid models was given by Tanenbaum (ref. 16) for equal electron and ion temperatures. His calculations included collisions between the charged particles and the neutrals and gave results that agreed very well with the calculations of Dougherty and Farley (ref. 5) who had included collisions in their kinetic theory.

The advantages of the continuum theories over the kinetic theories are that they are inherently simpler and collisional effects are easily included. Indeed, the continuum theories are only valid when the mean free path is less than the characteristic length of the probing wave and the collisional damping outweighs the Landau damping.

In this report the continuum theory of Thomson scattering is extended to include scattering from plasmas with unequal electron and ion temperatures, a constant magnetic field, and electron drift. Electron-neutral and ion-neutral collisions are included in the theory but collisions between charged particles are neglected. Thus the theory is not applicable to plasmas with very low electron temperatures where electron-ion collisions become dominant. The method used follows that of reference 3 (also see ref. 17), which used the Born scattering formula and the fluctuation-dissipation theorem which gives the spectrum of the fluctuations of a linear system in thermal equilibrium. The new results presented herein make it possible to analyze Thomson scattering experiments in the lower ionosphere and in other weakly ionized plasmas where collisional effects are important and where kinetic theory calculations have not yet been made. Furthermore, in those cases where kinetic theory calculations have been made and where the continuum theory is also valid, the continuum theory has the advantage of being easier to apply.

GENERAL THEORY

The phenomenon of Thomson scattering in a plasma is dependent on fluctuations in the number density of the electrons. If the electrons were stationary and randomly distributed, the total scattered power would be zero; however, because of the thermal motion of the electrons, fluctuations in the electron number density do exit. Furthermore, when the Debye length is less than the wavelength of the probing wave, collective plasma effects come into play and the fluctuations of the ions influence the electron fluctuations. The scattering may be thought of as an interaction between the incident wave, with wave vector $\underset{\sim}{k}_o$, and a thermally excited wave in the plasma. The scattered wave, with wave vector $\underset{\sim}{k}_s$, is related to $\underset{\sim}{k}_o$ by $\underset{\sim}{k}_s = \underset{\sim}{k}_o + \underset{\sim}{k}$ (as shown in fig. 1) where $\underset{\sim}{k}$ is the wave vector of the spatial Fourier component of the electron number density fluctuations that causes the scattering.

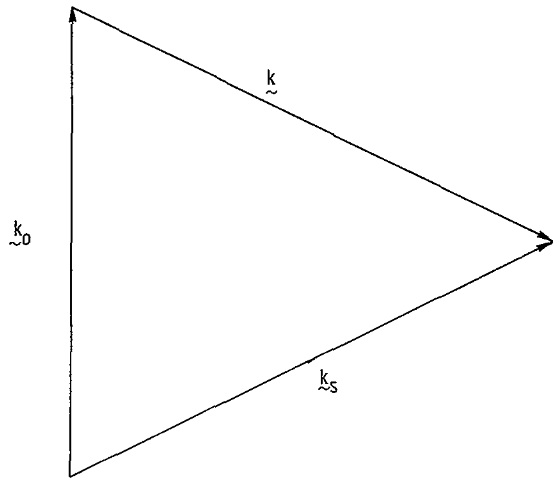


Figure 1. - Relation between wave vectors of incident, scattering, and scattered waves.

Thomson Scattering Cross Section

It can be shown (ref. 18) that the differential scattering cross section for a medium with a fluctuating dielectric constant is

$$\sigma_b(\omega_o - \omega)\, d\omega = \frac{k_o^4 L^3 \sin^2 \rho}{(4\pi)^2 \epsilon^2} \left\langle \left| \Delta\epsilon(-\underset{\sim}{k}, \omega) \right|^2 \right\rangle d\omega \tag{1}$$

where σ_b is the power scattered per unit solid angle per unit volume per unit frequency per unit incident power. (All symbols are defined in appendix A.) In this formula ω_o and k_o are the angular frequency and wave number of the incident wave, ρ is the angle between the electric vector of the incident wave and the direction of propagation of the scattered wave, ω is the deviation of the frequency of the scattered wave from the incident wave, and $\underset{\sim}{k}$ is the wave vector of the scattering wave ($\underset{\sim}{k} = -2\underset{\sim}{k}_o$ for backscattering). Also, $\left\langle \left| \Delta\epsilon(\omega) \right|^2 \right\rangle$ is the power spectrum of the fluctuating part of the permittivity $\Delta\epsilon$ and is defined as the Fourier transform of the auto correlation fluctuation of $\Delta\epsilon$; that is,

$$\int_{-\infty}^{\infty} \left\langle \left| \Delta\epsilon(-\underset{\sim}{k}, \omega) \right|^2 \right\rangle e^{+i\omega\tau}\, d\omega = \left\langle \Delta\epsilon^*(-\underset{\sim}{k}, t)\Delta\epsilon\,(-\underset{\sim}{k}, t + \tau) \right\rangle \tag{2}$$

where the angular brackets on the right hand side denote a time average. Note that the $\underset{\sim}{k}$ dependence of $\Delta\epsilon$ arises because $\Delta\epsilon$ has been expanded in a Fourier series in a box of sides L, where L has been taken to be much larger than the scattering volume.

To calculate $\Delta\epsilon$, recall that for transverse electromagnetic waves in a plasma with plasma frequency $\omega_e = (N_o e^2/m_e\epsilon_o)^{1/2}$ the dispersion relation may be written as

$$k_o^2c^2 = \omega_o^2 - \omega_e^2 \tag{3}$$

provided ω_o is much greater than the electron collision and cyclotron frequencies (ref. 19). Using this dispersion relation for the plasma and noting that $k_o^2 = \mu_o\epsilon(\underline{r},t)\,\omega_o^2$, the scattering cross section becomes

$$\sigma_b(\omega_o - \omega)\, d\omega = L^3 r_e^2 \sin^2\rho \left\langle \left|\Delta N(-\underset{\sim}{k}, \omega)\right|^2 \right\rangle d\omega \tag{4}$$

where ΔN is the fluctuating part of the electron number density and $r_e = e^2/4\pi\,\epsilon_o m_e c^2$ is the classical electron radius. Thus only the electron density fluctuations cause scattering.

Fluctuation-Dissipation Theorem

To use equation (4) the power spectrum of the fluctuations of the electron number density must be found. A form of the fluctuation-dissipation theorem due to Callen, Barasch, and Jackson (ref. 20), who generalized the original form of the theorem of Nyquist (ref. 21) to any linear system in thermal equilibrium is used. This theorem has previously been used by Farley, Dougherty, and Barron (ref. 4) to calculate Thomson scattering in a collisionless plasma. For a system in thermal equilibrium, the theorem may be stated as follows.

Suppose that a set of generalized forces $V_i(t)$ are applied to the system. The system responses are represented by $I_i(t)$ where the spectral components of V_i and I_i are related by

$$I_i(\omega) = \sum_j Y_{ij}(\omega)\, V_j(\omega) \tag{5}$$

and where Y_{ij} is a generalized admittance tensor. The rate of change of energy of the system is

$$\frac{dW}{dt} = \sum_i V_i(t)\, I_i(t) \tag{6}$$

For high temperatures (or low frequencies) where $\kappa T >> \hbar\omega$, the fluctuation-dissipation theorem may be stated as

$$\left\langle \left| I_i(\omega)\, I_j(\omega) \right| \right\rangle d\omega = \frac{\kappa T}{2\pi}\, (Y_{ij}^* + Y_{ji})\, d\omega \tag{7}$$

where the spectral function on the left hand side is defined by

$$\int_{-\infty}^{\infty} \left\langle \left| I_i(\omega)\, I_j(\omega) \right| \right\rangle e^{i\omega\tau}\, d\omega = \left\langle I_i^*(t)\, I_j(t+\tau) \right\rangle \tag{8}$$

that is, $\left\langle \left| I_i(\omega)\, I_j(\omega) \right| \right\rangle$ is the Fourier transform of the cross correlation function of the fluctuating quantities $I_i(t)$ and $I_j(t)$. (For high frequencies where $\hbar\omega \geq \kappa T$, eq. (7) must be modified to include quantum effects by replacing κT with $1/2\, \hbar\omega \coth(\hbar\omega/2\kappa T)$; e.g., for $T = 300$ K this change must be made if $\omega \geq 4\times10^{13}$ rad/sec.)

The fluctuation-dissipation theorem is applied to a specific system which is in equilibrium by choosing the I_i to correspond to the desired macroscopic variable. The relation for the rate of change of energy of the system is then used to identify the quantities that correspond to the V_i. Finally, the equations of motion of the system are examined to find the proper admittance tensor Y_{ij} that relates the I_i and the V_i. The fluctuation-dissipation theorem may then be applied to determine the fluctuations of the quantity corresponding to the I_i. The fluctuations of the I_i may also be thought of as arising from fluctuations of stochastic forces V_i. The fluctuation-dissipation theorem may then be alternatively written as

$$\left\langle \left| V_i(\omega)\, V_j(\omega) \right| \right\rangle d\omega = \frac{\kappa T}{2\pi}\, (Y_{ij}^{-1*} + Y_{ji}^{-1})\, d\omega \tag{9}$$

A simple example is to find the voltage fluctuations across a resistor R at temperature T. The system is one dimensional, so the admittance tensor reduces to one element of value 1/R, and the fluctuation-dissipation theorem gives the power spectrum of the voltage fluctuations

$$\left\langle \left| V(\omega) \right|^2 \right\rangle d\omega = \frac{\kappa T}{\pi}\, R\, d\omega \tag{10}$$

Note that since we allow ω to take on both positive and negative values, this expression is 1/2 times the standard form of Nyquist's theorem.

EFFECT OF COLLISIONS ON THOMSON SCATTERING WITH UNEQUAL ELECTRON AND ION TEMPERATURES (B = 0)

The first case that is examined is the scattering of electromagnetic waves from electron density fluctuations in a weakly ionized plasma with no magnetic field. Dougherty and Farley (ref. 5) have shown that for equal electron and ion temperatures, collisions do not affect the total backscattered power. Here it is shown that when the electrons and ions are not at the same temperature, the total backscattered power is dependent on the ion-neutral and electron-neutral collision frequencies.

Farley (ref. 6) has calculated the scattering cross section for unequal electron and ion temperatures, but only for the collisionless case. Here it is found that for $T_e/T_i \geq 10$, the total scattering cross section is substantially less than that of the collisionless calculation, but always greater than a cross section that varies with T_e/T_i as $[1 + (T_e/T_i)]^{-1}$. As the collision frequencies ν_{en} and ν_{in} increase, the total scattered power approaches the $(1 + T_e/T_i)^{-1}$ curve, although it does not reach it even in the limit ν_{en}, $\nu_{in} \to \infty$. Also, for $T_e/T_i \geq 5$, the ratio of the collision frequencies ν_{en}/ν_{in}, with ν_{in} fixed, has a large effect on the total scattered power, although the shape of the spectrum is essentially unchanged.

Evaluation of Backscattering Cross Section With Unequal Electron and Ion Temperatures

The backscattering cross section, from equation (4), is

$$\sigma_b(\omega_o - \omega)\, d\omega = L^3 r_e^2 \left\langle \left|\Delta N(2\underset{\sim}{k}_o, \omega)\right|^2 \right\rangle d\omega \tag{11}$$

so the scattering may be thought of as being an interaction between the incident wave, which is taken to be proportional to $e^{i(\omega_o t - k_o z)}$, and the spatial Fourier component of the fluctuating part of the electron number density ΔN whose wave vector is $2\underset{\sim}{k}_o$. (In the remainder of this report only backscattering is treated; however, the theory may easily be extended to other scattering angles by using equation (4) in place of equation (11) with $\underset{\sim}{k} = \underset{\sim}{k}_s - \underset{\sim}{k}_o$. The magnitude of $\underset{\sim}{k}$ is then given by $|\underset{\sim}{k}| = 2k_o \sin \theta/2$ where θ is the angle between $\underset{\sim}{k}_o$ and $\underset{\sim}{k}_s$.)

To calculate the thermal fluctuations of the electrons, a weakly ionized plasma is considered in which the electrons and ions each have Maxwellian distributions but not necessarily at the same temperature. It is assumed that the temperature of the neutrals and ions are equal. Because of the large difference between the electron and neutral masses, the electrons tend to exchange energy slowly with the heavy particles so that each species may achieve a Maxwellian distribution, but at different temperatures. In order that this temperature difference may be sustained under steady-state conditions, it is hypothesized that some external energy source, such as a constant electric field, is acting on the plasma.

The model used for the plasma is that of two fluids of charged particles, the electrons and ions, each of which is coupled to the neutral particles by collisions and which interact because of the self-consistent electric field. Short range electron-ion collisions are neglected.

The fluctuations of the electrons and ions are calculated separately by applying the fluctuation-dissipation theorem to fictitious sets of particles that are identical to the real electrons and ions except that no forces due to the electric field are included. The actual fluctuations on the electron number density are then found by using Maxwell's equations to include the interaction between the electrons and ions due to the self-consistent electric field.

In order to calculate the response function of each fictitious fluid, a longitudinal oscillating force $\underset{\sim}{F}_s$ is applied to the s species (s takes on the value e for electrons and i for ions). The rate of change of energy of the s species in a volume L^3 is

$$\frac{dW}{dt} = L^3 \underset{\sim}{F}_s \cdot \underset{\sim}{\Gamma}_s \tag{12}$$

where $\underset{\sim}{\Gamma}_s = N_s \underset{\sim}{u}_s$ is the flux density of the s species. With no applied magnetic field, the response $\underset{\sim}{\Gamma}_s$ is in the same direction as the applied force $\underset{\sim}{F}_s$, so the vector notation can be dropped. A scalar admittance function Y_s is introduced that relates the response to the applied force:

$$\Gamma_s = Y_s F_s \tag{13}$$

(Application of the fluctuation-dissipation theorem to a system with a tensor response function is treated in appendix B.) These admittance functions will be obtained from the equations of motion of each set of fictitious particles.

To apply the fluctuation-dissipation theorem, identify F_s with V and then, comparing equations (6) and (12), note that the response I must be identified with $L^3\Gamma_s$ and Y with $Y_s L^3$. The fluctuation-dissipation theorem (eqs. (7) and (9)) may thus be written in either of the forms

$$\left\langle |\Gamma_s|^2 \right\rangle d\omega = \frac{\kappa T_s}{L^3 \pi} \operatorname{Re} Y_s \, d\omega \tag{14}$$

or

$$\left\langle |F_s|^2 \right\rangle d\omega = \frac{\kappa T_s}{L^3 \pi} \operatorname{Re} \frac{1}{Y_s} \, d\omega \tag{15}$$

The quantities F_e and F_i may be considered to be stochastic forces that act on the electrons and ions, respectively.

The self-consistent electric field, given by Maxwell's equations, will be an additional force that acts on the charged particles. The Maxwell equation

$$\nabla \times \underset{\sim}{H} = \frac{\partial \underset{\sim}{D}}{\partial t} + \underset{\sim}{J} \tag{16}$$

is used. Here $\nabla \times \underset{\sim}{H} = 0$, since only longitudinal oscillations cause fluctuations in the electron number density.

Equation (16) may be expressed in terms of the flux densities as

$$\Gamma_e - \Gamma_i = Y_I(eE) \tag{17}$$

where

$$Y_I = i\omega \frac{\epsilon_o}{e^2} \tag{18}$$

With the inclusion of the force due to the electric field, the equations for the electron and ions become

$$\begin{aligned} \Gamma_e &= Y_e(F_e - eE) \\ \Gamma_i &= Y_i(F_i + eE) \end{aligned} \tag{19}$$

These two equations together with equation (17) form a closed set of equations for the electron and ion flux densities that can be solved once we know F_e, F_i, Y_e, and Y_i. Solving equations (17) and (19) for Γ_e gives

$$\Gamma_e = \frac{F_e Y_e (Y_i + Y_I) + F_i Y_e Y_i}{Y_e + Y_i + Y_I} \tag{20}$$

Assuming that the stochastic forces on the electrons and ions are independent, that is, that $\left\langle |F_e(\omega) \; F_i(\omega)| \right\rangle = 0$, the power spectrum of Γ_e is

$$\left\langle |\Gamma_e(\omega)|^2 \right\rangle d\omega = \left| \frac{Y_e(Y_i + Y_I)}{Y_e + Y_i + Y_I} \right|^2 \left(\left\langle |F_e(\omega)|^2 \right\rangle + \left| \frac{Y_i}{Y_i + Y_I} \right|^2 \left\langle |F_i(\omega)|^2 \right\rangle \right) d\omega \tag{21}$$

Using equation (15) for $\left\langle |F_e|^2 \right\rangle$ and $\left\langle |F_i|^2 \right\rangle$ and the continuity equation for the electrons

$$\frac{\partial \Delta N}{\partial t} + \nabla \cdot \underset{\sim}{\Gamma}_e = 0 \tag{22}$$

yields the power spectrum of the electron number density fluctuations

$$\left\langle |\Delta N(k,\omega)|^2 \right\rangle d\omega = \frac{k^2}{\omega^2} \frac{\kappa T_e}{\pi L^3} \left| \frac{Y_e(Y_i + Y_I)}{Y_e + Y_i + Y_I} \right|^2 \operatorname{Re}\left(\frac{Y_e}{|Y_e|^2} + \frac{T_i}{T_e} \frac{Y_i}{|Y_i + Y_I|^2} \right) d\omega \tag{23}$$

To this point the analysis is essentially that of Farley (ref. 6) who then calculated the admittance functions for a collisionless plasma. Here Farley's work is extended by calculating the Y_e and Y_i when collisions are important. For this purpose the approach of Tanenbaum (ref. 16) is followed and use is made of the transport equations for mass, momentum, and energy with the heat flow term and pressure tensor as derived by Goldman and Sirovich (ref. 22). Applying these equations to small amplitude disturbances in a weakly ionized gas with neutral velocity $u_n = 0$ gives (to first order in the fluctuating quantities)

$$\frac{\partial \rho_s}{\partial t} + \rho_{so} \nabla \cdot \underset{\sim}{u}_s = 0 \tag{24a}$$

$$\frac{\partial \underset{\sim}{u}_s}{\partial t} + \frac{\nabla p_s}{\rho_{so}} - \frac{\underset{\sim}{F}_s}{m_s} - \left[\nabla^2 \underset{\sim}{u}_s + \frac{1}{3} \nabla (\nabla \cdot \underset{\sim}{u}_s) \right] \frac{\eta_{os}}{\rho_{so}} = -\nu_{sn} \underset{\sim}{u}_s \tag{24b}$$

$$\frac{\partial p_s}{\partial t} - \frac{5}{3} \frac{p_{so}}{\rho_{so}} \frac{\partial \rho_s}{\partial t} - \frac{2}{3} \lambda_{os} \nabla^2 T_s = -N_o \kappa \nu'_{sn} (T_s - T_{so}) \tag{24c}$$

The energy equations imply that there is some external force that causes the electron and ion temperatures to be different under steady-state conditions. The temperatures of the ions and neutrals are assumed to be equal. Note that s takes values e (for electrons) and i (for ions) and that n denotes neutrals. Also, ρ_s, q_s, m_s, p_s, and T_s are the mass density, charge, mass, pressure, and temperature, respectively, for species s;

$\underset{\sim}{F}_s$ is the force whose response we are calculating; ν_{sn} and $\nu'_{sn} = 2m_s\,\nu_{sn}/(m_s + m_n)$ are, respectively, the effective collision frequencies for momentum and energy transfer with the neutrals; and η_{os} and λ_{os}, the viscosity and thermal conductivity for charged particles in a weakly ionized gas, are given by

$$\eta_{os} = \frac{p_s}{D_s \nu_{sn}} \tag{25a}$$

$$\lambda_{os} = \frac{15\,\kappa p_s}{4 m_s C_s \nu_{sn}} \tag{25b}$$

where D_s and C_s are constants (of order 1 or 2) that depend on the interparticle force law and the mass ratio between the electrons or ions and neutrals.

It is assumed that $\rho_s = \rho_{so} + \rho'_s$, $p_s = p_{so} + p'_s$, and $T_s = T_{so} + T'_s$ with the fluctuation quantities ρ'_s, p'_s, and T'_s proportional to $e^{i(\omega t - kz)}$. Since longitudinal disturbances have been specified, $\underset{\sim}{F}_s$ and $\underset{\sim}{u}_s$ are in the z direction.

Using the mass and energy equations allows the momentum transport equations to be written in the following form:

$$N_o u_s = \frac{i\omega N_o}{k^2 \kappa T_s} \frac{F_s}{z_s} \tag{26}$$

where

$$z_s = \Delta_s + 2i\theta_s[\psi_s + 2(3D_s\psi_s)^{-1}] - 2\theta_s^2 \tag{27}$$

$$\Delta_s = \frac{1 + i\left(\dfrac{5\omega}{3\sigma_s}\right)}{1 + i\left(\dfrac{\omega}{\sigma_s}\right)} \tag{28a}$$

$$\sigma_s = \nu'_{sn} + \frac{5k^2\kappa T_s}{2C_s \nu_{sn} m_s} \tag{28b}$$

and θ_s and ψ_s are the normalized Doppler shift frequencies and collision frequencies

$$\theta_s = \frac{\omega}{k}\left(\frac{m_s}{2\kappa T_s}\right)^{1/2} \tag{29a}$$

$$\psi_s = \frac{\nu_{sn}}{k}\left(\frac{m_s}{2\kappa T_s}\right)^{1/2} \tag{29b}$$

Comparison of equations (26) and (13) shows that the admittance functions for the electrons and ions considered separately are

$$Y_e = \frac{i\omega N_o}{k^2 \kappa T_e}\,\frac{1}{z_e}$$

and

$$Y_i = \frac{i\omega N_o}{k^2 \kappa T_i}\,\frac{1}{z_i} \tag{30}$$

Thus the power spectrum of the electron density fluctuations (eq. (23)) may be expressed in terms of the macroscopic parameters of the plasma and the differential backscattering cross section (eq. (11)) may be written as

$$\sigma_b(\omega_o - \omega)\,d\omega = N_o r_e^2 \left|\frac{\mu\alpha^2 + z_i}{\alpha^2(z_i + \mu z_e) + z_e z_i}\right|^2 \operatorname{Im}\left[z_e + \frac{\alpha^2}{\mu}\,\frac{z_i}{\mu\alpha^2 + z_i}\right]\frac{d\omega}{\pi\omega} \tag{31}$$

where $\mu = T_e/T_i$ and $\alpha = (k\lambda_D)^{-1}$, with the electron Debye length, λ_D, equal to $(\kappa T_e \epsilon_o / N_o e^2)^{1/2}$. This is essentially the same expression, with slightly different notation as Farley's (ref. 6). However, the z_e and z_i used here have been derived from continuum equations rather than from the kinetic equations Farley used.

Examination of equations (27) and (29a) shows that $z_s^*(\omega) = z_s(-\omega)$. Any function $F(z_e, z_i)$ with constant real coefficients also has this property, that is, $F^*(\omega) = F(-\omega)$. It follows that $|F(\omega)|^2$ is an even function of ω and that $\operatorname{Im}F(\omega)$ is an odd function of ω. Hence the differential scattering cross section (eq. (31)) is an even function of ω.

For the case of equal electron and ion temperatures, the cross section simplifies to

$$\sigma_b(\omega_o - \omega)\, d\omega = -N_o r_e^2 \,\mathrm{Im}\left[\frac{\alpha^2 + z_i}{\alpha^2(z_i + z_e) + z_e z_i}\right]\frac{d\omega}{\pi\omega} \tag{32}$$

Provided that the ion-neutral mean free path is less than the incident wavelength, equation (32) gives results (Tanenbaum, ref. 16) that agree very well with those obtained by Dougherty and Farley (ref. 5) using the Boltzmann equation with the Bhatnagar-Gross-Krook (BGK) model for collisions with neutrals.

Numerical Results

The normalized differential cross section, as obtained from equation (31), is evaluated for four values of (T_e/T_i) in figure 2 with $\alpha = 12.7$ and $m_i = 31$ amu (this value of m_i is chosen to represent a mixture of NO^+ and O_2^+ ions). Note that a typical Thomson scatter spectrum (ref. 3) has a peak at $\omega = kU_p$ (where U_p is the plasma sound speed)

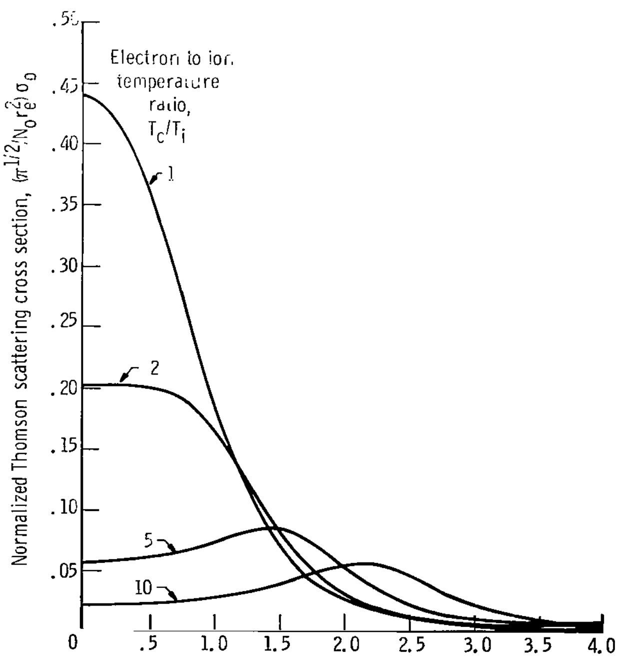


Figure 2. - Thomson scattering cross section as function of Doppler shift frequency. α = 12.7; normalized ion-neutral collision frequency, ψ_i, 1; normalized electron-neutral collision frequency, ψ_e, 0.1; ion mass, m_i, 31 atomic mass units; $C_i = D_i = 2$; $C_e = D_e = 1$.

which is due to scattering from the so-called ion acoustic waves in the plasma. In addition, the peak is sharp when there is little damping of the ion acoustic waves, but broadens and eventually disappears when the damping increases. Increasing the electron to ion temperature ratio causes the plasma sound speed to increase and the damping of ion acoustic waves due to collisions to decrease. This broadens the spectrum and increases the relative amplitude of the resonance, as can be seen in figure 2. The shape of the spectrum is essentially unaffected by a change in ψ_e/ψ_i although the total power is changed. This is shown in figure 3 where the spectrum (normalized so that $\sigma_b(\omega_o) = 1$) is plotted for two values of ψ_e/ψ_i.

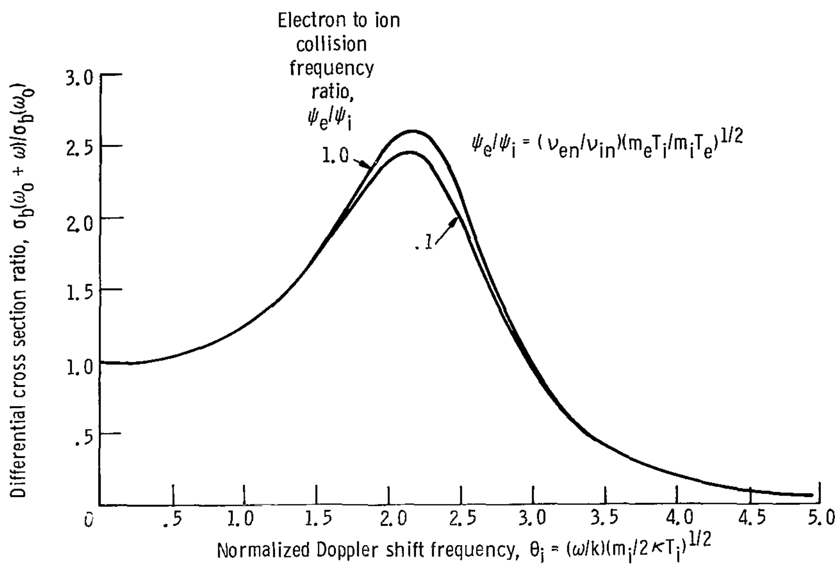


Figure 3. - Comparison of spectral shape for two values of normalized electron to ion collision frequency ratio. $\alpha = 12.7$; normalized ion-neutral collision frequency, ψ_i, 1; electron to ion temperature ratio, T_e/T_i, 10; ion mass, m_i, 31 atomic mass units; $C_i = D_i = 2$; $C_e = D_e = 1$.

Discussion of Total Power

The total backscattered power is given by

$$\sigma_{tot} = \int_{-\infty}^{\infty} \sigma_b(\omega_o - \omega)\, d\omega \tag{34}$$

Although this integral may easily be calculated numerically (see fig. 5), an approximate analytic form can be derived that is valid for $\alpha >> 1$, $\mu \leq 10$, and $\psi_i \geq 1$. Equation (34) may be written using equation (31) as

$$\frac{\sigma_{tot}}{N_o r_e^2} = -\mathrm{Im} \int_{-\infty}^{\infty} \frac{\mu\alpha^2 + z_i}{\alpha^2(z_i + \mu z_e) + z_e z_i} \frac{d\omega}{\pi\omega} - \mathrm{Im} \int_{-\infty}^{\infty} \frac{\alpha^4(\mu - 1)(z_i + \mu\beta')}{|\alpha^2 + z_e|^2 |z_i + \mu\beta'|^2} \frac{d\omega}{\pi\omega}$$

$$+ \mathrm{Im} \int_{-\infty}^{\infty} \frac{\alpha^4(\mu - 1)\mu\beta'}{|\alpha^2 + z_e|^2 |z_i + \mu\beta'|^2} \frac{d\omega}{\pi\omega} \tag{35a}$$

where

$$\beta' = \alpha^2 \frac{z_e}{(\alpha^2 + z_e)} \tag{35b}$$

The first integral in equation (35) is easily evaluated using residue theory to find the Cauchy principal value, which is just given by $-\pi i \times$ (Residue at $\omega = 0$) since there are no poles in the lower half plane. Since $z_e \approx 1$, $|\alpha^2 + z_e|^2$ may be approximated by $(\alpha^2 + 1)^2$, and the second integral can be evaluated in the same manner as the first. Thus equation (35a) becomes

$$\frac{\sigma_{tot}}{N_o r_e^2} = \frac{1}{1 + \alpha^2} + \frac{\alpha^4}{(1 + \alpha^2)[1 + \alpha^2(1 + \mu)]} + \frac{(\mu - 1)\alpha^4}{(1 + \alpha^2)^2} \int_{-\infty}^{\infty} \frac{\mathrm{Im}(\mu\beta')}{|z_i + \mu\beta'|^2} \frac{d\omega}{\pi\omega} \tag{36}$$

If $z_e = 1$, the remaining integral gives zero contribution, and the total cross section is given by the first two terms in equation (36). This is equivalent to Farley's approximate solution (ref. 6) and gives the identical result. This expression is also given by Salpeter (ref. 12) and Buneman (ref. 14). Note that there is no dependence on the collision frequencies in this approximation.

The case of the limit $\alpha \to \infty$ (Debye length $\to 0$) may be easily evaluated if it is assumed that Δ_s may be approximated by its low-frequency limit

$$\Delta_s \approx 1 + \frac{2i\omega}{3\sigma_s} \tag{37}$$

Equations (28) show that the approximation is certainly valid for Δ_e and should be fairly good for Δ_i when $\psi_i \geq 1$, provided μ is not too large, since the main contribution to the total power then comes from the region $\theta_i \leq 1$ (see fig. 2). Using equation (37) allows the integral in equation (36) to be evaluated

$$\frac{\sigma_{tot}}{N_o r_e^2} = \frac{1}{1+\mu} + \left(\frac{\mu - 1}{\mu + 1}\right)\frac{1}{1 + Q} \tag{38}$$

where

$$Q = \gamma \mu^{1/2}\left(\frac{\psi_e}{\psi_i}\right)\left(\frac{K_e}{K_i}\right) \tag{39}$$

with

$$K_i = 1 + \frac{2}{3D_i\psi_i^2} + \left(3\psi_i^2 + \frac{15}{4C_i}\right)^{-1} \tag{40a}$$

$$K_e = 1 + \frac{2}{3D_e\psi_e^2} + \left(6\gamma^2\psi_e^2 + \frac{15}{4C_e}\right)^{-1} \tag{40b}$$

and

$$\gamma = \left(\frac{m_e}{m_i}\right)^{1/2} \tag{40c}$$

For small Debye length ($\alpha \to \infty$), equation (38) shows that the total backscattered power is generally less than that calculated from the collisionless theory, but it is always greater than $(1 + \mu)^{-1}$ and is dependent on both the electron and ion collision frequencies. This approximate solution for $\psi_i = 1$ is shown in figure 4 along with $(1 + \mu)^{-1}$, the numerical integration of equation (31) for $\alpha >> 1$, and the large T_e/T_i approximate solution of Farley (ref. 6) for the collisionless case. For $\psi_i > 1$ the approximate solution should be even closer to the exact solution.

Figure 5 shows the total backscattering cross section (obtained by numerical integration) for three values of ψ_i and two values of ψ_e/ψ_i. For large values of μ the total backscattered power is less than that found from the collisionless theory, and it more closely follows the $(1 + \mu)^{-1}$ curve which is commonly used by experimentalists. The total power approaches the $(1 + \mu)^{-1}$ curve closely as ψ_i becomes larger, although it does not equal it in the limit as ψ_e and $\psi_i \to \infty$.

It should be pointed out that ψ_e and ψ_i are independent of temperature only when the interparticle force is that of hard spheres (see ref. 23, p. 251). In general, this is

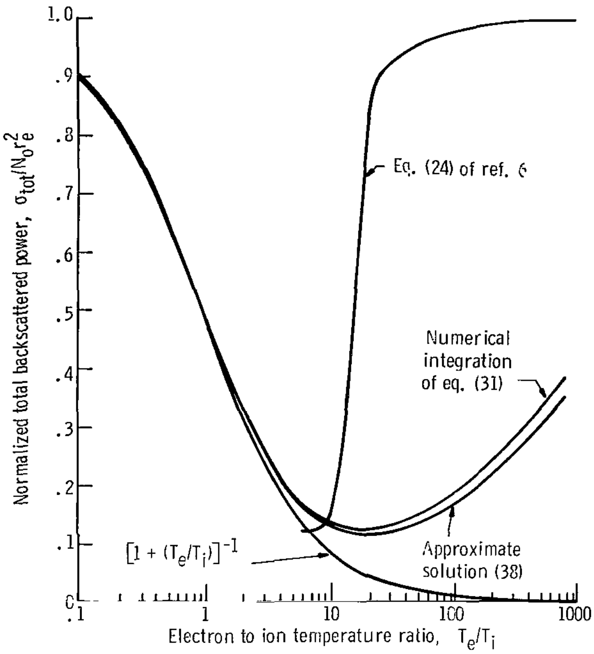


Figure 4. - Comparison of approximate solution (eq. (38)) and the true value obtained from the numerical integration of equation (31) for the total backscattered power. (Also shown are the approximate solution of Farley (ref. 6) for the collisionless case and the $(1 + T_e/T_i)^{-1}$ curve.) $\alpha = \infty$; normalized ion-neutral collision frequency, ψ_i, 1; normalized electron-neutral collision frequency, ψ_e, 0.1; ion mass, m_i, 31 atomic mass units; $C_i = D_i = 2$; $C_e = D_e = 1$.

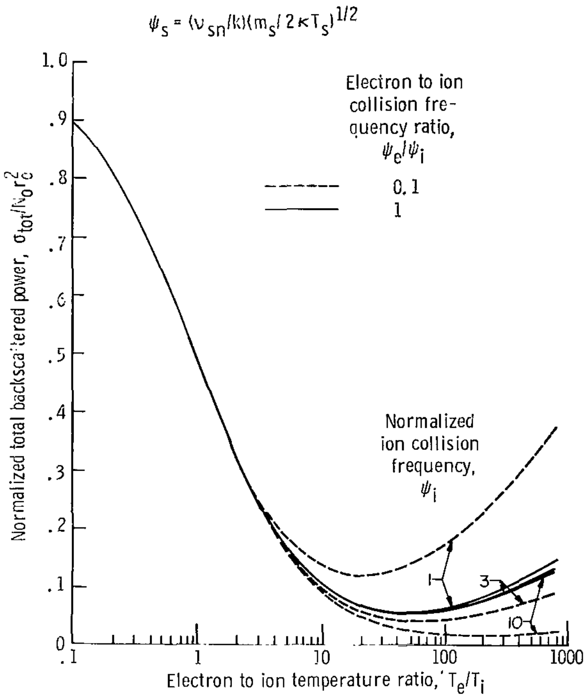


Figure 5. - Total backscattered power as function of electron to ion temperature ratio from numerical integration of equation (31). $\alpha = 12.7$; ion mass, m_i, 31 atomic mass units; $C_i = D_i = 2$; $C_e = D_e = 1$.

not the case, and an additional temperature dependence would be introduced if the proper interaction were known and taken into account. However, this additional temperature dependence is not important to the calculations presented in this report.

EFFECT OF COLLISIONS ON THOMSON SCATTERING IN A MAGNETIC FIELD

In this section the theory of Thomson scattering is extended to include the effects of electron-neutral and ion-neutral collisions on the signal scattered from a plasma in a uniform constant magnetic field.

It has been shown (refs. 4 and 10) that a magnetic field appreciably influences the Thomson backscattered signal only when the field is normal or nearly normal to the direction of propagation of the probing wave. Farley (ref. 6) has shown that for a collisionless plasma the total backscattered power is markedly altered by a magnetic field when the electron temperature T_e is not equal to the ion temperature T_i. Since, for the case of no magnetic field, the scattered power has been shown to be dependent on the electron-neutral and ion-neutral collision frequencies for $T_e \neq T_i$, it is not surprising that in the calculations presented here the total backscattered power depends on the electron and ion temperatures, the frequency of collisions, and also the magnitude and relative angle of the magnetic field.

In the experimentally important range of T_e/T_i between 1 and 10, the total backscattered power is increased by the magnetic field - an effect also shown in Farley's collisionless calculations. However, increasing the collision frequencies is shown to reduce the influence of the magnetic field and thus causes the scattered power to be closer to the field free case. It should be noted that when magnetic-field effects are important, the total scattering cross section is not proportional to $(1 + T_e/T_i)^{-1}$ as it is (to a good approximation for small T_e/T_i) in the field-free case.

Evaluation of Backscattering Cross Section With Magnetic Field

Providing that the incident frequency is much larger than the electron plasma, collision, and cyclotron frequencies, the differential backscattering cross section can then be written as (derived in appendix B)

$$\sigma_b(\omega_o - \omega)\, d\omega = \frac{N_o r_e^2}{2\pi i \omega} \left[\underset{\approx}{Z}^{-1} \cdot \left(\underset{\approx}{B} - \underset{\approx}{B}^{\dagger}\right) \cdot \underset{\approx}{Z}^{-1\dagger}\right]_{zz} d\omega \tag{B14}$$

The coordinate system is chosen so that the incident wave vector $\underset{\sim}{k}_o$ is alined with the z axis and the magnetic field $\underset{\sim}{B}_o$ lies in the x-z plane and forms an angle β with $\underset{\sim}{k}_o$. The symbol $\dagger$ denotes the Hermetian conjugate, and $[\ ]_{zz}$ means that the expression is the zz component of the tensor. The tensors $\underset{\approx}{Z}$ and $\underset{\approx}{B}$ are defined by

$$\underset{\approx}{Z} = \underset{\approx}{Z}_e + \left(\mu \underset{\approx}{Z}_i^{-1} + \underset{\approx}{Z}_I^{-1}\right)^{-1} \tag{B15}$$

$$\underset{\approx}{B} = \underset{\approx}{Z} - \frac{\mu - 1}{\mu} \left(\mu \underset{\approx}{Z}_i^{-1} + \underset{\approx}{Z}_I^{-1}\right)^{-1} \tag{B16}$$

where $\mu = T_e/T_i$, $\underset{\approx}{Z}_e$ and $\underset{\approx}{Z}_i$ are normalized impedance tensors that characterize the response of the electrons and ions, respectively, to an applied force proportional to $e^{i\omega t - ikz}$, and $\underset{\approx}{Z}_I^{-1}$ is a tensor (closely related to the usual conductivity tensor) given by

$$\underset{\approx}{Z}_I^{-1} = \frac{1}{\alpha^2}\begin{bmatrix} 1 - n^2 & 0 & 0 \\ 0 & 1 - n^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{41}$$

where $n = kc/\omega$. Note that, for equal electron and ion temperatures, $\underset{\approx}{B} = \underset{\approx}{Z}$ and the cross section (eq. (B14) is then just proportional to the imaginary part of the zz component of $\underset{\approx}{Z}^{-1}$.

It is only necessary to calculate $\underset{\approx}{Z}_e$ and $\underset{\approx}{Z}_i$ (evaluated at $k = 2k_o$) to find the scattering cross section given by equation (B14). Other authors (refs. 4, 10, and 13) have used kinetic theory to calculate Thomson scattering when magnetic field effects are important. In appendix C the continuum equations for a weakly ionized plasma in a constant magnetic field are used to calculate $\underset{\approx}{Z}_e$ and $\underset{\approx}{Z}_i$ for the case where electron-neutral and ion-neutral collisions are important.

An approximation used by other authors (refs. 4 and 10) may be used to simplify the amount of matrix algebra needed to evaluate the cross section as given by equation (B14). It is to assume that the phase velocity of any wave which causes scattering is much less than the velocity of light c. To use the approximation, the limit is taken as c approaches infinity in the evaluation of equation (B14). The result of this is to decouple the transverse and longitudinal waves. (The longitudinal oscillations are the only ones that are important here since the scattering is caused by the fluctuations in the electron number density.) Applying this approximation leads to the following form of the backscattering cross section

$$\sigma_b(\omega_o - \omega)\, d\omega = N_o r_e^2 \left|(\underset{\approx}{Z}^{-1})_{zz}\right|^2 \operatorname{Im}\left\{\frac{1}{(\underset{\approx}{Z}^{-1})_{zz}} - \frac{\mu - 1}{\mu}\left(\mu\underset{\approx}{Z}_i^{-1} + \underset{\approx}{Z}_I^{-1}\right)_{zz}^{-1}\right\}\frac{d\omega}{\pi\omega} \tag{42}$$

where

$$\left(\mu\underset{\approx}{Z}_i^{-1} + \underset{\approx}{Z}_I^{-1}\right)_{zz}^{-1} = \frac{\alpha^2}{1 + \frac{\mu\alpha^2}{\operatorname{Det}(\underset{\approx}{Z}_i)}\left(Z_{xx}^i Z_{yy}^i - Z_{xy}^i Z_{yx}^i\right)} \tag{43}$$

$$(\underset{\approx}{Z}^{-1})_{zz} = \frac{Z_{xx}^e Z_{yy}^e - Z_{xy}^e Z_{yx}^e}{\operatorname{Det}(\underset{\approx}{Z})} \tag{44}$$

and the elements of $\underset{\approx}{Z}$ are

$$Z_{ij} = Z_{ij}^{e} + \left(\mu \underset{\approx}{Z}_{i}^{-1} + \underset{\approx}{Z}_{I}^{-1}\right)_{zz}^{-1} \delta_{iz}\delta_{jz} \tag{45}$$

This is equivalent to the cross section used by Farley (ref. 6) for his kinetic theory calculations.

Cross Section for $\beta = 90^\circ$

As discussed previously, the magnetic field affects the backscattering cross section only when the incident wave is normal or nearly normal to the magnetic field. In this section the special case of $\beta = 90^{\circ}$ is examined because at this angle the effect of the magnetic field is maximized. For other angles the full equations presented in the previous section can be used if an evaluation of the cross section is desired.

For $\beta = 90^{\circ}$, the tensors $\underset{\approx}{Z}_s$ defined by equation (C15) simplify to

$$\underset{\approx}{Z}_s = \begin{bmatrix} x_s & 0 & 0 \\ 0 & y_s & \pm iA_s \\ 0 & \mp iA_s & z_s \end{bmatrix} \tag{46}$$

where the upper sign is to be taken for the electrons and the lower sign for the ions. The quantities x_s, y_s, z_s, and A_s are now defined as

$$x_s = 2i\theta_s \left[\psi_s + \frac{1}{2D_s\psi_s} \frac{1}{1 + \left(\frac{\phi_s}{\psi_s}\right)^2}\right] - 2\theta_s^2 \tag{47}$$

$$y_s = 2i\theta_s \left[\psi_s + \frac{1}{2D_s\psi_s} \frac{1}{1 + 4\left(\frac{\phi_s}{\psi_s}\right)^2}\right] - 2\theta_s^2 \tag{48}$$

$$z_s = \Delta_s + 2i\theta_s \left[\psi_s + \frac{2}{3D_s\psi_s} \frac{1 + \left(\frac{\phi_s}{\psi_s}\right)^2}{1 + 4\left(\frac{\phi_s}{\psi_s}\right)^2} \right] - 2\theta_s^2 \tag{49}$$

$$A_s = 2\phi_s\theta_s \left[1 - \frac{1}{D_s\psi_s^2} \frac{1}{1 + 4\left(\frac{\phi_s}{\psi_s}\right)^2} \right] \tag{50}$$

The thermal conductivity λ_s, given by equation (C14) becomes $\lambda_{os}/[1 + (\phi_s/\psi_s)^2]$ for this special case of $\beta = 90^\circ$. Note that the effect of a large magnetic field is to reduce the thermal conductivity in the direction normal to the field lines. This change in the definition of σ_s (and hence Δ_s) may be made by replacing the C_s, which appears in equation (28b) with $C_s[1 + (\phi_s/\psi_s)^2]$.

Equation (46) may now be used to find $\underset{\approx}{Z}_e$ and $\underset{\approx}{Z}_i$, and these in turn may be used to evaluate the exact expression for the backscattering cross section by using equations (B14) to (B16). As in the general case, a simpler expression for the cross section may be found by using the approximation that the velocity of light is large compared with the phase velocity of the plasma waves that cause the scattering. With this approximation, the backscattering cross section for $\beta = 90^\circ$ becomes

$$\sigma_b(\omega_o - \omega)\, d\omega = N_o r_e^2 \left| \frac{\mu\alpha^2 + \frac{z_i}{P_i}}{\alpha^2\left(\frac{z_i}{P_i} + \frac{\mu z_e}{P_e}\right) + \left(\frac{z_e}{P_e}\right)\left(\frac{z_i}{P_i}\right)} \right|^2 \operatorname{Im}\left\{ \frac{z_e}{P_e} + \frac{\alpha^2}{\mu} \frac{\frac{z_i}{P_i}}{\mu\alpha^2 + \frac{z_i}{P_i}} \right\} \frac{d\omega}{\pi\omega} \tag{51}$$

where

$$P_s = 1 + \frac{A_s^2}{y_s z_s - A_s^2} \tag{52}$$

and y_s, z_s, and A_s are given by equations (48), (49), and (50).

When the magnetic field approaches 0, $A_s = 0$ and $P_s = 1$, thus equation (51) reduces to equation (31) developed for the cross section with no magnetic field. Furthermore, for $\beta = 0^{\circ}$ ($\underset{\sim}{k} \parallel \underset{\sim}{B}_o$), $P_s = 1$ and hence the magnetic field does not change the scattering cross section when the magnetic-field lines are parallel to the incident wave vector.

Spectral Magnitude at Zero Doppler Shift for $\beta = 90^{\circ}$

In order to better understand the effect of the magnetic field on the Thomson scattering spectrum, the magnitude of the spectrum can be evaluated at zero Doppler shift for $\beta = 90^{\circ}$. The case considered is that where the cyclotron radius of the electrons is much smaller than both the electron-neutral mean free path and the wavelength of the incident wave λ_o but where the ion cyclotron radius is much larger than both the ion-neutral mean free path and λ_o. In terms of the normalized variables defined by equations (C6), (C7), and (C8) these conditions may be expressed by

$$\phi_e^2 >> \psi_e^2,\ 1$$

and

$$\phi_i^2 << \psi_i^2,\ 1$$

Furthermore, it is assumed that the electron and ion temperatures are equal and also that $\alpha^2 >> 1$. Then equation (51) is evaluated in the limit as $\omega \to 0$ to find

$$\sigma_b(\omega_o)\ d\omega = \frac{N_o r_e^2}{2\pi}\left[\frac{\phi_e \phi_i}{\psi_e}\left(1 + \frac{4C_e}{15}\right) + \psi_i + \frac{2}{3D_i\psi_i} + \frac{1}{3\left(\psi_i + \dfrac{5}{4C_i\psi_i}\right)}\right] d\theta_i \qquad (53)$$

Several observations can be made about the spectrum based on this expression. First, with $\psi_e/\psi_i \sim 0.1$ and $\psi_i \sim 1$, $\sigma_b(\omega_o)$ is primarily a function of ψ_e and ϕ_e (or ϕ_i since $\phi_e = (m_i/m_e)^{1/2}\ \phi_i$) if $\phi_e\phi_i \gtrsim 1$. Increasing the magnetic field strength causes $\sigma_b(\omega_o)$ to become larger, which implies a narrowing of the spectrum since the total cross section is independent of both collisions and the magnetic field for $T_e = T_i$. With zero magnetic field, increasing the collision frequencies causes the spectrum to be narrowed; however, here, where magnetic field effects are important, increasing the collision frequencies (up to the point where the ion collisions begin to dominate) causes the spectrum to be broadened. The physical explanation is that, as the electron-neutral collision frequency is increased, the electrons are somewhat freer to move across the field lines since their spiraling motion is broken up by the collisions; however, as the collision frequency is further increased, the collisions eventually dominate the effect of the field, and the spectrum again becomes sharply peaked (as when $B_o = 0$). Also the viscosity effects on the electron motion disappear for this case. The term involving the thermal conductivity remains, although the role of ψ_e is replaced by ϕ_e. Thus, the continuum equations for the electrons retain their validity even for $\psi_e < 1$, provided that $\phi_e^2 + \psi_e^2 > 1$. Finally, note that $\sigma_b(\omega_o)$ is a sensitive function of ψ_e/ψ_i. This differs from the field-free case (for $\alpha^2 >> 1$) where ψ_e/ψ_i has only a slight effect on the spectrum (refs. 5 and 16).

Numerical Evaluation of Cross Section

In figure 6 equation (51) has been numerically evaluated to show the influence of collisions on the spectrum for $\beta = 90^o$. The values chosen for the other parameters are typical of conditions found in the ionosphere at altitudes of about 100 kilometers if the in-

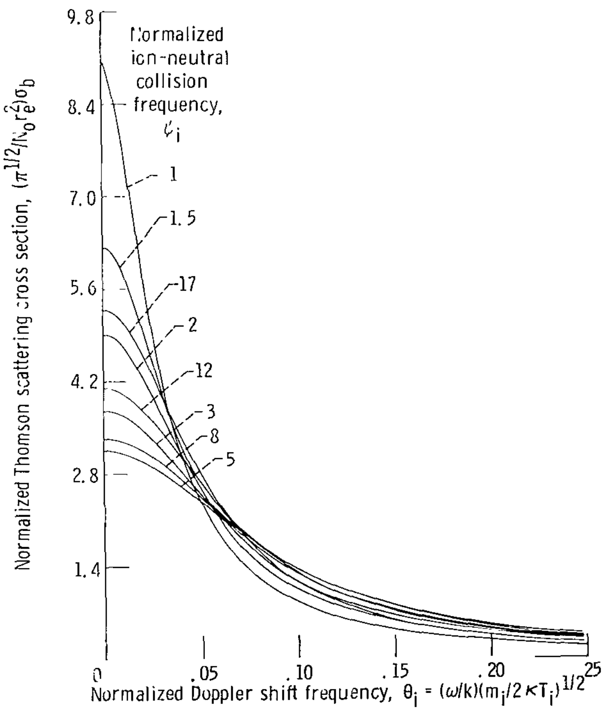
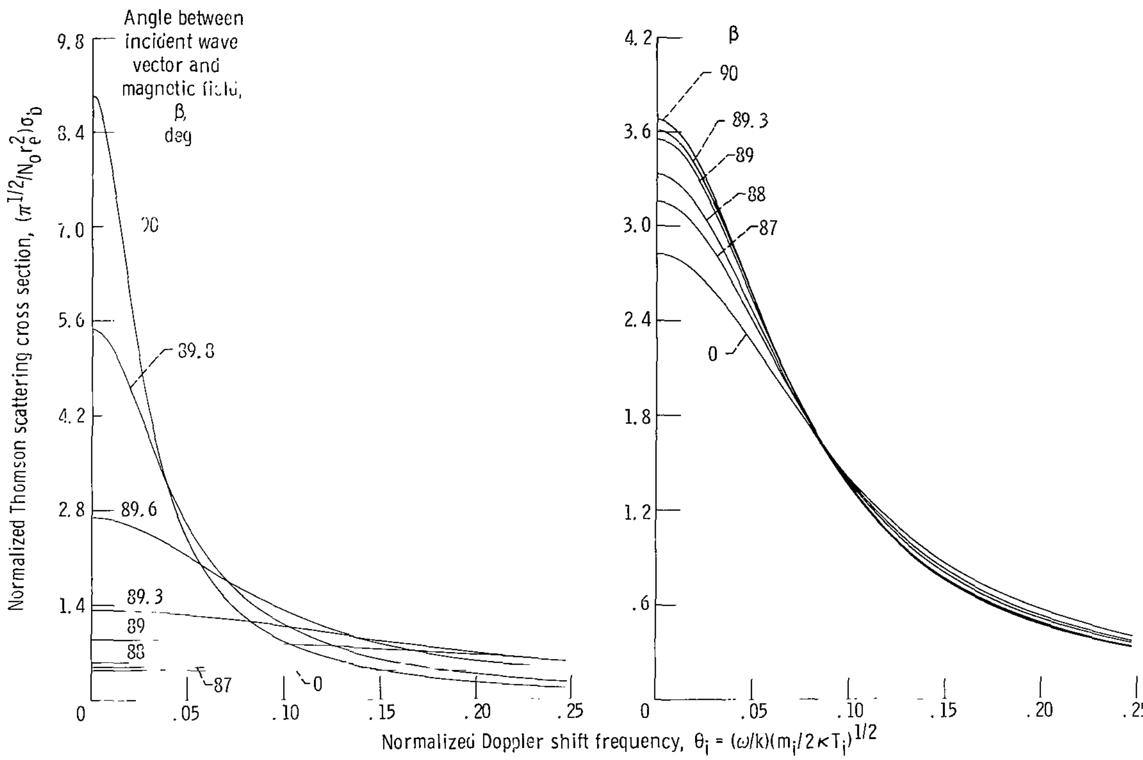


Figure 6. - Normalized Thomson scattering cross section as function of Doppler shift frequency for various values of normalized ion-neutral collision frequencies ($\psi_i = (\nu_{in}/k)(m_i/2\kappa T_i)^{1/2}$). Normalized ion cyclotron frequency $\varphi_i = (eB_o/km_i)(m_i/2\kappa T_i)^{1/2}$, 0.1; angle ($\beta$) between the incident wave vector and the magnetic field, 90°; α = 12.7; electron to ion temperature ratio, T_e/T_i, 1; normalized electron-neutral collision frequency, ψ_e, 0.1 ψ_i; ion mass, m_i, 31 atomic mass units; $C_i = D_i = 2$; $C_e = D_e = 1$.

cident wavelength λ_0 equals 6 meters. As discussed, we see that as ψ_i increases, the spectrum is broadened up to about $\psi_i = 5$. Then as ψ_i is increased further, the collisions begin to dominate the magnetic field, and the spectrum begins to peak about the origin.

The effect of the angle β on the spectrum is shown in figure 7 for two different collision frequencies. This shows that the magnetic field affects the spectrum only for β within a few degrees of 90° and that at the higher collision frequencies the relative change due to the field is less.



(a) Normalized ion-neutral collision frequency, ψ_i, 1.

(b) Normalized ion-neutral collision frequency, ψ_i, 10.

Figure 7. - Normalized Thomson scattering cross section as function of Doppler shift frequency for various values of β. Other parameters are the same as those in figure 6.

For equal electron and ion temperatures, neither collisions nor magnetic field affects the total cross section, which for this case is given by reference 4 as

$$\sigma_{tot} = N_o r_e^2 \frac{1 + \alpha^2}{1 + 2\alpha^2} \tag{54}$$

However, when the electron and ion temperatures are not equal, both collisions and magnetic field strength influence the total cross section. In figure 8 the effect of a magnetic field normal to the incident wave on the total cross section calculated by numerically integrating equation (51) is shown. This figure points out that in the relatively common experimental range when T_e/T_i is between 1 and 10, the total cross section increases with B_0. Also, for large B field (electron cyclotron radius on the order of, or smaller than, the wavelength of the incident wave), the total cross section may increase with T_e/T_i rather than falling off in accord with the usual $(1 + T_e/T_i)^{-1}$ relation. Farley (ref. 6) has published curves for the case of no collisions which show a similar effect of the magnetic field.

In figure 9 the effect of collisions on the total cross section in the presence of a magnetic field ($\phi_i = 0.1$) is shown. For T_e/T_i between 1 and 10, the effect of collisions is to drive the curve of total scattered power toward the field-free case.

The effect of the angle β on the total cross section is shown in figure 10. This again points out the fact that the magnetic field has an appreciable effect on the cross section only when the incident wave is nearly normal ($\beta > 80^{\circ}$) to the magnetic field for $T_e/T_i < 100$.

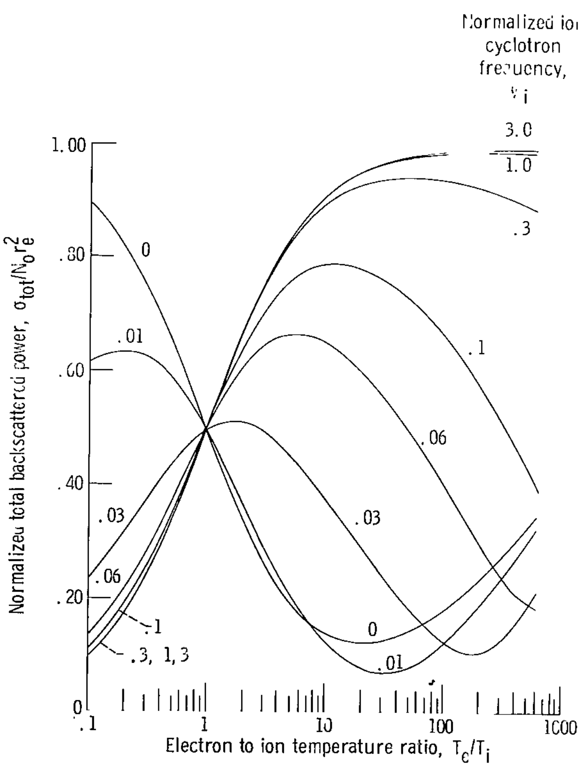


Figure 8. - Total backscattered power as function of electron to ion temperature ratio for eight values of normalized ion cyclotron frequency. Normalized ion-neutral collision frequency, ψ_i, 1; other parameters are the same as in figure 6.

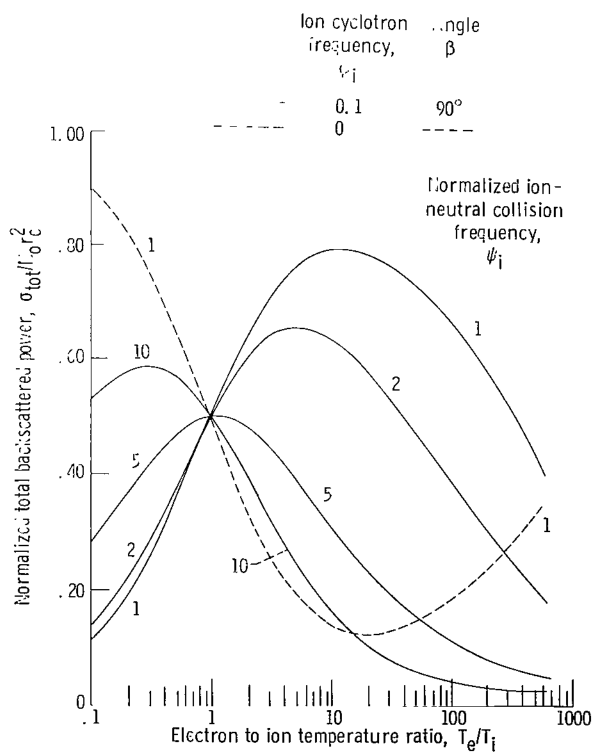


Figure 9. - Total backscattered power as function of electron to ion temperature ratio for several values of normalized ion-neutral collision frequency. Other parameters are the same as those in figure 6.

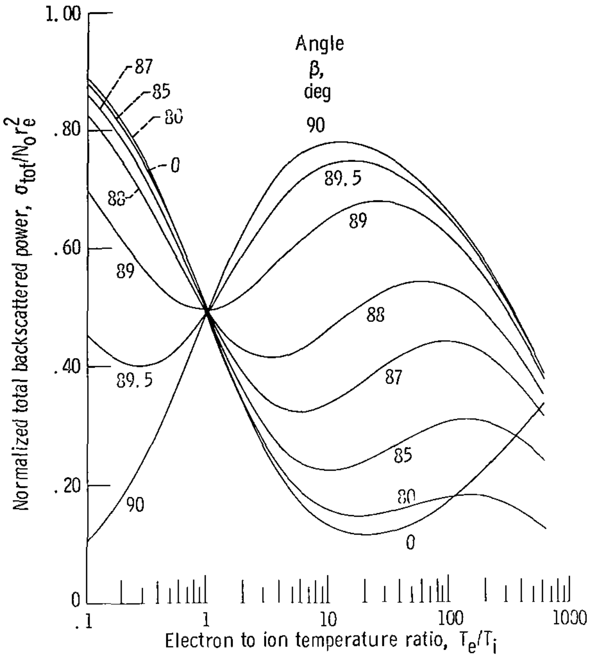


Figure 10. - Total backscattered power as function of electron to ion temperature ratio for various values of the angle between the incident wave and the magnetic field. Normalized ion cyclotron frequency, φ_i, 0.1; normalized ion collision frequency, ψ_i, 1. Other parameters are the same as those in figure 6.

SCATTERING FROM A PLASMA WITH ELECTRON DRIFT

Thomson scattering from a plasma in which the electrons have a net drift velocity relative to the ions has been discussed by Rosenbluth and Rostoker (ref. 13) for a collisionless plasma. They calculated the scattering for relative drift velocities below the critical value at which the instability of the ion acoustic wave occurs. They found that, as the drift velocity approaches the critical velocity at which the instability occurs, the spectrum develops a peak that becomes infinite as the critical velocity is reached. (The appearance of this infinity in the spectrum results from a breakdown of the linear theory.)

Here, the work of Rosenbluth and Rostoker is extended to include the effect of collisions, and backscattering is calculated (in the stable regime) using the linearized continuum equations and the fluctuation-dissipation theorem. It is assumed that the electrons and ions have Maxwellian velocity distributions and, for the sake of simplicity, that the ions have zero net velocity. Our model is thus a plasma in which the ions and neutral particles have no net fluid velocity and the electrons, which are assumed to have a Maxwellian velocity distribution, are drifting in the direction of the incident wave with a velocity below the critical velocity at which the two-stream instability occurs. Also, it is assumed that the thermal velocity of the electrons is much greater than the drift velocity. In general this means that, for drift velocities close to the critical velocity, the electron temperature must be greater than the ion temperature. This seems to be reasonable, since for many physical situations in which the electrons are drifting with respect to the ions (e.g., in a gas discharge device) the electrons will be hotter than the ions.

The critical drift velocity is dependent on the electron to ion temperature ratio, the collision frequencies, and the magnitude and relative angle of the magnetic field. In general, the critical velocity is found to be relatively independent of the electron temperature. With higher collision frequencies (which increase the net damping) larger drift velocities are required to produce instability. If a magnetic field is present, the critical drift velocity is smaller if the electrons are drifting perpendicular to the field.

When the wavelength is much smaller than the Debye length, ($\alpha \ll 1$), the only change in the spectrum of the scattered signal is the Doppler shift corresponding to the electron drift velocity. Furthermore, instabilities will not develop at these short wavelengths even for high electron drift velocities since there is essentially no coupling between the electrons and ions over distances much less than the Debye length.

When the wavelength is much larger than the Debye length ($\alpha \gg 1$), the differential backscattering cross section becomes unsymmetric about the frequency of the incident wave if the electrons have a net drift velocity. The spectrum develops a sharp peak which corresponds to scattering from the progressively more weakly damped ion acoustic waves as the drift velocity approaches its critical value.

The total backscattered power is obtained by numerical integration and is greatly enhanced as the drift velocity approaches the onset of instability.

Backscatter Cross Section With Electron Drift

The electrons and ions are both assumed to have Maxwellian velocity distributions, with the electrons having a net fluid velocity V_d in the direction of propagation of the incident wave. The power spectrum of the components of the stochastic force $\underset{\sim}{F}_s$, which may be thought of as driving the fluctuations, is given by equation (B3) in a reference frame in which the s species has zero net fluid velocity. Thus, in the laboratory reference frame, the spectrum of $\underset{\sim}{F}_s$ is Doppler shifted by a frequency kV_d and becomes

$$\left\langle \left| F_i^s(\omega)\ F_j^s(\omega) \right| \right\rangle d\omega = \frac{\kappa T_s}{2\pi L^3} \frac{k^2 \kappa T_s}{i(\omega - kV_d)} \left(Z_{ji}^s - Z_{ij}^{s\,*} \right) d\omega \tag{55}$$

where $\underset{\approx}{Z}_s$ is the impedance tensor as measured in the laboratory frame.

This shows that, for scattering from a plasma in which both species of particles have Maxwellian distributions and are drifting with the same velocity, the scattered signal is merely Doppler shifted by a frequency equal to kV_d. It will be shown that this is also the result for the case where the electrons are drifting through the plasma while the ions remain stationary if the wavelength is much smaller than the Debye length.

The fluid velocity of the electrons is now assumed to be the sum of the constant drift velocity plus a small part proportional to $e^{i\omega t - ikz}$; that is

$$\underset{\sim}{u}_e = \underset{\sim}{V}_d + \underset{\sim}{u}_e' e^{i\omega t - ikz} \tag{56}$$

When the transport equations are linearized, with the zero-order terms assumed to add to zero and the impedance tensor $\underset{\approx}{Z}_e$ solved for in the same manner as done previously, the form remains identical with that given by equation (C17) or equation (46) except that the frequency ω is replaced by its Doppler shifted value $(\omega - kV_d)$. Or, equivalently, θ_e is replaced by $(\theta_e - \chi_e)$ where χ_e is the electron drift velocity normalized with respect to the electron thermal velocity; that is,

$$\chi_e = V_d \left(\frac{m_e}{2\kappa T_e}\right)^{1/2} \tag{57}$$

The differential backscattering cross section may formally be written in the same form as equation (B14) but with the tensor $\underset{\approx}{B}$ now defined as

$$\underset{\approx}{B} = \frac{\theta_e}{\theta_e - \chi_e} \underset{\approx}{Z}_e + \frac{1}{\mu}\left(\mu \underset{\approx}{Z}_i^{-1} + \underset{\approx}{Z}_I^{-1}\right)^{-1} \tag{58}$$

in place of equation (B16). The tensors $\underset{\approx}{Z}_e$ and $\underset{\approx}{Z}_i$ remain of the same form as given by equation (C17) except that, in $\underset{\approx}{Z}_e$, θ_e must be replaced by its Doppler shifted value, $\theta_e - \chi_e$.

The approximate expression for $\beta = 90^\circ$ (eq. (51)) is likewise altered to read

$$\sigma_b(\omega_o - \omega)\, d\omega = N_o r_e^2 \left| \frac{\mu\alpha^2 + \frac{z_i}{P_i}}{\alpha^2\left(\frac{z_i}{P_i} + \frac{\mu z_e}{P_e}\right) + \frac{z_e}{P_e}\frac{z_i}{P_i}} \right|^2 \mathrm{Im}\left\{\frac{\theta_e}{\theta_e - \chi_e}\frac{z_e}{P_e} + \frac{\alpha^2}{\mu}\frac{\frac{z_i}{P_i}}{\mu\alpha^2 + \frac{z_i}{P_i}}\right\}\frac{d\omega}{\pi\omega} \tag{59}$$

with z_e/P_e given by equations (49) and (52) but with θ_e again to be replaced by $(\theta_e - \chi_e)$.

For the case of $\alpha \to 0$ (Debye length $\to \infty$) the differential cross section (eq. (59)) reduces to

$$\sigma_b(\omega_o - \omega)\, d\omega = -N_o r_e^2 \, \mathrm{Im}\left(\frac{P_e}{z_e}\right) \frac{d\omega}{\omega - kV_d} \tag{60}$$

which is the same spectrum as for the zero drift case except that the scattered signal is Doppler shifted by kV_d. This is what one would expect for very large Debye lengths since the electrons then move independently of the ions, and the ions have no influence over the electrons, which actually cause the scattering.

As the magnetic field strength B_o goes to zero, $P_e = P_i = 1$ and the z_s are again given by equation (27) (with θ_e replaced by $\theta_e - \chi_e$ in z_e).

It must be emphasized that in order to legitimately use these expressions for the backscattering cross section, one must be certain that the drift velocity of the electrons is less than the critical velocity at which the two-stream instability sets in.

Numerical Evaluation of Cross Section

The differential backscattering cross section is evaluated in figure 11 for the case of zero net electron drift velocity and for two drift velocities less than the critical drift velocity which corresponds to $\chi_e = 0.0055$ for this particular set of parameters. This figure shows that the spectrum changes from its symmetric shape for $\chi_e = 0$ to an unsymmetric form when the electrons are drifting with respect to the ions. As the drift velocity approaches the onset of instability, the spectrum becomes very sharply peaked at a frequency corresponding to the ion acoustic wave in the plasma.

The total scattering cross section is enhanced when the electrons are drifting with respect to the ions. As the electron drift velocity approaches the point at which the instability commences, the predicted total cross section becomes infinitely large. (In practice, the scattered power is limited by nonlinear effects.) This effect is shown in figure 12 for four values of the normalized collision frequency ψ_i.

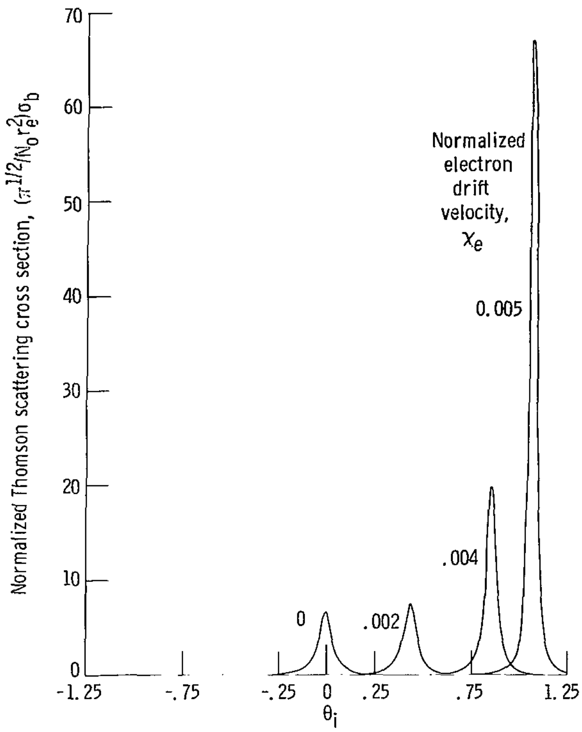


Figure 11. - Normalized Thomson scattering cross section as function of Doppler shift for various values of normalized electron drift velocity. $\alpha = 170$; normalized ion-neutral collision frequency, ψ_i, 2; normalized electron-neutral collision frequency, ψ_e, 0.2; electron to ion temperature ratio, 1; normalized ion cyclotron frequency, φ_i, 0.12; angle between incident wave vector and magnetic field, β, 90°; $C_e = D_e = 1$; $C_i = D_i = 2$.

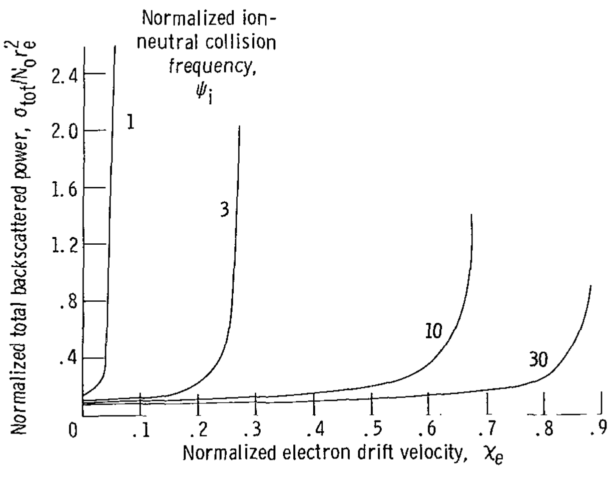


Figure 12. - Total scattered power as function of normalized drift velocity for several values of normalized ion-neutral collision frequencies. Magnetic-field strength, zero; $\alpha = 12.7$; normalized electron-neutral collision frequency, ψ_e, 0.1 ψ_i; electron to ion temperature ratio, 10; ion mass, m_i, 31 atomic mass units; $C_i = D_i = 2$; $C_e = D_e = 1$.

SUMMARY OF RESULTS

The theory of Thomson scattering has been extended to include the effects of collisions for scattering from a plasma with unequal electron and ion temperatures including the influence of a constant magnetic field and electron drift. The theory presented here is based on the two-fluid continuum equations for a plasma and should provide a good description when the wavelength of the probing wave is larger than the ion-neutral mean free path.

When the ions and electrons are not at the same temperature, the total backscattered power is shown to be dependent on the ion-neutral and electron-neutral collision frequencies, as well as the magnitude and relative angle of the magnetic field if one is present. For no magnetic field and electron to ion temperature ratio T_e/T_i greater than or equal to 10, the total backscattered power is shown to be substantially less than that of collisionless calculations but always greater than a cross section that varies with T_e/T_i as $(1 + T_e/T_i)^{-1}$. For example, with $\alpha >> 1$ ($\alpha = \lambda_o/4\pi\lambda_D$ where λ_o is the wavelength of the incident wave and λ_D is the Debye length) $T_e/T_i = 20$, and the ion-neutral mean free path $\lambda_{in} \sim 1/5\ \lambda_o$, the normalized total backscattered power is about 1/7 of that predicted for no collisions and about $2\frac{1}{2}$ times the value given by $(1 + T_e/T_i)^{-1}$.

As the electron-neutral and ion-neutral collision frequencies ν_{en} and ν_{in} increase, the scattered power approaches the $(1 + T_e/T_i)^{-1}$ curve, although it does not reach it even in the limit $\nu_{en}, \nu_{in} \rightarrow \infty$. Also, for $T_e/T_i \geq 5$, a change in the ratio of the collision frequencies ν_{en}/ν_{in} can have a large effect on the total scattered power although the shape of the spectrum is essentially unchanged.

A constant magnetic field affects Thomson scattering only when the field is normal to, or nearly normal to, the direction of propagation of the incident wave. For T_e/T_i in the common experimental range of 1 to 10, the total scattered power is increased by the magnetic field. In fact, the scattered power may increase with T_e/T_i rather than falling off in accord with the usual $(1 + T_e/T_i)^{-1}$ relation. For instance, with both the ion gyroradius and the ion-neutral mean free path equal to $1/5\ \lambda_o$, with the angle between the incident wave and the magnetic field β equal to 90^o, and with $T_e/T_i = 5$, the total backscattered power is almost five times that calculated for the field-free case. Increasing the collision frequencies lessens the influence of the field and causes the scattered power to become closer to the field-free case. In general, the total scattered power is a function of the collision frequencies and the magnetic field strength except for the special case of $T_e = T_i$ when it is <u>totally</u> independent of both the collisions and the magnetic field. When electrons are drifting with respect to the ions, only the component of the drift velocity in the direction of the incident wave affects the backscattered signal. The spectrum is no longer symmetric about the incident frequency (as it is for no drift). If the Debye length is large compared with the wavelength of the incident wave, the only effect on the spectrum is a Doppler shift corresponding to the component of the drift velocity in the direction of the incident wave. However, if the Debye length is small compared with the wavelength, an instability of the ion acoustic wave develops as the drift velocity reaches a certain critical value.

As the drift velocity increases, the spectrum of the backscattered signal develops a sharp peak at a frequency shift (from the incident wave frequency) corresponding to the weakly damped ion acoustic wave. The total scattered power is enhanced by a nonzero drift velocity and, as the electron drift nears the critical velocity, becomes very much greater than that calculated for a quiescent plasma. For example, with no magnetic field, $T_e/T_i = 10$, $\alpha >> 1$, and $\lambda_{in} = \frac{1}{5}\lambda_o$, an electron drift velocity equal to 1/20 of the electron thermal velocity causes the total backscattered power to be increased by a factor of 14 over the zero-drift case.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, February 26, 1971,
120-26.

APPENDIX A

SYMBOLS

A_s	see eq. (50)
$\underset{\approx}{A}_s$	see eq. (C13)
$\underset{\sim}{B}_o$	constant magnetic flux density
$\underset{\approx}{B}$	see eq. (B16)
C_s	constant of order 1 or 2
c	velocity of light
c_s	see eq. (C9)
$\underset{\sim}{D}$	electric flux density
D_s	constant of order 1 or 2
$\underset{\sim}{E}$	electric field intensity
e	electronic charge
$\underset{\sim}{F}$	applied force
$\underset{\approx}{G}_s$	see eq. (C12)
$\underset{\sim}{H}$	magnetic-field intensity
$\hbar$	Planck's constant/2π
$\underset{\sim}{I}$	generalized system response
Im	imaginary part
i	$\sqrt{-1}$
$\underset{\sim}{J}$	current density
K_s	see eqs. (40a) and (40b)
$\underset{\sim}{k}$	wave vector of scattering wave
$\underset{\sim}{k}_o$	wave vector of incident wave
$\underset{\sim}{k}_s$	wave vector of scattered wave
L	linear dimension of scattering volume
m_s	mass
N_o	ambient electron number density

N_s	number density
n	index of refraction (for scattering wave)
P_s	see eq. (52)
$\underset{\approx}{P}_s$	traceless pressure tensor
p_s	scalar pressure
Q	see eq. (39)
q_s	charge
Re	real part
r_e	classical electron radius
s_s	see eq. (C10)
T_{os}	equilibrium temperature
T_s	temperature
$\underset{\approx}{T}_s$	see eq. (C11)
U_p	plasma sound speed
$\underset{\sim}{u}_s$	mean velocity
$\underset{\sim}{V}$	generalized force
V_d	electron drift velocity
W	energy in L^3
x_s	see eq. (47)
Y_I	$i\omega\epsilon_o/e^2$
$\underset{\approx}{Y}_s$	generalized admittance tensor
$\underset{\approx}{Z}_s, \underset{\approx}{Z}, \underset{\approx}{Z}_I$	normalized impedance tensors (see eq. (B11))
z_s	normalized impedance function (see eq. (27))
α	$1/(k\lambda_D)$
β	angle between $\underset{\sim}{k}_o$ and $\underset{\sim}{B}_o$
β'	see eq. (35b)
$\underset{\sim}{\Gamma}_s$	flux density
γ	square root of electron to ion mass ratio
Δ_s	see eq. (C19)
ΔN	fluctuating part of electron number density

$\Delta\epsilon$	fluctuating part of permittivity
δ_{ij}	Kronecker delta
$\epsilon(r,t)$	permittivity of medium
ϵ_o	permittivity of free space
η_{os}	coefficient of viscosity
θ	scattering angle
θ_s	normalized Doppler shift frequency (see eq. (29a))
κ	Boltzmann's constant
λ_D	electron Debye length
λ_{in}	ion-neutral mean free path
λ_{os}	thermal conductivity (with $\underset{\sim}{B}_o = 0$)
λ_o	wavelength of incident wave
λ_s	thermal conductivity ($\underset{\sim}{B}_o \neq 0$)
μ	electron to ion temperature ratio
ν_{sn}	effective collision frequency for momentum transfer
ν'_{sn}	effective collision frequency for energy transfer
ρ	angle between incident electric field vector and $\underset{\sim}{k}_s$
ρ_s	mass density
σ_s	see eq. (C20)
$\sigma_b(\omega_o - \omega)$	differential backscattering cross section
σ_{tot}	total backscattering cross section
ϕ_s	normalized cyclotron frequency (see eq. (29b))
χ_e	normalized electron drift velocity (see eq. (57))
ψ_s	normalized collision frequency (see eq. (C7)
Ω_s	cyclotron frequency
ω	Doppler shift frequency
ω_e	plasma frequency
ω_o	frequency of incident wave

Subscripts:

e	electrons
i	ions
s	either electrons or ions
so	zero-order quantity; either electrons or ions
n	neutrals
i,j,k,l,m,n	components of vectors or tensors

Superscripts:

*	complex conjugate
†	Hermitian conjugate

APPENDIX B

BACKSCATTERING CROSS SECTION FOR ANISOTROPIC MEDIUM

A general tensor admittance function $\underset{\approx}{Y}_s$ is defined in the same manner as Dougherty and Farley (ref. 3):

$$\underset{\sim}{\Gamma}_s = \underset{\approx}{Y}_s \cdot \underset{\sim}{F}_s \tag{B1}$$

where $\underset{\sim}{F}_s$ is an applied force and $\underset{\sim}{\Gamma}_s$ is the resulting flux density. The rate of change of energy in a volume L^3 is

$$\frac{dW}{dt} = L^3 \underset{\sim}{\Gamma}_s \cdot \underset{\sim}{F}_s \tag{B2}$$

Applying the fluctuation-dissipation theorem (ref. 17) to the s species (for $\kappa T >> \hbar\omega$) gives the cross spectral density of the components of $\underset{\sim}{F}_s$.

$$\left\langle \left| F_i^s(\omega) F_j^s(\omega) \right| \right\rangle d\omega = \frac{\kappa T_s}{2\pi L^3} \left\{ \left[\underset{\approx}{Y}_s^{-1}(\omega) \right]_{ji} + \left[\underset{\approx}{Y}_s^{-1}(\omega) \right]_{ij}^* \right\} d\omega \tag{B3}$$

This may be thought of as the spectrum of the stochastic forces that are driving the s species when no interaction due to the electric fields is taken into account. The cross spectral density is the Fourier transform of the cross correlation function and is defined by

$$\int_{-\infty}^{\infty} \left\langle \left| F_i^s(\omega) F_j^s(\omega) \right| \right\rangle e^{i\omega\tau} d\omega = \left\langle F_i^{s*}(t) F_j^s(t+\tau) \right\rangle \tag{B4}$$

With the inclusion of the force due to the electric field, the equation of motion for the electrons (and ions) is

$$\underset{\sim}{\Gamma}_s = \underset{\approx}{Y}_s \cdot \left(\underset{\sim}{F}_s + q_s \underset{\sim}{E} \right) \tag{B5}$$

Maxwell's equations provide another relation

$$\underset{\sim}{\Gamma}_e - \underset{\sim}{\Gamma}_i = \underset{\approx}{Y}_I \cdot (e\underset{\sim}{E}) \tag{B6}$$

with

$$\underset{\approx}{Y}_I = \frac{i\omega\epsilon_o}{e^2} \begin{bmatrix} 1 - n^2 & 0 & 0 \\ 0 & 1 - n^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B7}$$

and where the coordinate system has been chosen so that the wave vector $\underset{\sim}{k}$ lies along the z axis.

Using equations (B5) and (B6) and assuming that the stochastic forces $\underset{\sim}{F}_e$ and $\underset{\sim}{F}_i$ are uncorrelated lead to the cross spectral density of the electron fluctuations

$$\left\langle \left| \Gamma_i^e \Gamma_\ell^e \right| \right\rangle d\omega = Y_{ij} Y_{\ell m}^* \left[\left\langle \left| F_j^e F_m^e \right| \right\rangle + Y'_{jk} Y'^*_{mn} \left\langle \left| F_k^i F_n^i \right| \right\rangle \right] d\omega \tag{B8}$$

where

$$\underset{\approx}{Y} = \underset{\approx}{Y}_e \cdot \left(\underset{\approx}{Y}_e + \underset{\approx}{Y}_i + \underset{\approx}{Y}_I\right)^{-1} \cdot \left(\underset{\approx}{Y}_i + \underset{\approx}{Y}_I\right) \tag{B9}$$

$$\underset{\approx}{Y}' = \left(\underset{\approx}{Y}_i + \underset{\approx}{Y}_I\right)^{-1} \cdot \underset{\approx}{Y}_i \tag{B10}$$

Normalized impedance tensors $\underset{\approx}{Z}_s$, $\underset{\approx}{Z}$, and $\underset{\approx}{Z}_I$ are defined by

$$\left.\begin{aligned} \underset{\approx}{Y}_s &= \frac{i\omega N_o}{k^2 \kappa T_s} \underset{\approx}{Z}_s^{-1} \\ \underset{\approx}{Y} &= \frac{i\omega N_o}{k^2 \kappa T_e} \underset{\approx}{Z}^{-1} \\ \underset{\approx}{Y}_I &= \frac{i\omega N_o}{k^2 \kappa T_e} \underset{\approx}{Z}_I^{-1} \end{aligned}\right\} \tag{B11}$$

The differential backscattering cross section is proportional to the power spectrum of the Fourier component of the electron number density fluctuations with wave vector $2\underset{\sim}{k}_o$ (ref. 3).

$$\sigma_b(\omega_o - \omega)\, d\omega = L^3 r_e^2 \left\langle \left| \Delta N(2\underset{\sim}{k}_o, \omega) \right|^2 \right\rangle d\omega \tag{B12}$$

The backscattering cross section is found by using equations (B7) to (B12) together with the continuity equation for the electrons

$$\frac{\partial \Delta N}{\partial t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{\Gamma}_e = 0 \tag{B13}$$

to be

$$\sigma_b(\omega_o - \omega)\, d\omega = \frac{N_o r_e^2}{2\pi i \omega} \left[\underset{\approx}{Z}^{-1} \cdot (\underset{\approx}{B} - \underset{\approx}{B}^{\dagger}) \cdot \underset{\approx}{Z}^{-1^{\dagger}} \right]_{zz} d\omega \tag{B14}$$

where

$$\underset{\approx}{Z} = \underset{\approx}{Z}_e + (\mu \underset{\approx}{Z}_i^{-1} + \underset{\approx}{Z}_I^{-1})^{-1} \tag{B15}$$

and

$$\underset{\approx}{B} = \underset{\approx}{Z} - \frac{\mu - 1}{\mu} (\mu \underset{\approx}{Z}_i^{-1} + \underset{\approx}{Z}_I^{-1})^{-1} \tag{B16}$$

APPENDIX C

CALCULATION OF IMPEDANCE TENSORS

In this appendix the normalized impedance tensors $\underset{\approx}{Z}_e$ and $\underset{\approx}{Z}_i$ are calculated for the electrons and ions considered separately; that is, without including the force due to the self-consistent electric field. The impedance tensor $\underset{\approx}{Z}_s$ for the s species is defined by

$$\underset{\approx}{Z}_s \cdot \underset{\sim}{\Gamma}_s = \frac{i\omega N_o}{k^2 \kappa T_s} F_s \tag{C1}$$

where $\underset{\sim}{\Gamma}_s$ is the flux density resulting from an applied force $\underset{\sim}{F}_s$ that is proportional to $e^{i\omega t - ikz}$.

The approach of Tanenbaum (ref. 11) is followed where the transport equations for mass, momentum, and energy are used. For small amplitude disturbances in a weakly ionized gas with the mean velocity of the neutral particles $\underset{\sim}{u}_n$ equal to 0, it is found to first order in the fluctuating quantities, that

$$\frac{\partial \rho_s}{\partial t} + \rho_{so} \nabla \cdot \underset{\sim}{u}_s = 0 \tag{C2}$$

$$\frac{\partial \underset{\sim}{u}_s}{\partial t} + \frac{\nabla p_s}{\rho_{so}} - \frac{\underset{\sim}{F}_s}{m_s} - \frac{q_s}{m_s} (\underset{\sim}{u}_s \times \underset{\sim}{B}_o) + \frac{\nabla \cdot \underset{\approx}{P}_s}{\rho_{so}} = -\nu_{sn} \underset{\sim}{u}_s \tag{C3}$$

$$\frac{\partial p_s}{\partial t} - \frac{5}{3} \frac{p_{so}}{\rho_{so}} \frac{\partial \rho_s}{\partial t} - \frac{2}{3} \lambda_s \nabla^2 T_s = -N_o \kappa \nu'_{sn} (T_s - T_{so}) \tag{C4}$$

where ρ_s, q_s, m_s, p_s, and T_s are the mass density, charge, mass, pressure, and temperature, respectively, for species s; $\underset{\sim}{B}_o$ is a constant magnetic field; and ν_{sn} and $\nu'_{sn} = 2m_s \nu_{sn}/(m_s + m_n)$ are, respectively, the effective collision frequencies for momentum and energy transfer with the neutrals. The form of the traceless pressure tensor and heat flow term developed by Chapman and Cowling (ref. 24) is used. For all fluctuating quantities varying as $e^{i\omega t - ikz}$, the divergence of the traceless pressure tensor $\underset{\approx}{P}_s$ may be written in the form

$$\nabla \cdot \underset{\approx}{P}_s = k^2 \eta_{os} \underset{\approx}{T}_s \cdot \underset{\sim}{u}_s \tag{C5}$$

Normalized frequencies for the Doppler shift frequency, collision frequency, and cyclotron frequency are defined by

$$\theta_s = \frac{\omega}{k}\left(\frac{m_s}{2\kappa T_s}\right)^{1/2} \tag{C6}$$

$$\psi_s = \frac{\nu_{sn}}{k}\left(\frac{m_s}{2\kappa T_s}\right)^{1/2} \tag{C7}$$

$$\phi_s = \frac{\Omega_s}{k}\left(\frac{m_s}{2\kappa T_s}\right)^{1/2} \tag{C8}$$

Then $\underset{\approx}{T}_s$ is a matrix whose elements are functions only of

$$c_s = \mp\left(\frac{\phi_s}{\psi_s}\right)\cos\beta \tag{C9}$$

$$s_s = \mp\left(\frac{\phi_s}{\psi_s}\right)\sin\beta \tag{C10}$$

where β is the angle between the incident wave vector $\underset{\sim}{k}_o$ and the magnetic field $\underset{\sim}{B}_o$. The upper sign is to be taken for the electrons, and the lower sign for the ions. The tensor $\underset{\approx}{T}_s$ is then given by

$$\underset{\approx}{T}_s = \underset{\approx}{G}_s^{-1} \cdot \underset{\approx}{A}_s \tag{C11}$$

where

$$\underset{\approx}{G}_s = \begin{bmatrix} 1 + \dfrac{s_s^2}{1 + 4c_s^2} & -c_s\left(1 - \dfrac{4s_s^2}{1 + 4c_s^2}\right) & \dfrac{c_s s_s}{1 + 4c_s^2} \\ c_s\left(1 - \dfrac{2s_s}{1 + 4c_s^2}\right) & 1 + \dfrac{2s_s^2}{1 + 4c_s^2} & -s_s\left(1 + \dfrac{2c_s^2}{1 + 4c_s^2}\right) \\ 0 & 2s_s & 1 \end{bmatrix} \tag{C12}$$

and

$$\underset{\approx}{A}_s = \begin{bmatrix} 1 & 0 & \dfrac{4}{3}\,\dfrac{c_s s_s}{1 + 4c_s^2} \\ 0 & 1 & \dfrac{2}{3}\,\dfrac{s_s}{1 + 4c_s^2} \\ 0 & 0 & \dfrac{4}{3} \end{bmatrix} \tag{C13}$$

The thermal conductivity is given by

$$\lambda_s = \lambda_{os} \frac{1 + \left(\dfrac{\phi_s}{\psi_s}\right)^2 \cos^2\beta}{1 + \left(\dfrac{\phi_s}{\psi_s}\right)^2} \tag{C14}$$

The coefficients for viscosity and thermal conductivity for charged particles in a weakly ionized gas which appear in equations (C5) and (C14) are given by

$$\eta_{os} = \frac{p_s}{D_s \nu_{sn}} \tag{C15}$$

and

$$\lambda_{os} = \frac{15\kappa p_s}{4m_s C_s \nu_{sn}} \tag{C16}$$

where D_s and C_s are constants (of order 1 or 2) that depend on the interparticle force law and the mass ratio between the electrons or ions and neutrals.

By using the mass and energy transport equations, we may write the momentum equations in the same form as equation (C1) with the impedance tensors $\underset{\approx}{Z}_s$ given by

$$\underset{\approx}{Z}_s = \frac{i\theta_s}{D_s\psi_s}\underset{\approx}{T}_s + \begin{bmatrix} h_s & \pm 2i\phi_s\theta_s \cos\beta & 0 \\ \mp 2i\phi_s\theta_s \cos\beta & h_s & \pm 2i\phi_s\theta_s \sin\beta \\ 0 & \mp 2i\phi_s\theta_s \sin\beta & \Delta_s + h_s \end{bmatrix} \tag{C17}$$

where

$$h_s = 2i\theta_s\psi_s - 2\theta_s^2 \tag{C18}$$

$$\Delta_s = \frac{1 + i\left(\dfrac{5\omega}{3\sigma_s}\right)}{1 + i\left(\dfrac{\omega}{\sigma_s}\right)} \tag{C19}$$

and

$$\sigma_s = \nu'_{sn} + \frac{5k^2\kappa T_s\left[1 + \left(\dfrac{\phi_s}{\psi_s}\right)^2 \cos^2\beta\right]}{2C_s\nu_{sn}m_s\left[1 + \left(\dfrac{\phi_s}{\psi_s}\right)^2\right]} \tag{C20}$$

Now equation (C17) may be used to evaluate $\underset{\approx}{Z}_e$ and $\underset{\approx}{Z}_i$ in terms of plasma parameters.

REFERENCES

1. Gordon, W. E.: Incoherent Scattering of Radio Waves by Free Electrons with Applications to Space Exploration by Radar. IRE Proc., vol. 46, no. 11, Nov. 1958, pp. 1824-1829.

2. Bowles, K. L.: Observation of Vertical-Incidence Scatter from the Ionosphere at 41 Mc/sec. Phys. Rev. Letters, vol. 1, no. 12, Dec. 15, 1958, pp. 454-455.

3. Dougherty, J. P.; and Farley, D. T.: A Theory of Incoherent Scattering of Radio Waves by a Plasma. Proc. Roy. Soc. (London), Ser. A, vol. 259, no. 1296, Nov. 22, 1960, pp. 79-99.

4. Farley, D. T.; Dougherty, J. P.; and Barron, D. W.: A Theory of Incoherent Scattering of Radio Waves by a Plasma. II. Scattering in a Magnetic Field. Proc. Roy. Soc. (London), Ser. A, vol. 263, no. 1313, Sept. 5, 1961, pp. 238-258.

5. Dougherty, J. P.; and Farley, D. T., Jr.: A Theory of Incoherent Scattering of Radio Waves by a Plasma. 3. Scattering in a Partly Ionized Gas. J. Geophys. Res., vol. 68, no. 19, Oct. 1, 1963, pp. 5473-5486.

6. Farley, D. T.: A Theory of Incoherent Scattering of Radio Waves by a Plasma. 4. The Effect of Unequal Ion and Electron Temperatures. J. Geophys. Res., vol. 71, no. 17, Sept. 1, 1966, pp. 4091-4098.

7. Gerry, Edward T.; and Rose, D. J.: Plasma Diagnostics by Thomson Scattering of a Laser Beam. J. Appl. Phys., vol. 37, no. 7, June 1966, pp. 2715-2724.

8. Rolfe, Edward; Fiocco, Giorgio; and Silk, J. K.: Investigation of Thomson Scattering from the Free Electrons in a Plasma. Raytheon Corp. (RADC-TR-65-416, DDC No. AD-487312), June 1966.

9. Salpeter, E. E.: Electron Density Fluctuations in a Plasma. Phys. Rev., vol. 120, no. 5, Dec. 1, 1960, pp. 1528-1535.

10. Salpeter, E. E.: Plasma Density Fluctuations in a Magnetic Field. Phys. Rev., vol. 122, no. 6, June 15, 1961, pp. 1663-1674.

11. Fejer, J. A.: Scattering of Radio Waves by an Ionized Gas in Thermal Equilibrium. Can. J. Phys., vol. 38, no. 8, Aug. 1960, pp. 1114-1133.

12. Salpeter, E. E.: Density Fluctuations in a Nonequilibrium Plasma. J. Geophys. Res., vol. 68, no. 5, Mar. 1, 1963, pp. 1321-1333.

13. Rosenbluth, M. N.; and Rostoker, N.: Scattering of Electromagnetic Waves by a Nonequilibrium Plasma. Phys. Fluids, vol. 5, no. 7, July 1962, pp. 776-788.

14. Buneman, O.: Scattering of Radiation by the Fluctuations in a Nonequilibrium Plasma. J. Geophys. Res., vol. 67, no. 5, May 1962, pp. 2050-2053.

15. Cohen, Marshall H.: Hydrodynamic Theory of Plasma Density Fluctuations. J. Geophys. Res., vol. 68, no. 20, Oct. 15, 1963, pp. 5675-5679.

16. Tanenbaum, B. Samuel: Continuum Theory of Thomson Scattering. Phys. Rev., vol. 171, no. 1, July 5, 1968, pp. 215-221.

17. Sitenko, A. G. (Morris D. Friedman, trans.): Electromagnetic Fluctuations in Plasma. Academic Press, 1967.

18. Booker, H. G.: A Theory of Scattering by Noniostropic Irregularities with Application to Radar Reflections from the Aurora. J. Atmos. Terr. Phys., vol. 8, no. 4/5, 1956, pp. 204-221.

19. Holt, E. H.; and Haskell, R. E.: Foundations of Plasma Dynamics. Macmillan Co., 1965, p. 373.

20. Callen, Herbert B.; Barasch, Murray L.; and Jackson, Julius L.: Statistical Mechanics of Irreversibility. Phys. Rev., vol. 88, no. 6, Dec. 15, 1952, pp. 1382-1386.

21. Nyquist, H.: Thermal Agitation of Electric Charge in Conductors. Phys. Rev., vol. 32, no. 1, July 1928, pp. 110-113.

22. Goldman, Edward; and Sirovich, Lawrence: Equations for Gas Mixtures. Phys. Fluids, vol. 10, no. 9, Sept. 1967, pp. 1928-1940.

23. Tanenbaum, B. Samuel: Plasma Physics. McGraw-Hill Book Co., Inc., 1967.

24. Chapman, Sydney and Cowling, T. G.: The Mathematical Theory of Non-uniform Gases. Second ed., Cambridge Univ. Press, 1952, p. 337.

NASA-Langley, 1971 —— 25 E-6051

ATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546

OFFICIAL BUSINESS
PENALTY FOR PRIVATE USE $300

FIRST CLASS MAIL

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS A
SPACE ADMINISTRATION

09U 001 50 51 3DS 71147 00903
AIR FORCE WEAPONS LABORATORY /WLOL/
KIRTLAND AFB, NEW MEXICO 87117

ATT E. LOU BOWMAN, CHIEF,TECH. LIBRARY

POSTMASTER: If Undeliverable (Section 158 Postal Manual) Do Not Retur

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION OFFICE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546