FINAL REPORT

PROJECT A-1167

USE OF GREEN'S FUNCTIONS IN THE NUMERICAL SOLUTION OF TWO-POINT BOUNDARY VALUE PROBLEMS

L. J. GALLAHER AND I. E. PERLIN

GRANT NUMBER NGR 11-002-101

1 APRIL 1970 to 31 MARCH 1971

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Engineering Experiment Station
GEORGIA INSTITUTE OF TECHNOLOGY
Atlanta, Georgia

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ABSTRACT

This study investigates the use of Green's functions in the numerical solution of the two-point boundary value problem. The first part deals with the role of the Green's function in solving both linear and nonlinear second order ordinary differential equations with boundary conditions and systems of such equations. The second part describes procedures for numerical construction of Green's functions and considers briefly the conditions for their existence. Finally, there is a description of some numerical experiments using nonlinear problems for which the known existence, uniqueness or convergence theorems do not apply. Examples here include some problems in finding rendezvous and periodic orbits of the restricted three body system.

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I. INTRODUCTION

This report is devoted to the investigation of the use of Green's functions for the numerical solution of second order ordinary differential equations with boundary conditions.

If L_2 represents a second order differential operator, say

$$L_2 \equiv \frac{d}{dx}p(x)\frac{d}{dx} + r(x),$$

then the solution to the linear equation

$$L_2 u = f$$

can be represented by

$$u = L_2^{-1} f = -Gf,$$

where this last equation represents

$$u(x) = -\int_a^b G(x,y)f(y)dy.$$

The function G is called the Green's function and plays the role of the inverse of the operator $-L_2$. The function G is not unique, but depends on the boundary conditions. The existence of G needs also to be investigated in each individual case and depends on the character of p, r and the boundary conditions.

Historically, the first and still most common method of solving second order differential equations with boundary conditions (the two point boundary value problem) is the shooting method. That is, one starts at one

end and assumes enough information about the function and its derivative at that end to be able to integrate the differential equation as an initial value problem. By repeatedly integrating to the other end and correcting the assumed initial conditions, one hunts for a solution that satisfies the boundary conditions at the other end. While this method works well for some problems, for others it does not. It has a tendency to fail badly when the solution is extremely sensitive to the assumed initial conditions.

The Green's function method can be useful either in place of the shooting method or as an adjunct or preliminary step to the use of the shooting method. That is, the Green's function method might be used to obtain an initial estimate for the shooting method.

Another useful technique is to introduce a discrete approximation for the operator L_2. The function G then is the inverse of the matrix approximating $-L_2$. Low order approximations to L_2 cause no special trouble except that large matrices result if high accuracy is desired. High order approximations for L_2 tend to introduce extraneous solutions and must be treated with care. (Varga and co-workers [3-7] have given new high order methods for the nonlinear two point boundary value problem.)

The Green's function method, on the other hand, can be of an arbitrarily high order, depending only on the order and quality of the methods available for the integration of differential equations (the initial value problem).

The first part (Chapter II) of this paper is concerned with the definition and use of the Green's function. Both single equations and systems of equations are discussed. Applications to nonlinear problems where iterative techniques are needed, are discussed. Here, problems of existence, uniqueness and convergence or stability of the solutions arise.

Chapter III describes procedures for generating Green's functions numerically. Some discussion concerning the question of existence of G is included here. A table of Green's functions in analytic form for some of the simpler forms of L_2 is also given.

Chapter IV deals with specific numerical experiments in the use of Green's functions for solving second order nonlinear ordinary differential equations, and systems of such equations, with various boundary conditions. Several types of problems were chosen. The first type was single simple nonlinear equations where multiple solutions were known to exist and problems in stability occur.

The other class of problems investigated is associated with the search for orbits in the restricted three body system. Here one deals with a pair of rather complicated nonlinear equations. Two types of orbits were sought. The first are rendezvous type orbits, that is, point to point in a fixed time. Multiple solutions can exist in this case and some solutions are more stable than others.

A second type of problem for the restricted three body system is that of periodic orbits. Searches for these orbits were also carried out. The difficulty that arises here is that solutions of this type appear to be densely packed. That is, for this problem there is in every neighborhood of a solution another solution. While some of these solutions are especially stable, efforts at finding periodic orbits by the Green's function method were not in general as successful as had been hoped.

II. USE OF GREEN'S FUNCTIONS

A. Single Equations

Consider a single second order linear ordinary differential equation of the form

$$\left(\frac{d}{dx}p(x)\frac{d}{dx} + r(x)\right)u(x) = f(x), \tag{1}$$

with boundary conditions

$$u(a) = u(b) = 0.$$

This will also be written as

$$L_2 u = f. \tag{1'}$$

Provided that sufficient restrictions are placed on p, r, and f, this equation can be solved for u as

$$u = L_2^{-1} f = -Gf, \tag{2}$$

where the above is a shorthand notation for

$$u(x) = -\int_a^b G(x,y)f(y)dy. \tag{2'}$$

The function G(x,y) is known as the Green's function for the operator $\frac{d}{dx}p(x)\frac{d}{dx} + r(x)$ and boundary conditions $u(a) = u(b) = 0$. The restrictions on p and r sufficient for the existence of G will be discussed later. The function -G plays the role of the inverse of the operator L_2. The function G

must satisfy the condition

$$L_2 G = -\delta,$$

meaning

$$\left(\frac{d}{dx}p(x)\frac{d}{dx} + r(x)\right) G(x,y) = -\delta(x-y), \tag{3}$$

where $\delta(x-y)$ is the Dirac delta. The function G also satisfies the boundary conditions

$$G(a,y) = G(b,y) = 0 \tag{4}$$

for all y, $a \leq y \leq b$. We note that while G itself is a function of two variables, $L_2 G$ is not a function, but is a distribution.

There are straightforward methods for constructing G that will be discussed later (Chapter III). We point out here that the inverse of L_2 is not unique, but depends on the boundary conditions; different boundary conditions give different Green's functions.

Consider now that the function f is also a (nonlinear) function of u and its derivatives; i.e., let

$$L_2 u = f(u,u'), \tag{5}$$

or

$$\left(\frac{d}{dx}p(x)\frac{d}{dx} + r(x)\right) u(x) = f(x,u(x),u'(x)), \tag{5'}$$

where $u'(x) = \frac{d}{dx}u(x)$. In this case, one can still construct an inverse or Green's function for L_2, but the function u(x) is exhibited as the solution

of a (nonlinear) integral equation:

$$u(x) = -\int_a^b G(x,y)f(y,u(y),u'(y))dy, \tag{6}$$

or

$$u = -Gf(u,u'). \tag{6'}$$

Still, in numerical work this integral equation can be useful in solving for u by iteration. Some theory exists for the existence, uniqueness and convergence of integral equations such as (6). Even when existence, uniqueness or convergence theorems are not available, solutions to equation (6) can sometimes be obtained by iteration of

$$u^{n+1}(x) = -\int_a^b G(x,y)f(y,u^n(y),u'^n(y))dy, \tag{7}$$

also written as

$$u^{n+1} = -Gf(u^n,u'^n). \tag{7'}$$

There are two kinds of convergence theorems of interest for equation (7), that will be referred to as local and global convergence.

A global convergence theorem states the circumstances under which, for any initial u^o, iteration of equation (7) converges to a solution of equation (6).

A local convergence theorem states the circumstances under which there exists a neighborhood of the solution, such that if u^o is chosen to lie in this neighborhood, iteration of (7) converges to u, a solution of (6).

Global convergence is much stronger than local convergence. Local convergence requires that the initial trial solution u^o lie close to the true solution before convergence can be guaranteed.

Bailey, Shampine and Waltman [1] examine in considerable detail the convergence properties of the iterative scheme:

$$u^{n+1}(x) = -\int_a^b G(x,y)f(y,u^n(y),u'^n(y))dy,$$

where G is the Green's function for the operator $\frac{d^2}{dx^2}$ and zero boundary conditions. Contraction mapping techniques are used to obtain convergence conditions on f. The same techniques can be used to establish similar results for other Green's functions provided that the quantities

$$\max_{a \le x \le b} \int_a^b |G(x,y)|dy,$$

and

$$\max_{a \le x \le b} \int_a^b \left|\frac{d}{dx}G(x,y)\right|dy,$$

can be established or bounded. The theorems of Bailey, Shampine and Waltman are of the global type; that is, they give the conditions on f such that convergence is obtained for any initial u^o.

There are also some local stability conditions of interest. Local stability implies the existence of a finite neighborhood about a solution u such that if u^o lies in this neighborhood the iteration scheme converges to the solution.

With local convergence theorems, uniqueness of the solution is not required. For example, it can happen that there are many solutions and that in the neighborhood of some solution there is a convergence region for that particular solution, but other solutions have no regions of convergence.

For example, consider the differential equation

$$\frac{d^2}{dx^2}u(x) = -\sin u(x)$$

with boundary conditions

$$u(a) = u(b) = 0.$$

The equivalent integral equation is

$$u(x) = \int_a^b G(x,y)\sin u(y)dy,$$

where G is the Green's function for $\frac{d^2}{dx^2}$ with the zero boundary conditions at a and b. Iteration gives

$$u^{n+1}(x) = \int_a^b G(x,y)\sin u^n(y)dy.$$

The methods of Bailey, Shampine and Waltman can be used to show that this iteration scheme can be guaranteed to converge if

$$\max_x \int_a^b |G(x,y)|dy = \left(\frac{b-a}{\pi}\right)^2 < 1.$$

Under these circumstances the solution is unique ($u(x) = 0$), and global convergence to this solution is guaranteed.

However, if $\left(\frac{b-a}{\pi}\right)^2 > 1$ then there are multiple solutions. Furthermore, the iteration scheme is then unstable in the neighborhood of the solution $u(x) = 0$ but may be stable in the neighborhood of some of the other solutions.

B. Multiple Equations

For systems of second order linear differential equations much the same sort of results can be obtained. Consider the system

$$\sum_{1 \le k \le m} \left(\frac{d}{dx} p_{jk}(x) \frac{d}{dx} + r_{jk}(x) \right) u_k(x) = f_j(x), \quad j = 1,2..n, \tag{8}$$

with boundary conditions

$$u_k(a) = u_k(b) = 0, \quad k = 1,2..n.$$

This will also be written as

$$L_2 u = f. \tag{8'}$$

Again provided that sufficient restrictions are placed on p and r, an inverse for L_2 will exist so that

$$u = L_2^{-1} f = -Gf, \tag{9}$$

where this stands for

$$u_k(x) = -\int_a^b \sum_{i \le j \le n} G_{kj}(x,y) f_j(y) dy, \quad k = 1,2..n. \tag{9'}$$

The Green's function, G, is now an n-by-n matrix of functions, and satisfies the system of equations

$$\sum_{1 \le k \le n} \left(\frac{d}{dx} p_{jk}(x) \frac{d}{dx} + r_{jk} \right) G_{km}(x,y) = -\delta_{jm} \delta(x-y), \tag{10}$$

and the boundary conditions

$$G_{km}(a,y) = G_{km}(b,y) = 0, \; k,m = 1,2 \ldots n.$$

Again there are straightforward but more involved methods for constructing G. These methods will be described in Chapter III.

If the vector f is also a function of vector u and its derivatives, that is

$$L_2 u = f(u,u'), \tag{11}$$

or

$$\sum_{1 \le k \le n} \left(\frac{d}{dx} p_{jk}(x) \frac{d}{dx} + r_{jk}(x) \right) u_k(x) = f(x,u(x),u'(x)), j = 1,2..n, \tag{11'}$$

(here u represents the set $u_1, u_2 \ldots u_n$ and u' represents $\frac{du_1}{dx}, \frac{du_2}{dx} \ldots\ldots$),

then the Green's function for L_2 does not give a solution for u but gives the integral equation

$$u = -Gf(u,u'), \tag{12}$$

or

$$u_k(x) = -\int_a^b \sum_{1 \le k \le n} G_{kj}(x,y) f(y,u(y),u'(y)) dy, k = 1,2..n. \tag{12'}$$

Again this equation can be used in numerical work by iterating with u^n on the right side and obtaining u^{n+1} on the left. Most of the existence, uniqueness, and convergence theorems can be extended to cover the vector case.

One interesting case that occurs in connection with the vectors case is when p and r of equation (11') are both diagonal. Then G is also diagonal and is much easier to find than if there is coupling between the equations in L_2. Then

$$u_k(x) = -\int_a^b G_{kk}(x,y) f_k(y,u(y),u'(y))dy, k = 1,2..n, \qquad (13)$$

and all of the coupling between the equations occurs in the functions f only. Because of the simplicity of (13) compared to (12') and the ease in finding G when it is diagonal as opposed to when it is not diagonal, it is often desirable even when dealing with linear equations to remove the coupling from L_2 and put it in f, and iterate equation (13) with a diagonal Green's function. That is, suppose one has an equation of the form

$$L_2 u = f, \qquad (14)$$

where u and f are vectors, (f does not depend on u or u') and L_2 is a non-diagonal matrix operator. Split L_2 so that $L_2 = L^{Diag} + L^{Off}$, where L^{Diag} contains the diagonal terms of L_2 and L^{Off} contains the off diagonal terms. Then

$$L^{Diag} u = f - L^{Off} u \qquad (15)$$

and

$$u = -G^{Diag}(f - L^{Off}u), \tag{16}$$

where G^{Diag} now is the diagonal Green's function for the operator for L^{Diag}. Iteration of (16) may be easier and faster than finding the non-diagonal Green's function of the original operator L_2.

III. CONSTRUCTION OF GREEN'S FUNCTIONS (NUMERICAL)

A. Single Equations

Here we will consider how to construct, or calculate numerically, the Green's function for the differential operator

$$L_2 \equiv \left(\frac{d}{dx} p(x) \frac{d}{dx} + r(x) \right), \tag{1}$$

with boundary conditions

$$G(a,y) = G(b,y) = 0, \quad a \le y \le b.$$

The Green's function can be constructed from the solutions of the equation

$$\left(\frac{d}{dx} p(x) \frac{d}{dx} + r(x) \right) g(x) = 0, \tag{2}$$

or

$$L_2 g = 0, \tag{2'}$$

together with the appropriate boundary conditions on g.

Introducing the definition

$$J(x) \equiv p(x) \frac{d}{dx} g(x),$$

equation (2) can be written as the pair of coupled ordinary differential equations,

$$\begin{aligned} g'(x) &= J(x)/p(x), \\ J'(x) &= -r(x)g(x), \end{aligned} \tag{3}$$

(prime indicating derivative with respect to x). There are two families of

solutions of interest for this system of equations; one family satisfies the boundary condition $g(a) = 0$; the other satisfies the condition $g(b) = 0$. The solution satisfying $g(a) = 0$ will be designated $g_a(x)$, and to make this solution unique, the condition $J_a(a) \equiv p(a)g_a'(a) = 1$ will be appended to this solution. Likewise the solution satisfying $g(b) = 0$ will be designated $g_b(x)$, and the condition $J_b(b) \equiv p(b)g_b'(b) = 1$ will be appended to the solution. This gives the two sets of simultaneous equations and boundary conditions:

$$\begin{aligned} g'_a(x) &= J_a(x)/p(x), \\ J'_a(x) &= -r(x)g_a(x), \\ g_a(a) &= 0,\ J_a(a) = 1, \end{aligned} \tag{4}$$

and

$$\begin{aligned} g'_b(x) &= J_b(x)/p(x), \\ J'_b(x) &= -r(x)g_b(x), \\ g_b(b) &= 0,\ J_b(b) = 1. \end{aligned} \tag{5}$$

Now provided that sufficient restrictions are placed on p and r, each set of equations and boundary conditions has a solution. From the theory of ordinary differential equations, both existence and uniqueness of these solutions can be guaranteed. Furthermore, there are straightforward numerical methods, such as Euler's method, or Adams', or Runge-Kutta methods [28] that can be used to generate these solutions in tabular form on the interval [a,b].

The solutions $g_a(x)$ and $g_b(x)$ are of course also solutions to equation (2), and it is straightforward to show that any pair of solutions of (2) satisfies the Wronskian condition,

$$J_a(x)g_b(x) - J_b(x)g_a(x) = A,$$

where A is a constant. From the boundary conditions chosen here for equations (4) and (5)

$$A = g_b(a) = -g_a(b). \tag{6}$$

It can now be shown that the Green's function can be constructed from the solutions $g_a(x)$ and $g_b(x)$ as follows:

$$G(x,y) = A^{-1}\begin{cases} g_a(x)g_b(y), & x \le y, \\ g_a(y)g_b(x), & y \le x. \end{cases} \tag{7}$$

To show this we note that the conditions on the G(x,y) are:

$$\left(\frac{d}{dx}p(x)\frac{d}{dx} + r(x)\right)G(x,y) = -\delta(x-y),$$
$$G(a,y) = G(b,y) = 0, \quad (a \le y \le b). \tag{8}$$

First, note that the boundary conditions on G are satisfied by virtue of the boundary conditions $g_a(a) = g_b(b) = 0$, and that G(x,y) is continuous at $x = y$.

Next, note that since both $g_a(x)$ and $g_b(x)$ satisfy equation (2), then

$$\left(\frac{d}{dx}p(x)\frac{d}{dx} + r(x)\right)G(x,y) = 0, \qquad \text{if } x \neq y \ ;$$

that is, G satisfies equation (8) everywhere except (possibly) at the points $x = y$, where $L_2 G$ is not defined except in the sense of a distribution.

Finally, in the vicinity of the points $x = y$, G must satisfy the condition

$$\lim_{\epsilon \to 0} \int_{y-\epsilon}^{y+\epsilon} \left(\frac{d}{dx} p(x) \frac{d}{dx} + r(x) \right) G(x,y) dx = -1.$$

This reduces to

$$\lim_{\epsilon \to 0} \; p(x) \frac{d}{dx} G(x,y) \Bigg|_{x = y-\epsilon}^{x = y+\epsilon} = -1,$$

or

$$A^{-1} p(y) \left(g_b'(y) g_a(y) - g_a'(y) g_b(y) \right) = -1.$$

By the Wronskian condition, this is satisfied.

From the above account it can be seen that the conditions for the existence of the Green's function then are essentially the same as the conditions for the existence of solutions to the differential equations (4) and (5). Sufficient conditions for the existence of g_a and g_b are known from the theory of ordinary differential equations and are that $r(x)$ and $1/p(x)$ be piecewise continuous and finite on the internal $a \leq x \leq b$.

One additional condition is required for the existence G and that is that the Wronskian constant, A, have an inverse, i.e., $A \neq 0$. This is equivalent to requiring that the two solutions g_a and g_b be linearly independent. If g_a and g_b are not linearly independent, then $g_b(a) = 0$, $g_a(b) = 0$, $A = 0$, and there will be no Green's function for these boundary conditions.

If g_a and g_b are linearly independent, then $g_a(b) \neq 0$, $g_b(a) \neq 0$, $A \neq 0$, and there will be a Green's function for these boundary conditions.

Green's functions for other boundary conditions can be calculated in a similar way. All that changes are the boundary conditions associated with the equations determining $g_a(x)$ and $g_b(x)$, although it should be pointed out that only those boundary conditions of the form $g_a(a) = 0$, $g_b(b) = 0$, $g_a'(a) = 0$, $g_b'(b) = 0$, or some linear combination of these, will necessarily have an associated G.

However, it should be noted that with boundary conditions of the form $u(a) = u_a$, $u(b) = u_b$, solutions to the equation $L_2 u = f$ can be obtained in the form

$$u = -Gf + v,$$

that is,

$$u(x) = -\int_a^b G(x,y)f(y)dy + v(y).$$

Here G is the same Green's function as for the zero boundary conditions. The function v, called the particular solution, satisfies $L_2 v = 0$ with $v(a) = u_a$ and $v(b) = u_b$. One can see that

$$v(x) = \frac{g_a(x)u_b}{g_a(b)} + \frac{g_b(x)u_a}{g_b(a)} \quad ,$$

$$= A^{-1}\left(g_b(x)u_a - g_a(x)u_b\right) \quad ,$$

where the g's are the same as those defined for the zero boundary condition.

B. Systems of Equations

The Green's function for the matrix operator L_2 can be obtained in a fashion similar to that described above for the scalar case.

The function G satisfies the system of equations and boundary conditions

$$\sum_{1 \le k \le n} \left(\frac{d}{dx} p_{jk}(x) \frac{d}{dx} + r_{jk}(x) \right) G_{km}(x,y) = -\delta_{jm} \delta(x-y), \qquad (9)$$

$$G_{jm}(a,y) = G_{jm}(b,y) = 0, \quad j,m = 1,2..n,$$

which will be shortened to

$$(p(x)G'(x,y))' + r(x)G(x,y) = I\delta(x-y),$$

where p, r and G are n-by-n matrices and the primes indicate differentiation with respect to x.

Introduce now the square n-by-n matrix functions g_a, g_b, J_a, and J_b that satisfy the matrix equations and boundary conditions

$$g_a'(x) = p^{-1}(x)J_a(x),$$

$$J_a'(x) = -r(x)g_a(x), \qquad (10)$$

$$g_a(a) = 0, \quad J_a(a) = I,$$

and

$$g_b'(x) = p^{-1}(x)J_b(x),$$

$$J_b'(x) = -r(x)g_b(x), \qquad (11)$$

$$g_b(b) = 0, \quad J_b(b) = I.$$

It is clear that g_a and g_b both satisfy the equation $L_2 g = 0$ or

$$(p(x)g'(x))' + r(x)g(x) = 0. \tag{12}$$

These equations can be solved as systems $2n^2$ coupled ordinary differential equations, integrating from a on the first set, and from b on the second set. The standard theorems for systems of ordinary differential equations apply and can be used to guarantee existence and uniqueness of the solutions provided p and r meet the appropriate conditions discussed later. Numerical methods can be used to tabulate g_a and g_b.

Also needed are the adjoint equations for the same boundary conditions. Let

$$(h'(x)p(x))' + h(x)r(x) = 0, \tag{13}$$

where h is an n-by-n matrix. Introduce h_a, h_b, K_a and K_b that satisfy the matrix equations and boundary conditions

$$\begin{aligned} h_a'(x) &= K_a(x)p^{-1}(x), \\ K_a'(x) &= -h_a(x)r(x), \\ h_a(a) &= 0,\ K_a(a) = I, \end{aligned} \tag{14}$$

and

$$\begin{aligned} h_b'(x) &= K_b(x)p^{-1}(x), \\ K_b'(x) &= -h_b(x)r(x), \\ h_b(b) &= 0,\ K_b(b) = I, \end{aligned} \tag{15}$$

so that h_a and h_b satisfy the adjoint equation (13) and indicated boundary conditions.

Solutions of (12) and (13) satisfy Wronskian conditions. For g, J, h, and K these are

$$\begin{aligned} h_b(x)J_a(x) - K_b(x)g_a(x) &= A_1, \\ K_a(x)g_b(x) - h_a(x)J_b(x) &= A_2, \\ h_a(x)J_a(x) - K_a(x)g_a(x) &= A_a, \\ h_b(x)J_b(x) - K_b(x)g_b(x) &= A_b \end{aligned} \tag{16}$$

where the A's are constant matrices and from the given boundary conditions

$$\begin{aligned} A_1 &= h_b(a) = -g_a(b), \\ A_2 &= g_b(a) = -h_a(b), \\ A_a &= A_b = 0. \end{aligned} \tag{17}$$

It is clear that equations (14) and (15) can be integrated numerically as systems of ordinary differential equations with initial conditions, and tables of the h's constructed on the interval [a,b]. The existence and uniqueness conditions are straightforward and are the same as those for the g's.

The Green's function for the matrix operator L_2 and these boundary conditions can now be constructed as

$$G(x,y) = \begin{cases} g_a(x)A_1^{-1}h_b(y), & x \le y, \\ g_b(x)A_2^{-1}h_a(y), & x \ge y. \end{cases} \tag{18}$$

It is easily seen that:

1) G satisfies the boundary conditions,

2) $(p(x)G'(x,y))' + r(x)G(x,y) = 0$, if $x \neq y$.

It is a bit more complicated but only an exercise in algebraic manipulation to show that the third and fourth conditions required are satisfied, i.e.,

$$3) \quad \lim_{\epsilon \to 0} p(x) \frac{d}{dx} G(x,y) \Big|_{x = y-\epsilon}^{x = y+\epsilon} = -I,$$

and

4) $g_a(y)A_1^{-1}h_b(y) = g_b(y)A_2^{-1}h_a(y)$, (the continuity condition for $G(x,y)$ at $x = y$).

Consider now the questions of existence of the matrix G. One needs restrictions on the matrices p and r to guarantee the existence and uniqueness of the g's and h's. A sufficient condition for the existence and uniqueness of the g's and h's is that r(x) and $p^{-1}(x)$ exist and be piecewise continuous on the interval [a,b].

One further condition is necessary for the existence of G and this is that the Wronskian constants, the matrices A_1 and A_2 have inverses. It is not clear how these conditions reflect back on p and r since $A_1 = h_b(a) = -g_a(b)$ and $A_2 = g_b(a) = -h_a(b)$, and the existence of the inverses of the g's and h's at the end points is difficult to determine except possibly numerically.

Thus, one can state sufficient conditions for the existence and uniqueness of G as being

a) existence and piecewise continuity of r and p^{-1},

b) existence of A_1^{-1} and A_2^{-1},

but note that these may be difficult to establish in any particular case--especially the existence of A_1^{-1} and A_2^{-1}.

It was noted before that other boundary conditions give different Green's functions for the same L_2 operator. While a particular set of boundary conditions were considered here, the same techniques apply to finding the G matrices for the various other boundary conditions.

We conclude this section by noting that the numerical calculation of the matrix G can be a monumental task. One integrates four sets of $2n^2$ simultaneous differential equations (or 4 n sets of 2n equations) to obtain the g's and h's, which then must be stored or tabulated at each point appropriate to the interval [a,b].

In practice it is usually convenient to deal with the matrix Green's function only when G is diagonal. This will occur if it can be arranged that p and r are diagonal. The g's, h's, J's, K's and A's then are also diagonal and all commute. Thus if L_2 is diagonal, the problem decomposes into an uncoupled collection of n, one-dimensional problems. That is, one can look for the $G_{kk}(x,y)$ independently by integrating for n separate scalar g's. Also the corresponding h and g functions are equal and the adjoint equations need not be solved separately.

For L_2 diagonal, G is of the form

$$G_k(x,y) = A_k^{-1}\begin{cases} g_{ka}(x)g_{kb}(y) & x \le y, \\ g_{kb}(x)g_{ka}(y) & x \ge y, \end{cases}$$

$$k = 1,2..n,$$

where the g's are solutions of the equations

$$g'_{ka}(x) = J_{ka}(x)/p_k(x),$$

$$J'_{ka}(x) = -r_k(x)g_{ka}(x),$$

$$g_{ka}(a) = 0,\ J_{ka}(x) = 1,$$

$$k = 1,2..n,$$

and

$$g'_{kb}(x) = J_{kb}(x)/p_k(x),$$

$$J'_{kb}(x) = -r_k(x)g_{kb}(x),$$

$$g_{kb}(b) = 0,\ J_{kb}(x) = 1,$$

$$k = 1,2...n,$$

and the A's are given by the Wronskian condition

$$J_{ka}(x)g_{kb}(x) - J_{kb}(x)g_{ka}(x) = A_k,\ k = 1,2..n.$$

Here the k index designates the diagonal element of the corresponding diagonal matrix.

Unless the number of dimensions is small, only the diagonal Green's functions are of much practical use in numerical work.

C. Analytic Form for Some Simple Operators

It has been shown that the scalar Green's function can be written as

$$G(x,y) = A^{-1}\begin{cases} g_a(x)g_b(y), & x \le y, \\ g_b(x)g_a(y), & x \ge y, \end{cases}$$

where the g's satisfy the equation

$$L_2 g = 0$$

and $g_a(x)$ satisfies the boundary conditions at a, while $g_b(x)$ satisfies the boundary conditions at b. Thus, if one can find analytic solutions of $L_2 g = 0$ that satisfy the boundary conditions, establish linear independence of g_a and g_b, and find the Wronskian constant, one can give analytic forms for G.

Table I lists some of the simpler but commonly encountered forms of L_2, some typical boundary conditions, and the corresponding Green's functions in analytic form.

Table I

A tabulation of Green's functions for some simple forms of $L_{\approx} \equiv \frac{d}{dx}p(x)\frac{d}{dx} + r(x)$ and boundary conditions.

L_2	Boundary Conditions	$G(x,y)$ =	Notes
$\frac{d^2}{dx^2}$	$G(a,y) = 0$ $G(b,y) = 0$	$\frac{1}{b-a}\begin{cases}(x-a)(b-y), & x \le y\\ (b-x)(y-a), & x \ge y\end{cases}$	
$\frac{d^2}{dx^2}$	$\frac{d}{dx}G(x,y)\Big\|_{x=a} = 0$ $G(b,y) = 0$	$b - y, \quad x \le y$ $b - x, \quad x \ge y$	
$\frac{d^2}{dx^2} + 1$	$G(a,y) = 0$ $G(b,y) = 0$	$\frac{1}{\sin(b-a)}\begin{cases}\sin(x-a)\sin(b-y), & x \le y\\ \sin(b-x)\sin(y-a), & x \ge y\end{cases}$	$b - a \neq \pi, 2\pi, 3\pi$...etc.
$\frac{d^2}{dx^2} + 1$	$\frac{d}{dx}G(x,y)\Big\|_{x=a} = 0$ $G(b,y) = 0$	$\frac{1}{\cos(b-a)}\begin{cases}\cos(x-a)\sin(b-y), & x \ge y\\ \sin(b-x)\cos(y-a), & x \ge y\end{cases}$	$b - a \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$...

L_2	Boundary Conditions	$G(x,y)$ =	Notes
$\frac{d^2}{dx^2} - 1$	$G(b,y) = 0$	$\frac{1}{\sinh(b-a)} \begin{cases} \sinh(x-a)\sinh(b-y), & x \le y \\ \sinh(b-x)\sinh(y-a), & x \ge y \end{cases}$	$a,b \neq \pm \infty$
$\frac{d^2}{dx^2} - 1$	$\frac{d}{dx}G(x,y)\Big\|_{x = a} = 0$ $G(b-y) = 0$	$\frac{1}{\cosh(b-a)} \begin{cases} \cosh(x-a)\sinh(b-y), & x \le y \\ \sinh(b-x)\cosh(y-a), & x \ge y \end{cases}$	$a,b \neq \pm \infty$
$\frac{d^2}{dx^2} - 1$	$G(0,y) = 0$ $G(\infty,y) = 0$	$\sinh x\, e^{-y}, \quad x \le y$ $e^{-x}\sinh y, \quad x \ge y$	
$\frac{d^2}{dx^2} - 1$	$\frac{d}{dx}G(x,y)\Big\|_{x = 0} = 0$ $G(\infty,y) = 0$	$\cosh x\, e^{-y}, \quad x \le y$ $e^{-x}\cosh y, \quad x \ge y$	

L^2	Boundary Conditions	$G(x,y)$ =	Notes
$\frac{d}{dx}x\frac{d}{dx} + x - \frac{n^2}{x}$	$G(0,y) = 0$ $G(\infty,y) = 0$	$-\frac{\pi}{2}\begin{cases} J_n(x)Y_n(y), & x \le y \\ Y_n(x)J_n(y), & x \ge y \end{cases}$	J and Y are the Bessel functions of 1st and 2nd kind. $n \neq 0$
$\frac{d}{dx}x\frac{d}{dx} + x$	$\frac{d}{dx}G(x,y)\Big\|_{x=0} = 0$ $G(\infty,y) = 0$	$-\frac{\pi}{2}\begin{cases} J_0(x)Y_0(y), & x \le y \\ Y_0(x)J_0(x), & x \ge y \end{cases}$	J_0 and Y_0 are zero order Bessel and Neumann functions.
$\frac{d}{dx}x\frac{d}{dx} - x - \frac{n^2}{x}$	$G(0,y) = 0$ $G(\infty,y) = 0$	$I_n(x)K_n(y), \quad x \le y$ $K_n(x)I_n(y), \quad x \le y$	I and K are the modified Bessel and Neumann functions (Watson's normalization of K). $n \neq 0$
$\frac{d}{dx}x\frac{d}{dx} - x$	$\frac{d}{dx}G(x,y)\Big\|_{x=0} = 0$ $G(\infty,y) = 0$	$I_0(x)K_0(y), \quad x \le y$ $K_0(x)I_0(y), \quad x \ge y$	I_0 and K_0 are the zero order modified Bessel functions of 1st and 2nd kind. (Watson's normalization of K).

L_2	Boundary Conditions	$G(x,y)$ =	Notes
$\frac{d^2}{dx^2} + 1 - \frac{\ell(\ell+1)}{x^2}$	$G(0,y) = 0$ $G(\infty,y) =$ finite	$(-1)^{\ell}\frac{\pi}{2}\begin{cases} J_{\ell+\frac{1}{2}}(x)\sqrt{xy}\,J_{-\ell-\frac{1}{2}}(y), & x \le y \\ J_{-\ell-\frac{1}{2}}(x)\sqrt{xy}\,J_{\ell+\frac{1}{2}}(y), & x \ge y \end{cases}$	$\ell \ge 0$, $\ell \ne \frac{1}{2}$, 3/2,...etc. J is the Bessel function.
$\frac{d}{dx}x^2\frac{d}{dx} + x^2 - \ell(\ell+1)$	$G(0,y) = 0$ $G(\infty,y) = 0$	$j_\ell(x)n_\ell(y)$ $x \le y$, $n_\ell(x)j_\ell(y)$ $x \ge y$,	j and n are the spherical Bessel and Neumann functions. $\ell > 0$, and $\ell \ne \frac{1}{2}$, 3/2...etc.
$\frac{d}{dx}x^2\frac{d}{dx} + x^2$	$\frac{d}{dx}G(x,y)\Big\vert_{x=0} = 0$ $G(\infty,y) = 0$	$j_0(x)n_0(y)$, $x \le y$ $n_0(x)j_0(y)$, $x \ge y$	$j_0(x) = \frac{\sin x}{x}$ $n_0(x) = -\frac{\cos x}{x}$

IV. COMPUTER EXPERIMENTS

A. Introduction

A series of computer experiments were carried out to investigate convergence of various iteration schemes for solution of the two-point boundary value problem. Particular attention was paid to cases where the standard convergence theorems did not apply; that is, the cases where there were multiple solutions, or where existence of even one solution was an open question.

Three groups of problems were examined. The first of these is typified by the equation

$$\frac{d^2}{dx^2}u(x) = -2\pi^2 \sin u(x), \qquad u(0) = u(1) = 0 . \tag{1}$$

This problem is characterized by having one solution at $u(x) = 0$, and at least two other solutions. The iterative scheme

$$u^{n+1}(x) = 2\pi^2 \int_0^1 G(x,y) \sin u^n(y)\,dy \tag{2}$$

is unstable in the neighborhood of $u(x) = 0$ but could be expected to be locally stable in the neighborhood of each of the other two solutions.

The second group of problems is associated with finding orbits of the restricted three body system characterized by a fixed time between two fixed end points. Here one has a pair of differential equations (that can be written as a single second order differential equation of a complex variable) with two-point boundary conditions.

The equations represent the motion of a very light body in the gravitational field of two massive bodies. The two massive bodies are at a constant separation (circular orbit) and the lighter body is restricted to move in the plane of their rotation. In the rotating coordinate system in which the two massive bodies appear to be at rest, the equations of motion for the restricted three-body problem are

$$\ddot{x} = x + 2\dot{y} - \mu' \frac{(x + \mu)}{((x+\mu)^2+y^2)^{3/2}} - \mu \frac{(x - \mu')}{((x-\mu')^2+y^2)^{3/2}} , \qquad (3a)$$

$$\ddot{y} = y - 2\dot{x} - \mu' \frac{y}{((x+\mu)^2+y^2)^{3/2}} - \mu \frac{y}{((x-\mu')^2+y^2)^{3/2}} . \qquad (3b)$$

Here the two massive bodies are located on the x axis with the center of mass of the system at the origin, μ is the ratio of the mass of the body located on the positive x axis to the mass of the entire system, and μ' is the ratio of the mass of the body located on the negative x axis to the mass of the entire system ($\mu + \mu' = 1$). The units of distance here are chosen so that the distance between the two massive bodies is unity, and the unit of time is chosen so that the angular velocity of the rotating reference frame is unity (period = 2π).

The third group of problems is that of finding <u>periodic</u> orbits of the restricted three body system. Here one has the same equations as in the second group but with different boundary conditions.

Arenstorf [29] has shown the existence of periodic orbits for this system but there are practical problems in actually finding such orbits. This problem was considered in the hope that the Green's function method would be useful in finding these Arenstorf orbits. But one of the characteristics

of this problem is that not only are there multiple solutions, but these solutions are densely packed. That is, some solutions have the property that in every neighborhood of the solution there are other solutions. Thus, instead of converging to a particular solution, the iterative scheme has a tendency to wander or drift through a family of solutions. This wandering continues until a solution is encountered that is more stable than any of its neighbors. The more stable solutions seem to be ones with the largest radii of curvature or the ones with the least number of axis crossings.

B. Some Simple Nonlinear Problems

The first group of problems run are some simple examples of nonlinear two-point boundary value problems where the usual existence and uniqueness theorems are not valid but for which local stability might be expected. These are of the type

$$\frac{d^2}{dx^2}u(x) = f(x,u(x)),$$
$$u(0) = u(1) = 0\ , \tag{4}$$

with various forms of f. The Green's function is elementary.

A computer program was written (in Algol for the B 5500) to solve equation 4 by iterating on the Green's function integral

$$u^{n+1}(x) = -\int_0^1 G(x,y)f(y,u^n(y))dy. \tag{5}$$

A variation of this with a relaxation parameter was also used; that is,

$$u^{n+1}(x) = (1-\omega)u^n(x) - \omega\int_0^1 G(x,y)f(y,u^n(y))dy. \tag{5'}$$

ω is called the relaxation parameter ($\omega > 1$ is called over-relaxation, $\omega < 1$ is called under-relaxation) and can be used to control the speed of convergence.

Starting with an initial trial solution $u^0(x)$ the program iterates to find successive $u^n(x)$ stopping when the maximum difference in two consecutive iterations drop below a given threshold. The function $u^n(x)$ is constructed as a table of values on the interval $[0,1]$ and, through interpolation, values at points between tabulated values are obtained.

The results for various f functions are given in Figure 1.

A technique used to reduce computation time is the progressive refinement of the mesh size, interpolation, and quadrature procedures. One starts with a coarse mesh and crude interpolation and quadrature procedures and, as convergence progresses, proceeds to a finer mesh, higher order interpolation and more accurate quadrature.

Figure 1a shows the sequence of approximations for the case

$$f(x,u) = -2\pi^2 \sin u.$$

This case is known to have multiple solutions. The solution $u(x) = 0$, $0 \le x \le 1$, is unstable, but at least two others are locally stable, and the initial trial solution, $u^0(x) = \sin \pi x$, is shown here converging to one of these stable solutions.

Figure 1b shows the sequence of approximations for the case

$$f(x,u) = -\pi^2 (2-\cos \pi x)\sin u .$$

Again there is an unstable solution, $u(x) = 0$. The initial trial solution $u^0(x) = \sin \pi x$ converges to a locally stable solution.

Figure 1c shows the case

$$f(x,u) = -\pi^2((u-3u^{-1/3})/4 + 2\sin^{3/2} \pi x).$$

Here an analytic solution is known, $u(x) = \sin^{3/2} \pi x$. The initial trial solution of $u^0(x) = \sin \pi x$ convergence to this solution even though f is singular at the boundaries where $u(0) = u(1) = 0$.

Figure 1

Figure 1 shows the sequence of iterates for solutions of the equations

$$\frac{d^2}{dx^2}u(x) = f(x,u(x)); \; u(0) = u(1) = 0.$$

Figure 1a is for $f(x,u) = -2\pi^2 \sin u$.

Figure 1b is for $f(x,u) = -\pi^2(2 - \cos \pi x)\sin u$.

Figure 1c is for $f(x,u) = -\pi^2((u - 3u^{-1/3})/4 + 2\sin^{3/2}\pi x)$.

Initially a coarse mesh, linear interpolation and a crude quadrature scheme was used. As convergence increases, the program switches to progressively finer mesh, more accurate quadrature, and spline-like interpolation. For clarity, successive iterations are displaced with respect to each other and scaled down slightly. The relaxation parameter, $\omega = \frac{1}{2}$, was used.

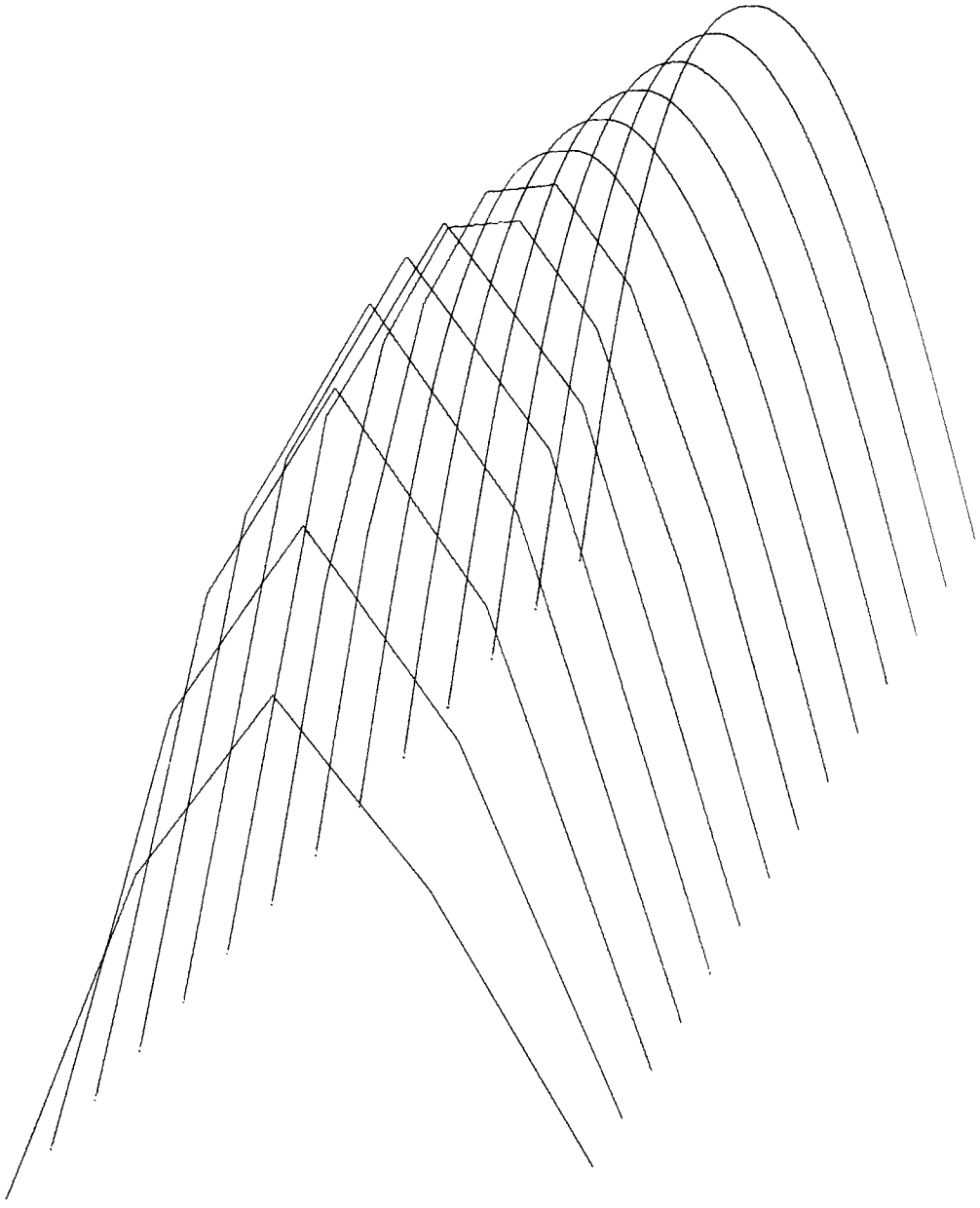

Figure 1a

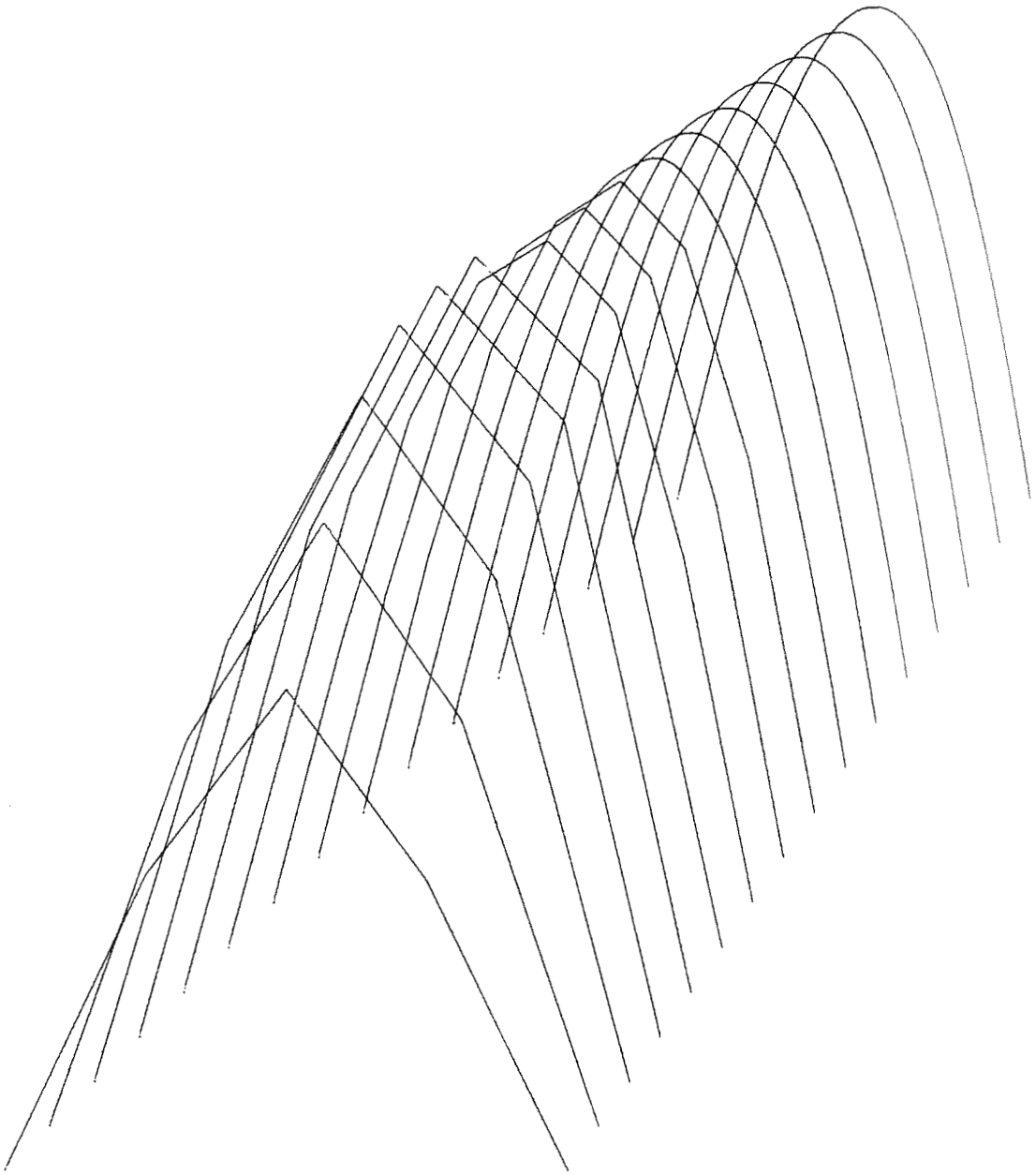

Figure 1b

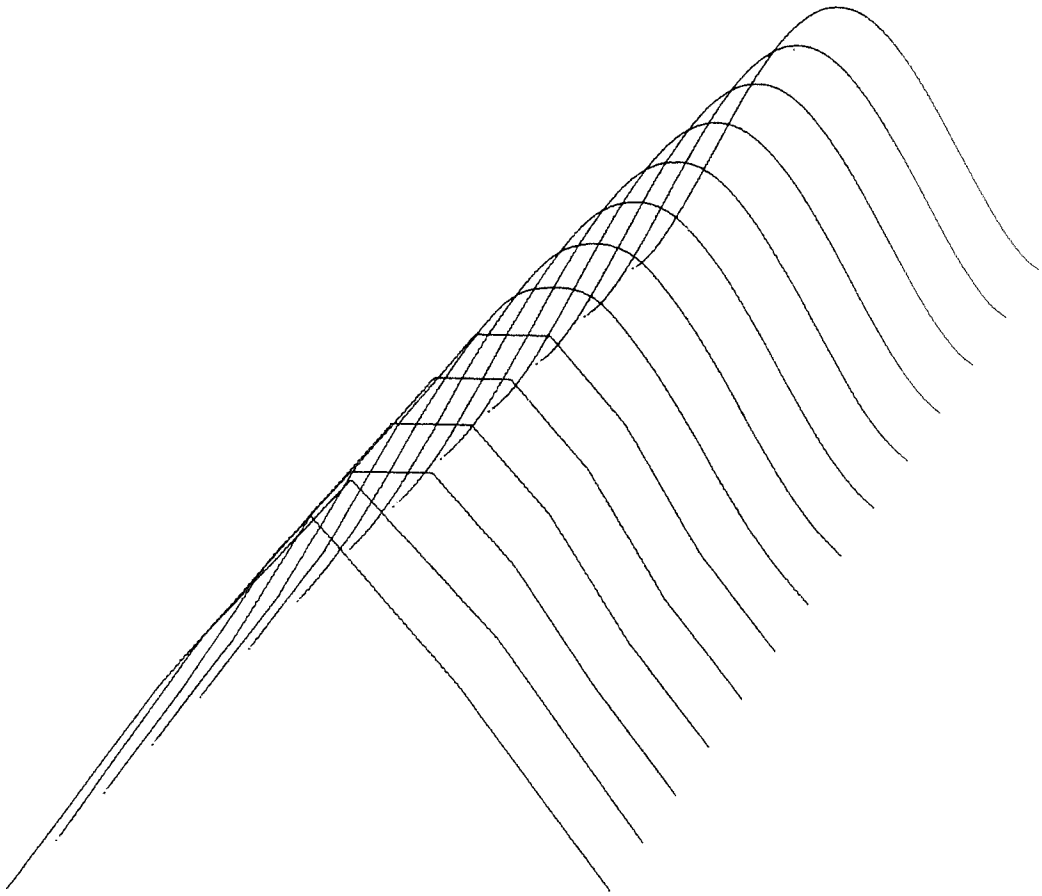

Figure 1c

C. Three Body Orbits

The second group of problems is associated with finding solutions of the restricted three-body problem that pass through two given points in a fixed time (rendezvous problem).

The differential equation [29] is

$$\ddot{z}(t) + 2i\dot{z}(t) - \gamma^2 z(t) = -f(z(t)), \tag{6}$$

where

$$f(z) = \frac{\mu'(z+\mu)}{|z+\mu|^2} + \frac{\mu(z-\mu')}{|z-\mu'|} - (1-\gamma^2)z, \tag{7}$$

with the boundary conditions

$$z(a) = z_a, z(b) = z_b.$$

Here γ, u, and μ' ($\mu + \mu' = 1$) are constants and $i = \sqrt{-1}$. $z = x + iy$ is a complex variable so that this equation represents a pair of coupled real second order differential equations with two-point boundary conditions.

1) The Green's function for the restricted three body orbits

The Green's function for the operator

$$L_2 \equiv \frac{d^2}{dt^2} + 2i\frac{d}{dt} - \gamma^2 , \tag{8}$$

with boundary conditions

$$G(a,s) = G(b,s) = 0 , \tag{9}$$

can be written as

$$G(t,s) = \frac{1}{W(s)} \begin{cases} v(t-a)\ v(s-b), & t \le s, \\ v(t-b)\ v(s-a), & t \ge s, \end{cases} \tag{10}$$

where

$$W(s) = \dot{v}(s-a)\ v(s-b) - \dot{v}(s-b)\ v(s-a). \tag{11}$$

(The Wronskian is not constant if L_2 is not self-adjoint.)

The function v is a solution of $L_2 v = 0$; $v(0) = 0$; i.e., $v(t) = e^{\alpha t} - e^{\beta t}$; $\alpha,\beta = -i(1 \pm \sqrt{1-\gamma^2})$. The parameter γ is artificially introduced. Its value can be adjusted to control the rate of convergence of the iteration.

Since the boundary conditions are not zero, the integral equation form of (6) is

$$z(t) = V(t) + \int_a^b G(t,s) f(z(s)) ds. \tag{12}$$

Here G is the Green's function given by (10) above, V(t) is the particular solution satisfying $L_2 V = 0$, and the boundary conditions $V(a) = z_a$, $V(b) = z_b$, i.e.,

$$V(t) = \frac{v(t-a)}{v(b-a)} z_b + \frac{v(t-b)}{v(a-b)} z_a. \tag{13}$$

2) <u>The computer program</u>

A computer program was written in Fortran for the UNIVAC 1108 to integrate the equation

$$z^{n+1}(t) = V(t) + \int_a^b G(t,s) f(z^n(s)) ds, \tag{14}$$

where V, G and f are given in (13), (10), and (7) respectively. Starting with an initial guess of $z^0(t)$, the program iterates to find successive $z^n(t)$, stopping when the maximum difference in two consecutive iterations drops below a prescribed threshold.

The function $z^n(t)$ is approximated by constructing a table of its values on the interval [a,b] and using cubic splines [24-27] to interpolate for the in-between points. These splines are also used to do the quadrature.

A relaxation parameter was also introduced to help convergence; that is, instead of (14), one uses

$$z^{n+1}(t) = (1-\omega)z^n(t) + \omega\left(V(t) + \int_a^b G(t,s)f^n(z(s))ds\right). \qquad (15)$$

where ω is the relaxation parameter. The case $\omega < 1$ is referred to as under-relaxation; the case $\omega > 1$ is referred to as over-relaxation. This gives in effect two parameters, γ and ω, to be adjusted to speed and control convergence.

3) Results

Figure 2 shows a typical example of the sequence of approximations from this iteration. The boundary conditions are $z(0) = 1.2$, $z(3.06) = -1.5$. The system constants μ and μ' ($\mu + \mu' = 1$) characterize the earth moon system, $\mu = 0.012277471$; here $\gamma = 0.95$ and $\omega = 0.5$. The initial orbit is marked with an I, and the final orbit with an F.

One notes here the dramatic and rather violent departure from the initial approximation (the first iteration leaves the page for most of the orbit). However, succeeding iterates come back on the page and quickly settle down to an almost circular uniform speed orbit.

Figure 2

Figure 2 shows the convergence of the iteration scheme for a rendezvous type orbit (fixed time between two fixed points) of the restricted three body system.

The three body parameters are chosen so that the earth-moon system is represented ($\mu \approx 0.012$). The orbit is represented in the rotating reference frame, normalized to unit angular velocity (period = 2π) and unit earth-moon distance. The length of the dashes is proportional to the speed in that part of the orbit. The time for the orbit is 3.06.

The earth, moon, initial and final orbits are indicated with the letters E, M, I, F respectively.

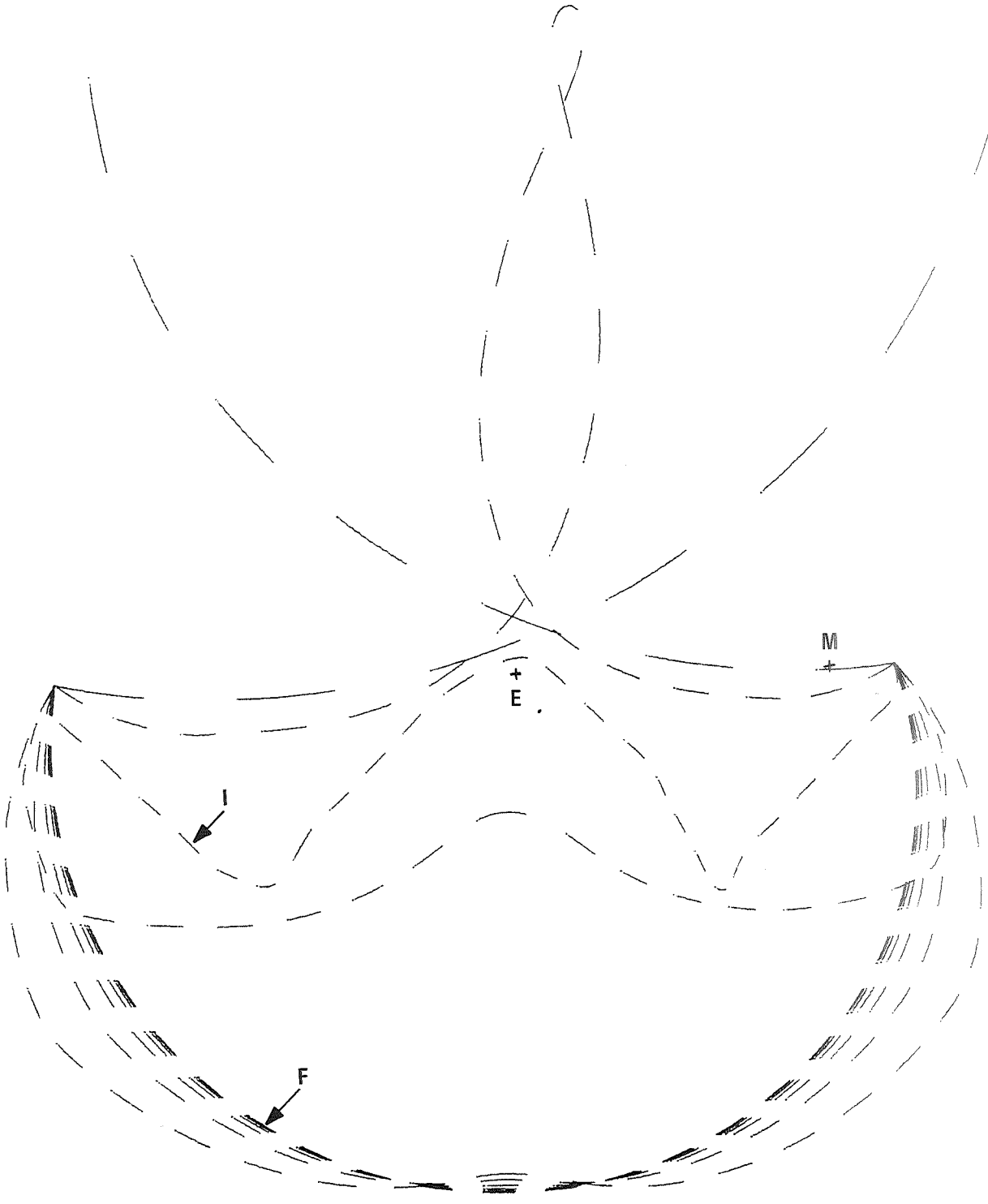

Figure 2

There is an orbit satisfying the boundary conditions in the neighborhood of the initial trial solution, but it is either highly unstable for this iterative scheme or else the initial guess was not close enough to have been in the stable region.

This behavior is typical of this particular problem, that is, orbits having a fixed time between two fixed points in the neighborhood of the earth-moon system. Only those orbits that were very smooth appeared to be stable. The more complex orbits between the same two points appeared to be unstable for this iterative scheme.

D. Arenstorf Orbits

A third group of problems is associated with finding periodic orbits of the restricted three body system [29]. Here one uses the pair of coupled differential equations

$$\left(\frac{d^2}{dt^2} - \gamma^2\right) x(t) = -f_1(\dot{y}(t), x(t), y(t)), \tag{16a}$$

$$\left(\frac{d^2}{dt^2} - \gamma^2\right) y(t) = -f_2(\dot{x}(t), x(t), y(t)), \tag{16b}$$

where

$$f_1(\dot{y}, x, y) = -2\dot{y} + \frac{\mu'(x+\mu)}{r^3} + \frac{\mu(x-\mu')}{r'^3} + (\gamma^2 - 1)x, \tag{17a}$$

$$f_2(\dot{x}, x, y) = 2\dot{x} + \frac{\mu' y}{r^3} + \frac{\mu y}{r'^3} + (\gamma^2 - 1)y, \tag{17b}$$

$$r \equiv \sqrt{(x+u)^2 + y_2} \quad , \quad r' \equiv \sqrt{(x-\mu')^2 + y^2} \quad ,$$

The boundary conditions for an orbit of period T are

$$\dot{x}(0) = 0, \; y(0) = 0,$$

$$\dot{x}(T/2) = 0, \; y(T/2) = 0.$$

Here γ, μ and μ' $(\mu + \mu' = 1)$ are constants.

1) The Green's functions

The Green's functions for the operator $\frac{d^2}{dt^2} - \gamma^2$ are

$$G_1(t,s) = \frac{1}{\gamma \sinh(\gamma(b-a))} \begin{cases} \cosh(\gamma(t-a))\cosh(\gamma(b-s)), & t \le s, \\ \cosh(\gamma(b-t))\cosh(\gamma(s-a)), & t \ge s, \end{cases} \tag{18a}$$

and

$$G_2(t,s) = \frac{1}{\gamma \sinh(\gamma(b-a))} \begin{cases} \sinh(\gamma(t-a))\sinh(\gamma(b-s)), & t \le s, \\ \sinh(\gamma(b-t))\sinh(\gamma(s-a)), & t \ge s. \end{cases} \tag{18b}$$

The first of these satisfies the boundary conditions

$$\frac{d}{dt}G_1(t,s)\Big|_{t=a} = \frac{d}{dt}G_1(t,s)\Big|_{t=b} = 0 , \tag{19a}$$

the second, the conditions

$$G_2(a,s) = G_2(b,s) = 0. \tag{19b}$$

In terms of these Green's functions, equations (16) become

$$x(t) = \int_a^b G_1(t,s) f_1(\dot{y}(s), x(s), y(s)) ds, \tag{20a}$$

$$y(t) = \int_a^b G_2(t,s) f_2(\dot{x}(s), x(s), y(s)) ds. \tag{20b}$$

The parameter γ in these equations is an artificially introduced parameter whose value can be adjusted to control the rate of convergence of the iteration. Values of γ in the neighborhood of 0.95 to 1.00 were generally satisfactory.

2) The computer program

A computer program was written (in Fortran for the UNIVAC 1108) to integrate the pair of equations

$$x^{n+1}(t) = \int_a^b G_1(t,s) f_1{}^n(s)\,ds, \tag{21a}$$

$$y^{n+1}(t) = \int_a^b G_2(t,s) f_2{}^n(s)\,ds, \tag{21b}$$

$$f_1{}^n(s) \equiv f_1(\dot{y}^n(s), y^n(s), x^n(s)),$$

$$f_2{}^n(s) \equiv f_2(\dot{x}^n(s), x^n(s), y^n(s)),$$

where the G's and f's are given in (18) and (17). Starting with initial guesses of x^o and y^o, the program iterates to find successive x^n and y^n, stopping when the maximum difference in two consecutive iterations drops below a prescribed threshold.

The functions $x^n(t)$ and $y^n(t)$ are approximated by constructing tables of their values over the interval $[a,b]$ and using cubic splines to interpolate for the in-between points. These splines are also used to do the quadrature over each interval and to approximate $\dot{x}$ and $\dot{y}$.

A relaxation parameter was also introduced to help convergence; that is, instead of (21), one uses

$$x^{n+1} = (1-\omega)x^n + \omega\int_a^b G_1(t,s) f_1{}^n(s)\,ds, \tag{21'a}$$

$$y^{n+1} = (1-\omega)y^n + \omega\int_a^b G_2(t,s) f_2{}^n(s)\,ds, \tag{21'b}$$

where ω is the relaxation parameter. The case $\omega < 1$ is referred to as under-relaxation; $\omega > 1$ is referred to as over-relaxation. This gives two parameters ω and γ to control or speed convergence, and various strategies were used to hunt for the optimum value of ω.

3) Results

Figure 3 shows the results of one of the apparently successful hunts for an Arenstorf orbit. Figure 3a shows the initial orbit, 3b the successive iterations, and 3c the final orbit. The system parameters here are for the earth moon system ($\mu \approx 0.012$) with $\gamma = 0.95$ and $\omega \approx 0.12$.

One notes that convergence appears to take place and that the final orbit is not too different from the initial guess. This convergence may be an illusion, and, if the iterations were allowed to continue, a very slow drifting away from 3c would eventually take place.

The difficulty is that with these boundary conditions the solutions are densely packed. That is, not only are there multiple solutions, but in every neighborhood of a periodic orbit there is another (having the same period). If the iterative scheme is allowed to continue, drifting around among these solutions continues until some especially stable solution is found. This behavior appears in other experiments (Figure 4, for example).

Figure 4 also shows an attempt to find a periodic orbit. The orbit parameters and initial trial orbit are exactly the same as in Figure 3. The only difference in the iteration scheme is that a slightly different method of hunting on ω was used. This sequence of orbits in Figure 4 has not converged yet, and if it is going to converge, it is clearly not going to be the same orbit as Figure 3c.

Figure 3

Figure 3 shows the convergence of the iteration scheme for a periodic, or Arenstorf, orbit of the restricted three body system. Figure 3a is the initial guess at the orbit, Figure 3b shows the sequence of orbits obtained by the iteration scheme, and Figure 3c is the final orbit to which the iteration appears to have converged.

The three body parameters are chosen so that the earth-moon system is represented ($\mu \approx 0.012$). The orbit is represented in the rotating reference frame, normalized to unit angular velocity (period = 2π), and unit earth-moon distance. Only half the orbit is shown, the other half is symmetric with that given. The length of the dashes is proportional to the speed in that part of the orbit. The full period of this orbit is 6.12.

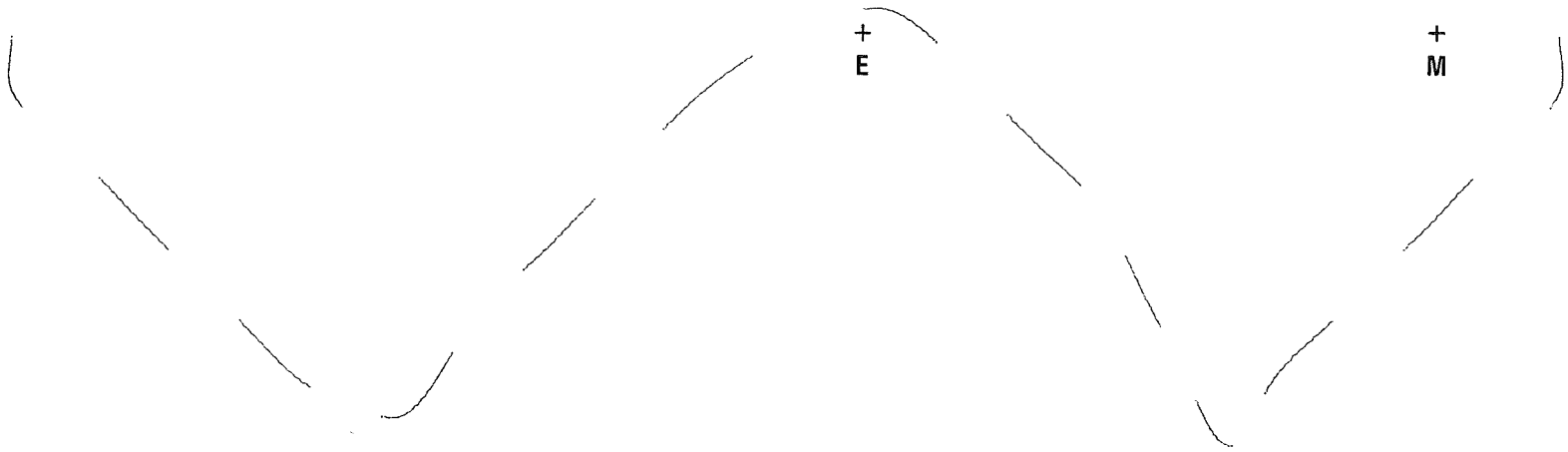

Figure 3a

Figure 3b

Figure 3c

Figure 4

Figure 4 shows a sequence of iterates in search of a periodic orbit of the restricted three body system.

The three body parameters are chosen so that the earth-moon system is represented ($\mu \approx 0.012$). The orbit is represented in the rotating reference frame, normalized to unit angular velocity (period = 2π), and unit earth-moon distance. Only half the orbit is shown; the other half is symmetric with that given. The length of the dashes is proportional to the speed in that part of the orbit.

The sequence of iterations has not yet converged.

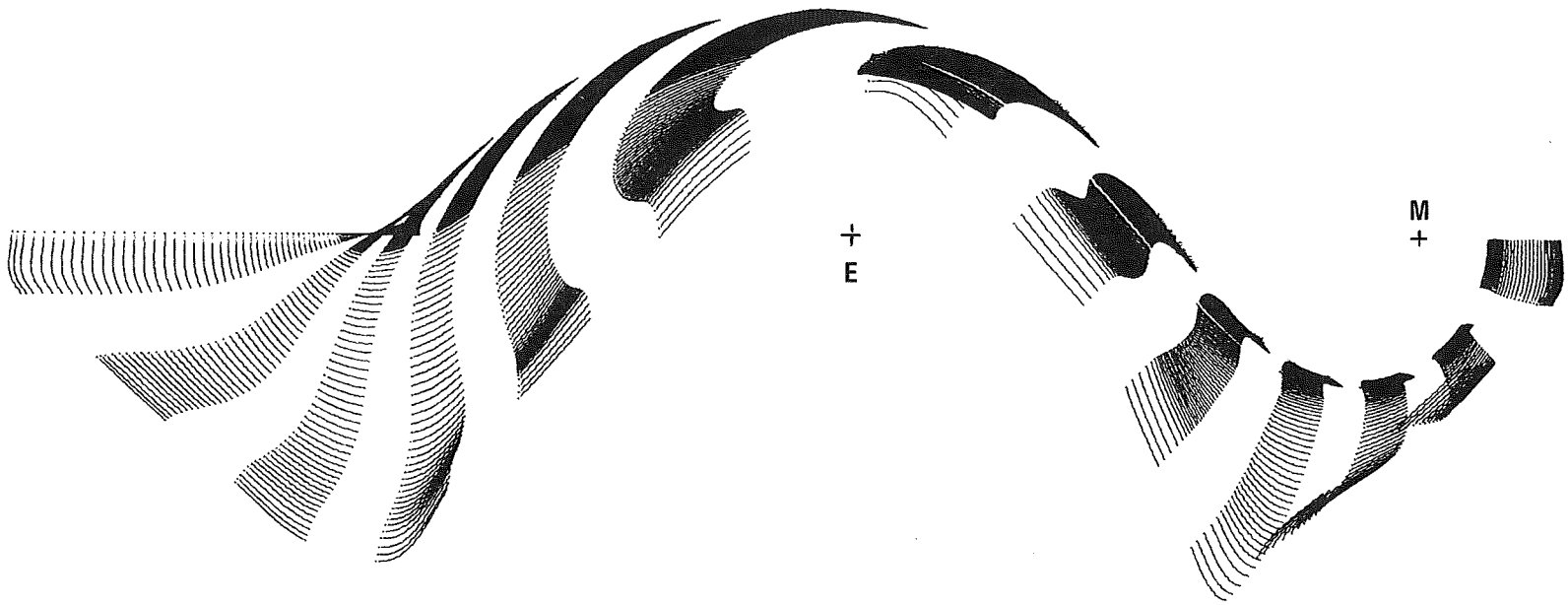

Figure 4

This example shows the difficulties associated with finding a particular periodic orbit. Because of the dense packing of these orbits, the final orbit to which convergence takes place depends on the details of the iteration scheme, such as ω, the value of the relaxation parameter, or the method of searching for an optimum ω.

Experiments of this type suggest that the most stable of the periodic orbits are those showing the greatest radii of curvature, or that have the fewest axis crossings, or that stay as far as possible from the singular points at E and M.

V. RESULTS AND CONCLUSIONS

A. Summary of Results

The primary result of this study is to show that the Green's function method of solving the two-point boundary value problem can be an effective tool in numerical work.

There is a straightforward prescription for producing the Green's function for both the single equation and for the system of equations case. It can be given in terms of solutions to sets of initial value problems which in turn can be generated to arbitrarily high order and accuracy by standard techniques such as Runge-Kutta, Adams, or other methods.

For linear differential equations the solution can be given directly in terms of the integral over the Green's function. For the nonlinear case, the Green's function provides an iterative scheme only. Convergence must be investigated in each individual case. There exists a literature on convergence theorems for a variety of classes of problems, but even when convergence cannot be guaranteed a priori, the method can often be used when combined with a relaxation method or other devices.

For single second order ordinary differentials the work involved in finding the Green's function can be considered nominal. For systems of such equations the work involved goes up as the square of the number of equations and may be considered excessive if the number of equations is large. In this case, a technique of splitting the original differential operator into a diagonal and off-diagonal component can be used. The Green's function for the diagonal component is then just the set of Green's functions for the individual (uncoupled) diagonal elements and the work

involved is only linear in the number of equations. The solution now involves a sequence of iterations, even in the linear case, and raises additional questions of convergence, but this could still turn out to be less work than finding the entire Green's function for the original matrix operator, especially if the problem is nonlinear, and iteration will be required anyway.

The particular numerical experiments carried out involved problems for which there were multiple solutions or for which the standard existence, uniqueness, and convergence theorems were not applicable. A search for orbits of the restricted three body system was investigated for both the rendezvous type orbits and periodic orbits. The orbits showing most stability with respect to the iteration scheme were those having the largest radii of curvature or that stayed farthest from the singular points. Searches for particular periodic orbits did not prove fruitful, since these solutions are densely packed. The problem of singling out a special one by this method needs further investigation.

B. Recommendations for Further Study

Continued work is needed in the theoretical area of convergence and stability of iterative methods for the two point boundary value problem. In the cases where multiple solutions exist, methods need to be developed to determine which solutions are locally stable, which are more stable than others and what are the regions of convergence or stability. These investigations could also examine the role of the relaxation parameter in the convergence process.

In the area of numerical experiments, more work should be done in comparing the Green's function method directly to the shooting method of solving the two-point boundary problem to see which takes less computer time, storage space, etc.

There is another way in which the Green's function method can be used, but there has been little or no numerical experimentation undertaken. The theory is reasonably straightforward and goes as follows:

Let L_2 be a second order linear differential operator and the function u satisfy the equation

$$L_2 u = f(u) \tag{1}$$

with zero end point boundary conditions, $u(a) = u(b) = 0$. Assume that $f(u)$ is a reasonably well behaved function, that $\frac{\partial}{\partial u} f(u)$ exists and can be computed for functions u in some neighborhood of the solutions to (1), and let $\bar{u}$ be a solution in this neighborhood. Then

$$\left(L_2 - \frac{\partial f}{\partial \bar{u}}(\bar{u})\right) u = f(u) - \frac{\partial f}{\partial \bar{u}}(\bar{u}) u,$$

or

$$\bar{L}_2 u = f(u) - \bar{f}' u,$$

where $\bar{L}_2 = L_2 - \bar{f}'$, $\bar{f}' = \frac{\partial f}{\partial \bar{u}}(\bar{u})$. If there now exists a Green's function for $\bar{L}_2$, say $\bar{G}$, then

$$u = -\bar{G}(f(u) - \bar{f}'u).$$

One now looks at the iterative equation

$$u^{n+1} = -\bar{G}^n\left(f(u^n) - \bar{f}'^n u^n\right), \tag{2}$$

where

$$\bar{f}'^n = \frac{\partial f}{\partial \bar{u}}(\bar{u})\Big|_{\bar{u} = u^n}, \qquad \bar{L}_2^n = L_2 - \bar{f}'^n,$$

and $\bar{G}^n$ is the Green's function for $\bar{L}_2^n$.

The iterative scheme (2) can be shown to converge quadradically. That is, it is always locally stable provided that the indicated entities exist. The system (2) is analogous to the Newton-Raphson method.

While this method has the advantage of being locally stable, it has the disadvantage of requiring the recalculation of $\bar{G}^n$, the Green's function, at every step. Whether this is practical or not needs to be determined by numerical experiments.

A technique similar to (2) above exists if f also depends on the derivatives of u; i.e., if f = f(u,u').

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