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ON A CRITERION FOR DETERMINING THE COMPLETE CONTINUITY OF URYSOHN'S INTEGRAL OPERATOR

by T. Nurekenov

Translated from the Russian by
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Abstract

The original of this paper appeared in Izvestia Akademii Nauk Kazahskoi SSR, Seria Fiziko-Matematiceskih Nauk, Matematika i Mehanika 15, 1963. The author considers a nonlinear integral operator

$$Ax(t) = \int_a^b K(t,s,x(s))ds$$

from L_p to L_q. By use of a result, herein called Helly's theorem, which is actually related to Helly's Principle of Choice, he proves the complete continuity of this operator under mild restrictions on the kernel. Suggestions are made about application of the theorem to other types of nonlinear equations.

ON A CRITERION FOR DETERMINING THE COMPLETE CONTINUITY OF URYSOHN'S INTEGRAL OPERATOR*

by

T. Nurekenov

In the present work we investigate the integral operator

$$(1) \qquad Ax(t) = \int_a^b K(t,s,x(s))ds,$$

where the kernel $K(t,s,u)$ is a function of the variables t,s,u; $t,s \in [a,b]$ and $-\infty < u < \infty$, which is measurable for all t,s and continuous in u for almost all t,s.

Various criteria for determining the continuity and complete continuity of this operator are known [1,2,3]. It turns out that the analysis of the operator (1) is considerably simplified if the kernel $K(t,s,u)$ is monotonic in the variable t.

In what follows, we give a new criterion by which the complete continuity of operator (1) may be determined. The fundamental result of this work may be formulated as follows:

Theorem. Let A be a continuous operator from L_p into L_q. Let $K(t,s,u)$ be a nondecreasing function in t for fixed s and u satisfying the inequality

$$(2) \qquad |K(t,s,u)| \leqslant K_1(t,s,u).$$

Assume that the operator B defined by

$$(3) \qquad Bx(t) = \int_a^b K_1(t,s,x(s))ds$$

*Russian original appeared in IZVESTIA AKADEMII NAUK KAZAHSKOI SSR, Seria Fiziko-Matematiceskih Nauk, Matematika i Mehanika 15, 3, 1963,

is a completely continuous operator from L_p into L_q. Then the operator A is completely continuous.

For the proof of this theorem we shall need the following

<u>Lemma</u>. Let $F=\{\phi(t)\}$ be an infinite family of nondecreasing functions defined on the interval $[a,b]$. Then there exists in F a sequence $\{\phi_k(t)\}_{k=1}^{\infty}$, which converges almost everywhere to some function $\phi(t)$, which may assume an infinite value on some set and which does not decrease in the boundary region.

<u>Proof of the lemma</u>. Choose in F some sequence of functions

$$\phi_1(t),\phi_2(t),\ldots,\phi_n(t),\ldots \tag{4}$$

From this sequence, $\{\phi_n\}_{n=1}^{\infty}$, form another sequence of functions $\{\phi_n^{(1)}\}_{n=1}^{\infty}$ as follows:

$$\phi_n^{(1)}(t) = \begin{cases} \phi_n(t), \text{ if } |\phi_n(t)| < 1 \\ \\ \operatorname{sgn} \phi(t), \text{ if } |\phi_n(t)| \geqslant 1. \end{cases} \tag{5}$$

By Helly's theorem [4] we may choose a subsequence of functions $\Phi_1 = \{\phi_{n_i}^{(1)}\}_{i=1}^{\infty}$, which converges everywhere to the nondecreasing function $\phi^{(1)}$.

Consider now the subsequence of functions $\{\phi_{n_i}\}_{i=1}^{\infty}$ from which we shall construct a sequence of "truncated" functions $\{\phi_{n_i}^{(2)}\}_{i=1}^{\infty}$ as follows:

$$\phi_{n_i}^{(2)}(t) = \begin{cases} \phi_{n_i}(t), \text{ if } |\phi_{n_i}(t)| < 2 \\ \\ 2\cdot\operatorname{sgn} \phi_{n_i}, \text{ if } |\phi_{n_i}(t)| \geqslant 2. \end{cases} \tag{6}$$

Applying Helly's theorem to the sequence $\{\phi_{n_i}^{(2)}\}_{i=1}^{\infty}$, we obtain the subsequence $\Phi_2 = \{\phi_{n_{i_k}}^{(2)}\}_{k=1}^{\infty}$ converging everywhere to the function $\phi^{(2)}(t)$.

Continuing this process, we may construct a countable set of converging subsequences $\Phi_1,\Phi_2,\Phi_3,\ldots,\Phi_n,\ldots$ converging respectively to the functions $\phi^{(1)}(t),\phi^{(2)}(t),\ldots$, i.e.,

$$\tag{7}\begin{aligned}
\Phi_1 &= \{\phi_{n_1}^{(1)}(t),\phi_{n_2}^{(1)}(t),\ldots,\phi_{n_k}^{(1)}(t),\ldots\},\\
\Phi_2 &= \{\phi_{n_{i_1}}^{(2)}(t),\phi_{n_{i_2}}^{(2)}(t),\ldots,\phi_{n_{i_k}}^{(2)}(t),\ldots\},\\
&\quad\cdot\ \cdot\ \cdot\\
\Phi_m &= \{\phi_{n_{i_{j\ldots 1}}}^{(m)}(t),\phi_{n_{i_{j\ldots 2}}}^{(m)}(t),\ldots,\phi_{n_{i_{j\ldots k}}}^{(m)}(t),\ldots\}.\\
&\quad\cdot\ \cdot\ \cdot
\end{aligned}$$

The sequence of functions $\{\phi^{(k)}(t)\}_{k=1}^{\infty}$ has the following properties:

1^0. $\phi^{(k)}(t)$ is nondecreasing in t for all k,

2^0. $\phi^{(k)}(t) \leqslant \phi^{(k+1)}(t)$, $k = 1,2,\ldots$.

It is easy to see that this sequence converges everywhere to the function $\phi(t)$, which is defined by the equation

$$\tag{8}\phi(t) = \sup\phi^{(k)}(t).$$

Obviously, the function $\phi(t)$ satisfies the conditions of the lemma.

It is not difficult to verify that the subsequence $\Phi = \{\phi_{n_1}(t),\phi_{n_2}(t),\ldots\}$ also converges to the function $\phi(t)$ at every point of the interval $[a,b]$. Thus, the lemma is proved.

We turn to the <u>proof of the theorem</u>. Let $x_1(t),x_2(t),\ldots,x_n(t)\ldots$ be a sequence of functions taken from the unit ball in L_p. Form the sequence

$\{\phi_n(t)\}_{n=1}^{\infty}$ by defining

$$\phi_n(t) = Ax_n(t),\ n = 1,2,\ldots \tag{9}$$

It is easy to see that, since the function $K(t,s,u)$ is nondecreasing in t, the functions $\phi_n(t)$ are nondecreasing in t for any n.

By the preceding lemma, from the sequence $\{\phi_n(t)\}_{n=1}^{\infty}$ we may choose some subsequence converging everywhere to the function $\phi(t)$. We shall also refer to this sequence as $\{\phi_n(t)\}_{n=1}^{\infty}$.

We now introduce the sequence $\{\psi_n(t)\}_{n=1}^{\infty}$, defined by

$$\psi_n(t) = Bx_n(t),\ n = 1,2,\ldots \tag{10}$$

Since the operator B is compact, we may assume that this sequence converges to some function $\psi(t)$ in the norm of the space L_q.

From inequality (2), we have

$$|\phi_n(t)| \leqslant \psi_n(t),\ n = 1,2,\ldots \tag{11}$$

Since $\psi_n(t)$ converges to $\psi(t)$ in the norm of L_p, it follows that the convergence is almost everywhere for some subsequence. Call this subsequence $\psi_n(t)$ once again. Therefore, the inequality

$$|\phi(t)| \leqslant \phi(t) \tag{12}$$

holds almost everywhere. From inequalities (11) and (12), it follows that the functions $\phi(t)$ and $\phi_n(t)$ are measurable and finite almost everywhere. By Lebesgue's theorem, the $\phi_n(t)$ converge to $\phi(t)$ in measure.

Consider now the sequence $\{U_n(t)\}_{n=1}^{\infty}$, defined by the equations

$$U_n(t) = |\phi_n(t)-\phi(t)|^q, \quad n = 1,2,\ldots \tag{13}$$

It is easy to see that $U(t)$ converges to zero and the functions $U_n(t)$ have equicontinuous and absolutely continuous integrals. By Vitali's theorem, they converge to zero in the norm of the space L_q. Q.E.D.

Corollary. If the kernel $K(t,s,u)$ of A is of bounded variation in t and also satisfies

$$|K(t,s,u)| \leqslant K_1(t,s,u),$$

where the operator

$$Bx(t) = \int_a^b K_1(t,s,x(s))ds$$

is a completely continuous operator from L_p into L_q, then the operator A is completely continuous, if it is continuous.

This theorem may be applied in the usual way to the investigation of nonlinear integral equations

$$x(t) = \int_a^b K(t,s,x(s);\lambda)ds$$

or integro-differential equations

$$\frac{\partial x(t,s)}{\partial t} = \int_a^b K(t,s,x(s);\lambda)ds + f(t).$$

Note also that the theorem may be generalized to integral operators in spaces of vector functions.

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