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A METHOD FOR APPROXIMATING THE EIGENVALUES OF NON SELF-ADJOINT ORDINARY DIFFERENTIAL OPERATORS

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A METHOD FOR APPROXIMATING THE EIGENVALUES OF NON SELF-ADJOINT ORDINARY DIFFERENTIAL OPERATORS

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§1. Introduction.

In this paper we develop a method for approximating the eigenvalues of a class of non self-adjoint ordinary differential operators.

There is an extensive literature dealing with the problem of finding upper and lower bounds for large classes of self-adjoint operators on a Hilbert space. This work covers many ordinary and partial differential operators of interest in applications. For the case of self-adjoint operators which are bounded below upper bounds are provided by the well-known Rayleigh-Ritz method. The method of intermediate problems introduced by Weinstein [36,37] provides lower bounds. This method was extended by Weinstein [38,39,40,41], Aronszajn and and Weinstein [5], Aronszajn [3,4], Weinberger [34,35], Bazley [7], Bazley and Fox [8,9], and Stenger [28]. For an exposition of these results and further references see Gould [19] and Fox and Rheinboldt [17]. Fichera [15,16] has developed a method for finding lower bounds which is an alternative to the method of intermediate problems.

Throughout the paper we consider an eigenvalue problem of the form

$$\begin{cases} \tilde{L}f = Lf + Af = \lambda f, \\ U_k f = 0, \quad k = 1,\cdots,m, \end{cases}$$

where

$$Lf = i^m f^{(m)} + cf, \quad c \text{ real},$$

the $m^{\underline{th}}$ order boundary conditions $U_k f = 0$ are self-adjoint relative to L, and A is any ordinary differential operator of order less than m. Thus we consider non self-adjoint operators which are lower order perturbations of the self-adjoint operator L. Assuming the eigenvalues and eigenvectors of L are known we consider the problem of approximating the eigenvalues of $\tilde{L}$. This problem is completely formulated in Section 2.

The main results of Section 3, Theorems 3.2 and 3.3, locate the eigenvalues of $\tilde{L}$ in circles centered at the eigenvalues of L. These basic location results are used in the formulation of the method for obtaining improvable approximations which is developed in Section 4. These results involve a method for estimating the perturbation of eigenvalues which depends on certain estimates for the resolvent operator of L. Variations of this method have been used by many authors to estimate eigenvalues. The result of Theorem 3.2 is proved by Schwartz [27] for the case where A is a bounded operator. Agmon [1] used such estimates, especially estimates similar to that given in Lemma 3.5 which depend on the higher dimensional version of the inequality in Lemma 3.3, to derive results on the angular distribution of eigenvalues of non self-adjoint elliptic partial differential operators. Clark [12] shows that the eigenvalues μ_k of $\tilde{L}$ are related to the eigenvalues λ_k of L by $|\lambda_k - \mu_k| = O(|\lambda_k|^{j/m})$ where j is the order of A.

The main results of this paper are in Section 4 in which the Galerkin method is applied to $\tilde{L}$ using the eigenvectors of L as a basis. We describe the method in the special case where the eigenvalues

$$\lambda_1 < \lambda_2 < \cdots$$

of L are all positive and distinct. For $n = 1,2,\cdots$ we use the eigenvalues of $\tilde{L}_n = P_n\tilde{L}|_{S_n}$, where S_n is the span of the first n eigenvectors of L and P_n is the projection onto S_n, as approximations for the eigenvalues of $\tilde{L}$. Assuming the circles discussed in the previous paragraph are mutually disjoint, there will be one eigenvalue of $\tilde{L}$ in each circle: let μ_p be the one in the $p^{\underline{th}}$ circle. Also, one of the eigenvalues of $\tilde{L}_n$ will be in the $p^{\underline{th}}$ circle if $p \leq n$; denote it by $\eta_p(n)$. Theorem 4.4 and the lemmas which follow it establish an estimate of the form

$$|\mu_p - \eta_p(n)| \leq \varepsilon_p(n), \quad n \geq p.$$

$\varepsilon_p(n)$ can be explicitly calculated in many cases.

Sections 5 and 6 contain the proof of two results from Section 4.

In Section 7 we present several examples which illustrate the results of the paper. For three of the examples we have computed the Galerkin eigenvalues and corresponding error estimates for $n = 1,\cdots,20$. Selected results from these computations are presented. A basic idea in the paper is that explicit estimates for the eigenvalues can be obtained if explicit values can be found for the constants in certain inequalities, such as (3.1), (4.3), and (4.4), which hold for all functions in some function space. For the examples treated in Section 7 values are given for the appropriate constants.

This paper is a continuation of previous work by the author [23,24]. In [23] a method was developed for the case of a bounded perturbation of an unbounded self-adjoint operator and in [24] the method was extended to cover, in its application to ordinary differential operators, perturbations of lower order which were subject to rather restrictive conditions. The purpose of this paper is to extend and improve this method so as to apply to a large class of ordinary differential operators which are lower order perturbations of the self-adjoint operator L. Example 7.1 was also treated in [24] and the numerical estimates obtained by the methods of this paper are substantially better than those reported there.

In [23] there is a discussion of other methods for approximating the eigenvalues of various types of non self-adjoint operators. In addition to these methods Vaǐnikko [31,32] has obtained the following result. Consider the eigenvalue problem

$$(1.1) \quad \begin{cases} \tilde{L}f = f^{(m)} + \sum_{k=0}^{m-1} a_k(\tau)f^{(k)} = \lambda b(\tau)f, \\ U_k f = 0, \qquad k = 1,\cdots,m \end{cases} .$$

Here $U_k f = 0$, $k = 1,\cdots,m$ are arbitrary boundary conditions and we assume each of the coefficients is in some Sobolev space: $b \in H_{r_0}$, $a_k \in H_{r_k}$, $k = 0,\cdots,m-1$. (See Section 2 for a definition of these spaces.) Let $\phi_1,\phi_2,\cdots$ be a sequence of polynomials of orders $m,m+1,\cdots$ which satisfy the boundary conditions and for $n = 1,2,\cdots$ define the n-dimensional eigenvalue problem

$$(1.2) \qquad \det(\tilde{L}\phi_\ell - \lambda M\phi_\ell, L\phi_k)_{\ell,k=1,\cdots,n} = 0,$$

where $Lf = f^{(m)}$ and $Mf = bf$. Let an eigenvalue μ^0 of (1.1) be a pole of order h of the resolvent of $\tilde{L}$ with respect to $M:(\tilde{L} - \lambda M)^{-1}$. For $n = 1,2,\cdots$ let μ^n be an eigenvalue of (1.2). Then if $\lim_{n\to\infty} \mu^n = \mu^0$ we have the asymptotic estimate

$$|\mu^n - \mu^0| = O(n^{-(r+r')/h}),$$

where $r = \min_{0\le k\le m-1} r_k$ and $r' = \min_{0\le k\le m-1} [r_k+m-k]$. See [33] for similar asymptotic error estimates for a more general class of operators.

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§2. Formulation of the Eigenvalue Problem.

For m a nonnegative integer let $H_m = H_m[a,b]$ denote the $m^{\underline{th}}$ Sobolev space on the interval $[a,b]$, i.e., the set of all complex valued functions f such that f has $m-1$ continuous derivatives, $f^{(m-1)}$ is absolutely continuous, and $f^{(m)}$ is square integrable on $[a,b]$. For $k = 0,1,\cdots,m$ let $|\cdot|_k$ be the seminorm on H_m defined by

$$|f|_k = \left(\int_a^b |f^{(k)}|^2 \, dt\right)^{1/2}.$$

Let $\|\cdot\|_m$ be the norm on H_m defined by

$$\|f\|_m = \left(\sum_{k=0}^m |f|_k^2\right)^{1/2}$$

and $(\cdot,\cdot)_m$ be the inner product defined by

$$(f,g)_m = \sum_{k=0}^m \int_a^b f^{(k)} \, \overline{g^{(k)}} \, dt.$$

H_m with this inner product is a Hilbert space and $H_0 = L_2$.

Let L be the $m^{\underline{th}}$ order differential operator defined by

$$Lf = i^m f^{(m)} + cf,$$

where c is a real constant and let

$$Af = \sum_{k=0}^j a_k(\tau) \, f^{(k)}$$

be any differential operator of order $j < m$ with continuous coefficients. Let $M = (M_{k\ell})$ and $N = (N_{k\ell})$ be complex $m \times m$ matrices such that $\text{rank}(M,N) = m$ and define the boundary forms $U_1, \cdots, U_m$ by

$$U_k f = \sum_{\ell=1}^{m} [M_{k\ell} f^{(\ell-1)}(a) + N_{k\ell} f^{(\ell-1)}(b)].$$

We wish to consider the eigenvalue problem determined by $\tilde{L} = L + A$ and the boundary forms $U_1, \cdots, U_m$, i.e., the problem of determining those complex values of λ for which there exists a nonzero function $f \in H_m$ which satisfies

$$\begin{cases} \tilde{L}f = \lambda f, \\ U_k f = 0, \qquad k = 1, \cdots, m \end{cases}$$

If we define V_m by

$$V_m = \{f \mid f \in H_m, U_k f = 0, k = 1, \cdots, m\}$$

the problem can be restated as that of finding values of λ for which there exist $f \in V_m$, $f \neq 0$ such that $\tilde{L}f = \lambda f$. For the remainder of the paper we will consider L and $\tilde{L}$ to be restricted to V_m. The operator L is formally self-adjoint. We now assume the boundary conditions are self-adjoint relative to L, i.e.,

$$\sum_{q=1}^{m} (-1)^{q-1} M_{\ell q} \overline{M}_{k,m+1-q} = \sum_{q=1}^{m} (-1)^{q-1} N_{\ell q} \overline{N}_{k,m+1-q}, \qquad \ell, k = 1, \cdots, m.$$

Then L will be a self-adjoint operator in the space L_2 with domain V_m:

$$(Lf, g)_0 = (f, Lg)_0$$

for all $f,g \in V_m$. The spectrum of L will consist of a countable set of real eigenvalues, each with geometric multiplicity less than or equal to m, which has no finite limit point. Numbering them in increasing order by magnitude and taking account of geometric multiplicities we have

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \nearrow \infty.$$

The eigenvectors $x_1, x_2, \cdots$ can be chosen to be orthonormal in L_2; they are complete in L_2. The spectrum of $\tilde{L}$ consists of eigenvalues, each with geometric multiplicity less than or equal to m, with no finite limit point.

Assuming the eigenvalues and eigenvectors of L are known we consider the problem of approximating the eigenvalues of $\tilde{L}$.

Throughout the paper we denote the spectrum, the resolvent set, and the resolvent of an operator B by $\sigma(B)$, $\rho(B)$, and $R_\lambda(B)$ respectively.

§3. Basic Location Theorems.

In this section we present a result which gives bounds on the perturbations of the eigenvalues of L when L is perturbed by the operator A. This result depends on certain estimates for the resolvent operator $R_\lambda(L) = (\lambda-L)^{-1}$.

Lemma 3.1. If $\lambda \in \rho(L)$ and $f \in L_2$, then

$$|R_\lambda(L)f|_0 \leq |f|_0 \quad (\mathrm{dist}(\lambda,\sigma(L)))^{-1}$$

where $\mathrm{dist}(\lambda,\sigma(L))$ denotes the distance from λ to $\sigma(L)$.

Proof. Since

$$R_\lambda(L)f = \sum_{k=1}^{\infty} (\lambda-\lambda_k)^{-1}(f,x_k)x_k$$

we see that

$$|R_\lambda(L)f|_0^2 \leq \sum_{k=1}^{\infty} |(\lambda-\lambda_k)^{-1}(f,x_k)|^2$$

$$\leq \max_k |(\lambda-\lambda_k)^{-2}| \sum_{k=1}^{\infty} |(f,x_k)|^2.$$

This gives the result.

Lemma 3.2. If $\lambda \in \rho(L)$ and $f \in L_2$, then

$$|R_\lambda(L)f|_m \leq \sup_k |(\lambda_k-c)(\lambda_k-\lambda)^{-1}| \, |f|_0.$$

Proof. For $f \in L_2$ we have

$$f = (\lambda-L)(\lambda-L)^{-1}f$$

$$= \lambda(\lambda-L)^{-1}f - L(\lambda-L)^{-1}f$$

$$= \lambda(\lambda-L)^{-1}f - i^m[(\lambda-L)^{-1}f]^{(m)} - c(\lambda-L)^{-1}f.$$

Hence

$$|R_\lambda(L)f|_m^2 = |(\lambda-c)(\lambda-L)^{-1}f - f|_0^2$$

$$= \sum_{k=1}^{\infty} |(\lambda_k-c)(\lambda_k-\lambda)^{-1}(f,x_k)|^2$$

$$\leq \sup_k |(\lambda_k-c)(\lambda_k-\lambda)^{-1}|^2 |f|_0^2.$$

We will need the following inequality. Its proof can be found in Agmon [2].

Lemma 3.3. There are constants γ and δ depending only on a, b, and m such that

$$|f|_k \leq \gamma |f|_0^{1-\frac{k}{m}} |f|_m^{\frac{k}{m}} + \delta |f|_0 \tag{3.1}$$

for all $f \in H_m$ and $k = 0,1,\cdots,m$.

Lemma 3.4. If $\lambda \in \rho(L)$ and $k = 0,1,\cdots,m$, then

$$|R_\lambda(L)f|_k \leq |f|_0 (\mathrm{dist}(\lambda,\sigma(L)))^{-1} \times$$

$$[\delta + \gamma(\mathrm{dist}(\lambda,\sigma(L)) \sup_q |(\lambda_q-c)(\lambda_q-\lambda)^{-1}|)^{k/m}]$$

for all $f \in L_2$ where γ and δ are the constants introduced in Lemma 3.3.

Proof. Using Lemmas 3.1, 3.2, and 3.3 we obtain

$$|R_\lambda(L)f|_k \leq \gamma |R_\lambda(L)f|_0^{1-\frac{k}{m}} |R_\lambda(L)f|_m^{\frac{k}{m}} + \delta |(\lambda-L)^{-1}f|_0$$

$$\leq |f|_0 (\mathrm{dist}(\lambda,\sigma(L))) \times$$

$$[\delta + \gamma(\mathrm{dist}(\lambda,\sigma(L)) \sup |(\lambda_q-c)(\lambda_q-\lambda)^{-1}|)^{k/m}].$$

Lemma 3.5. If $\lambda \in \rho(L)$ and $k = 0,\cdots,m$ then

$$\|R_\lambda(L)f\|_k \le |f|_0(\mathrm{dist}(\lambda,\sigma(L)))^{-1} \times$$
$$\{\sum_{\ell=0}^{k} [\delta + \gamma(\mathrm{dist}(\lambda,\sigma(L)) \sup_q |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|)^{\ell/m}]^2\}^{1/2}$$

for all $f \in L_2$.

Proof. This is an immediate consequence of Lemma 3.4.

Remark. The values of γ and δ in Lemma 3.3 are such that (3.1) holds for all $f \in H_m$. In Lemmas 3.4 and 3.5 we only used this inequality for functions of the form $(\lambda-L)^{-1}f$ for $f \in L_2$, i.e., for functions in V_m. Hence when considering a particular set of boundary conditions it may be possible to choose smaller values for γ and δ. See Section 7 for further remarks on the choice of γ and δ.

Lemma 3.6. If $f \in H_j$, where j is the order of A, and $\lambda \in \rho(L)$ then

$$\|R_\lambda(L)Af\|_j \le (\sum_{\ell=0}^{j} \max_{a\le t\le b} |a_\ell|^2)^{1/2} \|f\|_j (\mathrm{dist}(\lambda,\sigma(L)))^{-1} \times$$
$$\{\sum_{\ell=0}^{j} [\delta + \gamma(\mathrm{dist}(\lambda,\sigma(L)) \sup_q |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|)^{\ell/m}]^2\}^{1/2}.$$

Proof. This follows directly from Lemma 3.5 and the inequality

$$|Af|_0 \le (\sum_{\ell=0}^{j} \max |a_\ell|^2)^{1/2} \|f\|_j$$

for all $f \in H_j$.

Let

$$h_j(\lambda) = (\mathrm{dist}(\lambda,\sigma(L)))^{-1} \times$$

$$\{\sum_{\ell=0}^{j} [\delta + \gamma(\mathrm{dist}(\lambda,\sigma(L)) \sup_q |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|)^{\ell/m}]^2\}^{1/2}$$

and

$$\beta = (\sum_{\ell=0}^{j} \max|a_\ell|^2)^{1/2}.$$

Theorem 3.1. If λ is an eigenvalue of $\tilde{L}$ then either

$$\beta h_j(\lambda) \geq 1 \tag{3.2}$$

or $\lambda \in \sigma(L)$.

Proof. Suppose, for the sake of contradiction, that $\beta h_j(\lambda) < 1$ and $\lambda \in \rho(L)$. From Lemma 3.6 we have

$$||R_\lambda(L)Af||_j \leq \beta h_j(\lambda)||f||_j$$

for all $f \in H_j$. This, together with $\beta h_j(\lambda) < 1$, implies that $(\lambda-L)^{-1}A\colon H_j \longrightarrow H_j$ has norm less than 1. Hence $I_{H_j} - (\lambda-L)^{-1}A$ is invertible. Thus the restriction of this operator to V_m will be one-to-one. Since

$$\lambda I_{V_m} - \tilde{L} = \lambda I_{V_m} - L - A|_{V_m}$$

$$= (\lambda I_{V_m} - L)[I_{V_m} - (\lambda I_{V_m} - L)^{-1}A|_{V_m}]$$

this implies that $\lambda - \tilde{L}$ is one-to-one and hence that λ is not an eigenvalue of $\tilde{L}$ which contradicts the hypothesis. This completes the proof.

Inequality (3.2) is the basic result of this section. We will now show more explicitly what it implies about the location of the eigenvalues of $\tilde{L}$. Let p be such that $\mathrm{dist}(\lambda,\sigma(L)) = |\lambda-\lambda_p|$. Let λ_p^+ be the next element in the sequence $\{\lambda_p\}$ to the right of λ_p if such an element exists and let λ_p^- be the next element to the left if such an element exists. With p chosen in this way (3.2) implies

$$(3.3) \qquad |\lambda-\lambda_p| \leq \beta\{\textstyle\sum_{\ell=0}^{j} [\delta + \gamma(|\lambda-\lambda_p| \sup|(\lambda_q-c)(\lambda_q-\lambda)^{-1}|)^{\ell/m}]^2\}^{1/2}.$$

If

$$(3.4) \qquad \sup |(\lambda_q-c)(\lambda_q-\lambda)^{-1}| = |(\lambda_p-c)(\lambda_p-\lambda)^{-1}|$$

this becomes

$$(3.5) \qquad |\lambda-\lambda_p| \leq \beta\{\textstyle\sum_{\ell=0}^{j} [\delta + \gamma|\lambda_p-c|^{\ell/m}]^2\}^{1/2}.$$

In case (3.4) is not satisfied we derive a different bound for $\sup |(\lambda_q-c)(\lambda_q-\lambda)^{-1}|$. It is clearly sufficient to consider those values of q which belong to

$$A = \{q \mid |\lambda_q-c| > |\lambda_p-c|\}.$$

Let $B = \{q \mid \lambda_q < \lambda_p\}$ and $C = \{q \mid \lambda_q > \lambda_p\}$. Suppose $q \in A \cap B$. Then

$$|(\lambda_q-c)(\lambda_q-\lambda)^{-1}| \leq |\lambda_q-c|\,|\lambda_q - (\lambda_p+\lambda_p^-)/2|^{-1}.$$

A study of the function $|x-c|\,|x - (\lambda_p+\lambda_p^-)/2|^{-1}$ for

$x < (\lambda_p + \lambda_p^-)/2$ shows that

$$|(\lambda_q - c)(\lambda_q - \lambda)^{-1}| \leq \max(1, 2(c-\lambda_p^{-1})(\lambda_p - \lambda_p^-)^{-1}).$$

Similarly we find that

$$|(\lambda_q - c)(\lambda_q - \lambda)^{-1}| \leq \max(1, 2(\lambda_p^+ - c)(\lambda_p^+ - \lambda_p)^{-1})$$

for $q \in A \cap C$. Combining these estimates we obtain

$$\text{(3.6)} \quad \sup|\lambda_q - c||\lambda_q - \lambda|^{-1} \leq \begin{cases} \max(1, 2(c-\lambda_p^-)(\lambda_p - \lambda_p^-)^{-1}) \\ \qquad \text{if } \lambda_p^+ \text{ does not exist,} \\ \max(1, 2(\lambda_p^+ - c)(\lambda_p^+ - \lambda_p)^{-1} \\ \qquad \text{if } \lambda_p^- \text{ does not exist,} \\ \max(2(c-\lambda_p^-)(\lambda_p - \lambda_p^-)^{-1}, 2(\lambda_p^+ - c)(\lambda_p^+ - \lambda_p)^{-1}) \\ \qquad \text{if } \lambda_p^- \text{ and } \lambda_p^+ \text{ both exist.} \end{cases}$$

$$\equiv \quad g(p).$$

In the frequently occurring special case where $0 < \lambda_1 < \lambda_2 < \cdots$ and $c \geq 0$ this bound reduces to

$$g(p) = \begin{cases} \max(1, 2(\lambda_2 - c)(\lambda_2 - \lambda_1)^{-1}, & p = 1, \\ \max(2(c-\lambda_{p-1})(\lambda_p - \lambda_{p-1})^{-1}, 2(\lambda_{p+1} - c)(\lambda_{p+1} - \lambda_p)^{-1}), \\ & p \geq 2. \end{cases}$$

If $c = 0$ we have $g(p) = 2\lambda_{p+1}(\lambda_{p+1} - \lambda_p)^{-1}$. Combining (3.3) and (3.6) we get

$$\text{(3.7)} \quad |\lambda - \lambda_p| \leq \beta\{\sum_{\ell=0}^{j} [\delta + \gamma(|\lambda - \lambda_p| g(p))^{\ell/m}]^2\}^{1/2}.$$

We can solve (3.7) for $|\lambda - \lambda_p|$, i.e., (3.7) implies

$$|\lambda-\lambda_p| \leq \tau_p$$

where τ_p is the unique positive solution of

$$\tau = \beta\{\sum_{\ell=0}^{j} [\delta + \gamma(\tau g(p))^{\ell/m}]^2\}^{1/2}.$$

Let

$$(3.8) \qquad r_p = \max(\beta\{\sum_{\ell=0}^{j} [\delta + \gamma|\lambda_p - c|^{\ell/m}]^2\}^{1/2}, \tau_p).$$

These results are summarized in

Theorem 3.2. If λ is an eigenvalue of $\tilde{L}$ then for some p we have

$$|\lambda-\lambda_p| \leq r_p.$$

Thus we have shown that the eigenvalues of $\tilde{L}$ are contained in $\bigcup_{\ell=1}^{\infty} C_\ell$, where

$$C_\ell = \{\lambda \mid |\lambda-\lambda_\ell| \leq r_\ell\}.$$

It will be useful to have an explicit upper bound for τ_p. First suppose

$$2g(p)\beta(\delta+\gamma) \leq 1.$$

Then

$$(g(p))^{-1} \geq \beta\{\sum_{\ell=0}^{j} [\delta + \gamma(\tau g(p))^{\ell/m}]^2\Big|_{\tau=(g(p))^{-1}}$$

which implies $\tau_p \leq (g(p))^{-1}$. Hence

$$\tau_p \leq \beta(\delta+\gamma)(1+j)^{1/2}.$$

Next suppose

$$2g(p)\beta(\delta+\gamma) > 1.$$

In this case we have $\tau_p > (g(p))^{-1}$ and hence

$$\tau_p < \beta(\delta+\gamma)(\tau_p g(p))^{j/m}(j+1)^{1/2}$$

which implies

$$\tau_p < [\beta(\delta+\gamma)(j+1)^{1/2}]^{m/(m-j)}(g(p))^{j/(m-j)}.$$

Combining these two estimates we get

(3.9)

$$\tau_p \leq \max(\beta(\delta+\gamma)(1+j)^{1/2}, [\beta(\delta+\gamma)(j+1)^{1/2}]^{m/(m-j)}(g(p))^{j/(m-j)}).$$

Next we wish to count the number of eigenvalues of $\tilde{L}$ in each circle C_ℓ; this result will be contained in Theorem 3.3. To this end we will need to consider the family of operators

$$L_t \equiv L + tA, \quad 0 \leq t \leq 1.$$

Clearly $L_0 = L$ and $L_1 = \tilde{L}$. If λ is an eigenvalue of L_t the null spaces $\lambda - L_t$, $(\lambda-L_t)^2, \cdots$ form an increasing sequence of linear manifolds. Theorem 3.1, applied to L_t, shows that not all complex numbers are eigenvalues of L_t. Thus for some value of μ, $(\mu-L_t)^{-1}$ is compact. Hence there is an integer α, called the ascent of $\lambda - L_t$, such that $(\lambda-L_t)^\alpha$ and $(\lambda-L_t)^{\alpha+1}$ have the same null space. The dimension of the null space of $(\lambda-L_t)^\alpha$ is finite and is called the algebraic multiplicity of λ. As is well-known, the algebraic multiplicity of an eigenvalue is at least as great as its geometric multiplicity and the two are equal if L_t is self-adjcint. If Γ is a simple closed rectifiable curve lying in $\rho(L_t)$ and containing several eigenvalues of L_t in its interior, the projection operator associated with L_t and the part of the spectrum of L_t within Γ is defined by

$$P(t) = \frac{1}{2\pi i} \int_\Gamma R_\lambda(L_t)\, d\lambda \ ;$$

this is a contour integral of the operator valued analytic function $R_\lambda(L_t)$ defined for $\lambda \in \rho(L_t)$. The dimension of the range of $P(t)$ is equal to the number of eigenvalues of L_t within Γ counted according to algebraic multiplicities. We will use this characterization of algebraic multiplicity in the proof of Theorem 3.3. A discussion of these ideas can be found in Taylor [30].

Lemma 3.7. Let S and T be two linear operators defined on a linear manifold of a Hilbert space. If $\lambda \in \rho(S) \cap \rho(T)$, then

a) $R_\lambda(T) - R_\lambda(S) = R_\lambda(S)(T-S)R_\lambda(S)[I + (S-T)R_\lambda(S)]^{-1}$,

b) $R_\lambda(T) = [I + R_\lambda(S)(S-T)]^{-1}R_\lambda(S)$, and

c) $$R_\lambda(T) - R_\lambda(S) = R_\lambda(S)(T-S)R_\lambda(S) + R_\lambda(S)(T-S)R_\lambda(S)(T-S)R_\lambda(S)[I+(S-T)R_\lambda(S)]^{-1}.$$

Proof. For $\lambda \in \rho(S) \cap \rho(T)$ we have

$$(3.10) \qquad R_\lambda(T) - R_\lambda(S) = R_\lambda(S)(T-S)R_\lambda(T),$$

$$\lambda - T = [I + (S-T)R_\lambda(S)](\lambda-S),$$

and

$$\lambda - T = (\lambda-S)[I + R_\lambda(S)(S-T)],$$

and hence

$$(3.11) \qquad R_\lambda(T) = R_\lambda(S)[I + (S-T)R_\lambda(S)]^{-1},$$

and

$$(3.12) \qquad R_\lambda(T) = [I + R_\lambda(S)(S-T)]^{-1}R_\lambda(S).$$

Substitution of (3.11) into (3.10) gives (a), (3.12) is (b), and substitution of (a) in (3.10) gives (c).

Lemma 3.8. Let B be a Banach space. Suppose S is a bounded operator defined on B and suppose T is a closed operator with domain in B. If the domain of T contains the range of S, then TS is bounded.

Proof. TS is defined on all of B. Suppose $x_n \longrightarrow x$ and $TSx_n \longrightarrow y$, where convergence is with respect to the norm in the Banach space. Since S is continuous we have $Sx_n \longrightarrow Sx$. Since T is closed we have $T(Sx) = y$. This shows that TS is closed. This, together with the fact that TS is defined on all of B, shows that TS is bounded.

Lemma 3.9. Let P and Q be continuous not necessarily orthogonal projections on a Hilbert space such that $\|P-Q\| < \|P\|^{-1}$. Then the dimension of the range of P is not greater than the dimension of the range of Q.

Proof. Suppose the range of P has larger dimension than the range of Q. Since the dimension of the range of PQ is less than or equal to the dimension of the range of Q we see that the range of PQ is properly contained in the range of P. The range of P is closed since P is a continuous projection and the range of PQ is closed since the range of Q and hence that of PQ is finite dimensional. Thus there is a unit vector f in the range of P which is orthogonal to the range of PQ. In particular f is orthogonal to PQf and hence

$$\|f - PQf\|^2 = 1 + \|PQf\|^2 \geq 1.$$

Now $f = Pf$ and hence $f = P^2f$. Thus

$$\begin{aligned} 1 &\leq \|P^2f - PQf\| \\ &= \|P(Pf - Qf)\| \\ &\leq \|P\|\,\|P - Q\|, \end{aligned}$$

which contradicts the hypothesis. This completes the proof.

<u>Theorem 3.3</u>. Suppose q of the circles C_ℓ form a connected set C which does not intersect any of the other circles:

$$C = \bigcup_{k=1}^{q} C_{\ell_k} .$$

Then, counting according to algebraic multiplicities, there are q eigenvalues of $\tilde{L}$ in C.

<u>Proof</u>. Consider the operator

$$L_t = L + tA$$

for $0 \leq t \leq 1$. Theorem 3.2, applied to L_t instead of $\tilde{L}$, shows that the eigenvalues of L_t are contained in the circles

$$C_\ell^t \equiv \{\lambda \mid |\lambda-\lambda_\ell| \leq tr_\ell\}.$$

Here we have used the fact that if $\tau_{p,t}$ is the positive solution of

$$\tau = (t\beta)\{\textstyle\sum_{\ell=0}^{j} [\delta + \gamma(\tau g(p))^{\ell/m}]^2\}$$

then $\tau_{p,t} \leq t\tau_p$ if $t \leq 1$.

Let $C^*_{\ell_k}$ be obtained from C_{ℓ_k} by increasing the radii of the circles by a number r which is so small that $C^* = \bigcup_{k=1}^{q} C^*_{\ell_k}$ does not intersect C_j, $j \neq \ell_1,\cdots,\ell_q$. Now the number of eigenvalues of L_t in C^*, counted according to algebraic multiplicities, is equal to the dimension of the range of the projection

$$P(t) = \frac{1}{2\pi i} \int_{\partial C^*} R_\lambda(L_t)\, d\lambda.$$

For $0 \leq t,t' \leq 1$ we have

$$P(t) - P(t') = \frac{1}{2\pi i} \int_{\partial C^*} [R_\lambda(L_t) - R_\lambda(L_{t'})]\, d\lambda.$$

Using formula (a) of Lemma 3.7 we obtain

$$R_\lambda(L_t) - R_\lambda(L_{t'}) = R_\lambda(L_t)(t-t')AR_\lambda(L_t)[I + (t-t')AR_\lambda(L_t)]^{-1}.$$

Since $A\colon H_j \longrightarrow L_2$ is closed and the range of $R_\lambda(L_t)$ is contained in H_j we see, using Lemma 3.8, that $A(\lambda - L_t)^{-1}$ is bounded. Thus for $t - t'$ sufficiently small

$$\|P(t) - P(t')\| \leq \frac{1}{2\pi} \text{length } (\partial C^*)\, \|R_\lambda(L_t)\|\, |t-t'|\, \|AR_\lambda(L_t)\| \times$$
$$(1 - |t-t'|\, \|AR_\lambda(L_t)\|)^{-1},$$

where $\|\cdot\|$ denotes the operator norm associated with the L_2 norm. This inequality shows that $P(t)$ is continuous in t with respect to the operator norm.

The continuity of $P(t)$ together with Lemma 3.9 shows that the dimension of the range of $P(t)$ is continuous in t. Since the dimension is an integer we see that the dimension of the range of $P(t)$ is constant. Hence the number of eigenvalues of $\tilde{L} = L_1$ in C^* is equal to the number of eigenvalues of $L = L_0$ in C^*. Since L is self-adjoint it has q eigenvalues in C^*. Thus, since r is arbitrary, we have shown that there are q eigenvalues of $\tilde{L}$ in C^*.

<u>Corollary 1</u>. If $\bigcup_{k=1}^{\infty} C_k$ is not connected, then $\tilde{L}$ has at least one eigenvalue.

<u>Proof</u>. In this situation there will be an integer p such that $C_1 \cup \cdots \cup C_p$ is a connected set not intersecting any of the other circles. Theorem 3.3 applied to this set gives the result.

<u>Corollary 2</u>. If $\tilde{L}$ and the boundary operators U_k have real coefficients and the circle C_{ℓ_0} does not intersect the other circles, then the eigenvalue of $\tilde{L}$ in C_{ℓ_0} is real.

<u>Proof</u>. By Theorem 3.3 there is exactly one eigenvalue of $\tilde{L}$ in C_{ℓ_0}. If this eigenvalue is not real then its complex conjugate will also be an eigenvalue of $\tilde{L}$, will be in C_{ℓ_0}, and will be different from it. This contradicts the fact that there is only one eigenvalue in C_{ℓ_0}.

§4. Improvable Approximations to the Initial Eigenvalues of $\tilde{L}$.

Let $S_n = \mathrm{span}(x_1,\cdots,x_n)$ where $x_1,\cdots,x_n$ are the first n eigenvectors of L. For any operator B whose domain contains S_n define $B_n = (P_n B)|_{S_n}$ where P_n is the projection onto S_n. We now give several results which parallel the results of Section 3.

Lemma 4.1. If $\lambda \in \rho(L_n)$ and $f \in S_n$, then

$$|R_\lambda(L_n)f|_0 \leq |f|_0(\mathrm{dist}(\lambda,\sigma(L_n)))^{-1}.$$

Proof. The proof here is similar to that of Lemma 3.1.

Lemma 4.2. If $\lambda \in \rho(L_n)$ and $f \in S_n$, then

$$|R_\lambda(L_n)f|_m \leq \max_{1\leq k\leq n} |(\lambda_k-c)(\lambda_k-\lambda)^{-1}|\,|f|_0.$$

Proof. For $\lambda \in \rho(L_n)$ and $f \in S_n$ we have

$$\begin{aligned} f &= (\lambda-L_n)(\lambda-L_n)^{-1}f \\ &= \lambda(\lambda-L_n)f - i^m[(\lambda-L_n)^{-1}f]^{(m)} - c(\lambda-L_n)^{-1}f, \end{aligned}$$

and hence

$$\begin{aligned} |R_\lambda(L_n)f|_m^2 &= \sum_{k=1}^{n} |(\lambda_k-c)(\lambda_k-\lambda)^{-1}(f,x_k)|^2 \\ &\leq \max_k |(\lambda_k-c)(\lambda_k-\lambda)^{-1}|\ \ |f|_0^2. \end{aligned}$$

Lemma 4.3. If $\lambda \in \rho(L_n)$ and $k = 0,1,\cdots,m$, then

$$|R_\lambda(L_n)f|_k \leq |f|_0(\text{dist}(\lambda,\sigma(L_n)))^{-1} \times$$
$$[\delta + \gamma(\text{dist}(\lambda,\sigma(L_n)) \max_{1\leq q\leq n} |(\lambda_q - c)(\lambda_q-\lambda)^{-1}|)^{k/m}]$$

for all $f \in S_n$.

Proof. This follows immediately from Lemmas 3.3, 4.1, and 4.2.

Lemma 4.4. If $\lambda \in \rho(L_n)$ and $k = 0,1,\cdots,m$, then

$$\|R_\lambda(L_n)f\|_k \leq |f|_0(\text{dist}(\lambda,\sigma(L_n)))^{-1} \times$$
$$\{\textstyle\sum_{\ell=0}^k [\delta+\gamma(\text{dist}(\lambda,\sigma(L_n)) \max_{1\leq q\leq n} |(\lambda_q-c)(\lambda_q-\lambda)^{-1}|)^{\ell/m}]^2\}^{1/2}.$$

Proof. This follows directly from Lemma 4.3.

Lemma 4.5. If $\lambda \in \rho(L_n)$ and $f \in S_n$, then

$$\|R_\lambda(L_n)A_nf\|_j \leq (\textstyle\sum_{\ell=0}^j \max|a_\ell|^2)^{1/2}\|f\|_j(\text{dist}(\lambda,\sigma(L_n)))^{-1} \times$$
$$\{\textstyle\sum_{\ell=0}^j [\delta+\gamma(\text{dist}(\lambda,\sigma(L_n)) \max_{1\leq q\leq n} |(\lambda_q-c)(\lambda_q-\lambda)^{-1}|)^{\ell/m}]^2\}^{1/2},$$

where j is the order of A.

Proof. The proof is essentially the same as that of Lemma 3.6.

Let

$$h_{j,n}(\lambda) = (\text{dist}(\lambda,\sigma(L_n)))^{-1} \times$$
$$\{\textstyle\sum_{\ell=0}^j [\delta+\gamma(\text{dist}(\lambda,\sigma(L_n)) \max_{1\leq q\leq n} |(\lambda_q-c)(\lambda_q-\lambda)^{-1}|)^{\ell/m}]^2\}^{1/2}.$$

Theorem 4.1. If λ is an eigenvalue of $\tilde{L}_n$ then either

$$\beta h_{j,n}(\lambda) \geq 1$$

or $\lambda \in \sigma(L_n)$.

Proof. Suppose $\lambda \in \rho(L_n)$ and $\beta h_{j,n}(\lambda) < 1$. This, together with Lemma 4.5, implies that $(\lambda - L_n)^{-1}A_n$, considered as an operator on S_n with the j norm, has norm less than one. As in the proof of Theorem 3.1 this implies that λ is not an eigenvalue of $\tilde{L}_n$ which is a contradiction.

Using the same reasoning which led to Theorem 3.2 we obtain

Theorem 4.2. If λ is an eigenvalue of $\tilde{L}_n$ then for some p satisfying $1 \leq p \leq n$ we have

$$|\lambda - \lambda_p| \leq r_p.$$

Theorem 4.3. Suppose q of the circles $C_1, \cdots, C_n$ form a connected set C not intersecting the rest of the circles. Then, counting according to algebraic multiplicities, there are q eigenvalues of $\tilde{L}_n$ in C.

Proof. The concept of algebraic multiplicity is defined in this finite dimensional context just as in Section 3. The continuity argument used in the proof of Theorem 3.3 now applies here and gives the desired result.

For the remainder of the paper we assume the circles C_ℓ are mutually disjoint. Theorems 3.2 and 3.3 then assert that $\tilde{L}$ has a countable set of eigenvalues which are contained in the circles C_ℓ, one eigenvalue to each circle; denote the one in C_ℓ by μ_ℓ. Also, for n fixed, Theorem 4.3 asserts that the eigenvalues of $\tilde{L}_n$ are contained in the circles $C_1,\cdots,C_n$, one eigenvalue to each circle; denote the one in C_ℓ by $\eta_\ell(n)$. The eigenvalues $\eta_1(n),\cdots,\eta_n(n)$ will be called the n^{th} stage Galerkin eigenvalues of $\tilde{L}$.

We now derive estimates on the error which arises when $\eta_p(n)$, the p^{th} of the n^{th} stage Galerkin eigenvalues of $\tilde{L}$, is used as an approximation for μ_p, the p^{th} eigenvalue of $\tilde{L}$, $p = 1,\cdots,n$. Let y_p be a unit eigenvector of $\tilde{L}$ corresponding to μ_p and let y_p be orthogonally decomposed as follows: $y_p = y_p(n) + w_p(n)$, $y_p(n) \in S_n$, $w_p(n) \in S_n^\perp$. Let $z_k(n)$ be a unit eigenvector of $\tilde{L}_n$ corresponding to $\eta_k(n)$. The vectors $z_1(n),\cdots,z_n(n)$ span S_n.

<u>Lemma 4.6</u>. Let $\alpha_{p,1}(n),\cdots,\alpha_{p,n}(n)$ be the coefficients in the expansion of $y_p(n)$ in the basis $z_1(n),\cdots,z_n(n)$. Then

$$|\textstyle\sum_{\ell=1}^{n} \alpha_{p,\ell}(n) z_\ell(n)|_0 = (1 - |w_p(n)|_0^2)^{1/2},$$

and

$$|\textstyle\sum_{\ell=1}^{n} \alpha_{p,\ell}(n)(\mu_p - \eta_\ell(n)) z_\ell(n)|_0 = |P_n A w_p(n)|_0 .$$

Proof. The first result follows from

$$y_p(n) = \sum_{\ell=1}^{n} \alpha_{p,\ell}(n) z_\ell(n),$$

and

$$1 = |y_p|_0^2 = |y_p(n)|_0^2 + |w_p(n)|_0^2.$$

The second is a consequence of

$$\begin{aligned}\sum_{\ell=1}^{n} \alpha_{p,\ell}(n)(\mu_p - \eta_\ell(n)) z_\ell(n) &= \mu_p y_p(n) - \tilde{L}_n y_p(n) \\ &= \mu_p y_p(n) - P_n \tilde{L}(y_p - w_p(n)) \\ &= \mu_p y_p(n) - \mu_p P_n y_p + P_n(L+A) w_p(n) \\ &= P_n A w_p(n).\end{aligned}$$

Lemma 4.7. Let $\xi_1, \cdots, \xi_n$ be a basis for an n-dimensional inner product space. Let $\zeta = \sum_{\ell=1}^{n} \tau_\ell \xi_\ell$. Then

$$m(\xi_1, \cdots, \xi_n) \sum_{\ell=1}^{n} |\tau_\ell|^2 \leq \|\zeta\|^2 \leq M(\xi_1, \cdots, \xi_n) \sum_{\ell=1}^{n} |\tau_\ell|^2,$$

where $M(\xi_1, \cdots, \xi_n)$ and $m(\xi_1, \cdots, \xi_n)$ are respectively the greatest and least eigenvalues of the Gram matrix $((\xi_k, \xi_\ell))$.

Proof. We have $\|\zeta\|^2 = (\zeta, \zeta) = \sum_{k,\ell=1}^{n} \tau_k \bar{\tau}_\ell (\xi_k, \xi_\ell)$. Now $((\xi_k, \xi_\ell))$ is a Hermitian matrix and thus $\|\zeta\|^2$ is a Hermitian form in $(\bar{\tau}_1, \cdots, \bar{\tau}_n)$. The result follows from the fact that the value of a Hermitian form lies between the least eigenvalue of its matrix times the square of the Euclidean length of the vector and the greatest eigenvalue of its matrix times the square of the Euclidean length of the vector.

Lemma 4.8.

$$1 - |w_p(n)|_0^2 \leq M(z_1(n),\cdots,z_n(n)) \sum_{\ell=1}^{n} |\alpha_{p,\ell}(n)|^2,$$

and

$$m(z_1(n),\cdots,z_n(n)) \sum_{\ell=1}^{n} |\alpha_{p,\ell}(n)(\mu_p-\eta_\ell(n))|^2 \leq |P_n Aw_p(n)|_0^2.$$

Proof. This follows directly from Lemmas 4.6 and 4.7.

Theorem 4.4. For some ℓ, say ℓ_0, where $1 \leq \ell_0 \leq n$, we have

(4.1)

$$|\mu_p-\eta_{\ell_0}(n)| \leq (M(z_1(n),\cdots,z_n(n))/m(z_1(n),\cdots,z_n(n)))^{1/2} \times$$
$$|P_n Aw_p(n)|_0 (1 - |w_p(n)|_0^2)^{-1/2}.$$

Proof. We write M_n and m_n for $M(z_1(n),\cdots,z_n(n))$ and $m(z_1(n),\cdots,z_n(n))$ for the remainder of the paper. Suppose

$$|\mu_p-\eta_\ell(n)| > (M_n/m_n)^{1/2} |P_n Aw_p(n)|_0 (1 - |w_p(n)|_0^2)^{-1/2},$$

$\ell = 1,\cdots,n$. Then using the first inequality in Lemma 4.8 we have

$$m_n \sum_{\ell=1}^{n} |\alpha_{p,\ell}(n)(\mu_p - \eta_\ell(n))|^2$$

$$> M_n |P_n A w_p(n)|_0^2 (1 - |w_p(n)|_0^2)^{-1/2} \sum_{\ell=1}^{n} |\alpha_{p,\ell}(n)|^2$$

$$\geq |P_n A w_p(n)|_0^2$$

which contradicts the second inequality in Lemma 4.8. Hence we get the desired result.

Theorem 4.4 provides the basic estimate for $|\mu_p - \eta_{\ell_0}(n)|$. In the remainder of this section we will first discuss the convergence to zero of the right hand side of (4.1) and then derive computable bounds for $|w_p(n)|_0$ and $|P_n A w_p(n)|_0$. Since $w_p(n) = \sum_{\ell=n+1}^{\infty} (y_p, x_\ell)_0 x_\ell$ we see that $\lim_{n\to\infty} |w_p(n)|_0 = 0$ for p fixed. We prove that $\lim_{n\to\infty} |P_n A w_p(n)|_0 = 0$ with the help of the following lemma, whose proof can be found in Agmon [2].

Lemma 4.9. There is a constant K which depends on $a,b,m,L,$ and A such that

$$|Af|_0 \leq K(|Lf|_0 + |f|_0)$$

for all $f \in H_m$.

Using this lemma we can write

$$|P_n A w_p(n)|_0 \leq |A w_p(n)|_0$$

$$\leq K(|L w_p(n)|_0 + |w_p(n)|_0).$$

This, together with

$$Lw_p(n) = \sum_{\ell=n+1}^{\infty} \lambda_\ell (y_p, x_\ell)_0 ,$$

shows that $\lim_{n\to\infty} |P_n A w_p(n)|_0 = 0$ for p fixed. Thus the right side of (4.1) converges to zero as $n \longrightarrow \infty$ for p fixed provided M_n/m_n is bounded in n. In Section 6 we give sufficient conditions for M_n/m_n to be bounded.

Remark. The factor $(M_n/m_n)^{1/2}$ is present in inequality (4.1) because we are estimating the change in the eigenvalues when the (possibly) non self-adjoint operator $\tilde{L}_n$ is perturbed to obtain $\tilde{L}$. $(M_n/m_n)^{1/2}$ equals 1 if and only if $\tilde{L}_n$ is normal. This factor also arises in a fundamental result on the perturbation of eigenvalues of diagonalizable matrices; see Bauer and Fike [6].

Computable bounds on $|w_p(n)|_0$ and $|Aw_p(n)|_0$ will now be developed.

Lemma 4.10. If $n \geq p$, then

$$|w_p(n)|_0 \leq |Ay_p|_0 (\min_{\ell \geq n+1} |\mu_p - \lambda_\ell|)^{-1}.$$

Proof. Since

$$w_p(n) = \sum_{\ell=n+1}^{\infty} (y_p, x_\ell)_0 x_\ell$$

we have

$$\begin{aligned} |w_p(n)|_0^2 &= \sum_{\ell=n+1}^{\infty} |(y_p, x_\ell)_0|^2 \\ &\leq (\min_{\ell \geq n+1} |\mu_p - \lambda_\ell|)^{-1} \sum_{\ell=n+1}^{\infty} |(y_p, x_\ell)_0 (\mu_p - \lambda_\ell)|^2 . \end{aligned}$$

Now

$$
\begin{aligned}
(\mu_p-\lambda_\ell)(y_p,x_\ell)_0 &= (\mu_p y_p,x_\ell)_0 - (y_p,\lambda_\ell x_\ell)_0 \\
&= (\tilde{L}y_p,x_\ell)_0 - (y_p,Lx_\ell)_0 \\
&= (Ay_p,x_\ell)_0
\end{aligned}
$$

and hence

$$
\begin{aligned}
\sum_{\ell=1}^{\infty} |(y_p,x_\ell)_0(\mu_p-\lambda_\ell)|^2 &\le \sum_{\ell=1}^{\infty} |(Ay_p,x_\ell)_0|^2 \\
&= |Ay_p|_0^2 .
\end{aligned}
$$

Lemma 4.11.

(4.2) $$|Ay_p|_0 \le \sum_{\ell=0}^{j} \max|a_\ell|[\delta+\gamma(|\mu_p-c| + |Ay_p|_0)^{\ell/m}].$$

Proof. Using Lemma 3.3 we have

$$
\begin{aligned}
|Ay_p|_0 &\le (\sum_{\ell=0}^{j} \max|a_\ell|\,|y_p|_\ell) \\
&\le \sum_{\ell=0}^{j} \max|a_\ell|(\gamma|y_p|_0^{1-\frac{\ell}{m}}|y_p|_m^{\frac{\ell}{m}} + \delta|y_p|_0) \\
&= \sum_{\ell=0}^{j} \max|a_\ell|(\gamma|Ly_p-cy_p|_0^{\ell/m} + \delta) \\
&= \sum_{\ell=0}^{j} \max|a_\ell|(\gamma|\tilde{L}y_p-Ay_p-cy_p|^{\ell/m} + \delta) \\
&\le \sum_{\ell=0}^{j} \max|a_\ell|[\delta+\gamma(|\mu_p-c| + |Ay_p|_0)^{\ell/m}].
\end{aligned}
$$

We can solve (4.2) for $|Ay_p|_0$, i.e., (4.2) implies

$$|Ay_p|_0 \le t_p$$

where t_p is the unique positive solution of

$$t = \sum_{\ell=0}^{j} \max|a_\ell|[\delta + \gamma(|\mu_p - c| + t)^{\ell/m}].$$

Lemmas 4.10 and 4.11 provide a computable bound for $|w_p(n)|_0$ since we have a preliminary bound on $|\mu_p|$.

We now assume m is even (let $\mu = m/2$) and that the boundary conditions are of Sturm type, i.e., are of the form

$$\begin{aligned}
&\alpha_{10}f(a) + \alpha_{11}f^{(1)}(a) + \cdots + \alpha_{1,\kappa_1-1}f^{(\kappa_1-1)}(a) + f^{(\kappa_1)}(a) = 0,\\
&\alpha_{20}f(a) + \alpha_{21}f^{(1)}(a) + \cdots + \alpha_{2,\kappa_2-1}f^{(\kappa_2-1)}(a) + f^{(\kappa_2)}(a) = 0,\\
&\quad\vdots\\
&\alpha_{\mu 0}f(a) + \alpha_{\mu 1}f^{(1)}(a) + \cdots + \alpha_{\mu,\kappa_\mu-1}f^{(\kappa_\mu-1)}(a) + f^{(\kappa_\mu)}(a) = 0,\\
&\beta_{10}f(b) + \beta_{11}f^{(1)}(b) + \cdots + \beta_{1,\kappa'_1-1}f^{(\kappa'_1-1)}(b) + f^{(\kappa'_1)}(b) = 0,\\
&\quad\vdots\\
&\beta_{\mu 0}f(b) + \qquad \cdots \qquad + \beta_{\mu,\kappa'_\mu-1}f^{(\kappa'_\mu-1)}(b) + f^{(\kappa'_\mu)}(b) = 0,
\end{aligned}$$

where

$$0 \le \kappa_\mu < \kappa_{\mu-1} < \cdots < \kappa_2 < \kappa_1 \le m-1,$$

$$0 \le \kappa'_\mu < \kappa'_{\mu-1} < \cdots < \kappa'_2 < \kappa'_1 \le m-1.$$

Birkhoff [10] and Tamarkin [29] obtained asymptotic formulas for the eigenvalues of a wide class of differential operators and boundary conditions. Under the assumption that the boundary conditions are of Sturm type, they imply that the eigenvalues of

$$(-1)^\mu f^{(2\mu)} = \lambda f,$$

$$U_k f = 0, \quad k = 1,\cdots,m$$

are bounded below. This implies that c can be chosen such that L is positive definite; we assume for the rest of this section that c is so chosen. There is further explanation and additional applications of these asymptotic formulas in Section 6. For a complete discussion of these formulas see Naimark [22].

A proof of the following lemma can be found in Chapter XIX of Dunford and Schwartz [13].

Lemma 4.12. If $k = 1,\cdots,m-1$, then the domain of $L^{k/m}$ is contained in H_k where $L^{k/m}$ denotes the k/m power of L.

Lemma 4.13. There is a constant K which is independent of n such that

$$|Aw_p(n)|_0 \leq K|\mu_p-\lambda_{n+1}|^{(j-m)/m}$$

for $n \geq p$.

Proof. From Lemma 4.12 we see that H_j contains the domain of $L^{j/m}$. Hence, using Lemma 3.8, we see that $AL^{-j/m}$ is bounded as an operator on L_2. Thus

$$(4.3) \qquad \begin{aligned} |Af|_0 &= |AL^{-j/m}L^{j/m}f|_0 \\ &\leq \|AL^{-j/m}\| |L^{j/m}f|_0 \end{aligned}$$

for all f in the domain of $L^{j/m}$; in particular this holds for all $f \in V_m$ and we have

$$|Aw_p(n)|_0 \leq \|AL^{-j/m}\| |L^{j/m}w_p(n)|_0 .$$

Now

$$(L^{j/m}w_p(n),x_k)_0 = \lambda_k^{j/m}(w_p(n),x_k)_0$$

$$= \begin{cases} 0, & k \leq n, \\ \lambda_k^{j/m}(y_p,x_k)_0, & k > n, \end{cases}$$

and

$$(\mu_p-\lambda_k)(y_p,x_k)_0 = (Ay_p,x_k)_0$$

and hence

$$|L^{j/m}w_p(n)|_0^2 = \sum_{k=n+1}^{\infty} \lambda_k^{2j/m} |(Ay_p,x_k)_0(\mu_p-\lambda_k)^{-1}|^2$$

$$\leq \{\max_{k\geq n+1} (\lambda_k|\mu_p-\lambda_k|^{-1})^{2j/m}\}|\mu_p-\lambda_{n+1}|^{2(j-m)/m}|Ay_p|_0^2.$$

Thus

$$|Aw_p(n)|_0 \leq$$

$$\|AL^{-j/m}\|\{\max_{k\geq p+1} (\lambda_k|\mu_p-\lambda_k|^{-1})^{j/m}\}|Ay_p|_0|\mu_p-\lambda_{n+1}|^{(j-m)/m}.$$

Under the assumption that $j \leq \mu$ we present an additional estimate for $|Aw_p(n)|_0$ which is in some cases more readily computed than the bound given in Lemma 4.13.

Lemma 4.14. There are constants $c_1 > 0$, $c_2 \geq 0$ which depend on a, b, m, L, and the boundary conditions such that

$$(4.4) \qquad \|f\|_\mu^2 \leq c_1(Lf,f)_0 + c_2|f|_0^2$$

for all $f \in V_m$.

Inequality (4.4) for functions which vanish near a and b is Gärding's inequality. Our concern here is to prove it for functions in V_m and to show how the constants can be calculated. This is done in Section 5.

<u>Lemma 4.15</u>. If $j \leq m/2$, then

$$|Aw_p(n)|_0 \leq \beta\{\max_{k\geq n+1} (c_1\lambda_k+c_2)^{1/2}|\mu_p-\lambda_k|^{-1}\}|Ay_p|_0 .$$

<u>Proof</u>. Using Lemma 4.14 we have

$$(4.5) \qquad \begin{aligned} |Af|_0^2 &\leq \beta^2[c_1(Lf,f)_0 + c_2|f|_0^2] \\ &= \beta^2[c_1|L^{1/2}f|_0^2 + c_2|f|_0^2] \end{aligned}$$

for any $f \in V_m$. If we now estimate $|Aw_p(n)|_0$, using (4.5) instead of (4.3) we obtain

$$\begin{aligned} |Aw_p(n)|_0 &\leq \beta\{\textstyle\sum_{k=n+1}^{\infty}(c_1\lambda_k+c_2)|(\mu_p-\lambda_k)^{-1}(Ay_p,x_k)_0|^2\}^{1/2} \\ &\leq \beta\{\max_{k\geq n+1} (c_1\lambda_k+c_2)^{1/2}|\mu_p-\lambda_k|^{-1}\}|Ay_p|_0 . \end{aligned}$$

Theorem 4.4 and Lemmas 4.10, 4.11, 4.13 and 4.15 give an explicit estimate for $|\mu_p-\eta_{\ell_0}(n)|$ since M_n/m_n can be calculated. In many cases we can conclude that $\ell_0 = p$ and hence we have an estimate for $|\mu_p-\eta_p(n)|$ for any $n \geq p$. See Section 7 for an explanation of the use of these inequalities in numerical computation.

§5. Proof of Lemma 4.14.

In this section we assume that m is even, the boundary conditions are of Sturm type, and $j \leq m/2$. We do not assume that c is chosen such that L is positive definite.

Lemma 5.1. Suppose $\mu \leq i \leq 2\mu-1$, $0 \leq k \leq \mu-1$ and neither i nor k is equal to κ_ℓ for any ℓ. Let D be any constant. Then there exists a family of functions $f_\varepsilon(x)$, $0 < \varepsilon < 1$, such that

1) $f_\varepsilon \in V_m$ for all ε,

2)
$$f_\varepsilon^{(p)}(a) = \begin{cases} 1, & p = k \\ D\varepsilon^{-1}, & p = i \\ 0, & p \neq i,k,\kappa_1,\cdots,\kappa_\mu \end{cases}$$

3) $f_\varepsilon^{(p)}(b) = 0$ for all p,

4) $|f_\varepsilon'|_0$ and $|f_\varepsilon|_\mu$ are bounded in ε.

Proof. For $0 \leq p \leq 2\mu-1$ define $f_p(x) = (x-a)^p/p!$. Let $\gamma(x)$ be an infinitely differentiable function satisfying $0 \leq \gamma(x) \leq 1$, $\gamma(x) = 1$ if $x \leq a + (b-a)/3$, and $\gamma(x) = 0$ if $x \geq (a+b)/2$. For $\mu \leq q \leq 2\mu-1$ define

$$g_{q,\varepsilon}(x_q) = \int_a^{x_q} \cdots \left(\int_a^{x_1} D\gamma(a + (x_0-a)\varepsilon^{-2})\varepsilon^{-1}\, dx_0\right) \cdots dx_q.$$

Then

$$f_p^{(q)}(a) = \begin{cases} 1, & q = p, \\ 0, & q \neq p, \end{cases}$$

and

$$g_{q,\varepsilon}^{(p)}(a) = \begin{cases} D\varepsilon^{-1}, & p = q, \\ 0, & p \neq q. \end{cases}$$

Also

$$|g_{q,\varepsilon}^{(p)}|_0 \leq |D|[\int_a^b \gamma^2(x)\, dx]^{1/2}(b-a)^{q-p} \text{ if } 0 < \varepsilon < 1, \quad p \leq \mu.$$

The boundary conditions at a can be solved for $f^{(\kappa_1)}(a)$, $\cdots, f^{(\kappa_\mu)}(a)$ in terms of $f^{(p)}(a)$, $p \neq \kappa_1, \cdots, \kappa_\mu$:

$$f^{(\kappa_\ell)}(a) = \sum_{\substack{p=0 \\ p\neq\kappa_\mu,\cdots,\kappa_{\ell+1}}}^{\kappa_\ell - 1} c_{\ell p} f^{(p)}(a),$$

where the coefficients $c_{\ell p}$ depend only on the boundary conditions. If $k < \kappa_\ell < i$ define

$$h_{\kappa_\ell}(x) = c_{\ell k} f_{\kappa_\ell}(x);$$

if $i < \kappa_\ell$ define

$$h_{\kappa_\ell}(x) = c_{\ell k} f_{\kappa_\ell}(x) + c_{\ell i} g_{\kappa_\ell,\varepsilon}(x).$$

Now let

$$f_\varepsilon(x) = \gamma(x)\{g_{i,\varepsilon}(x) + f_k(x) + \sum_{\kappa_\ell > k} h_{\kappa_\ell}(x)\}.$$

f_ε' clearly satisfies conditions 1-4.

The following two lemmas are proved in Agmon [2]. Lemma 5.3 is closely related to Lemma 3.3.

Lemma 5.2. (Sobolev) There is a constant γ depending only on a, b, and ℓ such that

$$|f^{(k)}(\tau)| \leq \gamma r^{(\ell - \frac{1}{2} - k)}(|f|_\ell + r^{-\ell}|f|_0)$$

for all $f \in H_\ell$, $0 < r \leq 1$, $k = 0,\cdots,\ell-1$, $a \leq \tau \leq b$.

Lemma 5.3. There is a constant γ depending only on a, b, and ℓ such that

$$|f|_k^2 \leq \gamma(r^{\ell-k}|f|_\ell^2 + r^{-k}|f|_0^2)$$

for all $f \in H_\ell$, $0 < r \leq 1$, $k = 0,\cdots,\ell-1$.

If $f \in V_m$, then repeated integration by parts gives

$$(5.1)\quad (Lf,f)_0 = ((-1)^\mu f^{(2\mu)} + cf, f)_0$$
$$= |f|_\mu^2 + c|f|_0^2 + (-1)^\mu \sum_{k=0}^{\mu-1}(-1)^k f^{(2\mu-1-k)}\overline{f}^{(k)}\Big|_a^b .$$

Since the eigenvalues of L are bounded below we have

$$(Lf,f)_0 \geq \lambda|f|_0^2$$

for all $f \in V_m$ where λ is the least eigenvalue of L. Using this and (5.1) we obtain

$$(5.2)\quad (-1)^{\mu+1}\sum_{k=0}^{\mu}(-1)^k f^{(2\mu-1-k)}\overline{f}^{(k)}\Big|_a^b \leq |f|_\mu^2 + (c-\lambda)|f|_0^2$$

for $f \in V_m$. As indicated in the proof of Lemma 5.1 the boundary conditions at a can be solved for $f^{(\kappa_\ell)}(a)$:

$$f^{(\kappa_\ell)}(a) = \sum_{\substack{p=0 \\ p\neq\kappa_\mu,\cdots,\kappa_{\ell+1}}}^{\kappa_\ell - 1} c_{\ell p} f^{(p)}(a).$$

Similarly the boundary conditions at b can be solved for $f^{(\kappa'_\ell)}(b)$:

$$f^{(\kappa'_\ell)}(b) = \sum_{\substack{p=0 \\ p\neq\kappa'_\mu,\cdots,\kappa'_{\ell+1}}}^{\kappa'_\ell-1} c'_{\ell p}\, f^{(p)}(b)$$

Let $S(x) = (-1)^\mu \sum_{k=0}^{\mu-1} (-1)^k f^{(2\mu-1-k)}(x)\overline{f}^{(k)}(x)$. In terms of S inequality (5.2) is

$$(5.3) \qquad -S(b) + S(a) \leq |f|_\mu^2 + (c-\lambda)|f|_0^2.$$

If in $S(a)$ we substitute the above expression for $f^{(\kappa_\ell)}(a)$, $\ell = 1,\cdots,\mu-1$, we will get terms of the form

$$d_{ik}\, f^{(i)}(a)\, \overline{f}^{(k)}(a)$$

where $0 \leq i \leq 2\mu-1$, $0 \leq k \leq \mu-1$, $i+k \leq 2\mu-1$, and neither i nor k is equal to κ_ℓ for any ℓ. Likewise for $S(b)$ we get terms of the form

$$d'_{ik}\, f^{(i)}(b)\, \overline{f}^{(k)}(b)$$

where $0 \leq i \leq 2\mu-1$, $0 \leq k \leq \mu-1$, $i+k \leq 2\mu-1$, and neither i nor k is equal to κ'_ℓ for any ℓ. Let i, k be fixed with $i \geq \mu$. Now let f_ε be the family of functions shown to exist in Lemma 5.1 where $D = d_{ik}$. For these functions (5.3) reduces to

$$|d_{ik}|^2\varepsilon^{-1} \leq |f_\varepsilon|_\mu^2 + (c-\lambda)|f_\varepsilon|_0^2.$$

Since $|f_\varepsilon|_\mu$ and $|f_\varepsilon|_0$ are bounded in ε we see that

$d_{ik} = 0$. Similarly $d'_{ik} = 0$ if $i \geq \mu$. Hence (5.1) becomes

$$(5.4)\quad (Lf,f)_0 = |f|^2_\mu + c|f|^2_0 + S(b) - S(a)$$

$$= |f|^2_\mu + c|f|^2_0 + \sum_{\substack{0\leq i,k\leq \mu-1 \\ i,k\neq\kappa'_\ell,\ \ell=1,\cdots,\mu}} d'_{ik} f^{(i)}(b)\overline{f}^{(k)}(b)$$

$$- \sum_{\substack{0\leq i,k\leq \mu-1 \\ i,k\neq\kappa_\ell,\ \ell=1,\cdots,\mu}} d_{ik} f^{(i)}(a)\overline{f}^{(k)}(b)$$

and thus

$$(Lf,f)_0 \geq |f|^2_\mu + c|f|^2_0 - \sum |d'_{ik}|\,|f^{(i)}(b)|\,|\overline{f}^{(k)}(b)|$$

$$- \sum |d_{ik}|\,|f^{(i)}(a)|\,|\overline{f}^{(k)}(a)|.$$

Using Lemma 5.2 with $\ell = \mu$ we have

$$|f^{(i)}(a)| \leq \gamma r^{\mu - \frac{1}{2} - i}(|f|_\mu + r^{-\mu}|f|_0)$$

$$\leq \gamma(r^{1/2}|f|_\mu + r^{\frac{1}{2}-\mu}|f|_0), \quad 0 \leq i \leq \mu-1.$$

Using this and a similar estimate for $|f^{(i)}(b)|$ we obtain

$$(Lf,f)_0 \geq |f|^2_\mu + c|f|^2_0 - \sum |d'_{ik}|\gamma^2(r^{1/2}|f|_\mu + r^{\frac{1}{2}-\mu}|f|_0)^2$$

$$- \sum |d_{ik}|\gamma^2(r^{1/2}|f|_\mu + r^{\frac{1}{2}-\mu}|f|_0)^2$$

$$\geq |f|^2_\mu + c|f|^2_0 - 2\gamma^2(r|f|^2_\mu + r^{1-2\mu}|f|^2_0)$$

$$+ (\sum |d'_{ij}| + \sum |d_{ij}|)$$

$$= |f|_\mu^2\{1 - 2\gamma^2 r(\sum |d'_{ij}| + \sum |d_{ij}|)\}$$
$$+ |f|_0^2\{c - 2\gamma^2 r^{1-2\mu}(\sum |d_{ij}| + \sum |d'_{ij}|)\}$$
$$= \tfrac{1}{2} |f|_\mu^2 + \tilde{c}|f|_0^2$$

for r chosen appropriately. Finally, using Lemma 5.3 we have

$$\begin{aligned}(Lf,f) &\geq \tfrac{1}{2}\|f\|_\mu^2 - \tfrac{1}{2}\sum_{i=0}^{\mu-1} |f|_i^2 + \tilde{c}|f|_0^2 \\ &\geq \tfrac{1}{2}\|f\|_\mu^2 - \tfrac{1}{2}\sum_{i=0}^{\mu-1} \gamma(r^{\mu-i}|f|_\mu^2 + r^{-i}|f|_0^2) + \tilde{c}|f|_0^2 \\ &\geq \tfrac{1}{2}\|f\|_\mu^2 - \tfrac{\gamma}{2}\left(\sum_{i=0}^{\mu-1} r^{\mu-i}\right)\|f\|_\mu^2 + |f|_0^2\left(\tilde{c} - \tfrac{\gamma}{2}\sum_{i=0}^{\mu-1} r^{-i}\right) \\ &= \|f\|_\mu^2\left\{\tfrac{1}{2} - \tfrac{\gamma}{2}\left(\sum_{i=0}^{\mu-1} r^{\mu-i}\right)\right\} + |f|_0^2\left(\tilde{c} - \tfrac{\gamma}{2}\sum_{i=0}^{\mu-1} r^{-i}\right).\end{aligned}$$

Now by choosing r appropriately we arrive at Lemma 4.14.

Remarks. 1. The constants c_1 and c_2 in Lemma 4.14 are computable assuming the constants in Lemmas 5.2 and 5.3 are computable. See Section 7, especially Example 7.4, for further comments on finding explicit values for these constants.

2. Let $m = 2$ and write the boundary conditions as

$$\alpha_0 f(a) + \alpha_1 f^{(1)}(a) = 0,$$
$$\beta_0 f(b) + \beta_1 f^{(1)}(b) = 0.$$

Then, if $\alpha_0\alpha_1 \leq 0$ and $\beta_0\beta_1 \geq 0$, (5.4) implies

$$(Lf,f)_0 \geq |f|_1^2 + c|f|_0^2.$$

Hence in this case we obtain (4.4) with $c_1 = 1$ and $c_2 = \max(0,1-c)$.

§6. Proof that M_n/m_n is bounded.

We return here to the problem mentioned in Section 4 of showing that

$$M(z_1(n),\cdots,z_n(n))/m(z_1(n),\cdots,z_n(n))$$

is bounded in n. For the remainder of this section assume that $m = 2\mu$ is even, the boundary conditions are of Sturm type, L is positive definite, and the circles

$$C_\ell = \{\lambda \mid |\lambda-\lambda_\ell| \leq r_\ell\}$$

are mutually disjoint. We also assume that j, the order of A, is less than or equal to $m-2$. Let $d_\ell = \min_{k\neq\ell} |\lambda_k-\lambda_\ell|$.

Lemma 6.1. $\sum_{k=1}^{\infty} (\lambda_k^{\nu}/d_k)^2 < \infty$ if $0 < \nu < 1 - \frac{3}{2m}$.

Proof. Under the assumptions of this section the asymptotic formula for the eigenvalues of L which was mentioned in Section 4 is as follows. There is an integer p and a constant α such that

$$\lambda_k = \left(\frac{\pi(k-p)}{b-a}\right)^m \{1 + \frac{\alpha}{k} + O\left(\frac{1}{k^2}\right)\} \tag{6.1}$$

for all k. An immediate consequence of this is an asymptotic formula for d_k:

$$d_k = \left(\frac{\pi}{b-a}\right)^m (k-p)^{m-1} \{1 + O\left(\frac{1}{k}\right)\}. \tag{6.2}$$

Hence

$$\lambda_k^{2\nu}/d_k^2 = \left(\frac{\pi}{b-a}\right)^{2m(\nu-1)} (k-p)^{2(1+m\nu-m)} O(1)$$

and the result follows.

Lemma 6.2. For sufficiently large ℓ, say $\ell \geq \ell_1$,

$$r_\ell < d_\ell/3.$$

Proof. It follows from (3.6), (3.8), (3.9) and (6.1) that

$$r_\ell \leq d\ell^j, \qquad d > 0,$$

and from (6.2) that

$$d_\ell/3 \geq \tilde{d}\ell^{m-1}, \qquad \tilde{d} > 0.$$

The result now follows from the fact that $j < m-1$.

Lemma 6.3. Let $\phi_1,\cdots,\phi_n$ be any orthonormal basis for $S_n = \mathrm{span}(x_1,\cdots,x_n)$ and let $T: S_n \longrightarrow S_n$ be the linear operator defined by

$$T\phi_k = z_k(n), \qquad k = 1,\cdots,n.$$

Then

$$(M_n/m_n)^{1/2} = \|T\|\|T^{-1}\|$$

where $\|\cdot\|$ denotes the operator norm which is associated with the inner product on S_n which is induced by the L_2 inner product.

<u>Proof</u>. $\|T\|^2$ is equal to the largest eigenvalue of T^*T and the matrix of T^*T with respect to $\phi_1,\cdots,\phi_n$ is the transpose of the Gram matrix $((z_k(n),z_\ell(n))_0)$. Hence $\|T\|^2 = M_n$. Similar reasoning shows that $\|T^{-1}\|^2 = m_n$.

$\tilde{L}_n$ is a diagonalizable operator on S_n. Let $Q_1^n,\cdots,Q_n^n$ be the projections associated with $\tilde{L}_n$ and let $I_n = \{1,2,\cdots,n\}$. The following lemma is due to Lorch [21].

<u>Lemma 6.4</u>. $\|T\|\,\|T^{-1}\| \le 4\{\max_{I\subset I_n} \|\sum_{k\in I} Q_k^n\|\}^2$.

Combining Lemmas 6.3 and 6.4 we see that

$$(6.3) \qquad (M_n/m_n)^{1/2} \le 4\{\max_{I\subset I_n} \|\sum_{k\in I} Q_k^n\|\}^2 .$$

Let $P_1^n,\cdots,P_n^n$ be the projections associated with L_n. Then

$$(6.4) \qquad \begin{aligned} \|\textstyle\sum_{k\in I} Q_k^n\| &= \|\textstyle\sum_{k\in I} P_k^n + \sum_{k\in I} (Q_k^n - P_k^n)\| \\ &\le 1 + \|\textstyle\sum_{k\in I} (Q_k^n - P_k^n)\| \end{aligned}$$

since the P_k^n are orthogonal. Let $\Gamma_\ell = \{\lambda \mid |\lambda-\lambda_\ell| \le d_\ell/3\}$. Using Lemma 6.2 and formula (c) of Lemma 3.7 we have

$$(6.5) \qquad \begin{aligned} Q_k^n - P_k^n &= \frac{1}{2\pi i}\int_{\partial\Gamma_k} [R_\lambda(\tilde{L}_n) - R_\lambda(L_n)]\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\partial\Gamma_k} R_\lambda(L_n)A_nR_\lambda(L_n)\, d\lambda \\ &\quad + \frac{1}{2\pi i}\int_{\partial\Gamma_k} R_\lambda(L_n)A_nR_\lambda(L_n)A_nR_\lambda(L_n)[I-A_nR_\lambda(L_n)]^{-1}\, d\lambda \end{aligned}$$

if $k > \ell_1$. We now estimate the second term in the right side of (6.5).

Lemma 6.5. If $\lambda \in \rho(L_n)$, then

$$\|A_n R_\lambda(L_n)\| \le \|AL^{(2-m)/m}\| \max_{k=1,\cdots,n} \lambda_k^{(m-2)/m} |\lambda-\lambda_k|^{-1}.$$

Proof. From Lemma 4.12 we see that the range of $L^{(2-m)/m}$ is contained in H_{m-2} and hence it follows from Lemma 3.8 that $AL^{(2-m)/m}$ is bounded. It follows directly from the definition of a power of an operator that $L_n^{(2-m)/m} = (L^{(2-m)/m})_n$. Using this we find that

$$A_n L_n^{(2-m)/m} f = P_n AL^{(2-m)/m} f$$

for all $f \in S_n$ and hence that

$$\|A_n L_n^{(2-m)/m}\| \le \|AL^{(2-m)/m}\| .$$

For any $f \in S_m$

$$L_n^{(m-2)/m} R_\lambda(L_n) f = \sum_{k=1}^{n} (f,x_k)_0 \lambda_k^{(m-2)/m} (\lambda-\lambda_k)^{-1} x_k,$$

and hence

$$\|L_n^{(m-2)/m} R_\lambda(L_n) f\|_0^2 = \sum_{k=1}^{n} \lambda_k^{2(m-2)/m} |\lambda-\lambda_k|^{-2} |(f,x_k)_0|^2$$

$$\le \max_k \lambda_k^{2(m-2)/m} |\lambda-\lambda_k|^{-2} \|f\|_0^2 .$$

Thus

$$\|L_n^{(m-2)/m} R_\lambda(L_n)\| \le \max_k \lambda_k^{(m-2)/m} |\lambda-\lambda_k|^{-1}.$$

Combining these estimates we have

$$\begin{aligned}
\|A_n R_\lambda(L_n)\| &= \|A_n L_n^{(2-m)/m} L_n^{(m-2)/m} R_\lambda(L_n)\| \\
&\le \|A_n L_n^{(2-m)/m}\| \, \|L_n^{(m-2)/m} R_\lambda(L_n)\| \\
&\le \|AL^{(2-m)/m}\| \max_k \lambda_k^{(m-2)/m} |\lambda-\lambda_k|^{-1}.
\end{aligned}$$

Lemma 6.6. If $0 < \nu < 1$ and $\lambda \in \partial\Gamma_\ell$ then

$$\max_{k=1,\cdots,n} \lambda_k^\nu |\lambda-\lambda_k|^{-1} \le \max(3\,,\, 3\lambda_2/2\lambda_1)\lambda_\ell^\nu/d_\ell.$$

Proof. If $k \ne \ell$, then

$$\begin{aligned}
\lambda_k^\nu|\lambda-\lambda_k|^{-1} &\le \lambda_k^\nu|\lambda_k-\lambda_\ell|^{-1} \big|\, |\lambda-\lambda_\ell| \; |\lambda_k-\lambda_\ell| - 1\big|^{-1} \\
&\le (3/2)\lambda_k^\nu|\lambda_k-\lambda_\ell|^{-1} \\
&= (3/2)|\lambda_k-\lambda_\ell|^{\nu-1}(\lambda_k/|\lambda_k-\lambda_\ell|)^\nu \\
&\le (3/2)|\lambda_k-\lambda_\ell|^{\nu-1}(1 + \lambda_\ell/|\lambda_k-\lambda_\ell|)^\nu \\
&\le (3/2)(\lambda_\ell+d_\ell)^\nu/d_\ell.
\end{aligned}$$

Since $\lambda_k \ge d_k$ for $k \ge 2$ we have

$$(\lambda_\ell + d_\ell)^\nu \le \max(2,\lambda_2/\lambda_1)\lambda_\ell^\nu.$$

Hence

$$\lambda_k^\nu|\lambda-\lambda_k|^{-1} \le \max(2,\lambda_2/\lambda_1)\lambda_\ell^\nu/d_\ell$$

if $k \ne \ell$. In the case $k = \ell$ we have

$$\lambda_k^\nu |\lambda-\lambda_k|^{-1} \;=\; 3\ \lambda_\ell^\nu/d_\ell.$$

These two estimates give the result.

Lemma 6.7. There is an integer n_1 such that

$$\left\|\frac{1}{2\pi i}\int_{\partial\Gamma_\ell} R_\lambda(L_n)A_nR_\lambda(L_n)A_nR_\lambda(L_n)[I - A_nR_\lambda(L_n)]^{-1}d\lambda\right\|$$
$$\leq\ 2\,\|AL^{(2-m)/m}\|^2 \max(3\,,\,3\lambda_2/2\lambda_1)^2(\lambda_\ell^{(m-2)/m}/d_\ell)^2$$

for $\ell \geq n_1$.

Proof. Let

$$d \;=\; \max(3\,,\,3\lambda_2/2\lambda_1).$$

Suppose $\lambda \in \Gamma_\ell$. Lemmas 6.5 and 6.6 imply

$$\|A_nR_\lambda(L_n)\| \;\leq\; \|AL^{(2-m)/m}\|\,d\lambda_\ell^{(m-2)/m}/d_\ell \tag{6.6}$$

if $\ell > \ell_1$. Let n_1 be an integer such that $n_1 > \ell_1$ and

$$\|AL^{(2-m)/m}\|\,d\lambda_\ell^{(m-2)/m}/d_\ell \;<\; 1/2$$

if $\ell \geq n_1$. Then if $\ell \geq n_1$, $I - A_nR_\lambda(L\ )$ is invertible and

$$\|[I - A_nR_\lambda(L_n)]^{-1}\| \;\leq\; (1 - \|A_nR_\lambda(L_n)\|)^{-1} \;<\; 2. \tag{6.7}$$

Now, using (6.6), (6.7) and the fact that $\|R_\lambda(L_n)\| = 3/d_\ell$, we find

$$\|R_\lambda(L_n)A_nR_\lambda(L_n)A_nR_\lambda(L_n)[I - A_nR_\lambda(L_n)]^{-1}\|$$

$$\leq \|R_\lambda(L_n)\| \, \|A_nR_\lambda(L_n)\|^2 \|[I - A_nR_\lambda(L_n)]^{-1}\|$$

$$\leq (6/d_\ell^3)\|AL^{(2-m)/m}\|^2 d^2 \lambda_\ell^{2(m-2)/m}$$

if $n \geq n_1$.

Next we consider the problem of estimating

$$\left\|\sum_{k\in I} \frac{1}{2\pi i} \int_{\Gamma_k} R_\lambda(L_n)A_nR_\lambda(L_n)\, d\lambda\right\|,$$

where $I \subset I_n$. Let $1 \leq k \leq n$ and consider the Laurent expansion of $R_\lambda(L_n)$ about λ_k:

$$R_\lambda(L_n) = \sum_{\ell=-\infty}^{\infty} B_\ell^k(\lambda-\lambda_k)^\ell, \quad \lambda \in \Gamma_k,$$

where

$$B_\ell^k = \frac{1}{2\pi i} \int_{\Gamma_k} R_\lambda(L_n)(\lambda-\lambda_k)^{-\ell-1}\, d\lambda.$$

Since L_n is compact λ_k is a pole of $R_\lambda(L_n)$. The order of λ_k as a pole of $R_\lambda(L_n)$ is equal to the ascent of $\lambda_k - L_n$ and since λ_k has algebraic multiplicity one we see that λ_k is a simple pole. Thus the expansion has the form

$$R_\lambda(L_n) = B_{-1}^k(\lambda-\lambda_k)^{-1} + B_0^k + B(\lambda),$$

where $B(\lambda) = \sum_{\ell=1}^{\infty} B_\ell^k(\lambda-\lambda_k)^\ell$. Now $B_{-1}^k = P_k^n$, the $k^{\underline{th}}$ projection associated with L_n. Also

$$B_0^k = \frac{1}{2\pi i} \int_{\Gamma_k} R_\lambda(L_n)(\lambda-\lambda_k)^{-1}\, d\lambda.$$

Now

$$
\begin{aligned}
R_\lambda(L_n)A_nR_\lambda(A_n) \;=\; & P_k^nA_nP_k^n(\lambda-\lambda_k)^{-2} + B_0^kA_nP_k^n(\lambda-\lambda_k)^{-1} \\
& + B(\lambda)A_nP_k^n(\lambda-\lambda_k)^{-1} + P_k^nA_nB_0^k(\lambda-\lambda_k)^{-1} \\
& + B_0^kA_nB_0^k + B(\lambda)A_nB_0^k + P_k^nA_nB(\lambda)(\lambda-\lambda_k)^{-1} \\
& + B_0^kA_nB(\lambda) + B(\lambda)A_nB(\lambda)
\end{aligned}
$$

for $\lambda \in \Gamma_k$ and hence, since

$$\int_{\Gamma_k} (\lambda-\lambda_k)^{-2}\, d\lambda \;=\; 0,$$

$$\int_{\Gamma_k} (\lambda-\lambda_k)^{-1}\, d\lambda \;=\; 2\pi i,$$

$$\int_{\Gamma_k} B(\lambda)(\lambda-\lambda_k)^{-1}\, d\lambda \;=\; 2\pi i\, B(\lambda_k) \;=\; 0,$$

$$\int_{\Gamma_k} d\lambda \;=\; 0,$$

$$\int_{\Gamma_k} B(\lambda)\, d\lambda,$$

and

$$\int_{\Gamma_k} B^2(\lambda)\, d\lambda \;=\; 0,$$

$$\frac{1}{2\pi i}\int_{\Gamma_k} R_\lambda(L_n)A_nR_\lambda(L_n)\, d\lambda \;=\; B_0^kA_nP_k^n + P_k^nA_nB_0^k.$$

Using this formula we obtain

$$
\begin{aligned}
(6.8)\qquad & \Big\|\sum_{k\in I}\frac{1}{2\pi i}\int_{\Gamma_k} R_\lambda(L_n)A_nR_\lambda(L_n)\, d\lambda\Big\| \\
& \quad\le \Big\|\sum_{k\in I} B_0^kA_nP_k^n\Big\| + \Big\|\sum_{k\in I} P_k^nA_nB_0^k\Big\|.
\end{aligned}
$$

Now, since

$$R_\lambda(L_n) \;=\; \sum_{k=1}^{n} (\lambda-\lambda_k)^{-1}P_k^n,$$

and

$$\frac{1}{2\pi i}\int_{\Gamma_k} (\lambda-\lambda_\ell)^{-1}(\lambda-\lambda_k)^{-1}\,d\lambda = \begin{cases} 0, & \ell = k \\ (\lambda_k-\lambda_\ell)^{-1}, & \ell \neq k \end{cases},$$

we get

$$B_0^k = \sum_{\ell=1}^n P_\ell^n \frac{1}{2\pi i}\int_{\Gamma_k} (\lambda-\lambda_\ell)(\lambda-\lambda_k)^{-1}\,d\lambda$$

$$= \sum_{\substack{\ell=1 \\ \ell\neq k}}^n (\lambda_k-\lambda_\ell)^{-1} P_\ell^n.$$

For $x \in S_n$ we have

$$P_k^n x = (x,x_k)_0 x_k,$$

$$A_n P_k^n x = (x,x_k)_0 P_n A x_k$$

$$= (x,x_k)_0 \sum_{q=1}^n (Ax_k,x_q)_0 x_q,$$

and hence

$$B_0^k A_n P_k^n x = \sum_{\substack{\ell=1 \\ \ell\neq k}}^n (\lambda_k-\lambda_\ell)^{-1} P_\ell^n A_n P_k^n x$$

$$= \sum_{\substack{\ell=1 \\ \ell\neq k}}^n (\lambda_k-\lambda_\ell)^{-1} P_\ell^n (x,x_k)_0 \sum_{q=1}^n (Ax_k,x_q)_0 x_q$$

$$= \sum_{\substack{\ell=1 \\ \ell\neq k}}^n (\lambda_k-\lambda_\ell)^{-1} (x,x_k)_0 (Ax_k,x_\ell)_0 x_\ell,$$

and thus

$$\left\|\sum_{k\in I} B_0^k A_n P_k^n x\right\|_0 = \left\|\sum_{k\in I} (x,x_k)_0 \sum_{\substack{\ell=1 \\ \neq k}}^n (\lambda_k-\lambda_\ell)^{-1} (Ax_k,x_\ell)_0 x_\ell\right\|_0$$

$$\leq \sum_{k\in I} \|\sum_{\substack{\ell=1\\ \neq k}}^{n} (\lambda_k-\lambda_\ell)^{-1}(Ax_k,x_\ell)_0 x_\ell\|_0 \,|(x,x_k)_0|$$

$$\leq [\sum_{k\in I} \|\sum_{\substack{\ell=1\\ \neq k}}^{n} (\lambda_k-\lambda_\ell)^{-1}(Ax_k,x_\ell)_0 x_\ell\|_0^2]^{1/2}[\sum_{k\in I} |(x,x_k)_0|^2]^{1/2}$$

$$= [\sum_{k\in I} \sum_{\substack{\ell=1\\ \neq k}}^{n} |(\lambda_k-\lambda_\ell)^{-1}(Ax_k,x_\ell)_0|^2]^{1/2}\|x\|_0 .$$

Hence

$$(6.9)\quad \|\sum_{k\in I} B_0^k A_n P_k^n\|_0$$

$$\leq [\sum_{k\in I} \sum_{\substack{\ell=1\\ \neq k}}^{n} |(\lambda_k-\lambda_\ell)^{-1}(Ax_k,x_\ell)_0|^2]^{1/2}$$

$$= [\sum_{k\in I} \sum_{\substack{\ell=1\\ \neq k}}^{n} (AL^{(2-m)/m}L^{(m-2)/m}x_k,x_\ell)_0(\lambda_k-\lambda_\ell)^{-1}|^2]^{1/2}$$

$$= [\sum_{k\in I} \sum_{\substack{\ell=1\\ \neq k}}^{n} |\lambda_k^{(m-2)/m}(\lambda_k-\lambda_\ell)^{-1}(AL^{(2-m)/m}x_k,x_\ell)_0|^2]^{1/2}$$

$$\leq [\sum_{k\in I} \max_{\substack{\ell=1,\cdots,n\\ \neq k}} |\lambda_k^{(m-2)/m}(\lambda_k-\lambda_\ell)^{-1}|^2 \sum_{\substack{\ell=1\\ \neq k}}^{n} |(AL^{(2-m)/m}x_k,x_\ell)_0|^2]^{1/2}$$

$$\leq [\sum_{k\in I} \max_{\ell\neq k}|\lambda_k^{(m-2)/m}(\lambda_k-\lambda_\ell)^{-1}|^2 \; \|AL^{(2-m)/m}x_k\|_0^2]^{1/2}$$

$$\leq \|AL^{(2-m)/m}\| \; [\sum_{k\in I} \max_{\ell\neq k}|\lambda_k^{(m-2)/m}(\lambda_k-\lambda_\ell)^{-1}|^2]^{1/2}$$

$$\leq \|AL^{(2-m)/m}\| \; [\sum_{k\in I} (\lambda_k^{(m-2)/m}/d_k)^2]^{1/2}$$

$$\leq \|AL^{(2-m)/m}\| \; [\sum_{k=1}^{\infty} (\lambda_k^{(m-2)/m}/d_k)^2]^{1/2}.$$

Similarly we find that

$$P_k^n A_n B_0^k x = \sum_{\substack{\ell=1\\ \neq k}}^{n} (x,x_\ell)_0(\lambda_k-\lambda_\ell)^{-1}(Ax_\ell,x_k)_0 x_k$$

for $x \in S_n$ and thus

$$\|\sum_{k\in I} P_k^n A_n B_0^k x\|_0$$

$$= \|\sum_{k\in I} \sum_{\substack{\ell=1\\ \neq k}}^{n} (x,x_\ell)_0(\lambda_k-\lambda_\ell)^{-1}(Ax_\ell,x_k)_0 x_k\|_0$$

$$= \|\sum_{\ell=1}^{n} (\sum_{\substack{k\in I\\ \neq\ell}} (Ax_\ell,x_k)_0(\lambda_k-\lambda_\ell)^{-1}x_k)(x,x_\ell)_0\|_0$$

$$\leq \sum_{\ell=1}^{n} \|\sum_{\substack{k\in I\\ \neq\ell}} (Ax_\ell,x_k)_0(\lambda_k-\lambda_\ell)^{-1}x_k\|_0|(x,x_\ell)_0|$$

$$\leq [\sum_{\ell=1}^{n} \|\sum_{\substack{k\in I\\ \neq\ell}} (Ax_\ell,x_k)_0(\lambda_k-\lambda_\ell)^{-1}x_k\|_0^2 \sum_{\ell=1}^{n} |(x,x_\ell)_0|^2]^{1/2}$$

$$\leq [\sum_{\ell=1}^{n} \sum_{\substack{k\in I\\ \neq\ell}} |(Ax_\ell,x_k)_0(\lambda_k-\lambda_\ell)^{-1}|^2]^{1/2}\|x\|_0.$$

Hence

(6.10)

$$\|\sum_{k\in I} P_k^n A_n B_0^k\| \leq [\sum_{\ell=1}^{n} \sum_{\substack{k\in I\\ \neq\ell}} |(Ax_\ell,x_k)_0(\lambda_k-\lambda_\ell)^{-1}|^2]^{1/2}$$

$$\leq \|AL^{(2-m)/m}\| [\sum_{\ell=1}^{n} \max_{\substack{k\in I\\ \neq\ell}} |\lambda_\ell^{(m-2)/m}(\lambda_k-\lambda_\ell)^{-1}|^2]^{1/2}$$

$$\leq \|AL^{(2-m)/m}\| [\sum_{\ell=1}^{\infty} (\lambda_\ell^{(m-2)/m}/d_\ell)^2]^{1/2}.$$

Combining (6.8), (6.9) and (6.10) gives

<u>Lemma 6.8</u>. For any n and any $I \subset I_n$,

$$\|\sum_{k\in I} \frac{1}{2\pi i}\int_{\Gamma_k} R_\lambda(L_n)A_nR_\lambda(L_n)\,d\lambda\|$$

$$\leq 2\,\|AL^{(2-m)/m}\|\,[\sum_{k=1}^{\infty} (\lambda_k^{(m-2)/m}/d_k)^2]^{1/2}.$$

The method used to prove Lemmas 6.7 and 6.8 is an adaptation of a method used by Kramer [20] to prove a result on perturbation of spectral operators.

Lemma 6.9. Let k be fixed. Then $\|Q_k^n\|$ is bounded in n for $n \geq k$.

Proof. Using Lemma 3.5 we find that

$$\begin{aligned} \|R_\lambda(L_n)A_n f\|_j &= \|R_\lambda(L)A_n f\|_j \\ &\leq h_j(\lambda)|A_n f|_0 \\ &\leq \beta h_j(\lambda)\|f\|_j \end{aligned}$$

for $f \in S_n$. Let C_k^* be obtained from C_k by increasing the radius by such an amount that C_k^* does not intersect C_ℓ, $\ell \neq k$. If $\lambda \in \partial C_k^*$, then

$$\beta h_j(\lambda) < 1.$$

Thus $R_\lambda(L_n)A_n$ considered as an operator on S_n with the j norm has norm less than one. From this it follows that $I - R_\lambda(L_n)A_n$ is invertible and

$$(6.11) \qquad \|[I - R_\lambda(L_n)A_n]^{-1}f\|_j \leq (1 - \beta h_j(\lambda))^{-1}\|f\|_j .$$

Another application of Lemma 3.5 gives

$$(6.12) \qquad \|R_\lambda(L_n)f\|_j \leq \beta h_j(\lambda)|f|_0 .$$

From (6.11) and (6.12) together with formula (b) of Lemma 3.7 we obtain

$$\begin{aligned}|R_\lambda(\tilde{L}_n)f|_0 &\le \|R_\lambda(\tilde{L}_n)f\|_j \\ &= \|[I - R_\lambda(L_n)A_n]^{-1}R_\lambda(L_n)f\|_j \\ &\le (1 - \beta h_j(\lambda))^{-1}\,\|R_\lambda(L_n)f\|_j \\ &\le (1 - \beta h_j(\lambda))^{-1}\beta h_j(\lambda)|f|_0.\end{aligned}$$

for $f \in S_n$, $\lambda \in \partial C_k^*$. Hence

$$\|Q_k^n\| = \Big\|\frac{1}{2\pi i}\int_{\partial C_k^*} R_\lambda(\tilde{L}_n)\, d\lambda\Big\|$$

$$\le \text{Radius }(\partial C_k^*) \max_{\lambda\in\partial C_k^*} \beta(1 - \beta h_j(\lambda))^{-1}h_j(\lambda),$$

which shows that $\|Q_k^n\|$ is bounded in n.

Now from (6.3) and (6.4) we have

$$(M_n/m_n)^{1/2} \le 4\{1 + \max_{I\subset I_n}\|\sum_{k\in I} (Q_k^n - P_k^n)\|\}^2.$$

Using (6.5) and Lemmas 6.7 and 6.8 we find

$$\begin{aligned}&\|\sum_{k\in I} (Q_k^n - P_k^n)\| \\ &\le \|\sum_{\substack{k\in I\\ k\ge n_1}} \frac{1}{2\pi i}\int_{\partial\Gamma_k} R_\lambda(L_n)A_nR_\lambda(L_n)\, d\lambda\| \\ &\quad + \sum_{\substack{k\in I\\ k\ge n_1}} \Big\|\frac{1}{2\pi i}\int_{\partial\Gamma_k} R_\lambda(L_n)A_nR_\lambda(L_n)A_nR_\lambda(L_n)[I - A_nR_\lambda(L_n)]^{-1}d\lambda\Big\| \\ &\quad + \sum_{\substack{k\in I\\ <n_1}} \|Q_k^n\| + \sum_{\substack{k\in I\\ <n_1}} \|P_k^n\|\end{aligned}$$

$$\leq 2\,\|AL^{(2-m)/m}\|\,[\sum_{k=1}^{\infty}(\lambda_k^{(m-2)/m}/d_k)^2]^{1/2}$$

$$+\ 2\,\|AL^{(2-m)/m}\|^2 \max(3,\ 3\lambda_2/2\lambda_1)^2 \sum_{k=1}^{\infty}(\lambda_k^{(m-2)/m}/d_k)^2$$

$$+\ \sum_{k=1}^{n_1}\|Q_k^n\| + n_1,$$

where n_1 is as in Lemma 6.7. This inequality together with Lemmas 6.1 and 6.9 shows that M_n/m_n is bounded in n. This is summarized in

Theorem 6.1. Under the assumptions listed in the first paragraph of this section

$$M(z_1(n),\cdots,z_n(n))/m(z_1(n),\cdots,z_n(n))$$

is bounded in n.

§7. Applications.

Example 7.1. Consider the eigenvalue problem

$$\begin{cases} f^{(4)} + 10(\sin\tau)f^{(1)}(\tau) = \lambda f(\tau), & 0 \le \tau \le 1, \\ f(0) = f^{(2)}(0) = f(1) = f^{(2)}(1) = 0. \end{cases}$$

This is a problem of the type discussed in Sections 2-6 where

$$Lf = f^{(4)},$$

and

$$Af = 10(\sin\tau)f^{(1)}.$$

The problem determined by L and the boundary conditions is self-adjoint and L has eigenvalues and eigenvectors

$$\lambda_k = \pi^4 k^4,$$

$$x_k(\tau) = \sqrt{2}\sin k\pi\tau, \qquad k = 1,2,\cdots.$$

For these boundary conditions integration by parts shows that we can take $\gamma = 1$ and $\delta = 0$ in Lemma 3.3. We find that $\beta = 10 \sin 1$. Thus from (3.8) we have

$$r_p = \max(\beta(1 + p^2\pi^2)^{1/2}, \tau_p)$$

where τ_p is the unique positive solution of

$$\tau = \beta\left[1 + \left(\frac{2\tau(p+1)^4}{(p+1)^4 - p^4}\right)^{1/2}\right]^{1/2}.$$

An elementary calculation shows that

$$r_p = \beta(1 + p^2\pi^2)^{1/2}.$$

The circles $C_p = \{\lambda \mid |p^4\pi^4 - \lambda| \leq r_p\}$ are mutually disjoint. Thus from Theorems 3.2 and 3.3 we see that the μ_p, the eigenvalues of $\tilde{L} = L + A$, satisfy

$$|p^4\pi^4 - \mu_p| \leq 10(\sin 1)(1 + p^2\pi^2)^{1/2}.$$

By Corollary 2 of Theorem 3.3 they are all real.

Using Lemma 4.11 we have

$$|Ay_p|_0 \leq t_p$$

where t_p is the positive solution of

$$t = \beta(|\mu_p| + t)^{1/4}.$$

Thus

$$|w_p(n)|_0 \leq \frac{t_p}{|\mu_p - \lambda_{n+1}|}, \quad n \geq p,$$

by Lemma 4.10. Since

$$\begin{aligned} |Aw_p(n)|_0 &\leq \beta\, |w_p(n)^{(1)}|_0 \\ &= \beta\left(\int_0^1 |w_p(n)^{(1)}|^2\, d\tau\right)^{1/2} \\ &= \beta\left(\int_0^1 w_p(n)^{(2)}\overline{w_p(n)}\, d\tau\right)^{1/2} \\ &= \beta\,(L^{1/2}w_p(n), w_p(n))_0^{1/2} \\ &= \beta\,|L^{1/4}w_p(n)|_0, \end{aligned}$$

$$|L^{1/4}w_p(n)|_0^2 = \sum_{k=n+1}^{\infty} \lambda_k^{1/2} |(y_p,x_k)_0|^2,$$

and

$$(\mu_p - \lambda_k)(y_p,x_k)_0 = (Ay_p,x_k)_0,$$

we have

$$\begin{aligned} |Aw_p(n)|_0 &\le \beta\left(\sum_{k=n+1}^{\infty} \frac{\lambda_k^{1/2}}{|\mu_p-\lambda_k|^2} |(Ay_p,x_k)_0|^2\right)^{1/2} \\ &\le \beta \max_{k\ge n+1} \frac{\lambda_k^{1/4}}{|\mu_p-\lambda_k|} |Ay_p|_0 \\ &\le \beta t_p \frac{\lambda_{n+1}^{1/4}}{|\mu_p-\lambda_{n+1}|} . \end{aligned}$$

This estimate for $|Aw_p(n)|_0$ is essentially that of Lemma 4.13.

Thus for p fixed and $n \ge p$ there is by Theorem 4.4 an ℓ_0, $1 \le \ell_0 \le n$, such that

$$|\mu_p - \eta_{\ell_0}(n)| \le \frac{\beta(M_n/m_n)^{1/2}\lambda_{n+1}^{1/4}t_p}{[|\mu_p-\lambda_{n+1}|^2 - t_p^2]^{1/2}} \equiv \tilde{\varepsilon}_p(n).$$

We must require that $|\mu_p-\lambda_{n+1}|^2-t_p^2 > 0$ for this result to hold. Similar assumptions must be made relative to other error estimates in this paragraph; we will not mention them explicitly. For $n = p = 1$ we can use $|\mu_1-\lambda_2| \ge \lambda_2-\lambda_1-r_1$ to obtain

$$|\mu_1 - \eta_1(1)| \le \frac{\beta(M_1/m_1)^{1/2}\lambda_2^{1/4}t_1}{[|\mu_1-\lambda_2|^2 - t_1^2]^{1/2}}$$

$$\leq \frac{\beta(M_n/m_n)^{1/2}\lambda_2^{1/4}t_1'}{[(\lambda_2-\lambda_1-r_1)^2 - (t_1')^2]^{1/2}} \equiv \varepsilon_1(1),$$

where t_1' is the positive solution of

$$t = \beta(\lambda_1+r_1+t)^{1/4}.$$

Now suppose we have inequalities of the form

$$|\mu_p - \eta_p(n-1)| \leq \varepsilon_p(n-1), \qquad p = 1,\cdots,n-1.$$

These, together with $|\mu_n - \lambda_n| \leq r_n$, yield

$$\frac{\beta(M_n/m_n)^{1/2}\lambda_{n+1}t_p}{[|\mu_p-\lambda_{n+1}|^2-t_p^2]^{1/2}} \leq \begin{cases} \dfrac{\beta(M_n/m_n)^{1/2}\lambda_{n+1}t_p''}{[(\lambda_{n+1}-\eta_p(n-1)-\varepsilon_p(n-1)]^2-(t'')^2]^{1/2}}, & p<n, \\[2ex] \dfrac{\beta(M_n/m_n)^{1/2}\lambda_{n+1}t_p''}{[(\lambda_{n+1}-\lambda_n-r_n)^2 - (t_p'')^2]^{1/2}}, & p=n, \end{cases}$$

$$\equiv \varepsilon_p(n),$$

where t_p'' is the positive solution of

$$t = \begin{cases} \beta(\eta_p(n-1) + \varepsilon_p(n-1) + t)^{1/4}, & p < n, \\ \beta(\lambda_n + r_n + t)^{1/4}, & p = n. \end{cases}$$

The numbers t_1', t_p'' etc., can be computed numerically. Thus we have

$$|\mu_p - \eta_{\ell_0}(n)| \leq \varepsilon_p(n)$$

for some ℓ_0, $1 \leq \ell_0 \leq n$. We can conclude $\ell_0 = p$ in this example as follows. From the computed values of

$\eta_1(n),\cdots,\eta_n(n)$ and $\varepsilon_p(n)$ it will be clear that $|\mu_p-\eta_k(n)| > \varepsilon_p(n)$, $k = 1,\cdots,p-1,p+1,\cdots,n$. Hence the only choice for ℓ_0 is p. This will also be true in many other cases and the inequalities $|\mu_p-\eta_k(n)| > \varepsilon_p(n)$, $k \neq p$, are simple to verify if $\varepsilon_p(n)$ is sufficiently small since $\eta_k(n) \in C_k$, $k = 1,\cdots,n$. Proceeding inductively we define $\varepsilon_p(n)$, $1 \leq p \leq n$, $n = 1,2,\cdots$ and establish

$$|\mu_p-\eta_p(n)| \leq \varepsilon_p(n).$$

We obtain $\varepsilon_p(n)$ from $\tilde{\varepsilon}_p(n)$ by using the estimate for μ_p which we had in the $(n-1)^{st}$ stage of the estimating procedure. Clearly $\lim\limits_{n\to\infty} \varepsilon_p(n) = 0$ for p fixed. This discussion on the definition of $\varepsilon_p(n)$ and the proof that $\ell_0 = p$ pertains to the other examples in this section.

Values for $\eta_p(n)$, $\varepsilon_p(n)$ were computed for $1 \leq p \leq n$, $n = 1,2,\cdots,20$. Tables 1 and 2 contain the computed values for $\eta_p(n)$ and $\varepsilon_p(n)$ for $p = 1,2,$ and 5. Table 3 contains the values of $\eta_p(20)$ and $\varepsilon_p(20)$ for $p = 1,\cdots,20$. The values in Table 3 give the best approximations for μ_p, $p = 1,\cdots,20$, that were obtained. $\eta_p(n)$, $p = 1,\cdots,n$, are the eigenvalues of the matrix $((\tilde{L}_n x_k, x_\ell)_0)$, $1 \leq \ell,k \leq n$. We will denote this matrix as well as the corresponding operator by $\tilde{L}_n$. This is a real non-symmetric matrix. These eigenvalues were computed as follows. Using the QR method of Francis [18], as implemented in a program originally written by Professor B.N. Parlett, the eigenvalues were computed; we will call these numbers the first approximations to the eigenvalues of $\tilde{L}_n$. Next the eigenvectors were computed. Then a matrix P was formed with these approximate eigenvectors as columns. Now $P^{-1}\tilde{L}_nP$ has the same eigenvalues as $\tilde{L}_n$ and, assuming that the column vectors of P are close to the eigenvectors of $\tilde{L}_n$, $P^{-1}\tilde{L}_nP$ should be approximately diagonal and hence the diagonal elements of $P^{-1}\tilde{L}_nP$ should be approximately equal to the eigenvalues of $\tilde{L}_n$. We will call the diagonal elements of $P^{-1}\tilde{L}_nP$ the $2^{\underline{nd}}$ approximations to the eigenvalues of $\tilde{L}_n$. The difference between the eigenvalues of $\tilde{L}_n$ and their $2^{\underline{nd}}$ approximations can be estimated with Gershgorin's theorem. We use these computed diagonal elements of $P^{-1}\tilde{L}_nP$ for $\eta_1(n),\cdots,\eta_n(n)$.

In general the off diagonal elements of $P^{-1}\tilde{L}_nP$ were small. Consider for example $n = 4$. The QR method produces

93.2076349,

1,554.28815,

7,885.89093

and

24,932.4448.

The diagonal elements of $P^{-1}\tilde{L}_nP$ are

$$93.2076495$$

$$1{,}544.28838,$$

$$7{,}885.89194,$$

and

$$24{,}932.4458$$

and the row sums $\sum_{\ell \neq k} |a_{k\ell}|$ where $(a_{k\ell}) = P^{-1}\tilde{L}_n P$ are

$$.274 \times 10^{-9},$$

$$.411 \times 10^{-3},$$

$$.736 \times 10^{-1},$$

and

$$.813 \times 10^{-3}.$$

Thus, for instance,

$$|\eta_1(4) - 93.2076495| \leq .274 \times 10^{-9}.$$

These estimates can be refined by scaling in the following way. Suppose we want a better estimate for $\eta_3(4)$. The eigenvalues we are seeking are also the eigenvalues of the matrix $(b_{k\ell})$ which we obtain from $(a_{k\ell})$ by multiplying the $3^{\underline{rd}}$ row by τ and the $3^{\underline{rd}}$ column by τ^{-1}. The diagonal elements of $(b_{k\ell})$ and $(a_{k\ell})$ are the same but the row sums for $(b_{k\ell})$ depend on τ. Now choose τ as small as possible while keeping the $3^{\underline{rd}}$ Gershgorin disk separate from the other disks. Here we can choose $\tau = 10^{-10}$; with this choice for τ the $3^{\underline{rd}}$ row sum for $(b_{k\ell})$ is $.736 \times 10^{-11}$. We can perform such scaling for each eigenvalue. For $n = 10$ the radii of the Gershgorin disks before scaling were

$$.197 \times 10^{-6},$$
$$.881 \times 10^{-5},$$
$$.124 \times 10^{-3},$$
$$.723 \times 10^{-2},$$
$$.253 \times 10^{-2},$$
$$.279,$$
$$.209 \times 10^{2},$$
$$.784 \times 10^{2},$$
$$.789 \times 10^{1},$$
$$.451 \times 10^{-4}.$$

The ordering here is such that the 3rd radii gives an estimate for $\eta_3(10)$ for example. After scaling the corresponding radii are

$$.197 \times 10^{-6},$$
$$.603 \times 10^{-8},$$
$$.196 \times 10^{-7},$$
$$.424 \times 10^{-6},$$
$$.704 \times 10^{-7},$$
$$.427 \times 10^{-5},$$
$$.194 \times 10^{-3},$$
$$.475 \times 10^{-3},$$
$$.329 \times 10^{-4},$$
$$.451 \times 10^{-4}.$$

These calculations are affected by the round off error which occurs in the inversion of P and in the calculation of $P^{-1}\tilde{L}_n P$. P^{-1} was computed in single precision; for $n = 4$ no element of $P^{-1}P - I$ exceeds in absolute value $.3 \times 10^{-7}$. Double precision was used to multiply P^{-1}, $\tilde{L}_n$, and P. In addition there is the error which arises when the matrix $\tilde{L}_n$ is entered in the computer. To reduce this the entries in $\tilde{L}_n$ were computed in double precision and then truncated to single precision. The remaining calculations were all done in single precision.

For a complete discussion of this method of computing a $2^{\underline{nd}}$ set of approximations for the eigenvalues of a matrix from an initial set of approximations for the eigenvalues and eigenvectors and estimation of the resulting accuracy using Gershgorin's theorem together with scaling see Wilkenson [42].

In the tables of this section all figures are rounded to the number of digits presented. The notation $.487|2$ means $.487 \times 10^2$. A dash in the table indicates the corresponding number is the same as the number directly above it. All of the computations were performed on the UNIVAC 1108 Computer of the Computer Science Center of the University of Maryland.

Table 1

$\eta_p(n)$

n\p	1		2		5		15	
1	.9309239	2						
2	.9320139	2	.15542023	4				
3	.9320682	2	.15542827	4				
4	.9320765	2	.15542884	4				
5	.9320786	2	.15542895	4	.608764151	5		
6	.9320792	2	.15542898	4	.608764551	5		
7	.9320795	2	.15542899	4	.608764592	5		
8	.9320796	2	.15542900	4	.608764597	5		
9	.9320797	2	——	4	.608764601	5		
10	——	2	——	4	.608764605	5		
11	——	2	——	4	.608764602	5		
12	——	2	——	4	.608764603	5		
13	——	2	——	4	.608764605	5		
14	——	2	——	4	.608764606	5		
15	——	2	——	4	.608764606	5	.493133098	7
16	——	2	——	4	.608764603	5	.493133097	7
17	——	2	——	4	.608764602	5	.493133101	7
18	——	2	——	4	.608764607	5	.493133101	7
19	——	2	——	4	.608764603	5	.493133100	7
20	——	2	——	4	.608764606	5	.493133100	7

Table 2

$\varepsilon_p(n)$

n\p	1		2		5		15	
1	.109510	1						
2	.28955	0	.6910	0				
3	.12111	0	.2455	0				
4	.6187	-1	.1210	0				
5	.3577	-1	.691	-1	.3279	0		
6	.2252	-1	.432	-1	.1442	0		
7	.1508	-1	.289	-1	.843	-1		
8	.1059	-1	.203	-1	.555	-1		
9	.772	-2	.148	-1	.390	-1		
10	.580	-2	.111	-1	.287	-1		
11	.447	-2	.85	-2	.218	-1		
12	.351	-2	.67	-2	.170	-1		
13	.281	-2	.54	-2	.135	-1		
14	.229	-2	.44	-2	.110	-1		
15	.188	-2	.36	-2	.90	-2	.12	0
16	.157	-2	.30	-2	.75	-2	.6	-1
17	.132	-2	.25	-2	.63	-2	.4	-1
18	.113	-2	.21	-2	.54	-2	.3	-1
19	.97	-3	.18	-2	.46	-2	.2	-1
20	.83	-3	.16	-2	.40	-2	.2	-1

Table 3

p	$\eta_p(20)$		$\varepsilon_p(20)$	
1	.932079727	2	.834	-3
2	.155429005	4	.159	-2
3	.788589868	4	.237	-2
4	.249325000	5	.317	-2
5	.608764606	5	.396	-2
6	.126237964	6	.477	-2
7	.233875011	6	.560	-2
8	.398983423	6	.645	-2
9	.639096829	6	.735	-2
10	.974086689	6	.832	-2
11	.142616228	7	.939	-2
12	.201987069	7	.106	-1
13	.278209682	7	.120	-1
14	.374206342	7	.138	-1
15	.493133100	7	.160	-1
16	.638379791	7	.191	-1
17	.813570038	7	.235	-1
18	.102256125	8	.309	-1
19	.126944459	8	.455	-1
20	.155854502	8	.891	-1

Example 7.2. Consider the eigenvalue problem for the Mathieu equation

$$\begin{cases} -f^{(2)}(\tau) + 2q(\cos 2\tau)f(\tau) = \lambda f(\tau), \\ f(0) = f(\pi) = 0, \end{cases}$$

where q is a complex constant. The eigenvalues of this problem will be the characteristic values of the Mathieu equation associated with odd periodic solutions. Here we let $Lf = -f^{(2)}$ and $Af = 2q(\cos 2\tau)f$. The eigenvalues and eigenvectors of L are

$$\lambda_k = k^2,$$

$$x_k(\tau) = \sqrt{\frac{2}{\pi}} \sin k\tau, \qquad k = 1,2,\cdots.$$

$\tilde{L}$ will be non self-adjoint unless q is real.

For this example we easily find that

$$C_p = \{\lambda \mid |\lambda - p^2| \leq 2|q|\}.$$

If $|q| < 3/4$ these circles will be mutually disjoint and there will be one eigenvalue of $\tilde{L}$ in each C_p.

Using Theorem 4.4 and Lemma 4.10 we find that for p fixed and $n \geq p$ there is an ℓ_0, $1 \leq \ell_0 \leq n$, such that

$$|\mu_p - \eta_{\ell_0}(n)| \leq \frac{(M_n/m_n)^{1/2} 4|q|^2}{[|\mu_p - \lambda_{n+1}|^2 - 4|q|^2]^{1/2}} \equiv \tilde{\varepsilon}_p(n).$$

As in Example 7.1 we can define error estimates $\varepsilon_p(n)$, using at the n^{th} stage the estimates for μ_p from the

$(n-1)^{\underline{st}}$ stage in order to estimate $|\mu_p - \lambda_{n+1}|$. We can show that $\ell_0 = p$ as in Example 7.1.

The computations were carried out for $q = \frac{1+i}{2\sqrt{2}}$ with n going from 1 to 20. The ℓ, k element of the matrix of $\tilde{L}_n$ with respect to $x_1, \cdots, x_n$ is

$$\delta_{\ell k} k^2 + \frac{4q}{\pi} \int_0^{\pi} \cos 2\tau \ \sin k\tau \ \sin \ell\tau \ d\tau .$$

The values for $\eta_1(n), \cdots, \eta_n(n)$ were computed using the power method as developed by E.E. Osborne [25]; the computer program was written by Ehrlich [14]. These values were not refined as in Example 7.1. Table 4 gives the real and imaginary parts of $\eta_p(n)$ and Table 5 gives the values of $\varepsilon_p(n)$ for $p = 1$, 2, and 10. Table 6 contains the values of $\eta_p(20)$ and $\varepsilon_p(20)$ for $p = 1, \cdots, 20$. The characteristic values for the Mathieu equation for complex values of q have recently been tabulated by Blanch and Clemm [11].

Table 4

$\eta_p(n)$

n/p	1				2				10			
1	.64645	0	-.35355	0								
2	.64645	0	-.35355	0	.400000	1	.000000	0				
3	.64508	0	-.38341	0	.400000	1	.454909	-27				
4	.64508	0	-.38341	0	.399996	1	-.208332	-1				
5	.64511	0	-.38342	0	.399996	1	-.208332	-1				
6	——	0	——	0	.399998	1	-.208333	-1				
7	——	0	——	0	——	1	——	-1				
8	——	0	——	0	——	1	——	-1				
9	——	0	——	0	——	1	——	-1				
10	——	0	——	0	——	1	——	-1	.10000000	3	.69444442	-2
11	——	0	——	0	——	1	——	-1	——	3	.69444444	-2
12	——	0	——	0	——	1	——	-1	——	3	.12626263	-2
13	——	0	——	0	——	1	——	-1	——	3	.12626262	-2
14	——	0	——	0	——	1	——	-1	.99999999	2	.12626266	-2
15	——	0	——	0	——	1	——	-1	——	2	.12626265	-2
16	——	0	——	0	——	1	——	-1	——	2	.12626265	-2
17	——	0	——	0	——	1	——	-1	——	2	.12626266	-2
18	——	0	——	0	——	1	——	-1	——	2	.12626265	-2
19	——	0	——	0	——	1	——	-1	——	2	.12626265	-2
20	——	0	——	0	——	1	——	-1	——	2	.12626266	-2

Table 5

$\varepsilon_p(n)$

n\p	1		2		10	
1	.57735	0				
2	.12955	0	.25820	0		
3	.7132	-1	.9264	-1		
4	.4466	-1	.5190	-1		
5	.3106	-1	.3433	-1		
6	.2269	-1	.2439	-1		
7	.1733	-1	.1830	-1		
8	.1366	-1	.1426	-1		
9	.1105	-1	.1144	-1		
10	.912	-2	.938	-2	.5495	-1
11	.747	-2	.765	-2	.2438	-1
12	.636	-2	.649	-2	.1554	-1
13	.548	-2	.557	-2	.1115	-1
14	.477	-2	.484	-2	.857	-2
15	.419	-2	.425	-2	.686	-2
16	.371	-2	.376	-2	.566	-2
17	.331	-2	.335	-2	.478	-2
18	.297	-2	.300	-2	.410	-2
19	.268	-2	.270	-2	.356	-2
20	.243	-2	.244	-2	.313	-2

Table 6

p	$\eta_p(20)$				$\varepsilon_p(20)$	
1	.64511279	0	-.38341700	0	.24276410	-2
2	.39999774	1	-.20833277	-1	.24462794	-2
3	.90013346	1	.14238454	-1	.24746011	-2
4	.16000023	2	.83332760	-2	.25153524	-2
5	.24999999	2	.52084830	-2	.25697720	-2
6	.36000000	2	.35714295	-2	.26395698	-2
7	.49000001	2	.26041665	-2	.27271085	-2
8	.64000000	2	.19841267	-2	.28356169	-2
9	.81000000	2	.15625002	-2	.29695245	-2
10	.99999999	2	.12626266	-2	.31349869	-2
11	.12100000	3	.10416668	-2	.33407274	-2
12	.14400000	3	.87412561	-3	.35994473	-2
13	.16900000	3	.74404793	-3	.39302948	-2
14	.19600000	3	.64102585	-3	.43634552	-2
15	.22500000	3	.55803573	-3	.49493354	-2
16	.25600000	3	.49019597	-3	.57787746	-2
17	.28899999	3	.43402798	-3	.70335894	-2
18	.32400000	3	.38699679	-3	.91383116	-2
19	.36100000	3	.34722220	-2	.13368434	-1
20	.40000000	3	.32894737	-2	.26733725	-1

<u>Example 7.3</u>. Let $Lf = f^{(4)}$ and $Af(\tau) = \tau f^{(2)}(\tau)$ and consider the problem

$$\begin{cases} Lf + Af = \lambda f, \\ f(0) = f^{(1)}(0) = f(1) = f^{(1)}(1) = 0. \end{cases}$$

The eigenvalues of L are $\lambda_n = \rho_n^4$ where $\rho_1, \rho_2, \cdots$ are the positive solutions of $1 = \cosh\rho \cos\rho$. The unnormalized eigenvectors of L are

$$x_n(\tau) = (\cos\rho_n - \sin\rho_n - e^{-\rho_n})e^{\rho_n\tau} + (-\cos\rho_n - \sin\rho_n + e^{\rho_n})e^{-\rho_n\tau}$$

$$+ 2(\cosh\rho_n - \cos\rho_n)\sin\rho_n\tau + 2(\sin\rho_n - \sinh\rho_n)\cos\rho_n\tau.$$

In (3.1) we can let $\gamma = 1, \delta = 0$ and we find that

$$r_p = (1 + \lambda_p^{1/2} + \lambda_p)^{1/2}.$$

Thus the eigenvalues of $\tilde{L}$ occur one each in the circles $C_1, C_2, \cdots$ and they are real.

By Lemma 4.11 we have

$$|Ay_p|_0 \leq t_p$$

where t_p is the positive solution of $t = (|\mu_p|+t)^{1/2}$, i.e.,

$$t_p = \frac{1 + (1 + 4|\mu_p|)^{1/2}}{2}.$$

Using Lemma 4.10 we find

$$|w_p(n)|_0 \leq \tau_p|\mu_p - \lambda_{n+1}|^{-1}.$$

Integration by parts shows that

$$|Aw_p(n)|_0 \leq |Lw_p(n)|_0$$

and, proceeding as in the proof of Lemma 4.13, we obtain

$$|Aw_p(n)|_0 \leq \frac{\lambda_{n+1}^{1/2}\tau_p}{|\mu_p - \lambda_{n+1}|} .$$

Combining these results with Theorem 4.4 shows that

$$|\mu_p - \eta_{\ell_0}(n)| \leq \frac{(M_n/m_n)^{1/2}\lambda_{n+1}^{1/2}(1+(1+4|\mu_p|)^{1/2})}{[4|\mu_p-\lambda_{n+1}|^2 - (1+(1+4|\mu_p|)^{1/2})^2]^{1/2}} \equiv \tilde{\varepsilon}_p(n)$$

if $n \geq p$ for some $\ell_0 \leq n$. Just as in Example 7.1 we estimate $\tilde{\varepsilon}_p(n)$ by $\varepsilon_p(n)$, using at the $n^{\underline{th}}$ stage the estimates for μ_p obtained at the $(n-1)^{\underline{st}}$ stage, and show that $\ell_0 = p$.

The computational procedures used in this example are the same as those used in Example 7.1. Table 7 lists the values of λ_p and r_p for $p \leq 10$, Tables 8 and 9 list the values of $\eta_p(n)$ and $\varepsilon_p(n)$ for $n \leq 20$ and $p = 1$ and 5, and Table 10 lists the values of $\eta_p(20)$ and $\varepsilon_p(20)$ for $p \leq 20$.

Table 7

p	λ_p		r_p	
1	.5006	3	.229	2
2	.38035	4	.622	2
3	.146176	5	.1214	3
4	.399438	5	.2004	3
5	.891354	5	.2991	3
6	.1738813	6	.4175	3
7	.3082084	6	.5557	3
8	.5084815	6	.7136	3
9	.7934031	6	.8912	3
10	.11840136	7	.10886	4

Table 8

$\eta_p(n)$

n\p	1		5	
1	.4950003	3		
2	.4949835	3		
3.	.4949857	3		
4	.4949863	3		
5	.4949868	3	.89007252	5
6	.4949870	3	.89007080	5
7	.4949872	3	.89007077	5
8	.4949872	3	.89007074	5
9	.4949873	3	.89007073	5
10	.4949873	3	.89007073	5
11	.4949874	3	.89007074	5
12	——	3	——	5
13	——	3	——	5
14	——	3	——	5
15	——	3	——	5
16	——	3	——	5
17	.4949875	3	——	5
18	——	3	——	5
19	——	3	——	5
20	——	3	——	5

Table 9

$\varepsilon_p(n)$

n\p	1		5	
1	.4397	0		
2	.1963	0		
3	.1162	0		
4	.773	-1		
5	.552	-1	.1491	1
6	.414	-1	.763	0
7	.322	-1	.512	0
8	.258	-1	.381	0
9	.211	-1	.299	0
10	.176	-1	.243	0
11	.149	-1	.203	0
12	.128	-1	.172	0
13	.111	-1	.148	0
14	.97	-2	.129	0
15	.85	-2	.114	0
16	.76	-2	.101	0
17	.68	-2	.90	-1
18	.61	-2	.81	-1
19	.55	-2	.73	-1
20	.50	-2	.66	-1

Table 10

p	$\eta_p(20)$		$\varepsilon_p(20)$	
1	.494987465	3	.5027	-2
2	.378185872	4	.1370	-1
3	.145702886	5	.2680	-1
4	.398608925	5	.4431	-1
5	.890070740	5	.6631	-1
6	.173697687	6	.9297	-1
7	.307959661	6	.1246	0
8	.508157720	6	.1615	0
9	.792994409	6	.2047	0
10	.118351009	7	.2550	0
11	.170308296	7	.314	0
12	.237742900	7	.385	0
13	.323460204	7	.471	0
14	.430499370	7	.578	0
15	.562133345	7	.718	0
16	.721868858	7	.909	0
17	.913446414	7	.119	1
18	.114084031	8	.165	1
19	.140825862	8	.256	1
20	.172014325	8	.529	1

Example 7.4. Consider an eigenvalue problem of the form

$$\begin{cases} -f^{(2)}(\tau) + a(\tau)f^{(1)}(\tau) = \lambda f(\tau), \\ f^{(1)}(0) + f(0) = 0, \\ f^{(1)}(1) - f(1) = 0. \end{cases}$$

If we let $Lf = -f^{(2)}$ then L has one negative eigenvalue. If L is redefined by $Lf = -f^{(2)} + 5f$ then L is positive definite. This will enable us to apply Lemma 4.15. Hence we consider the eigenvalue equation $-f^{(2)} + af^{(1)} + 5f = \lambda f$. The eigenvalues of the original problem are found by subtracting 5 from the eigenvalues of this problem. For this example we show how explicit values for the constants in inequalities (3.1) and (4.4) can be obtained.

Redheffer [26] proved a version of the inequality in Lemma 5.3 when $\ell = 2$ in which the constants are, in a certain sense, best possible. He showed that

$$|f|_1^2 \leq r|f|_2^2 + (12 + \tfrac{1}{r})|f|_0^2$$

for all $f \in H_2$ and $r > 0$. If $|f|_2 \leq |f|_0$ we can let $r = 2$ in this inequality to obtain

$$|f|_1 \leq \sqrt{2}\,|f|_0^{1/2}|f|_2^{1/2} + \sqrt{12.5}\,|f|_0.$$

If $|f|_2 > |f|_0$ we can let $r = |f|_0|f|_2^{-1}$ and find

$$|f|_1 \leq \sqrt{2}\,|f|_0^{1/2}|f|_2^{1/2} + \sqrt{12}\,|f|_0.$$

Thus in inequality (3.1) we can let $\gamma = \sqrt{2}$ and $\delta = \sqrt{12.5}$.

Now we consider inequality (4.4). Proceeding as in Section 5 we have

$$(Lf,f)_0 = \int_0^1 (-f^{(2)} + 5f)\overline{f}\, d\tau$$
$$= |f|_1^2 + 5|f|_0^2 - f^{(1)}\overline{f}\,\big|_0^1,$$

and using the boundary conditions,

$$(Lf,f)_0 = |f|_1^2 + 5|f|_1^2 - |f(1)|^2 - |f(0)|^2.$$

Following the proof of Lemma 5.2 in [2] we find

$$|f(\tau)|^2 \le h|f|_1^2 + \frac{2}{h}|f|_0^2$$

for $f \in H_1$ and $0 < h \le 1/2$. Using this to bound $|f(0)|^2$ and $|f(1)|^2$ we find

$$(Lf,f)_0 \ge (1-2h)|f|_1^2 + (5 - \frac{4}{h})|f|_0^2$$
$$= (1-2h)\,\|f\|_1^2 + (4 + 2h - \frac{4}{h})|f|_0^2.$$

Letting $h = 1/4$ we obtain

$$\|f\|_1^2 \le 2(Lf,f)_0 + 23|f|_0^2$$

for all $f \in H_2$ which satisfy the boundary conditions. Thus in (4.4) we can let $c_1 = 2$ and $c_2 = 23$. These values can then be used in Lemma 4.15.

Example 7.5. The eigenvalue problem

$$\begin{cases} -f^{(2)} + af^{(1)} = \lambda f, \\ f(0) = f(1) = 0, \end{cases}$$

can be treated by introducing a new independent variable:

$$g(\tau) = f(\tau)\exp\left(-\frac{1}{2}\int_0^{\tau} a(s)\,ds\right).$$

This transforms the above problem into

$$\begin{cases} -g^{(2)}(\tau) + \left(\frac{(a(\tau))^2}{4} - \frac{a^{(1)}(\tau)}{4}\right) g(\tau) = \lambda g(\tau), \\ g(0) = g(1) = 0. \end{cases}$$

Hence we can consider the problem as a zero order perturbation of a $2^{\underline{nd}}$ order problem rather than as a $1^{\underline{st}}$ order perturbation. This transformation, applied to Example 7.4, would produce non self-adjoint boundary conditions unless $a(0)$ and $a(1)$ are real.

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