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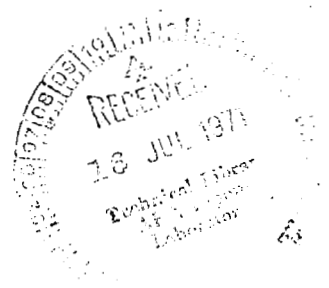
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# FORMATION AND PROPAGATION OF A SHOCK WAVE FROM A CONCAVE BOUNDARY

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# FORMATION AND PROPAGATION OF A SHOCK WAVE FROM A CONCAVE BOUNDARY

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## SUMMARY

This report shows when and where a shock wave forms in a two-dimensional flow field and presents a method for predicting the shape and strength of the shock wave. Knowledge of the point of formation, the strength, and the shape of shock waves are important in the design of inlets and the design of blading for turbomachines.

The present method consists of first finding the envelope of characteristics forming in the flow field from the bounding streamline, as suggested by Tollmien. In a simple wave field, where one family of characteristics consists of straight lines, the envelope can be given in closed form; in a complex wave field, where both families are generally curved, the envelope must result from a solution of the whole flow field. Once the envelope is found, criteria and methods are presented for locating the shock within the envelope. In simple waves, the shock path results from a point by point integration; in complex waves, the principle of minimum entropy production may be used to pick out the most likely shock location. Specific examples of the simple wave case are presented.

## INTRODUCTION

The object of this report is to solve the following problem: For given upstream conditions and a given boundary profile, determine whether a shock wave will occur and, if so, find the location, strength, and shape of the shock wave.

The motivation for this study comes from the current interest in supersonic inlets and supersonic operation of turbines and turbocompressors. Focused characteristic inlets have been successfully designed for some time; however, at high supersonic Mach number hidden shocks appear. These unexpected shock waves, which decrease

the efficiency of the inlet, have not yet been analyzed. Furthermore, there is interest in finding the path of shock waves that may occur during off-design operation. There is a similar need to study the formation of shock waves within turbomachinery blade rows and the propagation of shocks between stages. Impingement of a shock wave on a blade could radically alter blade loading and flow incidence. This off-design loading may enhance blade flutter and lead to early blade failure. Off-design flow incident would impair performance.

M. B. Abbott (ref. 1) shows that shock-wave formation always results from the convergence of characteristics. As a compression wave processes the gas, the speed of sound increases behind the wave. In a train of compression waves, for instance, those coming from a concave boundary or an accelerating piston, trailing waves tend to overtake the leading wave. If these trailing waves coalesce into the lead wave, the wave front steepens, and its speed increases. As coalescence continues, the process through the wave finally becomes discontinuous; that is, a shock wave forms.

In steady, plane, supersonic flow, characteristics are propagation paths for Mach waves. So the coalescence of trailing waves shows up graphically as the formation of an envelope of characteristics (refs. 2 and 3). This process is shown in figure 1. Usually the envelope will be cusped in its initial point. The initial point of the envelope, the cusp, is also the initial point of the shock wave. Courant and Friedrichs (ref. 2) Oswatitsch and Kuerti (ref. 3) and others discuss the general nature of the envelope but do not give details on its calculation. Tollmien (ref. 4) gives more detail but then concentrates on a series extension of the analytical solution downstream of the envelope. He does not calculate the shock shape within the envelope.

The present treatment considers a characteristic envelope and the resulting shock wave that form in the external flow over a concave profile. The envelope forms in a

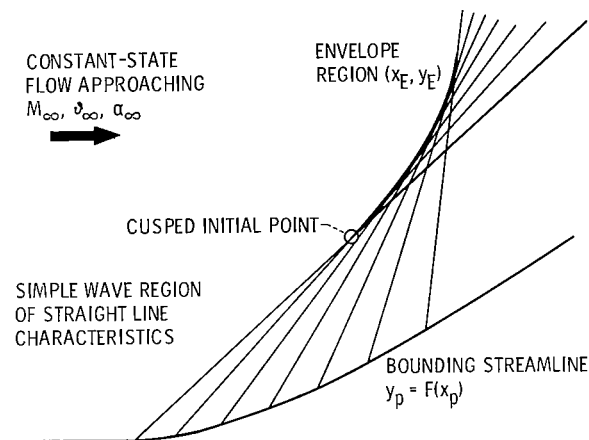


Figure 1. - Formation of envelope in simple wave flow.

simple wave region, so its shape and its starting point can be given in closed form. A point by point integration fixes the shock path once the initial point of the envelope is known.

## SYMBOLS

A	mean slope along shock wave, defined after eq. (7a)
B	slope of characteristic, defined after eq. (7a)
$E_1, E_2$	parametric form of locus of characteristic envelope
F	function giving location of bounding streamline
G	parametric form of family of characteristics
K	$\sqrt{(\gamma + 1)/(\gamma - 1)}$
M	Mach number
m	slope
q	magnitude of flow velocity
R	channel radii of curvature <sup>1</sup>
s	entropy
x, y	spatial coordinates, physical plane <sup>1</sup>
$\alpha$	Mach angle
$\gamma$	ratio of specific heats
$\xi, \eta$	characteristic net variables
$\pi$	pressure ratio across shock wave
$\vartheta$	flow deflection angle
$\omega$	Prandtl-Meyer function
Subscripts:	
A	upstream border of characteristic envelope
B	return flow region of characteristic envelope
C	downstream zone of characteristic envelope
e	envelope

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<sup>1</sup>In analysis of channel flow, spatial distances are measured in entrance half heights.

- p boundary profile  
 $\infty$  condition of undisturbed flow

## ANALYSIS

### Envelope Conditions and Shape

If, as in figure 1, a compression region borders a region of constant state, at least one family of characteristics will consist of straight lines. This family of lines can be written as

$$G(x, y; x_p) = y - F(x_p) - (x - x_p) \left[ \tan(\vartheta_p + \alpha_p) \right] = 0 \quad (1)$$

where  $F(x_p)$  is the contour of the bounding profile or bounding streamline. For given upstream conditions, the angle  $\alpha_p$  of each Mach line to the approaching flow direction is a function of profile position  $x_p$  through

$$\vartheta_p = \tan^{-1} \left( \frac{dy_p}{dx_p} \right) = - \left\{ K \left[ \tan^{-1} \left( \frac{\cot \alpha_p}{K} \right) - \tan^{-1} \left( \frac{\cot \alpha_\infty}{K} \right) \right] + (\alpha_p - \alpha_\infty) \right\}$$

If an envelope exists, then neighboring characteristics will intersect along the envelope. And a characteristic will intersect the envelope at a point of tangency. According to Courant (ref. 5), these conditions are both necessary and sufficient, subject to the constraint that both the envelope and the members of the family in fact have well defined tangents. Thus, since a well defined characteristic field is given, the conditions for the existence of an envelope are in fact conditions on the shape of the bounding streamline.

Assume that an envelope exists and is given by

$$\left. \begin{aligned} x_e &= E_1(x_p) \\ y_e &= E_2(x_p) \end{aligned} \right\} \quad (2)$$

Some characteristic intersects the envelope in the point  $(x^0, y^0)$  such that  $(x^0, y^0)$  satisfies the equation

$$C \left( \begin{matrix} x^0 \\ y^0 \end{matrix}; x_p \right) = 0$$

So we may write

$$\frac{\partial G}{\partial x} \left( \frac{dE_1}{dx_p} \right) + \frac{\partial G}{\partial y} \left( \frac{dE_2}{dx_p} \right) + \left( \frac{\partial G}{\partial x_p} \right) = 0 \quad (3)$$

The slope of the characteristic through the point of intersection is

$$\frac{dy}{dx} = - \frac{(\partial G / \partial x)}{(\partial G / \partial y)} \quad (4)$$

The slope of the envelope is

$$\left( \frac{dy_e}{dx_e} \right) = \frac{(dE_2/dx_p)}{(dE_1/dx_p)} \quad (4a)$$

If this is in fact an envelope; these slopes must be equal. Consequently,

$$\frac{\partial G}{\partial x} \left( \frac{dE_1}{dx_p} \right) + \frac{\partial G}{\partial y} \left( \frac{dE_2}{dx_p} \right) = 0$$

The result is that

$$\left( \frac{\partial G}{\partial x_p} \right) = 0 \quad (5)$$

which along with  $G(x, y; x_p) = 0$  presents two equations to relate  $x$  and  $y$  to  $x_p$  along the envelope.

For the characteristic family from equation (1), the envelope is generally given by

$$\left. \begin{aligned} (x_e - x_p) &= \frac{\tan(\vartheta_p + \alpha_p) - \tan \vartheta_p}{\left[ \sec^2(\vartheta_p + \alpha_p) \right] \left[ 1 + \left( \frac{d\alpha_p}{d\vartheta_p} \right) \left( \frac{d\vartheta_p}{dx_p} \right) \right]} \\ (y_e - y_p) &= \frac{\left[ \tan(\vartheta_p + \alpha_p) - \tan \vartheta_p \right] \tan(\vartheta_p + \alpha_p)}{\left[ \sec^2(\vartheta_p + \alpha_p) \right] \left[ 1 + \left( \frac{d\alpha_p}{d\vartheta_p} \right) \left( \frac{d\vartheta_p}{dx_p} \right) \right]} \end{aligned} \right\} \quad (6)$$

Notice that the curvature ( $d^2y_p/dx_p^2$ ) must be positive (i. e., a concave profile) for an envelope to exist in a supersonic compression flow. If the curvature is infinite, as in a corner, the envelope begins on the profile itself.

## Shock Wave Region

Tollmien (ref. 4) shows that the shock wave and the envelope have a common starting point: the point for which, in this case,

$$\frac{\partial^2 G}{\partial x_p^2} = 0 \quad (6a)$$

Given this initial point, it is possible to integrate the equation

$$\left(\frac{dy}{dx}\right)_s = \tan \vartheta_s \quad (7)$$

point by point along the shock wave -  $\vartheta_s$  being a known function of the upstream Mach number and the flow angle  $\vartheta_p$ .

An alternate difference technique could use a simultaneous solution of the equations

$$\left. \begin{aligned} \frac{y_{s,n+1} - y_{s,n}}{x_{s,n+1} - x_{s,n}} &= \frac{\tan \vartheta_{s,n+1} + \tan \vartheta_{s,n}}{2} \\ \frac{y_{s,n+1} - y_{p,n+1}}{x_{s,n+1} - x_{p,n+1}} &= \tan(\vartheta_{p,n+1} + \alpha_{p,n+1}) \end{aligned} \right\} \quad (7a)$$

which amounts to about the same thing.

Figure 2 shows the envelope and shock wave which develop in a flow that approaches a profile of the form  $F(x_p) = \left(1 - e^{-ax_p^2}\right)$  up to its point of inflexion. This curve has an inflexion at  $x_p^2 = 1/2a$ ; beyond this point the profile is continued as a straight line at the angle

$$\vartheta_p = \tan^{-1}\left(\sqrt{2a} e^{-1/2}\right)$$



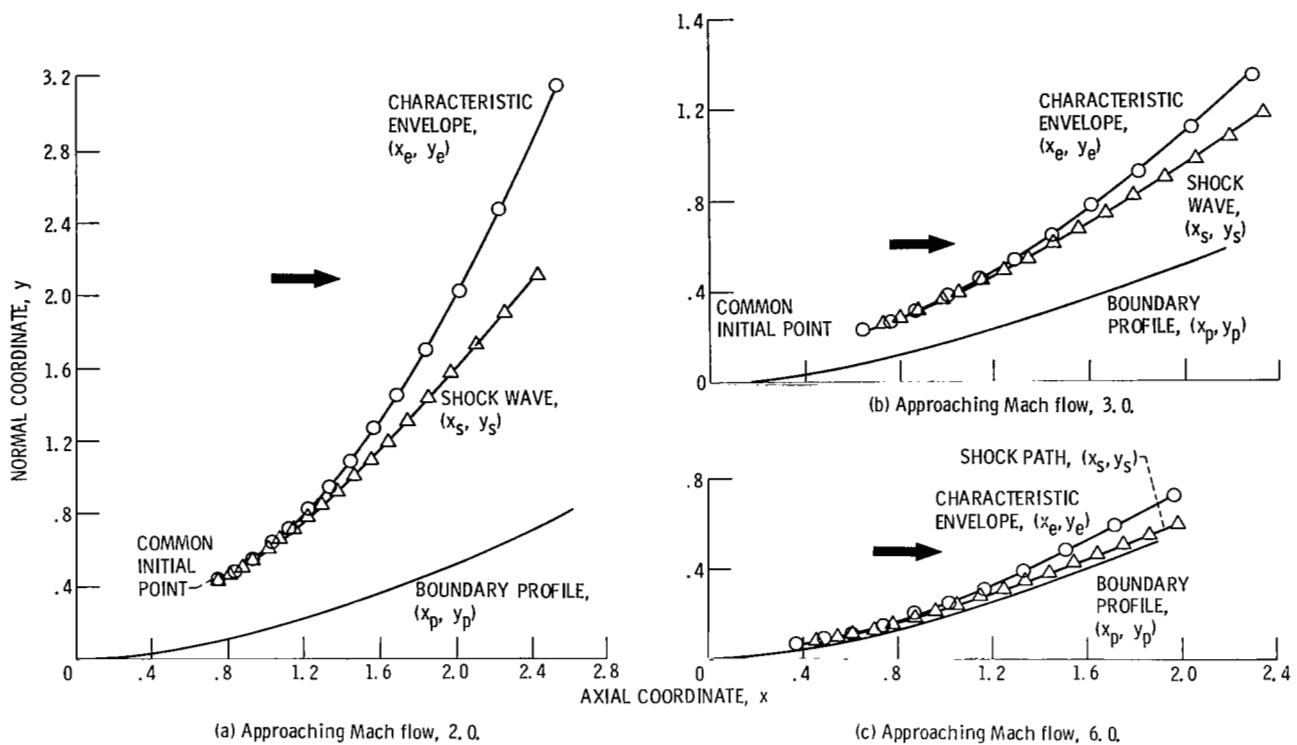


Figure 2. - Comparison of envelope and shock wave paths.

For the conditions given in figure 2, the shock and the envelope agree closely up to flow deflections of about  $5^\circ$  to  $8^\circ$ . This is equivalent to the region where the Prandtl-Meyer epicycloid approximates the shock polar. As a measure of the accuracy of the shock path, the far field shock angle, using the present method, was within  $1^\circ$  of the tabulated shock angle for two-dimensional flow (ref. 6) for the range of Mach numbers considered.

The main contribution of this section has been the integration for the shock path using the initial point of the envelope as the starting point of the integration.

## Complex Wave Region

Simple wave shock formation occurs most often in external flows. Internal flows usually lead to complex wave interactions since all boundaries affect the flow. However, in a complex wave field, neither family of characteristics is a set of straight lines. Thus, the existence of an envelope can only be known from a complete solution of the flow problem, that is, from a solution of the fundamental characteristic equations

$$\left. \begin{aligned} \frac{\partial y}{\partial \xi} &= \left( \frac{\partial x}{\partial \xi} \right) \left[ \tan(\varphi + \alpha) \right] \\ \frac{\partial y}{\partial \eta} &= \left( \frac{\partial x}{\partial \eta} \right) \left[ \tan(\varphi - \alpha) \right] \end{aligned} \right\} \quad (8)$$

Along with appropriate initial or boundary data, these equations furnish the functions  $x(\xi, \eta)$  and  $y(\xi, \eta)$  for mapping the flow variable  $q$  onto the physical plane

$$\left. \begin{array}{ccc} \text{Characteristic net} & & \text{Physical plane} \\ \left. \begin{array}{l} q(\xi, \eta) \\ \varphi(\xi, \eta) \end{array} \right\} & \longrightarrow & \left. \begin{array}{l} q(x, y) \\ \varphi(x, y) \end{array} \right\} \end{array} \right\} \quad (9)$$

The techniques for finding the mapping functions are given in references 1 to 4. Therefore, assume that such a mapping exists. And further, following Tollmien (ref. 4), assume that the envelope forms because of the convergence of left running characteristics ( $\eta = \text{constant}$ ). (See fig. 3.) Along such a characteristic,  $\xi$  is the current variable. Each such characteristic will touch the envelope in one point  $\xi_e = \xi_e(\eta)$ . Then the mapping of the envelope in the physical plane is

$$\begin{aligned} x_e &= x[\xi_e(\eta), \eta] \\ y_e &= y[\xi_e(\eta), \eta] \end{aligned}$$

Where each characteristic touches the envelope, the slope of the characteristic will equal the slope of the envelope. Thus, the slope along the characteristic is

$$\left( \frac{dy}{dx} \right)_{\eta=\text{const}} = \frac{(\partial y / \partial \xi)}{(\partial x / \partial \xi)}$$

This must equal the slope of the envelope at that point

$$\frac{dy_e}{dx_e} = \frac{(\partial y / \partial \xi)(d\xi_e/d\eta) + (\partial y / \partial \eta)}{(\partial x / \partial \xi)(d\xi_e/d\eta) + (\partial x / \partial \eta)} \quad (10)$$

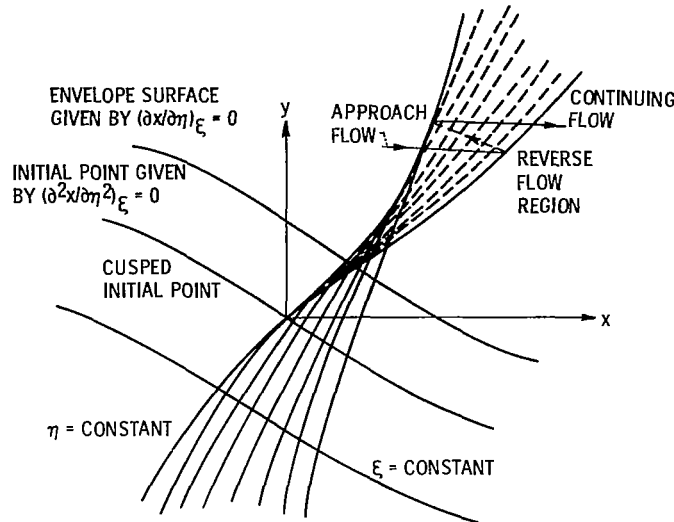


Figure 3. - Envelope in complex wave region.

Crossmultiply to obtain

$$\left(\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right) - \left(\frac{\partial x}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right) = 0$$

This equation may be written as

$$\left(\frac{\partial x}{\partial \eta}\right)\left[\left(\frac{\partial y}{\partial \xi}\right) - \left(\frac{\partial x}{\partial \xi}\right)\tan(\vartheta - \alpha)\right] = 0 \quad (10a)$$

Exclude the exceptional case  $(\partial x / \partial \xi) = 0$ , which simply implies vertical characteristics. Then, the general condition for the envelope is

$$\left(\frac{\partial x}{\partial \eta}\right) = 0 \quad (11)$$

This condition shows the steepening process at the wave front. It states that two Mach waves coincide at one  $(x, y)$  position.

For cases of interest, the envelope will form within the flow field. Thus, there will be some extreme value beyond which  $\xi_e(\eta)$  will not go (fig. 3). Whether this extreme value is a minimum or maximum (as in ref. 4) depends on the curvature of the

boundary generating the wave train. In any event, at the extreme value, following reference 4, a sufficient condition is

$$\left. \begin{aligned} \left( \frac{d\xi_e}{d\eta} \right) &= 0 \\ \left( \frac{d^2\xi_e}{d\eta^2} \right) &\neq 0 \end{aligned} \right\} \quad (12)$$

Let the boundary of the envelope be given by

$$\frac{\partial x}{\partial \eta} = F(\xi_e(\eta), \eta) = 0$$

Then

$$\frac{dF}{d\eta} = \left( \frac{\partial F}{\partial \xi} \right) \left( \frac{d\xi_e}{d\eta} \right) + \left( \frac{\partial F}{\partial \eta} \right) = 0$$

So at the extreme value, the initial point

$$\left. \begin{aligned} \frac{dF}{d\eta} = \frac{\partial^2 x}{\partial \eta^2} &= 0 \\ \frac{\partial^3 x}{\partial \eta^3} &\neq 0 \end{aligned} \right\} \quad (13)$$

(This result excludes the unlikely case in which  $(\partial x / \partial \eta)$  becomes infinite.) Now the location of the envelope is known in both simple and complex wave regions. But within the envelope region the path of the shock wave is still unknown.

The final determination of the shock wave rests on the minimum entropy production theorem (ref. 5). The reasoning is as follows (see fig. 4). Within the envelope, the mapping  $x = x(\xi, \eta)$ ,  $y = y(\xi, \eta)$  is triple valued. The envelope region B would be a field of return flow. At any value of  $(x, \xi)$  there are three states available to the gas:  $\eta_A$ ,  $\eta_B$ ,  $\eta_C$ . According to the second law, the shock process will take the gas to the available state with the locally maximum entropy. Hence, from branch A to branch C.

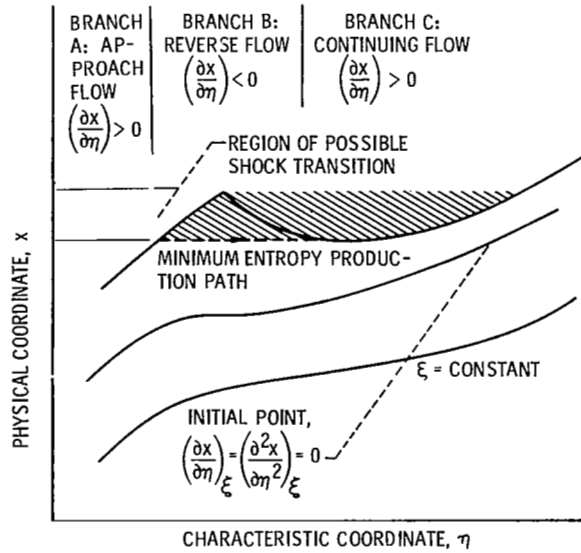


Figure 4. - Typical plot used to determine shock wave path.

Branch B is not allowed since it would imply a returning flow. But it is still necessary to choose one out of a continuum of points such that  $x_A$  and  $x_C$  coincide. The minimum entropy production theorem will pick out the most likely value of  $x_A - x_C$  in which the shock will occur.

Since the Mach angle  $\alpha$  increases through a shock wave,  $(\partial \eta / \partial x)_C > (\partial \eta / \partial x)_A$ . Consequently, the jump in  $\eta$  gets larger at larger downstream positions  $x$ . But the jump in  $\eta$  is proportional to the flow deflection if  $\xi$  is constant.

$$\eta_C - \eta_A = \frac{\eta_C - \eta_A}{2}$$

Since shock strength increases with flow deflection, minimum entropy production implies that the shock will occur at the earliest possible location. Hence, the shock wave follows the envelope.

The entropy jump across the wave is found from

$$\frac{\Delta S}{R} = -\frac{7}{2} \ln \left[ \frac{6\pi + 1}{\pi(\pi + 6)} \right] - \ln \pi$$

where

$$\pi = 1 + \frac{\gamma M_A^2 \delta}{\sqrt{M_A^2 - 1}} + \gamma M_A^2 \left[ \frac{(\gamma + 1)M_A^4 - 4(M_A^2 - 1)}{4(M_A^2 - 1)^2} \right] \delta^2 + \dots$$

$$\delta = \vartheta_C - \vartheta_A = \frac{\eta_C - \eta_A}{2}$$

and

$$\omega(M_A) = \frac{\xi_A - \eta_A}{2}$$

As an example, consider the steady plane flow contained between two circular arc walls (fig. 5). The inlet conditions are  $\vartheta_\infty = 0$  and  $M_\infty = 1.503$ . The characteristic

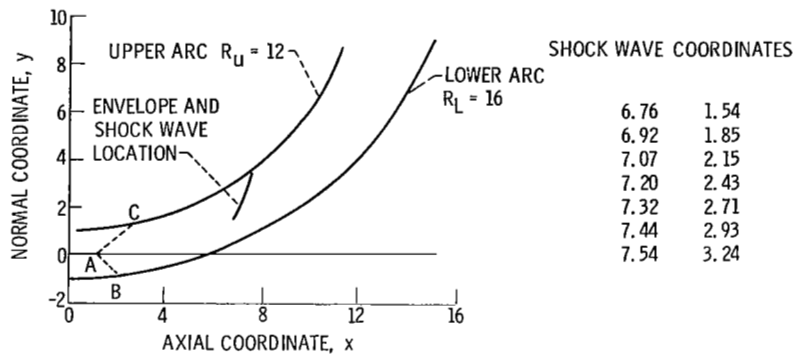


Figure 5. - Physical layout of channel showing shock wave location. Note: arc BAC is the beginning of the complex wave field shown in figure 6. Upstream flow conditions: Mach 1.503; flow deflection angle,  $\theta$ .

map (a point plot) is also shown (fig. 6) as it was obtained from a machine solution and electronic plotter. The point plot clearly shows the convergence of the left running waves. This convergence occurs in spite of a reflection of the compression waves off the upper expansion surface. Indeed, the envelope forms in a generally expanding flow; the inlet Mach number is 1.503, but the envelope forms at  $M_A = 1.92$  and  $\vartheta_A = 25^\circ$ .

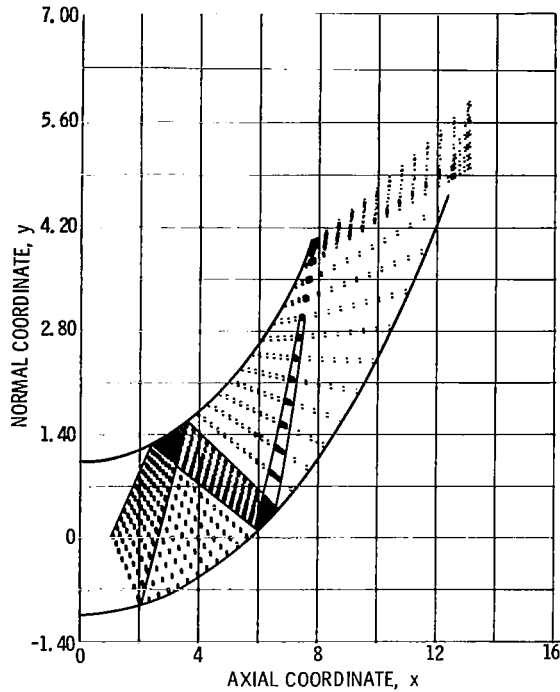


Figure 6. - Computer point-plot showing characteristics converging. Upper arc, 12; lower arc, 16; Mach 1.5; flow deflection angle, 0.

Figure 7 is a plot of the physical coordinate  $x$  as a function of the characteristic coordinates  $\xi$  and  $\eta$ . The initial point of the envelope is the point where

$$\left(\frac{\partial x}{\partial \eta}\right) = 0$$

$$\left(\frac{\partial^2 x}{\partial \eta^2}\right) = 0$$

and

$$\left(\frac{\partial^3 x}{\partial \eta^3}\right) \neq 0$$

which figure 7 verifies. For the local flow conditions of this example ( $M_A = 1.92$ ) there is a very small entropy change in the vicinity of the initial point,  $(\Delta S/R) = 10^{-4}$ . Never-

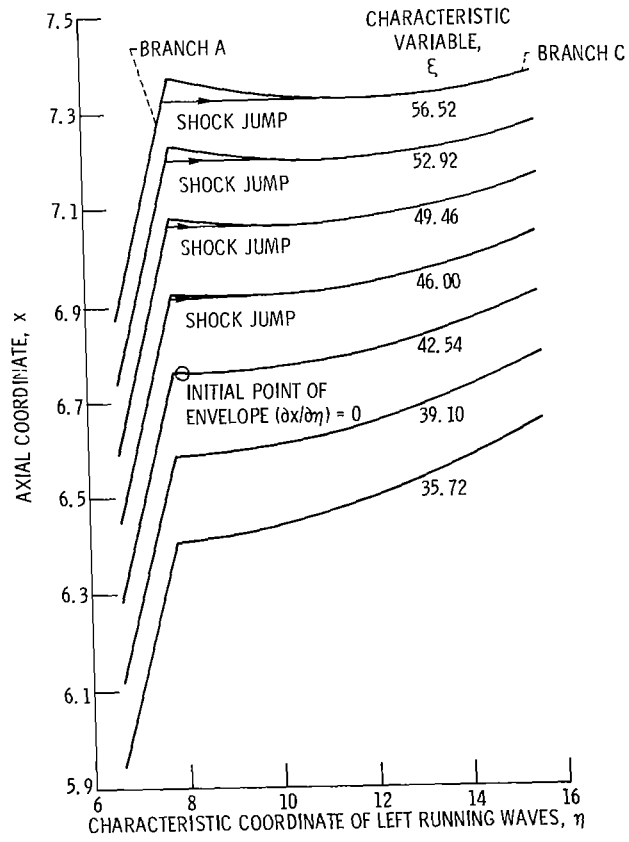


Figure 7. - Determination of shock location from plot of  $x = x(\xi, \eta)$ . Inlet height, 2; Mach 1.5; flow deflection angle, 0.

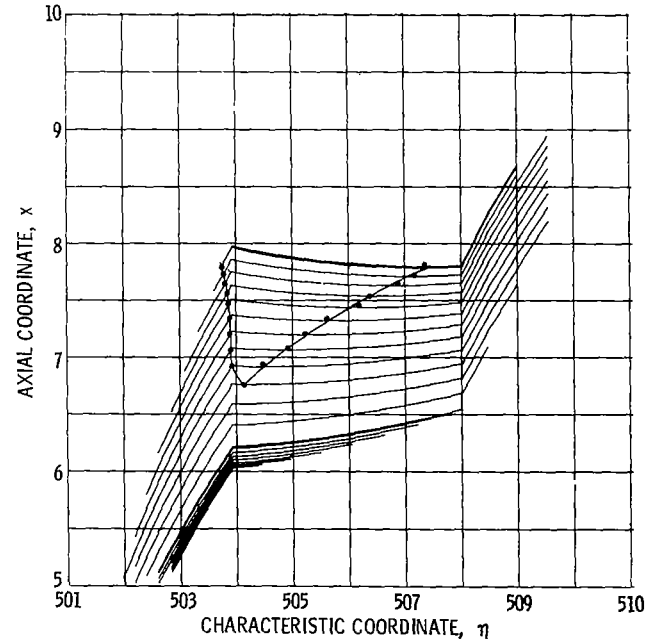


Figure 8. - Locus of shock wave. Upper arc, 12; lower arc, 16; Mach 1.5; flow deflection angle, 0.



theless, the pressure change across the jump is not negligible ( $\pi = 1.15$ ). Figure 8 shows the locus of the shock wave in detail.

## SUMMARY AND CONCLUSIONS

This report presents methods for predicting shock formation and shock-wave trajectory in simple and complex wave fields. In simple wave regions, say for external flows, the method consists of point by point integration, given the shock slope by either method, starting out from the initial point of the characteristic envelope. In complex wave fields, minimum entropy production principle indicates that the shock wave lies along the upstream boundary of the characteristic envelope.

At the present time these methods are being extended to technical applications. Integral boundary-layer solutions are being used (refs. 7 and 8), along with the simple wave method, to predict the shock trajectory that causes a laminar boundary layer to separate as it approaches a concave bend. The method for complex wave flows is being extended to include the effect of envelope interaction with a boundary and to include the effect of channel exit on the shock wave path.

Lewis Research Center,  
National Aeronautics and Space Administration,  
Cleveland, Ohio, March 31, 1971,  
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