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A UNIFIED FORMALISM FOR POSITION DETERMINATION FROM VARIOUS NAVIGATION MEASUREMENTS

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16. Abstract

A technique has been developed for coordinate determination of moving vehicles. Its application offers a unified description for extracting position information from various combinations of measurements such as range, range rate, angle, distance sum, and distance difference. Specifically, the desired coordinates are obtained by a matrix multiplication of a column matrix obtained from given measurements. Various combinations of measurements affect only the method of constructing the column matrix. The matrix that transforms the column matrix into desired coordinates depends only on the known coordinates of reference points. This matrix also transforms the error covariance matrix in measurements into an error covariance matrix in computed coordinates.

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SYMBOLS

a	column matrix consisting of the coordinates of the position vector in the x-basis
Δa	change in a coordinates between measurements, a 3×1 column matrix
b_1,b_2,b_3	range-difference measurements
c_1,c_2,c_3	orthonormal triplet
$\bar{c}$	column matrix consisting of a triplet of c vectors
f_{jk}^{-1}	jk element of F^{-1}
F	Grammian of x_1, x_2, a 2×2 matrix
G	Grammian of x_1,x_2, x_3, a 3×3 matrix
g_{jk}^{-1}	jk element of G^{-1}
J	3×4 matrix consisting of T^{-1} and row sum of T^{-1}
l_1,l_2,l_3	vectors from corresponding reference points to vehicle position
m	4×1 matrix consisting of product of measurement and the corresponding error in the same measurement
n	2×1 column matrix consisting of the first two elements of matrix a
P_1,P_2,P_3	reference points
R	vehicle position
r	position vector of vehicle
Δr	change in position vector between different time of measurement
$r \cdot \bar{x}$	projection vector; a column matrix consisting of $r \cdot x_1$, $r \cdot x_2$, and $r \cdot x_3$ for simultaneous measurement or of $r^{(1)} \cdot x_1$, $r^{(2)} \cdot x_2$, $r^{(3)} \cdot x_3$ for nonsimultaneous measurements
T	linear transformation between the x and c bases, a 3×3 matrix
t_{kl}^{-1}	kl element of T^{-1}

v	column matrix consisting of displacement in the w coordinates to account for vehicle motion between measurements
w	column matrix consisting of coordinates of the position vector
Δw	change in w coordinates between different time of measurements
$\bar{x}$	column matrix consisting of a triplet of x vectors
x_1, x_2, x_3	position vectors of corresponding reference points
α	2×1 column matrix consisting of the first two elements of the projection vector $r \cdot \bar{x}$
β	3×1 matrix consisting of coordinates of x_4 expressed in the x basis
$\beta_1, \beta_2, \beta_3$	coordinates of x_4 expressed in the x basis
θ	angle between line of sight with respect to the horizon; numerically positive if measured in the counterclockwise direction
Λ_r	longitude of vehicle
λ_r	latitude of vehicle
ξ_k	length of vector x_k
ρ	length of position vector r

Operators

c( )	cosine of ( )
cov( )	covariance of ( )
d( )	derivative of ( )
s( )	sine of ( )
$(\ )^t$	transpose of a matrix
$(\ ) \cdot (\ )$	inner product of two vectors
$(\ )^{-1}$	inverse of a matrix

Superscripts and Subscripts

$(\ )_{i,j}$	sum of ( ) for all values of i and j, unless specified $i = 1, 2, 3$, and $j = 1, 2, 3$
$(\ )^{(k)}$	index to the time of measurement
$(\ )_{k,l}$	k, l element of ( ), or k component of $(\ )^{l}$
$(\ )_r$	See explanation for λ_r and Λ_r
λ_{c_1}	latitude of vector c_1

A UNIFIED FORMALISM FOR POSITION DETERMINATION FROM VARIOUS NAVIGATION MEASUREMENTS

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SUMMARY

A new computational technique has been developed for coordinate determination of moving vehicles. This technique is applicable to a wide class of measurements, such as range, range rate, angle, distance sum, distance difference, or combinations of these. The generality of the method and the simplicity of the required computation make the technique ideally suitable for navigation of aircraft or spacecraft.

Coordinate determination by the new technique proceeds in the following sequence. First, a vector quantity called the projection is computed by simple arithmetic operations on the given measurements. Second, a matrix is constructed that depends only on some chosen Cartesian coordinate system and on the location of the reference points to which measurements are being made. Then the coordinates of the vehicle in the chosen Cartesian system are obtained by multiplication of this matrix by the projection vector. The computed Cartesian coordinates of the vehicle can readily be converted into longitude and latitude. A further advantage of this technique is that errors in measurements are related directly to errors in computed coordinates by the same matrix.

Different types of measurements affect only the method of constructing the projection vector. In this report, construction of the projection vector for range measurements, angle measurements, distance-sum measurements, and distance-difference measurements is explicitly carried out. It was found that the number of measurements, and not the type of measurements, determines the uniqueness of the solution. For instance, four measurements, be they range, range rate, etc., determine the position uniquely, whereas three measurements always yield two solutions to the coordinate-determination problem. In all cases, multiple solutions, if present, are exhibited in the process of constructing the projection.

INTRODUCTION

The fundamental problem in navigation is determining the position of a vehicle by means of measurements relative to known points of reference. Many types of measurements can be used. Measurements of range, range rate, angle, and differences in distance are commonly employed. Position is determined by processing these data in an appropriate manner.

The analytical methods used to process the data are as varied as the measurements themselves. In one method (refs. 1, 2) position is established from the solution of simultaneous quadratic algebraic equations. Another method (ref. 3) has the desirable quality of expressing the coordinates of position explicitly in terms of measured and known quantities, but requires many trigonometric operations that consume a considerable amount of computation time. Furthermore, this method is restricted because it assumes that the reference points lie in the equatorial plane. Still another method (ref. 4), to be used with measurements of distance differences, requires an iterative process or spherical trigonometry. This method too is restricted by the assumption that the reference points and the vehicle whose position is to be determined lie at the same altitude.

The purpose of this report is to develop another method of processing the measurements. This method relies on linear algebra to yield a single formalism by which any of the usual measurements can be processed but is not limited to special navigation situations. Thus, it can be used to determine the position of space vehicles as well as ships and airplanes. It relates the desired position parameters directly to the measured quantities through a transformation whose terms are already known.

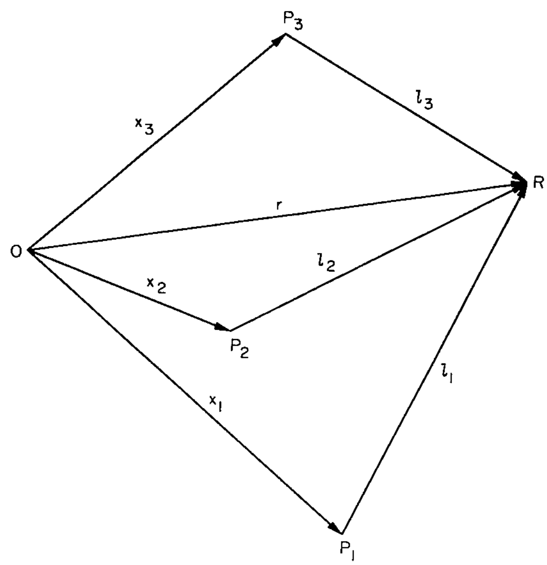
The idea of the method is very simple and is given in the first section below. Any navigation system relies on the use of known points of reference for making measurements and on the existence of a standard coordinate system for expressing the position of the vehicle. The reference points form a basis for a coordinate system and the position of the vehicle is given in this basis by the various possible measurements. The reference points are said to be known in the sense that their positions are established in terms of the standard coordinate basis. Hence the same transformation that relates the basis formed by the reference points to the standard also relates the measurements to the desired standard expression for vehicle position. The difference between the various navigation schemes is shown to reduce to different modes of expressing the vehicle's position in the reference basis.

The basic scheme works if a set of measurements is made essentially at the same time and if the set is sufficient to determine position uniquely. The modification to the scheme needed to account for motion of the vehicle between measurements is shown to be minor. A more significant modification is then developed for the common case where the measurements do not uniquely specify the vehicle's position, but determine it only to within a choice of sign. Once this modification is given, the usual navigation systems can be treated.

Since the formalism developed in this report separates the measurements and the transformation between the reference and the standard bases, measurement errors are similarly separate. The final section of the report develops this condition to show how the covariance of position determination and of measurement errors are related by a precomputable transformation.

DESCRIPTION OF THE BASIC PROCEDURE

The basic formalism for position determination can most easily be developed by reference to the situation illustrated in sketch (a). Three reference points, P_1, P_2, and P_3, and their positions are located relative to an origin O, say the center of the earth, by the vectors x_1, x_2, and x_3. The vehicle's position R is located by the vector r from O. The vectors l_1, l_2, and l_3 locate R relative to the three reference points. Not illustrated are three unit vectors c_1, c_2, and c_3 forming an orthonormal triplet at O. These vectors form the basis of the standard coordinate system; for example, c_3 pointing north from the center of the earth, c_1 lying in the equatorial plane, and together with c_3, determining the Greenwich meridian.



Sketch (a)

Since the x's and c's are sets of independent vectors, the following relations hold:

$$r = a^t\bar{x} = w^t\bar{c} \tag{1}$$

Here the bar over x and c indicate that these quantities are triplets of vectors. The a and w are triplets of scalars. The superscript t denotes transpose. The a's will be seen to be related to the measurements, and the w's are the desired coordinates of the point R. Written out more fully, equation (1) appears as

$$r = a_1x_1 + a_2x_2 + a_3x_3 = w_1c_1 + w_2c_2 + w_3c_3$$

The two bases, the vector sets $\bar{x}$ and $\bar{c}$, are related by a linear transformation

$$\bar{x} = T\bar{c} \tag{2}$$

The elements in the kth row of T, k = 1,2,3, are the coordinates of x_k expressed in the c-basis. If the reference points are known points on the earth, then the elements of T are fixed numbers. If they are moving points, such as satellites, then the t_{kl} are known functions of the ephemerides. The relation between a and w is

$$a^tT = w^t \ ; \qquad w = T^ta \tag{3}$$

Writing equation (1) in the form $r = \bar{x}^ta$, one obtains

$$r \cdot \bar{x} = \bar{x} \cdot \bar{x}^ta = Ga \tag{4}$$

where $r \cdot \bar{x}$ is the (column) vector $(r \cdot x_1, r \cdot x_2, r \cdot x_3)^t$, G is the so-called Grammian $\bar{x} \cdot \bar{x}^t$, and can be written in matrix form

$$\bar{x} \cdot \bar{x}^t = \begin{bmatrix} x_1 \cdot x_1 & x_2 \cdot x_1 & x_3 \cdot x_1 \\ x_1 \cdot x_2 & x_2 \cdot x_2 & x_3 \cdot x_2 \\ x_1 \cdot x_3 & x_2 \cdot x_3 & x_3 \cdot x_3 \end{bmatrix}$$

Reference to sketch (a) gives rise to the following equations:

$$(r - x_k)^t (r - x_k) = l_k{}^t l_k = \rho^2 + \xi_k{}^2 - 2r \cdot x_k = \lambda_k{}^2 ; \qquad k = 1, 2, 3$$

where $\rho^2 = r \cdot r$; $\xi_k{}^2 = x_k \cdot x_k$; and $\lambda_k{}^2 = l_k \cdot l_k$. Solving for $r \cdot x_k$ gives the equation

$$r \cdot x_k = \frac{1}{2}\left[\rho^2 + \xi_k{}^2 - \lambda_k{}^2\right] \tag{5}$$

In order to illustrate the formalism simply, consider an example in which the values of the λ_k are given by measurements of range, and in which the altitude ρ of the vehicle R is known. Since the ξ_k are the lengths of the vectors to the reference points, they are known. Hence, the right-hand side of equation (5) can be evaluated.

Equations (3) and (4) yield the equations

$$w = T^t a = T^t G^{-1}(r \cdot \bar{x}) \tag{6}$$

Now the Grammian can be expressed in the following way

$$G = \bar{x}\bar{x}^t = T\bar{c}\bar{c}^t T^t = TT^t$$

or

$$G^{-1} = (TT^t)^{-1} \tag{7}$$

because $\bar{c}\bar{c}^t$ is equal to the identity matrix. Using equations (6) and (7), then, one obtains the equation

$$w = T^t (TT^t)^{-1}(r \cdot \bar{x}) = T^{-1}(r \cdot \bar{x}) \tag{8}$$

Equation (8) is the desired result. It shows that the desired coordinates of the vehicle w are related to the measurements contained in what will be referred to as the projection vector $r \cdot \bar{x}$ by a linear transformation whose elements are already known. The coordinates of the vehicle can be expressed in many ways using the w_k. If its position is to be expressed in terms of latitude λ_r and longitude Λ_r, for example, and the standard coordinate is the Greenwich meridian, then

$$\lambda_r = \tan^{-1} \frac{w_2}{w_1} \quad ; \quad \Lambda_r = \tan^{-1} \frac{w_3}{\sqrt{w_1^2 + w_2^2}}$$

In any case, the w_i are given in equation (8) by using only arithmetic operations. Furthermore, the procedure summarized by equation (8), although developed with the simplest navigation situation in mind, will be shown to hold whenever four independent measurements are given at a time. By the modifications to the procedure discussed next, any of the usual navigation measurements can be related to desired coordinate expressions in a unified way.

MODIFICATIONS TO THE BASIC PROCEDURE

Nonsimultaneous Measurements

In deriving the changes to be made to equation (8) when the measurements are not given simultaneously, one should keep in mind an example similar to that already used. Thus, three fixed reference points are assumed to be given and range measurements taken with respect to them. If an airplane is the vehicle from which the measurements are made, its motion between range measurements is assumed to be measured. The range measurements are made in the order 1, 2, then 3. Position will be determined as of the time of the last measurement. Quantities will be indexed as to the time of measurement by a superscripted number in parentheses. Then the position of the vehicle at the various times can be written:

$$\left.\begin{aligned} r^{(1)} &= r - \Delta r^{(1)} = a^t\bar{x} - \Delta a^{(1)t}\bar{x} = w^t\bar{c} - \Delta w^{(1)t}\bar{c} \\ r^{(2)} &= r - \Delta r^{(2)} = a^t\bar{x} - \Delta a^{(2)t}\bar{x} = w^t\bar{c} - \Delta w^{(2)t}\bar{c} \\ r^{(3)} &= r = a^t\bar{x} = w^t\bar{c} \end{aligned}\right\} \tag{9}$$

As in the previous section, $a^t\bar{x} = w^t\bar{c}$ and $\bar{x} = T\bar{c}$. Hence, equations (9) can be written in the form

$$\left.\begin{aligned} r^{(1)} &= a^t\bar{x} - \Delta w^{(1)t}\bar{c} \\ r^{(2)} &= a^t\bar{x} - \Delta w^{(2)t}\bar{c} \\ r &= a^t\bar{x} = \bar{x}^t a \end{aligned}\right\} \tag{10}$$

The projection vector for this case is similar to that given before:

$$(r \cdot \bar{x})^t = \left(r^{(1)} \cdot x_1,\ r^{(2)} \cdot x_2,\ r \cdot x_3\right)$$

The terms on the right are given by

$$r^{(k)} \cdot x_k = \frac{1}{2}\left(\rho_k{}^2 + \xi_k{}^2 - \lambda_k{}^2\right)$$

wherein the ξ_k are known, and the ρ_k and λ_k are measured. Then equation (10) can be written in the form

$$r \cdot \bar{x} = \bar{x}\bar{x}^t a - v \tag{11}$$

Here $v^t = (v_1, v_2, v_3)$ and v_1 can be written as

$$\begin{aligned} v_1 &= \Delta w^{(1)t}\bar{c} \cdot x_1 \\ &= \left(\Delta w_1{}^{(1)}c_1 + \Delta w_2{}^{(1)}c_2 + \Delta w_3{}^{(1)}c_3\right) \cdot (t_{11}c_1 + t_{12}c_2 + t_{13}c_3) \\ &= t_{1j}\ \Delta w_j \qquad j = 1, 2, 3 \end{aligned}$$

Hereafter, whenever a subscript is repeated it means summation and unless specified $j = 1, 2, 3$, that is,

$$t_{1j}\ \Delta w_j = \sum_{j=1}^{3} t_{1j}\ \Delta w_j$$

The t_{1k}, $k = 1, 2, 3$ are the elements in the first row of the known transformation matrix T. The three values $\Delta w_k{}^{(1)}$ refer to the measured motion of the airplane between times (1) and (3). The component v_2 is similarly expressed as

$$v_2 = t_{2j}\ \Delta w_j{}^{(2)} \qquad j = 1, 2, 3$$

The third component v_3 is zero. Using equations (6) and (7) one can write equation (11) as

$$r \cdot \bar{x} = T\bar{c}\bar{c}^t T^t (T^t)^{-1} w - v = Tw - v$$

Solving for w, one gets

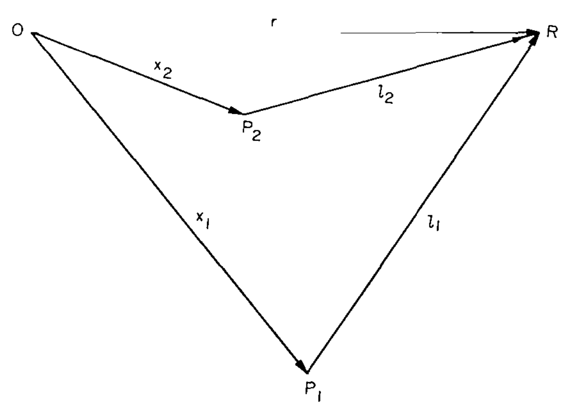
$$w = T^{-1}(r \cdot \bar{x} + v) \tag{12}$$

Equation (12) differs from equation (8) only by the additional term which modifies the projection vector by adding the motion of the aircraft between the times when range is measured.

Indirect Expression

Equation (12) shows that the procedure for expressing the position of the vehicle in a desired coordinate system, when the measurements are not made simultaneously, is nearly the same as that when all the measurements are made at once. The desired coordinates are expressed directly in terms of the measured quantities. This direct expression was shown to arise when range measurements are made to three independent reference points and the altitude of the vehicle is known (or measured). It will be shown later that this form holds whenever four independent measurements are made. If only three are available, the position of the vehicle can still be determined, but not uniquely, and it cannot be expressed quite so directly as before.

The modification to the basic procedure necessary when only three measurements are available will be developed using the example of two given reference points. Range to these points and altitude will be the measurements made. Sketch (b) illustrates the situation. The reference points P_1 and P_2 are located by the known vectors x_1 and x_2 with respect to point of origin. The vehicle is located at R by the vector r. The vectors l_1 and l_2 locate R relative to the reference points. The lengths of x_1 and x_2, namely ξ_1 and ξ_2, are known. The lengths of l_1, l_2, and r, namely λ_1, λ_2, and ρ, are measured simultaneously.



Sketch (b)

The two vectors x_1 and x_2 by themselves do not form a basis for specifying the position of the vehicle. They can be used to generate the third vector x_3 and thus produce a triplet of independent vectors $\bar{x}$. The vector x_3 is constructed from the cross product of x_1 and x_2:

$$x_3 = \frac{(x_1) \times (x_2)}{\|(x_1) \times (x_2)\|} \tag{13}$$

Since the denominator in equation (13) is the magnitude of the numerator, the vector x_3 has unit length.

Equations (1) through (4) hold:

$$r = a^t\bar{x} = w^t\bar{c} \tag{1}$$

$$\bar{x} = T\bar{c} \tag{2}$$

$$a^tT = w^t \; ; \qquad w = T^ta \tag{3}$$

$$r \cdot \bar{x} = \bar{x}\bar{x}^t a = Ga \tag{4}$$

The two quantities α and n are defined for convenience by

$$\alpha^t = (r \cdot x_1,\; r \cdot x_2) \tag{13a}$$

$$n^t = (a_1,\; a_2) \tag{13b}$$

Then equation (4) can be written

$$r \cdot \bar{x} = \begin{bmatrix} \alpha \\ r \cdot x_3 \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n \\ a_3 \end{bmatrix} \tag{14}$$

where F is given by

$$F = \begin{bmatrix} x_1 \cdot x_1 & x_1 \cdot x_2 \\ x_2 \cdot x_1 & x_2 \cdot x_2 \end{bmatrix} = F^t \tag{15}$$

The following equation comes directly from equation (14):

$$n = F^{-1}\alpha \tag{16}$$

Neither $r \cdot x_3$ nor a_3 is known directly. Either can be calculated by considering $r \cdot r$:

$$r \cdot r = \rho^2 = a^t x x^t a = [n^t,\; a_3] \begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n \\ a_3 \end{bmatrix} \tag{17}$$

Hence, follow the equations

$$a_3{}^2 = \rho^2 - n^t F n$$
$$= \rho^2 - \alpha^t F^{-1} \alpha$$

or

$$a_3 = \pm[\rho^2 - \alpha^t F^{-1} \alpha]^{1/2} \tag{18}$$

Since the correct sign must be chosen for a_3 in equation (18), the solution is not unique. The choice of sign means that R could be at either of two places located symmetrically with respect to the plane formed by the vectors x_1 and x_2.

Equation (18) completes the information required for position determination. The steps are, in summary:

$$n = F^{-1}\alpha \tag{16}$$

$$a_3 = \pm[\rho^2 - \alpha^t F^{-1}\alpha]^{1/2} \tag{18}$$

$$w = T^t a \tag{3}$$

Whereas equation (8) expresses w directly in terms of the measurements in $r \cdot \bar{x}$, equations (16), (18), and (3) form an indirect expression, requiring calculations of intermediate quantities.

If $r \cdot x_1$ and $r \cdot x_2$ are measured at different times, equation (16) changes to the form

$$n = F^{-1}\alpha + v$$

where

$$v_1 = t_{1j}\Delta w_j^{(1)} \qquad (j = 1, 2, 3)$$

and

$$v_2 = 0$$

CONSTRUCTION OF PROJECTION VECTORS FOR VARIOUS NAVIGATION SYSTEMS

An important advantage of this formalism is that different types of measurements affect only the method of constructing the projection vector, a component of which is expressed in equation (5). Once the equations that relate the projection vector and the measurements are known, the basic procedure or its modification can be applied to solve all position determination problems. In the previous section, ρ and λ were the measured quantities in the projection. This section is concerned with constructing the projection when other types of measurements are used.

The following list classifies navigation systems according to the type of measurements provided to construct the projection:

1. Altitude, and any combination of two or three range or angle measurements.

2. Four independent range measurements.

3. One altitude and three distance difference measurements.

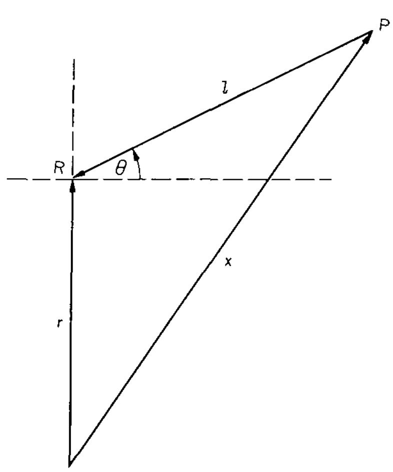
4. One altitude and two distance difference measurements.

5. Three independent range measurements, where the plane containing the reference points either (a) does not contain the origin, or (b) does contain the origin.

Calculating the projection vector from the data of each of these systems will be shown in the order listed.

Projection Using Angle Measurements

Equation (5) expresses the projection vector in terms of the known altitude ρ and ξ and the range, λ. It will now be shown how an angle measurement can be substituted for range. Sketch (c1) shows that the equation $r - l = x$ holds. Taking the dot product of x with itself and rearranging terms give the equations



Sketch (c1)

$$\left.\begin{aligned} 0 &= \lambda^2 - 2\rho\lambda c(90° + \theta) + \rho^2 - \xi^2 \\ &= \lambda^2 + 2\rho\lambda s\theta + \rho^2 - \xi^2 \end{aligned}\right\} \tag{19a}$$

where θ, the angle between the horizon and the line of sight, is considered positive if the reference point, P, lies above the horizon. Solving equation (19a) for λ gives

$$\lambda = -\rho s\theta \pm [(\rho s\theta)^2 + (\xi^2 - \rho^2)]^{1/2} \tag{19b}$$

Note that λ, being a length, must always be positive. To choose the correct sign in equation (19b) one must consider the two cases $\xi > \rho$ and $\xi < \rho$. Considering first the condition $\xi > \rho$, it will be shown that

$$\lambda = -\rho s\theta + [(\rho s\theta)^2 + (\xi^2 - \rho^2)]^{1/2} \tag{19c}$$

both for $\theta \geq 0$ and for $\theta \leq 0$. It can be seen from sketch (c2) that the following relations hold:

$$\overline{QR} = \rho s\theta = -\rho s\theta' \quad \text{since} \quad \theta' < 0$$

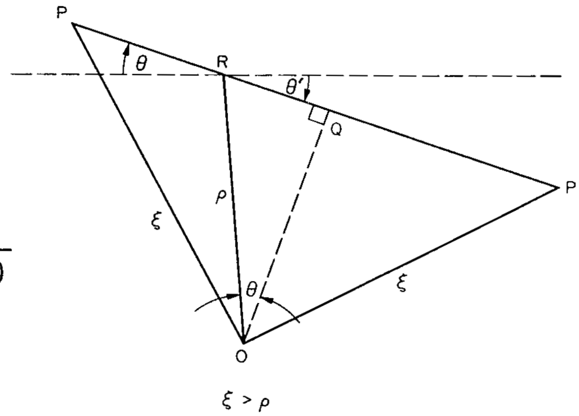
$$\overline{PQ} = \overline{P'Q} = \sqrt{\xi^2 - (\rho c\theta)^2}$$

$$= \sqrt{(\rho s\theta)^2 + (\xi^2 - \rho^2)}$$

$$\overline{PR} = \overline{PQ} - \overline{QR} = -\rho s\theta + \sqrt{(\rho s\theta)^2 + (\xi^2 - \rho^2)}$$

$$\overline{P'R} = \overline{P'Q} + \overline{QR}$$

$$= -\rho s\theta' + \sqrt{(\rho s\theta')^2 + (\xi^2 - \rho^2)} \qquad (20)$$



Sketch (c2)

In both cases where $\theta \geq 0$ or $\theta \leq 0$, equation (20) is the required solution for λ for the condition $\xi > \rho$. If, on the other hand, $\xi < \rho$, then there are two possible solutions for λ. Sketch (c3) and equation (20) show that the following relations hold:

$$\overline{QR} = |\rho s\theta| = -\rho s\theta \quad \text{since} \quad \theta < 0$$

$$\overline{PR} = \overline{QR} - \overline{PQ} = -\rho s\theta - \sqrt{(\rho s\theta)^2 + (\xi^2 - \rho^2)}$$

$$\overline{P'R} = \overline{QR} + \overline{P'Q} = -\rho s\theta + \sqrt{(\rho s\theta)^2 + (\xi^2 - \rho^2)}$$

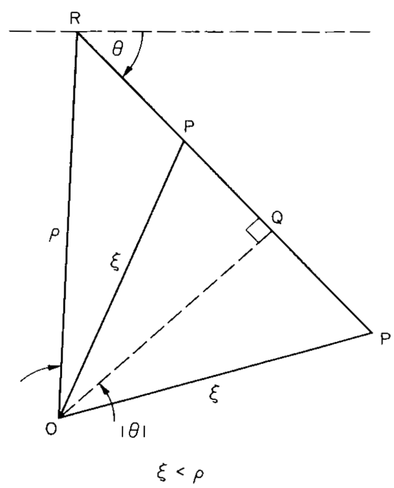
Therefore, there are two possible solutions:

$$\lambda = -\rho s\theta + [(\rho s\theta)^2 + (\xi^2 - \rho^2)]^{1/2}$$

or

$$\lambda = -\rho s\theta - [(\rho s\theta)^2 + (\xi^2 - \rho^2)]^{1/2}$$

To know which solution to choose under the condition $\xi < \rho$ requires additional information.



Sketch (c3)

To calculate the projection vector in terms of angle measurements one can "dot" through the equation $\mathbf{r} - \boldsymbol{l} = \mathbf{x}$ with $\mathbf{r}$ to get $\mathbf{r} \cdot \mathbf{x} = \rho^2 - \rho\lambda s\theta$. Then using equation (19b) gives

$$\mathbf{r} \cdot \mathbf{x} = \rho^2 c\theta \pm \rho s\theta(\xi^2 - \rho^2 c^2\theta)^{1/2} \qquad (21)$$

After the proper sign in the equation is selected, equation (21) is a variation in form of equation (5). Specifically, the equation that relates the kth component of the projection vector and its corresponding angle measurement is given by

$$(r \cdot x)_k = \begin{cases} \rho^2 c\theta_k + \rho s\theta_k [\xi_k{}^2 - \rho^2 c\theta_k]^{1/2} \\ \rho^2 c\theta_k - \rho s\theta_k [\xi_k - \rho^2 c\theta_k]^{1/2} \end{cases} \tag{5a}$$

Projection Using Four Independent Range Measurements

When measurements of range are given with respect to four independent reference points, the following procedure can be used to calculate the projection vector. Let x_1, x_2, x_3, and x_4 be the position vector of the four reference points relative to a common origin. One of them, say x_4, can be written as a linear combination of the other three;

$$x_4 = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = \beta^t x = x^t \beta \tag{22}$$

Since all the x's are known, the β's are also known. The problem is to eliminate the unknown altitude, ρ, from the expression for the projection. Now equation (1) relates the vehicle's position to three reference points by the expression $r = a^t \bar{x}$. The projection onto x_4 then is given by

$$r \cdot x_4 = a^t x x^t \beta = (r \cdot \bar{x})^t \beta$$

where equation (4) has been used. This equation can be expanded by means of equation (5) to give

$$\begin{bmatrix} \rho^2 + \xi_1{}^2 - \lambda_1{}^2 \\ \rho^2 + \xi_2{}^2 - \lambda_2{}^2 \\ \rho^2 + \xi_3{}^2 - \lambda_3{}^2 \end{bmatrix}^t \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \rho^2 + \xi_4{}^2 - \lambda_4{}^2 \tag{23a}$$

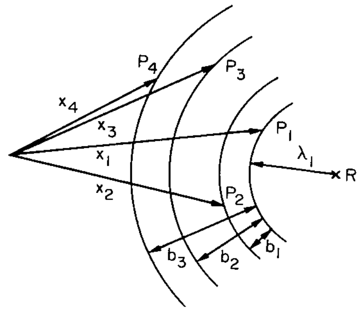
The solution of this equation for ρ^2 is

$$\rho^2 = \left[\lambda_4{}^2 - \xi_4{}^2 + \sum_i \beta_i \left(\xi_i{}^2 - \lambda_i{}^2\right)\right] \Big/ \left(1 - \sum_i \beta_i\right) \tag{23b}$$

The denominator of equation (23b) cannot be zero so long as the four reference points do not lie on a plane. After ρ^2 is determined from equation (23b) all the quantities in the right side of equation (5) are known and the projection vector is completely specified.

Projection Using Altitude and Three Distance Difference Measurements

Distance-difference measurements arise in using hyperbolic intersection methods of navigation such as Loran C. Sketch (d) shows the point R to be determined and four reference stations P_1 through P_4. These known points are at positions x_1 through x_4 from an origin. A constant difference in distance of R from two reference stations gives the locus of a hyperbola. This difference is determined by measuring the time difference between pulses from each station. The length λ of the vector to the first reference of any pair of stations is not known. Only the difference in their distances is known. With P_1 as the primary references, sketch (d) illustrates the following relations between the vehicle at R and four known stations:



Sketch (d)

$$\left.\begin{aligned} \lambda_1 &= \lambda \\ \lambda_2 &= \lambda_1 + b_1 \\ \lambda_3 &= \lambda_1 + b_2 \\ \lambda_4 &= \lambda_1 + b_3 \end{aligned}\right\} \tag{24}$$

As in equation (22) of the previous section, x_4 can be written in terms of the other vectors

$$x_4 = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

where the β's are known. Equation (23a) results. Substituting for the λ_i's by equation (24) and expanding gives the following quadratic equation for λ:

$$\lambda^2\left(\sum_i \beta_i - 1\right) + 2\lambda(\beta_2 b_1 + \beta_3 b_2 - b_3) + \sum_i \beta_i(\rho^2 + \xi_i^{\,2}) - (\rho^2 + \xi_4^{\,2})$$

$$+ b_1^{\,2} + b_2^{\,2} + b_3^{\,2} = 0 \tag{25}$$

After λ is computed from equation (25), then equation (24) can be used to compute λ_1, λ_2, and λ_3. Therefore, all quantities in the right side of equation (5) are known and so is the projection vector.

Projection Using Altitude and Two Distance Differences

If one altitude and two distance differences are known, the equation for λ becomes a quartic. Two solutions are possible; the proper one must be selected. The equation to be solved is

$$\rho^2 = \frac{1}{4}\begin{bmatrix} \rho^2 + \xi_1{}^2 - \lambda_1{}^2 \\ \rho^2 + \xi_2{}^2 - (\lambda_1 + b_1)^2 \\ \rho^2 + \xi_3{}^2 - (\lambda_1 + b_2)^2 \end{bmatrix} G^{-1} \begin{bmatrix} \rho^2 + \xi_1{}^2 - \lambda_1{}^2 \\ \rho^2 + \xi_2{}^2 - (\lambda_1 + b_1)^2 \\ \rho^2 + \xi_3{}^2 - (\lambda_1 + b_2)^2 \end{bmatrix} \tag{26}$$

Equation (26) comes from equations (1) and (4) and taking a dot product

$$r \cdot r = a^t\bar{x} \cdot x^t a = a^t Ga = (r \cdot \bar{x})^t (G^{-1})^t GG^{-1}(r \cdot \bar{x})$$

$$= (r \cdot \bar{x})^t G^{-1}(r \cdot \bar{x})$$

With the measured quantities ρ, b_1, and b_2, and the notation g_{ij}^{-1} for the ijth element of G^{-1} and defining the matrix P as

$$P = \begin{bmatrix} \rho^2 + \xi_1{}^2 \\ \rho^2 + \xi_2{}^2 - b_1{}^2 \\ \rho^2 + \xi_3{}^2 - b_2{}^2 \end{bmatrix}$$

equation (26) can be written as the quartic

$$\lambda^4 \sum_{ij} g_{ij}^{-1} - 4\lambda^3 [b_1, b_2] \begin{bmatrix} g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} \end{bmatrix} P$$

$$+ 4\lambda^2 \left\{ [b_1, b_2] \begin{bmatrix} g_{22}^{-1} & g_{23}^{-1} \\ g_{32}^{-1} & g_{33}^{-1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \left[\sum_i g_{1i}^{-1}, \sum_i g_{2i}^{-1}, \sum_i g_{3i}^{-1} \right] \right\} P$$

$$- 4\lambda [b_1, b_2] \begin{bmatrix} g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} \end{bmatrix} P + P^t G^{-1} P - 4\rho^2 = 0 \tag{27}$$

Computation of the coefficients for equation (27), although tedious by hand, requires a simple digital computer program involving only matrix multiplication and basic arithmetic operations. After the coefficients are computed, methods, such as Ferrari's or Descarte's are available for extracting the required root for equation (27). Again after λ is computed from equation (27), the first three expressions in equation (24) can be applied to compute λ_1, λ_2, and λ_3. This computation completely specifies all quantities in the right side of equation (5).

Projection Using Three Independent Range Measurements

Two cases will be considered here. In the first, the reference points form a plane that does not contain the origin. The three range measurements λ_1, λ_2, and λ_3 are given, and ρ is to be determined. Then equation (26) can be expanded to give the following biquadratic in ρ:

$$\rho^4 \sum_{ij} g_{ij}^{-1} + 2\left[\rho^2 \left(\sum_i g_{1i}^{-1}, \sum_i g_{2i}^{-1}, \sum_i g_{3i}^{-1}\right)\begin{pmatrix}\xi_1^2 - \lambda_1^2 \\ \xi_2^2 - \lambda_2^2 \\ \xi_3^2 - \lambda_3^2\end{pmatrix} + 2\right]$$

$$+ \begin{pmatrix}\xi_1^2 - \lambda_1^2 \\ \xi_2^2 - \lambda_2^2 \\ \xi_3^2 - \lambda_3^2\end{pmatrix}^t G^{-1} \begin{pmatrix}\xi_1^2 - \lambda_1^2 \\ \xi_2^2 - \lambda_2^2 \\ \xi_3^2 - \lambda_3^2\end{pmatrix} = 0 \qquad (28)$$

In the second case, three range measurements are made again and ρ is to be determined, but this time the plane formed by the reference points does contain the origin. This development is similar to that in the section labeled "Indirect Expression" wherein the reference vectors were not linearly independent. In the present case, only two of the reference vectors are independent. Expressing the third in terms of the first two gives the expression

$$x_3 = (\beta_1, \beta_2)(x_1, x_2)^t$$

Following the arguments used in deriving equation (23a) gives the following expression for the projection of r along x_3:

$$[(\rho^2 + \xi_1^2 - \lambda_1^2), (\rho^2 + \xi_2^2 - \lambda_2^2)][\beta_1, \beta_2]^t = \rho^2 + \xi_3^2 - \lambda_3^2$$

Solving this equation for ρ^2 gives

$$\rho^2 = \left[\sum_i \beta_i(\xi_i^2 - \lambda_i^2) - (\xi_3^2 - \lambda_3^2)\right] \Big/ \left(1 - \sum_i \beta_i\right) \qquad i = 1, 2 \qquad (29)$$

Since the reference points are not colinear, the denominator of equation (29) is not zero. Again, equation (29) can be used to compute ρ^2. Once ρ^2 is known, the right side of equation (5) and thus the projection vector is completely specified.

Table 1 summarizes all typical coordinate-determination problems and their corresponding solution.

TABLE 1.- TYPICAL COORDINATE-DETERMINATION PROBLEMS

Given set of measurements	Number of solutions	Brief description of solution
One altitude measurement and (a) 3 independent ranges (b) 2 independent ranges and 1 angle (c) 1 independent range and 2 angles (d) 3 independent angle measurements	1	Direct expression Use equations (5) and (21) to construct the projection vector
Four independent range measurements or (a) 3 independent ranges and 1 angle (b) 2 independent ranges and 2 angles (c) 1 independent range and 3 angles	1	Direct expression Use equation (23b) to solve for ρ^2 and use equations (21) and (5) to solve variations of this problem
One altitude and (a) 3 distance differences (b) 3 distance sums (c) 2 distance sums, 1 distance difference (d) 1 distance sum, 2 distance differences Also other combinations of distance difference, distance sum, and range rate	1	Direct expression Use equation (25) to solve for λ^2
Three independent ranges (a) 2 ranges, 1 angle (b) 1 range, 2 angles	2	Direct expression Use equation (28) to solve for ρ^2, and equation (21) for variation of this problem
One altitude and (a) 2 distance differences (b) 1 distance difference, 1 distance sum (c) 2 distance sums Also other combinations of distance sum, distance difference, and range rate	2	Direct expression Use equation (27) to solve for λ^2
One altitude and (a) 2 ranges (b) 1 range, 1 angle (c) 2 angles	2	Indirect expression

RELATION BETWEEN COVARIANCES OF NAVIGATION AND MEASUREMENT ERRORS

The procedure developed in the preceding sections converts measured quantities into the quantities desired for navigation by means of a transformation whose terms are already known. Since the procedure separates the measurements and the transformation, it also separates measurement errors from the particular geometry summarized by the transformation. The separation will be shown in this section wherein expressions for the navigation errors and their covariances will be related to measurement errors and their covariances.

Error Covariance Matrix

The objective here is to show that error of the computed coordinates is a linear transformation of the errors in the measured quantities. Furthermore, the error covariance matrix of the computed coordinates is also some linear combination of the error covariance of the measured quantities. The relation between errors in computed coordinates and errors in measured quantities for the one altitude and three range measurements case will be derived. Assuming that the position of the reference points is completely known, we may take the partial derivatives of the computed coordinates in equation (8) with respect to the measured quantities as follows:

$$dw = \begin{bmatrix} \sum_j t_{1j}^{-1} \\ \sum_j t_{2j}^{-1} \\ \sum_j t_{3j}^{-1} \end{bmatrix} \rho \; d\rho - T^{-1} \begin{bmatrix} \lambda_1 \; d\lambda_1 \\ \lambda_2 \; d\lambda_2 \\ \lambda_3 \; d\lambda_3 \end{bmatrix} \qquad (30a)$$

Next, we define the following two matrices

$$J = \begin{bmatrix} \sum_j t_{1j}^{-1} & \\ \sum_j t_{2j}^{-1} & T^{-1} \\ \sum_j t_{3j}^{-1} & \end{bmatrix} \qquad (30b)$$

$$m = (\rho \; d\rho, \; -\lambda_1 \; d\lambda_1, \; -\lambda_2 \; d\lambda_2, \; -\lambda_3 \; d\lambda_3) \qquad (30c)$$

Equation (30a) may be rewritten in terms of the two matrices defined in equations (30b) and (30c) as

$$dw = Jm \qquad (30d)$$

The error of computed coordinates, therefore, is a linear transformation of m, the product of the measured quantities and the error in measurement of the corresponding quantity. The linear transformation matrix J depends only on the location of the reference points. By virtue of equation (30d), the covariance matrix of the computed coordinates is

$$\mathrm{cov}(dw) = J\ \mathrm{cov}(m) J^t \tag{30e}$$

In many coordinate determination problems, the desired coordinates are latitude and longitude. Then the rectangular coordinates must be converted into polar coordinates. For the direct expression, if the c-vectors are chosen such that c_1, c_3 are vectors from the center of the earth to Greenwich and to the North Pole, respectively, then the longitude and latitude of the aircraft position are, respectively,

$$\lambda_r = \tan^{-1} \frac{w_2}{w_1} , \qquad \Lambda_r = \tan^{-1} \frac{w_3}{(w_1{}^2 + w_2{}^2)^{1/2}} \tag{31}$$

The error in computed latitude and longitude is

$$\begin{bmatrix} d\lambda_r \\ d\Lambda_r \end{bmatrix} = \begin{bmatrix} \dfrac{\cos \lambda_r}{\sqrt{w_1{}^2 + w_2{}^2}} & \dfrac{-\sin \lambda_r}{\sqrt{w_1{}^2 + w_2{}^2}} & 0 \\ \dfrac{-w_3 \sin \lambda_r}{\rho^2} & \dfrac{-w_3 \cos \lambda_r}{\rho^2} & \dfrac{\sqrt{w_1{}^2 + w_2{}^2}}{\rho^2} \end{bmatrix} dw \tag{32}$$

Again, we define the matrix L

$$L = \begin{bmatrix} \dfrac{\cos \lambda_r}{\sqrt{w_1{}^2 + w_2{}^2}} & \dfrac{-\sin \lambda_r}{\sqrt{w_1{}^2 + w_2{}^2}} & 0 \\ \dfrac{-w_3 \sin \lambda_r}{\rho^2} & \dfrac{-w_3 \cos \lambda_r}{\rho^2} & \dfrac{\sqrt{w_1{}^2 + w_2{}^2}}{\rho^2} \end{bmatrix} \tag{33}$$

Next, we write equation (32) in terms of equations (30b) and (33)

$$\begin{bmatrix} d\lambda_r \\ d\Lambda_r \end{bmatrix} = LJm \tag{34}$$

The error covariance matrix for the computed latitude and longitude are related to the error covariance matrix of the measured quantities in the following expression

$$\operatorname{cov}\begin{bmatrix} d\lambda_r \\ d\Lambda_r \end{bmatrix} = LJ\ \operatorname{cov}(m) J^t L^t \tag{35}$$

It is to be emphasized that matrix J in equations (30d), (30e), (34), and (35) depends only on the location of the reference points and that the matrix L in equations (34) and (35), which becomes singular at both the north and south poles, depends on aircraft position.

The relation between the errors of computed coordinates and the errors of measured quantities will be derived for the indirect expression. First, write equation (3) in component form in light of equations (16), (13a), and (5) and as follows:

$$w = T^t \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = T^t \begin{bmatrix} F^{-1} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \cdot x_1 \\ r \cdot x_2 \\ a_3 \end{bmatrix}$$

$$= T^t \begin{bmatrix} F & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho^2 + \xi_1^2 - \lambda_1^2 \\ \rho^2 + \xi_2^2 - \lambda_2^2 \\ a_3 \end{bmatrix} \tag{36}$$

Second, take the partial derivatives of the computed Cartesian coordinate in equation (36) with respect to the measured quantities

$$dw = T^t \begin{bmatrix} \sum_j f_{1j}^{-1} & & \\ \sum_j f_{2j}^{-1} & F^{-1} & \\ \dfrac{1 - a_1 - a_2}{a_3} & \dfrac{-a_1}{a_3} & \dfrac{-a_2}{a_3} \end{bmatrix} \begin{bmatrix} \rho\ d\rho \\ -\lambda_1\ d\lambda_1 \\ -\lambda_2\ d\lambda_2 \end{bmatrix} \tag{37}$$

Next we define the matrices

$$K = \begin{bmatrix} \sum_j f_{1j}^{-1} & & \\ & & F^{-1} \\ \sum_j f_{2j}^{-1} & & \\ \dfrac{1 - a_2 - a_1}{a_3} & \dfrac{-a_1}{a_3} & \dfrac{-a_2}{a_3} \end{bmatrix} \tag{38a}$$

$$m' = (\rho \; d\rho, \; -\lambda_1 \; d\lambda_1, \; -\lambda_2 \; d\lambda_2) \tag{38b}$$

Then upon substituting equations (38a and b) into equation (37), we have

$$dw = T^t K m' \tag{39}$$

As exhibited by equation (39), the error of the computed coordinates is again a linear transformation of errors in measured quantities. However, in this indirect expression case, the linear transformation matrix depends on both the location of the reference points and on the position of the aircraft. The covariance matrix of the computed coordinates is

$$\mathrm{cov}(dw) = T^t K \; \mathrm{cov}(m') K^t T$$

To express the computed coordinates for the indirect case in latitude and longitude, choose c_1 as the vector pointing from the center of the earth to the intersection of the orbital plane with the equatorial plane so that the longitudinal distance from Greenwich and the projection of c_1 on the surface of the Earth is no greater than 180°. Choose c_3 as in the direct expression case. Then the latitude and longitude of the aircraft are, respectively,

$$\lambda_r = \lambda_{c_1} + \tan^{-1} \frac{w_2}{w_1}$$

$$\Lambda_r = \tan^{-1} \frac{w_3}{\sqrt{w_1^2 + w_2^2}}$$

The errors in computed longitude and latitude are related to the errors in measurements as

$$d\begin{bmatrix} \lambda_r \\ \Lambda_r \end{bmatrix} = L T^t K m'$$

The covariance matrix of the computed coordinates is given by

$$\mathrm{cov}\begin{bmatrix} d\lambda_r \\ d\Lambda_r \end{bmatrix} = L T^t K \; \mathrm{cov}(m') K^t T L^t$$

CONCLUSION

The coordinates of the position of a vehicle, say an aircraft, can be determined by a matrix multiplication of the projection vectors constructed from measurements. Measurements may be range, range rate, angle, distance difference, etc., and methods for construction of the projection vector from typical combinations of measurements were presented. The matrix displaying the coordinates by a linear transformation of the projection vector depends on the reference points from which the measurements were made. Since the position of the reference points is already known, this matrix is precomputable.

The basic procedure developed for coordinate determination assumes that measurements were made essentially at the same time and that the set of measurements is sufficient to determine the vehicle position uniquely. The scheme also works for nonsimultaneous measurements, provided a component is added in the projection vector to account for motions of the vehicle between measurements. If the set of measurements is insufficient to determine the position uniquely, then the procedure can determine the position to within a choice of sign.

The advantages of this technique are: (1) it is applicable to any combination of different types of measurements; (2) the measurements and the transformation are separated and related by a linear transformation; (3) the error in computed coordinates and that in measured quantities are separated and related by another linear transformation; (4) the covariance matrix of the computed coordinates and that of the measurements are also separated and related by some variation of the linear transformation in (3); and (5) the transformation in (2) is already known and is therefore precomputable.

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