

N71-30209

NASA-CR-119190

NONLINEAR OSCILLATIONS AND STABILITY BOUNDS
IN A NUCLEAR REACTOR WITH LINEAR REACTIVITY FEEDBACK

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~~NASA~~ 03-002-006

Semi-Annual Report - Part B

January 1971

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ENGINEERING EXPERIMENT STATION
COLLEGE OF ENGINEERING
THE UNIVERSITY OF ARIZONA
TUCSON, ARIZONA

NONLINEAR OSCILLATIONS AND STABILITY BOUNDS
IN A NUCLEAR REACTOR WITH LINEAR REACTIVITY FEEDBACK

by

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NASA 03-002-006
Semi-Annual Report, - Part B
January 1971

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ABSTRACT

Systems of autonomous nonlinear differential equations describing the stability of a nuclear reactor with two reactivity feedbacks have been analyzed to determine stability bounds. The system parameters used are representative of a fast reactor with in-core thermionic converters. A property common to all fast reactors is the possibility of one positive temperature coefficient which complicates control problems. This study may have application in the solution of such problems. Oscillation boundaries of the linearized system are established as an aid to the controls designer and to provide preliminary information in nonlinear stability analysis.

Bounded regions are determined for the effective lifetime model and coupled temperature equations by means of computer surveys of exact solutions. The nature of the bounded trajectories indicates the existence of limiting planes in state space. A mathematical description of these planes yields results in close agreement with numerical calculations. All trajectories outside the bounded regions exhibit finite escape time. Stable domains in state space cannot be determined by observing the effects of power perturbations alone. All three variables, power, and the two temperatures, must be varied. Concentric limit cycles have not been found for the effective lifetime model.

Bounded regions are determined for the prompt jump model and essentially the same results and conclusions are obtained as for the

effective lifetime model. The limiting planes and concentric limit cycles are not considered in this case.

An analytical (perturbation expansion) method is presented and applied to the effective lifetime model and coupled temperature equations. In the case of a stable limit cycle, this method gives results in precise agreement with the exact computer solution. In the case of a stable focus, the method gives a subregion of the stable region, but which overlaps into the unstable region. However, for higher orders of approximation the overlap decreases. It is concluded that the method is better suited to the analysis of systems which are not globally asymptotically stable or do not have infinite regions of bounded solutions. The formulation of the method as used here is not the most general, as it fails for the resonant case (critical power). A more general formulation is possible which treats the perturbation as a function of phase angle as well as amplitude.

Obvious extensions of this study are (1) derive the more general equations for the perturbations expansion method, (2) apply the method to the prompt jump model, and (3) apply the method to higher order systems, e.g., the one-delay group model.

CHAPTER 1

INTRODUCTION

Purpose

The purpose of this study is to examine stability bounds of sets of autonomous nonlinear differential equations that describe the behavior of a nuclear reactor with two reactivity feedbacks. The nature of unbounded solutions beyond the stable region is also examined.

The type of reactor considered in this study is a fast reactor with in-core thermionic converters. The advantages of this reactor in space applications would be not only the weight savings in shield and core due to high power density but also the elimination of the external electrical conversion system. Also of current interest is the development of fast breeder reactors for stationary power plant application. These are necessary if we are to make full use of our natural resources of nuclear fuels.

All of the temperature coefficients of fast reactors may not be negative. In particular the fuel Doppler coefficient, which is prompt, may be positive. This property of fast reactors increases the complexity of control compared to thermal reactors. However, there is no reason to believe that fast reactors cannot be designed and constructed to operate safely. Study of the stability behavior of such reactor systems with a positive temperature coefficient is essential to the eventual development of effective controls. Hence the present study is

not only relevant to current efforts in the design and development of fast reactors but may have application in the not too distant future when they become a reality.

Background

Two nonlinear analyses of the two-temperature, or two-path reactivity feedback, reactor model, without delayed neutrons, are presented by Gyftopoulos (1961) and Smets (1961). Their results are in agreement on the nonlinearly asymptotically stable region; however, their results are incomplete in the region of linearly oscillatory solutions, and in the case of existence of periodic solutions, their results disagree. Shotkin (1964) extends and clarifies their work. Schmidt (1969) considers both the effective lifetime (no delayed neutrons) and prompt jump neutron kinetics models for the thermionic reactor. He presents a method for predicting limit cycles. One of the important results of his study is the prediction and demonstration of unstable limit cycles in the prompt jump model. He also discusses linear stability in terms of the system transfer function and Routh criterion, as well as nonlinear stability.

Summary

In Chapter 2 of this study the oscillation boundaries in parameter space are determined for the linearized thermionic reactor model, using the prompt jump equation. This complements the linear analysis of Schmidt (1969). In nonlinear analysis it is often useful to know beforehand whether or not a case to be studied exhibits

linearly oscillatory behavior. This information is also useful to the controls designer.

Stable regions of the effective lifetime model are explored in Chapter 3 by making computer surveys of exact solutions. The nature of bounded trajectories suggests the existence of limiting planes in state space. A mathematical model by Kendall (1971) is examined and found to be in agreement with the numerical data. The nature of unbounded trajectories outside the stable region is explored, and it is found that they exhibit finite escape time. The possibility that concentric limit cycles exist in the effective lifetime model is explored. The search for such a configuration was unsuccessful.

In Chapter 4 stable regions of the prompt jump model are explored. Unbounded trajectories outside the stable region are found to exhibit finite escape time. Trajectories of a one delay group model are compared to corresponding ones of the prompt jump model to determine their behavior at prompt critical and super prompt critical, where the prompt jump model is no longer valid.

In Chapter 5 the perturbation expansion method of Shotkin (1969) is presented and applied to various problems. Two cases are considered for the two-temperature model using effective lifetime neutron kinetics. The resulting third-order system requires the elimination of a nondominant canonical coordinate in terms of the dominant ones. Second, third- and fourth-order approximations are obtained in both cases. The amount of algebra increases rapidly with higher order, which in practice is usually limited to the third.

Conclusions of this study are presented as Chapter 6.

CHAPTER 2

OSCILLATION BOUNDARIES IN PARAMETER SPACE

Introduction

In this chapter we present a method for calculating oscillation boundaries of a linearized reactor model and give the numerical results for four cases. The knowledge of these boundaries is a useful supplement to the stability information for the control designer and also has application in nonlinear stability analysis, for much about the behavior of a nonlinear system can be inferred from the study of the associated linear system (Hetrick, in press).

In the linear stability studies reported for the two-feedback thermionic reactor model (Brehm et al. 1967, Brehm et al. 1968), the stable and unstable regions were plotted in parameter space. The space consists of a plane whose coordinate axes are the two reactivity coefficients, each multiplied by the equilibrium power level. The stability boundary of greatest interest is called the resonance line (otherwise known as the critical line or the dynamic stability boundary).

A radial line through the origin in this plane corresponds to a fixed ratio of the two reactivity coefficients. Increasing the equilibrium power then corresponds to moving outward along one such radial line. The interesting cases are stable at low power and unstable at high power, and the value of equilibrium power that corresponds to crossing the resonance line is called the critical power.

Just below the resonance line there are two complex conjugate characteristic roots with negative real parts (damped oscillations), and just above the resonance line the real parts are positive (growing oscillations); the resonance line itself corresponds to pure imaginary roots.

Stability boundaries in a parameter space, such as shown in Figs. 2, 3, and 4 by Brehm et al. (1967), contain much information about the dynamic behavior of a system. However, except for the above observations about the neighborhood of the resonance line, these diagrams do not indicate whether the system response is oscillatory or not.

This information is easily added to the parameter plane in the form of a boundary line between oscillatory and non-oscillatory regions.

Oscillation Boundaries and Departure Points

An oscillation boundary is a locus of points in the parameter plane where the system characteristic equation has two equal real roots. Near this boundary, there are two complex roots with small imaginary parts on one side of the boundary and two nearly equal real roots on the other side. One way to locate the boundary is by trial and error, interpolating among numerical roots as functions of the parameters.

Another procedure is to seek algebraic conditions for the existence of two equal real roots. This is most efficiently done by solving for the points at which the root locus departs from the real axis (departure points).

We will now consider the two-region system of Eqs. (19) through (27) and Fig. 1 given by Brehm et al. (1967). These are reproduced here for convenience as follows:

$$G(s) = \frac{\delta n(s)}{\delta \rho(s)} = \frac{n_0}{\beta} \frac{s + \lambda}{s} \quad (2-1)$$

$$\frac{T_1(s)}{\delta n(s)} = \frac{b}{s + \eta_1} \quad (2-2)$$

$$\frac{T_2(s)}{\delta n(s)} = \frac{b\eta_2}{(s + \eta_1)(s + \eta_3)} \quad (2-3)$$

$$H(s) = \frac{\alpha_1 T_1(s) + \alpha_2 T_2(s)}{\delta n(s)} = \alpha_1 b \frac{s + \eta_3 + \eta_2(\alpha_2/\alpha_1)}{(s + \eta_1)(s + \eta_3)} \quad (2-4)$$

$$G(s)H(s) = K \frac{(s + \lambda)(s + \xi)}{s(s + \eta_1)(s + \eta_3)} \quad (2-5)$$

$$K = \frac{\alpha_1 b n_0}{\beta} \quad (2-6)$$

$$\xi = \eta_3 + \eta_2 \frac{\alpha_2}{\alpha_1} \quad (2-7)$$

$$D(s) = s^3 + (K + \eta_1 + \eta_3)s^2 + [\eta_1\eta_3 + K(\lambda + \xi)]s + K\lambda\xi \quad (2-8)$$

$$K + \eta_1 + \eta_3 > 0 \quad (2-9)$$

$$\eta_1\eta_3 + K(\lambda + \xi) - \frac{K\lambda\xi}{K + \eta_1 + \eta_3} > 0 \quad (2-10)$$

$$K\lambda\xi > 0 \quad (2-11)$$

$$x \equiv \frac{\alpha_1 b n_0}{\beta} = K \quad (2-12)$$

$$y \equiv \frac{\eta_2}{\eta_3} \frac{\alpha_2 b n_0}{\beta} \quad (2-13)$$

$$\xi = \eta_3 \left(1 + \frac{y}{x}\right) \quad (2-14)$$

$$K\xi = \eta_3 (x + y) \quad (2-15)$$

$$x > -(\eta_1 + \eta_3) \quad (2-16)$$

$$\begin{aligned} &(\lambda + \eta_3)x^2 + \eta_3 xy + [\eta_3^2 + \eta_1(\lambda + 2\eta_3)]x + \eta_3(\eta_1 + \eta_3 - \lambda)y \\ &\quad + \eta_1 \eta_3 (\eta_1 + \eta_3) > 0 \end{aligned} \quad (2-17)$$

$$x + y > 0 \quad (2-18)$$

$G(s)$ and $H(s)$ are the forward-loop transfer function and the feedback transfer function respectively. The relationship between these two quantities and those of Eqs. (2-2) and (2-3) are shown in Fig. 2.1. The open-loop transfer function is given by Eq. (2-5). K and ξ may be either positive or negative, depending on the signs and magnitudes of the two reactivity coefficients.

The characteristic polynomial, Eq. (2-8), is a cubic in s . Stability may be determined by the Routh-Hurwitz criterion. Using the parameters x and y defined in Eqs. (2-12) and (2-13) and Eq. (2-14), the vanishing of each Routh number yields a stability boundary in the x, y plane.

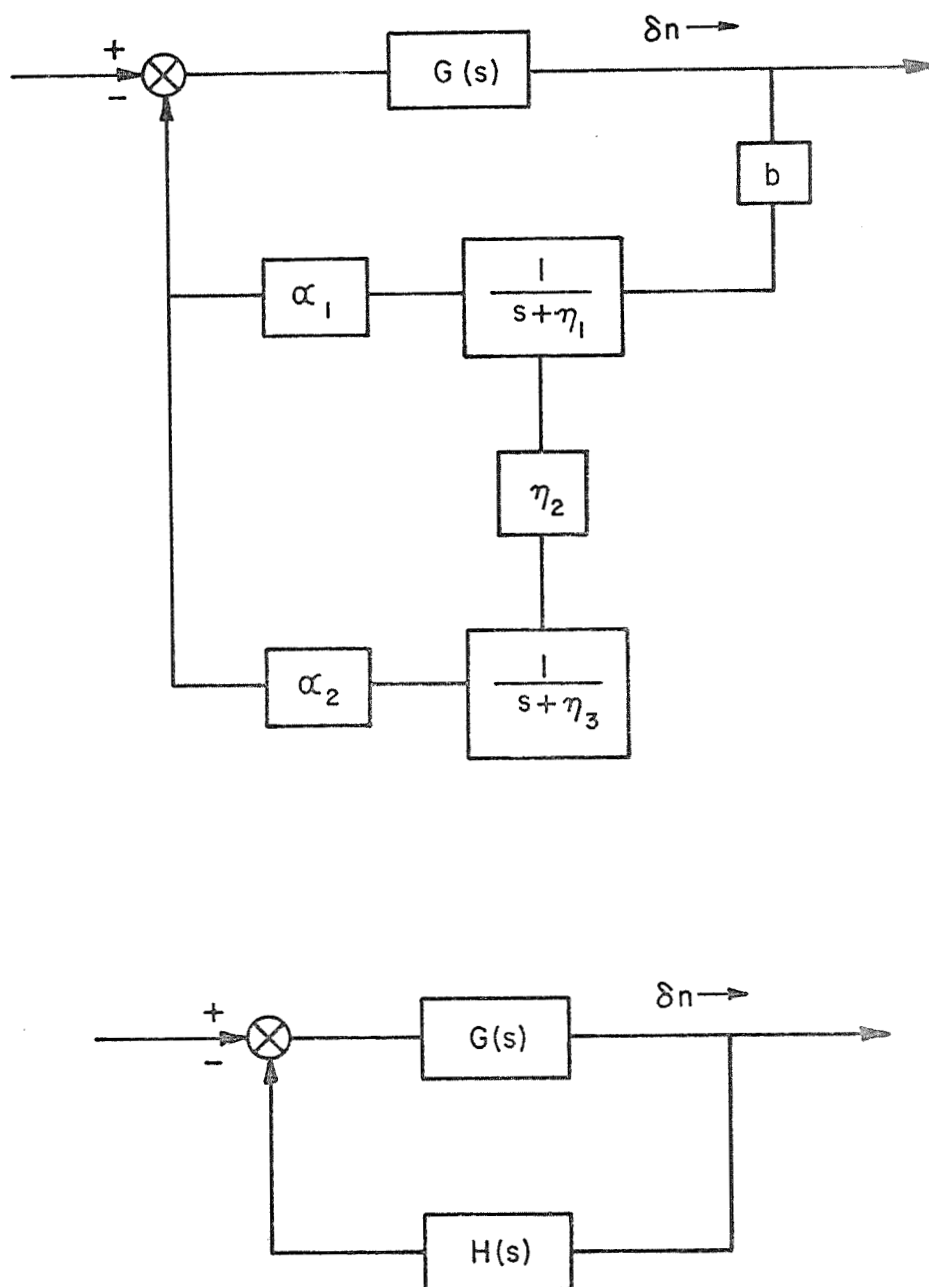


Figure 2.1 Block-Diagrams of Two-Temperature Feedback.

Equations (2-16) through (2-18) are obtained from Eqs. (2-9) through (2-11), respectively, by using Eqs. (2-12) through (2-14).

The symbols are

- λ = delayed-neutron decay constant
- η_1 = reciprocal time constant (Region 1)
- η_2 = coupling coefficient
- η_3 = reciprocal time constant (Region 2)
- α_1 = reactivity coefficient (Region 1)
- α_2 = reactivity coefficient (Region 2)
- b = reciprocal heat capacity (Region 1)
- β = delayed-neutron fraction
- n_0 = equilibrium power

Heat is generated only in Region 1 (the prompt region). The sign convention is such that positive α means negative reactivity feedback. Note that K , x , and y are proportional to the equilibrium power n_0 and that ξ = constant (fixed zero-location) corresponds to a fixed radial line in the parameter plane.

The departure points are the real solutions of

$$\frac{1}{s} + \frac{1}{s + \eta_1} + \frac{1}{s + \eta_3} - \frac{1}{s + \lambda} - \frac{1}{s + \xi} = 0 . \quad (2-19)$$

Kuo (1967) gives a derivation and proof of Eq. (2-19). He refers to departure points as breakaway points. Eq. (2-19) yields a quartic in s :

$$\begin{aligned}
s^4 + 2(\lambda + \xi)s^3 + [(\eta_1 + \eta_3)(\lambda + \xi) - \eta_1\eta_3 + 3\lambda\xi]s^2 \\
+ 2(\eta_1 + \eta_3)\lambda\xi s + \eta_1\eta_3\lambda\xi = 0
\end{aligned}
\tag{2-20}$$

For each real solution of Eq. (2-20), K is computed from Eq. (2-5) with $GH = -1$, and the point (x,y) on the oscillation boundary is obtained from Eqs. (2-12) and (2-14).

Numerical Results

Cases I and III (Brehm et al. 1967) were chosen for illustration, along with one variation of each, denoted as Case IA or IIIA. The fixed parameters are

$$\begin{aligned}
\lambda &= 0.1 \text{ sec}^{-1} \\
\eta_2 &= 0.25 \text{ sec}^{-1} \\
\alpha_1 &= -\frac{1}{3} \times 10^{-6} \text{ per } ^\circ\text{C} \\
\alpha_2 &= 10^{-6} \text{ per } ^\circ\text{C} \\
b &= 100^\circ\text{C per mw-sec} \\
\beta &= 10^{-2}
\end{aligned}$$

The four cases are as follows:

<u>Case</u>	<u>$\eta_1 (\text{sec}^{-1})$</u>	<u>$\eta_3 (\text{sec}^{-1})$</u>
I	0.5	0.2
IA	0.3	0.12
III	0.5	0.05
IIIA	0.15	0.05

The resonance line in each case is in the second quadrant of the x, y plane, corresponding to prompt-positive and delayed-negative reactivities. The systems that possess critical power levels are non-minimum-phase, negative-gain, conditionally stable systems.

Case I, which is from Fig. 2 (Brehm et al. 1967), is shown in Fig. 2.2. The stable region is bounded by the resonance line (second last Routh number is zero) and the line $\xi = 0$ (last Routh number is zero). The radial line labeled $\xi = -0.2$ is illustrated by a root locus for $K < 0$ in Fig. 2.3. At low gain (small n_0) there is no oscillation. As the gain is raised, the system encounters the first departure point and enters the region of damped oscillations. The stability boundary is crossed when the root locus enters the right half plane. Oscillations eventually cease, but at high gain, far inside the unstable region.

Below the line $\xi = 0$ in the second quadrant of Fig. 2.2, the system is unstable; the zero is in the left half plane but the gain is still negative. The oscillation boundary recedes to infinity when $\xi = \lambda$, because the root locus returns to the real axis at the double zero only for $|K| \rightarrow \infty$.

Just below the line $\xi = 0$, an interesting phenomenon appears. This is shown in Fig. 2.4, which is a magnified view of a portion of Fig. 2.2. The oscillation boundary has a sharp point extending below the line $\xi = 0$. Thus for small positive ξ , a system could experience two distinct oscillatory regimes as n_0 is increased. This possibility is illustrated in Figs. 2.5, 2.6, and 2.7, which are a sequence

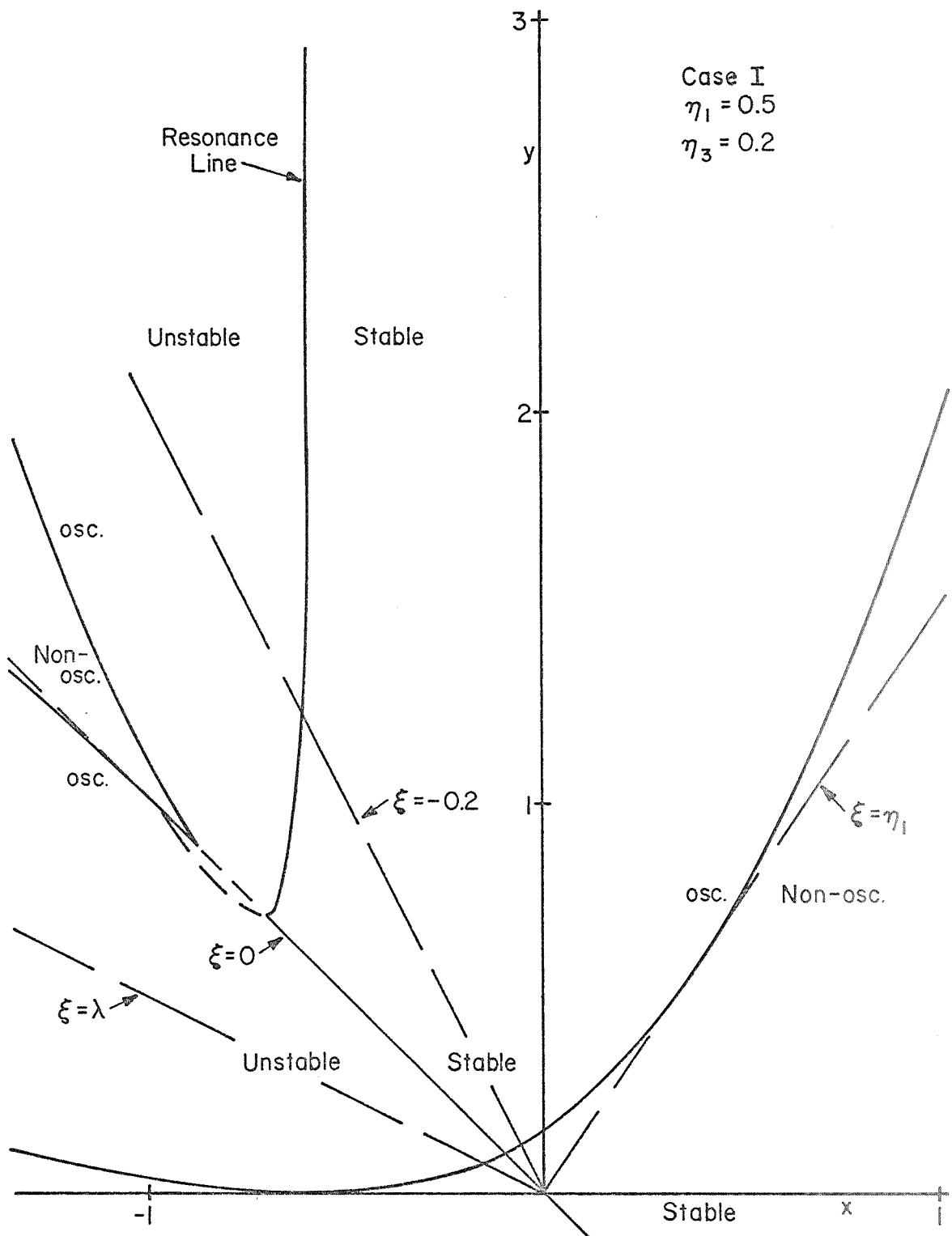


Figure 2.2 Oscillation Boundary in Parameter Space for Case I.

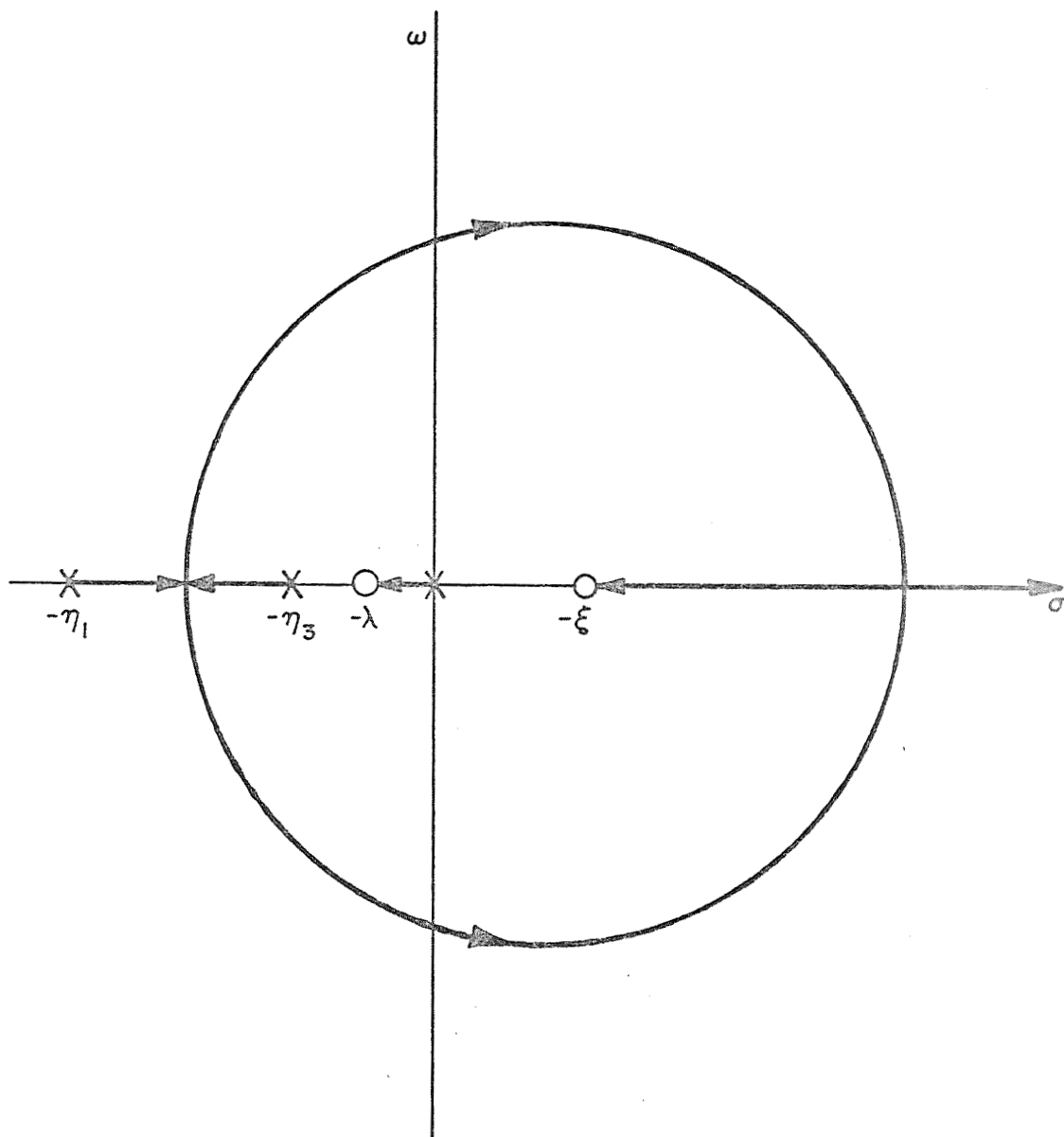


Figure 2.3 Root Locus for Case I ($\xi = -0.2$).

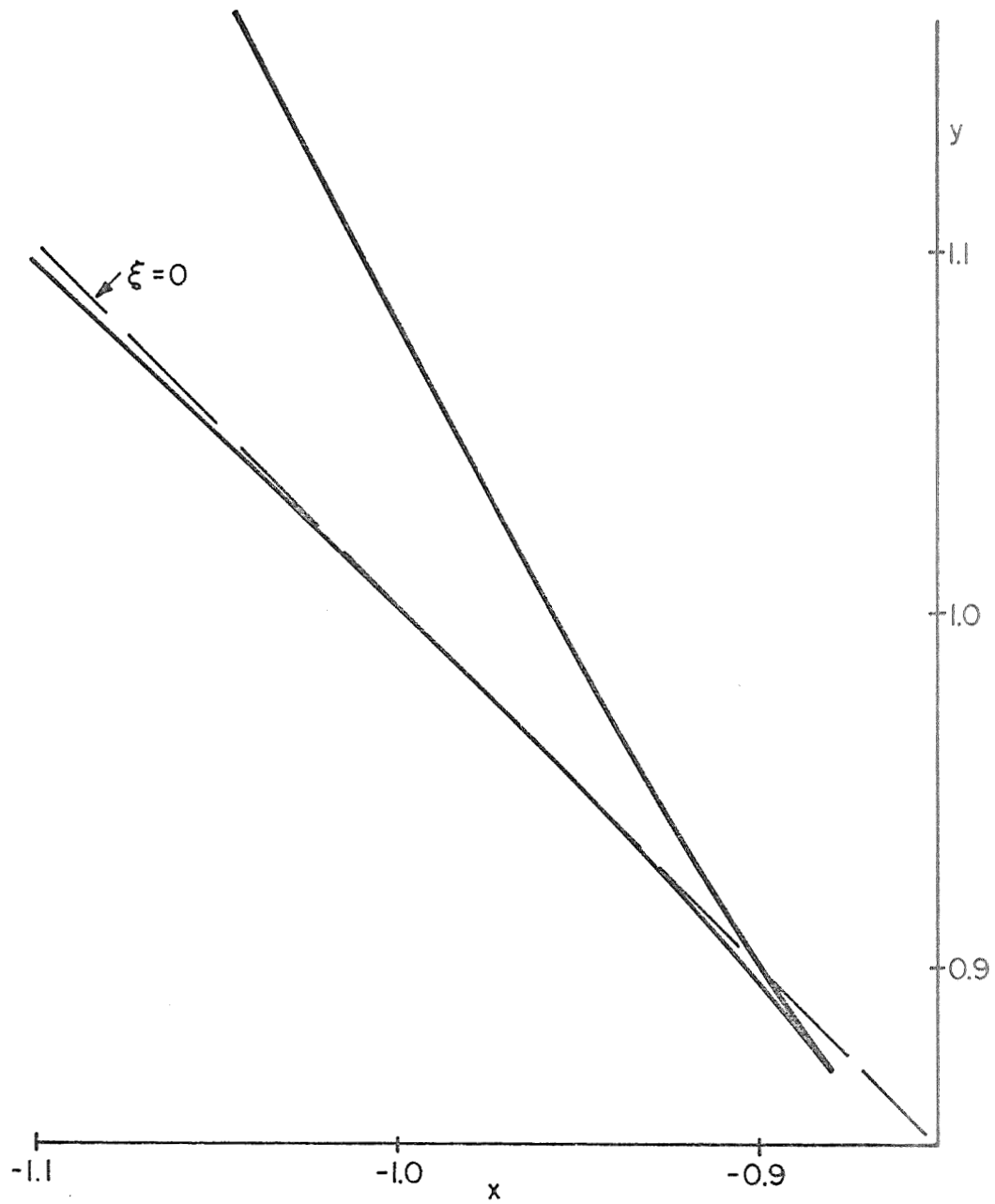


Figure 2.4 Partial Boundary in Parameter Space for Case I.

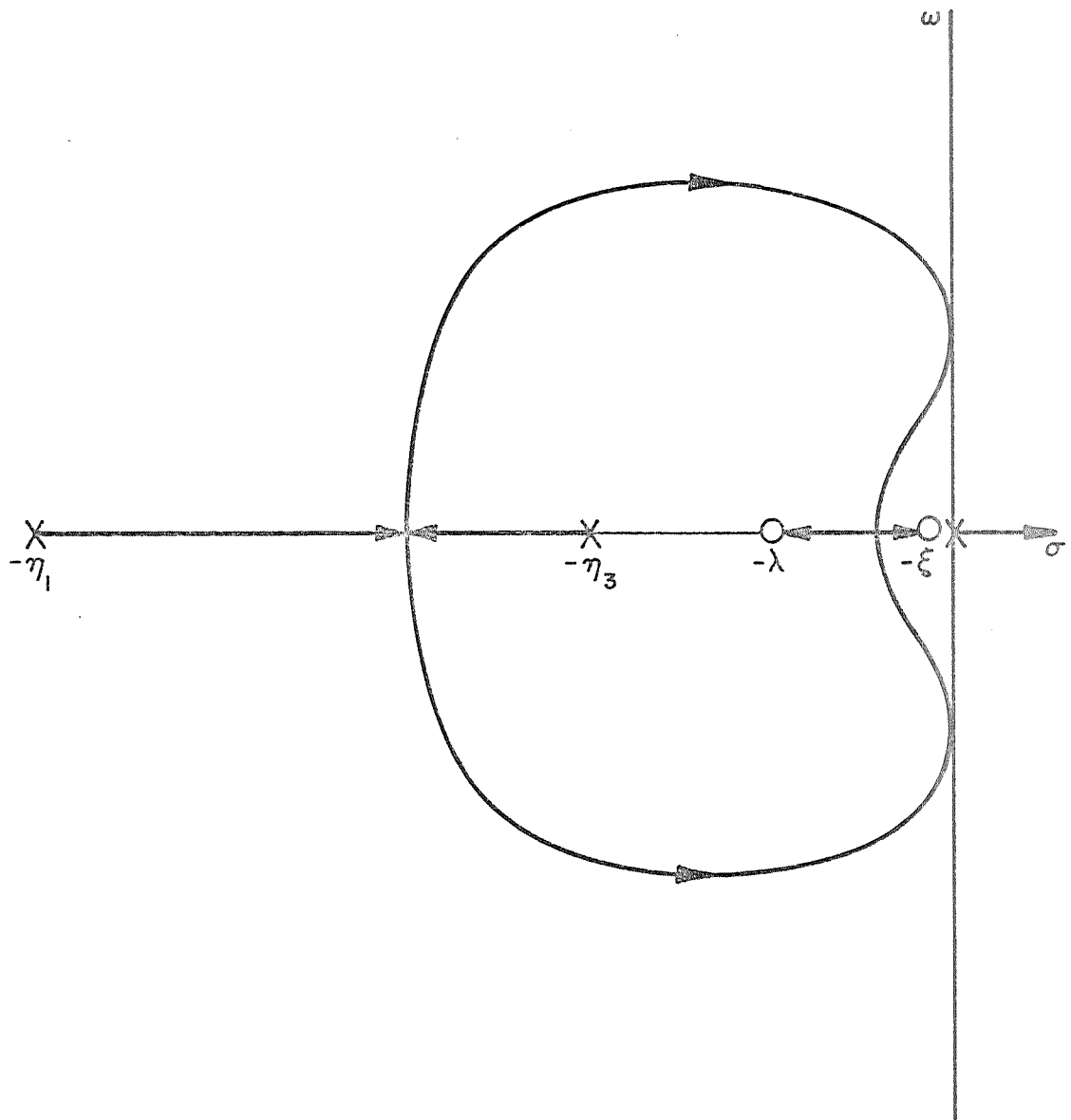


Figure 2.5 Root Locus for Case I ($\xi = 0.0125$).

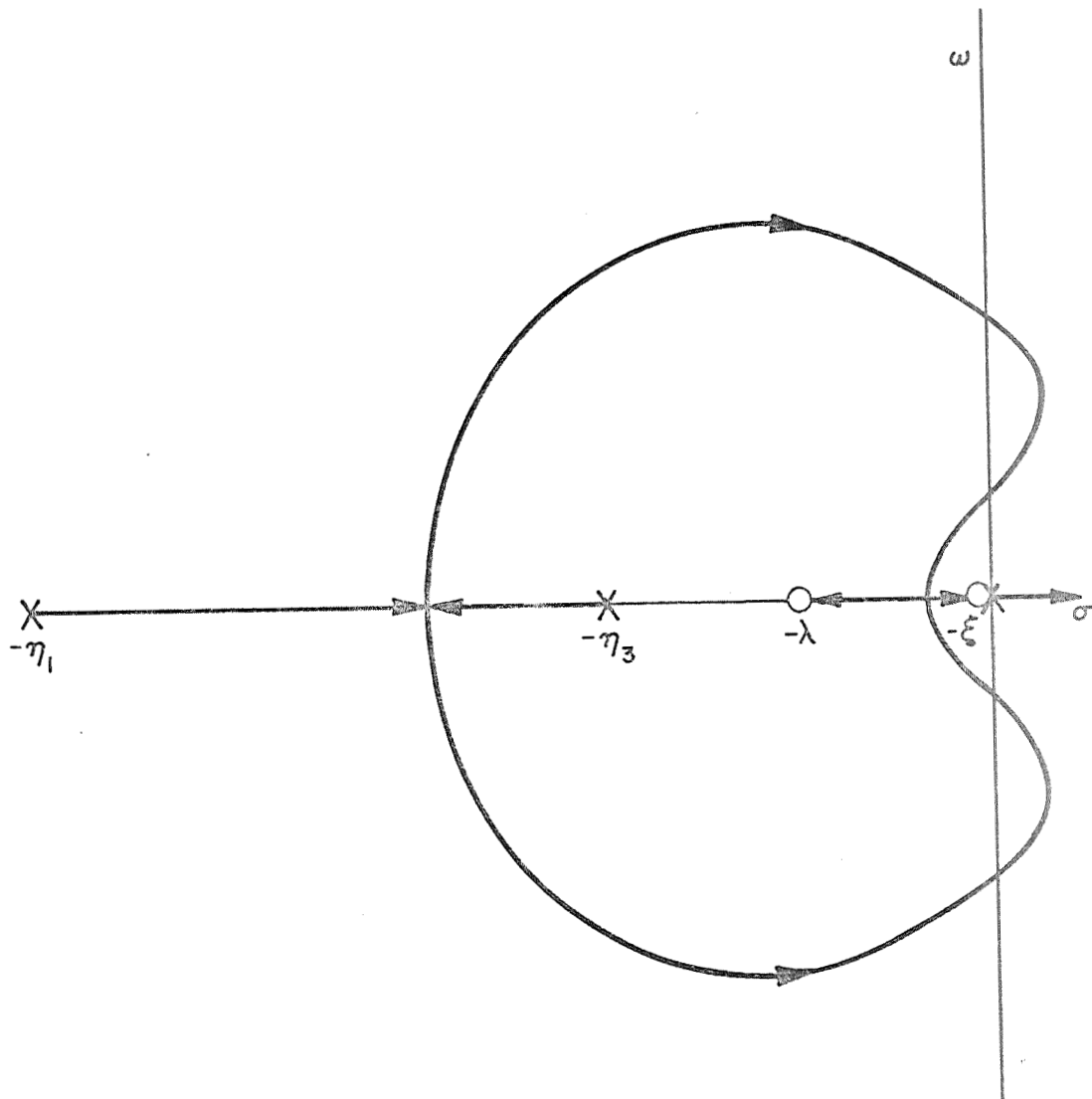


Figure 2.6 Root Locus for Case I ($\xi = 0.008$).

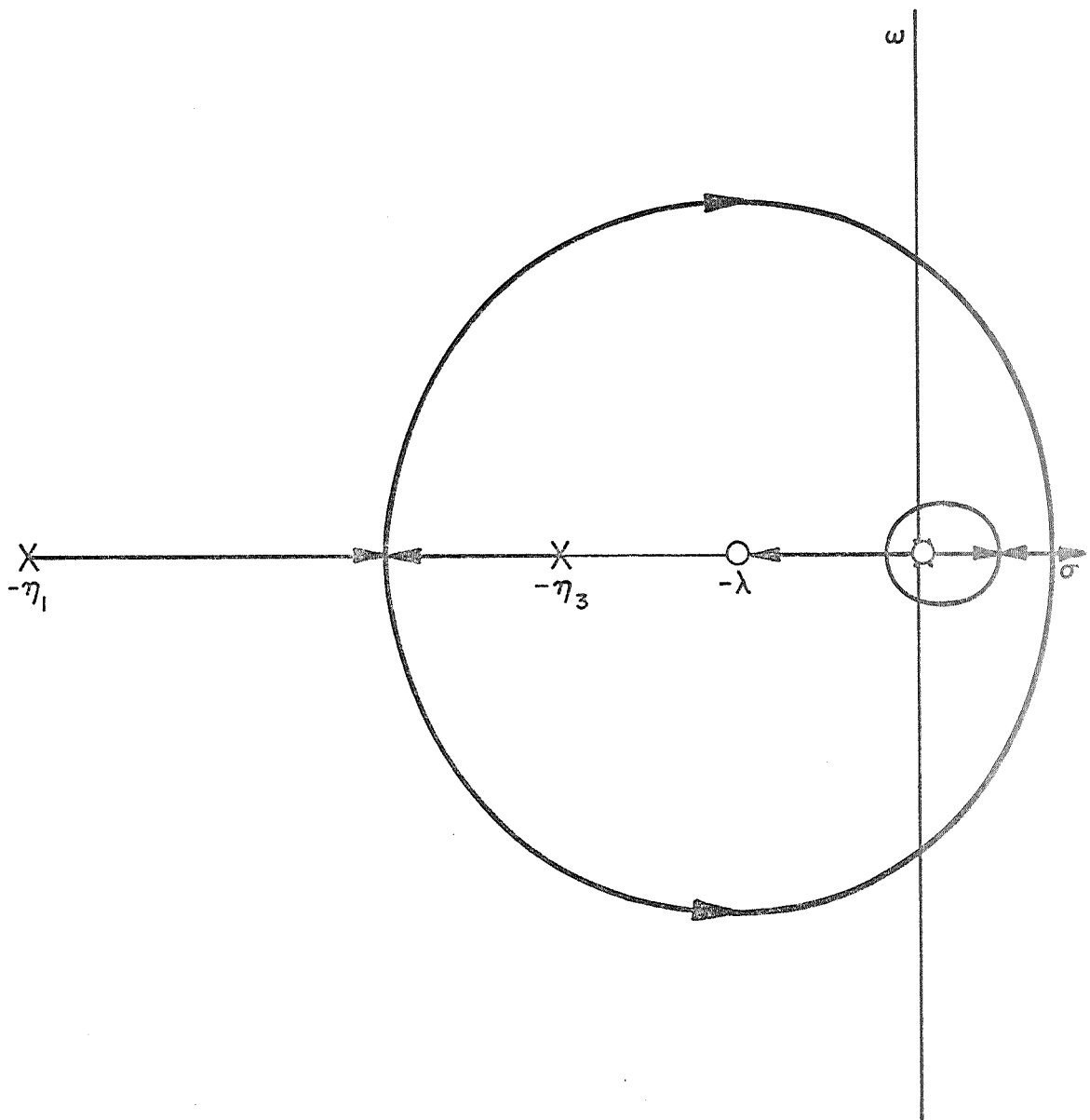


Figure 2.7 Root Locus for Case I ($\xi = 0.002$).

of root-locus plots for successively smaller positive values of ξ . When ξ is sufficiently small, the oscillation boundary is crossed four times (four real roots of Eq. (2-20), corresponding to four departure points). This type of root locus (two concentric loops) was also noted in Fig. 7 (Brehm, Hetrick, Schmidt 1969).

The cases shown in Figs. 2.5 through 2.7 are unstable for all n_0 . To illustrate the analogous behavior for conditionally stable systems, we contrived Case IA (see Figs. 2.8 and 2.9). Here the oscillation boundary has its sharp point in the stable region. A root locus plot for small negative ξ is shown in Fig. 2.10. The system enters the oscillatory region, leaves it, and reenters, remaining stable throughout this range of n_0 .

Case III, which is from Fig. 4 of Brehm et al. (1967), is shown in Fig. 2.11. Here the unusual systems are in the first quadrant (stable at all power levels n_0). The points for $\xi = \eta_1$ are easily obtained, because the cancellation of a pole and a zero makes the system quite simple. (The oscillation boundary is not asymptotic to the line $\xi = \eta_1$, but is tangent to it at a point not shown in the figure). For ξ slightly greater than η_1 (a radial line slightly closer to the y-axis), the root locus has two loops as shown in Fig. 2.12. Note that $K > 0$ in this case.

Case IIIA, shown in Fig. 2.13, is similar, except that the oscillation boundary has its sharp point below the line $\xi = \eta_1$. A magnified portion of the parameter plane is shown in Fig. 2.14, and a root locus for ξ slightly less than η_1 is shown in Fig. 2.15.

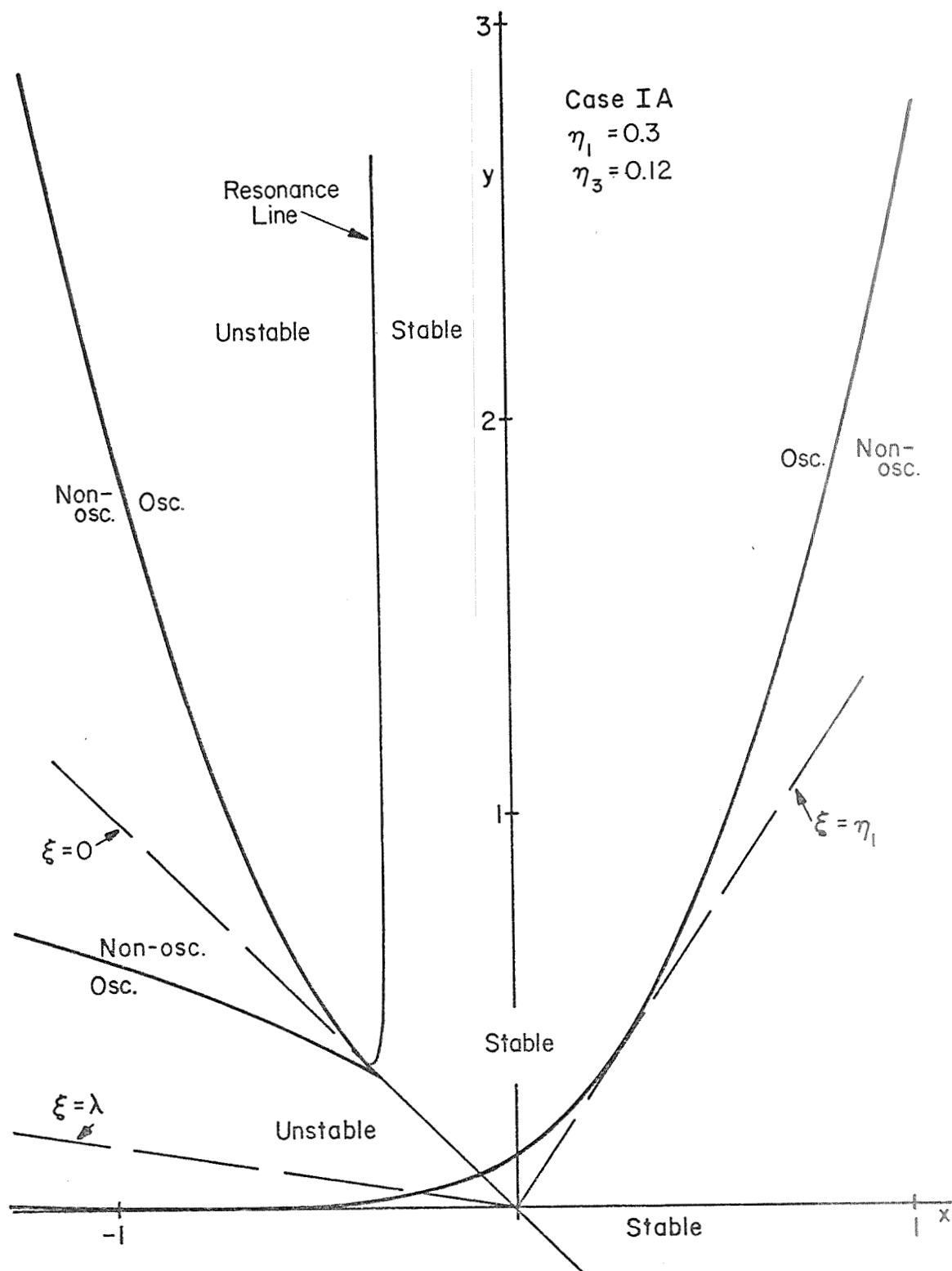


Figure 2.8 Oscillation Boundary in Parameter Space for Case IA.

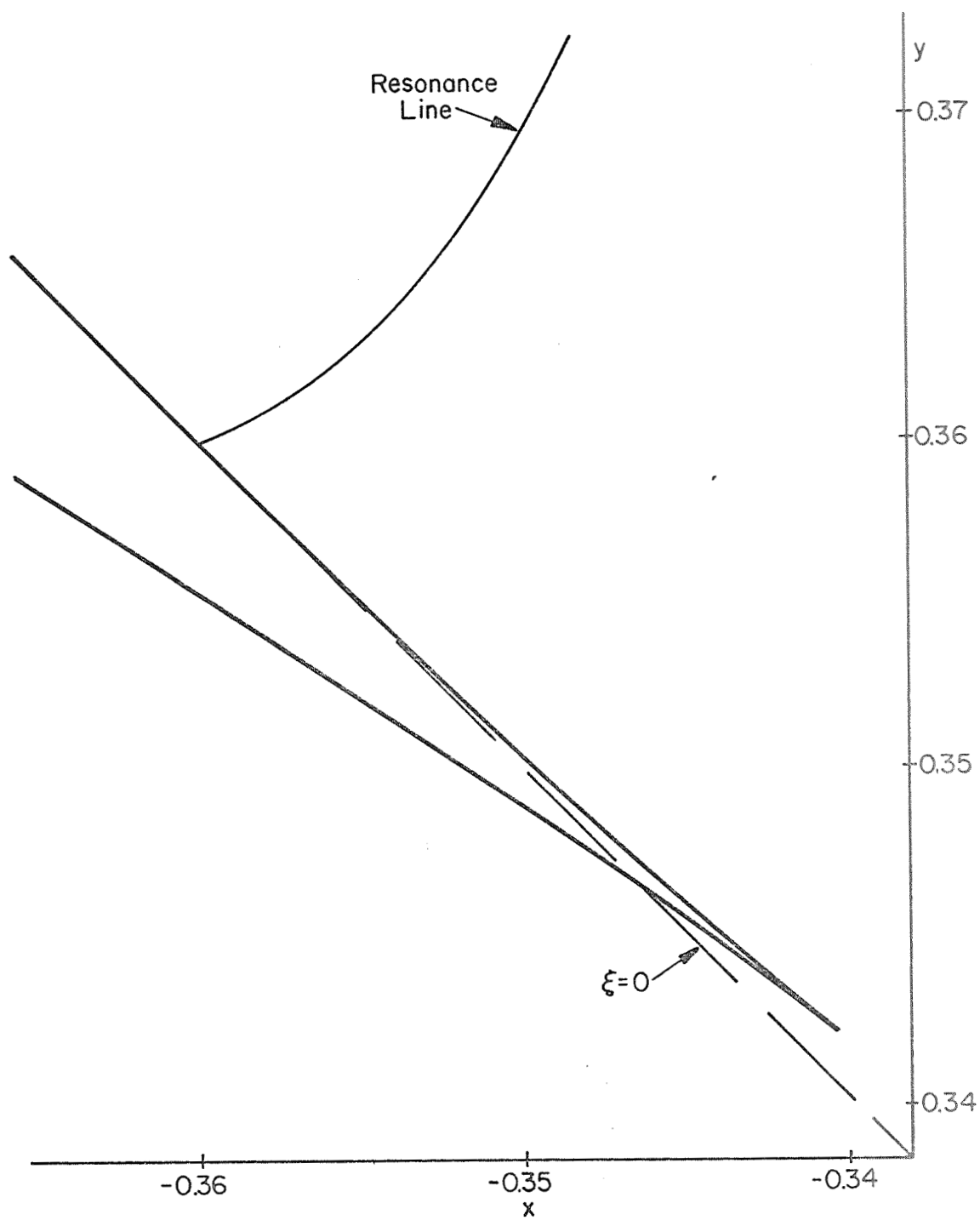


Figure 2.9 Partial Boundary in Parameter Space for Case IA.

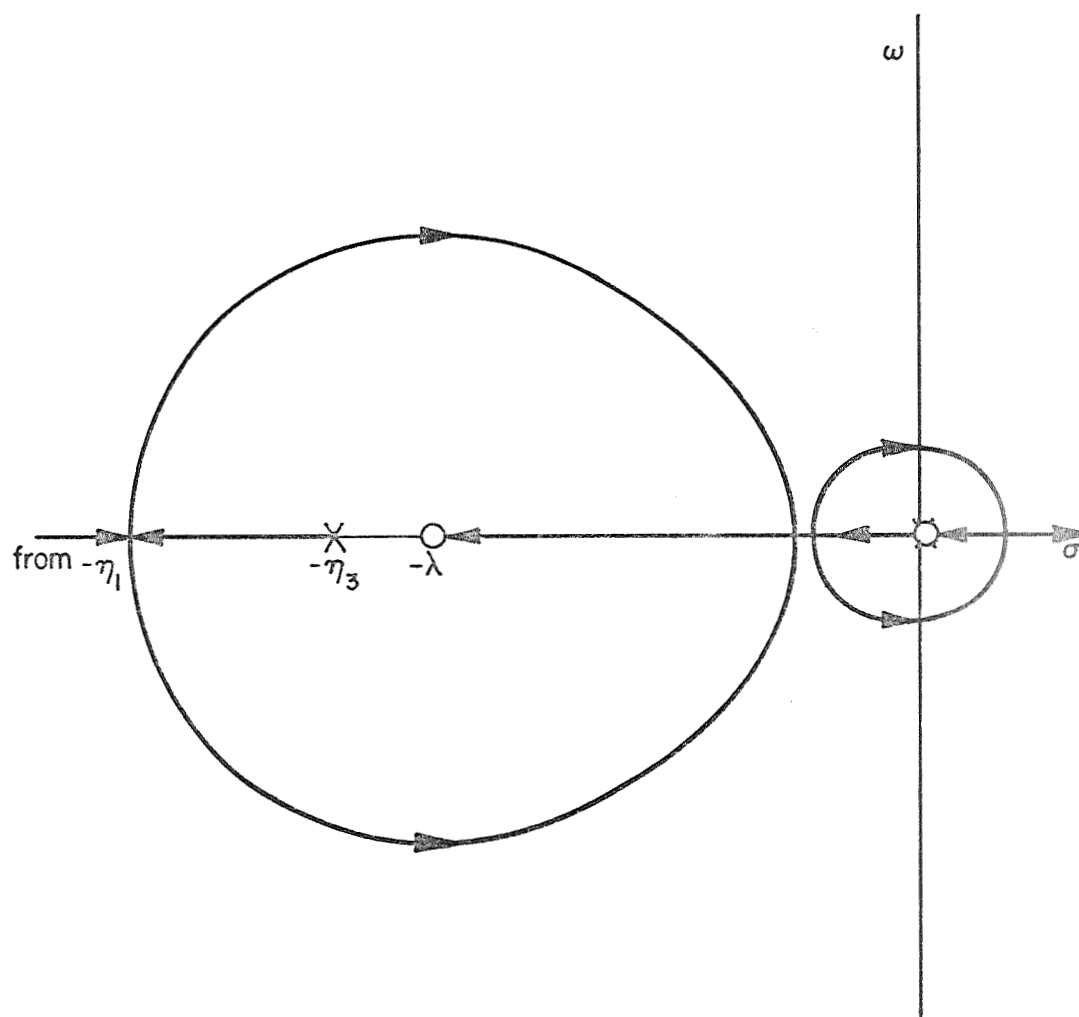


Figure 2.10 Root Locus for Case IA ($\xi = -6 \times 10^{-4}$).

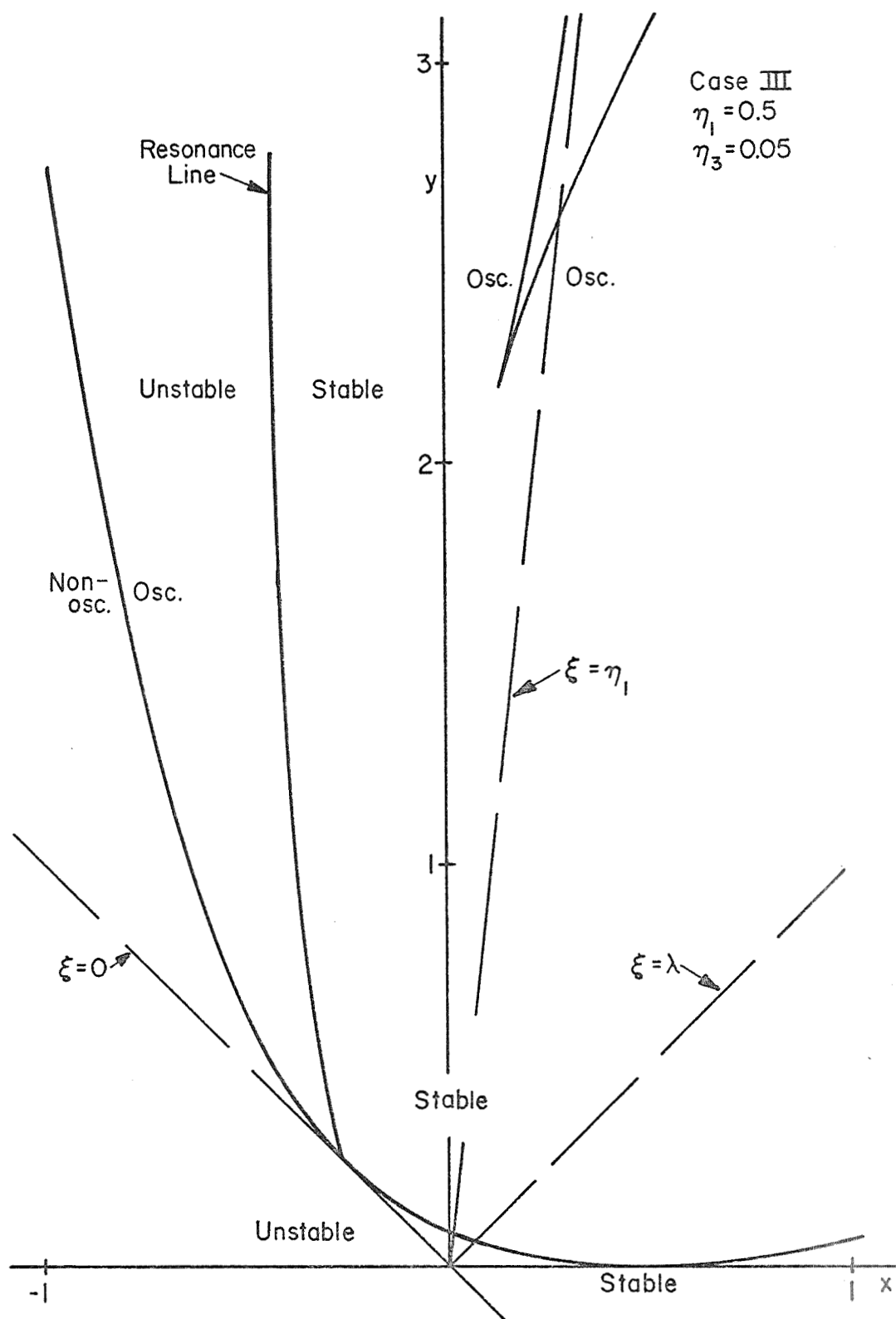


Figure 2.11 Oscillation Boundary in Parameter Space for Case III.

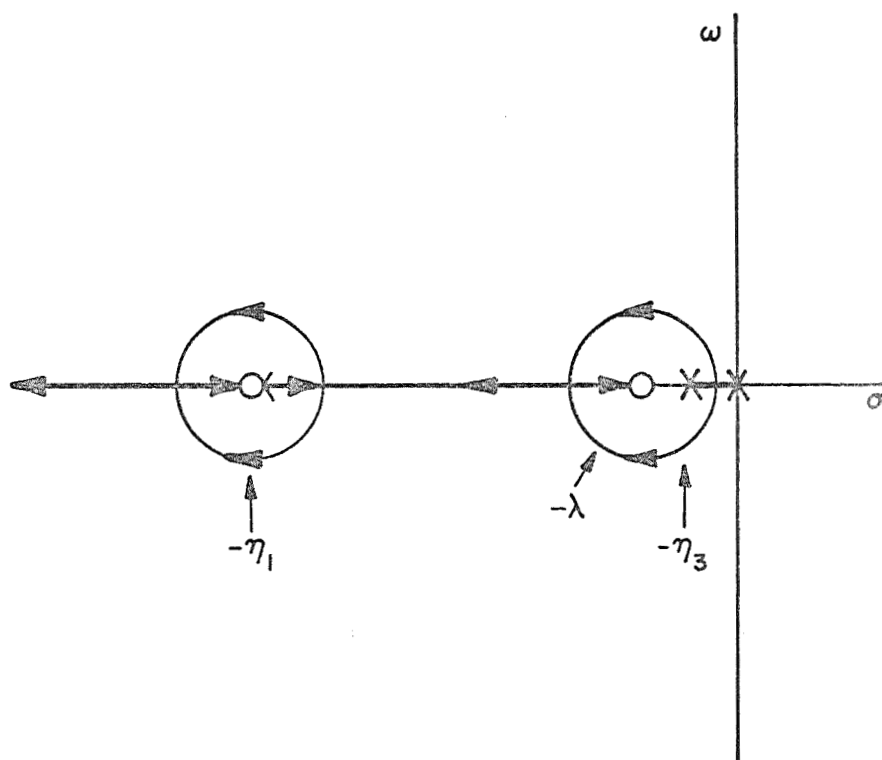


Figure 2.12 Root Locus for Case III ($\xi = 0.51$).

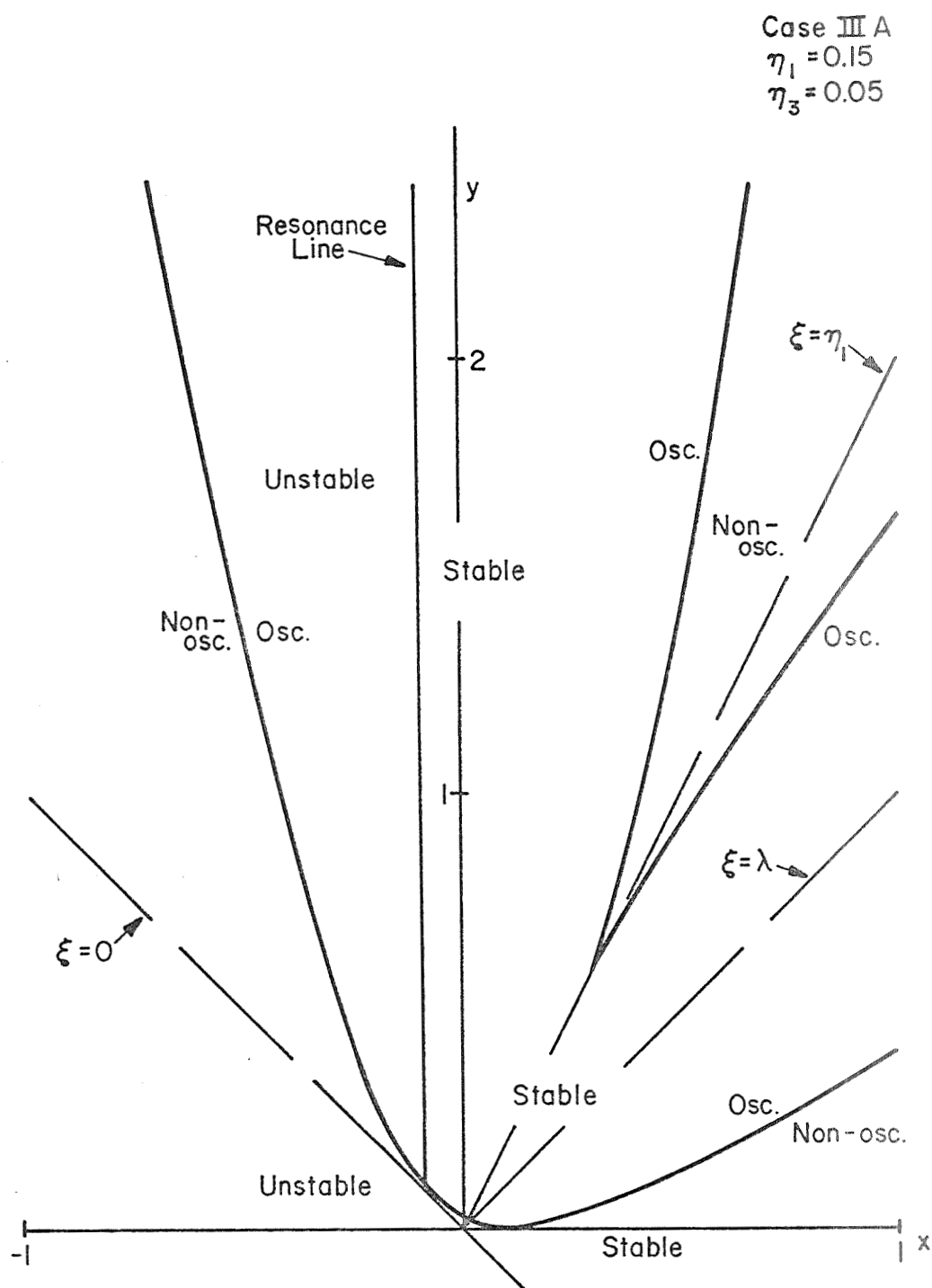


Figure 2.13 Oscillation Boundary in Parameter Space for Case IIIA.

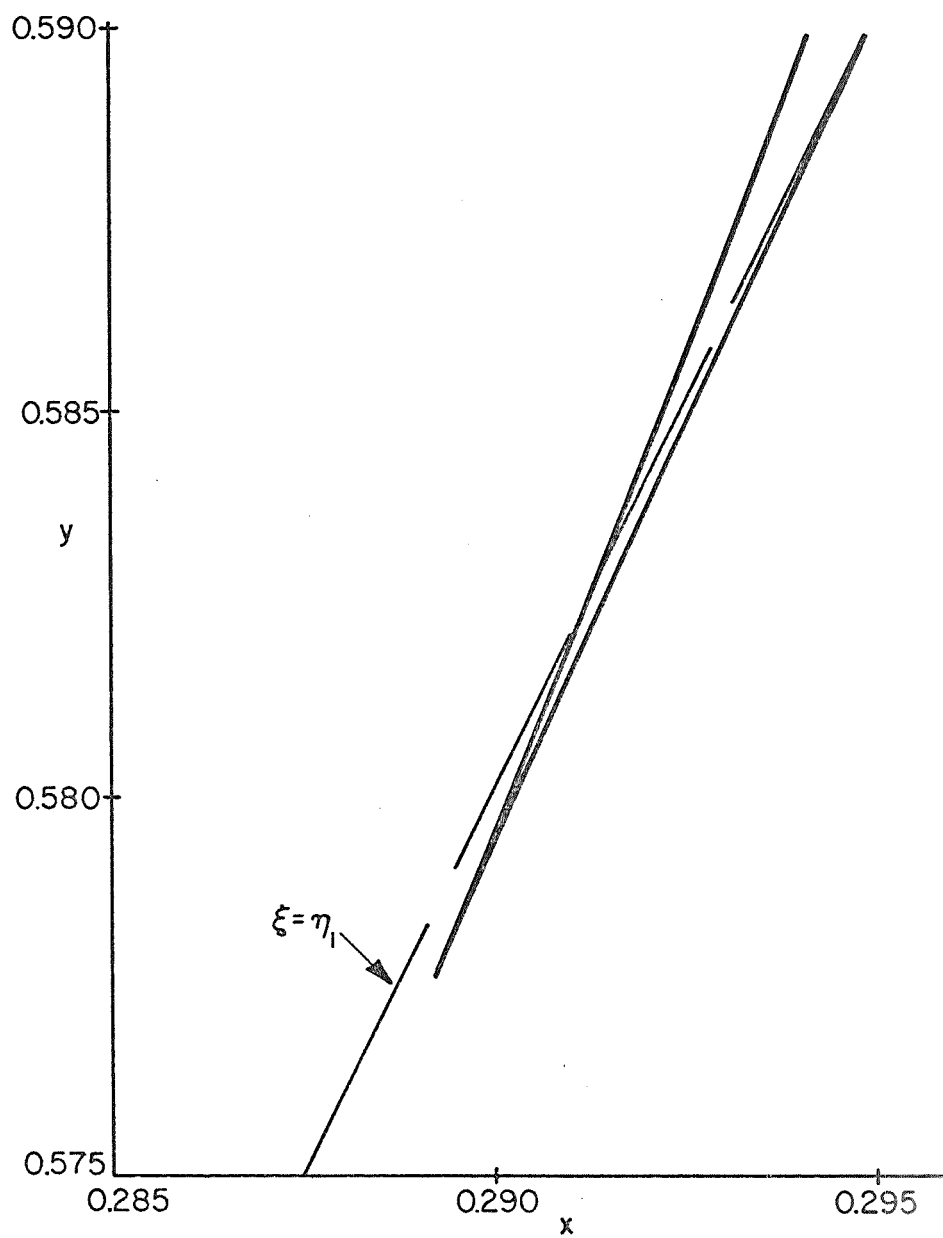


Figure 2.14 Partial Boundary in Parameter Space for Case IIIA.

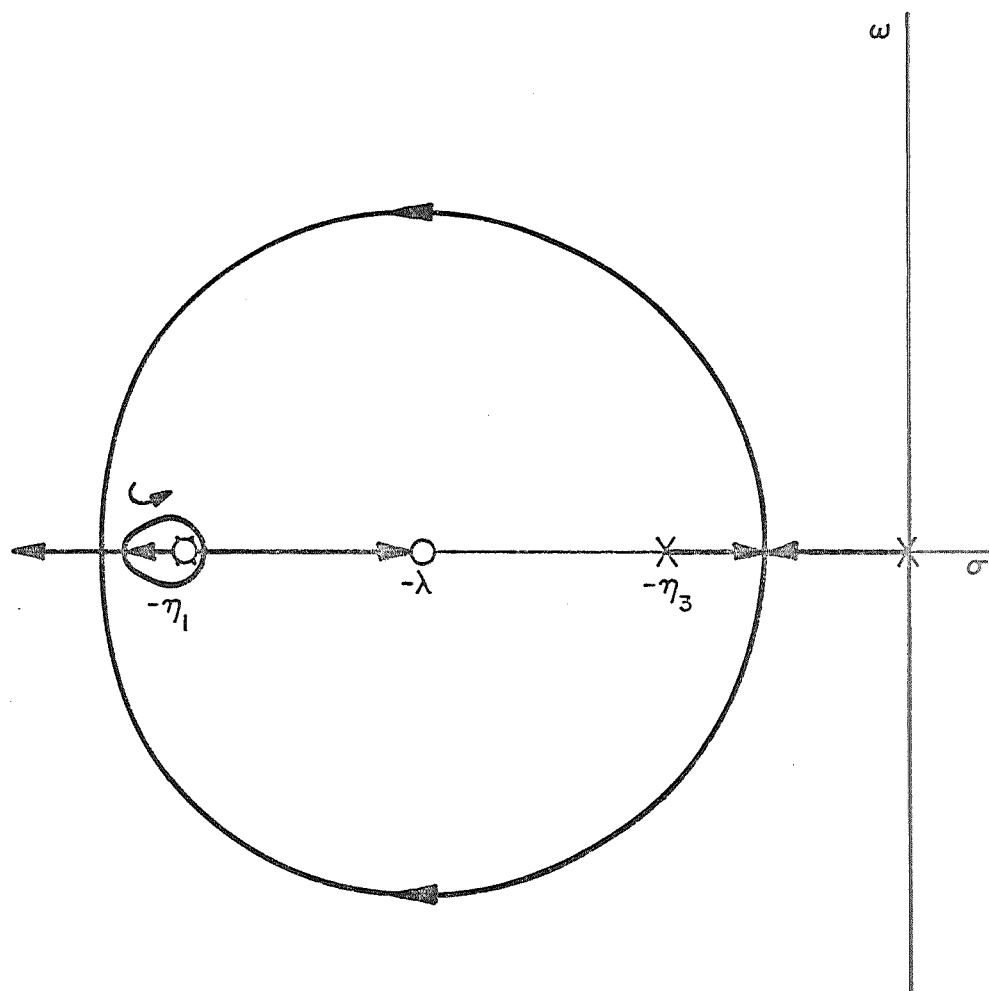


Figure 2.15 Root Locus for Case IIIA ($\xi = 0.14986$).

Third-order systems with four crossings of the oscillation boundary (four departure points with the same sign of K) are not encountered very frequently. In general, the oscillation boundary may be roughly located by making a few root-locus sketches and correlating them with the x, y plane. The peculiar systems we used for illustration are not very important in practice. However, when more detail is desired, it can be obtained qualitatively. A double-loop root locus such as Fig. 2.15 can be anticipated by sketching limiting cases with the zero very close to the pole. The two loops then have very little effect on each other, and the smaller loop must exist for a zero very near, and to the right of, the pole of $-\eta_1$. Once this is established, the details of the oscillation boundary are easily obtained.

CHAPTER 3

STABLE REGIONS WITH THE EFFECTIVE LIFETIME MODEL

Introduction

We now explore the stability regions of the thermionic reactor using the effective lifetime model and the coupled temperature equations. We do so by making computer surveys of exact solutions of the system of differential equations. The observed stability bounds are then compared to some which have been derived mathematically. The characteristics of the various stability regions are discussed, and finally the search for concentric limit cycles is described.

System Equations

The equations are

$$\frac{dn}{dt} = \frac{\lambda}{\beta} \rho n = \frac{\rho}{\ell}, n \quad (3-1)$$

$$\frac{dT_1}{dt} = b(n - n_0) - \eta_1 T_1 \quad (3-2)$$

$$\frac{dT_2}{dt} = \eta_2 T_1 - \eta_3 T_2 \quad (3-3)$$

$$\rho = -\alpha_1 T_1 - \alpha_2 T_2 \quad (3-4)$$

Most of the symbols have been defined in Chapter 2. The quantity $\ell' = \beta/\lambda$, as shown, is the effective lifetime. T_1 is the temperature (departure from equilibrium) of the fuel-emitter (Region 1), where the heat is

generated. T_2 is the temperature (departure from equilibrium) of the collector (Region 2), where no heat is generated. The symbol ρ stands for net reactivity. Since we consider only autonomous systems there is no externally impressed reactivity. The sign convention that we observe is that a positive value of α means negative reactivity feedback. For a more detailed description of the thermionic reactor, see Brehm et al. (1967) or Schmidt (1969).

Equation (3-1) is the well-known effective lifetime model. Because of its simplicity it has been used extensively (Gyftopoulos 1961, Shotkin 1964, Smets 1964, Akcasu and Noble 1966, Schmidt 1969). This model does not describe the effect of delayed neutrons on stability behavior but gives the correct time scale for neutron kinetics for small ρ .

Equations (3-2) and (3-3) are the coupled temperature equations of the physical system. They are, basically, the heat balance equations for Region 1 and Region 2, respectively. Equation (3-4) represents the temperature reactivity feedbacks from Region 1 and Region 2. In the thermionic reactor the feedback from Region 1 is prompt positive, while that from Region 2 is delayed negative.

Stability Regions

The system of equations of the previous section was programmed for solution on the CDC 6400 digital computer of the University of Arizona Computer Center. A brief description of the program and a listing are given in the Appendix. The mapping of stability regions for

power perturbations is accomplished by varying the initial power for a given set of equilibrium conditions and examining the computer solutions.

Such a mapping was done for the following case, designated as Case I, $\xi = -0.1$,

$$\begin{aligned} \lambda &= 0.1 \text{ sec}^{-1} , & \beta &= 0.01 , \\ b &= 100 \text{ }^{\circ}\text{C/Mw} - \text{sec} , & \eta_1 &= 0.5 \text{ sec}^{-1} \\ \eta_2 &= 0.25 \text{ sec}^{-1} , & \eta_3 &= 0.2 \text{ sec}^{-1} \\ \alpha_1 &= -\frac{1}{3} \times 10^{-6} \text{ per } ^{\circ}\text{C} , & \alpha_2 &= 0.4 \times 10^{-6} \text{ per } ^{\circ}\text{C} . \end{aligned}$$

The critical power, n_c , for this case is 262.5 Mw. The equilibrium power, n_0 , was varied and, for each n_0 , the maximum power perturbation (both positive and negative) was determined for which the system remained stable. The results are shown in Fig. 3.1 with the threshold for unbounded solutions plotted as a function of equilibrium power. The bounded region does not disappear at critical power, but extends to 300 Mw where it ends abruptly. All solutions for greater n_0 are unbounded and exhibit finite escape time. The 300 Mw limit in Fig. 3.1 is precisely the cutoff for the existence of stable limit cycles observed by Shotkin (1964) and verified by Schmidt (1969) in his computer studies. A further discussion of this limit appears in a following section of this chapter. It is noted that the regions designated "bounded" in Fig. 3.1 are those of stable limit cycles.

Recently, Curtis (in preparation) investigated a similar system (effective lifetime model, power perturbation only) using the Popov (1963) criterion. The stable region that he obtains is smaller than the total bounded region (stable + bounded) of Fig. 3.1. This underestimation

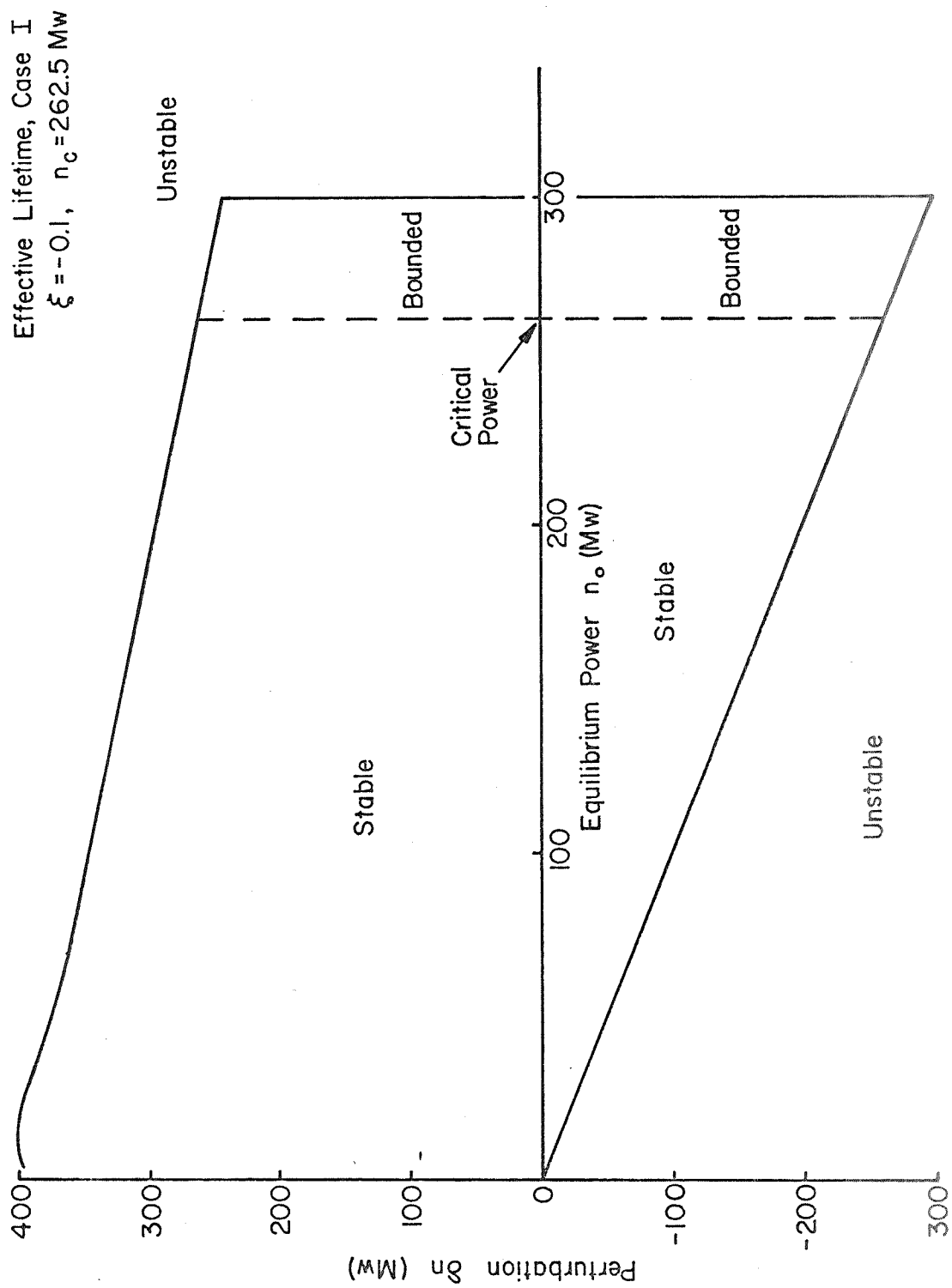


Figure 3.1 Effective Lifetime Stability Regions.

is to be expected because the method is based on sufficient conditions for stability. Also in contrast to Fig. 3.1, his stable region disappears at critical power. Note, too, that frequency-domain stability criteria of this type are restricted to solutions of exponential order, because the existence of Laplace or Fourier transforms is tacitly assumed. Hence stability conclusions are tentative whenever the possibility of finite escape exists.

Finally we note that in both cases the procedure of perturbing the power alone is not sufficient for mapping the stable domain in state space. To do this one would have to study perturbations in both temperatures, as well as those in power.

The Bounded Region

We now examine the nature of trajectories in the bounded region of Fig. 3.1. We select an equilibrium power of 265 Mw for these studies. A typical trajectory in the n, T_1 plane is shown in Fig. 3.2. Note that this is actually the projection of a trajectory in n, T_1, T_2 space onto the n, T_1 plane. The initial conditions for this run are $T_1 = T_2 = 0$, and n_i (initial power) = 525.306 Mw, i.e., a δn (power perturbation) = 260.306 Mw. As a result of the positive δn and the prompt positive temperature coefficient, n, T_1 and T_2 all increase initially. The power peaks at 6036 Mw with corresponding $T_1 = 1.016 \times 10^6$ °C and $T_2 = 0.8464 \times 10^6$ °C. The delayed negative temperature coefficient has taken effect, so the power decreases as the temperatures continue to increase. With a further decrease in power the temperatures also decrease, hence the curve in Fig. 3.2

Effective Lifetime, Case I, $\xi = -0.1$
 $n_o = 265 \text{ Mw}$, $n_I = 525.306$

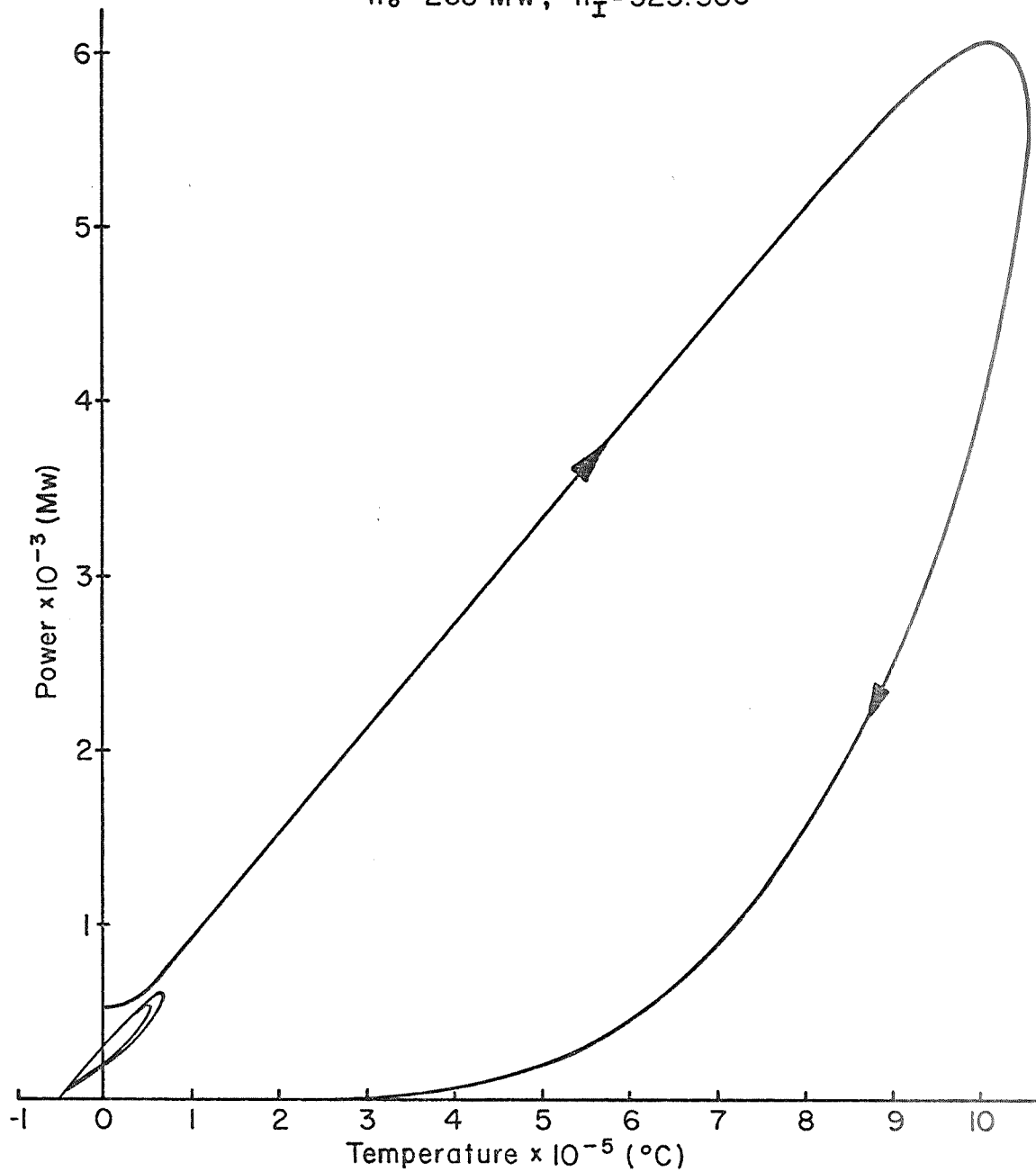


Figure 3.2 Projection: Effective Lifetime Stable Limit Cycle.

moves back toward the origin for decreasing power. The power remains at a very low level (but not zero) as T_1 becomes negative (below equilibrium), and finally as n and T_1 increase again the trajectory winds onto a stable limit cycle. Trajectories that originate inside this limit cycle, off the equilibrium point, also wind onto the limit cycle. We will demonstrate an analytical method for calculating this limit cycle in Chapter 5. If n_i is increased to a value of 525.3075 Mw (initial $T_1 = \text{initial } T_2 = 0$) the trajectory diverges with finite escape time. Below this limit the trajectories make increasingly large traverses before winding onto the limit cycle, with increasing n_i . Note that there are an infinite number of choices below the limiting value, i.e., one may choose n_i values as close as he likes to 525.3075 Mw.

A three-dimensional view of the foregoing trajectory is shown in Fig. 3.3. The dotted line in the n, T_1 plane is the trajectory projection of Fig. 3.2. The initial rise of the trajectory in n, T_1, T_2 space is along a straight line which lies in the maximum reactivity plane. After the trajectory passes its peak it cuts across planes of lower reactivity (toward the n, T_2 plane) as n and T_1 decrease. It moves just above the $n = 0$ plane into the region of negative temperatures before rising again to wind onto the stable limit cycle situated in n, T_1, T_2 space.

The Stable Region

To investigate the stable region of Fig. 3.1 we choose an equilibrium power of 100 Mw which is far below the critical power. A computation with negative initial values of T_1 and T_2 and $n_i = 2.9$ Mw

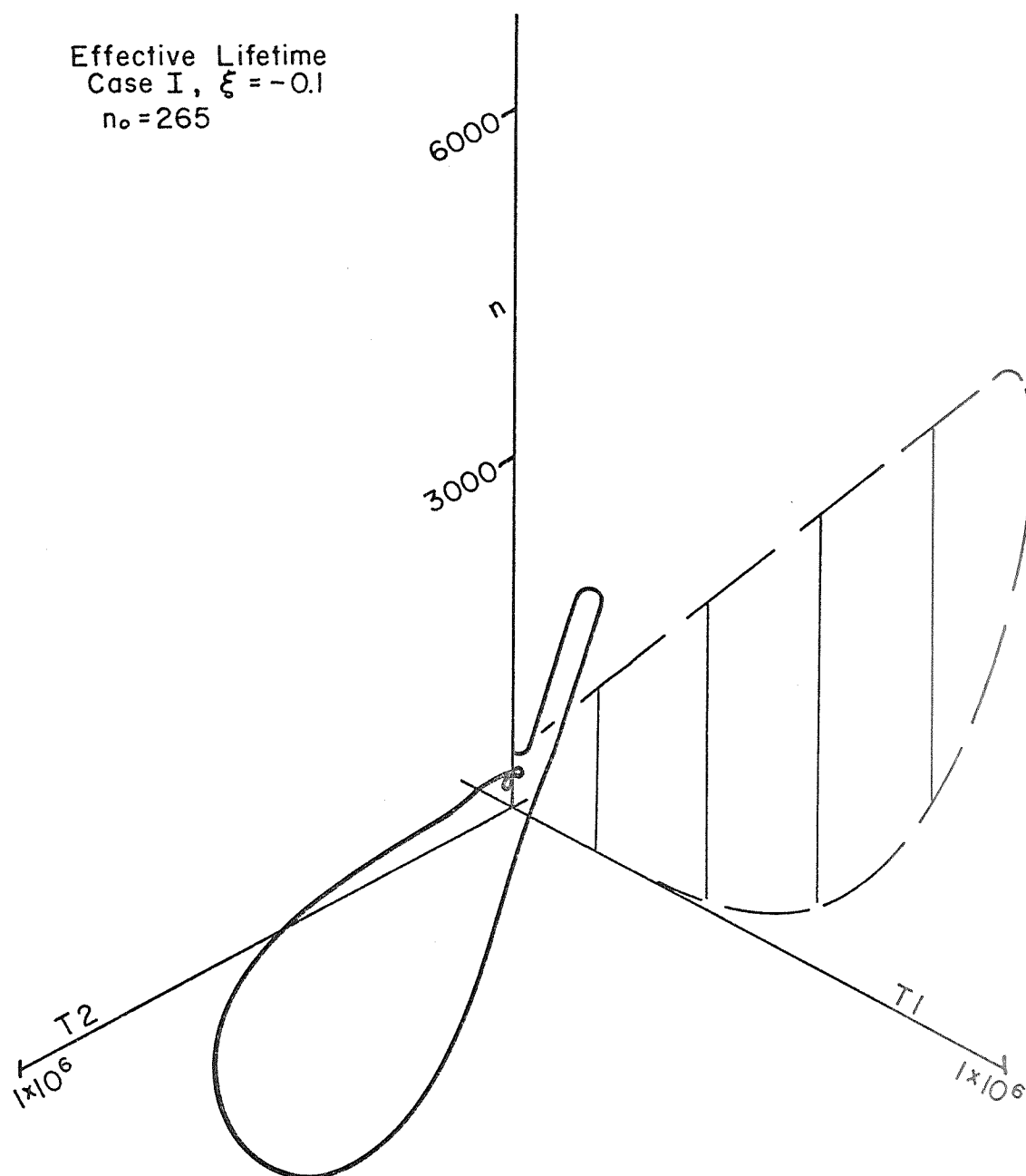


Figure 3.3 Effective Lifetime Stable Limit Cycle.

gives the projected trajectory of Fig. 3.4. Qualitatively this trace is similar to that of Fig. 3.2, the essential difference being that here the trajectory ends at the equilibrium point, not on a limit cycle about that point. Thus the equilibrium point is a stable focus and the region traversed by the trajectory is one of asymptotic stability. Trajectories with n_1 values of 2.92 Mw ($\delta n = -97.08$ Mw) and above ($T_{1I} = -300,000$, $T_{2I} = -200,000$) exhibit finite escape time.

Limits of Stability Regions

A study of the nature of the bounded and stable trajectories suggests the existence of limiting planes in the n, T_1, T_2 space. Kendall (1971) derives mathematical expressions for these planes which are shown in Fig. 3.5 for the stable limit cycle case ($n_0 = 265$ Mw). Plane A is a separatrix plane between bounded trajectories and those exhibiting finite escape time. Those below are bounded while those above diverge. Plane B is the maximum reactivity plane and Plane C is the $n = 0$ plane. Note the intersection of the A and C planes which is shown as the dotted line behind the n, T_2 plane. The triangular region interior to the three planes A, B, and C, extending to infinity, is the bounded region. The equation of plane A is

$$n = \frac{-\alpha_2 \eta_2}{\alpha_1 b} T_1 + \frac{\alpha_2 (\eta_3 - \eta_1)}{\alpha_1 b} T_2 - \frac{\alpha_2 \eta_2}{\alpha_1 \eta_3} n_0 + \frac{\eta_1 \ell'}{\alpha_1 b} \left[\left(\alpha_1 + \alpha_2 \frac{\eta_2}{\eta_3} \right) \frac{bn_0}{\eta_1 \ell'} + \eta_3 + \frac{\alpha_2}{\alpha_1} \eta_2 \right], \quad (3-5)$$

and that of plane B is

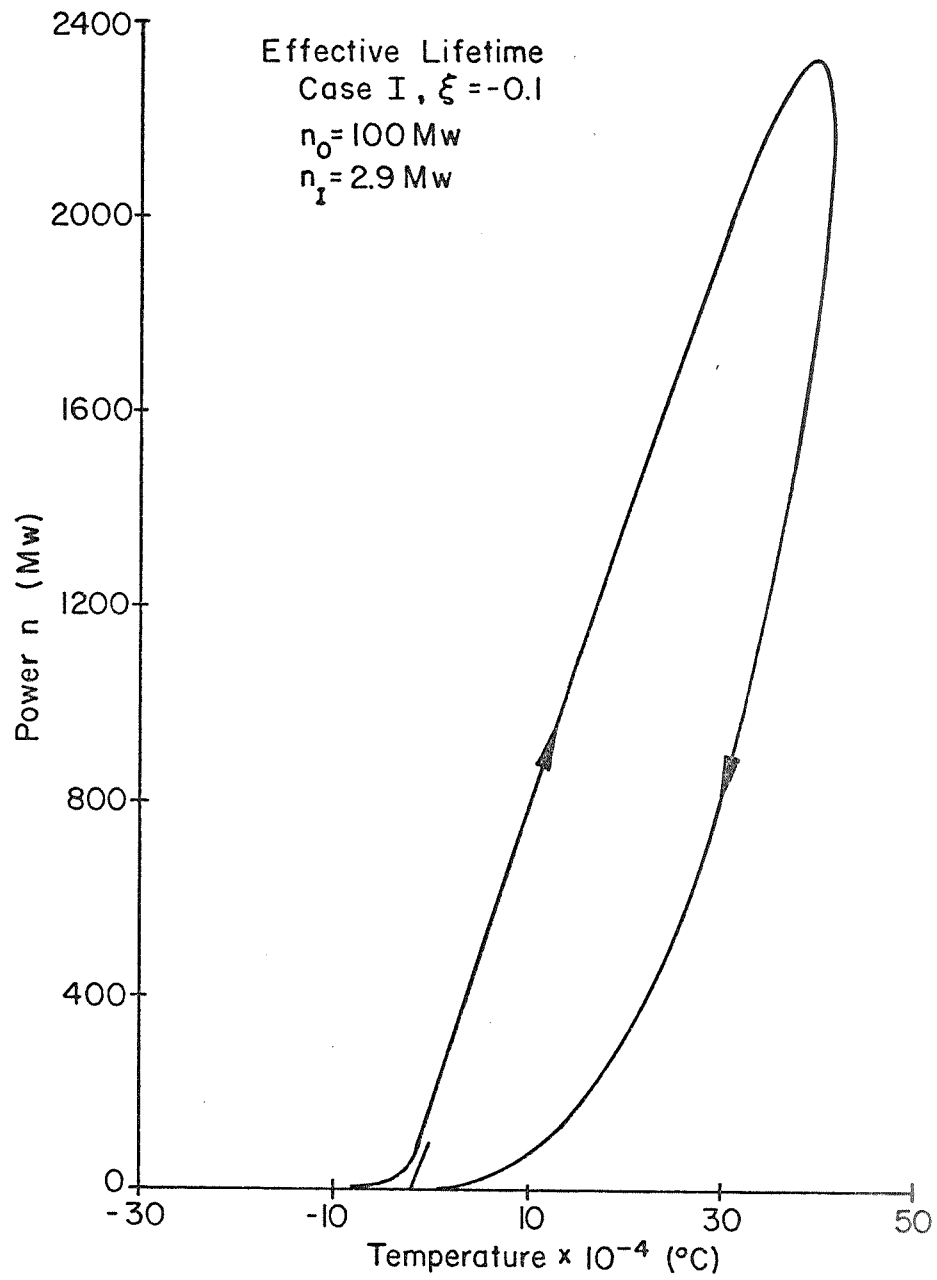


Figure 3.4 Effective Lifetime Stable Focus.

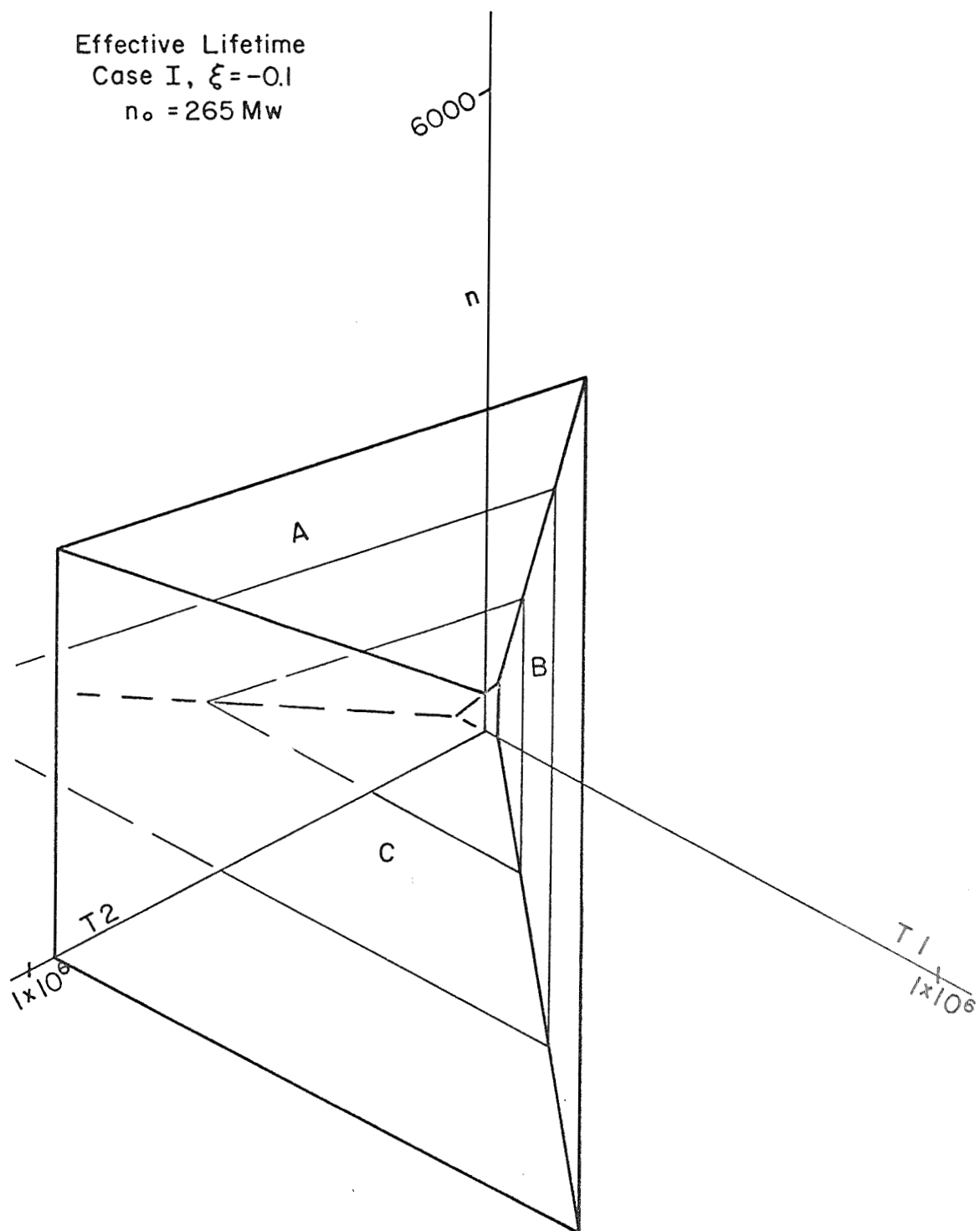


Figure 3.5 Effective Lifetime Limiting Planes.

$$T_1 = -\frac{\alpha_2}{\alpha_1} T_2 + \frac{\ell'}{\alpha_1} \left[(\alpha_1 + \alpha_2 \frac{\eta_2}{\eta_3}) \frac{bn_0}{\eta_1 \ell'} + \eta_3 + \frac{\alpha_2}{\alpha_1} \eta_2 \right] - \frac{bn_0}{\eta_1} \left[1 + \frac{\alpha_2}{\alpha_1} \frac{\eta_2}{\eta_3} \right] . \quad (3-6)$$

From Eq. (3-6) the maximum reactivity can be calculated to be 0.01.

The numerical computations give 0.0104. Computer solutions show that the line formed by the intersection of planes A and B is a line of attraction for both bounded and diverging trajectories. Bounded trajectories follow this line until just before they peak, as previously described, and those that originate above plane A in the vicinity of this line follow it before diverging with finite escape time.

Kendall (1971) gives a mathematical proof that the triangular region formed by Eqs. (3-5) and (3-6) and the $n = 0$ plane is one having the property that all trajectories cross the planar boundaries from the outside to the interior. However, boundedness is not proved because the region opens to infinity. His attempts to prove boundedness using the Liapunov method and geometric proofs were both unsuccessful, but he could not find by numerical computation any trajectory that originated within the region, or crossed the boundary planes, that became unbounded. He concludes that the region contains only bounded solutions.

One of the conditions of the proof leads to an interesting result. It is

$$n_0 \frac{(\alpha_1' K_1 + \alpha_2' K_2)}{\ell'} \geq -\gamma_1 \gamma_2 \quad (3-7)$$

where the following are for the uncoupled heat transfer equations,

$$\alpha_1' = \text{reactivity coefficient (Region 1)}$$

$\alpha_1' =$ reactivity coefficient (Region 2)

$K_1 =$ reciprocal heat capacity (Region 1)

$K_2 =$ reciprocal heat capacity (Region 2)

$\gamma_1 =$ reciprocal time constant (Region 1)

$\gamma_2 =$ reciprocal time constant (Region 2)

Using the proper transformations (Schmidt, 1969), we find in terms of our coupled heat transfer equations

$$\alpha_1' = \frac{1}{3} \times 10^{-6} \text{ per } ^\circ\text{C}$$

$$\alpha_2' = -\frac{2}{3} \times 10^{-6} \text{ per } ^\circ\text{C}$$

$$K_1 = K_2 = b = 100 \text{ } ^\circ\text{C/Mw-sec}$$

$$\gamma_1 = 0.25 \text{ sec}^{-1}$$

$$\gamma_2 = 0.4 \text{ sec}^{-1}$$

Substituting these values into Eq. (3-7) we find

$$n_0 \leq - \frac{(0.25) (0.4) (0.1)}{(100) \left(-\frac{1}{3}\right) (10^{-6})} = 300 \text{ Mw} .$$

This is precisely the limit shown in Fig. 3.1 for stable limit cycles.

For the stable focus case ($n_0 = 100 \text{ Mw}$) there are analogous limiting planes to those shown in Fig. 3.5. However, the bounded triangular region extending to infinity is one of asymptotic stability, as all trajectories in this region terminate at the equilibrium point. As before, this conclusion is based on numerical calculation and not on mathematical proof.

Finite Escape Trajectories

Let us suppose that for finite escape trajectories the power becomes proportional to the square of the reactivity at large values of reactivity, i.e.,

$$n = A\rho^2 \quad . \quad (3-8)$$

We can test this hypothesis by calculating A and comparing Eq. (3-8) with numerically calculated finite escape trajectories.

We use the following equation which is the second-order equivalent of Eqs. (3-1) through (3-4)

$$\begin{aligned} \ddot{\rho} + (\eta_1 + \eta_3) \dot{\rho} + \eta_1 \eta_3 \rho \\ = -\alpha_1 b \dot{n} - (\alpha_2 \eta_2 + \alpha_1 \eta_3) b (n - n_0) \quad . \end{aligned} \quad (3-9)$$

Substituting the proper derivatives of Eq. (3-8) into Eq. (3-9) we find

$$\begin{aligned} \frac{\rho^3}{2\ell^2} + (\eta_1 + \eta_3) \frac{\rho^2}{2\ell} + \eta_1 \eta_3 \rho \\ = -\alpha_1 b \frac{A\rho^3}{\ell} - (\alpha_2 \eta_2 + \alpha_1 \eta_3) b (A\rho^2 - n_0) \quad . \end{aligned} \quad (3-10)$$

Equating coefficients of ρ^3 (neglecting other ρ terms as small compared to ρ^3) we have

$$A = \frac{-1}{2\ell^2 \alpha_1 b} = 1.5 \times 10^5 \quad . \quad (3-11)$$

A plot of Eq. (3-8) using Eq. (3-11) is shown in Fig. 3.6 with typical numerical data points. Note that for large ρ , approximately 10 (1000\$), the data points converge to Eq. (3-8). This verifies the hypothesis. It is noted that this result is an asymptotic property of the mathematical model, having no real physical significance.

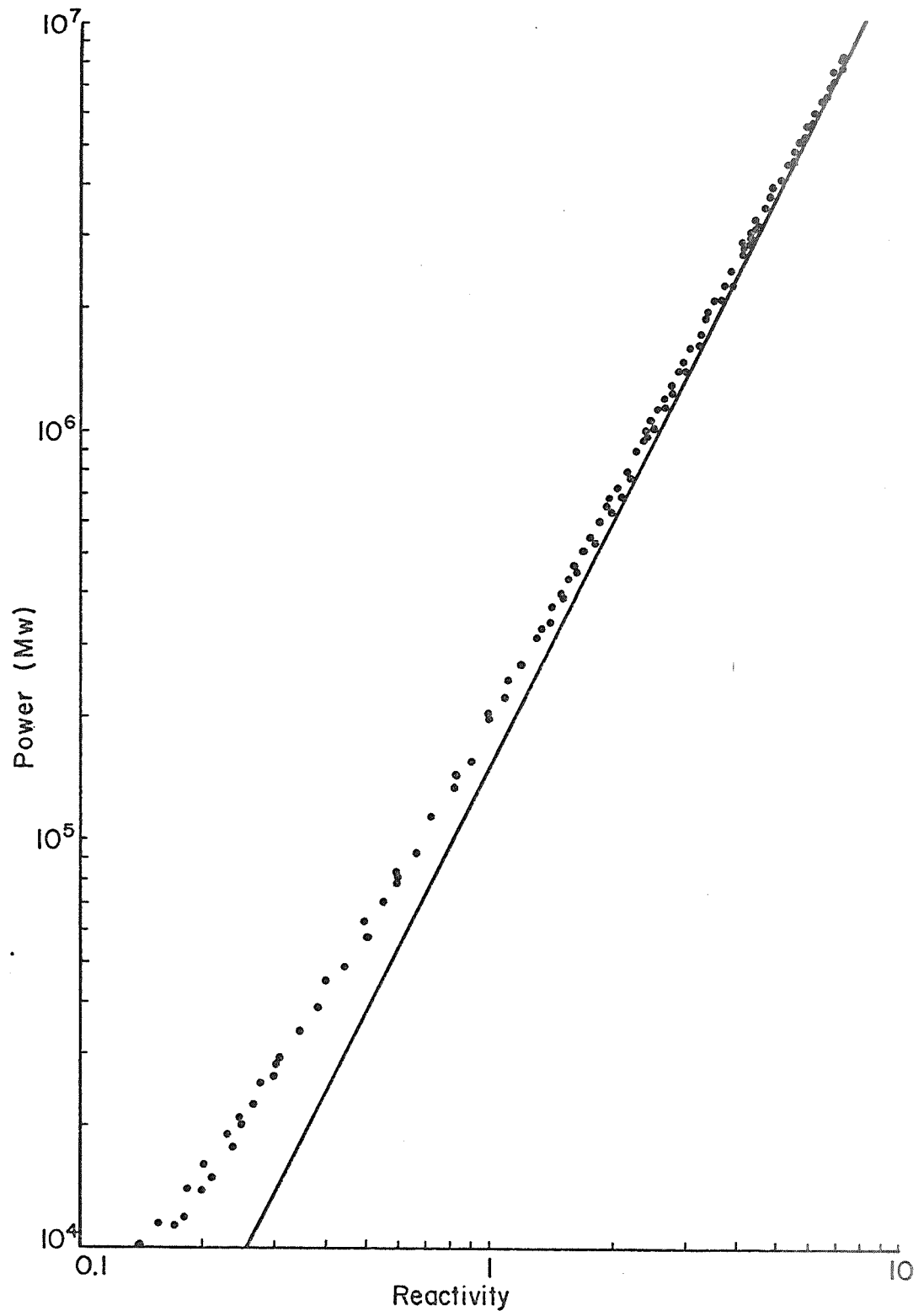


Figure 3.6 Effective Lifetime Finite Escape Trajectories.

The Search for Concentric Limit Cycles

It was suggested by H. B. Smets (private communication to T. R. Schmidt, 1970) that it is possible to have concentric limit cycles in the effective lifetime model at equilibrium powers slightly above critical. Although Schmidt (1969) observes only stable limit cycles with the effective lifetime model, there is no mathematical proof that precludes the existence of concentric limit cycles for this model. Consequently a search was made for this phenomenon using an equilibrium power of 265 Mw, which is 2.5 Mw above critical power. The stable limit cycle found has already been discussed and is shown in Figs. 3.2 and 3.3. The search for a larger surrounding limit cycle was unsuccessful. Even when the large initial value of 3×10^7 °C was used for both temperatures, the trajectory terminated on the single stable limit cycle. This tends to corroborate the conclusion of Kendall (1971) that the region of Fig. 3.5 is an infinite region of bounded solutions.

Summary

In this chapter we investigated the stability regions of the effective lifetime model and coupled heat transfer equations of the thermionic reactor. The observed boundaries of bounded and stable regions were correlated with a mathematical model. It was found that trajectories outside these regions exhibited finite escape time, and for large values of reactivity the power becomes proportional to the square of the reactivity. The search for concentric limit cycles in this model proved unsuccessful.

CHAPTER 4

STABILITY REGIONS WITH THE PROMPT JUMP MODEL

Introduction

In this chapter we explore some of the stable regions of the prompt jump model. This neutron kinetics model together with the coupled temperature equations of the thermionic reactor form the system of equations. The computation technique and computer program used here are the same as those of Chapter 3. The prompt jump model is then compared with a one-delay group model to determine if apparent finite escape trajectories of the prompt jump model are real or the result of the failure of the model near prompt critical.

The System Equations

The equations are

$$(\beta - \rho) \frac{dn}{dt} = (\lambda\rho + \frac{d\rho}{dt}) n \quad (4-1)$$

$$\frac{dT_1}{dt} = b(n - n_0) - \eta_1 T_1 \quad (4-2)$$

$$\frac{dT_2}{dt} = \eta_2 T_1 - \eta_3 T_2 \quad (4-3)$$

$$\rho = -\alpha_1 T_1 - \alpha_2 T_2 \quad (4-4)$$

Equation (4-1) is the familiar prompt jump model for one delay group.

The symbols have been defined previously. The other equations have been

discussed and are shown here for completeness. Equation (4-1) represents the simplest neutron kinetics model having some measure of realism (provided $\rho < \beta$), for it describes the effect of delayed neutrons on stability behavior. The derivation of Eq. (4-1) from the one-delay group equations represents a reduction of order, though there are pitfalls and ramifications that involve the theory of singular perturbations. These will not be treated here. For a derivation and a fuller discussion of Eq. (4-1) see Hetrick (in press). Equation (4-1) is somewhat more complicated than the effective lifetime model and has not enjoyed as wide a usage.

Stable Regions

The stable regions for three different cases are considered. We use the previously given Case I parameters with ξ values of -0.1, -0.2, and -0.6. The corresponding values of α_2 are 0.4×10^{-6} , 0.533×10^{-6} , and 1.067×10^{-6} (per °C), respectively. The maximum power perturbation to reach the threshold of unbounded solutions can be determined by solving Eqs. (4-1) through (4-4) for various initial powers. This was done for a series of equilibrium powers, using the digital computer program described in the appendix, to map stable regions of the prompt jump model.

The results for Case I, $\xi = -0.1$, are shown in Fig. 4.1. The stable region ends abruptly at the critical power of 190.9 Mw. The prompt jump model exhibits unstable limit cycles below but near critical power. The interior region of such a limit cycle is asymptotically stable. At critical power the unstable limit cycle (hence the stable

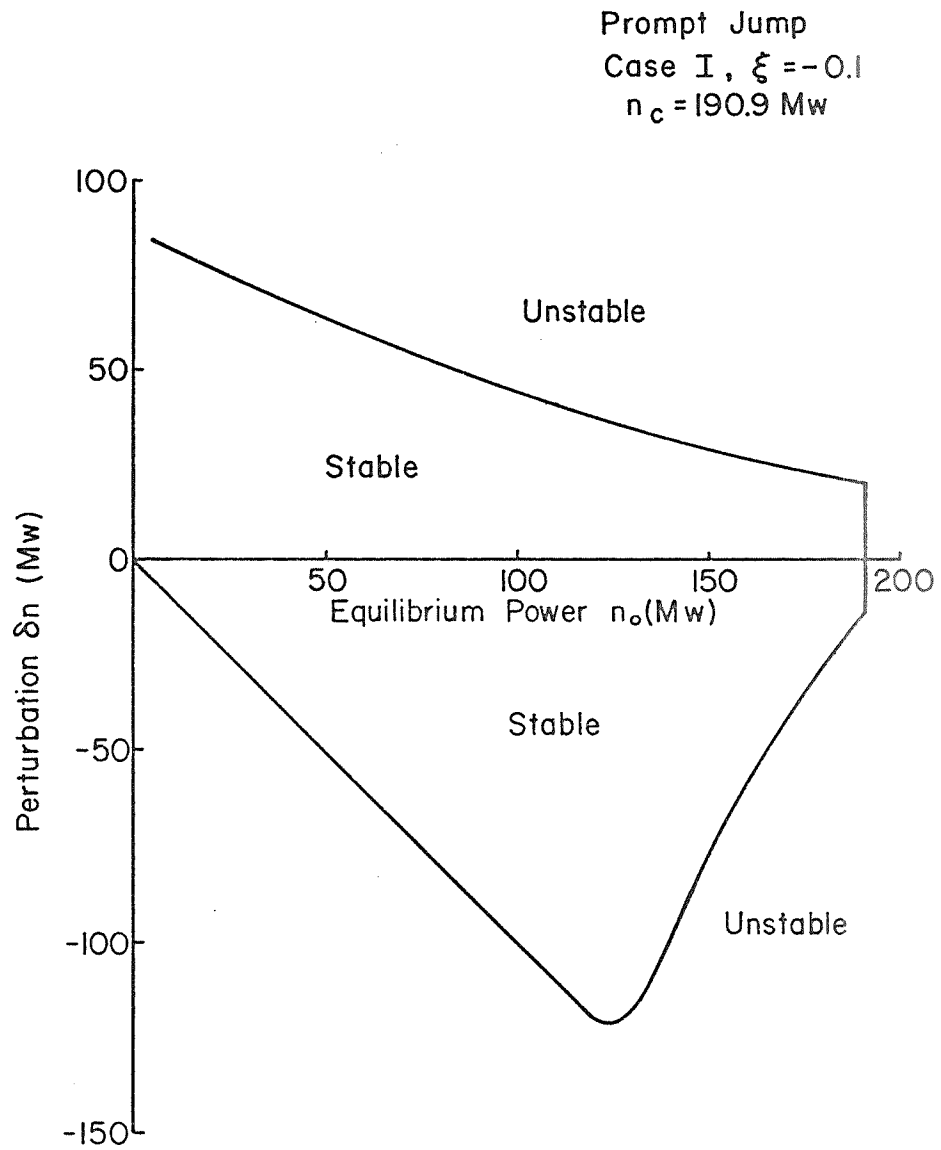


Figure 4.1 Prompt Jump Stability Regions, $\xi = -0.1$.

region) disappears. See Schmidt (1969) for limit cycle studies of the prompt jump model.

The results for Case I, $\xi = -0.2$ are shown in Fig. 4.2, and the stable region ends at the critical power of 186.9 Mw.

The results for Case I, $\xi = -0.6$ are shown in Fig. 4.3, and the stable region ends at the critical power of 182.9 Mw. In all three cases trajectories outside the stable region exhibit finite escape time.

Comparison of the three cases indicates that the stable region for positive power perturbations increases for decreasing ξ . Likewise the stable region for negative power perturbations tends to decrease with decreasing ξ . This is explained by the fact that decreasing ξ means increasing α_2 (negative feedback coefficient), hence a more stable system with increases of power. The reverse implication is true for decreases in power.

Curtis (in preparation) has investigated a similar system (one-delay group model, power perturbation only) using the Popov (1963) criterion. He obtains a comparatively smaller stable region because his method is based on sufficient conditions of stability.

Here, as with the effective lifetime model, one would have to study perturbations in both temperatures as well as the power to map out the stable domain in state space.

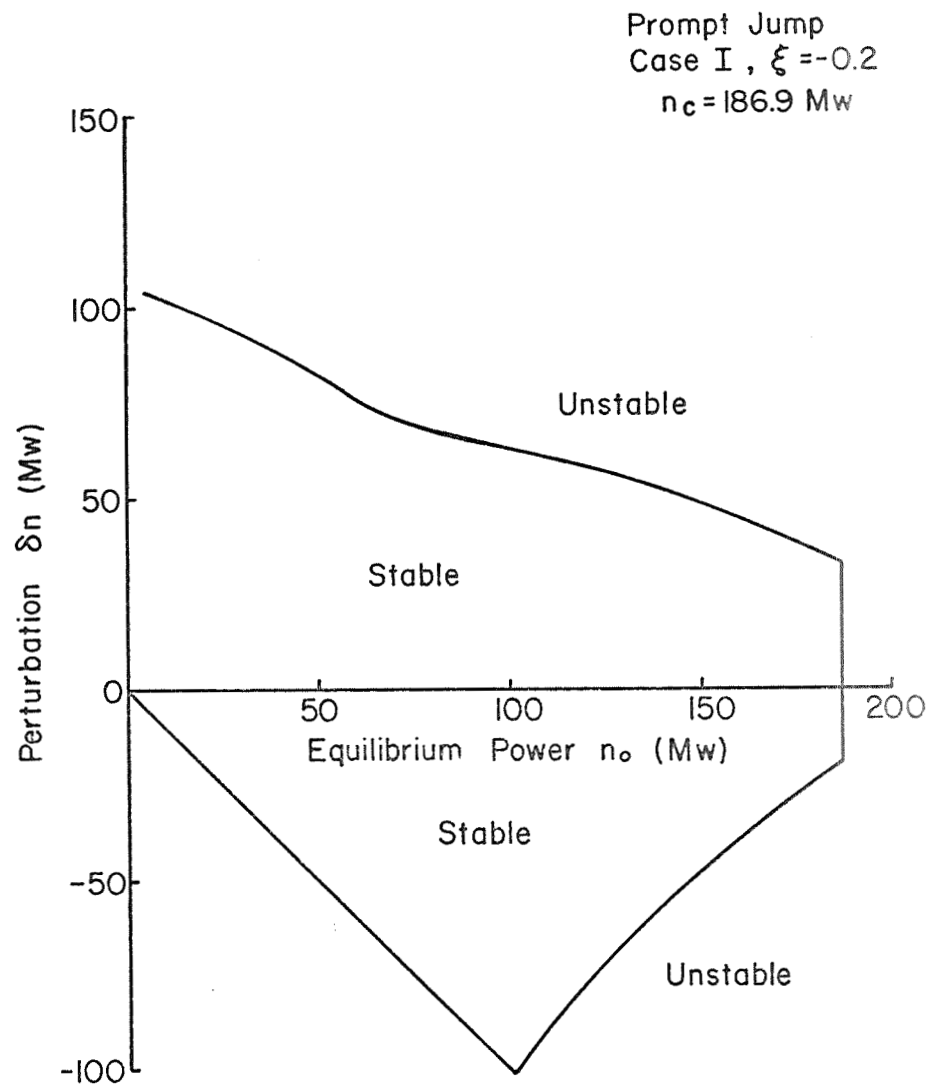


Figure 4.2 Prompt Jump Stability Regions, $\xi = -0.2$.

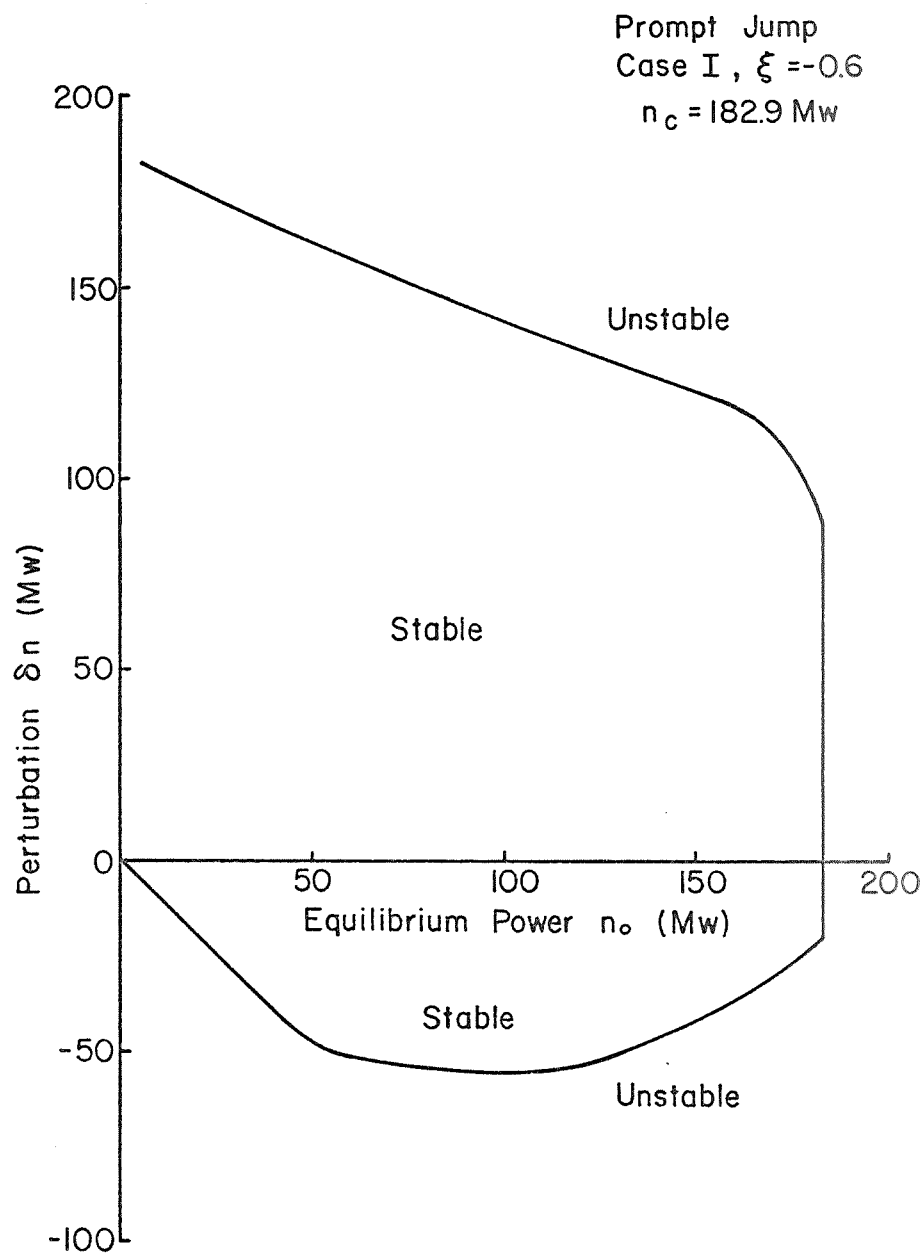


Figure 4.3 Prompt Jump Stability Regions, $\xi = -0.6$.

Comparison of the Prompt Jump Model
to the One-Delay Group Model

Equation (4-1) is no longer valid for $\rho = \beta$, i.e., the prompt critical condition. Hence the true nature of a prompt jump trajectory is not known when this condition is attained. One may ascertain the stability behavior by comparing the prompt jump results with those calculated using a one-delay group model.

The one delay group neutron kinetic equations are

$$\frac{dn}{dt} = \frac{\rho - \beta}{\ell} n + \lambda c \quad (4-5)$$

$$\frac{dc}{dt} = \frac{\beta}{\ell} n - \lambda c \quad (4-6)$$

The equations in conjunction with Eqs. (4-2) through (4-4) comprise the system equations. All symbols are as previously defined, and c is the delayed neutron precursor concentration. The initial value of c is found by equating Eqs. (4-1) and (4-5) (when these two models are compared). The neutron lifetime (generation time) is denoted by ℓ .

For this comparison we select the Case I parameters with $\xi = -0.2$. Since we wish to know whether a given trajectory is unbounded or bounded, the particular value of neutron lifetime used in the calculation is unimportant. We arbitrarily select 1×10^{-3} sec, in preference to a smaller value, to avoid possible computational difficulties in the numerical integration of the differential equations. The system of equations was programmed for solution on the CDC 6400 digital computer of the University of Arizona Computer Center, using the code developed by Secker (1969). The results of the computations

are shown in Fig. 4.4, which is taken from Fig. 5.30 of Schmidt (1969). The solid lines represent prompt jump trajectories while the two dotted ones are those calculated here (one delay group model). The reactivities along the dotted lines correspond to prompt critical and super prompt critical conditions, so the corresponding portions of the prompt jump curves are no longer valid. That is to say, the portion of curve A corresponding to A' is not valid, and the same is true for curves B and B'. The two curves A' and B' both attain infinite value in finite time. They are unbounded and exhibit finite escape time.

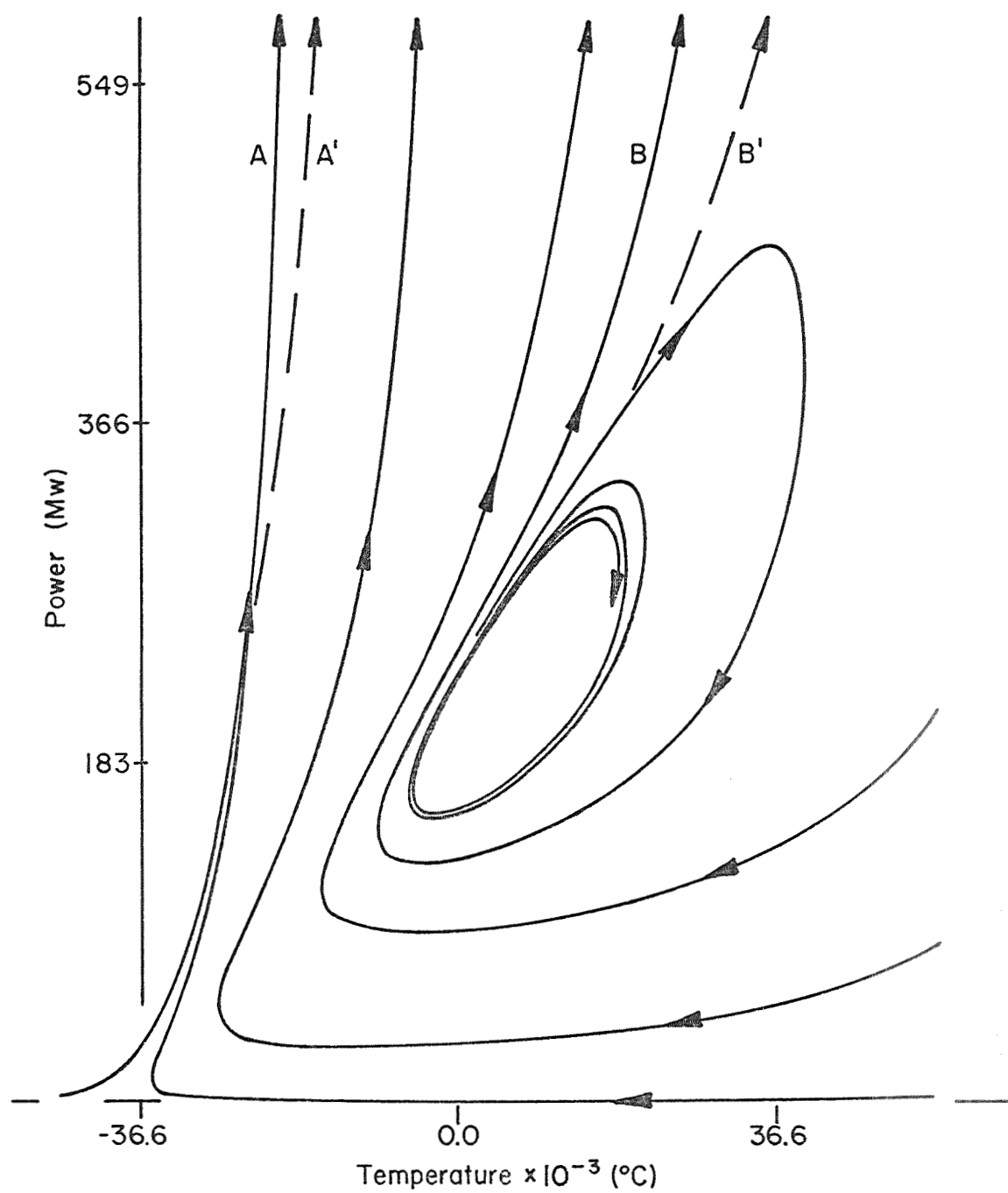


Figure 4.4 Comparison of the Prompt Jump and One Group Models.

CHAPTER 5

THE PERTURBATION EXPANSION METHOD

Introduction

In this study we consider only the point-reactor model. Strictly speaking, this model is reliably applicable only when the power level is reasonably close to the equilibrium value. For large power excursions, spatial effects may become significant, or the temperature dependence of feedback reactivity and heat-transfer coefficients may change in a radical way. It is therefore useful to have some means of estimating the bounds of applicability for point kinetic models. From an idea that originated with Malkin (1958), Shotkin (1969) developed a perturbation expansion method, based on the linearized system, that can be used in regions where the point model may not be globally asymptotically stable or may not even be bounded; i.e., it may have solutions with finite escape time. When the model is linearly asymptotically stable, the method can be used to calculate the largest disturbance in power, or other physical variables, for which the system still remains asymptotically stable for a given set of operating conditions.

Devooght and Smets (1967) summarized and compared the available literature on the local stability of equilibrium. They concluded that techniques which deal with linearized approximations to a system are easily applied but give much smaller bounds on allowable disturbances

than do techniques (topological or Liapunov) that explicitly include the nonlinear terms. However, these nonlinear techniques are difficult to apply generally and require varying amounts of ingenuity, depending on the particular problem. On the other hand, the perturbation expansion method is straightforward and generally applicable to a wide class of problems and in some cases gives results that compare favorably with the Liapunov method (Shotkin 1969). Furthermore, an analysis of nonlinear behavior, based on the linearized system, highlights two main practical questions of point reactor kinetics: (1) What are the reactor physical parameters and equilibrium values for which the reactor is linearly stable? and (2) What are bounds on allowable departures from these equilibrium values for which the linear analysis is still valid? The method is applicable to systems with nonlinear feedback functionals and can also be used to predict the amplitude and frequency of stable-limit cycles, when they exist. The latter property will be demonstrated below.

Outline of Method

The equations and techniques of the perturbation expansion method are derived in detail by Shotkin (1969). A brief outline will be given here to summarize those equations used in the present study. The essence of the method is that with any linearly stable system, there is associated a dominant canonical coordinate corresponding to the least negative eigenvalue (or a complex pair of canonical coordinates corresponding to a dominant pair of complex eigenvalues). These dominant eigenvalues belong to the term or pair of terms in the linearized

solution (an exponential decay or a damped oscillation) that is the last to disappear as time approaches infinity. In this sense, these dominant terms represent the asymptotic solutions of linearized problems.

Consider the general matrix system

$$\frac{d\Psi}{dt} = L\Psi + N(\Psi) \quad (5-1)$$

where $\Psi(t)$ is a single column n -row ($n \times 1$) matrix and L is an $n \times n$ square matrix with real constant elements. The elements of Ψ , $(\psi_1, \psi_2, \dots, \psi_n)$, represent departures from equilibrium. Equilibrium is defined as $\Psi = 0$, i.e., the perturbations in all physical quantities are equal to zero. $N(\Psi)$ is an $n \times 1$ matrix consisting of polynomials, functions of the elements of Ψ , beginning with terms at least of the second order with $N(0) = 0$.

For the case where Ψ has a single element, ψ_1 , $L = b$, and

$$N(\Psi) = \sum_{m=2} B_m \psi_1^m,$$

the system of equations, Eq. (5-1), is reduced to the single equation

$$\frac{d\psi_1}{dt} = b\psi_1 + B_2\psi_1^2 + B_3\psi_1^3 + \dots \quad (5-2)$$

It will be shown in the following how to transform the matrix Eq. (5-1) into a scalar equation of the form Eq. (5-2).

The stability of the equilibrium solution, $\psi_1 = 0$, in the linearized approximation ($\psi_1^2 \ll |\psi_1|$) depends upon the sign of b . If

$b > (<) 0$, the system is linearly unstable (stable). The sign of b depends on the physics of the linearized reactor system since it is a function of core size, power level, feedback coefficients, etc. Non-linear stability is determined by the coefficients of the higher order terms, the B_m in Eq. (5-2). These coefficients introduce the nonlinear effects in a straightforward way. Suppose that the higher order terms in Eq. (5-2) are given at least up to the third order, i.e., ψ_1^3 . Let

$$\psi_1 = \phi A, \phi = \pm 1, \quad (5-3)$$

where A is the magnitude of the disturbance. Then Eq. (5-2) becomes, after dividing through by ϕ ($\phi^2 = 1$), and defining $\bar{B}_2 = B_2 \phi$,

$$\frac{dA}{dt} = bA + \bar{B}_2 A^2 + B_3 A^3 + \dots \quad (5-4)$$

The stability behavior of Eq. (5-2) is identical with that of Eq. (5-4). For small A , the sign of $\dot{A}$ (where the dot signifies time derivative) is determined by the sign of b . As A increases, the real, positive values of A that will cause the sign of $\dot{A}$ to change are sought. The values may be found by setting $\dot{A}$ equal to zero and solving for the roots of the algebraic polynomial on the right-hand side. Thus for terms only up to and including A^3 , there are the three roots

$$\begin{aligned} A_0 &= 0 \\ A_1 &= -\frac{\bar{B}_2}{2B_3} + \frac{(\bar{B}_2^2 - 4bB_3)^{1/2}}{2B_3} \\ A_2 &= -\frac{\bar{B}_2}{2B_3} - \frac{(\bar{B}_2^2 - 4bB_3)^{1/2}}{2B_3} \end{aligned} \quad (5-5)$$

The first root corresponds to the equilibrium state (zero perturbation). The other two roots must satisfy the requirement that they be real and positive. If there is no real, positive solution of Eqs. (5-5), then higher order terms must be included to determine the bound A_i ($i=1$ or 2). When the system is globally asymptotically stable, an infinite number of terms in Eq. (5-4) would be required to predict the infinite bound. However, this perturbation approach of neglecting higher order nonlinear terms is better suited for determining realistic bounds in specifically those parameter regions where the system is not globally asymptotically stable. This feature will be amplified by two of the cases studied here.

Next, it will be shown that it is possible to transform the matrix system of nonlinear autonomous ordinary differential equations, Eq. (5-1), into a single scalar differential equation of the form of Eq. (5-4). The question of the stability of the solutions of Eq. (5-1) within a given disturbance region about equilibrium, can then be resolved in terms of the stability of Eq. (5-4). The constant b is the real part of the dominant linear root, and Eqs. (5-5) give the bound on the region about equilibrium for which the behavior predicted by the linear theory still holds. If $b < 0$, then for $A < A_i$ the system is asymptotically stable, and for $A > A_i$ the system diverges in time from its equilibrium value. The opposite is true for $b > 0$, i.e., for $A < A_i$ the amplitude grows in time, while for $A > A_i$ the system approaches A_i . These two cases are illustrated in Fig. 5.1 where there is a single $A_i > 0$. The arrows indicate the direction of

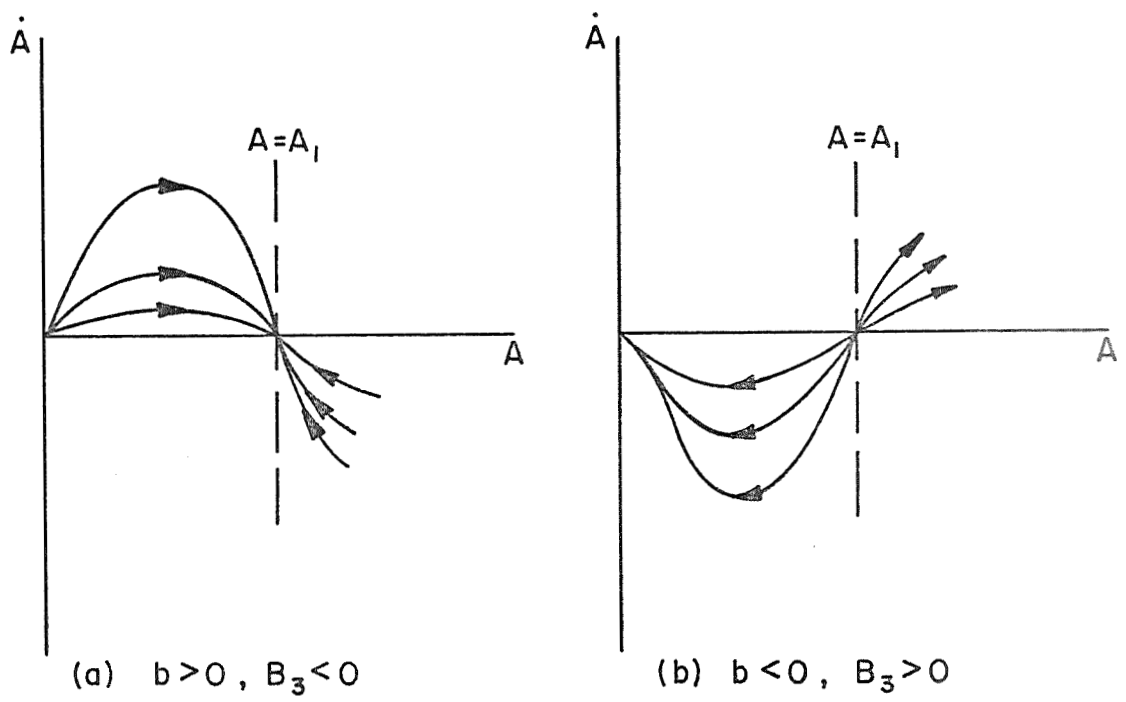


Figure 5.1 Trajectories for Two Types of Stability Behavior.

the solution point on the trajectory. It is to be noted that the stability behavior is independent of $\bar{B}_2$.

In carrying through the analysis, Shotkin (1969) avoids using matrix notation in order to emphasize the physical significance of the system parameters. His notation and procedure are followed closely here. The first step is to diagonalize the matrix L in Eq. (5-1) and to find the canonical coordinates associated with the characteristic roots of S_ℓ , which are the elements of the diagonalized matrix. The characteristic values of S_ℓ , with $\ell = 1, 2, \dots, n$, associated with the $n \times n$ matrix L are determined as the solutions of the characteristic determinant

$$|L - IS_\ell| = 0, \quad (5-6)$$

where I is the identity matrix. The characteristic vector, ρ_ℓ (an $n \times 1$ matrix), associated with the ℓ^{th} characteristic value is defined by

$$L\rho_\ell = IS_\ell\rho_\ell. \quad (5-7)$$

The matrix $\Psi(t)$ can be expressed as a linear combination of the ρ_ℓ with time-dependent coefficients, $y_\ell(t)$:

$$\Psi(t) = \sum_{\ell=1}^n y_\ell(t) \rho_\ell. \quad (5-8)$$

The $y_\ell(t)$ are scalar quantities and the ρ_ℓ are vectors. If Eq. (5-8) is substituted into Eq. (5-1), with $N=0$, and Eq. (5-7) is used, the following is obtained

$$\sum_{\ell=1}^n \dot{y}_\ell(t) \rho_\ell = \sum_{\ell=1}^n y_\ell(t) IS_\ell \rho_\ell. \quad (5-9)$$

Because the ρ_ℓ are linearly independent, Eq. (5-9) can be satisfied only when

$$\dot{y}_\ell(t) = S_\ell y_\ell(t) , \quad \ell = 1, 2, \dots, n . \quad (5-10)$$

The y_ℓ are called the canonical coordinates associated with the linear characteristic values, S_ℓ . It should be noted that Eqs. (5-10) are a set of n scalar equations.

An explicit method is now given for calculating the y_ℓ in terms of the elements of Ψ , the ψ_i . Given the linear system

$$\frac{d\Psi(t)}{dt} = L\Psi(t) , \quad (5-11)$$

a transformation row matrix, K_ℓ (associated with the ℓ^{th} characteristic root, S_ℓ),

$$K_\ell(S_\ell) = (K_{\ell 1}, K_{\ell 2}, \dots, K_{\ell n}) , \quad (5-12)$$

is sought to transform the physical variables, ψ_i , to a set of canonical coordinates, y_ℓ ,

$$y_\ell = K_\ell \Psi = \sum_{i=1}^n K_{\ell i} \psi_i , \quad (5-13)$$

such that

$$\dot{y}_\ell(t) = S_\ell y_\ell(t) , \quad \ell = 1, 2, \dots, n . \quad (5-14)$$

Since the linear transformation, Eq. (5-13), leads to a set of homogeneous equations, Eqs. (5-14), each y_ℓ contains an arbitrary constant multiplier. Thus $K_{\ell 1} = 1$ may be chosen without affecting the transformation. This will fix the initial values of the y_ℓ with respect to

the initial values of the ψ_i . Also, if $\psi_1 = n$, or $\psi_1 = r = \ln(n/n_0)$ is chosen, then the transformation is fixed to the most important variable, the power. If Eq. (5-11) and Eq. (5-13) are substituted into Eq. (5-14) the following is obtained

$$K_{\ell} \dot{\Psi} = K_{\ell} L \Psi = S_{\ell} K_{\ell} \Psi, \quad (5-15)$$

which means that the elements of $K_{\ell j}$ of the transformation must satisfy

$$\sum_{j=1}^n K_{\ell j} L_{ji} \psi_i = S_{\ell} K_{\ell i} \psi_i, \quad (5-16)$$

where L_{ji} are the elements of the matrix L . Because the ψ_i are linearly independent, Eq. (5-16) must hold separately for each ψ_i or

$$S_{\ell} K_{\ell i} = \sum_{j=1}^n K_{\ell j} L_{ji}. \quad (5-17)$$

Equation (5-17) determines the elements, $K_{\ell i}$, of the sought transformation,

$$y = K \Psi, \quad (5-18)$$

where y and Ψ are $n \times 1$ column matrices, and K is an $n \times n$ square matrix. With the assumptions that K is nonsingular and that the inverse transformation

$$\Psi = K^{-1} y \quad (5-19)$$

can always be found, the ψ_i can always be expressed in terms of the canonical coordinates, y_{ℓ} . Equation (5-19) can then be used to eliminate the ψ_i in Eq. (5-1), so that Eq. (5-1) may be re-expressed

in terms of the canonical system

$$\dot{y}_\ell = S_\ell y_\ell + Y_\ell(y_1, y_2, \dots, y_n), \quad \ell = 1, 2, \dots, n \quad (5-20)$$

where the Y_ℓ are polynomial functions of the y_ℓ beginning with terms of the second order. Equations (5-20) will be the subject of this analysis. If the roots, S_ℓ , are ordered as

$$\operatorname{Re} S_1 > \operatorname{Re} S_2 > \dots > \operatorname{Re} S_n, \quad (5-21)$$

then it is clear that the coordinate y_1 (or coordinates y_1 and y_2 if $\operatorname{Re} S_1 = \operatorname{Re} S_2$) will dominate the linear behavior for sufficiently long times after an initial disturbance. It is assumed that this coordinate will also be a key factor in the behavior for long times after large disturbances. This seems consistent with the perturbation approach, which is restricted to reasonable disturbances.

The next step is to eliminate the nondominant coordinates. Since the y_ℓ are linearly independent, any one of them cannot be expressed in terms of the others. However, they are related in a non-linear manner through Eqs. (5-20). Each of the nondominant coordinates can be expressed in terms of the dominant coordinates through a non-linear transformation beginning with terms of at least the second order. Such a transformation (Malkin 1958) for the case where the dominant roots are complex is now exhibited. If the dominant root is real, a similar result can be obtained by just setting one of the complex coordinates (say, y_2 below) equal to zero. Thus, for $S_1 = b + i\omega$ and $S_2 = b - i\omega$

$$\begin{aligned}
\dot{y}_1 &= S_1 y_1 + Y_1(y_1, y_2, \dots, y_n) \\
\dot{y}_2 &= S_2 y_2 + Y_2(y_1, y_2, \dots, y_n) \\
\dot{y}_j &= S_j y_j + Y_j(y_1, y_2, \dots, y_n) , \\
j &= 3, 4, \dots, n .
\end{aligned} \tag{5-22}$$

We seek a transformation

$$y_j = v_j(y_1, y_2) = v_j^{(1)}(y_1, y_2) + v_j^{(2)}(y_1, y_2) + \dots \tag{5-23}$$

with

$$\begin{aligned}
v_j^{(1)} &= C_{j0} y_1 + C_{j1} y_2 \\
v_j^{(2)} &= C_{j2} y_1^2 + C_{j3} y_1 y_2 + C_{j4} y_2^2 ,
\end{aligned}$$

where $v_j^{(n)}$ contains all terms of the n^{th} order in $y_1^a y_2^b$, $a+b = n$.

If Eq. (5-23) is differentiated and Eq. (5-22) is used, an equation is obtained that determines the transformation coefficients, C_{jk} , after the y_j are replaced by v_j :

$$S_j v_j + Y_j(y_1, y_2, v_3, \dots, v_n) = \frac{\partial v_j}{\partial y_1} \dot{y}_1 + \frac{\partial v_j}{\partial y_2} \dot{y}_2 . \tag{5-24}$$

Equation (5-24) can be solved for the C_{jk} by setting equal all those terms of the n^{th} order ($y_1^a y_2^b$, $a+b = n$). When Eq. (5-24) is satisfied to a given order, e.g., the n^{th} order approximation with $a+b = n$, it can be shown (Malkin 1958) that nondominant coordinates can be chosen which will have zero time derivatives up to the $n-1$ order. For the first order,

$$S_j(C_{j0}y_1 + C_{j1}y_2) = S_1y_1C_{j0} + S_2y_2C_{j1} , \quad (5-25)$$

which can be satisfied only for

$$C_{j0} = C_{j1} = 0 . \quad (5-26)$$

This verifies that the transformation, Eq. (5-23), must start with terms of the second order.

If the dominant root is real, the stability problem is determined by the single equation governing the canonical coordinate associated with that root:

$$\dot{y}_1 = S_1y_1 + Y_1(y_1) . \quad (5-27)$$

It is assumed that the nondominant roots have been eliminated by the transformation just discussed. If the dominant roots are complex, there are two canonical equations (once the nondominant roots have been eliminated):

$$\begin{aligned} \dot{y}_1 &= S_1y_1 + Y_1(y_1, y_2) \\ \dot{y}_2 &= S_2y_2 + Y_2(y_1, y_2) , \end{aligned} \quad (5-28)$$

with $y_1 = y_2^*$ and $S_1 = S_2^*$ (where * denotes complex conjugate). It follows that $Y_1 = Y_2^*$. A solution to Eqs. (5-28) is sought in terms of an amplitude, A , and a phase, θ :

$$y_1(A, \theta) = Ae^{i\theta} + A^2U_2(\theta) + A^3U_3(\theta) + \dots \quad (5-29a)$$

$$y_1(A, 0) = A , \quad U_n(0) = 0 , \quad (5-29b)$$

where the U_n are periodic functions of θ . If Eq. (5-29) is substituted into Eq. (5-28), then Y_1 may be expanded in a series of A^n :

$$Y_1(y_1, y_2) = Y_1(A, \theta) = A^2 Y_{12}(\theta) + A^3 Y_{13}(\theta) + \dots, \quad (5-30)$$

where the $Y_{1n}(\theta)$ depend on the coefficients of y_1 computed in the previous order $(n-1)$ approximation. The following is obtained from Eq. (5-28)

$$\dot{y}_1(A, \theta) = \frac{\partial y_1}{\partial A} \dot{A} + \frac{\partial y_1}{\partial \theta} \dot{\theta} = S_1 y_1 + A^2 Y_{12} + A^3 Y_{13} + \dots \quad (5-31)$$

If it is assumed that

$$\dot{A} = bA + B_2 A^2 + B_3 A^3 + \dots \quad (5-32a)$$

$$\dot{\theta} = \omega + \Omega_2 A + \Omega_3 A^2 + \dots, \quad (5-32b)$$

then stability is determined by the amplitude equation, Eq. (5-32a), which has the identical form of Eq. (5-4). By equating terms with equal powers of A in Eq. (5-31), the B_n and Ω_n in Eqs. (5-32) can be solved for. These are

$$\begin{aligned} B_2 + i\Omega_2 &= -b\alpha_1 \\ B_3 + i\Omega_3 &= \alpha_3 - \alpha_1(2B_2 + i\Omega_2) - 2b\alpha_2 \\ B_4 + i\Omega_4 &= \alpha_5 - \alpha_1(2B_3 + i\Omega_3) - \alpha_2(3B_2 + i\Omega_2) - 3b\alpha_4 \\ B_5 + i\Omega_5 &= \alpha_7 - \alpha_1(2B_4 + i\Omega_4) - \alpha_2(3B_3 + i\Omega_3) \\ &\quad - \alpha_4(4B_2 + i\Omega_2) - 4b\alpha_6 \end{aligned} \quad (5-33)$$

with

$$\alpha_1 = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} U_2(\theta) d\theta ; \quad \alpha_2 = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} U_3(\theta) d\theta ;$$

$$\alpha_3 = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} Y_{13}(\theta) d\theta ; \quad \alpha_4 = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} U_4(\theta) d\theta$$

$$\alpha_5 = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} Y_{14}(\theta) d\theta \quad \alpha_6 = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} U_5(\theta) d\theta$$

$$\alpha_7 = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} Y_{15}(\theta) d\theta .$$

In the following section one of the examples given by Shotkin (1969) is worked out in detail.

A Second Order Reactor Example

In this example the effective lifetime model is used with a prompt-power coefficient and a single-temperature feedback. The physical equations are

$$\begin{aligned} \dot{r}(t) &= -A_1 T(t) - \gamma [e^{r(t)} - 1] \\ \dot{T}(t) &= \epsilon [e^{r(t)} - 1 - T(t)] \end{aligned} \quad (5-34)$$

with $r \equiv \ln [n(t)/n_0]$. The system parameters result in

$S_1 = -4.25 + i6.5$, $A_1 = 7.5$, $\epsilon = 10$, and $\gamma = -1.5$.

The matrix L is

$$\begin{bmatrix} -\gamma & -A_1 \\ \epsilon & -\epsilon \end{bmatrix}$$

and the canonical matrix, K, is

$$\begin{bmatrix} 1 & \frac{-A_1}{S_1 + \epsilon} \\ 1 & \frac{-A_1}{S_2 + \epsilon} \end{bmatrix}$$

so that the canonical coordinates are,

$$y_\ell = r - \frac{A_1}{S_\ell + \epsilon} T, \quad \ell = 1, 2 \quad (5-35)$$

which satisfy the differential equations:

$$\dot{y}_\ell = S_\ell (y_\ell + e^r - 1 - r) . \quad (5-36)$$

The inverse matrix, K^{-1} , is

$$\begin{bmatrix} \frac{-(S_1 + \epsilon)}{(S_2 - S_1)} & \frac{(S_2 + \epsilon)}{(S_2 - S_1)} \\ \frac{-(S_2 + \epsilon)(S_1 + \epsilon)}{A_1(S_2 - S_1)} & \frac{(S_2 + \epsilon)(S_1 + \epsilon)}{A_1(S_2 - S_1)} \end{bmatrix}$$

so that

$$r = \frac{(\epsilon + S_1)y_1 - (\epsilon + S_2)y_2}{S_1 - S_2} \quad (5-37)$$

$$T = \frac{(\epsilon + S_1)(\epsilon + S_2)}{A_1(S_1 - S_2)} (y_1 - y_2) \quad (5-38)$$

where $S_1 = S_2^* = b + i\omega$. Since there are only two equations, there are no nondominant coordinates to eliminate. Equation (5-37) may be rewritten as

$$r = C_1 y_1 + C_2 y_2 ,$$

with

$$C_1 = C_2^* = \frac{1 - i[(b + \epsilon)/\omega]}{2} . \quad (5-39)$$

Using the power series expansion

$$e^r = 1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \dots$$

in Eq. (5-36) together with Eqs. (5-29a) and (5-37), we obtain

$$\begin{aligned}
\dot{y}_1(A, \theta) = & AS_1 e^{i\theta} \\
& + A^2 [S_1 U_2(\theta) + \frac{S_1}{2} C_1^2 e^{i2\theta} + S_1 C_1 C_2 + \frac{S_1 C_2^2}{2} e^{-i2\theta}] \\
& + A^3 [S_1 U_3(\theta) + S_1 C_1^2 e^{i\theta} U_2(\theta) + S_1 C_1 C_2 e^{-i\theta} U_2(\theta) \\
& + S_1 C_1 C_2 e^{i\theta} U_2^*(\theta) + S_1 C_2^2 e^{-i\theta} U_2^*(\theta) + \frac{S_1 C_1^3}{6} e^{i3\theta} \\
& + \frac{S_1 C_1^2 C_2}{2} e^{i\theta} + \frac{S_1 C_1 C_2^2}{2} e^{-i\theta} + \frac{S_1 C_2^3}{6} e^{-i3\theta}] \\
& + A^4 [S_1 U_4(\theta) + S_1 C_1^2 e^{i\theta} U_3(\theta) + \frac{S_1 C_1^2}{2} U_2^2(\theta) + S_1 C_2^2 e^{-i\theta} U_3^*(\theta) \\
& + \frac{S_1 C_2^2}{2} (U_2^*(\theta))^2 + S_1 C_1 C_2 e^{-i\theta} U_3(\theta) + S_1 C_1 C_2 e^{i\theta} U_3^*(\theta) \\
& + S_1 C_1 C_2 U_2(\theta) U_2^*(\theta) + \frac{S_1 C_1^3}{2} e^{i2\theta} U_2(\theta) \\
& + \frac{S_1 C_2^3}{2} e^{-i2\theta} U_2^*(\theta) + S_1 C_1 C_2^2 U_2^*(\theta) + \frac{S_1 C_1 C_2^2}{2} e^{-i2\theta} U_2(\theta) \\
& + S_1 C_1^2 C_2 U_2(\theta) + \frac{S_1 C_1^2 C_2}{2} e^{i2\theta} U_2^*(\theta) + \frac{S_1 C_1^4}{24} e^{i4\theta} \\
& + \frac{S_1 C_2^4}{24} e^{-i4\theta} + \frac{S_1 C_1^3 C_2}{6} e^{i2\theta} + \frac{S_1 C_1 C_2^3}{6} e^{-i2\theta} \\
& + \frac{S_1 C_1^2 C_2^2}{4}] .
\end{aligned} \tag{5-40}$$

Making use of Eqs. (5-29) through (5-32) we obtain another relationship

$$\begin{aligned}
\dot{y}(A, \theta) = & A[b + i\omega]e^{i\theta} \\
& + A^2[(B_2 + i\Omega_2)e^{i\theta} + 2bU_2(\theta) + \omega \frac{dU_2}{d\theta}] \\
& + A^3[(B_3 + i\Omega_3)e^{i\theta} + 2B_2U_2(\theta) + \Omega_2 \frac{dU_2}{d\theta} \\
& + 3bU_3(\theta) + \omega \frac{dU_3}{d\theta}] \\
& + A^4[(B_4 + i\Omega_4)e^{i\theta} + 2B_3U_2(\theta) + \Omega_3 \frac{dU_2}{d\theta} \\
& + 3B_2U_3(\theta) + \Omega_2 \frac{dU_3}{d\theta} + 4bU_4(\theta) + \omega \frac{dU_4}{d\theta}] . \quad (5-41)
\end{aligned}$$

Equating coefficients of A in Eqs. (5-40) and (5-41) simply gives

$S_1 = b + i\omega$. For higher powers, differential equations for $U_n(\theta)$ result, which may be solved for the $U_n(\theta)$. The quantities B_n and Ω_n can be obtained by using the auxiliary condition, Eq. (5-29b).

The $O(A^2)$ approximation is defined as that in which $U_3(\theta) = 0$. In this case $(B_3 + i\Omega_3)$ is calculated from Eq. (5-33) with $\alpha_2 = 0$. It can be shown that

$$\begin{aligned}
U_2(\theta) = & (0.0976, 0.6973)e^{i\theta} + (0.0272, -0.2212)e^{i2\theta} \\
& + (0.0539, -0.0679)e^{-i2\theta} - (0.1787, 0.4082), \quad (5-42)
\end{aligned}$$

where the ordered-pairs of numbers indicate complex constants. The following quantities can also be calculated,

$$B_2 + i\Omega_2 = (0.4148, 2.9635)$$

$$B_3 + i\Omega_3 = (2.2790, -2.2890) .$$

Using these values of B_2 and B_3 together with $b = -4.25$ in Eq. (5-32a) set equal to zero, we find that $A = 1.277$. Note that with b negative and B_3 positive this case is of the type (b) shown in Fig. 5.1. Now using Eq. (5-29a) we find,

$$\begin{aligned} y_1(A, \theta) = & 1.436 \cos \theta - 1.137 \sin \theta + 0.1323 \cos 2\theta \\ & + 0.25 \sin 2\theta - 0.2915 + i(1.137 \cos \theta + 1.436 \sin \theta \\ & - 0.4715 \cos 2\theta - 0.0435 \sin 2\theta - 0.6658) . \end{aligned}$$

Equations (5-37) and (5-38) can be used to obtain,

$$\begin{aligned} r &= \text{Re}(y_1) + 0.8846 \text{Im}(y_1) \\ T &= 11.59 \text{Im}(y_1) + 7.5 , \end{aligned} \tag{5-43}$$

where Re and Im stand for real part and imaginary part, respectively. By letting θ vary from 0 to 2π in Eq. (5-43) we obtain a closed curve (boundary) in the r, T plane.

A similar procedure is followed in the $O(A^3)$ calculations where $U_4(\theta) = 0$. The calculations give

$$\begin{aligned} U_3(\theta) = & (-0.1730, 0.0828)e^{i\theta} + (0.3137, -0.0052)e^{i2\theta} \\ & - (0.0503, 0.0173)e^{i3\theta} + (0.0032, 0.0917)e^{-i\theta} \\ & - (0.0842, 0.0885)e^{-i2\theta} + (0.0255, 0.0161)e^{-i3\theta} \\ & - (0.0349, 0.0796) , \end{aligned}$$

$$B_3 = 0.8016 , B_4 = 0.2609 ,$$

$$A = 1.70 , A^2 = 2.90 , A^3 = 4.93 .$$

The use of these data together with the previously calculated $U_2(\theta)$

gives

$$\begin{aligned}
 y_1(A, \theta) = & 1.1462 \cos \theta - 1.9618 \sin \theta + 1.3677 \cos 2\theta \\
 & + 0.0338 \sin 2\theta - 0.1218 \cos 3\theta + 0.1648 \sin 3\theta \\
 & - 0.6901 + i(2.8828 \cos \theta + 1.1158 \sin \theta \\
 & - 1.3002 \cos 2\theta + 1.8852 \sin 2\theta - 0.0062 \cos 3\theta \\
 & - 0.3734 \sin 3\theta - 1.5764) .
 \end{aligned}$$

Equations (5-43) are used to obtain the final results. The results for both $O(A^2)$ and $O(A^3)$ are plotted in Fig. 5.2. The completely closed curves are shown here whereas only those portions for positive temperature and power greater than 1 are shown by Shotkin (1969).

A Stable Limit Cycle Case

In this case we consider the effective lifetime model with two temperature feedbacks, as discussed in Chapter 3. The added complexity here is that we deal with a third-order system in which one of the canonical coordinates must be eliminated in terms of the dominant coordinates. We also extend the calculation to $O(A^4)$. The physical equations are

$$\begin{aligned}
 \dot{\mathbf{r}} &= -A_1 T_1 - A_2 T_2 \\
 \dot{T}_1 &= A_0(e^{\mathbf{r}} - 1) - \eta_1 T_1 \\
 \dot{T}_2 &= \eta_2 T_1 - \eta_3 T_2 ,
 \end{aligned} \tag{5-44}$$

where

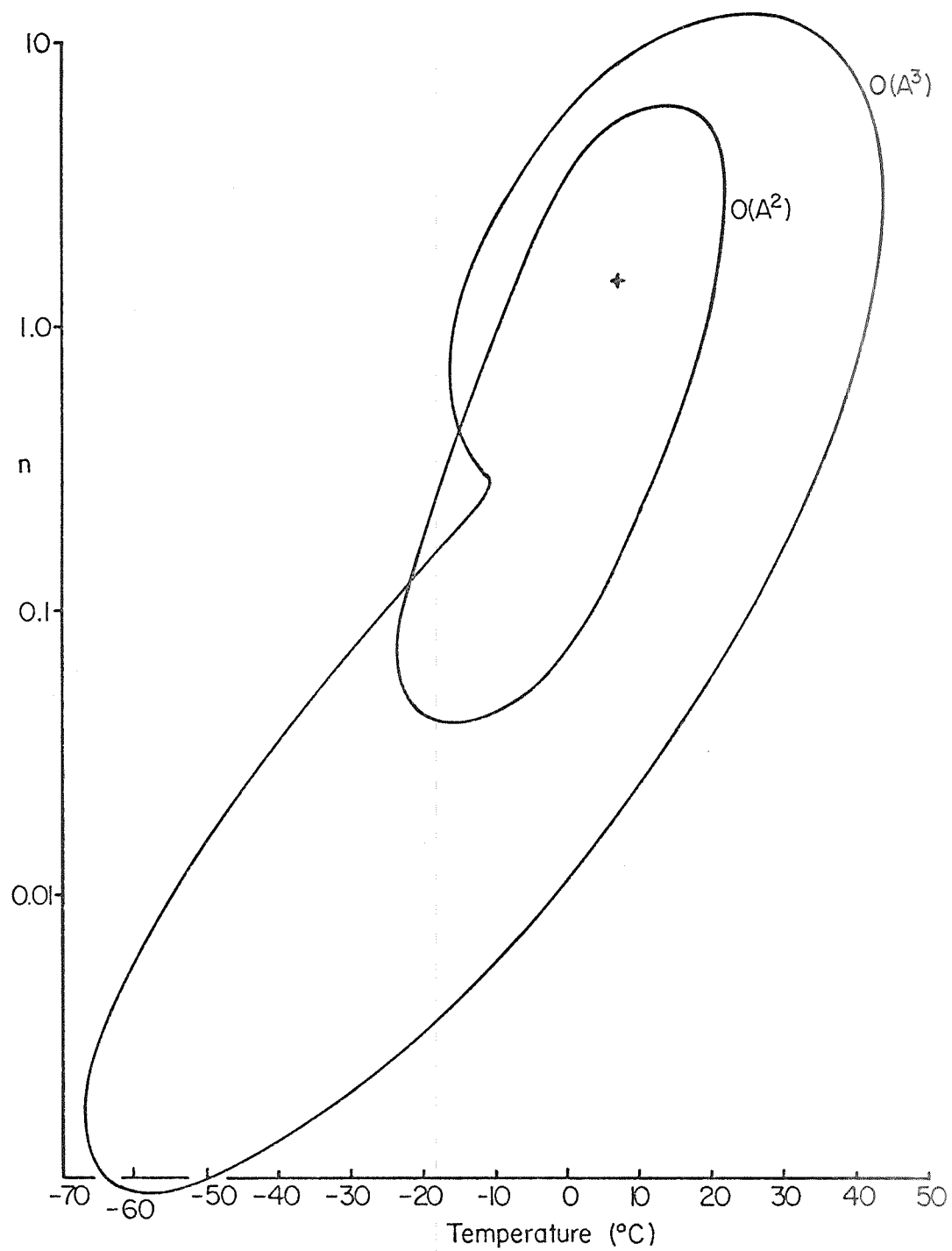


Fig. 5.2 Stable Region Approximations for Second Order Reactor Model.

$$r = \ln n/n_0 ; A_0 = bn_0$$

$$A_1 = \lambda\alpha_1/\beta ; A_2 = \lambda\alpha_2/\beta .$$

The system parameters are

$$n_0 = 265 \text{ Mw} ; n_c = 262.5 \text{ Mw}$$

$$\lambda = 0.1 \text{ sec}^{-1} ; b = 100^\circ\text{C/Mw-sec}$$

$$\eta_1 = 0.5 \text{ sec}^{-1} ; \beta = 0.01$$

$$\eta_2 = 0.25 \text{ sec}^{-1} ; \eta_3 = 0.2 \text{ sec}^{-1}$$

$$\alpha_1 = -\frac{1}{3} \times 10^{-6} \text{ per } ^\circ\text{C} ; \alpha_2 = 0.4 \times 10^{-6} \text{ per } ^\circ\text{C} .$$

The matrix, L, is

$$\begin{bmatrix} 0 & -A_1 & -A_2 \\ A_0 & -\eta_1 & 0 \\ 0 & \eta_2 & -\eta_3 \end{bmatrix} ,$$

from which are calculated

$$S_1 = 0.000667 + i0.1122$$

$$S_2 = 0.000667 - i0.1122$$

$$S_3 = -0.7013 .$$

This indicates that y_3 is the nondominant coordinate and should be eliminated in terms of y_1 and y_2 .

The canonical matrix, K, is

$$\begin{bmatrix} 1 & \frac{S_1}{A_0} & \frac{1}{\eta_2} \left[\frac{S_1^2}{A_0} + \frac{\eta_1 S_1}{A_0} + A_1 \right] \\ 1 & \frac{S_2}{A_0} & \frac{1}{\eta_2} \left[\frac{S_2^2}{A_0} + \frac{\eta_1 S_2}{A_0} + A_1 \right] \\ 1 & \frac{S_3}{A_0} & \frac{1}{\eta_2} \left[\frac{S_3^2}{A_0} + \frac{\eta_1 S_3}{A_0} + A_1 \right] \end{bmatrix}, \quad (5-45)$$

and the inverse matrix, K^{-1} , is

$$\begin{bmatrix} (0.4002, -0.6215) & (0.4002, 0.6215) & 0.1996 \\ (1.315, -3.584)(10^4) & (1.315, 3.584)(10^4) & (-2.629)(10^4) \\ (-0.655, -4.099)(10^4) & (-0.655, 4.099)(10^4) & (1.310)(10^4) \end{bmatrix}. \quad (5-46)$$

The nondominant coordinate is eliminated as follows:

Let

$$\begin{aligned} y_3 = v_3(y_1, y_2) = & C_{32}y_1^2 + C_{33}y_1y_2 + C_{34}y_2^2 + C_{35}y_1^3 \\ & + C_{36}y_1^2y_2 + C_{37}y_1y_2^2 + C_{38}y_2^3. \end{aligned} \quad (5-47)$$

Evaluate

$$\begin{aligned} \frac{\partial v_3}{\partial y_1} &= 2C_{32}y_1 + C_{33}y_2 + 3C_{35}y_1^2 + 2C_{36}y_1y_2 + C_{37}y_2^2 \\ \frac{\partial v_3}{\partial y_2} &= 2C_{34}y_2 + C_{33}y_1 + 3C_{38}y_2^2 + 2C_{37}y_1y_2 + C_{36}y_1^2. \end{aligned} \quad (5-48)$$

The derivative of y_1 is

$$\dot{y}_1 = S_1y_1 + S_1(e^r - 1 - r) = S_1y_1 + S_1\left(\frac{r^2}{2} + \frac{r^3}{6} + \dots\right),$$

where

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$$\begin{aligned}
\dot{y}_2 = & S_2 y_2 + \frac{S_2 C_1^2}{2} y_1^2 + \frac{S_2 C_2^2}{2} y_2^2 + S_2 C_1 C_2 y_1 y_2 \\
& + S_2 C_1 C_3 C_{32} y_1^3 + S_2 C_1 C_3 C_{33} y_1^2 y_2 + S_2 C_1 C_3 C_{34} y_1 y_2^2 \\
& + S_2 C_2 C_3 C_{32} y_1^2 y_2 + S_2 C_2 C_3 C_{33} y_1 y_2^2 + S_2 C_2 C_3 C_{34} y_2^3 \\
& + \frac{S_2 C_1^3}{6} y_1^3 + \frac{S_2 C_2^3}{6} y_2^3 + \frac{S_2 C_1^2 C_2}{2} y_1^2 y_2 + \frac{S_2 C_1 C_2^2}{2} y_1 y_2^2 .
\end{aligned}
\tag{5-51}$$

From Eqs. (5-48), (5-50), and (5-51) the quantity

$$\frac{\partial v_3}{\partial y_1} \dot{y}_1 + \frac{\partial v_3}{\partial y_2} \dot{y}_2$$

is

$$\begin{aligned}
& 2S_1 C_{32} y_1^2 + 2S_2 C_{34} y_2^2 + (2S_1 C_1 C_2 C_{32} + \frac{S_1 C_1^2 C_{33}}{2} + 2S_1 C_{36} \\
& + S_2 C_1^2 C_{34} + S_2 C_1 C_2 C_{33} + S_2 C_{36}) y_1^2 y_2 + (S_1 C_2^2 C_{32} + S_1 C_1 C_2 C_{33} \\
& + S_1 C_{37} + 2S_2 C_1 C_2 C_{34} + \frac{S_2 C_2^2 C_{33}}{2} + 2S_2 C_{37}) y_1 y_2^2 \\
& + (S_1 C_{33} + S_2 C_{33}) y_1 y_2 + (S_1 C_1^2 C_{32} + \frac{S_2 C_1^2 C_{33}}{2}) y_1^3 \\
& + (\frac{S_1 C_2^2 C_{33}}{2} + S_2 C_2^2 C_{34}) y_2^3 .
\end{aligned}
\tag{5-52}$$

By Eq. (5-24) the above quantity must be equal to $S_3 v_3 + Y_3(y_1, y_2, v_3)$,

which is

$$\begin{aligned}
& (S_3 C_{32} + \frac{S_3 C_1^2}{2}) y_1^2 + (S_3 C_{34} + \frac{S_3 C_2^2}{2}) y_2^2 \\
& + (S_3 C_{36} + S_3 C_1 C_3 C_{33} + S_3 C_2 C_3 C_{32} + \frac{S_3 C_1^2 C_2}{2}) y_1^2 y_2 \\
& + (S_3 C_{37} + S_3 C_1 C_3 C_{34} + S_3 C_2 C_3 C_{33} + \frac{S_3 C_1 C_2^2}{2}) y_1 y_2^2 \\
& + (S_3 C_{33} + S_3 C_1 C_2) y_1 y_2 + (S_3 C_{35} + S_3 C_1 C_3 C_{32} + \frac{S_3 C_1^3}{6}) y_1^3 \\
& + (S_3 C_{38} + S_3 C_2 C_3 C_{34} + \frac{S_3 C_2^3}{6}) y_2^3 . \tag{5-53}
\end{aligned}$$

By equating coefficients of corresponding terms in Eqs. (5-52) and (5-53) the constants of Eq. (5-47) are found to be

$$C_{32} = - \frac{S_3 C_1^2}{2(S_3 - 2S_1)}$$

$$C_{33} = \frac{S_3 C_1 C_2}{(S_1 + S_2 - S_3)}$$

$$C_{34} = \frac{S_3 C_2^2}{2(2S_2 - S_3)}$$

$$C_{35} = \frac{S_1 C_1^2 C_{32}}{S_3} + \frac{S_2 C_1^2 C_{33}}{2S_3} - C_1 C_3 C_{32} - \frac{C_1^3}{6}$$

$$\begin{aligned}
C_{36} = & (2S_1 C_1 C_2 C_{32} + \frac{S_1 C_1^2 C_{33}}{2} + S_2 C_1^2 C_{34} + S_2 C_1 C_2 C_{33} \\
& - S_3 C_1 C_3 C_{33} - S_3 C_2 C_3 C_{32} - \frac{S_3 C_1^2 C_2}{2}) / (S_3 - 2S_1 - S_2)
\end{aligned}$$

$$\begin{aligned}
C_{37} = & (S_1 C_2^2 C_{32} + S_1 C_1 C_2 C_{33} + 2S_2 C_1 C_2 C_{34} + \frac{S_2 C_2^2 C_{33}}{2} \\
& - S_3 C_1 C_3 C_{34} - S_3 C_2 C_3 C_{33} - \frac{S_3 C_1 C_2^2}{2}) / (S_3 - S_1 - 2S_2) \\
C_{38} = & \frac{S_1 C_2^2 C_{33}}{2S_3} + \frac{S_2 C_2^2 C_{34}}{S_3} - C_2 C_3 C_{34} - \frac{C_2^3}{6} .
\end{aligned}$$

We have thus determined the nonlinear transformation that specifies y_3 in terms of y_1 and y_2 .

Making use of $y_2 = y_1^*$, and substituting the fifth-order approximation of Eq. (5-29a) into Eq. (5-50), we obtain,

$$\begin{aligned}
\dot{y}_1 = & A[S_1 e^{i\theta}] \\
& + A^2[S_1 U_2(\theta) + \frac{S_1 C_1^2}{2} e^{i2\theta} + \frac{S_1 C_2^2}{2} e^{-i\theta} + S_1 C_1 C_2] \\
& + A^3[S_1 U_3(\theta) + S_1 C_1^2 e^{i\theta} U_2(\theta) + S_1 C_2^2 e^{-i\theta} U_2^*(\theta) \\
& + S_1 C_1 C_2 \{e^{i\theta} U_2^*(\theta) + e^{-i\theta} U_2(\theta)\} + (S_1 C_1 C_3 C_{32} + \frac{S_1 C_1^3}{6}) e^{i3\theta} \\
& + (S_1 C_2 C_3 C_{34} + \frac{S_1 C_2^3}{6}) e^{-i3\theta} + (S_1 C_1 C_3 C_{33} + S_1 C_2 C_3 C_{32} \\
& + \frac{S_1 C_1^2 C_2}{2}) e^{i\theta} + (S_1 C_1 C_3 C_{34} + S_1 C_2 C_3 C_{33} + \frac{S_1 C_1 C_2^2}{2}) e^{-i\theta}]
\end{aligned}$$

$$\begin{aligned}
& + A^4 \{ S_1 U_4(\theta) + \frac{S_1 C_1^2}{2} \{ 2e^{i\theta} U_3(\theta) + U_2^2(\theta) \} \\
& + \frac{S_1 C_2^2}{2} \{ 2e^{-i\theta} U_3^*(\theta) + (U_2^*(\theta))^2 \} + S_1 C_1 C_2 \{ e^{i\theta} U_3^*(\theta) \\
& + e^{-i\theta} U_3(\theta) + U_2(\theta) U_2^*(\theta) \} + \{ 3S_1 C_1 C_3 C_{32} + \frac{S_1 C_1^3}{2} \} e^{i2\theta} U_2(\theta) \\
& + \{ 3S_1 C_2 C_3 C_{34} + \frac{S_1 C_2^3}{2} \} e^{-i2\theta} U_2^*(\theta) + \{ S_1 C_1 C_3 C_{33} \\
& + S_1 C_2 C_3 C_{32} + \frac{S_1 C_1^2 C_2}{2} \} \{ 2U_2(\theta) + e^{i2\theta} U_2^*(\theta) \} \\
& + \{ S_1 C_1 C_3 C_{34} + S_1 C_2 C_3 C_{33} + \frac{S_1 C_1 C_2^2}{2} \} \{ 2U_2^*(\theta) + e^{-i2\theta} U_2(\theta) \} \\
& + \{ \frac{S_1 C_3^2 C_{33}^2}{2} + S_1 C_1 C_3 C_{35} + \frac{S_1 C_1^2 C_3 C_{32}}{2} + \frac{S_1 C_1^4}{24} \} e^{i4\theta} \\
& + \{ \frac{S_1 C_3^2 C_{34}^2}{2} + S_1 C_2 C_3 C_{38} + \frac{S_1 C_2^2 C_3 C_{34}}{2} + \frac{S_1 C_2^4}{24} \} e^{-i4\theta} \\
& + \{ S_1 C_3^2 C_{32} C_{33} + S_1 C_1 C_3 C_{36} + S_1 C_2 C_3 C_{35} + \frac{S_1 C_1^2 C_3 C_{33}}{2} \\
& + S_1 C_1 C_2 C_3 C_{32} + \frac{S_1 C_1^3 C_2}{6} \} e^{i2\theta} \\
& + \{ S_1 C_3^2 C_{33} C_{34} + S_1 C_1 C_3 C_{38} + S_1 C_2 C_3 C_{37} + \frac{S_1 C_2^2 C_3 C_{33}}{2} \\
& + S_1 C_1 C_2 C_3 C_{34} + \frac{S_1 C_1 C_2^3}{6} \} e^{-i2\theta}
\end{aligned}$$

$$\begin{aligned}
& + \{S_1 C_3^2 C_{32} C_{34} + \frac{S_1 C_3^2 C_{33}^2}{2} + S_1 C_1 C_3 C_{37} + S_1 C_2 C_3 C_{36} \\
& + \frac{S_1 C_1^2 C_3 C_{34}}{2} + \frac{S_1 C_2^2 C_3 C_{32}}{2} + S_1 C_1 C_2 C_3 C_{33} + \frac{S_1 C_1^2 C_2^2}{4}\}] \\
& + A^5 [S_1 U_5(\theta) + S_1 C_1^2 e^{i\theta} U_4(\theta) + S_1 C_1^2 U_2(\theta) U_3(\theta) + S_1 C_2^2 e^{-i\theta} U_4^*(\theta) \\
& + S_1 C_2^2 U_2^*(\theta) U_3^*(\theta) + S_1 C_1 C_2 e^{-i\theta} U_4(\theta) + S_1 C_1 C_2 U_2^*(\theta) U_3(\theta) \\
& + S_1 C_1 C_2 U_2(\theta) U_3^*(\theta) + S_1 C_1 C_2 e^{i\theta} U_4^*(\theta) + \{3S_1 C_1 C_3 C_{32} + \frac{S_1 C_1^3}{2}\} e^{i2\theta} U_3(\theta) \\
& + \{2S_1 C_1 C_3 C_{32} + \frac{S_1 C_1^3}{3}\} e^{i\theta} U_2^2(\theta) + \{3S_1 C_2 C_3 C_{34} + \frac{S_1 C_2^3}{2}\} e^{-i2\theta} U_3^*(\theta) \\
& + \{2S_1 C_2 C_3 C_{34} + \frac{S_1 C_2^3}{3}\} e^{-i\theta} (U_2^*(\theta))^2 + \{S_1 C_1 C_3 C_{33} + S_1 C_2 C_3 C_{32} \\
& + \frac{S_1 C_1^2 C_2}{2}\} \{2U_3(\theta) + 2e^{i\theta} U_2^*(\theta) U_2(\theta) + e^{i2\theta} U_3^*(\theta)\} \\
& + \{S_1 C_1 C_3 C_{34} + S_1 C_2 C_3 C_{33} + \frac{S_1 C_1 C_2^2}{2}\} \{2U_3^*(\theta) + 2e^{-i\theta} U_2^*(\theta) U_2(\theta) \\
& + e^{-i2\theta} U_3(\theta)\} + \{2S_1 C_3^2 C_{33}^2 + 4S_1 C_1 C_3 C_{35} + 2S_1 C_1^2 C_3 C_{32} + \frac{S_1 C_1^4}{6}\} e^{i3\theta} U_2(\theta) \\
& + \{2S_1 C_3^2 C_{34}^2 + 4S_1 C_2 C_3 C_{38} + 2S_1 C_2^2 C_3 C_{34} + \frac{S_1 C_2^4}{6}\} e^{-i3\theta} U_2^*(\theta) \\
& + \{S_1 C_3^2 C_{32} C_{33} + S_1 C_1 C_3 C_{36} + S_1 C_2 C_3 C_{35} + \frac{S_1 C_1^2 C_3 C_{33}}{2} + S_1 C_1 C_2 C_3 C_{32}
\end{aligned}$$

$$\begin{aligned}
& + \frac{S_1 C_1^3 C_2}{6} \{3e^{i\theta} U_2(\theta) + e^{i3\theta} U_2^*(\theta)\} + \{S_1 C_3^2 C_{33} C_{34} \\
& + S_1 C_1 C_3 C_{38} + S_1 C_2 C_3 C_{37} + \frac{S_1 C_2^2 C_3 C_{33}}{2} + S_1 C_1 C_2 C_3 C_{34} \\
& + \frac{S_1 C_1 C_2^3}{6} \{3e^{-i\theta} U_2^*(\theta) + e^{-i3\theta} U_2(\theta)\} + \{S_1 C_3^2 C_{32} C_{34} \\
& + \frac{S_1 C_3^2 C_{33}^2}{2} + S_1 C_1 C_3 C_{37} + S_1 C_2 C_3 C_{36} + \frac{S_1 C_1^2 C_3 C_{34}}{2} \\
& + \frac{S_1 C_2^2 C_3 C_{32}}{2} + S_1 C_1 C_2 C_3 C_{33} + \frac{S_1 C_1^2 C_2^2}{4} \{2e^{-i\theta} U_2(\theta) + 2e^{i\theta} U_2^*(\theta)\} \\
& + \frac{S_1 C_1 C_3^2 C_{32}^2}{2} e^{i5\theta} + \frac{S_1 C_1 C_3^2 C_{34}^2}{2} e^{-i3\theta} + S_1 C_1 C_3^2 C_{32} C_{33} e^{i3\theta} \\
& + \frac{S_1 C_1 C_3^2}{2} \{2C_{33} C_{34} + C_{33}^2\} e^{i\theta} + S_1 C_1 C_3^2 C_{33} C_{34} e^{-i\theta} \\
& + \frac{S_1 C_1^3 C_3 C_{32}}{6} e^{i5\theta} + \frac{S_1 C_1^3 C_3 C_{33}}{6} e^{i3\theta} + \frac{S_1 C_1^3 C_3 C_{34}}{6} e^{i\theta} \\
& + \frac{S_1 C_2^3 C_3 C_{32}}{6} e^{-i\theta} + \frac{S_1 C_2^3 C_3 C_{33}}{6} e^{-i3\theta} + \frac{S_1 C_2^3 C_3 C_{34}}{6} e^{-i5\theta} \\
& + \frac{S_1 C_1^2 C_2 C_3 C_{32}}{2} e^{i3\theta} + \frac{S_1 C_1^2 C_2 C_3 C_{33}}{2} e^{i\theta} + \frac{S_1 C_1^2 C_2 C_3 C_{34}}{2} e^{-i\theta} \\
& + \frac{S_1 C_1 C_2^2 C_3 C_{32}}{2} e^{i\theta} + \frac{S_1 C_1 C_2^2 C_3 C_{33}}{2} e^{-i\theta} + \frac{S_1 C_1 C_2^2 C_3 C_{34}}{2} e^{-i3\theta} \\
& + \frac{C_2 C_3^2 C_{32}^2}{2} e^{i3\theta} + \frac{C_2 C_3^2 C_{34}^2}{2} e^{-i5\theta} + C_2 C_3^2 C_{32} C_{33} e^{i\theta} \\
& + \frac{C_2 C_3^2}{2} \{2C_{32} C_{34} + C_{33}^2\} e^{-i\theta} + C_2 C_3^2 C_{33} C_{34} e^{-i3\theta} \} . \quad (5-54)
\end{aligned}$$

Equation (5-41) of the previous section is general and is applicable here. By equating coefficients of corresponding powers of A in Eqs. (5-41) and (5-54) we obtain differential equations for $U_2(\theta)$, $U_3(\theta)$, and $U_4(\theta)$. The coefficient of A^5 in Eq. (5-54) is used in the calculation of B_5 to $O(A^4)$ only, since the highest order calculated is the fourth, i.e., $U_5(\theta) = 0$. As before, the B_n and Ω_n may be obtained from Eq. (5-29b) when the corresponding $U_n(\theta)$ is determined.

It can be shown for $O(A^2)$ that

$$\begin{aligned} U_2(\theta) = & (0.622411, 0.325433)e^{i\theta} - (0.113049, 0.248721)e^{i2\theta} \\ & + (0.037026, -0.083204)e^{-i2\theta} + (-0.546388, 0.006492) . \end{aligned} \quad (5-55)$$

and

$$B_2 + i\Omega_2 = (-0.000415, -0.000217) .$$

From Eq. (5-33) we obtain

$$B_3 + i\Omega_3 = (-0.014382, -0.031297) .$$

Using these values of B_2 , B_3 , and $b = 0.000667$ in Eq. (5-32a) set equal to zero, we find that $A = 0.2014$ and $A^2 = 0.0405$. Using the latter values and Eqs. (5-29a) and (5-55) we obtain

$$\begin{aligned} y_1(A, \theta) = & 0.2266 \cos \theta - 0.0132 \sin \theta - 0.0031 \cos 2\theta \\ & + 0.0067 \sin 2\theta - 0.0221 + i(0.0132 \cos \theta \\ & + 0.2266 \sin \theta - 0.0134 \cos 2\theta - 0.0061 \sin 2\theta \\ & + 0.0003) . \end{aligned} \quad (5-56)$$

Let

$$y_1(A, \theta) = u + iv \quad (5-57)$$

where

$$u \equiv \text{Re}(y_1) \text{ and } v \equiv \text{Im}(y_1) .$$

Using Eqs. (5-46), (5-47), (5-56) and (5-57) it can be shown that

$$\begin{aligned} r &= 0.8004 u + 1.2429 v - 0.0393 u^2 - 0.1785 v^2 \\ &\quad - 0.1538 uv - 0.0057 u^3 + 0.0245 v^3 \\ &\quad - 0.0024 u^2v + 0.0164 uv^2 \\ T_1 \times 10^{-4} &= 2.6294 u + 7.1679 v + 0.5172 u^2 \\ &\quad + 2.351 v^2 + 2.0254 uv - 0.0755 u^3 \\ &\quad - 0.3231 v^3 + 0.0311 u^2v - 0.2161 uv^2 \\ T_2 \times 10^{-4} &= -1.3102 u + 8.1971 v - 0.2577 u^2 \\ &\quad - 1.1714 v^2 - 1.010 uv - 0.0376 u^3 \\ &\quad + 0.1610 v^3 - 0.0155 u^2v + 0.1077 uv^2 . \end{aligned} \quad (5-58)$$

We have thus completed the $O(A^2)$ calculation. Note that in this case we have b positive and B_3 negative which corresponds to type (a) of Fig. 5.1.

For $O(A^3)$ we find that

$$\begin{aligned} U_3(\theta) &= (0.579155, 0.365236)e^{i\theta} + (0.021160, -0.383189)e^{i2\theta} \\ &\quad + (-0.053363, 0.070971)e^{i3\theta} + (0.114272, 0.031321)e^{-i\theta} \\ &\quad - (0.008067, 0.127672)e^{-i2\theta} + (0.026997, 0.035247)e^{-i3\theta} \\ &\quad + (-0.680154, 0.008086) , \end{aligned}$$

and

$$B_3 + i\Omega_3 = - (0.007810, 0.025562) .$$

We obtain from Eq. (5-33)

$$B_4 + i\Omega_4 = - (0.003624, 0.028448) .$$

We then find for the third-order polynomial obtained from Eq. (5-32a) that $A = 0.2537$. Using the appropriate equations we find that

$$\begin{aligned} y_1(A, \theta) = & 0.3050 \cos \theta - 0.0264 \sin \theta - 0.0047 \cos 2\theta \\ & + 0.0148 \sin 2\theta - 0.0004 \cos 3\theta - 0.0006 \sin 3\theta \\ & - 0.0463 + i(0.0274 \cos \theta + 0.3013 \sin \theta \\ & - 0.0297 \cos 2\theta - 0.0092 \sin 2\theta + 0.0017 \cos 3\theta \\ & - 0.0013 \sin 3\theta + 0.0006) . \end{aligned} \quad (5-59)$$

We obtain the final results for $O(A^3)$ by using Eqs. (5-57), (5-58), and (5-59).

For $O(A^4)$ we have that

$$\begin{aligned} U_4(\theta) = & (0.418599, 0.477335)e^{i\theta} + (0.143263, -0.411830)e^{i2\theta} \\ & + (-0.168930, 0.080414)e^{i3\theta} + (0.311551, -0.004049)e^{i4\theta} \\ & + (0.223249, 0.020991)e^{-i\theta} - (0.068004, 0.143018)e^{-i2\theta} \\ & + (0.084819, 0.039455)e^{-i3\theta} + (-0.018648, 0.003129)e^{-i4\theta} \\ & - (0.925899, 0.062427) , \end{aligned} \quad (5-60)$$

and

$$B_4 + i\Omega_4 = -(0.004291, 0.029337) , \quad (5-60)$$

The quantity B_5 is calculated using Eq. (5-33), and is -0.009698 . It is noted that the quantity $Y_{15}(\theta)$ needed in this calculation is the coefficient of A^5 in Eq. (5-54), with $U_5(\theta) = 0$. Now using the known values of b , B_2 , B_3 , B_4 , and B_5 in Eq. (5-32a) set equal to zero, a fourth-order polynomial in A is obtained. The smallest positive real root is 0.2447 . The appropriate calculations show that

$$\begin{aligned}
y_1(A, \theta) = & 0.2945 \cos \theta - 0.0260 \sin \theta - 0.0041 \cos 2\theta \\
& + 0.0146 \sin 2\theta - 0.0007 \cos 3\theta - 0.0007 \sin 3\theta \\
& + 0.0011 \cos 4\theta + 0.00003 \sin 4\theta - 0.0460 \\
& + i(0.0271 \cos \theta + 0.2895 \sin \theta - 0.0294 \cos 2\theta \\
& - 0.0078 \sin 2\theta + 0.0020 \cos 3\theta - 0.0021 \sin 3\theta \\
& - 0.000004 \cos 4\theta + 0.0012 \sin 4\theta + 0.0003) .
\end{aligned}
\tag{5-61}$$

Equations (5-57), (5-58), and (5-61) provide the final result.

The results for this case in all three orders of approximation are shown in Fig. 5.3. The dotted line represents the limit cycle obtained by the exact digital computer solution of Eqs. (5-44). The computer plot of this limit cycle is shown in Fig. 5.4.

It is to be noted in Fig. 5.3 that the $O(A^2)$ boundary is within the limit cycle and that of $O(A^3)$ is outside. The $O(A^4)$ boundary is inside the limit cycle but very close to it, and it is anticipated that higher-order approximations will approach the limit cycle asymptotically. However, the degree of convergence is already high with the fourth-order approximation. Although this is not a case of asymptotic stability, the instability bound is finite, so that the method gives a precise calculation of the limit cycle. This is in contrast to the results of the following case in which the stable region is of infinite extent and the lower orders of approximation do not delineate the stable region as precisely. A further example also tends to illustrate this point.

Effective Lifetime, Case I, $\xi = -0.1$

$n_o = 265 \text{ Mw}$, $n_c = 262.5 \text{ Mw}$

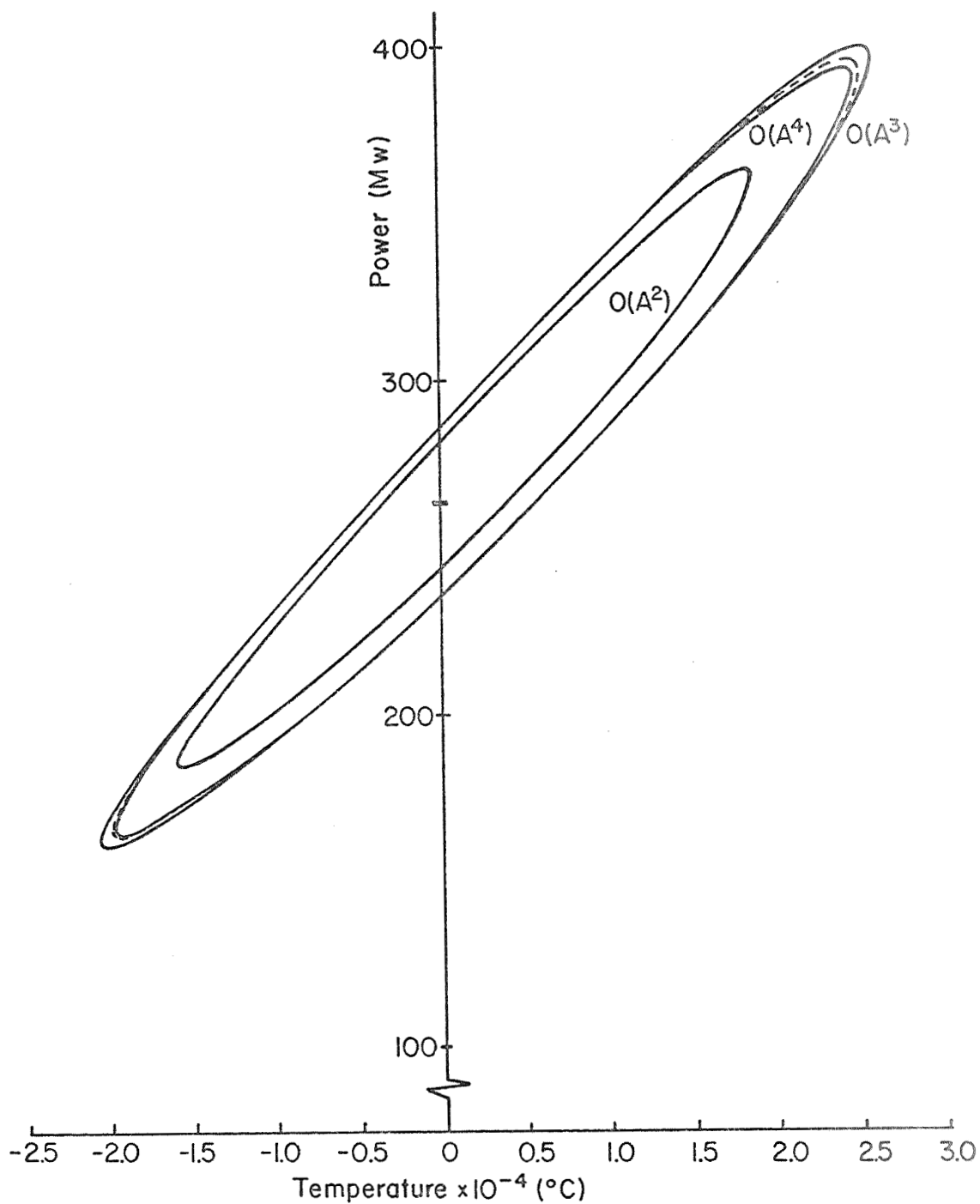


Figure 5.3 Approximations to the Effective Lifetime Limit Cycle.

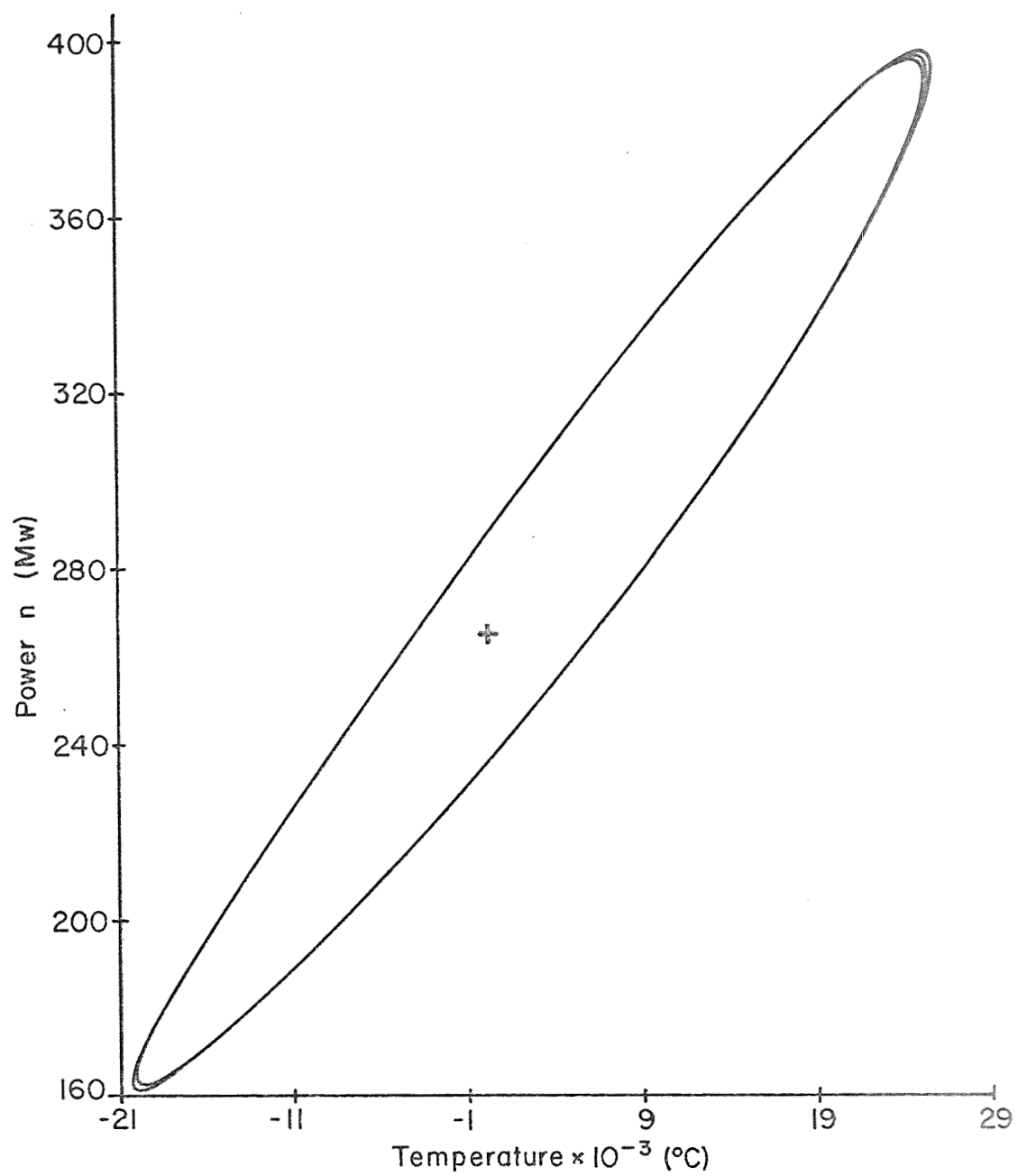


Figure 5.4 Exact Computation of Effective Lifetime Limit Cycle.

A Stable Focus Case

In the previous case the equilibrium power is above the critical power, and the trajectory limit is a stable limit cycle in n, T_1, T_2 space. A comparison case was considered in which the equilibrium power was slightly below critical. However, no solutions were obtained for approximations through the fourth order. It was concluded after some investigation that the formulation of the equations is not general enough to cover the resonant case and the near-resonance region below critical power. In this regime the difference between the frequency of the perturbation and the system resonance frequency is an important parameter. Hence, A , the perturbation is now a function of amplitude, and θ , a difference in phase angles. Thus Eqs. (5-32a) and (5-32b) become a coupled pair of equations, and the analysis becomes more complicated. The derivation of the more generalized equations is discussed by Bogoliubov and Mitropolsky (1961).

The generalized equations are not considered here. Instead a case below the resonance regime is studied. We select the stable focus case with an equilibrium power of 100 Mw which is discussed in Chapter 3. All other system parameters are identical to those in the immediately preceding section.

Here b is negative and B_3 is positive, so we are dealing with a case of the type (b) shown in Fig. 5.1. From the matrix, L , the following characteristic values are calculated,

$$S_1 = -0.051096 + i0.054448$$

$$S_2 = -0.051096 - i0.054448$$

$$S_3 = -0.597807 ,$$

which indicate that y_3 is the nondominant coordinate. The canonical matrix is,

$$\begin{bmatrix} 1 & (-0.51096 + i0.054448)(10^{-5}) & (-2.356668 + i0.866400)(10^{-5}) \\ 1 & (-0.51096 - i0.054448)(10^{-5}) & (-2.356668 - i0.866400)(10^{-5}) \\ 1 & (-5.97807)(10^{-5}) & (3.672112)(10^{-5}) \end{bmatrix}$$

and the inverse matrix is

$$\begin{bmatrix} 0.447597 - i0.995394 & 0.447597 + i0.995394 & 0.104806 \\ (0.540200 - i3.758941)(10^4) & (0.540200 + i3.758941)(10^4) & (-1.080400)(10^4) \\ (-0.339483 - i3.408740)(10^4) & (-0.339483 + i3.408740)(10^4) & (0.678966)(10^4) \end{bmatrix}.$$

The following equations may be derived,

$$\begin{aligned} r &= 0.8952 u + 1.9908 v - 0.1730 uv - 0.0316 u^2 - 0.2695 v^2 \\ &\quad - 0.0366 u^2v - 0.0449 uv^2 - 0.0074 u^3 - 0.0216 v^3 \end{aligned}$$

$$\begin{aligned} T_1 \times 10^{-4} &= 1.080 u + 7.518 v + 1.784 uv + 0.3262 u^2 \\ &\quad + 2.7783 v^2 + 0.3774 u^2v + 0.4626 uv^2 \\ &\quad + 0.0763 u^3 + 0.2231 v^3 \end{aligned}$$

$$\begin{aligned} T_2 \times 10^{-4} &= -0.6790 u + 6.8175 v - 1.1209 uv - 0.2050 u^2 \\ &\quad - 1.7460 v^2 - 0.2372 u^2v - 0.2907 uv^2 \\ &\quad - 0.0480 u^3 - 0.1402 v^3 . \end{aligned} \tag{5-62}$$

For $O(A^2)$ we find that

$$U_2(\theta) = (0.216780, 1.579699)e^{i\theta} - (0.395233, 0.445535)e^{i2\theta} \\ + (0.254037, 0.054588)e^{-i2\theta} - (0.075584, 1.188752) ,$$

and

$$B_2 + i\Omega_2 = (0.011077, 0.080716)$$

$$B_3 + i\Omega_3 = (0.102974, -0.227627)$$

$$A = 0.6527 \quad (b = -0.0511) .$$

Using these values we obtain

$$y_1(A, \theta) = 0.7450 \cos \theta - 0.6729 \sin \theta - 0.0601 \cos 2\theta \\ + 0.2131 \sin 2\theta - 0.0322 + i(0.6729 \cos \theta \\ + 0.7450 \sin \theta - 0.1665 \cos 2\theta - 0.2766 \sin 2\theta \\ - 0.5064) . \quad (5-63)$$

Equations (5-57), (5-62) and (5-63) define the solution for $O(A^2)$.

For $O(A^3)$ we have,

$$U_3(\theta) = (-0.477336, 1.530942)e^{i\theta} + (1.236247, -1.441863)e^{i2\theta} \\ + (0.040442, 0.332080)e^{i3\theta} + (-0.848358, 0.820366)e^{-i\theta} \\ + (0.282601, -0.778948)e^{-i2\theta} + (-0.200825, 0.052832)e^{-i3\theta} \\ + (-0.032771, -0.515409) ,$$

$$B_3 + i\Omega_3 = (0.054221, -0.071177)$$

$$B_4 + i\Omega_4 = (0.267332, -0.313246)$$

$$A = 0.4946$$

and

$$\begin{aligned}
y_1(A, \theta) = & 0.3872 \cos \theta - 0.4725 \sin \theta + 0.1493 \cos 2\theta \\
& + 0.2026 \sin 2\theta - 0.0194 \cos 3\theta - 0.0338 \sin 3\theta \\
& - 0.022458 + i(0.6710 \cos \theta + 0.5926 \sin \theta \\
& - 0.3644 \cos 2\theta - 0.0434 \sin 2\theta + 0.0466 \cos 3\theta \\
& + 0.0292 \sin 3\theta - 0.3532) .
\end{aligned}$$

For $O(A^4)$ we find that

$$\begin{aligned}
U_4(\theta) = & (-2.099054, 1.328758)e^{i\theta} + (1.929285, 0.224706)e^{i2\theta} \\
& + (-1.457886, 0.281575)e^{i3\theta} + (0.088037, -0.163331)e^{i4\theta} \\
& + (0.744196, 1.873708)e^{-i\theta} - (0.219933, 0.967076)e^{-i2\theta} \\
& + (0.119753, 0.986123)e^{-i3\theta} + (0.092968, -0.092731)e^{-i4\theta} \\
& + (0.802634, -3.471732) ,
\end{aligned}$$

$$B_4 + i\Omega_4 = (-0.054428, -0.109563)$$

$$B_5 = 0.362122$$

$$A = 0.5666 .$$

Use of these data gives

$$\begin{aligned}
y(A, \theta) = & 0.255433 \cos \theta - 0.580187 \sin \theta + 0.407071 \cos 2\theta \\
& + 0.158304 \sin 2\theta - 0.167064 \cos 3\theta + 0.021814 \sin 3\theta \\
& + 0.018652 \cos 4\theta + 0.007275 \sin 4\theta + 0.052488 \\
& + i(1.264775 \cos \theta + 0.410653 \sin \theta = 0.605922 \cos 2\theta \\
& + 0.186503 \sin 2\theta + 0.200644 \cos 3\theta - 0.118694 \sin 3\theta \\
& - 0.026387 \cos 4\theta - 0.000508 \sin 4\theta - 0.833110) .
\end{aligned}$$

The final results for all three orders are plotted in Fig. 5.5.

The dotted line is the projection onto the n, T_1 plane of the line of attraction in n, T_1, T_2 space, which is discussed in Chapter 3. The

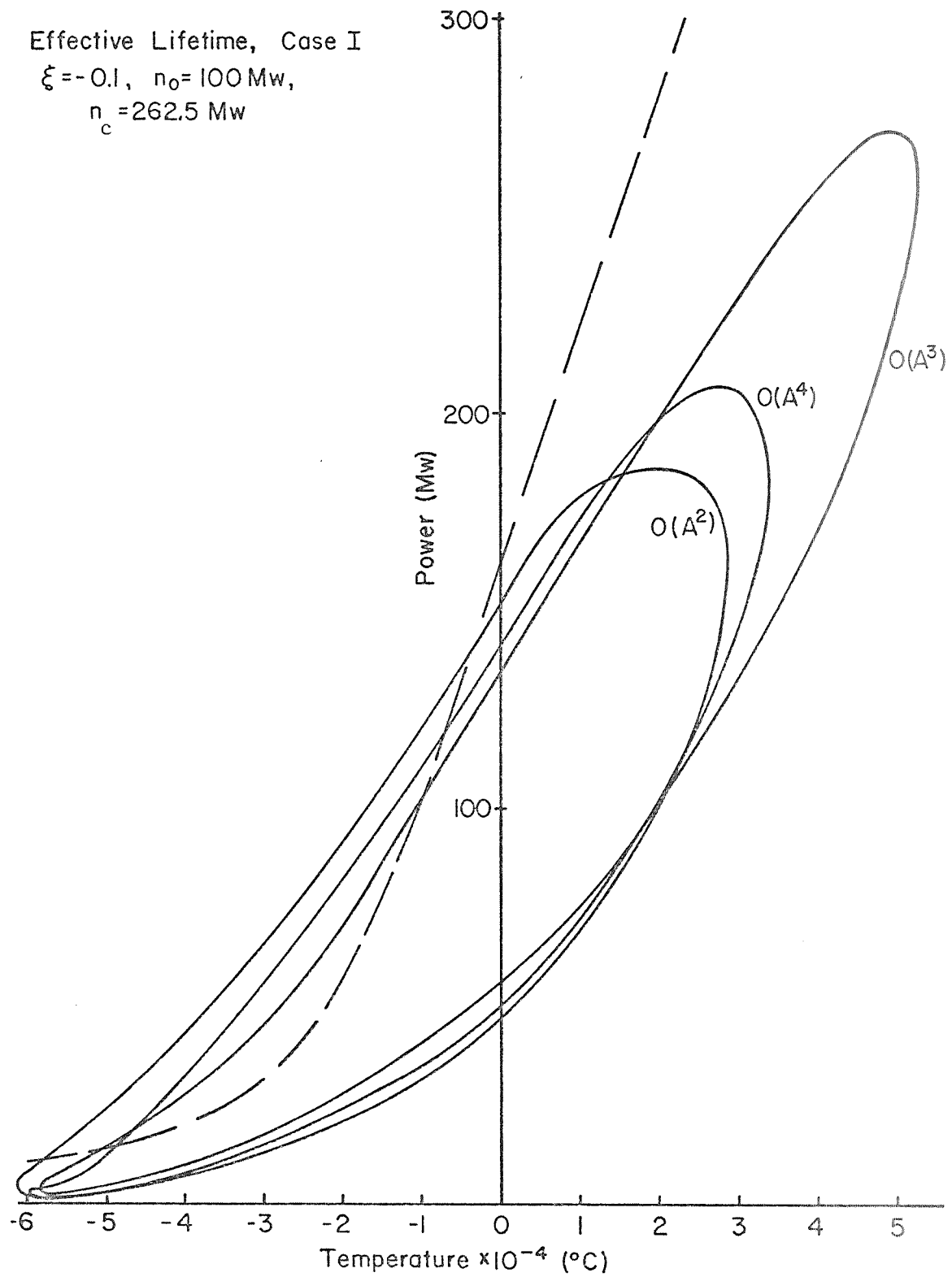


Figure 5.5 Stable Region Approximations for Effective Lifetime Stable Focus.

fact that this dotted line passes through the boundaries of all three approximations indicates that the predicted regions of stability include unstable regions. Note, however, that the stability regions tend to become longer and narrower with increasing order. It appears that for higher orders the predicted stability regions will lie entirely below the dotted line. It can be seen that the $O(A^3)$ approximation overestimates the size of the stability region as compared to the $O(A^4)$ approximation. This can be explained by examining Eq. (5-33). In estimating B_4 for the $O(A^3)$ calculation, it is assumed that $3b\alpha_4 = 0$, i.e., $U_4(0) = 0$. When this quantity is small, the approximation is good. When it is large, as in this case, the approximation is poor. Whether or not this type of oscillation damps out with increasing order depends on the magnitudes of the neglected terms in each step of the calculation. In this case we have a stable region of infinite extent as discussed in Chapter 3, although the system is not globally asymptotically stable. It would appear that the infinite order approximation would be required to give a good approximation to the entire stable region. As already pointed out, the technique employed is more useful for analyzing stability regions that are not of infinite extent. A comparison of the two cases just presented tends to bear this out.

A Simple Reactor Case

To study further the effects of stability regions of infinite extent, consider the following set of equations

$$\frac{dx}{dt} = -2x + 2y$$

$$\frac{dy}{dt} = -x - xy . \quad (5-64)$$

The nature of the trajectories of this system in the x,y plane is shown in Fig. 5.6. Sketches of this type can usually be obtained with the aid of a method called analysis of singular points (Cunningham, 1958). The $(0,0)$ point is a stable focus, and the $(-1,-1)$ point is a saddle point. The entire plane above the $y = -1$ line is stable.

The L matrix is

$$\begin{bmatrix} -2 & 2 \\ -1 & 0 \end{bmatrix} \quad (5-65)$$

and the characteristic values are

$$S_1 = -1+i ; \quad S_2 = -1-i .$$

The canonical matrix, K , is

$$\begin{bmatrix} 1 & \frac{2}{S_1} \\ 1 & \frac{2}{S_2} \end{bmatrix} \quad (5-66)$$

and the inverse matrix, K^{-1} , is

$$\begin{bmatrix} \frac{S_1}{S_1 - S_2} & \frac{-S_2}{S_1 - S_2} \\ \frac{-S_1 S_2}{2(S_1 - S_2)} & \frac{S_1 S_2}{2(S_1 - S_2)} \end{bmatrix} \quad (5-67)$$

The canonical variables are

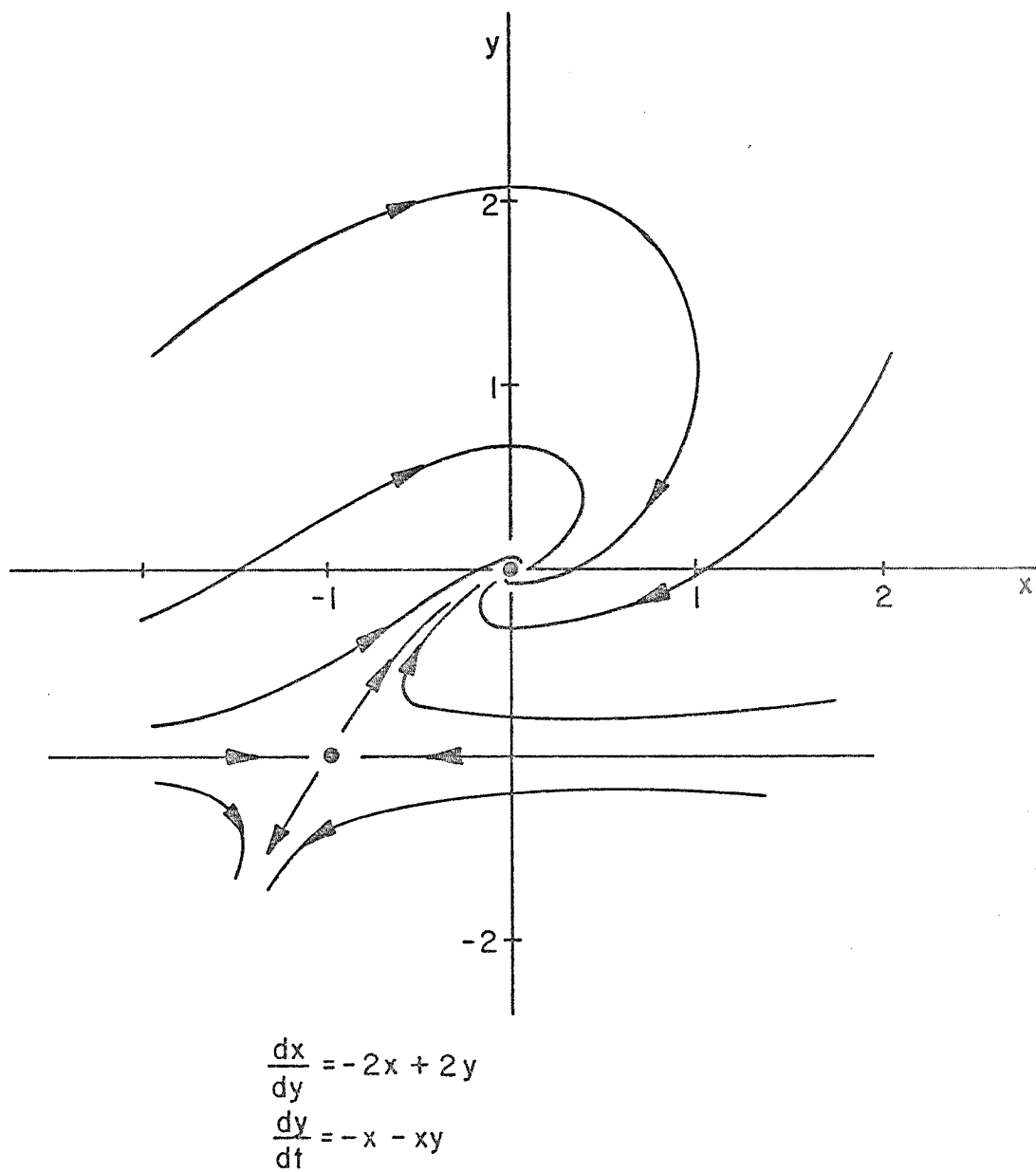


Figure 5.6 Solution Curves in State Space.

$$y_1 = x - (S_1 + 2)y$$

$$y_2 = x - (S_2 + 2)y \quad (5-68)$$

and they satisfy the differential equations

$$\dot{y}_1 = S_1 y_1 + (S_1 + 2)xy$$

$$\dot{y}_2 = S_2 y_2 + (S_2 + 2)xy \quad (5-69)$$

By means of the inverse transformation we obtain

$$\dot{y}_1 = S_1 y_1 - \frac{1}{2} y_1^2 + \frac{1+i}{2} y_1 y_2 - \frac{i}{2} y_2^2 \quad (5-70a)$$

$$\dot{y}_2 = S_2 y_2 + \frac{i}{2} y_1^2 + \frac{1-i}{2} y_1 y_2 - \frac{1}{2} y_2^2 \quad (5-70b)$$

If we substitute Eq. (5-29a) into Eq. (5-70a), remembering that

$y_2 = y_1^*$, we obtain

$$\begin{aligned} \dot{y}_1 = & A[-1+i]e^{i\theta} \\ & + A^2[(-1+i)U_2(\theta) - \frac{1}{2}e^{i2\theta} + \frac{1+i}{2}] \\ & + A^3[(-1+i)U_3(\theta) - e^{i\theta}U_2(\theta) + \frac{1+i}{2}e^{-i\theta}U_2(\theta) \\ & + \frac{1+i}{2}e^{i\theta}U_2^*(\theta) - ie^{-i\theta}U_2^*(\theta)] \\ & + A^4[(-1+i)U_4(\theta) - \frac{1}{2}e^{i\theta}U_3(\theta) - \frac{1}{2}U_2^2(\theta) + \frac{1+i}{2}e^{-i\theta}U_3(\theta) \\ & + \frac{1+i}{2}U_2(\theta)U_2^*(\theta) - \frac{i}{2}e^{-i\theta}U_3^*(\theta) - \frac{i}{2}(U_2^*(\theta))^2] \quad (5-71) \end{aligned}$$

By equating coefficients of like powers of A in Eqs. (5-41) and (5-71)

we obtain the differential equations that allow us to solve for the

$U_n(\theta)$. Then by using the auxiliary condition, Eq. (5-29b), we can evaluate the corresponding values of B_n and Ω_n .

The coefficients of A yield

$$S_1 = b + i\omega = -1 + i .$$

We have b negative and B_3 positive (to be shown), so this case is of the type (b) in Fig. 5.1.

For $O(A^2)$ we find that

$$\begin{aligned} U_2(\theta) = & (0.1, -0.3)e^{i\theta} + (0.25, 0.25)e^{i2\theta} \\ & + (0.15, 0.05)e^{-i2\theta} + (-0.5, 0.) \end{aligned}$$

and

$$\begin{aligned} B_2 + i\Omega_2 &= (0.1, -0.3) \\ B_3 + i\Omega_3 &= (0.27, -0.06) \\ A &= 1.75 , \end{aligned}$$

from which we find

$$\begin{aligned} y_1(A, \theta) = & 2.056 \cos \theta + 0.918 \sin \theta + 1.224 \cos 2\theta \\ & - 0.612 \sin 2\theta - 1.53 + i(-0.918 \cos \theta \\ & + 2.056 \sin \theta + 0.918 \cos 2\theta + 0.306 \sin 2\theta) . \end{aligned}$$

From the transformation represented by Eq. (5-67) we find that

$$\begin{aligned} x &= \text{Re}(y_1) - \text{Im}(y_1) = u - v \\ y &= -\text{Im}(y_1) = -v . \end{aligned} \tag{5-72}$$

This completes the solution for $O(A^2)$.

For $O(A^3)$ we have

$$\begin{aligned} U_3(\theta) = & (-0.125, 0.025)e^{i\theta} + (0.2, -0.1)e^{i2\theta} \\ & + (-0.0125, 0.0875)e^{i3\theta} + (-0.0125, -0.0875)e^{-i\theta} \\ & + (0., 0.1)e^{-i2\theta} + (0.05, -0.025)e^{-i3\theta} + (-0.1, 0.) \end{aligned}$$

and

$$B_3 + i\Omega_3 = (0.02, -0.01)$$

$$B_4 + i\Omega_4 = (0.021, 0.063)$$

$$A = 2.938 .$$

From the above we find that

$$\begin{aligned} y_1(A, \theta) = & 0.3142 \cos \theta - 0.2634 \sin \theta + 8.5248 \cos 2\theta \\ & + 3.3456 \sin 2\theta + 0.9510 \cos 3\theta - 2.8530 \sin 3\theta - 6.852 \\ & + i(-4.1746 \cos \theta + 0.9482 \sin \theta + 2.5896 \cos 2\theta \\ & + 5.9352 \sin 2\theta + 1.5850 \cos 3\theta - 1.5850 \sin 3\theta) . \end{aligned}$$

We obtain the final results by using Eq. (5-72).

The results of both orders of approximation are plotted in Fig. 5.7. Note that the boundary for $O(A^2)$ dips slightly into the unstable region, i.e., below the $y = -1$ line. The stable region for this approximation must therefore lie above $y = -1$ and within the plotted boundary. The $O(A^3)$ curve crosses into the stable region twice, indicating two stable regions for two ranges of the parameter θ . The reason for this behavior is that the higher order terms, corresponding to 2θ and 3θ , are dominating the first order terms. In other words, the perturbation terms are dominating the linear approximation. It is felt that in this case the solution is no longer valid, because the

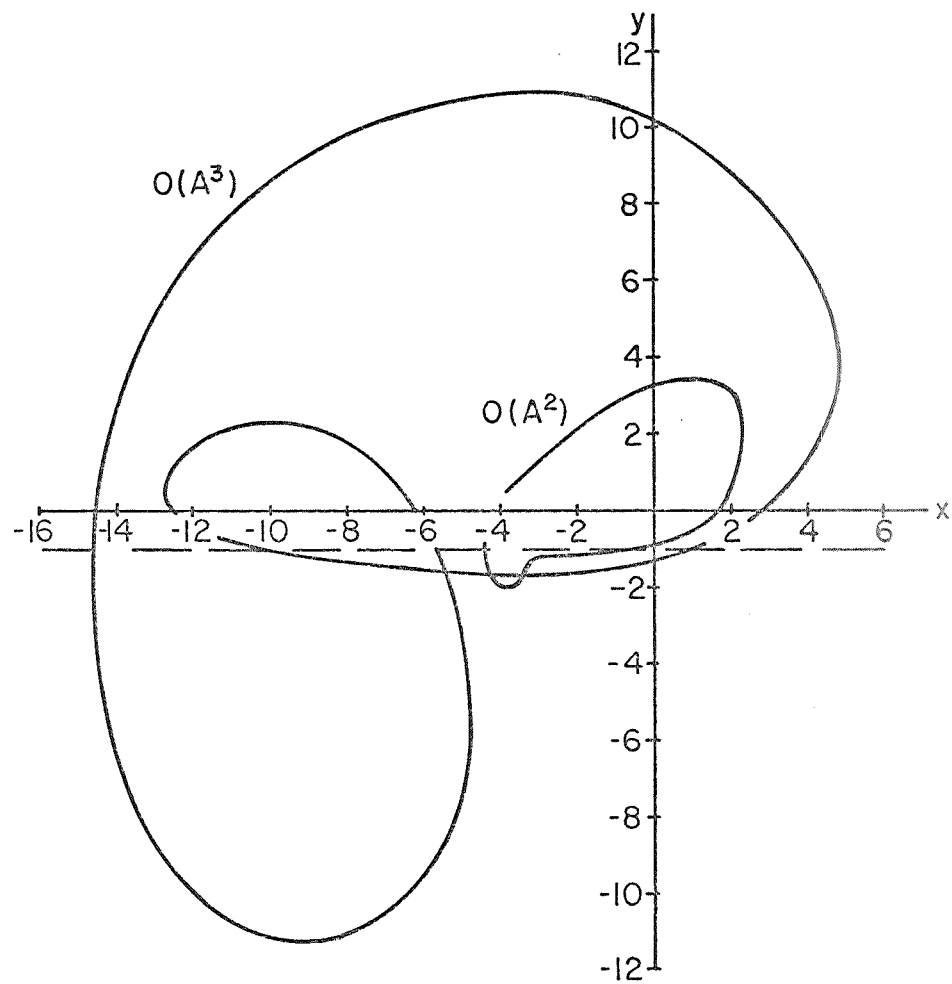


Figure 5.7 Approximations to the Stable Region.

underlying assumption in perturbation theories is that the perturbations remain small. When the combination of the magnitudes of A and the coefficients of the higher order terms is such that the perturbation terms are more important than the linear approximation, then the solution is no longer reliable. In cases where the stability regions are of infinite extent, it is speculated that the likelihood of A becoming large enough to cause the above effect would be greater than in cases of finite stability regions.

Summary

In Chapter 5 we have outlined the perturbation expansion method of Shotkin (1969) and worked one of his examples to verify our understanding of the method. We then applied it to a third-order system in which a nondominant coordinate was eliminated in terms of the dominant ones. Two cases were considered, (1) a stable limit cycle which represents a boundary of finite extent, and (2) a stable focus which has a stable domain of infinite extent. The method gave highly precise results for the finite region but not the other case. Another case of infinite region was studied in which the third-order approximation solution appears not to be valid. These results tend to indicate that the method is most effective with systems that are not globally asymptotically stable. We imply here that systems having stable domains of infinite extent are closely related to globally asymptotically stable systems with respect to the effectiveness of the method.

CHAPTER 6

CONCLUSIONS

The oscillation boundaries in parameter space for the linearized thermionic reactor model are most easily calculated using an algebraic condition on the roots of the characteristic equation. Much supplemental information can be obtained qualitatively by sketching root locus plots for the system. These boundaries complement the previously determined stability boundaries and are useful to the controls designer and in nonlinear stability analysis.

Stable and bounded regions in the effective lifetime model can be determined by making computer surveys of exact solutions. Such regions are defined by limiting planes in state space which can be described mathematically. The mathematical model agrees well with numerical results. All trajectories outside the stable and bounded regions exhibit finite escape time. On these trajectories, for large reactivity, the power is proportional to the square of the reactivity. Stable domains in state space cannot be completely determined by perturbing power alone, as was done here. Perturbations in both temperatures, as well as power, must be studied. Finally, although concentric limit cycles may exist in the effective lifetime model, they were not found in this study.

Stable regions of the prompt jump model can be determined by making computer surveys of exact solutions. The stable domain in state

space can be determined by considering perturbations in both temperatures as well as in power, but not perturbations in power alone.

The perturbation expansion technique of calculating instability bounds in linearly stable systems is a practical and straightforward method. It can often give results comparable to the Liapunov method, which is usually difficult to apply and requires varying amounts of ingenuity, depending on the particular problem. The method converges rapidly when the bounds are finite. It is not as precise for a condition of global stability, which implies an infinite number of expansion terms for a complete description. Regions of infinite extent share this property. Hence the method is better suited to the analysis of systems not having globally asymptotically stable regions, or infinite regions of bounded solutions. The formulation of the method used in this study is not the most general. It is not adequate for the analysis of systems at critical power or the resonance zone below critical power. The reason is that amplitude and phase angle of the perturbation are considered to be independent variables. This cannot be the case at critical power (undamped system), for the allowable amplitude depends on the phase angle (frequency) in a crucial way when the latter approaches the resonant frequency of the system. A more general formulation can be obtained by established methods. Obvious extensions of the work done here would be: (1) obtain this more general formulation, (2) apply the method, using the prompt jump model, to determine the effect of delayed neutrons, and (3) apply the method to higher order systems, e.g., one delay group model. In principle the method can be applied to systems of any order, although the algebra may become prohibitive.

APPENDIX

DIGITAL COMPUTER PROGRAM

The digital computer program consists of the main program, DIFEQN, and the subroutine, GSVDQ. The subroutines used in plotting output data are part of the computer system, and not supplied by the user. Hence they will not be discussed here.

The main program provides for the input and output of data, and the preparation of data for plotting by calling the plot subroutines. The user selects either the prompt jump or effective lifetime neutron kinetics model by setting the parameter PJ to 1 or 0, respectively. The temperature differential equations are the coupled ones for the thermionic reactor model. The temperatures are scaled (within the program) to increase the integration speed. The numerical integration of the differential equations is performed in subroutine GSVDQ.

GSVDQ was written by F. T. Krogh, Jet Propulsion Laboratory, Section 314, Pasadena, California, May 1, 1969. It was adapted for use on the CDC 6400 computer of the University of Arizona Computer Center by J. G. Guppy and K. D. Kearns, September, 1969.

```

C      PROGRAM DIFEQN(INPUT, OUTPUT, TAPE 6)
      EFFECTIVE LIFETIME  NONLINEAR
      COMMON/MG1/Y(10),F(10),KD(10),EP(10),YN(10),DT(20,10),
      *KQ(10)
      COMMON/MG2/NEQ,T,TFINAL,DELT,IFLAG,NPRINT
      COMMON/MG3/H,HMINA,HMAXA,MXSTEP,KSTEP,KEMAX,EMAX
      DIMENSION XN(4000),XT1(4000)
      DATA HMINA,HMAXA,MXSTEP,PJ/0.001,1.000,1000,0./
      CALL INITIAL(8H41003625,6,0,0,0)
      1 READ 2, B,ALPH1,ALPH2,ETA1,ETA2,ETA3,BETA,XLAMB,H,DELT
      *,T,PZERO,TFINAL,Y(1),Y(2),PWR
      2 FORMAT (8E10.5)
      IF ( B )3,3,4
      4 NEQ = 3
      KO = 0
      DO 5 I = 1,NEQ
      KD(I) = 1
      5 EP(I) = 0.0001
      PRINT 6
      6 FORMAT (//16X1HB15X5HALPH113X5HALPH214X4HETA113X4HETA2
      *14X4HETA313X4HBETA)
      PRINT 7,B,ALPH1,ALPH2,ETA1,ETA2,ETA3,BETA
      7 FORMAT (/2X,7(5XE13.5))
      PRINT 8
      8 FORMAT (//14X6HLAMBDA6X17HEQUILIBRIUM POWER3X13HINITIA
      *L POWER5X13HINITIAL TEMP15X13HINITIAL TEMP211X2HPJ)
      PRINT 9, XLAMB,PZERO,PWR,Y(1),Y(2),PJ
      9 FORMAT (/2X,6(5XE13.5))
      PRINT 10
      10 FORMAT(//9X4H1ME15X5HPOWER13X5HT ONE13X5HT TWO11X9HS
      *TEP SIZE7X5HKSTEP9X3HRRH0//)
      Y(1) = Y(1)/100.
      Y(2) = Y(2)/100.
      YZERO = ALOG(PZERO)
      Y(3) = ALOG(PWR)
      RHO = (100./BETA)*((-ALPH1)*Y(1)-ALPH2*Y(2))

```

```

U1 = ALPHA*(PJ*ETA1-XLAMB)-PJ*ALPH2*ETA2
U2 = ALPH2*(PJ*ETA3-XLAMB)
U3 = (-PJ)*ALPH1*B
CALL GSVDQ
GO TO 30
20 CALL GSVDQ1
30 GO TO(40,40,50,50,60,60,60,60), IFLAG
40 DELT = H
V = BETA*(1.-PJ*RHO)
U = U3*(EXP(Y(3))-EXP(YZERO))+100.*U1*Y(1)+100.*U2*Y(2
*)
F(1) = B*(EXP(Y(3))-EXP(YZERO))/100.-ETA1*Y(1)
F(2) = ETA2*Y(1)-ETA3*Y(2)
F(3) = U/V
RHO = (100./BETA)*((-ALPH1)*Y(1)-ALPH2*Y(2))
T1 = Y(1)*100.
T2 = Y(2)*100.
PWR = EXP(Y(3))
GO TO 20
50 PRINT 11, T,PWR,T1,T2,H,KSTEP,RHO
11 FORMAT (5(5XEL3.5),5X15,5XF10.6)
K0 = K0 + 1
XN(K0) = PWR
XT1(K0) = T1
IF (IFLAG.EQ.3) GO TO 20
IF (IFLAG.EQ.4) GO TO 12
GO TO 1
12 CALL PLOT(2,-1,-3)
CALL PLOT(5,2,-3)
CALL SCALE(XN,7,K0,1,10.)
CALL SCALE(XT1,5,K0,1,10.)
CALL AXIS(0,0,1H 1,7,90.,XN(K0+1),XN(K0+2),10.)
CALL AXIS(0,0,1H -1,5,90.,XT1(K0+1),XT1(K0+2),10.)
CALL LINE(XT1,XN,K0,1,0,1)
GO TO 1
60 PRINT 61, IFLAG

```

```
61 FORMAT (/10X17HBLEW IT   IFLAG =,I2)
   GO TO 50
   3 CALL PLOT(0.,0.,999)
     END
```

```

SUBROUTINE GSVDDQ
  DIMENSION DD(20), D(19), GAM(15,4), GAS(15), PT(16), FAC(3)
  DIMENSION ETA(13,13), ETA1(13), ETA2(13), ETA4(13), ETA5(13),
  1 ETA6(13), ETA7(13), ETA8(13), ETA9(13), ETA10(13), ETA11(13),
  2 ETA12(13), ETA13(13), ETA3(13)
  DIMENSION GAM1(15), GAM2(15), GAM3(15), GAM4(15)
  COMMON/MG1/ Y(10),F(10),KD(10),EP(10),YN(10),DT(20,10),KQ(10)
  COMMON/MG2/ NEQ,I,TFINAL,DELT,IFLAG,NPRINT
  COMMON/MG3/ H,MINA,HMAXA,MXSTEP,KSTEP,KEMAX,EMAX
  DATA KMAX0/4/
  DATA FAC/1.0E0,.5E0,.166666667E0/
  DATA KQVAX/14/
  DATA RND,KBIT2/2.84E-15,98/
  DATA P1,P01,P25,P3E1/.1,.01,.25,3./
  DATA PT/1.,2.,4.,8.,16.,32.,64.,128.,256.,512.,1024.,2048./
  1 4096.,8192.,16384.,32768./
  DATA GAS/1.,.5,.8,333333333333333E-02,
  1 -4.1666666666666E-02, -2.6388888888888E-02,
  2 -1.8750000000000E-02, -1.4269179894179E-02,
  3 -1.1367394179894E-02, -9.3565365961199E-03,
  4 -7.8925540123456E-03, -6.7858499846347E-03,
  5 -5.9240564123376E-03, -5.2366932579502E-03,
  6 -4.6774984070422E-03, -4.2149522390054E-03/
  DATA (GAM1(I), I= 1,15)/ 1.,.5,.41666666666667,.375,
  1 3.4861111111111E-01, 3.2986111111111E-01,
  2 3.1559193121693E-01, 3.0422453703703E-01,
  3 2.948680004091E-01, 2.8697544642857E-01,
  4 2.8018959644393E-01, 2.7426554003159E-01,
  5 2.6902884677364E-01, 2.6435134836661E-01,
  6 2.6013639612760E-01/
  DATA (GAM2(I), I= 1,15)/
  1 .5,.16666666666667,.125,.105555555555556,
  2 9.3750000000000E-02, 8.5615079365079E-02,
  3 7.9571759259259E-02, 7.4852292768959E-02,
  4 7.1032986111111E-02, 6.7858499846347E-02,
  5 6.5164620535714E-02, 6.2840319095403E-02,

```

6	6.0807479291549E-02,	5.90093331346077E-02,	SVDD
7	5.7403493298178E-02/		SVDD
	DATA (GAM3(I), I= 1,15)/		SVDD
1	1.66666666666667E-01,	4.16666666666667E-02,	SVDD
2	2.91666666666667E-02,	2.36111111111111E-02,	SVDD
3	2.0337301587301E-02,	1.8129960317460E-02,	SVDD
4	1.6518132716049E-02,	1.5277226631393E-02,	SVDD
5	1.4285188191438E-02,	1.3469396553164E-02,	SVDD
6	1.2783657921710E-02,	1.2197038823124E-02,	SVDD
7	1.1687961645573E-02,	1.1240866335288E-02,	SVDD
8	1.0844218294347E-02/		SVDD
	DATA (GAM4(I), I= 1,15)/		SVDD
1	4.16666666666667E-02,	8.33333333333333E-03,	SVDD
2	5.55555555555556E-03,	4.3650793650794E-03,	SVDD
3	3.6789021164021E-03,	3.2236552028219E-03,	SVDD
4	2.8954475308642E-03,	2.6454358398803E-03,	SVDD
5	2.4473749148228E-03,	2.2857954381805E-03,	SVDD
6	2.1509366948148E-03,	2.0363087102021E-03,	SVDD
7	1.9374130112343E-03,	1.8510261910607E-03,	SVDD
8	1.7747637472430E-03/		SVDD
	DATA (EIA1(I), I= 1,13)/		SVDD
1	1.13636363636363E-01,	3.33333333333333E-01,	SVDD
2	4.6052631578947E-02,	6.7307692307692E-02,	SVDD
3	2.7138157894737E-02,	3.4374999999999E-02,	SVDD
4	1.8748457551826E-02,	2.2254731075697E-02,	SVDD
5	1.4060299565271E-02,	1.6112321954813E-02,	SVDD
6	1.1080216892449E-02/	1.2419706453289E-02,	SVDD
	DATA (EIA2(I), I= 1,13)/		SVDD
1	3.4090909090909E-01,	2.E-01,	SVDD
2	1.3815789473684E-01,	2.0192307692308E-01,	SVDD
3	8.1414473684211E-02,	1.0312500000000E-01,	SVDD
4	5.6245372655479E-02,	6.6764193227092E-02,	SVDD
5	4.2180898695812E-02,	4.833696586440E-02,	SVDD
6	3.3240650677347E-02/	3.7259119359868E-02,	SVDD
	DATA (EIA3(I), I= 1,13)/		SVDD
1	3.4285714285714E-01,	1.4285714285714E-01,	SVDD
		2.8571428571429E-01,	SVDD
		3.4615384615385E-01,	SVDD

2	2.4561403508772E-01,	1.8750000000000E-01,	SVDQ
3	1.5030364372469E-01,	1.2462649402390E-01,	SVDQ
4	1.0587364264561E-01,	9.1585830058939E-02,	SVDQ
5	8.0344568944404E-02,	7.1278315297139E-02,	SVDQ
6	6.3822049300507E-02/		SVDQ
	DATA (ETA4(I), I= 1,13)/	1.1111111111111E-01, 2.2222222222222E-01,	SVDQ
1	2.8571428571429E-01,	2.5396825396825E-01,	SVDQ
2	3.0701754385965E-01,	2.5000000000000E-01,	SVDQ
3	2.0875506072874E-01,	1.7803784860558E-01,	SVDQ
4	1.5439906219151E-01,	1.3568271119843E-01,	SVDQ
5	1.2051685341661E-01,	1.0799744741961E-01,	SVDQ
6	9.7505908653552E-02/		SVDQ
	DATA (ETA5(I), I= 1,13)/	9.0909090909091E-02, 1.8181818181818E-01,	SVDQ
1	2.4242424242424E-01,	2.4242424242424E-01,	SVDQ
2	1.7316017316017E-01,	2.5000000000000E-01,	SVDQ
3	2.2773279352227E-01,	2.0542828685259E-01,	SVDQ
4	1.8527887462981E-01,	1.6760805500982E-01,	SVDQ
5	1.5223181484203E-01,	1.3885386096845E-01,	SVDQ
6	1.2718161998289E-01/		SVDQ
	DATA (ETA6(I), I= 1,13)/	7.6923076923076E-02, 1.5384615384615E-01,	SVDQ
1	2.0979020979020E-01,	2.2377622377622E-01,	SVDQ
2	1.8648018648018E-01,	1.1188811188811E-01,	SVDQ
3	1.9129554655870E-01,	1.9173306772908E-01,	SVDQ
4	1.8527887462981E-01,	1.7598845776031E-01,	SVDQ
5	1.6576353171687E-01,	1.5551632428466E-01,	SVDQ
6	1.4568076470767E-01/		SVDQ
	DATA (ETA7(I), I= 1,13)/	6.6666666666667E-02, 1.3333333333333E-01,	SVDQ
1	1.8461538461538E-01,	2.0512820512821E-01,	SVDQ
2	1.8648018648019E-01,	1.3426573426573E-01,	SVDQ
3	6.9619269619269E-02,	1.3944223107570E-01,	SVDQ
4	1.5202369200395E-01,	1.5643418467583E-01,	SVDQ
5	1.5601273573353E-01,	1.5278796771827E-01,	SVDQ
6	1.4799315779828E-01/		SVDQ
	DATA (ETA8(I), I= 1,13)/	5.8823529411765E-02, 1.1764705882353E-01,	SVDQ
1	1.6470588235294E-01,	1.8823529411765E-01,	SVDQ
2	1.8099547511312E-01,	1.4479638009050E-01,	SVDQ

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3 9.2143150966680E-02, 4.2122583299054E-02, SVDQ
4 9.7729516288253E-02, 1.1493123772102E-01, SVDQ
5 1.2536737692873E-01, 1.3096111518709E-01, SVDQ
6 1.3319384201845E-01, SVDQ
DATA (ETA9(I), I= 1,13)/
1 1.4860681114551E-01, 5.2631578947368E-02, 1.0526315789474E-01, SVDQ
2 1.7337461300310E-01, 1.7337461300310E-01, SVDQ
3 1.4860681114551E-01, 1.4860681114551E-01, SVDQ
4 1.0669206954037E-01, 6.0966896880210E-02, SVDQ
5 2.4941003269177E-02, 6.6306483300589E-02, SVDQ
6 8.3578251285819E-02, 9.6294937637564E-02, SVDQ
DATA (ETA10(I), I=1,13)/
1 1.0515303317246E-01, SVDQ
2 4.7619047619048E-02, 9.6238095238095E-02, SVDQ
3 1.3533834586466E-01, 1.6040100250627E-01, SVDQ
4 1.6511867905057E-01, 1.4860681114551E-01, SVDQ
5 1.1558307533539E-01, 7.5482824708831E-02, SVDQ
6 3.9193005137278E-02, 1.4515927828621E-02, SVDQ
7 4.3779084006858E-02, 5.8846906334067E-02, SVDQ
DATA (ETA11(I), I=1,13)/
1 1.2422360248447E-01, 4.3478260869565E-02, 8.6956521739130E-02, SVDQ
2 1.5691402419091E-01, 1.4906832298137E-01, SVDQ
3 1.2060842643694E-01, 1.4645308924485E-01, SVDQ
4 5.1689325615829E-02, 8.6148876026383E-02, SVDQ
5 8.3308803190348E-03, 2.4613964578967E-02, SVDQ
6 4.0320117294887E-02, 2.8246515040352E-02, SVDQ
DATA (ETA12(I), I=1,13)/
1 1.1478260869565E-01, 4.0000000000000E-02, 8.0000000000000E-02, SVDQ
2 1.4906832298137E-01, 1.3913043478261E-01, SVDQ
3 1.2302059496568E-01, 1.4310559006212E-01, SVDQ
4 6.2027190738996E-02, 9.3729977116705E-02, SVDQ
5 1.5162202180643E-02, 3.4459550410553E-02, SVDQ
6 1.7869142892052E-02, 4.7258811991616E-03, SVDQ
DATA (ETA13(I), I=1,13)/
1 1.0666666666667E-01, 3.7037037037037E-02, 7.4074074074074E-02, SVDQ
2 1.4170692431562E-01, 1.3037037037037E-01, SVDQ
3 1.2367149758454E-01, 1.3913043478261E-01, SVDQ
9.6937198067633E-02, 9.6937198067633E-02, SVDQ

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4 7.0297482837529E-02, 4.3393507924400E-02, SVDQ
5 2.2462521749101E-02, 9.1892134428142E-03, SVDQ
6 2.6546616612574E-03/ SVDQ
EQUIVALENCE (D(2),D(1)) SVDQ00208
EQUIVALENCE (PT(1),PTS1),(PT(2),PTS2),(PT(3),PTS3),(PT(4),PTS4), SVDQ00242
1 (PT(5),PTS5),(GAM(2,1),P5) SVDQ00243
DO 11 MV=1,15 SVDQ
GAM(MM,1)= GAM1(MM) SVDQ
GAM(MM,2)= GAM2(MM) SVDQ
GAM(MM,3)= GAM3(MM) SVDQ
GAM(MM,4)= GAM4(MM) SVDQ
11 CONTINUE SVDQ
DO 5 KEN=1,13
ETA(KEN,1)= ETA1(KEN)
ETA(KEN,2)= ETA2(KEN)
ETA(KEN,3)= ETA3(KEN)
ETA(KEN,4)= ETA4(KEN)
ETA(KEN,5)= ETA5(KEN)
ETA(KEN,6)= ETA6(KEN)
ETA(KEN,7)= ETA7(KEN)
ETA(KEN,8)= ETA8(KEN)
ETA(KEN,9)= ETA9(KEN)
ETA(KEN,10)= ETA10(KEN)
ETA(KEN,11)= ETA11(KEN)
ETA(KEN,12)= ETA12(KEN)
ETA(KEN,13)= ETA13(KEN)
5 CONTINUE
ERND= 0.0
KEMAX= 0
LFD= 0
E2HMAX= 0.0
KSTEP= 7
NE= NEG
IF(NE.LE.0) GO TO 1190
HH= H
NV= 0
SVDQ00309
SVDQ00310
SVDQ00311
SVDQ00312
SVDQ00313

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KDMAX = 0
KDD = KD(1)
KDS = KDD
DO 10 I=1,NE
  KQ(I) = 1
  DT(1,I) = 0.
  IF(KDS.LE.0) KDD = IABS(KD(I))
  IF((KDD.EQ.0).OR.(KDD.GT.KMAX0)) HH = 0.E0
  IF(KDD.GT.KDMAX) KDMAX = KDD
10 NV = NV + KDD
  IF((DELT*HH).LE.0.E0) GO TO 1190
  ERMX = P1
  EMAX = ERND
  RNDC = RND * P25
  LDOUB = 0
  E2HFAC = P25
  LSC = 8
  LSTC = 4
  KSOUT = MXSTEP
  TOUT = T
  IFL = 13
  IFLAG = 1
20 RETURN
  ENTRY GSVDQ1
  IF(2-IFL) 30,60,320
30 IF(IFL.GT.5) GO TO 1180
  KSOUT = KSTEP + MXSTEP
  IF(IFL-4) 40,1210,210
40 TOUT = T + DELT
50 TPS1 = ABS(AMOD((TOUT-T)/HH,PTS2)-PTS1)
  LFD = -1
  IF(TPS1.GE.P5) LFD = 1
  GO TO 200
60 KQM = 0
  DO 80 I=1,NE
    KQ(I) = KQ(I)

```

SVDQ00314
SVDQ00315
SVDQ00316
SVDQ00317
SVDQ00318
SVDQ00319
SVDQ00320
SVDQ00321
SVDQ00322
SVDQ00323
SVDQ00325
SVDQ00326
SVDQ00327
SVDQ00328
SVDQ00329
SVDQ00330
SVDQ00331
SVDQ00332
SVDQ00349
SVDQ00350
SVDQ00351
SVDQ00352
SVDQ00353
SVDQ00400
SVDQ00401
SVDQ00402
SVDQ00405
SVDQ00406
SVDQ00410
SVDQ00412
SVDQ00413
SVDQ00414
SVDQ00422
SVDQ00430
SVDQ00431
SVDQ00432

```

IF(KQQ.GT.KQM) KQM = KQQ
D(1) = F(I)
DO 70 K=1,KQQ
D(K+1) = D(K) - DT(K,I)
70 DT(K,I) = D(K)
DT(KQQ+1,I) = D(KQQ+1)
80 CONTINUE
DO 90 J=1,NV
YN(J) = Y(J)
90 CONTINUE
LFD = - LFD
TL = T
KSTEP = KSTEP + 1
110 IF(LSC.EQ.0) GO TO 140
LSC = LSC - LSTC
IF(LSC.EQ.0) GO TO 130
IF(LSTC.NE.(-1)) GO TO 140
IF(LDOUB.LT.0) RNDC = RND * P1
120 E2HAVE = E2HMAX
TPS1 = TPS1
GO TO 190
130 IF(ABS(HH).LT.HMINA) GO TO 1000
LSTC = 0
140 IF(LDOUB.NE.1) GO TO 150
IF((LFD.GT.0).AND.(ABS(HH+HH).LE.HMAXA)) GO TO 1030
GO TO 200
150 RQMAX = TPS1/FLOAT(KQM+3)
IF(LSTC.NE.0) GO TO 120
IF(ABS(E2HAVE).GT.1.0E-10) GO TO 155
TPS1 = E2HMAX/1.0E-10
GO TO 156
155 TPS1 = E2HMAX/E2HAVE
156 IF(TPS1-TPS1) 160,190,170
160 E2HFAC = AMAX1(.075E0,E2HFAC-RQMAX,E2HFAC*TPS1)
GO TO 180
170 TPS1 = TPS1*TPS1

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SVDQ00433
SVDQ00434
SVDQ00435
SVDQ00436
SVDQ00437
SVDQ00438
SVDQ00439
SVDQ00443
SVDQ00444
SVDQ00445
SVDQ00446
SVDQ00447
SVDQ00448
SVDQ00453
SVDQ00454
SVDQ00455
SVDQ00456
SVDQ00457
SVDQ00458
SVDQ00459
SVDQ00460
SVDQ00461
SVDQ00462
SVDQ00463
SVDQ00464
SVDQ00465
SVDQ00466
SVDQ00467

SVDQ00468
SVDQ00469
SVDQ00470
SVDQ00471
SVDQ00472

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E2HFAC = AMIN1(PTS1,E2HFAC*TPS1)
180 RNDC = (1.1-E2HFAC)*RND
E2HAVE = P5 * (E2HMAX+E2HAVE)
190 ERRMX = AMAX1(P1,ERRMX-RQMAX*TPS1)
200 TPD = (YOUT-TL)/HH
TPD1 = (TFINAL-TL)/HH
IF(TPD1.LI.FAC(1)) GO TO 1220
IF(TPD.LE.0.E0) GO TO 1280
IF(KSOUT.GT.KSTEP) GO TO 210
IFL = 5
GO TO 310
210 IF(EMAX.EQ.ERND) GO TO 220
IFL = 6
IF(EMAX.GE.P1) GO TO 310
IF((LSFC.GE.0).OR.(LD0UB.EQ.1)) ERRMX = PTS1
220 IFL = 1
230 T = TL + HH
240 J = 0
DO 290 I=1,NE
IF(KD.LE.0) KDD = IABS(KD(I))
KDC = KDD
250 KQ0 = KC(I)
TPD = 0.E0
K = KDC
260 TPD = TPD + DT(KQ0,I) * GAM(KQ0,KDC)
KQ0 = KQ0 - 1
IF(KQ0.GT.0) GO TO 260
270 K = K - 1
IF(K.LE.0) GO TO 280
L = J + K
TPD = YN(L+1)*FAC(K)+HH*TPD
GO TO 270
280 J = J + 1
Y(J) = YN(J) + HH*TPD
KDC = KDC - 1
IF(KDC.GT.0) GO TO 250
SVDQ00473
SVDQ00474
SVDQ00475
SVDQ00476
SVDQ00489
SVDQ00490
SVDQ00494
SVDQ00495
SVDQ00498
SVDQ00500
SVDQ00501
SVDQ00504
SVDQ00506
SVDQ00507
SVDQ00508
SVDQ00510
SVDQ00511
SVDQ00516
SVDQ00517
SVDQ00518
SVDQ00519
SVDQ00520
SVDQ00521
SVDQ00522
SVDQ00523
SVDQ00524
SVDQ00525
SVDQ00526
SVDQ00527
SVDQ00528
SVDQ00529
SVDQ00530
SVDQ00531
SVDQ00532
SVDQ00533
SVDQ00534

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290 CONTINUE
   IF (IFL) 20,320,300
300 CONTINUE
310 IFLAG = IFL
   IF (IFLAG.GE.4) GO TO 1430
   RETURN
320 EPS = EP(1)
   ERND = 0.
   EMAX = 0.
   EZHMAX = 0.
   J = 0
   IF (LDOUB.GE.0) LDOUB = 1
   DO 790 I=1,NE
   IF (KDS.LE.0) KDD = IABS(KD(I))
   KQ = KG(I)
   KQ1 = KQ + 1
   D(1) = F(I)
   DO 330 K=1,KQ1
   D(K+1) = D(K) - DT(K,I)
330 CONTINUE
   TPS3 = ABS(0(KQ+1))
   IF (LDOUB.LT.0) GO TO 720
340 IF (KQ.NE.1) GO TO 520
   EZH = PTS2
   TPS5 = DT(3,I)
   IF (LSIC.LT.2) GO TO 370
   IF (LSIC.NE.4) GO TO 350
   TPS4 = 0.
   IF (KDD.GT.1) TPS3 = AMAX1(TPS3,ABS(HH*D(1)))
   TPS3 = TPS3 * P1
   GO TO 510
350 DT(2,I) = D(2)
   D(2) = D(1) - DT(5,I)
   TPS2 = -D(2)
   TPS3 = PTS5 * ABS(TPS2)
360 DT(7,I) = PT(4)

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SVDQ00535
 SVDQ00540
 SVDQ00541
 SVDQ00546
 SVDQ00551
 SVDQ00552
 SVDQ00556
 SVDQ00557
 SVDQ00558
 SVDQ00559
 SVDQ00560
 SVDQ00561
 SVDQ00590
 SVDQ00591
 SVDQ00592
 SVDQ00596
 SVDQ00597
 SVDQ00599
 SVDQ00600
 SVDQ00601
 SVDQ00604
 SVDQ00605
 SVDQ00607
 SVDQ00610
 SVDQ00611
 SVDQ00612
 SVDQ00614
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 SVDQ00616
 SVDQ00617
 SVDQ00618
 SVDQ00619
 SVDQ00620
 SVDQ00621
 SVDQ00622
 SVDQ00624

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IF (LSTC-2) 420,380,380
370 IF (TPS5.EQ.0.) GO TO 360
IF (DT(6,I).EQ.0.) GO TO 400
TPS2 = DT(5,I) - DT(1,I)
380 TPS4 = DT(4,I)
TPS1 = ABS(TPS4)
TPS4 = IPS2*SIGN(PTS2,TPS4) - TPS5*TPS1
IF (TPS4.GT.(-TPS1)) GO TO 410
390 TPS6 = -PTS1
GO TO 450
400 DT(6,I) = PT(1)
TPS6 = 0.
GO TO 450
410 IF (TPS4.LT.TPS1) GO TO 440
IF (TPS1.EQ.0.) GO TO 390
420 TPS6 = PTS1
GO TO 450
430 KQ(I) = 2
IF (2-LSTC) 510,510,520
440 TPS6 = TPS4/TPS1
450 TPS4 = TPS5 + TPS6
IF (TPS4.LT.P25) GO TO 430
E2H = PTS4*TPS4
IF (2-LSTC) 460,470,480
460 LSC = 0
GO TO 510
470 IF (TPS5-P25) 430,460,460
480 IF (TPS4.GT.PTS2) GO TO 490
IF (TPS4.GT.P5) D(2) = D(2)*GAM(2,1)
GO TO 510
490 IF (TPS4.LT.PTS4) GO TO 500
TPS4 = PTS4
D(2) = D(2)/PT(3)
TPS3 = TPS3*DT(7,I)
GO TO 510
500 D(2) = D(2)*PTS2*(TPS4-PTS1)/(TPS4*TPS4)

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SVDQ00625
SVDQ00626
SVDQ00627
SVDQ00628
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SVDQ00659
SVDQ00661
SVDQ00662
SVDQ00663

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IF(TPS4*GE*P3E1) E2H = E2H*DT(7,I)
510 DT(5,I) = D(1)
    DT(4,I) = D(2)
    D(4) = TPS4
    DT(3,I) = D(4)
520 KDC = 0
    TPD = D(KQ1)
    J = J + KDD
    K = J
530 TPD = HM * TPD
    KDC = KDC + 1
    Y(K) = Y(K) + GAM(KQQ*1,KDC)*TPD
    K = K - 1
    IF(KDC*LT*KDD) GO TO 530
    IF(EPS) 540,550,560
540 EPS = EP(I)
    IF(EPS*NE*0.) GO TO 560
550 IF(HMAXA) 1190,780,1190
560 TPS4 = ABS(D(KQQ*2))
    TPS2 = ABS(D(KQQ))
    TPS6 = HM/EPS
    E = ABS(GAS(KQQ*1)*TPS3*TPS6)
    LRND = 1
    FRND = PT(KQQ*2)*RND*ABS(D(1))
    IF((TPS3*TPS4)*GT*FRND) GO TO 570
    LRND = 0
    IF((PTS4*TPS2)*LT*FRND) LRND = -1
570 IF(E*LE*ERND) GO TO 590
    IF(E*LE*EMAX) GO TO 580
    EMAX = E
    KEMAX = I
580 IF(LRND*LE*0) GO TO 590
    ERND = E
    IF(ERND*GT*ERRMX) LDOUB = 0
590 IF(LDOUB*LE*0) GO TO 780
    TPS1 = ABS(D(KQQ))

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SVDQ0664
SVDQ0668
SVDQ0669
SVDQ0670
SVDQ0679
SVDQ0682
SVDQ0683
SVDQ0684
SVDQ0685
SVDQ0686
SVDQ0687
SVDQ0688
SVDQ0689
SVDQ0690
SVDQ0693
SVDQ0694
SVDQ0695
SVDQ0696
SVDQ0697
SVDQ0698
SVDQ0699
SVDQ0701
SVDQ0704
SVDQ0710
SVDQ0712
SVDQ0713
SVDQ0714
SVDQ0716
SVDQ0717
SVDQ0718
SVDQ0719
SVDQ0720
SVDQ0721
SVDQ0722
SVDQ0723
SVDQ0724

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        TPS5 = TPS1
        IF(KQQ-2) 600,610,620
        E2H = E*E2H
600      IF(E2H.LI.P01) GO TO 780
        IF(D(4).LI.P3E1) GO TO 770
        LSTC = - 1
        LSC = - 5
        GO TO 770
610      TPS1 = TPS2
        IF(LSTC.NE.2) GO TO 620
        KQ(I) = 3
        TPS2 = 0.
        TPS4 = 0.
        LRND = 0
        E2H = TPS2 + TPS3 + TPS4
        E2H = ABS(GAS(KQQ-1)*PT(KQQ+1)*E2H*TPS6)
        IF(E2H.GT.E2HMAX) E2HMAX = E2H
        IF(LRND) 630,640,660
630      K = (KBIT2/KQQ) - 4
        IF(K.LE.3) GO TO 640
        IF(K.GT.KQMAX) K = KQMAX
        E2H = E2H/PT(K+1)
        GO TO 650
640      E2H = AMIN1(E2H,E2H*P3E1*E2HFAC)
650      E2H = E2H*P1
        TPS6 = PTS4
        GO TO 670
660      E2H = E2H*E2HFAC
        TPS6 = FLOAT(KQQ+2)
670      IF(IPS5.LI.(IPS3*TPS6)) GO TO 680
        IF(IPS2.LE.(IPS4*TPS6)) GO TO 760
        IF(KQQ.NE.KQMAX) KQ(I) = KQ1
        GO TO 760
680      TPS6 = IPS6*P25
        IF((IPS1.GT.(IPS3*TPS6)).OR.(IPS2.GT.(IPS4*TPS6))) GO TO 760
        IF(LSTC.LE.0) GO TO 750

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SVDQ0725
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SVDQ0751
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SVDQ0755
SVDQ0757
SVDQ0758
SVDQ0761
SVDQ0762
SVDQ0764
SVDQ0765
SVDQ0768
SVDQ0769
SVDQ0771

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```

IF(E2H.LT.P01) GO TO 750
IF(LSC-LSTC) 690,750,770
690 IF(KSTEP-4) 750,700,710
700 KQ1 = LSTC
710 LSC = KQ1
GO TO 770
720 IF(LDOUB.EQ.(-2)) GO TO 340
DT(KQQ+1,I) = D(KQQ+1)
IF(LDOUB.EQ.(-1)) DT(KQQ+1,I) = D(KQQ+2)
K = KQQ
730 IF(K.EQ.1) GO TO 740
IF((ABS(D(K-1)).GT.(PT(2)*ABS(D(K+1))))).OR.
1 (ABS(D(K)).GT.(PT(2)*ABS(D(K+2)))) GO TO 740
K = K - 1
GO TO 730
740 IF((K+K).GE.KQQ) GO TO 780
LDOUB = - 4
E2H = 0.
KQQ = K + 1
750 KQ(I) = KQQ - 1
IF(KQQ.EQ.2) DT(3,I) = 0.
760 IF(E2H.LT.P01) GO TO 780
770 LDOUB = 2
780 CONTINUE
790 CONTINUE
IF(IFL.LT.0) GO TO 1250
IF(LDOUB) 800,950,870
800 LDOUB = LDOUB + 1
IF(LDOUB+1) 810,870,820
810 IF(LDOUB.EQ.(-2)) GO TO 820
LDOUB = 0
GO TO 220
820 DO 860 I = 1,NE
KQQ = KC(I)
IP = DT(KQQ+1,I)
IF(KQQ.LE.3) GO TO 860
SVDQ00772
SVDQ00773
SVDQ00774
SVDQ00775
SVDQ00776
SVDQ00778
SVDQ00781
SVDQ00782
SVDQ00783
SVDQ00784
SVDQ00785
SVDQ00786
SVDQ00787
SVDQ00788
SVDQ00789
SVDQ00790
SVDQ00791
SVDQ00792
SVDQ00793
SVDQ00797
SVDQ00798
SVDQ00799
SVDQ0800
SVDQ0801
SVDQ0811
SVDQ0818
SVDQ0820
SVDQ0821
SVDQ0822
SVDQ0823
SVDQ0825
SVDQ0826
SVDQ0827
SVDQ0828
SVDQ0829
SVDQ0830

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      IF (LDOUT*NE.0) GO TO 840
      DO 830 K = 3,KQQ
      DT(K,I) = DT(K,I) + ETA(KQQ-1,K-2)*IP
830  CONTINUE
      GO TO 860
840  DO 850 K = 2,KQQ
      DT(K,I) = DT(K,I) + ETA(K-1,KQQ-1)*TP
850  CONTINUE
860  CONTINUE
      IFL = 0
      GO TO 240
870  IFL = 2
      IF (LSTC*LE.0) GO TO 300
      IF (2-LSTC) 880,900,940
880  LSTC = LSTC - 1
      IF (LSTC*EQ.3) GO TO 890
      IF (LSC) 920,960,920
890  IFL = 1
      GO TO 300
900  IF (LSC-2) 910,930,920
910  LSTC = 0
920  LDOUT = 2
      GO TO 60
930  LSTC = 1
      LSC = 0
      GO TO 60
940  IF (LSC) 300,60,300
950  HH = FAC(2)*HH
      IF (LSTC*LT.2) GO TO 990
      ERND = P25*ERND
      IF (ERND*GE.PI) GO TO 950
      LSTC = 4
960  LSC = 4
      DO 970 I = 1,NE
      KQ(I) = 1
970  CONTINUE

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SVDQ0831
SVDQ0832
SVDQ0833
SVDQ0834
SVDQ0835
SVDQ0836
SVDQ0837
SVDQ0838
SVDQ0839
SVDQ0840
SVDQ0841
SVDQ0843
SVDQ0844
SVDQ0845
SVDQ0846
SVDQ0847
SVDQ0848
SVDQ0849
SVDQ0850
SVDQ0851
SVDQ0852
SVDQ0853
SVDQ0854
SVDQ0855
SVDQ0856
SVDQ0857
SVDQ0858
SVDQ0861
SVDQ0862
SVDQ0863
SVDQ0865
SVDQ0866
SVDQ0867
SVDQ0868
SVDQ0869
SVDQ0870

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          IF(LSTC-3) 890,890,1170
980      IF(LDOUB.EQ.0) GO TO 990
          LSC = 1
          LSTC = 1
          GO TO 140
990      IF(ABS(HH).GE.HMINA) GO TO 1040
          IF(IFL.EQ.7) GO TO 1010
1000     IFL = 7
          GO TO 1020
1010     HH = HH + HH
          IFL = 2
1020     H = HH
          GO TO 310
1030     HH = HH + HH
          IF(LSTC.EQ.1) GO TO 1050
          LSC = - 3
1040     LSTC = - 1
1050     DO 1160 I = 1,NE
          KQ0 = KC(I)
          IF(KQ0.NE.1) GO TO 1070
          DT(6,I) = 0.
          D(3) = DT(3,I)*PT(2)
          IF(D(3).GT.PT(3)) LSC = - 6
          IF(LDOUB.NE.0) GO TO 1060
          KQM = 8
          IF(D(3).GE.PT(5)) DT(7,I) = DT(7,I)*PT(2)
          D(3) = D(3)/PT(3)
1060     DT(3,I) = D(3)
          GO TO 1160
1070     DO 1080 K = 1,KQ0
          D(K) = DT(K,I)/PT(K)
1080     IF(LDOUB.EQ.0) D(K) = D(K)/PT(K)
          KQ02 = KQ0 - 2
          IF(KQ02) 1160,1140,1090
1090     DO 1130 J = 1,KQ02
          IF(LDOUB.NE.0) GO TO 1110

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SVDQ00871
SVDQ00873
SVDQ00874
SVDQ00875
SVDQ00876
SVDQ00878
SVDQ00879
SVDQ00880
SVDQ00881
SVDQ00883
SVDQ00884
SVDQ00885
SVDQ00886
SVDQ00890
SVDQ00891
SVDQ00892
SVDQ00893
SVDQ00896
SVDQ00897
SVDQ00898
SVDQ00899
SVDQ00900
SVDQ00901
SVDQ00902
SVDQ00903
SVDQ00904
SVDQ00905
SVDQ00906
SVDQ00907
SVDQ00911
SVDQ00912
SVDQ00913
SVDQ00914
SVDQ00915
SVDQ00916
SVDQ00917

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      K = KQQ
1100 D(K-1) = D(K-1) + D(K)
      K = K - 1
      IF(K+J-KQQ) 1130,1130,1100
1110 DO 1120 K = J,KQQ2
      D(K+1) = D(K+1) - D(K+2)
1120 CONTINUE
1130 CONTINUE
1140 DO 1150 K = 2,KQQ
      IF(LDOUB.NE.0) D(K) = D(K)*PT(K)
      DT(K,1) = D(K)*PT(K)
1150 CONTINUE
1160 CONTINUE
1170 H = HH
      IF(LDOUB.NE.0) GO TO 50
      LFD = 1
      IF(LSTC.GE.0) GO TO 220
      LDOUB = - 3
      LSC = LSTC - KQM
      GO TO 220
1180 IF(7-IFL) 1181,980,220
1181 IF(IFL-8) 60,1200,60
1190 IFL = 8
      GO TO 310
1200 WRITE(6,1205)
1205 FORMAT(26H0IFLAG=8 IN CALL TO DVDQ1.)
      STOP
1210 IF(I-FINAL) 200,1190,200
1220 IFL = 4
      IF(KSTEP.NE.0) GO TO 1270
1230 HH = HH+TPD1*.75E0
      IFL = - 1
      GO TO 230
1250 HH = H
      IF(EMAX.LT.P01) GO TO 1260
      IF(TPD1.LT.0.E0) GO TO 1190

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SVDQ0920
 SVDQ0921
 SVDQ0922
 SVDQ0923
 SVDQ0926
 SVDQ0927
 SVDQ0928
 SVDQ0929
 SVDQ0930
 SVDQ0931
 SVDQ0932
 SVDQ0933
 SVDQ0936
 SVDQ0937
 SVDQ0938
 SVDQ0939
 SVDQ0940
 SVDQ0941
 SVDQ0942
 SVDQ0943
 SVDQ0949
 SVDQ0950
 SVDQ0956
 SVDQ0957
 SVDQ0958
 SVDQ0959
 SVDQ0960
 SVDQ0963
 SVDQ0974
 SVDQ0975
 SVDQ0977
 SVDQ0981
 SVDQ0982
 SVDQ0989
 SVDQ0990
 SVDQ0992

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LDOUT = 0
ERND = P/HH
ERND = ERND*ERND*PI
GO TO 950
1260 IFL = 4
1270 IF(TPD1.GT.TPD) GO TO 1280
T = TFINAL
TPD = TPD1
GO TO 1290
1280 T = TOUT
IFL = 3
1290 IF((TPD.EQ.0.E0).AND.(IFLAG.LE.2)) GO TO 310
1300 TP = TPD
D(2) = TP
KQ2 = 0
KDC = 0
D(1) = PT(1)
DD(1) = PT(1)
DO 1310 K=2,KQM
DD(1) = DD(1) + PT(1)
TP = TP + PT(1)
1310 D(K+1) = (D(K)*TP)/DD(1)
GO TO 1350
1320 KQ2 = 1
L = KQM - KDC
KDC = KDC + 1
1330 IF(L.LE.0) GO TO 1350
TP = 0.
K = L
J = L + KDC
1340 JS = J - K
TP = TP + GAS(K)*D(JS+1)
K = K - 1
IF(K.GT.0) GO TO 1340
D(J) = TP
L = L - 1

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SVDQ0993
SVDQ0994
SVDQ0995
SVDQ0996
SVDQ1000
SVDQ1004
SVDQ1005
SVDQ1006
SVDQ1007
SVDQ1008
SVDQ1009
SVDQ1010
SVDQ1013
SVDQ1014
SVDQ1015
SVDQ1016
SVDQ1017
SVDQ1018
SVDQ1019
SVDQ1020
SVDQ1021
SVDQ1022
SVDQ1023
SVDQ1026
SVDQ1027
SVDQ1028
SVDQ1029
SVDQ1030
SVDQ1031
SVDQ1032
SVDQ1033
SVDQ1034
SVDQ1035
SVDQ1036
SVDQ1037
SVDQ1042

SVDQ1043
SVDQ1047
SVDQ1048
SVDQ1049
SVDQ1050
SVDQ1051
SVDQ1052
SVDQ1053
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SVDQ1056
SVDQ1057
SVDQ1058
SVDQ1059
SVDQ1060
SVDQ1061
SVDQ1062
SVDQ1063
SVDQ1064
SVDQ1065
SVDQ1066
SVDQ1067
SVDQ1068
SVDQ1069
SVDQ1070
SVDQ1075

1AND2

3AND4

5T024

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      GO TO 1330
1350  J = 0
      DO 1420 I=1,NE
      IF(KDS.LE.0) KDD = IABS(KD(I))
      IF(KDC.GT.KDD) GO TO 1410
      TP = 0.
      KQ = KC(I) + KQ2
1360  L = KQ - KDC
      IF(L.LE.0) GO TO 1370
      TP = TP + D(KQ)*DT(L,I)
      KQ = KQ - 1
      IF(KQ) 1390,1390,1360
1370  K = J + KDD
      L = KDC
1380  L = L - 1
      IF(L.EQ.0) GO TO 1400
      TP = TP*HH + YN(K)*FAC(L)*TPD
      K = K - 1
      GO TO 1380
1390  F(I) = TP
      GO TO 1420
1400  Y(K) = YN(K) + HH*TP
1410  J = J + KDD
1420  CONTINUE
      IF(KDC.NE.KDMAX) GO TO 1320
      GO TO 310
1430  IF(NPRINT.EQ.0) GO TO 1460
      WRITE(6,1431)
1431  FORMAT(1H1///20X,12HGSVDQ OUTPUT//)
1435  WRITE(6,1435) (D(M), M=1,19)
      FORMAT(/3X,10E13.3)
      WRITE(6,1435) (DD(M), M=1,20)
      DO 1440 JGG = 1,10
      WRITE(6,1435) (DT(M,JGG), M=1,20)
1440  CONTINUE
      WRITE(6,1435) (EP(M), M=1,10)

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DO 1445 JGG = 1,13
WRITE(6,1435) (ETA(M,JGG), M=1,13)
CONTINUE
1445 WRITE(6,1435) (F(M), M=1,10)
WRITE(6,1435) (FAC(M), M=1,3)
DO 1450 JGG=1,4
WRITE(6,1435) (GAM(M,JGG), M=1,15)
CONTINUE
1450 WRITE(6,1435) (GAS(M), M=1,15)
WRITE(6,1435) (PT(M), M=1,16)
WRITE(6,1435) (Y(M), M=1,10)
WRITE(6,1435) (YN(M), M=1,10)
WRITE(6,1435) (DEL,E,EMAX,EPS,ERNDC,ERRMX,E2H,E2HFACE,E2HMAX
WRITE(6,1435) (FRND,H,HH,HMAXA,HMINA,PTS1,PTS2,PTS3,PTS4,PTS5
WRITE(6,1435) (P01,P1,P25,P3E1,P5,RND,RNDC,RQMAX,T,TFINAL
WRITE(6,1435) (TL,TOUL,TP,TPD,TPD1,TPS1,TPS2,TPS3,TPS4,TPS5
WRITE(6,1435) (TPS6
WRITE(6,1455) (KD(M), M=1,10)
1455 FORMAT(/3X,10I13)
WRITE(6,1455) (KQ(M), M=1,10)
WRITE(6,1455) (I,IFL,IFLAG,J,JS,K,KBIT2,KDC,KDD,KDMAX
WRITE(6,1455) (KDS,KEMAX,KMAX0,KQM,KQMAX,KQQ,KQQ2,KQ1,KSOUT,KSTEP
WRITE(6,1455) (L,LDOUB,LFD,LRND,LSC,LSTC,MXSTEP,NE,NEQ,NV
1460 RETURN
END

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26T051

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54T061

62AND63

64AND65

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