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SOLUTION OF AN INITIAL-VALUE PROBLEM IN LINEAR TRANSPORT THEORY: MONOENERGETIC NEUTRONS IN A SLAB WITH INFINITE REFLECTORS

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16. Abstract

The solution of an initial-value problem in linear transport theory is obtained by using the normal-mode expansion technique of Case. The problem is that of monoenergetic neutrons migrating in a thin slab surrounded by infinitely thick reflectors and the scattering is taken to be isotropic. The results obtained indicate that the reflector may give rise to a branch-cut integral term typical of a semi-infinite medium whereas the central slab may contribute a summation over discrete residue terms. Exact expressions are obtained for these discrete time eigenvalues, and numerical results showing the behavior of real time eigenvalues as a function of the material properties of the slab and reflector are presented. These eigenvalues are finite in number and may disappear into the branch cut or continuum as the material properties are varied; such disappearing eigenvalues correspond to exponentially time-decaying modes. The two largest eigenvalues can be compared with critical dimensions of slabs and spheres, and the numerical values are shown to agree with the criticality results of others. In the limit of purely absorbing reflectors or a bare slab, the present solution has the same properties as have been previously reported by others who used the approach of Lehner and Wing.

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SOLUTION OF AN INITIAL-VALUE PROBLEM IN LINEAR TRANSPORT THEORY: MONOENERGETIC NEUTRONS IN A SLAB WITH INFINITE REFLECTORS*

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SUMMARY

The solution of an initial-value problem in linear transport theory is obtained by using the normal-mode expansion technique of Case. The problem is that of monoenergetic neutrons migrating in a thin slab surrounded by infinitely thick reflectors and the scattering is taken to be isotropic. The results obtained indicate that the reflector may give rise to a branch-cut integral term typical of a semi-infinite medium whereas the central slab may contribute a summation over discrete residue terms. Exact expressions are obtained for these discrete time eigenvalues, and numerical results showing the behavior of real time eigenvalues as a function of the material properties of the slab and reflector are presented. These eigenvalues are finite in number and may disappear into the branch cut or continuum as the material properties are varied; such disappearing eigenvalues correspond to exponentially time-decaying modes. The two largest eigenvalues can be compared with critical dimensions of slabs and spheres, and the numerical values are shown to agree with the criticality results of others. In the limit of purely absorbing reflectors or a bare slab, the present solution has the same properties as have been previously reported by others who used the approach of Lehner and Wing.

INTRODUCTION

Linearized transport equations are encountered in a number of different areas such as neutron diffusion, radiative transfer, sound propagation, and plasma theory, and the extent to which they correspond to reality varies from problem to problem even within a

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given area. Since there are some problems where the approximations required for linearization are not unduly severe from a physical point of view, a substantial effort to develop methods for solving such equations has been made. However, analytical solutions have been obtained for only a small class of highly idealized problems and few of these solutions have been for the time-dependent situation. Therefore, it might be expected that any new time-dependent result should provide additional insight into the general character of such solutions. In any case, analytical solutions provide a set of check cases for comparison of the approximate methods (see, for example, refs. 1 and 2) which are used in practical applications. The present idealized neutron transport problem, being somewhat more complicated than those considered previously, gives such new results. In addition, the procedure used to obtain the present solution is generally applicable to other time-dependent problems in linear transport theory and is therefore of interest in its own right.

The basic assumptions customarily made in neutron transport theory (see, for example, refs. 1, 2, and 3) have been summarized by Wigner (ref. 4). Since the conventional derivations of the governing transport equation are based mainly on plausibility arguments, Osborn and Yip (ref. 5) have reexamined the situation starting from a microscopic point of view using quantum mechanics.* They conclude that they are only partially successful in justifying the conventional neutron transport equation because a number of required approximations are merely stated and not analyzed or evaluated; however, their approach "brings many aspects of the neutron problem into contact with other microscopic transport theories." For the present problem, the additional restrictions to monoenergetic (called one-group or constant cross-section approximation in ref. 1) neutrons, plane geometry, and isotropic scattering in the laboratory system are made. (See refs. 1 to 4.) It is pointed out in references 1 and 2 that even though the monoenergetic approximation is rather severe from a physical point of view and is made primarily in order to obtain analytical solutions which usually cannot be obtained for the general case, it forms the basis of the more physical multigroup approximations. Under these restrictions, the solution of the initial-value monoenergetic neutron-transport equation for a one-dimensional slab of finite thickness surrounded by infinitely thick reflectors of a different material is obtained in this report.

One mathematically rigorous approach which has been used to treat such problems is a spectral analysis. Lehner and Wing (refs. 6 and 7) used this approach to solve the initial value, monoenergetic neutron-transport problem for a bare slab where the

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*Osborn and Yip (ref. 5) give four reasons for using a quantum mechanical treatment: (1) a formalism for describing the creation and destruction of particles exists, (2) the neutron-nuclear interactions are truly quantum phenomena, (3) the interpretation of an observable density in phase space, and (4) other peculiarly quantum effects such as spin and associated statistics.

scattering is isotropic. Lehner (ref. 8) considered a slab of finite thickness surrounded by a purely absorbing medium which had the same total macroscopic cross section as the slab. Very recently Hintz (ref. 9) has generalized this problem by allowing the purely absorbing medium to have any cross section. Mika (ref. 10) has studied the initial-value problem for monoenergetic neutrons in a nonuniform slab surrounded by a vacuum but did not obtain results as complete as those of Lehner and Wing (refs. 6 and 7). In particular, theorems concerning the reality and number of discrete time eigenvalues were not established.

Another approach which has been used to solve a few time-dependent, monoenergetic neutron-transport problems in plane geometry is the normal-mode expansion technique of Case. (See refs. 2 and 11.) This method was used by Bowden and Williams (refs. 12 and 13) to analyze the problem which had been treated by Lehner and Wing (refs. 6 and 7). A second application of this technique was made by Kuščer and Zweifel (ref. 14) to the time-dependent, one-speed albedo problem for a semi-infinite medium. Finally, Erdmann and Lurie (refs. 15 and 16) have also utilized this approach in a two-media time-dependent problem, the time decay of a plane isotropic burst of monoenergetic neutrons introduced at the interface of two dissimilar semi-infinite media. In all these time-dependent solutions, contributions due to various parts of the spectrum of the transport operator have been indicated by suitably deforming the integration contour of the inverse Laplace transformation. In view of these successful applications of Case's technique, in particular references 15 and 16, this technique has been chosen to analyze the present problem. In this problem, one would expect to find discrete time eigenvalues (time constants) and obtain some insight concerning their behavior as a function of material properties. Since the reflectors can scatter as well as absorb neutrons, the solutions for the bare slab and slab surrounded by purely absorbing media are included and it is shown that the present solution agrees with such solutions (refs. 6 to 9) for these special cases. Some preliminary results of the present work were given in reference 17 and a summary of the present results is given in reference 18.

SYMBOLS

A	nondimensional slab half-thickness (see eq. (118))
A_m, a_m, b_m	expansion coefficients in Case's normal-mode expansion for medium m (see eq. (29))
$A_{m\pm}, a_{m\pm}$	definite parity expansion coefficients (see eq. (32))

a	slab half-thickness
$\overline{B}_{m\pm}, \overline{b}_{1\pm}$	expansion coefficients for solution of associated eigenvalue problem (see eq. (76))
C'	contour in z'-plane (see fig. 3)
C_m	contours in s-plane (see figs. 2 and 4)
$C_{m\pm}$	expansion coefficients in full-range normal-mode expansion of initial distribution in medium m (see eq. (47))
C_ρ	contour in s-plane (see fig. 4)
c_m	mean number of secondary neutrons per collision in medium m
$E_{m\pm}$	expansion coefficients defined by equations (50)
$F_{m\pm}, \widetilde{F}_\pm$	integrations over initial distribution $f(x,\mu)$ given by equation (46)
f	initial neutron angular flux, generally referred to as the initial distribution
$f_{m\pm}$	definite parity parts of initial distribution in medium m (see eq. (16))
g_m	solution of equation (42)
h_m	given by equation (80)
Im( ),Re( )	imaginary and real parts
$I_{m\pm}$	inhomogeneous terms given by equations (55) and (57)
$J_{m\pm}$	inhomogeneous terms given by equations (56) and (58)
k	given by equation (62)
$L_{m\pm}$	integrations over initial distribution $f(x,\mu)$ defined by equation (67)

l_m	lower limits given by equation (68)
$M_\pm$	integrations over initial distribution $f(x, \mu)$ given by equations (74) and (75)
N_m	given by equation (80)
P	denotes that Cauchy principal value is to be taken upon integration
S_{me}, S_{mi}	denote regions in s-plane (see figs. 2 and 4)
s	complex variable of Laplace transformation (see eq. (6))
s_n	denote values of s for which associated eigenvalue problem has nontrivial solutions
t	real time multiplied by constant neutron speed
X_0, X_{0m}, X_m	Case's X-functions given by equations (59) to (61), respectively
x	geometric coordinate perpendicular to slab faces (see fig. 1)
z	complex variable
α', β'	defined in equation (28)
$\alpha_{1\pm}, \alpha_{2\pm}$	given by equation (89)
$\beta_{1\pm}, \beta_{2\pm}$	given by equation (90)
γ	associated with inverse Laplace transformation (see eq. (7))
$\delta_m(s)$	defined by equation (30)
$\delta(\nu - \mu)$	Dirac delta function
ζ	given by equation (118)

ζ_n	denote values of ζ for which associated eigenvalue problem has nontrivial solutions
ζ_0	ζ_n with largest real part
λ_m	defined by equation (25)
μ	direction cosine (see fig. 1)
ν	introduced as complex separation parameter in equation (22) but used thereafter as real, z being used to denote complex values
$\pm\nu_{0m}$	values of z for which $\Omega_m(z,s) = 0$
σ_m	total macroscopic cross section for medium m
σ_R,σ_D	given by equation (118)
$\sigma_{min} = \min(\sigma_1,\sigma_2)$	
Φ	given by equation (110)
$\varphi_{m\nu}$	continuum mode in Case's method (see eq. (24))
$\varphi_{\pm\nu_{0m}}$	discrete mode in Case's method (see eq. (26))
Ψ	neutron angular flux
$\psi_\pm$	definite parity parts of Laplace transform of neutron angular flux, generally referred to as transformed solution
$\psi_{m\pm}$	$\psi_\pm$ in medium m (see eq. (17))
$\psi_{mc\pm},\psi_{mp\pm}$	parts of $\psi_{m\pm}$ (see eq. (31))
$\psi_{m\nu}$	an elementary solution (see eq. (22))
$\overline{\psi}_\pm$	solution of associated eigenvalue problem

Ω_m dispersion function (given by eq. (27))

Ω'_m given by equation (49)

Subscripts:

c complimentary solution

e,i regions exterior and interior to a curve

m physical medium, 1 for reflector and 2 for slab

n eigenvalue, 0 for the one with largest real part

p particular solution

u solution due to part of initial distribution which has not been scattered, that is, uncollided

ν associated with continuum modes

$\pm$ definite parity, + for even and - for odd

Superscript:

$\pm$ limiting values of a function on its branch cut as argument approaches cut from upper (+) and lower (-) half-planes.

A bar over a symbol denotes the associated eigenvalue problem quantity.

PROBLEM DEFINITION

Basic Equations

Consider a slab of material which scatters neutrons isotropically (in the laboratory system), extends from $x = -a$ to $x = a$, and is characterized by a total macroscopic cross section σ_2 and a mean number of secondary neutrons per collision c_2. This uniform slab is surrounded by uniform infinitely thick reflectors of another material characterized by the nuclear properties σ_1 and c_1. (See fig. 1.) For isotropic scattering of monoenergetic neutrons in a sourceless medium with plane geometry, the

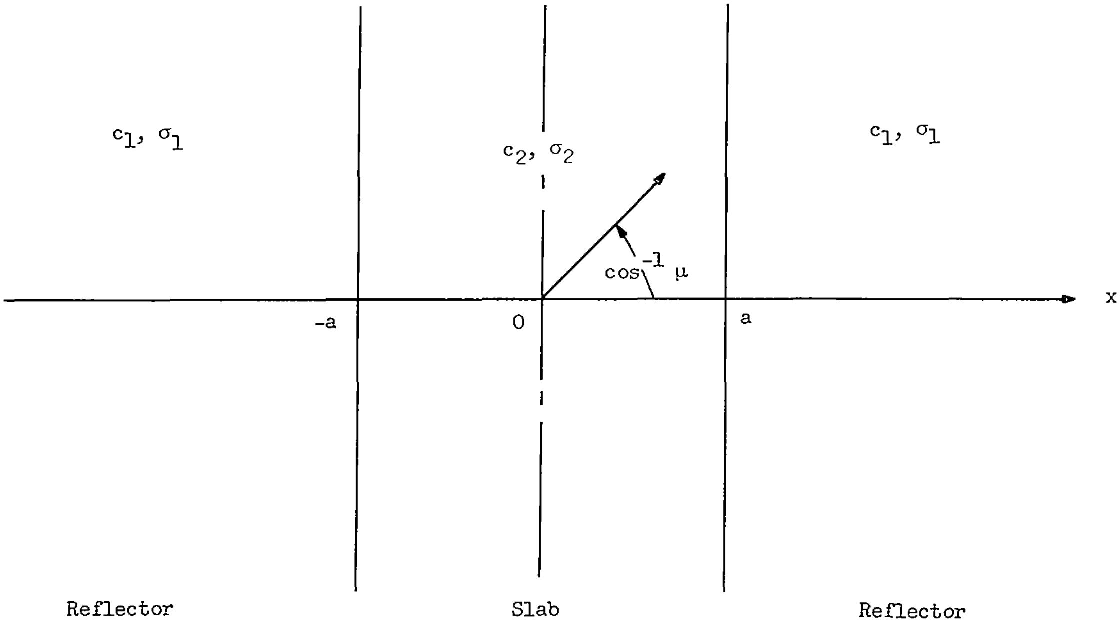

Figure 1.- Geometry of problem.

neutron angular flux $\Psi(x,\mu,t)$ satisfies the equation (see refs. 1, 2, and 3 for a complete list of the approximations and assumptions):

$$\frac{\partial\Psi(x,\mu,t)}{\partial t} + \mu\frac{\partial\Psi(x,\mu,t)}{\partial x} + \sigma(x)\ \Psi(x,\mu,t) = \frac{c(x)\ \sigma(x)}{2}\int_{-1}^{1} \Psi(x,\mu',t)\ d\mu' \tag{1}$$

where t is the real time multiplied by the constant neutron speed, x and μ are shown in figure 1, and $\sigma(x)$ and $c(x)$ are given by

$$\sigma(x),c(x) = \begin{cases} \sigma_1,c_1 & \left(|x| > a\right) \\ \sigma_2,c_2 & \left(|x| < a\right) \end{cases} \tag{2}$$

Since multiplying media are not of infinite extent, c_1 has been taken to be less than unity. Therefore, equation (1) is to be solved subject to the boundary conditions

$$\lim \Psi(x,\mu,t) = 0 \qquad \left(|x| \to \infty\right) \tag{3}$$

and the continuity conditions

$$\Psi(\pm a+,\mu,t) = \Psi(\pm a-,\mu,t) \tag{4}$$

given the initial condition

$$\Psi(x,\mu,0) = f(x,\mu) \tag{5}$$

which is assumed to satisfy equation (3) and be extendable without poles or branch cuts in the finite μ-plane except perhaps for a discontinuity across the imaginary axis. When the material properties of the reflectors are taken to be those of a vacuum, this problem reduces to that of references 6 and 7 whereas for purely absorbing medium, it reduces to that considered in references 8 and 9. When $\sigma(x)$ is constant and $c(x) = 1$ (that is, neutrons are conserved) equation (1) is the one-dimensional Boltzmann equation for the Lorentz model of kinetic theory (ref. 19).

The general procedure used here to solve the mathematical problem presented in equations (1) to (5) consists of the following steps:

(1) Remove the t-dependence with a Laplace transformation

(2) Solve the transformed equation by applying Case's technique

(3) Determine the analytic properties of this transformed solution in some right-half s-plane

(4) Recover the t-dependence and simplify by suitably deforming the integration path of the inverse transformation

(5) Calculate real discrete time eigenvalues as a function of material properties if and when they exist.

Since many details are involved in performing these few steps, a brief synopsis is given. Step (1) is easy and the transformed equation and boundary conditions are given by equations (8) to (10). Step (2) is accomplished by construction of a solution from Case's elementary solutions, or normal modes. Since the geometrical symmetry aids in this construction, symmetry properties are introduced immediately after the time removal. The elementary solutions of Case are given by equations (24) to (28). For details concerning these solutions and their completeness and orthogonality properties, the reader is referred to references 2, 11 to 16, and 20. Appendixes A and C summarize the important results taken from these references which are required in this report. Construction of the transformed solution is done in the section so titled and a few details are given in appendixes B and D. Equations (50) to (58) give the expansion coefficients of the transformed solution implicitly. Extension of these equations to the complex plane is presented in appendix E. Since the solution valid for all regions of the transform plane is needed for step (3), it is obtained in the same section and is given by equations (70) to (73). Step (3) is performed in the section "Properties of Transformed Solution" and many of the details are given in appendixes F, G, and H. In particular, one must examine where the associated eigenvalue problem has nontrivial solutions and show how these solutions enter the transformed solution. Details for step (4) are found in the section "Recovery of Time-Dependent Solution" and appendix I. The previously cited results (refs. 12 to 16) lead one to expect that the reflectors should contribute continuous-spectrum type terms typical of a semi-infinite medium whereas the central slab should give rise to some point-spectrum type terms and their corresponding discrete time eigenvalues. The solution $\Psi(x,\mu,t)$ is given by equation (114). Step (5), the calculation of real time eigenvalues, is outlined rather explicitly in appendix J and the numerical results are presented and discussed in the section "Calculation of Time Eigenvalues." The report is concluded with a short section showing how for special values of the nuclear properties the present solution reduces to those obtained previously by others (refs. 6 to 9) who used a different method.

Time Removal

If one takes the Laplace transformation of $\Psi(x,\mu,t)$ as

$$\psi(x,\mu,s) = \int_0^{\infty} e^{-st}\,\Psi(x,\mu,t)\;dt \tag{6}$$

then the inverse transformation required to recover the t-dependence is

$$\Psi(x,\mu,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\psi(x,\mu,s)\;ds \tag{7}$$

where γ is to the right of all singularities and branch cuts of $\psi(x,\mu,s)$ in the transform plane (s-plane). From previously cited work of others, it is expected that the path of integration in equation (7) can be deformed to indicate more precisely the character of $\Psi(x,\mu,t)$. When the transformation of equation (6) is applied to equation (1), integration by parts is performed in the usual manner, and use of the initial condition (5) is made, the following expression is obtained:

$$\mu\frac{\partial}{\partial x}\psi(x,\mu,s) + \left[s + \sigma(x)\right]\psi(x,\mu,s) = \frac{c(x)\;\sigma(x)}{2}\int_{-1}^{1}\psi(x,\mu',s)\;d\mu' + f(x,\mu) \tag{8}$$

Equations (3) and (4) become under the same transformation

$$\lim\,\psi(x,\mu,s) = 0 \qquad \left(|x| \rightarrow \infty\right) \tag{9}$$

and

$$\psi(\pm a+,\mu,s) = \psi(\pm a-,\mu,s) \tag{10}$$

Symmetry Considerations

Before applying Case's technique to solve equation (8) subject to conditions (9) and (10), it is useful to examine some symmetry properties of the transformed solution which follow directly from the governing equations. In references 12 and 13, these ideas were introduced at a later step, but here they aid in the construction of the solution. An arbitrary function of two variables $g(x,\mu)$ can be written as the sum of its even and odd parts, namely, $g_+(x,\mu)$ and $g_-(x,\mu)$. They are given, of course, by

$$g_\pm(x,\mu) = \frac{1}{2}\left[g(x,\mu) \pm g(-x,-\mu)\right] \tag{11}$$

and have the property

$$g_\pm(-x,-\mu) = \pm g_\pm(x,\mu) \tag{12}$$

Since $c(x)$ and $\sigma(x)$ are even functions of x, it is easily shown from equation (8) that the even and odd parts of $\psi(x,\mu,s)$ obey the equation

$$\mu \frac{\partial}{\partial x}\psi_\pm(x,\mu,s) + \left[s + \sigma(x)\right]\psi_\pm(x,\mu,s) = \frac{c(x)\ \sigma(x)}{2}\int_{-1}^{1} \psi_\pm(x,\mu',s)\ d\mu' + f_\pm(x,\mu) \tag{13}$$

The boundary conditions for $\psi_\pm$ corresponding to equations (9) and (10) are written as

$$\lim \psi_\pm(x,\mu,s) = 0 \qquad \left(|x| \to \infty\right) \tag{14}$$

and

$$\psi_\pm(a+,\mu,s) = \psi_\pm(a-,\mu,s) \tag{15}$$

where the $\pm$ subscripts denote definite parity parts of a function, that is, even and odd. (See eqs. (11) and (12).) Equations (13) to (15) indicate the following:

(1) All solutions of the homogeneous equation associated with equation (13) can be made to have a definite parity.

(2) The boundary conditions preserve the parity.

(3) The definite parity parts of an initial distribution excite inhomogeneous solutions of corresponding definite parity.

Therefore, this problem can be separated into two problems, one for ψ_+, the other for ψ_-, and the results can be combined at any stage of the calculation. The functions $f_\pm(x,\mu)$ and $\psi_\pm(x,\mu,s)$ are broken up as

$$f_\pm(x,\mu) = \begin{cases} f_{1\pm}(x,\mu) & \left(|x| > a\right) \\ f_{2\pm}(x,\mu) & \left(|x| < a\right) \end{cases} \tag{16}$$

and

$$\psi_\pm(x,\mu,s) = \begin{cases} \psi_{1\pm}(x,\mu,s) & \left(|x| > a\right) \\ \psi_{2\pm}(x,\mu,s) & \left(|x| < a\right) \end{cases} \tag{17}$$

so that equations (13) to (15) become

$$\mu \frac{\partial}{\partial x} \psi_{m\pm}(x,\mu,s) + \left(s + \sigma_m\right)\psi_{m\pm}(x,\mu,s) = \frac{c_m \sigma_m}{2} \int_{-1}^{1} \psi_{m\pm}(x,\mu',s)\, d\mu' + f_{m\pm}(x,\mu) \tag{18}$$

where throughout the subscript $m = 1$ denotes medium 1 and $m = 2$ denotes medium 2,

$$\lim \psi_{1\pm}(x,\mu,s) = 0 \qquad \left(|x| \to \infty\right) \tag{19}$$

and

$$\psi_{1\pm}(a,\mu,s) = \psi_{2\pm}(a,\mu,s) \tag{20}$$

The notation $g_{m\pm}(a,\mu)$ means the limit of $g_\pm(x,\mu)$ as $x \to a$ from medium m.

Elementary Solutions

Solutions of equations (18) are constructed from Case's elementary solutions which are denoted here as $\psi_{m\nu}(x,\mu,s)$. These elementary solutions are solutions of the homogeneous equation corresponding to equation (18); that is,

$$\mu \frac{\partial}{\partial x} \psi_{m\nu}(x,\mu,s) + \left(s + \sigma_m\right)\psi_{m\nu}(x,\mu,s) = \frac{1}{2} c_m \sigma_m \int_{-1}^{1} \psi_{m\nu}(x,\mu',s)\, d\mu' \tag{21}$$

in the form

$$\psi_{m\nu}(x,\mu,s) = \varphi_{m\nu}(\mu,s)\, e^{-\left(s+\sigma_m\right)x/\nu} \tag{22}$$

where ν is a complex parameter introduced in this separation of variables and $\varphi_{m\nu}(\mu,s)$ is normalized as

$$\int_{-1}^{1} \varphi_{m\nu}(\mu,s)\, d\mu = s + \sigma_m \tag{23}$$

These solutions have been investigated in references 12 to 16 and many of their results are given in appendix A and are used herein. They show that the values of ν for which solutions $\varphi_{m\nu}(\mu,s)$ can be found are ν real $(-1 \leqq \nu \leqq 1)$ and for some region of the s-plane $\nu = \pm\nu_{0m}$. For $-1 \leqq \nu \leqq 1$, the solutions are

$$\varphi_{m\nu}(\mu,s) = \frac{1}{2} c_m \sigma_m \nu P \frac{1}{\nu - \mu} + \lambda_m(\nu,s)\, \delta(\nu - \mu) \tag{24}$$

where P denotes that the Cauchy principal value is to be taken upon integration, $\delta(\nu - \mu)$ is the Dirac delta function, and $\lambda_m(\nu,s)$ is determined from the normalization as

$$\lambda_m(\nu,s) = s + \sigma_m - c_m \sigma_m \nu \tanh^{-1} \nu \tag{25}$$

These are called the continuum modes and exist for all values of s. There are two discrete solutions

$$\varphi_{\pm\nu_{0m}}(\mu,s) = \frac{1}{2} \frac{c_m \sigma_m \nu_{0m}}{\nu_{0m} \mp \mu} \tag{26}$$

at $\nu = \pm\nu_{0m}$ provided that the function $\Omega_m(z,s)$

$$\Omega_m(z,s) = s + \sigma_m - c_m \sigma_m z \tanh^{-1} \frac{1}{z} \tag{27}$$

of two complex variables s and z vanishes at the two points $z = \pm\nu_{0m}(s)$. This condition occurs when s lies inside the curve C_m ($s \in S_{mi}$, see fig. 2) defined by (see refs. 12 and 13)

$$C_m = \left\{ \frac{s + \sigma_m}{c_m \sigma_m} = \alpha' + i\beta' \;\middle|\; \alpha' = \frac{2\beta'}{\pi} \tanh^{-1}\left(\frac{2\beta'}{\pi}\right) \right\} \tag{28}$$

Note that ν_{0m} is an analytic function of s for $s \in S_{mi}$ except for a branch cut on the real s-axis between $-\sigma_m$ and $-\sigma_m(1 - c_m)$ and that $+\nu_{0m}$ denotes that zero of $\Omega_m(z,s)$ for which $\mathrm{Re}(\nu_{0m}) > 0$ when $\mathrm{Re}(s) > -\sigma_m(1 - c_m)$. The important result is that the general solution of equation (21) can be expressed as the linear combination

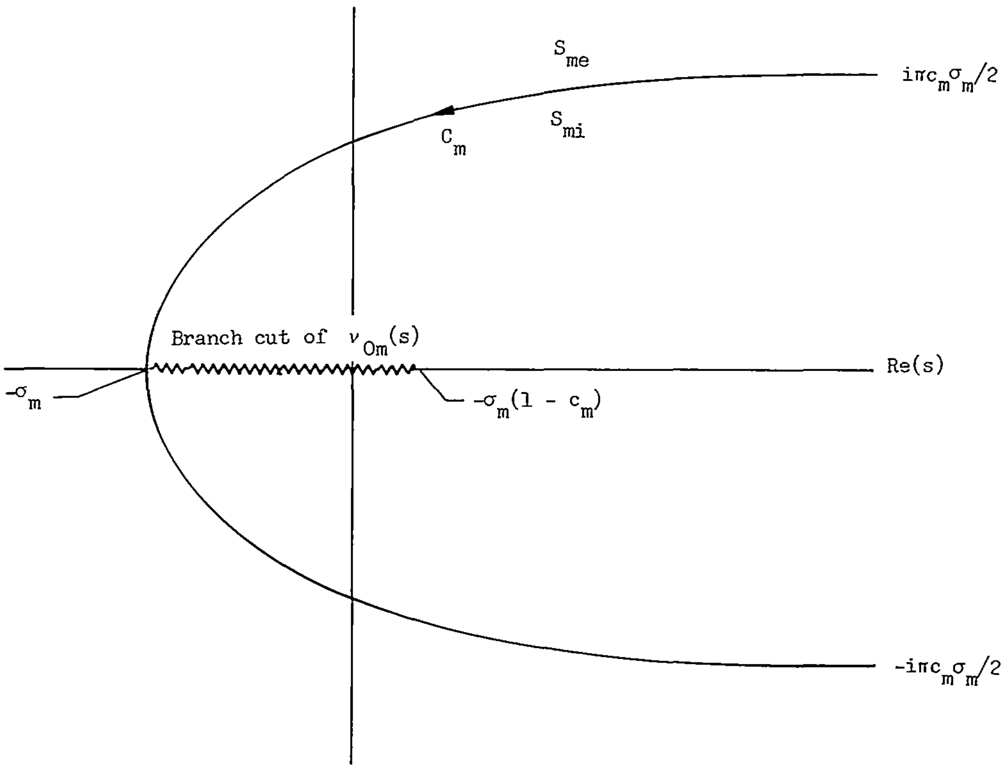

Figure 2.- Regions in a single-medium s-plane. Location of Im(s) axis depends on whether $c_m \gtrless 1$.

$$\psi_m(x,\mu,s) = \left[a_m \psi_{\nu_{0m}}(x,\mu,s) + b_m \psi_{-\nu_{0m}}(x,\mu,s)\right]\delta_m(s) + \int_{-1}^{1} A_m(\nu)\, \psi_{m\nu}(x,\mu,s)\, d\nu \tag{29}$$

where $\delta_m(s)$ is defined as

$$\delta_m(s) \equiv \begin{cases} 1 & \left(s \in S_{mi}\right) \\ 0 & \left(s \in S_{me}\right) \end{cases} \tag{30}$$

and the s-dependence of the expansion coefficients has not been determined. Note that the present notation is slightly different than that used by other authors.

CONSTRUCTION OF TRANSFORMED SOLUTION

Solutions of equation (18) are now obtained by constructing even and odd particular solutions $\psi_{mp\pm}(x,\mu,s)$ and adding to them solutions of the corresponding homogeneous equations $\psi_{mc\pm}(x,\mu,s)$ so that conditions (19) and (20) can be satisfied; that is,

$$\psi_{m\pm}(x,\mu,s) = \psi_{mc\pm}(x,\mu,s) + \psi_{mp\pm}(x,\mu,s) \qquad (m = 1, 2) \tag{31}$$

The functions $\psi_{mc\pm}$ and $\psi_{mp\pm}$ are constructed from Case's elementary solutions $\psi_{m\nu}(x,\mu,s)$. One must select the expansion coefficients in a general expansion, such as equation (29), so that the given boundary conditions are satisfied.

Explicit Form of $\psi_{mc\pm}$

The solution in the form of equation (29) does not have definite parity. However, for a medium which is connected and symmetric about $x = 0$ (such as the slab), even and odd solutions $\psi_{2c\pm}$ can be written as

$$\psi_{2c\pm}(x,\mu,s) = a_{2\pm}\left[\psi_{\nu_{02}}(x,\mu,s) \pm \psi_{-\nu_{02}}(x,\mu,s)\right]\delta_2(s) + \int_0^1 A_{2\pm}(\nu)\left[\psi_{2\nu}(x,\mu,s) \pm \psi_{2(-\nu)}(x,\mu,s)\right] d\nu \tag{32}$$

where the expansion coefficients have been redefined as

$$\left.\begin{aligned} a_{2\pm} &= \frac{1}{2}\left(a_2 \pm b_2\right) \\ A_{2\pm}(\nu) &= \frac{1}{2}\left[A_2(\nu) \pm A_2(-\nu)\right] \end{aligned}\right\} \tag{33}$$

Note that the properties

$$\left.\begin{aligned} \psi_{\pm\nu_{0m}}(-x,-\mu,s) &= \psi_{\mp\nu_{0m}}(x,\mu,s) \\ \psi_{m(\pm\nu)}(-x,-\mu,s) &= \psi_{m(\mp\nu)}(x,\mu,s) \end{aligned}\right\} \tag{34}$$

have been used.

For a medium which extends to infinity in the x-direction (such as the reflectors), the boundary conditions (9) require that

$$b_1 \equiv A_1(-\nu) \equiv 0 \qquad (0 \leqq \nu \leqq 1 \quad \text{if} \quad x \to +\infty) \tag{35a}$$

or

$$a_1 \equiv A_1(\nu) \equiv 0 \qquad (0 \leqq \nu \leqq 1 \quad \text{if} \quad x \to -\infty) \tag{35b}$$

for the expansion coefficients in equation (29) when $\mathrm{Re}(s) > -\sigma_1$. Thus, in the reflectors, the solution has the form

$$\psi_1(x,\mu,s) = \begin{cases} b_1 \psi_{-\nu_{01}}(x,\mu,s)\ \delta_1(s) + \int_{-1}^{0} A_1(\nu)\ \psi_{1\nu}(x,\mu,s)\ d\nu & (x < -a) \\ a_1' \psi_{\nu_{01}}(x,\mu,s)\ \delta_1(s) + \int_{0}^{1} A_1'(\nu)\ \psi_{1\nu}(x,\mu,s)\ d\nu & (x > a) \end{cases} \tag{36}$$

for $\mathrm{Re}(s) > -\sigma_1$. The continuity conditions (10) and the parity of the solutions $\psi_{2c\pm}$ are used next to relate the primed and unprimed coefficients in equation (36). One finds that an even solution inside the slab requires

$$\left.\begin{aligned} a_1' &= b_1 \\ A_1'(\nu) &= A_1(-\nu) \qquad (0 \leqq \nu \leqq 1) \end{aligned}\right\} \tag{37a}$$

whereas an odd solution inside the slab requires

$$\left.\begin{aligned} a_1' &= -b_1 \\ A_1'(\nu) &= -A_1(-\nu) \qquad (0 \leqq \nu \leqq 1) \end{aligned}\right\} \tag{37b}$$

In view of equations (37a) and (37b), one defines, respectively,

$$\left.\begin{aligned} a_{1+} &\equiv b_1 \\ A_{1+}(-\nu) &\equiv A_1(-\nu) \qquad (0 \leqq \nu \leqq 1) \end{aligned}\right\} \tag{38a}$$

and

$$\left.\begin{aligned} a_{1-} &\equiv b_1 \\ A_{1-}(-\nu) &\equiv A_1(-\nu) \qquad (0 \leqq \nu \leqq 1) \end{aligned}\right\} \tag{38b}$$

In terms of these coefficients, $\psi_{1c\pm}$ can be written from equation (36) as

$$\psi_{1c\pm}(x,\mu,s) = \begin{cases} a_{1\pm}\psi_{-\nu_{01}}(x,\mu,s)\ \delta_1(s) + \int_0^1 A_{1\pm}(-\nu)\ \psi_{1(-\nu)}(x,\mu,s)\ d\nu & (x < -a) \\ \pm a_{1\pm}\psi_{\nu_{01}}(x,\mu,s)\ \delta_1(s) \pm \int_0^1 A_{1\pm}(-\nu)\ \psi_{1\nu}(x,\mu,s)\ d\nu & (x > a) \end{cases} \tag{39}$$

for $\mathrm{Re}(s) > -\sigma_1$.

Explicit Form of $\psi_{mp\pm}$

The definite parity particular solutions $\psi_{mp\pm}$ are constructed from two particular solutions ψ_{mp} as

$$\psi_{mp\pm}(x,\mu,s) = \frac{1}{2}\left[\psi_{mp}(x,\mu,s) \pm \psi_{mp}(-x,-\mu,s)\right] \tag{40}$$

The solutions ψ_{mp} are obtained in a conventional way by integration of a Green's function g_m for medium m over all the medium; that is, as

$$\psi_{mp}(x,\mu,s) = \int_{\text{Medium } m} g_m(x,\mu;x_0)\ dx_0 \tag{41}$$

The function g_m satisfies the equation:

$$\mu\frac{\partial}{\partial x}g_m(x,\mu;x_0) + (s + \sigma_m)g_m(x,\mu;x_0) = \frac{c_m\sigma_m}{2}\int_{-1}^{1} g_m(x,\mu';x_0)d\mu' + \delta(x - x_0)\ f_m(x_0,\mu) \tag{42}$$

This equation is seen to be the homogeneous equation corresponding to equation (18) for $x \neq x_0$. When one integrates over all x_0 in medium m, it is seen that equation (18) is obtained. Note that g_m is not exactly what is customarily called the Green's function, since $\delta(x - x_0)$ has been weighted with $f_m(x_0,\mu)$. Upon integrating equation (42) on x from $x_0 - \epsilon$ to $x_0 + \epsilon$ and taking the limit $\epsilon \to 0$, one obtains the jump condition

$$g_m(x_0+,\mu;x_0) - g_m(x_0-,\mu,x_0) = \frac{f_m(x_0,\mu)}{\mu} \tag{43}$$

In appendix B, g_m is constructed from Case's elementary solutions and, as a result, the explicit forms of $\psi_{2p\pm}$ and $\psi_{1p\pm}$ can be written as

$$\psi_{2p\pm}(x,\mu,s) = \left[F_{2\pm}\left(x,\nu_{02},s\right)\psi_{\nu_{02}}(x,\mu,s) \pm F_{2\pm}\left(-x,\nu_{02},s\right)\psi_{-\nu_{02}}(x,\mu,s)\right]\delta_2(s)$$
$$+ \int_0^1 F_{2\pm}(x,\nu,s)\,\psi_{2\nu}(x,\mu,s)\,d\nu$$
$$\pm \int_0^1 F_{2\pm}(-x,\nu,s)\,\psi_{2(-\nu)}(x,\mu,s)\,d\nu \tag{44}$$

$$\psi_{1p\pm}(x,\mu,s) = \Big\{F_{1\pm}\left(x,\nu_{01},s\right)\psi_{\nu_{01}}(x,\mu,s)$$
$$+ \left[F_{1\pm}\left(x,-\nu_{01},s\right) - \tilde{F}_{\pm}\left(-a,\nu_{01},s\right)\right]\psi_{-\nu_{01}}(x,\mu,s)\Big\}\,\delta_1(s)$$
$$+ \int_0^1 F_{1\pm}(x,\nu,s)\,\psi_{1\nu}(x,\mu,s)\,d\nu$$
$$+ \int_0^1 \left[F_{1\pm}(x,-\nu,s) - \tilde{F}_{\pm}(-a,\nu,s)\right]\psi_{1(-\nu)}(x,\mu,s)\,d\nu \qquad (x < -a) \tag{45a}$$

for $\mathrm{Re}(s) > -\sigma_1$, and

$$\psi_{1p\pm}(x,\mu,s) = \Big\{\pm\left[-\tilde{F}_{\pm}\left(-a,\nu_{01},s\right) + F_{1\pm}\left(-x,-\nu_{01},s\right)\right]\psi_{\nu_{01}}(x,\mu,s)$$
$$\pm F_{1\pm}\left(-x,\nu_{01},s\right)\psi_{-\nu_{01}}(x,\mu,s)\Big\}\,\delta_1(s)$$
$$\pm \int_0^1 \left[-\tilde{F}_{\pm}(-a,\nu,s) + F_{1\pm}(-x,-\nu,s)\right]\psi_{1\nu}(x,\mu,s)\,d\nu$$
$$\pm \int_0^1 F_{1\pm}(-x,\nu,s)\,\psi_{1(-\nu)}(x,\mu,s)\,d\nu \qquad (x > a) \tag{45b}$$

for $\mathrm{Re}(s) > -\sigma_1$, where

$$\left.\begin{aligned}
\tilde{F}_{\pm}(-a,\omega,s) &\equiv F_{1\pm}(-a,-\omega,s) \mp F_{1\pm}(-a,\omega,s) \\
F_{2\pm}(x,\omega,s) &\equiv \int_{-a}^{x} C_{2\pm}\left(x_0,\omega\right) e^{(s+\sigma_2)x_0/\omega}\,dx_0 \\
F_{1\pm}(x,\omega,s) &\equiv \int_{-\infty}^{x} C_{1\pm}\left(x_0,\omega\right) e^{(s+\sigma_1)x_0/\omega}\,dx_0
\end{aligned}\right\} \tag{46}$$

where $\omega = \pm\nu_{0j}$ and ν, $-1 \leqq \nu \leqq 1$. Here the $C_{m\pm}$ are full-range expansion coefficients* of the function $f_{m\pm}(x,\mu)/\mu$ and are given by

$$C_{m\pm}(x_0,\nu) = \frac{1}{\nu\Omega_m^+(\nu,s)\ \Omega_m^-(\nu,s)} \int_{-1}^{1} f_{m\pm}(x_0,\mu)\ \varphi_{m\nu}(\mu,s)\ d\mu$$

and if $s \in S_{mi}$,

$$\left.\begin{aligned} C_{m\pm}(x_0,\nu_{0m}) &= \frac{2}{c_m\sigma_m\nu_{0m}^2\,\Omega_m'(\nu_{0m},s)} \int_{-1}^{1} f_{m\pm}(x_0,\mu)\ \varphi_{\nu_{0m}}(\mu)\ d\mu \\ C_{m\pm}(x_0,-\nu_{0m}) &= \frac{2}{c_m\sigma_m\nu_{0m}^2\,\Omega_m'(-\nu_{0m},s)} \int_{-1}^{1} f_{m\pm}(x_0,\mu)\ \varphi_{-\nu_{0m}}(\mu,s)\ d\mu \end{aligned}\right\} \tag{47}$$

Throughout, the + and - superscripts are used to denote the limiting values of a function on its branch cut as the argument approaches the cut from the upper (+) and lower (-) half-planes. The function $\Omega_m(z,s)$ of equation (27) has a branch cut along the real z-axis (-1,1) where its limiting values are given by

$$\Omega_m^{\pm}(\nu,s) = \lambda_m(\nu,s) \pm \frac{i\pi c_m\sigma_m\nu}{2} \qquad (-1 \leqq \nu \leqq 1) \tag{48}$$

The functions $\Omega_m'(z,s)$ are defined by

$$\Omega_m'(z,s) \equiv \frac{d}{dz}\Omega_m(z,s) \tag{49}$$

Equations for Expansion Coefficients

Solutions in medium 1, $|x| > a$, have been constructed so that the boundary condition (19) is satisfied. Application of the continuity condition (20) permits the determination of the unknown expansion coefficients of $\psi_{mc\pm}$ which are implicit in equation (31); that is, if one substitutes $x = a$ in equation (31), applies the continuity condition (20), and uses the explicit forms of $\psi_{mc\pm}$ given by equations (32) and (39), a two-media full-range expansion involving the $\varphi_{m\nu}$ which contains unknown coefficients $a_{m\pm}$ and $A_{m\pm}$

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*Note that the parity of these coefficients is opposite that indicated by the ± subscript because of the $1/\mu$ factor. Nevertheless, the solutions $\psi_{mp\pm}$ are easily seen to have the indicated parity.

is obtained. The same expansion is, of course, obtained for $x = -a$. This type of expansion and its orthogonality relations are discussed in appendix C and it is shown in appendix D that such an expansion is obtained for the present problem. Erdmann (ref. 15) proved completeness theorems which apply in such time-dependent problems while Kuščer, McCormick, and Summerfield (ref. 20) derived orthogonality relations which are applicable to two-media expansions which arise in time-independent problems. In appendix C, their results are extended to obtain orthogonality relations which are valid for all regions of the transform plane. As usual in problems involving a slab, one cannot obtain closed-form solutions for the expansion coefficients. However, the orthogonality relations (appendix C) can be used to obtain expressions which give the expansion coefficients implicitly; that is, the continuum coefficients $A_{2\pm}(\nu)$ are given as the solutions of Fredholm integral equations and all the other coefficients are obtained from $A_{2\pm}(\nu)$. For later convenience, however, expressions are obtained for these coefficients in the form

$$\left.\begin{aligned} E_{2\pm}(\nu,s) &\equiv A_{2\pm}(\nu)\ \Omega_2^+(\nu,s)\ \Omega_2^-(\nu,s)\ e^{(s+\sigma_2)a/\nu} \\ E_{1\pm}(\nu,s) &\equiv A_{1\pm}(-\nu)\ \Omega_1^+(\nu,s)\ \Omega_1^-(\nu,s)\ e^{-(s+\sigma_1)a/\nu} \end{aligned}\right\} \tag{50}$$

Use of the orthogonality relations leads, after some algebra, to the following equations:

$$\begin{aligned} E_{2\pm}(\nu,s) = I_{2\pm}(\nu) \pm \left[\frac{k}{2}\frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)} X_0(-\nu,s)\right]\Bigg[\int_0^1 E_{2\pm}(\mu,s)\frac{e^{-2(s+\sigma_2)a/\mu}\ X_0(-\mu,s)\mu\ d\mu}{\Omega_2^+(\mu,s)\ \Omega_2^-(\mu,s)\ (\mu+\nu)} \\ + \delta_2(s)\ a_{2\pm}\ e^{-(s+\sigma_2)a/\nu_{02}}\ X_0(-\nu_{02},s)\ \frac{\nu_{02}}{\nu_{02}+\nu}\Bigg] \qquad (0 \leqq \nu \leqq 1) \end{aligned} \tag{51}$$

$$\begin{aligned} &\frac{1}{2}c_2\sigma_2\nu_{02}\Omega_2'(\nu_{02},s)\ a_{2\pm}\ e^{(s+\sigma_2)a/\nu_{02}} \\ &\quad = J_{2\pm}(\nu_{02}) \pm \left[\frac{k}{2}\frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)} X_0(-\nu_{02},s)\right]\Bigg[\int_0^1 E_{2\pm}(\mu,s)\frac{e^{-2(s+\sigma_2)a/\mu}\ X_0(-\mu,s)\ \mu\ d\mu}{\Omega_2^+(\mu,s)\ \Omega_2^-(\mu,s)\ (\mu+\nu_{02})} \\ &\quad + \frac{1}{2}a_{2\pm}\ e^{-(s+\sigma_2)a/\nu_{02}}\ X_0(-\nu_{02},s)\Bigg] \qquad (s \in S_{2i}) \end{aligned} \tag{52}$$

$$E_{1\pm}(\nu,s) = I_{1\pm}(\nu) \pm \frac{c_2\sigma_2}{c_1\sigma_1} \frac{\Omega_1^+(\nu,s)\ \Omega_1^-(\nu,s)}{\Omega_2^+(\nu,s)\ \Omega_2^-(\nu,s)} E_{2\pm}(\nu,s)\ e^{-2(s+\sigma_2)a/\nu}$$

$$\pm \left[\frac{1}{2} k \frac{1}{X_0(-\nu,s)}\right]\left[\int_0^1 E_{2\pm}(\mu,s) \frac{e^{-2(s+\sigma_2)a/\mu} X_0(-\mu,s)\ \varphi_{1\nu}(\mu,s)\ 2\mu\ d\mu}{\Omega_2^+(\mu,s)\ \Omega_2^-(\mu,s)\ c_1\sigma_1\nu}\right.$$

$$\left. + \delta_2(s)\ a_{2\pm}\ e^{-(s+\sigma_2)a/\nu_{02}} X_0\left(-\nu_{02},s\right) \frac{\nu_{02}}{\nu - \nu_{02}}\right] \qquad (0 \leqq \nu \leqq 1) \quad (53)$$

and

$$\frac{1}{2} c_1\sigma_1\nu_{01}\Omega_1'\left(\nu_{01},s\right)\ a_{1\pm}\ e^{-(s+\sigma_1)a/\nu_{01}}$$

$$= J_{1\pm}\left(\nu_{01}\right) \mp \left[\frac{1}{2} k \frac{1}{X_0\left(-\nu_{01},s\right)}\right]\left[\int_0^1 E_{2\pm}(\mu,s) \frac{e^{-2(s+\sigma_2)a/\mu} X_0(-\mu,s)\,\mu\ d\mu}{\Omega_2^+(\mu,s)\ \Omega_2^-(\mu,s)\ \left(\mu - \nu_{01}\right)}\right.$$

$$\left. + \delta_2(s)\ a_{2\pm}\ e^{-(s+\sigma_2)a/\nu_{02}} X_0\left(-\nu_{02},s\right) \frac{\nu_{02}}{\nu_{02} - \nu_{01}}\right] \qquad \left(s \in S_{1i}\right) \quad (54)$$

The $I_{m\pm}$ and $J_{m\pm}$ terms in equations (51) to (54) which contain only integrations over the initial distribution are therefore known functions when $f(x,\mu)$ is specified and are given by

$$I_{2\pm}(\nu) = \frac{c_1\sigma_1}{c_2\sigma_2} F_{1\pm}(-a,\nu,s)\ e^{(s+\sigma_1)a/\nu}\ \Omega_2^+(\nu,s)\ \Omega_2^-(\nu,s)$$

$$\pm \left[\frac{1}{2} k \frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)} X_0(-\nu,s)\right]\left\{\left[\int_0^1 F_{2\pm}(a,\mu,s)\ e^{-(s+\sigma_2)a/\mu}\ X_0(-\mu,s) \frac{\mu\ d\mu}{\mu + \nu}\right.\right.$$

$$\left. + \delta_2(s)\ F_{2\pm}\left(a,\nu_{02},s\right)\ e^{-(s+\sigma_2)a/\nu_{02}}\ X_0\left(-\nu_{02},s\right) \frac{\nu_{02}}{\nu_{02} + \nu}\right]$$

$$\mp \frac{\Omega_1(\infty,s)}{\Omega_2(\infty,s)} \left[\int_0^1 F_{1\pm}(-a,\mu,s)\ e^{(s+\sigma_1)a/\mu} \frac{\varphi_{2\nu}(\mu,s)\ 2\mu\ d\mu}{X_0(-\mu,s)\ c_2\sigma_2\nu}\right.$$

$$\left.\left. + \delta_1(s)\ F_{1\pm}\left(-a,\nu_{01},s\right)\ e^{(s+\sigma_1)a/\nu_{01}} \frac{1}{X_0\left(-\nu_{01},s\right)} \frac{\nu_{01}}{\nu - \nu_{01}}\right]\right\} \qquad (0 \leqq \nu \leqq 1) \quad (55)$$

$$J_{2\pm}(\nu_{02}) = \pm\left[\frac{1}{2}k\frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)}X_0(-\nu_{02},s)\right]\left\{\left[\int_0^1 F_{2\pm}(a,\mu,s)\, e^{-(s+\sigma_2)a/\mu}\, X_0(-\mu,s)\frac{\mu\, d\mu}{\mu+\nu_{02}}\right.\right.$$

$$\left.+\frac{1}{2}F_{2\pm}(a,\nu_{02},s)\, e^{-(s+\sigma_2)a/\nu_{02}}\, X_0(-\nu_{02},s)\right]$$

$$\pm\frac{\Omega_1(\infty,s)}{\Omega_2(\infty,s)}\left[\int_0^1 F_{1\pm}(-a,\mu,s)\, e^{(s+\sigma_1)a/\mu}\frac{\mu\, d\mu}{X_0(-\mu,s)\,(\mu-\nu_{02})}\right.$$

$$\left.\left.+\delta_1(s)\, F_{1\pm}(-a,\nu_{01},s)\, e^{(s+\sigma_1)a/\nu_{01}}\frac{\nu_{01}}{X_0(-\nu_{01},s)\,(\nu_{01}-\nu_{02})}\right]\right\} \qquad (s \in S_{2i}) \quad (56)$$

$$I_{1\pm}(\nu) = \mp\left[F_{1\pm}(-a,\nu,s)\, e^{-(s+\sigma_1)a/\nu} - \frac{c_2\sigma_2}{c_1\sigma_1}F_{2\pm}(a,\nu,s)\, e^{-(s+\sigma_2)a/\nu}\right]\Omega_1^+(\nu,s)\,\Omega_1^-(\nu,s)$$

$$\pm\left[\frac{1}{2}k\frac{1}{X_0(-\nu,s)}\right]\left\{\left[\int_0^1 F_{2\pm}(a,\mu,s)\, e^{-(s+\sigma_2)a/\mu}\, X_0(-\mu,s)\,\varphi_{1\nu}(\mu,s)\frac{2\mu\, d\mu}{c_1\sigma_1\nu}\right.\right.$$

$$\left.+\delta_2(s)\, F_{2\pm}(a,\nu_{02},s)\, e^{-(s+\sigma_2)a/\nu_{02}}\, X_0(-\nu_{02},s)\frac{\nu_{02}}{\nu-\nu_{02}}\right]$$

$$\mp\frac{\Omega_1(\infty,s)}{\Omega_2(\infty,s)}\left[\int_0^1 F_{1\pm}(-a,\mu,s)\, e^{(s+\sigma_1)a/\mu}\frac{\mu\, d\mu}{X_0(-\mu,s)\,(\mu+\nu)}\right.$$

$$\left.\left.+\delta_1(s)\, F_{1\pm}(-a,\nu_{01},s)\, e^{(s+\sigma_1)a/\nu_{01}}\frac{\nu_{01}}{X_0(-\nu_{01},s)\,(\nu_{01}+\nu)}\right]\right\} \qquad (0 \leqq \nu \leqq 1) \quad (57)$$

and

$$J_{1\pm}(\nu_{01}) = \mp F_{1\pm}(-a,\nu_{01},s)\,\frac{1}{2}c_1\sigma_1\nu_{01}\Omega_1'(\nu_{01},s)\, e^{-(s+\sigma_1)a/\nu_{01}}$$

$$\mp\left[\frac{1}{2}k\frac{1}{X_0(-\nu_{01},s)}\right]\left\{\left[\int_0^1 F_{2\pm}(a,\mu,s)\, e^{-(s+\sigma_2)a/\mu}\, X_0(-\mu,s)\frac{\mu\, d\mu}{\mu-\nu_{01}}\right.\right.$$

$$\left.+\delta_2(s)\, F_{2\pm}(a,\nu_{02},s)\, e^{-(s+\sigma_2)a/\nu_{02}}\, X_0(-\nu_{02},s)\frac{\nu_{02}}{\nu_{02}-\nu_{01}}\right]$$

$$\pm\frac{\Omega_1(\infty,s)}{\Omega_2(\infty,s)}\left[\int_0^1 F_{1\pm}(-a,\mu,s)\, e^{(s+\sigma_1)a/\mu}\frac{\mu\, d\mu}{X_0(-\mu,s)\,(\mu+\nu_{01})}\right.$$

$$\left.\left.+\frac{1}{2}F_{1\pm}(-a,\nu_{01},s)\, e^{(s+\sigma_1)a/\nu_{01}}\frac{1}{X_0(-\nu_{01},s)}\right]\right\} \qquad (s \in S_{1i}) \quad (58)$$

In these equations, the X_{0m} functions which were shown in reference 14 to be continuous across the curves C_m in the s-plane (see appendix A and fig. 2) have been used. For two material media, one takes the ratio of these single-medium X_{0m} functions

$$X_0(z,s) = \frac{X_{02}(z,s)}{X_{01}(z,s)} \tag{59}$$

where

$$X_{0m}(z,s) = \begin{cases} (\nu_{0m} - z)X_m(z,s) & (s \in S_{mi}) \\ (1 - z)X_m(z,s) & (s \in S_{me}) \end{cases} \tag{60}$$

and

$$X_m(z,s) = \frac{1}{1 - z} \exp\left\{\frac{1}{2\pi i} \int_0^1 \log_e \left[\frac{\Omega_m^+(\nu,s)}{\Omega_m^-(\nu,s)}\right] \frac{d\nu}{\nu - z}\right\} \tag{61}$$

For $\operatorname{Re}(z) < 0$, $X_0(z,s)$ given by equation (59) is a nonvanishing analytic function of z and s provided $s \notin \left(-\sigma_m, -\sigma_m(1 - c_m)\right)$, the branch cut of $\nu_{0m}(s)$, $m = 1, 2$. The quantity

$$k = s\left(c_1\sigma_1 - c_2\sigma_2\right) + \sigma_1\sigma_2\left(c_1 - c_2\right) \tag{62}$$

is related to the difference between medium 1 and medium 2 continuum solutions; several equivalent expressions for k are given in appendix C.

Equations (50) to (58) determine the expansion coefficients $A_{m\pm}$ and $a_{m\pm}$ as follows. Recall that the inhomogeneous terms $I_{m\pm}$ and $J_{m\pm}$ are known functions for a given initial distribution $f(x,\mu)$. Equation (52) is used to eliminate $a_{2\pm}$ from equation (51) and upon using equation (50), one obtains an inhomogeneous Fredholm integral equation for the unknown coefficient $A_{2\pm}$. It is seen from equations (52), (53), and (54) that the remaining unknown coefficients are given in terms of $A_{2\pm}$ and other known functions. However, one needs to know the analytic properties of the transformed solution $\psi_\pm$ in the s-plane in order to invert the Laplace transform. For part of this investigation, another form of the solution is much more convenient.

Complex Representation of $\psi_\pm(x,\mu,s)$

In equations (50), the coefficient $E_{m\pm}(\nu,s)$ were introduced since they are the forms of the normal-mode expansion coefficients which are extendable to the complex plane.

(See ref. 21.) Thus, equations (51) to (58) can be written in a compact form valid for $\mathrm{Re}(s) > -\sigma_{min} = -\min(\sigma_1, \sigma_2)$. A brief outline of this extension to the complex plane is given in appendix E.

$$E_{2\pm}(z,s) = I_{2\pm}(z,s) \pm \frac{k}{c_2\sigma_2} \frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)} \frac{X_0(-z,s)}{2\pi i} \int_{C'} \frac{E_{2\pm}(z',s)\, X_0(-z',s)\, e^{-2(s+\sigma_2)a/z'}}{\Omega_2(z',s)\,(z'+z)}\, dz' \tag{63}$$

$$\begin{aligned} E_{1\pm}(z,s) = I_{1\pm}(z,s) &\pm \frac{c_1\sigma_1}{c_2\sigma_2} E_{2\pm}(z,s)\, e^{-2(s+\sigma_2)a/z} \\ &\mp \frac{k}{c_2\sigma_2 X_0(-z,s)\, 2\pi i} \int_{C'} \frac{E_{2\pm}(z',s)\, X_0(-z',s)\, e^{-2(s+\sigma_2)a/z'}}{\Omega_2(z',s)\,(z'-z)}\, dz' \end{aligned} \tag{64}$$

$$\begin{aligned} I_{2\pm}(z,s) = \frac{c_2\sigma_2}{c_1\sigma_1} L_{1\pm}(-a,z,s) &+ \left[\frac{k}{2\pi i} \frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)} X_0(-z,s)\right]\left[\pm \int_{C'} \frac{L_{2\pm}(a,z',s)\, X_0(-z',s)}{c_2\sigma_2\Omega_2(z',s)\,(z'+z)}\, dz'\right. \\ &\left. + \frac{\Omega_1(\infty,s)}{\Omega_2(\infty,s)} \int_{C'} \frac{L_{1\pm}(-a,z',s)\, dz'}{c_1\sigma_1 X_0(-z',s)\, \Omega_1(z',s)\,(z'-z)}\right] \end{aligned} \tag{65}$$

and

$$\begin{aligned} I_{1\pm}(z,s) = \mp L_{1\pm}(-a,z,s)\, e^{-2(s+\sigma_1)a/z} &\pm \frac{c_1\sigma_1}{c_2\sigma_2} L_{2\pm}(a,z,s) \\ &- \left[\frac{k}{2\pi i X_0(-z,s)}\right]\left[\pm \int_{C'} \frac{L_{2\pm}(a,z',s)\, X_0(-z',s)}{c_2\sigma_2\Omega_2(z',s)\,(z'-z)}\, dz'\right. \\ &\left. + \frac{\Omega_1(\infty,s)}{\Omega_2(\infty,s)} \int_{C'} \frac{L_{1\pm}(-a,z',s)\, dz'}{c_1\sigma_1 X_0(-z',s)\, \Omega_1(z',s)\,(z'+z)}\right] \end{aligned} \tag{66}$$

The functions $L_{m\pm}(x,z,s)$ which appear in equations (65) and (66) are given by

$$\begin{aligned} L_{m\pm}(x,z,s) = \int_{l_m}^{x} e^{-(s+\sigma_m)(x-x_0)/z} &\left[\frac{1}{2} c_m\sigma_m \int_0^1 f_{m\pm}(x_0,-\mu) \frac{d\mu}{\mu+z}\right. \\ &\left. - \frac{1}{2} c_m\sigma_m \int_0^1 f_{m\pm}(x_0,\mu) \frac{d\mu}{\mu-z} + \frac{1}{z} f_{m\pm}(x_0,z)\, \Omega_m(z,s)\right] dx_0 \end{aligned} \tag{67}$$

with

$$\left.\begin{aligned} l_1 &= -\infty \\ l_2 &= -a \end{aligned}\right\} \tag{68}$$

and are analytic for $\operatorname{Re}(s) > -\sigma_m$. These functions were introduced for $z = \nu$, $0 \leqq \nu \leqq 1$, as

$$L_{m\pm}(x,\nu,s) = F_{m\pm}(x,\nu,s)\ \Omega_m^+(\nu,s)\ \Omega_m^-(\nu,s)\ e^{-(s+\sigma_m)x/\nu} \tag{69}$$

in order to extend $F_{m\pm}$ to the complex plane. In equations (63) to (69), z does not lie outside the contour C' which encircles ν_{0m} as shown in figure 3. The restriction $\operatorname{Re}(s) > -\sigma_{min}$ is discussed in the next section.

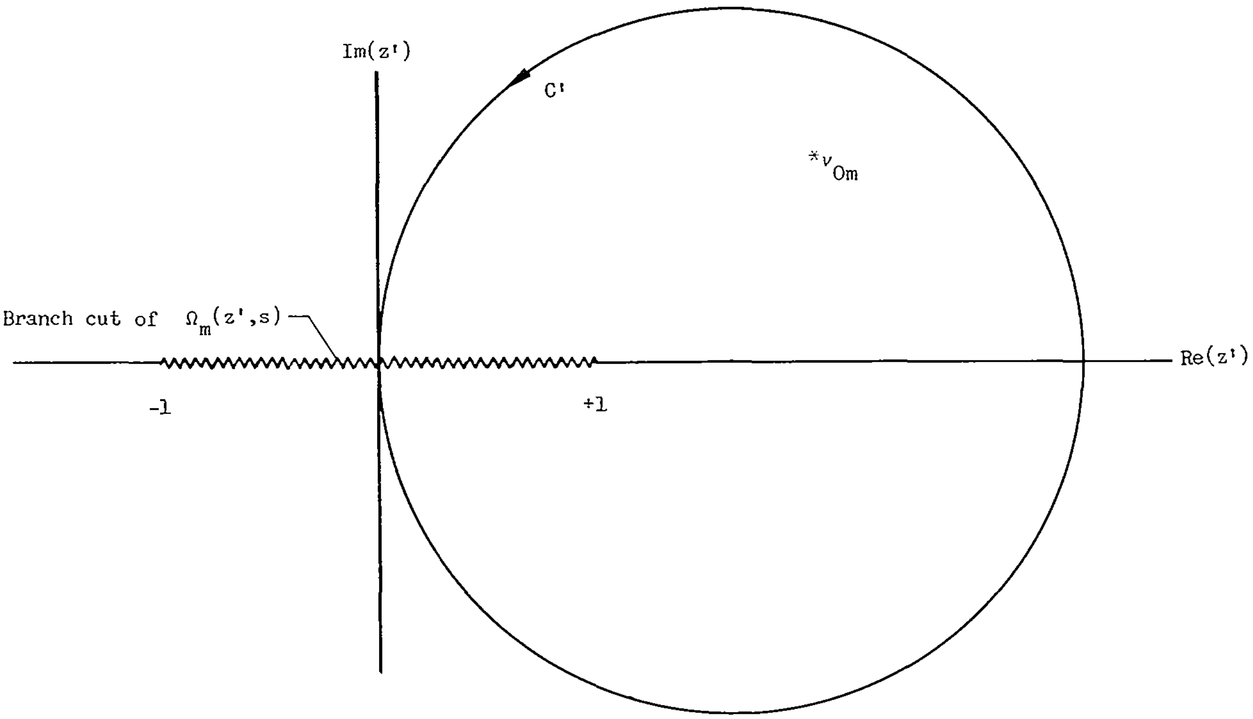

Figure 3.- Contour C' in z'-plane.

That equations (63) to (66) reduce to equations (51) to (58) as all contours C' are collapsed onto the branch cut $\nu \in (0,1)$ due to $\Omega_m(z,s)$ (refer to fig. 3) can be seen as follows. If $s \in S_{mi}$, $\left[\Omega_m(z)\right]^{-1}$ has a pole at $z = \nu_{0m}$ whose residue leads to a discrete term. When $s \in S_{me}$, $\Omega_m(z)$ does not vanish. The continuum terms are simply those due to the integration around the branch cut.

The solutions $\psi_{mc\pm}(x,\mu,s)$ and $\psi_{mp\pm}(x,\mu,s)$ can now be written similarly as

$$\psi_{2c\pm}(x,\mu,s) = \frac{1}{2\pi i}\left[\int_{C'} \frac{E_{2\pm}(z',s)\ e^{-(s+\sigma_2)(a+x)/z'}}{\Omega_2(z',s)\ (z'-\mu)}\, dz' \pm \int_{C'} \frac{E_{2\pm}(z',s)\ e^{-(s+\sigma_2)(a-x)/z'}}{\Omega_2(z',s)\ (z'+\mu)}\, dz'\right] \quad (|x| < a) \qquad (70)$$

for $Re(s) > -\sigma_{min}$,

$$\psi_{1c\pm}(x,\mu,s) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{C'} \frac{E_{1\pm}(z',s)\ e^{(s+\sigma_1)(x+a)/z'}}{\Omega_1(z',s)\ (z'+\mu)}\, dz' & (x < -a) \\[2ex] \dfrac{\pm 1}{2\pi i}\displaystyle\int_{C'} \frac{E_{1\pm}(z',s)\ e^{-(s+\sigma_1)(x-a)/z'}}{\Omega_1(z',s)\ (z'-\mu)}\, dz' & (x > a) \end{cases} \qquad (71)$$

for $Re(s) > -\sigma_{min}$,

$$\psi_{2p\pm}(x,\mu,s) = \frac{1}{2\pi i}\left[\int_{C'} \frac{L_{2\pm}(x,z',s)}{\Omega_2(z',s)\ (z'-\mu)}\, dz' \pm \int_{C'} \frac{L_{2\pm}(-x,z',s)}{\Omega_2(z',s)\ (z'+\mu)}\, dz'\right] \quad (|x| < a) \qquad (72)$$

for $Re(s) > -\sigma_2$, and

$$\psi_{1p\pm}(x,\mu,s) = \begin{cases} \dfrac{1}{2\pi i}\left[\displaystyle\int_{C'} \frac{L_{1\pm}(x,z',s)}{\Omega_1(z',s)\ (z'-\mu)}\, dz' + \int_{C'} \frac{M_{\pm}(x,z',s) \pm L_{1\pm}(-a,z',s)\ e^{-(s+\sigma_1)(a-x)/z'}}{\Omega_1(z',s)\ (z'+\mu)}\, dz'\right] & (x < -a) \\[2ex] \dfrac{1}{2\pi i}\left[\displaystyle\int_{C'} \frac{L_{1\pm}(-a,z',s)\ e^{-(s+\sigma_1)(a+x)/z'} \mp M_{\pm}(x,z',s)}{\Omega_1(z',s)\ (z'-\mu)}\, dz' \pm \int_{C'} \frac{L_{1\pm}(-x,z',s)}{\Omega_1(z',s)\ (z'+\mu)}\, dz'\right] & (x > a) \end{cases} \qquad (73)$$

for $Re(s) > -\sigma_1$.

The functions $M_{\pm}(x,z,s)$ are also integrations over the initial distribution $f_{1\pm}(x,\mu)$ and are given by

$$M_{\pm}(x,z,s) = -\int_{-x}^{-a} e^{-(s+\sigma_1)(x+x_0)/z}\left[\frac{1}{2}c_1\sigma_1\int_0^1 f_{1\pm}(x_0,\mu)\frac{d\mu}{\mu+z}\right.$$

$$\left.-\frac{1}{2}c_1\sigma_1\int_0^1 f_{1\pm}(x_0,-\mu)\frac{d\mu}{\mu-z}+\frac{1}{z}f_{1\pm}(x_0,-z)\ \Omega_1(z,s)\right]dx_0 \qquad (x > a) \qquad (74)$$

and

$$M_{\pm}(x,z,s) = \int_{x}^{-a} e^{-(s+\sigma_1)(x_0-x)/z}\left[\frac{1}{2}c_1\sigma_1\int_0^1 f_{1\pm}(x_0,\mu)\frac{d\mu}{\mu+z}\right.$$

$$\left.-\frac{1}{2}c_1\sigma_1\int_0^1 f_{1\pm}(x_0,-\mu)\frac{d\mu}{\mu-z}+\frac{1}{z}f_{1\pm}(x_0,-z)\ \Omega_1(z,s)\right]dx_0 \qquad (x < -a) \qquad (75)$$

for $\mathrm{Re}(s) > -\sigma_1$ and z not outside C'. Again, the discrete and continuum terms which appear in equations (32), (39), (44), and (45) are due to the zeros and branch cuts of $\Omega_m(z,s)$ which appear in the integrands of equations (70) to (73).

PROPERTIES OF TRANSFORMED SOLUTION

General Properties in s-Plane

Analytic properties of $\psi_{\pm}(x,\mu,s)$ as a function of s must be investigated before the time-dependent solution $\Psi(x,\mu,t)$ can be recovered according to the inverse Laplace transformation given by equation (7). To do this, the behavior of $\psi_{\pm}$ in some right-half s-plane is required. Before looking at the details, a review of some results of earlier cited work in which Case's method was used is in order. In these works, the analytic properties of the functions of s such as ν_{0m}, Ω_m, and the various X-functions are given.

In the semi-infinite medium problems considered in references 14 to 16, expansion coefficients could be found explicitly and this fact aided in the extraction of the s-dependence of the transformed solutions. These solutions were found to contain the branch cuts of $\nu_{0m}(s)$ so that the integration contour of the inverse Laplace transformation was deformed around these branch cuts. For the slab problem solved in references 12 and 13, expansion coefficients could not be found explicitly but the theorems of references 6 and 7 gave the analytic properties of the transformed solution in the s-plane. In that problem, the solution does not contain the branch cut of $\nu_0(s)$ even though $\nu_0(s)$

appears explicitly in it. Instead, there are a finite number of poles at values of s, for example, $s_0, \ldots, s_N$, which lie on the branch cut of $\nu_0(s)$, that is, on the real s-axis. These poles contribute a sum of residues as the integration contour is moved to the left of them in the s-plane. Furthermore, in these previously solved time-dependent problems, there is a real number, for example γ_1, such that the integration contour cannot be deformed into the region $\mathrm{Re}(s) < \gamma_1$ for arbitrary values of x. The present transformed solution should exhibit similar properties; that is, $\psi_\pm$ may not be analytic for $\mathrm{Re}(s)$ less than some number γ_1 when x is arbitrary whereas for $\mathrm{Re}(s)$ greater than γ_1, it should be analytic except for poles and/or branch cuts. Such singularities probably occur where $\nu_{0m}(s)$ has its branch cut.

First note that for arbitrary initial distributions $f(x,\mu)$, $\psi_\pm(x,\mu,s)$ is not analytic for $\mathrm{Re}(s) < -\sigma_{min}$. This statement is true since each of the inhomogeneous terms $I_{m\pm}$ of equations (63) and (64) contains both $L_{1\pm}$ and $L_{2\pm}$ as can be seen from equations (65) and (66) and one of the two will not be analytic for $\mathrm{Re}(s) < -\sigma_{min} = -\min(\sigma_1, \sigma_2)$. In particular, note that for $|x| > a$, $\psi_{1\pm}(x,\mu,s)$ never appears to be analytic for $\mathrm{Re}(s) < -\sigma_{min}$. However, for special cases of material properties and initial distributions, $\psi_{2\pm}(x,\mu,s)$ can be shown to be analytic for $-\sigma_2 < \mathrm{Re}(s) < -\sigma_1$ except perhaps for poles.

Consider now the behavior of $\psi_\pm$ for $\mathrm{Re}(s) > -\sigma_{min}$. Note that the transform plane for the present problem must be taken as a superposition of two "single-medium" planes, that is, one for each material medium in the problem. The expressions (32), (39), (44), and (45) for the transformed solution were not defined for $s \in C_m$ and outwardly appear to be discontinuous at $s \in C_m$. However, this is not the case. The complex representation of $E_{m\pm}$ given by equations (63) and (64) shows that such coefficients are continuous across the curves C_m. Thus, it is seen from the representation of $\psi_\pm$ given in equations (70) to (73) that $\psi_\pm$ is indeed continuous across the curves C_m.

The Associated Eigenvalue Problem

It is convenient to introduce at this time the solution of the associated eigenvalue problem, that is, the solution of equation (18) subject to the boundary conditions (19) and (20) with $f_{m\pm}(x,\mu) \equiv 0$. Such solutions, denoted with a bar, have the form

$$
\overline{\psi}_{\pm}(x,\mu,s) = \begin{cases} \overline{b}_{1\pm}\psi_{-\nu_{01}}(x,\mu,s)\ \delta_1(s) + \int_0^1 \overline{B}_{1\pm}(-\nu)\ \psi_{1(-\nu)}(x,\mu,s)\ d\nu & (x < -a) \\ \left[\psi_{\nu_{02}}(x,\mu,s) \pm \psi_{-\nu_{02}}(x,\mu,s)\right]\delta_2(s) & \\ + \int_0^1 \overline{B}_{2\pm}(\nu)\left[\psi_{2\nu}(x,\mu,s) \pm \psi_{2(-\nu)}(x,\mu,s)\right] d\nu & (|x| < a) \\ \pm\overline{b}_{1\pm}\psi_{\nu_{01}}(x,\mu,s)\ \delta_1(s) \pm \int_0^1 \overline{B}_{1\pm}(-\nu)\ \psi_{1\nu}(x,\mu,s)\ d\nu & (x > a) \end{cases} \tag{76}
$$

where obviously $\overline{B}_{m\pm}$ and $\overline{b}_{1\pm}$ can be obtained from the $E_{m\pm}$ given by equations (63) and (64) in the case $f_{m\pm}(x,\mu) \equiv 0$. It will be seen later that the solution $\psi_{\pm}$ has poles at those values of s for which the associated eigenvalue problem has nontrivial solutions. In appendix F, it is shown that as the slab thickness becomes very large, this eigenvalue problem has only trivial solutions for $\mathrm{Re}(s) > -\sigma_2$ except perhaps on the branch cuts of $\nu_{0m}(s)$. When the slab thickness is not large, one still expects that if the eigenvalue problem has nontrivial solutions for $\mathrm{Re}(s) > -\sigma_2$, they occur only when s is real. This statement has been proved rigorously by using the approach of Lehner and Wing (refs. 6 and 7) for several problems which are special cases of the present problem: the bare slab considered in references 6 and 7 and the slab surrounded by pure absorbers considered in references 8 and 9. In all these problems, there is no scattering in the reflector and, therefore, no branch cut of $\nu_{01}(s)$. As already indicated, the $X_0(z,s)$ function contains the branch cuts due to both $\nu_{01}(s)$ and $\nu_{02}(s)$ and these branch cuts lie on the real s-axis from $-\sigma_m$ to $-\sigma_m(1 - c_m)$ and may or may not overlap depending on the values of material properties. Note that c_1 has been taken less than unity and this insures that the branch cut of ν_{01} lies entirely to the left of $s = 0$. In previously solved time-dependent problems, singularities of the transformed solution always occur where the $\nu_{0m}(s)$ has branch cuts. Since the analysis of appendix F indicates that for large values of the slab half-thickness a, the singularities of $\psi_{\pm}$ for $\mathrm{Re}(s) > -\sigma_{min}$ also occur where the $\nu_{0m}(s)$ have branch cuts, it will be assumed for all values of a that the singularities of $\psi_{\pm}$ occur on the branch cuts of $\nu_{0m}(s)$. In any case, it is shown that the only other singularities of $\psi_{\pm}$, $\mathrm{Re}(s) > -\sigma_{min}$ which could occur off the branch cuts of $\nu_{0m}(s)$ are poles, whose residue could readily be added to the time-dependent solution.

In order to determine the behavior of $\psi_{\pm}$ on the branch cuts of $\nu_{0m}(s)$ one first considers $\overline{\psi}_{\pm}$ in the region $s \in S_{1i} \cap S_{2i}$. For this region, the expansion coefficients are given by the following equations. (See appendix G for a brief description of the manner in which these equations were obtained.)

$$\overline{B}_{2\pm}(\mu) = \pm\frac{k}{2}\,\frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)}\,\frac{\left(\nu_{02}^2 - \mu^2\right)}{\left(\nu_{01}^2 - \mu^2\right)}\,\frac{h_2(\mu)}{N_2(\mu)}\left[\frac{h_2(\nu_{02})}{\mu + \nu_{02}} \pm \frac{h_2(-\nu_{02})}{\mu - \nu_{02}} + \int_0^1 \overline{B}_{2\pm}(\nu)\, h_2(\nu)\frac{d\nu}{\nu + \mu}\right] \qquad (0 \leqq \mu \leqq 1) \tag{77}$$

$$\overline{B}_{1\pm}(-\mu) \mp \frac{c_2\sigma_2}{c_1\sigma_1}\overline{B}_{2\pm}(\mu)\, e^{(\sigma_1-\sigma_2)a/\mu}$$

$$= \pm\frac{k}{2}\,\frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)}\,\frac{h_1(\mu)}{N_1(\mu)}\left[\frac{h_2(\nu_{02})}{\mu - \nu_{02}} \pm \frac{h_2(-\nu_{02})}{\mu + \nu_{02}} + \int_0^1 \overline{B}_{2\pm}(\nu)\, h_2(\nu)\frac{2\varphi_{1\mu}(\nu,s)}{c_1\sigma_1\mu}\, d\nu\right] \tag{78}$$

and

$$\mp h_1(-\nu_{01})\, \overline{b}_{1\pm} = h_2(\nu_{02}) \pm h_2(-\nu_{02}) + \left(\nu_{02}^2 - \nu_{01}^2\right)\int_0^1 \overline{B}_{2\pm}(\nu)\, h_2(\nu)\frac{d\nu}{\nu^2 - \nu_{01}^2} \tag{79}$$

where

$$\left.\begin{aligned} h_2(\omega) &= \omega\frac{X_2(-\omega,s)}{X_1(-\omega,s)}\, e^{-(s+\sigma_2)a/\omega} \\ h_1(\omega) &= \frac{\Omega_1(\infty,s)}{\Omega_2(\infty,s)}\,\omega\frac{X_1(-\omega,s)}{X_2(-\omega,s)}\, e^{(s+\sigma_1)a/\omega} \\ N_m(\mu) &= \mu\Omega_m^+(\mu,s)\, \Omega_m^-(\mu,s) \end{aligned}\right\} \tag{80}$$

In addition, the eigenvalue condition

$$0 = \frac{h_2(\nu_{02})}{\nu_{01} + \nu_{02}} \pm \frac{h_2(-\nu_{02})}{\nu_{01} - \nu_{02}} + \int_0^1 \overline{B}_{2\pm}(\nu)\, h_2(\nu)\frac{d\nu}{\nu + \nu_{01}} \tag{81}$$

must be satisfied. As noted in appendix G, this equation is an additional constraint on the solutions of equation (77). Since the eigenvalue condition (81) has different limiting values as s approaches the branch cut of $\nu_{01}(s)$, it is concluded that there are only trivial solutions of the associated eigenvalue problem on the $\nu_{01}(s)$ cut. When s belongs to the branch cut of $\nu_{02}(s)$ which is not also part of the $\nu_{01}(s)$ cut, that is, when $\mathrm{Re}(\nu_{02}) = \mathrm{Im}(\nu_{01}) = 0$, it appears that nontrivial solutions of the associated eigenvalue problem may exist. From Bowden's results for the bare slab (refs. 12 and 13), it is expected that equations (77) and (81) are satisfied only at isolated points $\{s_n\}$. In the limit $c_2\sigma_2 a \to \infty$, these points lie on the branch cut of $\nu_{02}(s)$, that is, the s_n are real. The "thick-slab" eigenvalue condition is seen from equations (77) and (81) to be equation (81) with $\overline{B}_{2\pm}(\mu) = 0$.

If material properties are such that $-\sigma_2 < -\sigma_1$, then part of the branch cut of $\nu_{02}(s)$ lies in $s \in S_{2i} \cap S_{1e}$. In this region however, $s < -\sigma_{min} = -\sigma_1$ and for such values, the solution $\overline{\psi}_\pm(x,\mu,s)$, $|x| > a$, that is $\overline{\psi}_{1\pm}$, is not bounded as $|x| \to \infty$. However, $\overline{\psi}_{2\pm}$ may have nontrivial solutions on such a part of the branch cut of $\nu_{02}(s)$. The equation for $\overline{B}_{2\pm}$ and the additional constraint for this region are (again, see appendix G for some discussion)

$$\overline{B}_{2\pm}(\mu) = \pm\frac{k}{2}\frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)}\left(\nu_{02}^2 - \mu^2\right)\frac{h_2(\mu)}{N_2(\mu)}\frac{X_1(-\mu,s)}{X_{01}(-\mu,s)}\left[\frac{h_2(\nu_{02})}{\mu + \nu_{02}}\frac{X_1(-\nu_{02},s)}{X_{01}(-\nu_{02},s)} \pm \frac{h_2(-\nu_{02})}{\mu - \nu_{02}}\frac{X_1(\nu_{02},s)}{X_{01}(\nu_{02},s)} + \int_0^1 \overline{B}_{2\pm}(\nu)\, h_2(\nu)\frac{X_1(-\nu,s)}{X_{01}(-\nu,s)}\frac{d\nu}{\nu + \mu}\right] \quad (0 \leqq \mu \leqq 1) \tag{82}$$

and

$$0 = h_2(\nu_{02})\frac{X_1(-\nu_{02},s)}{X_{01}(-\nu_{02},s)} \pm h_2(-\nu_{02})\frac{X_1(\nu_{02},s)}{X_{01}(\nu_{02},s)} + \int_0^1 \overline{B}_{2\pm}(\nu)\, h_2(\nu)\frac{X_1(-\nu,s)}{X_{01}(-\nu,s)}\, d\nu \tag{83}$$

It will be seen later that the zeros of equation (83) can, under some conditions, be poles of $\psi_{2\pm}$ and therefore may contribute discrete modes in $\Psi(x,\mu,t)$, $|x| < a$. For this reason one is interested in where these zeros lie. They will be referred to as pseudo-eigenvalues.

Relationship Between $\psi_\pm$ and $\overline{\psi}_\pm$

It is now shown how the solution of the associated eigenvalue problem $\overline{\psi}_\pm$ is contained in the inhomogeneous solution $\psi_\pm$ by following a procedure similar to that of Bowden and Williams (ref. 13). In appendix H, it is shown that the original expansion coefficients of equations (32) and (39) can be written as

$$A_{m\pm}(\mu) = \left[a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\right]\overline{B}_{m\pm}(\mu) + B_{m\pm}(\mu) \tag{84a}$$

and

$$a_{1\pm} = \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]\overline{b}_{1\pm} + b_{1\pm} \qquad \left(s \in S_{1i} \cap S_{2i}\right) \qquad (84b)$$

where $\overline{B}_{m\pm}$ and $\overline{b}_{1\pm}$ are given by equations (77) to (79). The coefficients $B_{m\pm}$ and $b_{1\pm}$ are given by

$$\begin{aligned} B_{2\pm}(\nu) = {} & \frac{c_1\sigma_1}{c_2\sigma_2} F_{1\pm}(-a,\nu,s)\ e^{(\sigma_1-\sigma_2)a/\nu} \\ & \pm \frac{k}{2} \frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)} \frac{\left(\nu_{02}^{2} - \nu^2\right)}{\left(\nu_{01}^{2} - \nu^2\right)} \frac{h_2(\nu)}{N_2(\nu)} \left\{\int_0^1 B_{2\pm}(\mu)\ h_2(\mu)\ \frac{d\mu}{\mu + \nu}\right. \\ & + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\left[\frac{h_2(\nu_{02})}{\nu + \nu_{02}} \mp \frac{h_2(-\nu_{02})}{\nu - \nu_{02}}\right] + \int_0^1 F_{2\pm}(a,\mu,s)\ h_2(\mu) \frac{d\mu}{\mu + \nu} \\ & \left. \mp \int_0^1 F_{1\pm}(-a,\mu,s)\ h_1(\mu) \frac{\nu_{01}^{2} - \mu^2}{\nu_{02}^{2} - \mu^2} \frac{2\varphi_{2\nu}(\mu,s)}{c_2\sigma_2\nu}\ d\mu\right\} \end{aligned} \qquad (85)$$

$$\begin{aligned} & B_{1\pm}(-\nu) \mp \frac{c_2\sigma_2}{c_1\sigma_1} B_{2\pm}(\nu)\ e^{(\sigma_1-\sigma_2)a/\nu} \\ & \qquad = \pm \frac{k}{2} \frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)} \frac{h_1(\nu)}{N_1(\nu)} \left\{\frac{1}{2} F_{2\pm}(a,\nu_{02},s)\left[\frac{h_2(\nu_{02})}{\nu - \nu_{02}} \mp \frac{h_2(-\nu_{02})}{\nu + \nu_{02}}\right]\right. \\ & \qquad + \int_0^1 \left[B_{2\pm}(\mu) + F_{2\pm}(a,\mu,s)\right] h_2(\mu)\ \frac{2\varphi_{1\nu}(\mu,s)}{c_1\sigma_1\nu}\ d\mu \\ & \qquad \left. \mp \int_0^1 F_{1\pm}(-a,\mu,s)\ h_1(\mu)\ \frac{\left(\nu_{01}^{2} - \mu^2\right)}{\left(\nu_{02}^{2} - \mu^2\right)} \frac{d\mu}{\mu + \nu}\right\} \\ & \qquad \mp \left[F_{1\pm}(-a,\nu,s) - \frac{c_2\sigma_2}{c_1\sigma_1} F_{2\pm}(a,\nu,s)\ e^{(\sigma_1-\sigma_2)a/\nu}\right] \end{aligned} \qquad (86)$$

and

$$\mp h_1\left(-\nu_{01}\right)\left[b_{1\pm} - \widetilde{F}_{\pm}\left(-a,\nu_{01},s\right)\right] = \beta_{1\pm} \qquad (87)$$

The coefficient $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\right]$ is given by

$$\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\right] = \frac{-\nu_{01}\beta_{1\pm} + \beta_{2\pm}}{\left(\nu_{01}\alpha_{1\pm} - \alpha_{2\pm}\right)} \tag{88}$$

In these equations, $\alpha_{1\pm}$, $\alpha_{2\pm}$, $\beta_{1\pm}$, and $\beta_{2\pm}$ are

$$\left.\begin{aligned} \alpha_{1\pm} &= h_2\left(\nu_{02}\right) \pm h_2\left(-\nu_{02}\right) + \left(\nu_{02}^2 - \nu_{01}^2\right)\int_0^1 \overline{B}_{2\pm}(\mu)\ h_2(\mu)\ \frac{d\mu}{\mu^2 - \nu_{01}^2} \\ \alpha_{2\pm} &= \nu_{02}h_2\left(\nu_{02}\right) \mp \nu_{02}h_2\left(-\nu_{02}\right) + \left(\nu_{02}^2 - \nu_{01}^2\right)\int_0^1 \overline{B}_{2\pm}(\mu)\ h_2(\mu)\ \frac{\mu\ d\mu}{\mu^2 - \nu_{01}^2} \end{aligned}\right\} \tag{89}$$

$$\begin{aligned} \beta_{1\pm} = &\frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\left[h_2\left(\nu_{02}\right) \mp h_2\left(-\nu_{02}\right)\right] \\ &+ \left(\nu_{02}^2 - \nu_{01}^2\right)\int_0^1 \left[B_{2\pm}(\mu) + F_{2\pm}(a, \mu, s)\right] h_2(\mu)\ \frac{\mu\ d\mu}{\mu^2 - \nu_{01}^2} \\ &\pm \left(\nu_{01}^2 - \nu_{02}^2\right)\int_0^1 F_{1\pm}(-a, \mu, s)\ h_1(\mu)\ \frac{d\mu}{\mu^2 - \nu_{02}^2} \\ &\pm \left[F_{1\pm}\left(-a, \nu_{01}, s\right) h_1\left(\nu_{01}\right) + F_{1\pm}\left(-a, -\nu_{01}, s\right) h_1\left(-\nu_{01}\right)\right] \end{aligned} \tag{90a}$$

and

$$\begin{aligned} \beta_{2\pm} = &\frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\ \ \nu_{02}\left[h_2\left(\nu_{02}\right) \pm h_2\left(-\nu_{02}\right)\right] \\ &+ \left(\nu_{02}^2 - \nu_{01}^2\right)\int_0^1 \left[B_{2\pm}(\mu) + F_{2\pm}(a, \mu, s)\right] h_2(\mu)\ \frac{\mu\ d\mu}{\mu^2 - \nu_{01}^2} \\ &\mp \left(\nu_{01}^2 - \nu_{02}^2\right)\int_0^1 F_{1\pm}(-a, \mu, s)\ h_1(\mu)\ \frac{\mu\ d\mu}{\mu^2 - \nu_{02}^2} \\ &\mp \left[\nu_{01}F_{1\pm}\left(-a, \nu_{01}, s\right) h_1\left(\nu_{01}\right) - \nu_{01}F_{1\pm}\left(-a, -\nu_{01}, s\right)\ h_1\left(-\nu_{01}\right)\right] \end{aligned} \tag{90b}$$

In terms of these quantities, the solutions $\psi_{m\pm}$ can be written as

$$\psi_{2\pm}(x,\mu,s) = \left[a_{2\pm} + \frac{1}{2} F_{2\pm}\left(a,\nu_{02},s\right)\right] \overline{\psi}_{2\pm}(x,\mu,s) + \int_0^1 B_{2\pm}(\nu) \left[\psi_{2\nu}(x,\mu,s) \pm \psi_{2(-\nu)}(x,\mu,s)\right] d\nu$$

$$+ \int_0^1 \left[F_{2\pm}(x,\nu,s)\ \psi_{2\nu}(x,\mu,s) \pm F_{2\pm}(-x,\nu,s)\ \psi_{2(-\nu)}(x,\mu,s)\right] d\nu$$

$$+ \frac{1}{2}\left[F_{2\pm}\left(x,\nu_{02},s\right) \pm F_{2\pm}\left(-x,-\nu_{02},s\right)\right] \psi_{\nu_{02}}(x,\mu,s)$$

$$+ \frac{1}{2}\left[F_{2\pm}\left(x,-\nu_{02},s\right) \pm F_{2\pm}\left(-x,\nu_{02},s\right)\right] \psi_{-\nu_{02}}(x,\mu,s) \qquad (|x| < a) \quad (91)$$

and

$$\psi_{1\pm}(x,\mu,s) = \left[a_{2\pm} + \frac{1}{2} F_{2\pm}\left(a,\nu_{02},s\right)\right] \overline{\psi}_{1\pm}(x,\mu,s)$$

$$\pm \left[b_{1\pm} - \tilde{F}_{\pm}\left(-a,\nu_{01},s\right) + F_{1\pm}\left(-x,-\nu_{01},s\right)\right] \psi_{\nu_{01}}(x,\mu,s)$$

$$\pm F_{1\pm}\left(-x,\nu_{01},s\right) \psi_{-\nu_{01}}(x,\mu,s)$$

$$\pm \int_0^1 \left[B_{1\pm}(-\nu) - \tilde{F}_{\pm}(-a,\nu,s) + F_{1\pm}(-x,-\nu,s)\right] \psi_{1\nu}(x,\mu,s)$$

$$\pm \int_0^1 F_{1\pm}(-x,\nu,s)\ \psi_{1(-\nu)}(x,\mu,s)\ d\nu \qquad (x > a) \quad (92)$$

The solution $\psi_{1\pm}(x,\mu,s)$ for $x < -a$ has a similar form. In these equations, $\overline{\psi}_{m\pm}(x,\mu,s)$ are the parts of $\overline{\psi}_{\pm}(x,\mu,s)$ which are given by equation (76). Equation (79) is written in terms of $\alpha_{1\pm}$ as

$$\mp h_1\left(-\nu_{01}\right) \overline{b}_{1\pm} = \alpha_{1\pm} \qquad (93)$$

Properties of $\psi_{\pm}$ on the Branch Cuts of $\nu_{0m}(s)$

Consider now what happens on the branch cut of $\nu_{01}(s)$ where $\nu_{01} = i|\nu_{01}|$ for $\mathrm{Im}(s) = 0^-$ and $\nu_{01} = -i|\nu_{01}|$ for $\mathrm{Im}(s) = 0^+$. From these equations, it can be shown that the quantities $\overline{B}_{2\pm}$, $\overline{B}_{1\pm}$, $B_{2\pm}$, $B_{1\pm}$, $\alpha_{1\pm}$, $\alpha_{2\pm}$, $\beta_{1\pm}$, and $\beta_{2\pm}$ do not contain the branch cut of $\nu_{01}(s)$. Equations (93) and (87) show that $\overline{b}_{1\pm}$ and $b_{1\pm}$ have branch cuts due to that of $\nu_{01}(s)$. Equation (88) indicates that $a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)$ has the branch cut due to $\nu_{01}(s)$ unless $\alpha_{1\pm}/\alpha_{2\pm}$ is equal to $\beta_{1\pm}/\beta_{2\pm}$. In general, this statement will not be true since $\beta_{1\pm}/\beta_{2\pm}$ depends on the arbitrary initial distribution $f_{\pm}(x,\mu)$ whereas $\alpha_{1\pm}/\alpha_{2\pm}$ does not. Therefore, it is concluded that both $\psi_{1\pm}$ and $\psi_{2\pm}$ contain the branch cut of $\nu_{01}(s)$.

On the branch cut of $\nu_{02}(s)$, the quantities $B_{2\pm}$, $B_{1\pm}$, $b_{1\pm}$, $\beta_{1\pm}$, and $\beta_{2\pm}$ are single valued. Since the quantities $\alpha_{1\pm}$ and $\alpha_{2\pm}$ of equation (89) are related above and below the branch cut of $\nu_{02}(s)$ by

$$\left[\alpha_{j\pm}\right]^+ = \pm\left[\alpha_{j\pm}\right]^- \tag{94}$$

it follows then from equation (88) that on that part of the branch cut of $\nu_{02}(s)$ which is not also part of the $\nu_{01}(s)$ cut, that is, for $\mathrm{Re}(\nu_{02}) = \mathrm{Im}(\nu_{01}) = 0$, one has

$$\left[a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\right]^+ = \pm\left[a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\right]^- \tag{95}$$

if the denominator on the right-hand side of equation (88) does not vanish. It is seen from equations (76) to (79) that for this same region,

$$\left[\overline{\psi}_{\pm}(x,\mu,s)\right]^+ = \pm\left[\overline{\psi}_{\pm}(x,\mu,s)\right]^- \tag{96}$$

Hence, the product

$$\left[a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\right]\overline{\psi}_{\pm}(x,\mu,s) \tag{97}$$

which appears in $\psi_{\pm}$ does not contain the branch cut of $\nu_{02}(s)$. However, the denominator of $a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)$, namely $\nu_{01}\alpha_{1\pm} - \alpha_{2\pm}$, is equivalent to the eigenvalue condition (eq. (81)). Thus, if the associated eigenvalue problem has a nontrivial solution at $s = s_n$, $\mathrm{Re}(s) > -\sigma_{min}$ then, $\psi_{\pm}$ has a pole there.

The analytic properties of the transformed solution $\psi_{\pm}(x,\mu,s)$ may be summarized as follows. For arbitrary initial distributions $f_{\pm}(x,\mu)$, $\psi_{\pm}$ is not analytic to the left of $\mathrm{Re}(s) = -\sigma_{min}$ in the s-plane, whereas to the right of $\mathrm{Re}(s) = -\sigma_{min}$ it is analytic except for the branch cut along $\left(-\sigma_{min}, -\sigma_1(1 - c_1)\right)$ (due to the branch cut of $\nu_{01}(s)$) if $\sigma_{min} > \sigma_1(1 - c_1)$ and poles at the values of s at which the associated eigenvalue problem has nontrivial solutions $\overline{\psi}_{\pm}$. It has been assumed that for arbitrary slab thicknesses a, these poles, if they exist, lie on the branch cut of $\nu_{02}(s)$ since this result is the rigorous one obtained by others for several special cases of the present problem and obtained herein for the case when $c_2\sigma_2 a$ is large. For special values of material properties and initial data, $\psi_{\pm}(x,\mu,s)$ for $|x| < a$ (that is, $\psi_{2\pm}$) may be analytic in the region $-\sigma_2 < \mathrm{Re}(s) < -\sigma_1$ except perhaps for poles.

RECOVERY OF TIME-DEPENDENT SOLUTION

The time-dependent solution $\Psi(x,\mu,t)$ is obtained from the inverse Laplace transformation (eq. (7)) where γ is to the right of all singularities of $\psi(x,\mu,s)$ in the s-plane. From the preceding analysis, it is expected that one can choose any $\gamma > \max\left(-\sigma_1(1 - c_1), -\sigma_2(1 - c_2)\right)$. In order to show the time dependence of the solution $\Psi(x,\mu,t)$ more explicitly, the inversion contour should be deformed as far as possible to the left in the s-plane by making use of the analytic properties of $\psi(x,\mu,s)$ obtained in the previous section.

Behavior of $\psi_{\pm}$ on the Contour $\mathrm{Re}(s) = \gamma$

The behavior of $\psi_{\pm}$ on the contour $\mathrm{Re}(s) = \gamma$ must be examined as $|s| \to \infty$. (See fig. 4.) This contour crosses both of the curves C_1 and C_2 and it has been shown that $\psi_{\pm}$ is continuous across these curves. As $|s| \to \infty$ on such a contour, $s \in S_{1e} \cap S_{2e}$ and it is shown in appendix I that $\psi_{\pm}$ behaves as follows:

$$\psi_{2\pm}(x,\mu,s) \to \frac{1}{\mu}\int_{-a}^{x} e^{-(s+\sigma_2)(x-x_0)/\mu}\left[f_{2\pm}(x_0,\mu) + 0\left(\frac{1}{s}\right)\right] dx_0$$

$$+ \frac{e^{-(\sigma_2-\sigma_1)\left(\frac{a+x}{\mu}\right)}}{\mu}\int_{-\infty}^{-a} e^{-(s+\sigma_1)\left(\frac{x-x_0}{\mu}\right)}\left[f_{1\pm}(x_0,\mu) + 0\left(\frac{1}{s}\right)\right] dx_0 \qquad (98)$$

for $|x| < a$ and $\mu > 0$,

$$\psi_{2\pm}(x,\mu,s) \rightarrow \frac{1}{|\mu|}\int_x^a e^{-(s+\sigma_2)\left(\frac{x_0-x}{|\mu|}\right)}\left[f_{2\pm}\left(x_0,-|\mu|\right)+0\left(\frac{1}{s}\right)\right]dx_0$$

$$+\frac{e^{-(\sigma_2-\sigma_1)\left(\frac{a-x}{|\mu|}\right)}}{|\mu|}\int_a^\infty e^{-(s+\sigma_1)\left(\frac{x_0-x}{|\mu|}\right)}\left[f_{1\pm}\left(x_0,-|\mu|\right)+0\left(\frac{1}{s}\right)\right]dx_0 \tag{99}$$

for $|x| < a$ and $\mu < 0$,

$$\psi_{1\pm}(x,\mu,s) \rightarrow \frac{e^{-(\sigma_2-\sigma_1)\left(\frac{a-x}{\mu}\right)}}{\mu}\int_{-a}^{a} e^{-(s+\sigma_2)\left(\frac{x-x_0}{\mu}\right)}\left[f_{2\pm}\left(x_0,\mu\right)+0\left(\frac{1}{s}\right)\right]dx_0$$

$$+\frac{e^{-(\sigma_2-\sigma_1)\frac{2a}{\mu}}}{\mu}\int_{-\infty}^{-a} e^{-(s+\sigma_1)\left(\frac{x-x_0}{\mu}\right)}\left[f_{1\pm}\left(x_0,\mu\right)+0\left(\frac{1}{s}\right)\right]dx_0$$

$$+\frac{1}{\mu}\int_a^x e^{-(s+\sigma_1)\left(\frac{x-x_0}{\mu}\right)}\left[f_{1\pm}\left(x_0,\mu\right)+0\left(\frac{1}{s}\right)\right]dx_0 \tag{100}$$

for $x > a$ and $\mu > 0$, and

$$\psi_{1\pm}(x,\mu,s) \rightarrow \frac{1}{|\mu|}\int_x^\infty e^{-(s+\sigma_1)\left(\frac{x_0-x}{|\mu|}\right)}\left[f_{1\pm}\left(x_0,-|\mu|\right)+0\left(\frac{1}{s}\right)\right]dx_0 \tag{101}$$

for $x > a$ and $\mu < 0$. Expressions similar to equations (100) and (101) are obtained for $x < -a$. It is seen that $\psi_\pm$ is not necessarily $0\left(\frac{1}{s}\right)$. However, the parts which are not can be easily inverted as follows. Define for all s the function $\psi_{u\pm}(x,\mu,s)$ as that part of each of equations (98) to (100) which is not $0\left(\frac{1}{s}\right)$. It is shown in appendix I that upon making the substitution

$$\left.\begin{aligned} x - x_0 &= \mu t \qquad &(\mu > 0) \\ x_0 - x &= |\mu| t \qquad &(\mu < 0) \end{aligned}\right\} \tag{102}$$

$\psi_{u\pm}(x,\mu,s)$ can be written as

$$\psi_{u\pm}(x,\mu,s) = \int_0^\infty e^{-st}\left[\Psi_{u\pm}(x,\mu,t)\right]dt \tag{103}$$

that is, the parts of $\psi_{\pm}$ which do not behave as $0\left(\frac{1}{s}\right)$ as $|s| \to \infty$, $\mathrm{Re}(s) = \gamma$ can be inverted by inspection. The solution $\Psi_{u\pm}(x,\mu,t)$ is given by

$$\Psi_{u\pm}(x,\mu,t) = \begin{cases} e^{-\sigma_2 t} f_{2\pm}(x-\mu t,\mu) & \left(t < \frac{a+x}{\mu}\right) \\ e^{-\sigma_1 t} e^{-(\sigma_2-\sigma_1)\left(\frac{a+x}{\mu}\right)} f_{1\pm}(x-\mu t,\mu) & \left(t > \frac{a+x}{\mu}\right) \end{cases} \tag{104}$$

for $|x| < a$ and $\mu > 0$;

$$\Psi_{u\pm}(x,\mu,t) = \begin{cases} e^{-\sigma_2 t} f_{2\pm}(x-\mu t,\mu) & \left(t < \frac{a-x}{|\mu|}\right) \\ e^{-\sigma_1 t} e^{(\sigma_2-\sigma_1)\left(\frac{a-x}{\mu}\right)} f_{1\pm}(x-\mu t,\mu) & \left(t > \frac{a-x}{|\mu|}\right) \end{cases} \tag{105}$$

for $|x| < a$ and $\mu < 0$;

$$\Psi_{u\pm}(x,\mu,t) = \begin{cases} e^{-\sigma_1 t} f_{1\pm}(x-\mu t,\mu) & \left(t < \frac{x-a}{\mu}\right) \\ e^{-\sigma_2 t} e^{(\sigma_2-\sigma_1)\left(\frac{x-a}{\mu}\right)} f_{2\pm}(x-\mu t,\mu) & \left(\frac{x-a}{\mu} < t < \frac{x+a}{\mu}\right) \\ e^{-\sigma_1 t} e^{-(\sigma_2-\sigma_1)\left(\frac{2a}{\mu}\right)} f_{1\pm}(x-\mu t,\mu) & \left(t > \frac{x+a}{\mu}\right) \end{cases} \tag{106}$$

for $x > a$ and $\mu > 0$ and

$$\Psi_{u\pm}(x,\mu,t) = e^{-\sigma_1 t} f_{1\pm}(x-\mu t,\mu) \tag{107}$$

for $x > a$ and $\mu < 0$. That $\Psi_{u\pm}$ describes the motion of uncollided neutrons from the initial distribution can be seen by direct substitution; that is, $\Psi_{u\pm}$ satisfies the equation

$$\frac{\partial \Psi_{u\pm}}{\partial t} + \mu \frac{\partial \Psi_{u\pm}}{\partial x} + \sigma(x)\, \Psi_{u\pm} = 0 \tag{108}$$

In the limit $t \to 0$, note that

$$\Psi_{u\pm}(x,\mu,0) = f_{\pm}(x,\mu) \tag{109}$$

For arbitrary $f(x,\mu)$ which vanishes as $|x| \to \infty$, $\psi_{u\pm}(x,\mu,s)$ given by equations (103) and (104) to (107) is an analytic function of s for $\mathrm{Re}(s) > -\sigma_{min}$ for almost all x and μ. If $f_{1\pm} \equiv 0$ $\left(f_{2\pm} \equiv 0\right)$, then $\psi_{u\pm}$ is an analytic function of s for $\mathrm{Re}(s) > -\sigma_2$ $\left(\mathrm{Re}(s) > -\sigma_1\right)$. Therefore, the function $\Phi_\pm(x,\mu,s)$ defined as

$$\Phi_\pm(x,\mu,s) \equiv \psi_\pm(x,\mu,s) - \psi_{u\pm}(x,\mu,s) \qquad \left(\mathrm{Re}(s) > -\sigma_{min}\right) \tag{110}$$

has the same analytic properties as $\psi_\pm$ in the right-half plane $\mathrm{Re}(s) > -\sigma_{min}$ except that it is $0\left(\frac{1}{s}\right)$ as $|s| \to \infty$. If $\psi_\pm$ has a branch cut along $\left(-\sigma_{min}, -\sigma_1\left(1 - c_1\right)\right)$, that is, if $\sigma_{min} > \sigma_1\left(1 - c_1\right)$, then

$$\left[\Phi_\pm\right]^- - \left[\Phi_\pm\right]^+ = \left[\psi_\pm\right]^- - \left[\psi_\pm\right]^+ \tag{111}$$

on the branch cut. Similarly, if $\psi_\pm$ has a pole at $s = s_n$, then

$$\mathrm{Residue}\left(\Phi_\pm\right)_{s_n} = \mathrm{Residue}\left(\psi_\pm\right)_{s_n} \tag{112}$$

Explicit Form of $\Psi_\pm(x,\mu,t)$

The definite parity parts of the time-dependent solution therefore can be written from equation (7) as

$$\Psi_\pm(x,\mu,t) = \Psi_{u\pm}(x,\mu,t) + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Phi_\pm(x,\mu,s)\, e^{st}\, ds \tag{113}$$

By using the analytic properties, one can deform the contour to the left and obtain, in general,

$$\begin{aligned}\Psi_\pm(x,\mu,t) = \Psi_{u\pm}(x,\mu,t) &+ \sum_{s=s_n} \mathrm{Residue}\left[\psi_\pm(x,\mu,s)\, e^{st}\right] \\ &+ \frac{1}{2\pi i}\int_{-\sigma_{min}}^{-\sigma_1(1-c_1)} \left\{\left[\psi_\pm(x,\mu,s)\right]^- - \left[\psi_\pm(x,\mu,s)\right]^+\right\} e^{st}\, ds \\ &+ \frac{1}{2\pi i}\int_{-\sigma_{min}-i\infty}^{-\sigma_{min}+i\infty} \left[\psi_\pm(x,\mu,s) - \psi_{u\pm}(x,\mu,s)\right] e^{st}\, ds \\ &+ \frac{1}{2\pi i}\lim_{\rho\to 0}\int_{C_\rho} \psi_\pm(x,\mu,s)\, e^{st}\, ds \qquad \left(-\sigma_{min} < -\sigma_1\left(1 - c_1\right) < s_n\right)\end{aligned} \tag{114}$$

where C_ρ is a small circular contour of radius ρ with center at $s = -\sigma_1(1 - c_1)$. All the contours are indicated in figure 4. Generally, the point $s = -\sigma_1(1 - c_1)$ will not satisfy the eigenvalue conditions (eq. (81)) and the contribution from the contour C_ρ vanishes as $\rho \to 0$. If, however, $s = -\sigma_1(1 - c_1)$ happens to satisfy equation (81), the contribution from the contour C_ρ has the form of a discrete residue term. Details concerning this point are discussed in appendixes I and K.

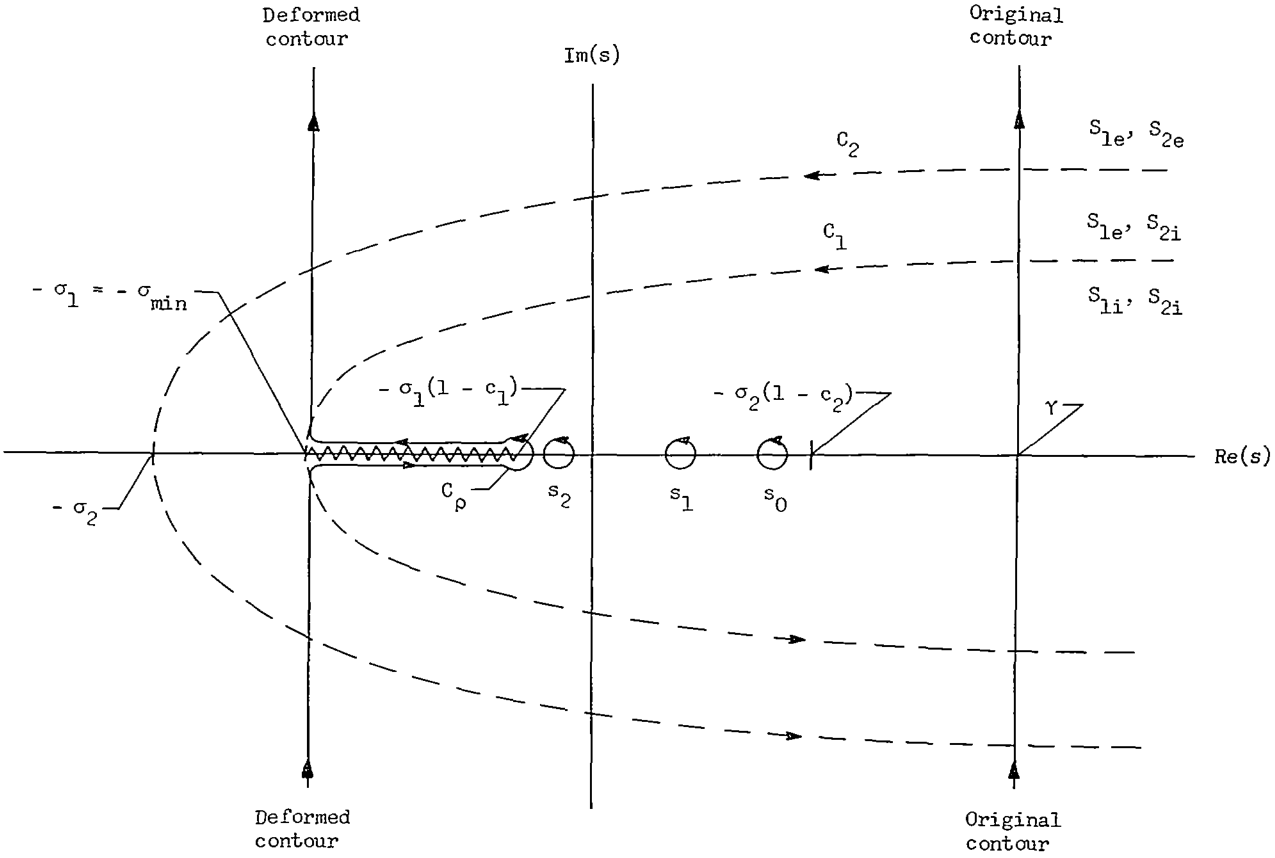

Figure 4.- Integration contours for the inverse Laplace transformation of $\psi(x,\mu,s)$ when $\nu_{01}(s)$ branch cut is embedded in the $\nu_{02}(s)$ branch cut, $c_2 > 1$, and the initial distribution is arbitrary.

Equation (114) is the solution of the time-dependent problem written in a form in which the uncollided portion of the initial distribution $f(x,\mu)$ has been separated. For arbitrary $f(x,\mu)$ the contour cannot be deformed further to the left. It is indicated in the final section that this solution reduces to those obtained previously by others for special cases of the present problem.

This section is concluded by indicating the form of some parts of equation (114). The uncollided term $\Psi_{u\pm}(x,\mu,t)$ is given explicitly by equations (104) to (107). The form of $\psi_\pm(x,\mu,s)$ on the branch cut $\left(-\sigma_{min}, -\sigma_1(1 - c_1)\right)$ was given in the previous section. From those results, it follows that on this branch cut, $\left[\psi_\pm(x,\mu,s)\right]^- - \left[\psi_\pm(x,\mu,s)\right]^+$ can be written from equations (91) and (92) as

$$\left[\psi_\pm(x,\mu,s)\right]^- - \left[\psi_\pm(x,\mu,s)\right]^+ = \left\{\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]^- - \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]^+\right\}\overline{\psi}_{2\pm}(x,\mu,s) \tag{115}$$

for $|x| < a$ and

$$\begin{aligned}\left[\psi_\pm(x,\mu,s)\right]^- - \left[\psi_\pm(x,\mu,s)\right]^+ = {} & \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]^- \left[\overline{\psi}_{1\pm}(x,\mu,s)\right]^- \\ & - \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]^+ \left[\overline{\psi}_{1\pm}(x,\mu,s)\right]^+ \\ & \pm \left\{\left[b_{1\pm} - \tilde{F}_\pm(-a,\nu_{01},s)\right]^- \psi_{\nu_{01}}(x,\mu,s)\right. \\ & \left. - \left[b_{1\pm} - \tilde{F}_\pm(-a,\nu_{01},s)\right]^+ \psi_{-\nu_{01}}(x,\mu,s)\right\}\end{aligned} \tag{116}$$

for $x > a$, where $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]$ is given by equation (88), $\overline{\psi}_{m\pm}(x,\mu,s)$ by equation (76) and $\left[b_{1\pm} - \tilde{F}_\pm(-a,\nu_{01},s)\right]$ by equation (87). The solution $\psi_\pm(x,\mu,s)$ has poles at $s = s_0, \ldots, s_N$ because of the poles of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]\nu_{02}^{(1\mp1)/2}$. Again, from the results given in the previous section, it follows that

$$\begin{aligned}&\text{Residue}\left[\psi_\pm(x,\mu,s)\ e^{st}\right]_{s_n} \\ &= e^{s_n t}\left\{\overline{\psi}_\pm(x,\mu,s)\left[\nu_{02}^{-(1\mp1)/2}\right]\right\}_{s_n} \text{Residue}\left\{\nu_{02}^{(1\mp1)/2}\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]\right\}_{s_n}\end{aligned} \tag{117}$$

Note that the factor $\nu_{02}^{(1\mp1)/2}$ is introduced so that $\overline{\psi}_\pm \nu_{02}^{-(1\mp1)/2}$ and $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]\nu_{02}^{(1\mp1)/2}$ are single valued on the branch cut of ν_{02} (see eqs. (95) and (96)). These terms have an exponential time dependence $e^{s_n t}$ and the implicit equations, namely, equations (77) and (81), from which the eigenvalues $\{s_n\}$ can be computed have been obtained. Since information concerning the behavior of eigenvalues (that is, number, location, etc.) as a function of material properties is not readily obtained analytically from such expressions, a numerical study of real time eigenvalues has been made and the results are discussed in the next section.

CALCULATION OF TIME EIGENVALUES

First note that the eigenvalues and pseudo-eigenvalues depend on five parameters $(c_1,\ \sigma_1,\ c_2,\ \sigma_2,$ and $a)$ and therefore many numerical computations would be required in order to determine the specific dependence on each parameter. It will be seen that the bare-slab results of references 12 and 13, the theorems of reference 9 for slabs surrounded by purely absorbing media, and some observations of the present numerical results for a few reflected slab cases allow some conclusions about the behavior of eigenvalues for reflected slabs as a function of the slab half-thickness a to be drawn. However, rather than compute eigenvalues $\{s_n\}$ in terms of c_1, σ_1, c_2, σ_2, and a, one defines a nondimensional variable ζ and nondimensional parameters σ_R, σ_D, and A as

$$\left.\begin{aligned} \zeta &= \frac{s + \sigma_2}{c_2\sigma_2} \\ \sigma_R &= \frac{c_1\sigma_1}{c_2\sigma_2} \\ \sigma_D &= \frac{\sigma_1 - \sigma_2}{c_2\sigma_2} \\ A &= c_2\sigma_2 a \end{aligned}\right\} \tag{118}$$

In terms of these quantities, the branch cut of ν_{02} becomes the real interval $(0,1)$ and the branch cut of ν_{01} becomes the real interval $(-\sigma_D,\ -\sigma_D + \sigma_R)$. Since σ_m and c_m are nonnegative, it follows that

$$-\sigma_D \leqq \frac{1}{c_2} \tag{119}$$

where the equality holds only if $\sigma_1 = 0$. Also c_1 has been restricted to less than unity so that $-\sigma_D + \sigma_R \geqq 1$ implies that $c_2 < 1$. Obviously, $\sigma_R = 0$ when the reflector is a purely absorbing medium or a vacuum and $\sigma_D = 0$ when the total macroscopic cross sections of the two media are the same. It has been shown in the previous section that, in general, the inversion contour can be deformed to the left only as far as $Re(s) = -\sigma_{min}$ which corresponds to $Re(\zeta) = \max\left(-\sigma_D, 0\right)$. However, there are no eigenvalues on the branch cut of ν_{01} so the region of the real ζ-axis where the eigenvalues $\left\{\zeta_n\right\}$ should appear is

$$\max\left(-\sigma_D + \sigma_R, 0\right) < \zeta_n < 1 \tag{120}$$

This interval corresponds to $s \epsilon S_{1i} \cap S_{2i}$ and equations (77) and (81), written in terms of the quantities of equations (118), are solved numerically to obtain the real eigenvalues $\left\{\zeta_n\right\}$ for specified σ_R, σ_D, and A. The pseudo-eigenvalues are obtained numerically by solving equations (82) and (83) also written in terms of the quantities of equations (118). In addition, numerical results are obtained in the thick-slab approximation, that is, equation (81) with $\overline{B}_{2\pm}(\mu) = 0$. Details concerning numerical procedures and computational equations are given in appendix J.

The time dependence of discrete modes is seen from equations (114) and (117) to be

$$e^{s_n t} = e^{\left(c_2 \zeta_n - 1\right)\sigma_2 t} \tag{121}$$

Now $\zeta_n = -\sigma_D + \sigma_R$ implies that $s_n = -\sigma_1\left(1 - c_1\right) \leqq 0$ since $c_1 < 1$ and the equality holds only if $\sigma_1 = 0$. Therefore such ζ_n values correspond to time-decaying modes regardless of the value of c_2. For values of ζ_n within the interval (120), the time decay or growth depends on whether $c_2\zeta_n$ is less than or greater than unity as can be seen from equation (121). A discrete mode represents a critical system if $c_2\zeta_n = 1$. The largest eigenvalue ζ_0 with an even parity eigenfunction corresponds to a critical slab problem with parameters

$$\left.\begin{aligned} c_{slab} &= \frac{1}{\zeta_0} \\ c_{reflector} &= \frac{\sigma_R}{\zeta_0 + \sigma_D} \\ \sigma_{slab} a_{critical} &= \zeta_0 A \end{aligned}\right\} \tag{122}$$

where $a_{critical}$ is the critical slab half-thickness. It is generally known (see, for example, ref. 2) that the critical radius for a bare sphere $\left(\sigma_R = 0\right)$ can be obtained from the largest slab eigenvalue ζ_1 with an odd parity eigenfunction. That is, when ζ_1 is used in equations (122) in place of ζ_0, the $a_{critical}$ is the bare-sphere critical radius.

Comparison With Published Numerical Results

Many different combinations of material parameters could be considered, but here the study of the eigenvalue behavior is restricted to the case of overlapping branch cuts. As σ_R departs from zero, one would like to see how the eigenvalues depart from those previously reported (refs. 12 and 13) for a bare slab. A comparison of the present eigenvalues $\{\zeta_n\}$ for vacuum reflectors, that is, $\sigma_R = 0$, with those of reference 12 is given in tables I and II. Results generally agree to three figures for slab half-thicknesses A from 0.4 to 20. In table II, eigenvalues calculated in the thick-slab approximation are also shown for bare slabs. For slabs with half-thicknesses $A > 1$, the thick-slab approximation generally agrees with the numerical solution of the exact eigenvalue condition to three figures. This agreement can be seen from table III where such results are compared as σ_R departs from zero with $\sigma_D = 0$. From the bare-slab results $(\sigma_R = 0)$ of tables I, II, and III, critical slab half-thicknesses are obtained from ζ_0 by using equations (122). These values are compared with the critical slab half-thickness results of Mitsis (taken from ref. 2) in figure 5 (open symbols). Closed symbols give critical sphere quarter-diameters obtained from equations (122) and ζ_1 whereas Mitsis' critical sphere results are taken from reference 22. The agreement is good to the scale of the figure. For $\sigma_R = 0$, the eigenvalues ζ_0 and ζ_1 have also been compared directly with numerical bounds computed by Mullikin (ref. 23) for bare slabs and spheres and again the agreement is good. Critical half-thicknesses for slabs

TABLE I.- EIGENVALUES $\{\zeta_n\}$ FOR BARE SLABS

n	Eigenvalues for –									
	A = 1		A = 5		A = 10		A = 15		A = 20	
	Present	Reference 12	Present	Reference 12	Present	Reference 12	Present	Reference 12	Present	Reference 12
0	0.703	0.705	0.975	0.975	0.993	0.993	0.997	0.997	0.998	0.998
1			.897	.897	.971	.971	.987	.987	.992	.992
2			.762	.762	.935	.935	.970	.970	.983	.983
3			.560	.560	.883	.883	.946	.946	.969	.969
4			.276	.276	.814	.816	.915	.915	.952	.951
5					.728	.727	.877	.877	.930	.930
6					.621	.621	.831	.831	.904	.905
7					.493	.493	.777	.777	.874	.874
8					.340	.340	.714	.714	.840	.840
9					.157	.157	.642	.644	.800	.800
10							.560	.560	.756	.756
11							.467	.467	.707	.707
12							.362	.361	.653	.653
13							.243	.243	.593	.593
14							.110	.110	.526	.526
15									.453	.453
16									.373	.373
17									.286	.286
18									.190	.190
19									.084	.084

TABLE II.- EIGENVALUE ζ_0 FOR THIN BARE SLABS

Slab thickness, A	Eigenvalues determined by –		
	Thick-slab approximation (eq. (J19))	Present (eq. (J6))	Bowden (ref. 12)
1.0	0.702	0.703	0.705
.8	.612	.615	.615
.6	.473	.483	.483
.4	.244	.282	.282
.2	(*)	.043	.048

*No solution found for $\zeta > 0.001$.

TABLE III.- EIGENVALUES ζ_0 AND ζ_1 FOR THIN REFLECTED SLABS

$[\sigma_D = 0]$

σ_R	Eigenvalues for –					
	A = 0.4		A = 0.7		A = 1.0	
	ζ_0		ζ_0		ζ_0	
	Thick-slab approximation (eq. (J19))	Present solution (eq. (J6))	Thick-slab approximation (eq. (J19))	Present solution (eq. (J6))	Thick-slab approximation (eq. (J19))	Present solution (eq. (J6))
0	0.244	0.282	0.550	0.556	0.702	0.703
.2	.376	.398	.600	.604	.727	.728
.4	.512	.522	.661	.663	.759	.759
.6	.656	.660	.739	.740	.803	.803
.8	.816	.817	.843	.844	.870	.870

σ_R	Eigenvalues for –							
	A = 1.4				A = 2.0			
	ζ_1		ζ_0		ζ_1		ζ_0	
	Thick-slab approximation (eq. (J19))	Present solution (eq. (J6))	Thick-slab approximation (eq. (J19))	Present solution (eq. (J6))	Thick-slab approximation (eq. (J19))	Present solution (eq. (J6))	Thick-slab approximation (eq. (J19))	Present solution (eq. (J6))
0	0.142	0.132	0.808	0.808	0.508	0.508	0.885	0.885
.2	.252	.247	.820	.820	.540	.539	.891	.891
.4	.403	.402	.836	.836	.585	.584	.898	.898
.6	(*)	(*)	.860	.860	.656	.656	.909	.909
.8	(*)	(*)	.900	.900	(*)	(*)	.930	.930

*$\sigma_R > \zeta_1$ (or ζ_1 in branch cut of ν_{01}).

with infinite reflectors have been recently computed by Kowalska (ref. 24) for a number of combinations of c_{slab} and $c_{reflector}$. Some present results ζ_0 for $\sigma_R \neq 0$ can be compared with the critical slab half-thicknesses of reference 24. The parameters are given in terms of ζ_0 and the present input quantities σ_R, σ_D, and A by equations (122). Figure 6 gives a few present cases (circles) for which c_{slab} was close to some of the points (diamonds) of reference 24; no attempt was made to compute exactly these points. The present cases for $c_{slab} \approx 1.11$ are from A = 2 and 1.4 in table III.

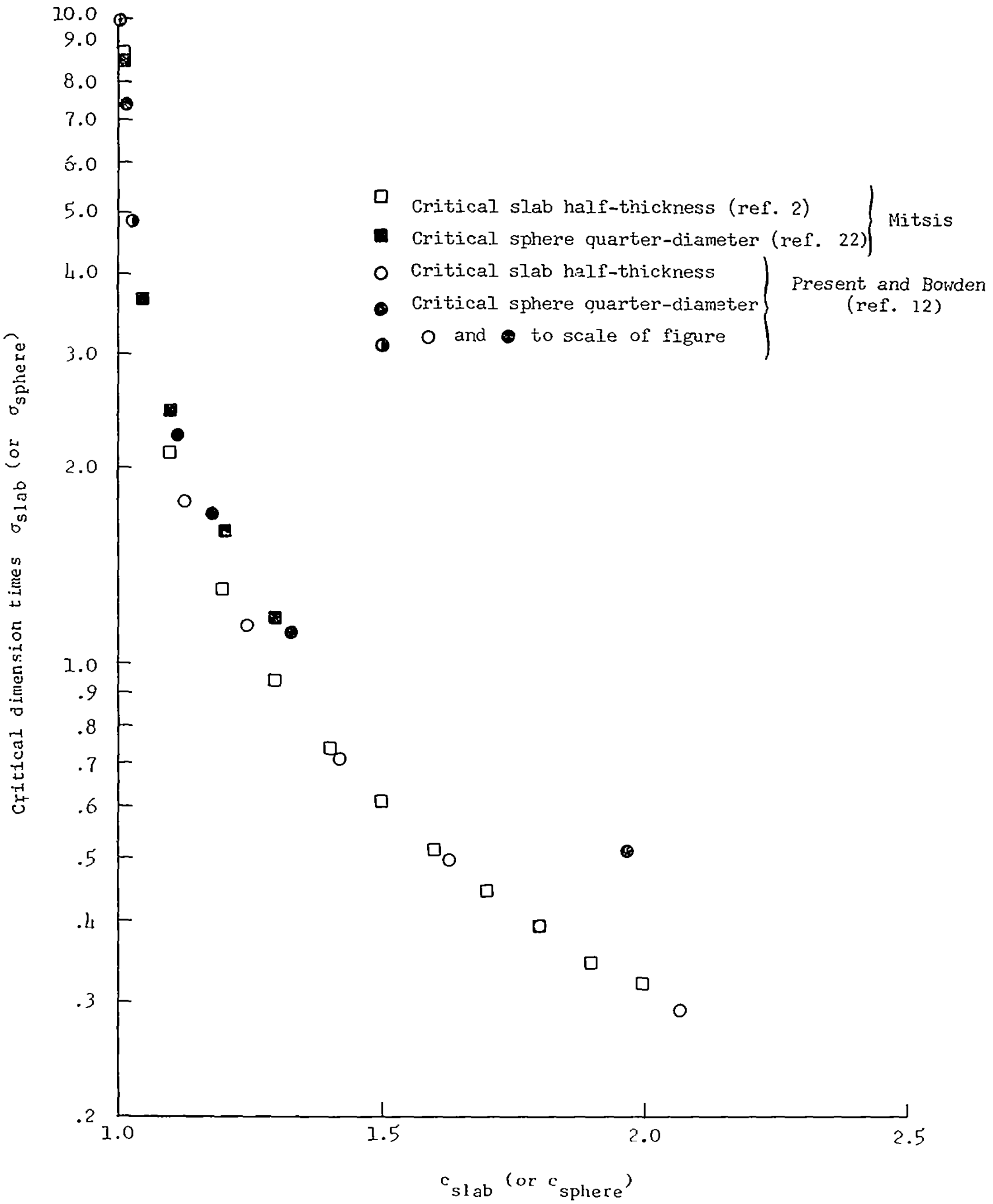

Figure 5.- Critical dimension of bare systems.

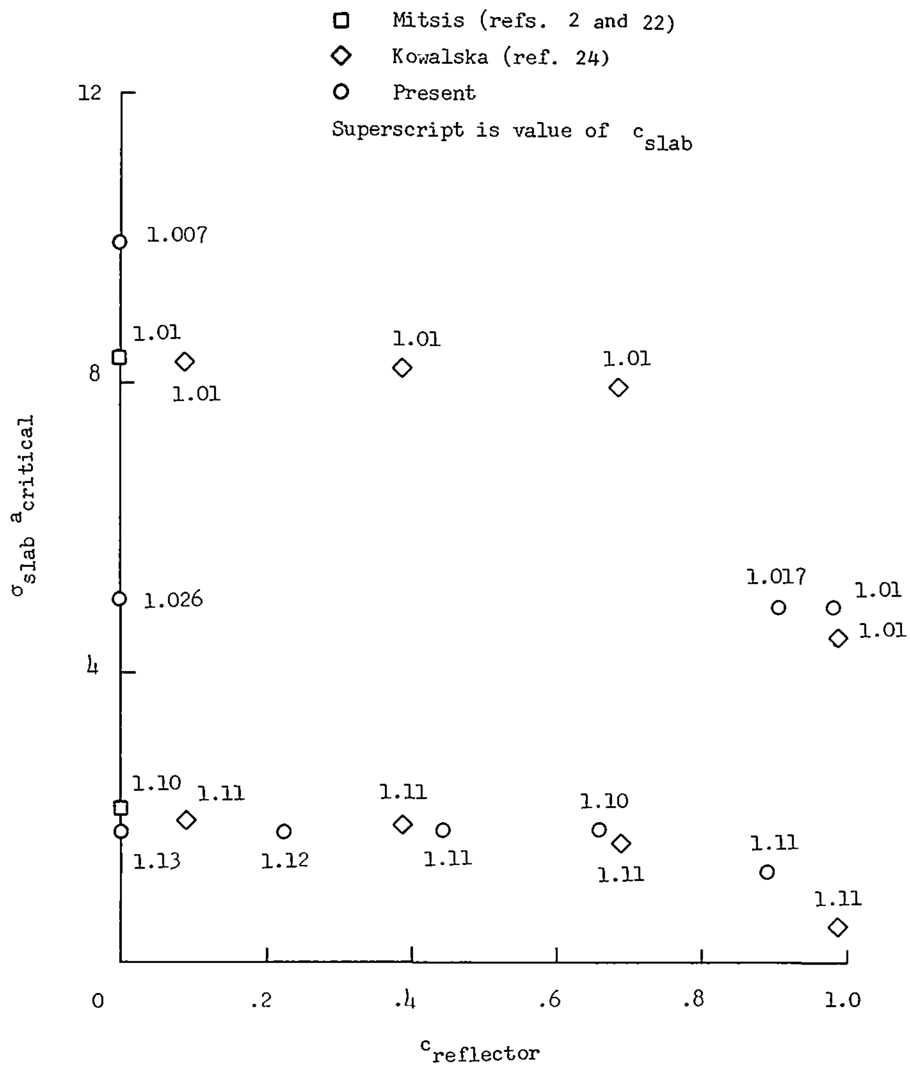

Figure 6.- Critical half-thickness for finite slabs with infinite reflectors.

Sample Reflected Slab Results

The remainder of the results have been computed for $A = 5$. For a bare slab with $A = 5$, it can be seen from table I that there are five eigenvalues. The behavior of these eigenvalues has been studied as σ_R departs from zero for several values of σ_D. In figure 7, results are given for $\sigma_D = 0$. The calculations show that the largest eigenvalue ζ_0 is present up to $\sigma_R = 0.9999$. Apparently, this eigenvalue remains up to $\sigma_R = 1$, which is only obtained for $c_2 < 1$. All other eigenvalues disappear into the branch cut of ν_{01} at $\zeta_n = \sigma_R$, labeled with an asterisk, which corresponds to a time-decaying mode, regardless of the value of c_2. An asterisk is used in figures 7 and 9 to 11 to indicate the points at which an eigenvalue or pseudo-eigenvalue coincides with

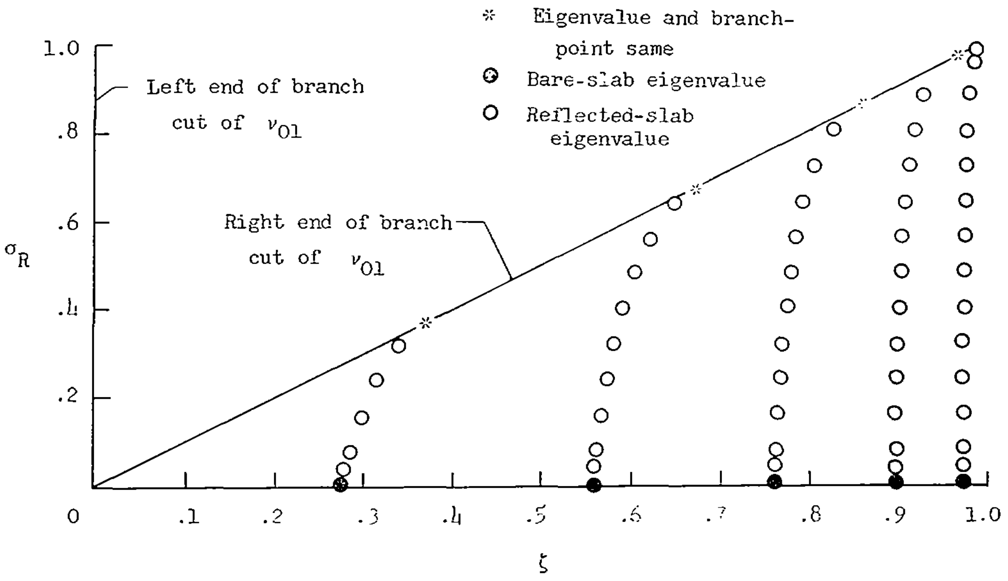

Figure 7.- Dependence of eigenvalues ζ_n on σ_R. $\sigma_D = 0$; $A = 5$.

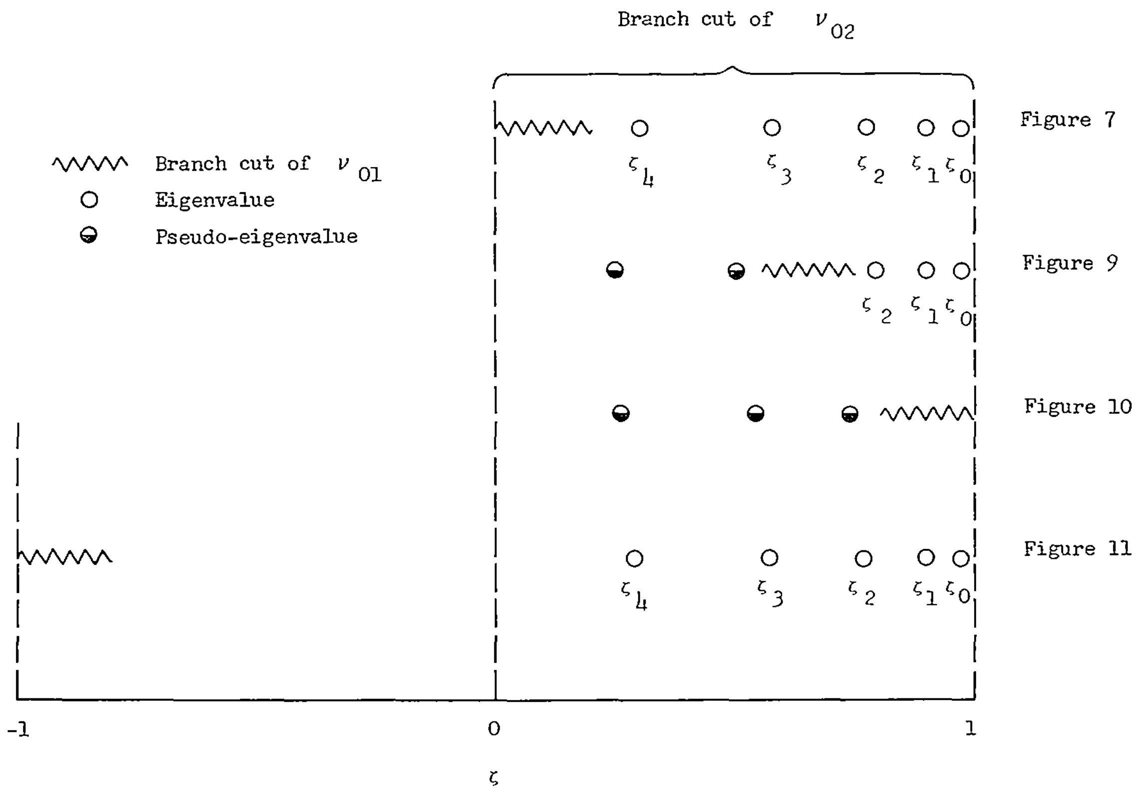

Figure 8.- Relative location on the real ζ-axis of the branch cuts of ν_{0m}, the eigenvalues ζ_n, and the pseudo-eigenvalues for $\sigma_R = 0.2$ as read from figures 7 and 9 to 11.

the branch points of ν_{01}. Even though such points appear to have a discrete eigenvalue type of time dependence, it is felt that they are properly part of the branch-cut integral contribution. Note that the branch points of ν_{01} are located at $\zeta = -\sigma_D$ and $\zeta = -\sigma_D + \sigma_R$ and that the limiting form of the condition which determines whether such points are eigenvalues (or pseudo-eigenvalues) no longer depends explicitly on σ_R or σ_D. (See appendixes J and K.) The theorems of Lehner (ref. 8) apply for $\sigma_R = 0$ in figure 7.

In figure 8, the manner in which the ν_{0m} branch cuts overlap and the location of the eigenvalues and pseudo-eigenvalues for $\sigma_R = 0.2$ in figures 7 and 9 to 11 is indicated.

In figure 9, results are presented for $\sigma_D = -0.65 + 0.5\sigma_R$. These results typify results for $-\sigma_D$ values in the range between zero and $(\zeta_0)_{\sigma_R=0}$, where the notation $(\zeta_n)_{\sigma_R=0}$ means bare-slab eigenvalue, which depends on c_2, σ_2, and a. The open and closed circles represent eigenvalues as in figure 7 whereas the half-closed circles are pseudo-eigenvalues corresponding to $s < -\sigma_{min} = -\sigma_1$. Again the largest eigenvalue ζ_0 appears to remain provided that $c_2 > 1$. Here, as in the next two figures, results for $\sigma_R = 0$ agree with the theorems of Hintz (ref. 9) which apply only for $c_1 = 0$. Basically, his result is that the strip $Re(\zeta)$ between 0 and $-\sigma_D$ belongs to the continuous spectrum and that the bare-slab eigenvalues lying in this interval are not eigenvalues of the

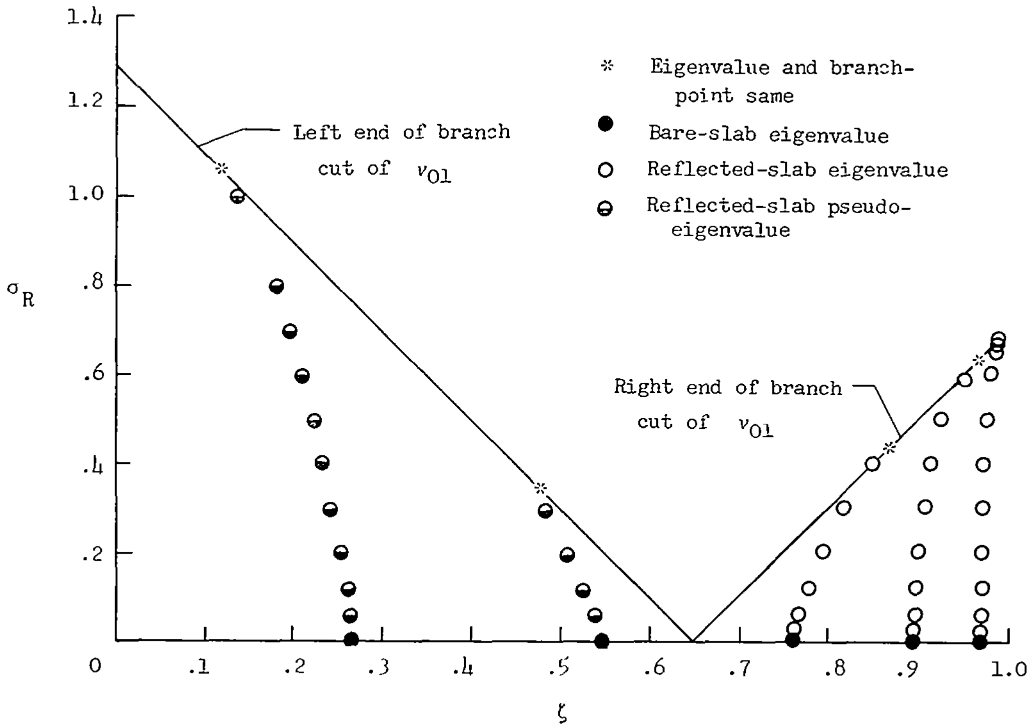

Figure 9.- Dependence of eigenvalues ζ_n on σ_R. $\sigma_D = -0.65 + 0.5\sigma_R$; A = 5.

slab surrounded by purely absorbing media. He finds that there are no eigenvalues if $-\sigma_D > (\zeta_0)_{\sigma_R=0}$, but does not discuss the physical significance. It is seen from equation (119) that for such cases, $\zeta_0 < 1/c_2$ and corresponds therefore to a time-decaying mode. In other words, stationary (critical) or time-increasing modes cannot disappear into the continuous spectrum as material properties are varied. In fact, when $\sigma_R \neq 0$, such modes could not disappear into the branch cut of ν_{01} either. In figure 10, results are given for $-\sigma_D + \sigma_R = 1$ which, it may be remembered, implies $c_2 < 1$. For this case, all the bare-slab eigenvalues lie in the continuous spectrum of reference 9 when $\sigma_R = 0$. In both figures 9 and 10, $s = -\sigma_{min}$ corresponds to $\zeta = -\sigma_D$. Figure 11 shows the behavior of the eigenvalues for $\sigma_D = 1$ and it is similar to that of figure 7. For $\sigma_R = 0$, the continuous spectrum of reference 9 lies in the strip $-\sigma_D = -1 < \mathrm{Re}(\zeta) < 0$. Here $s = -\sigma_{min}$ corresponds to $\zeta = 0$.

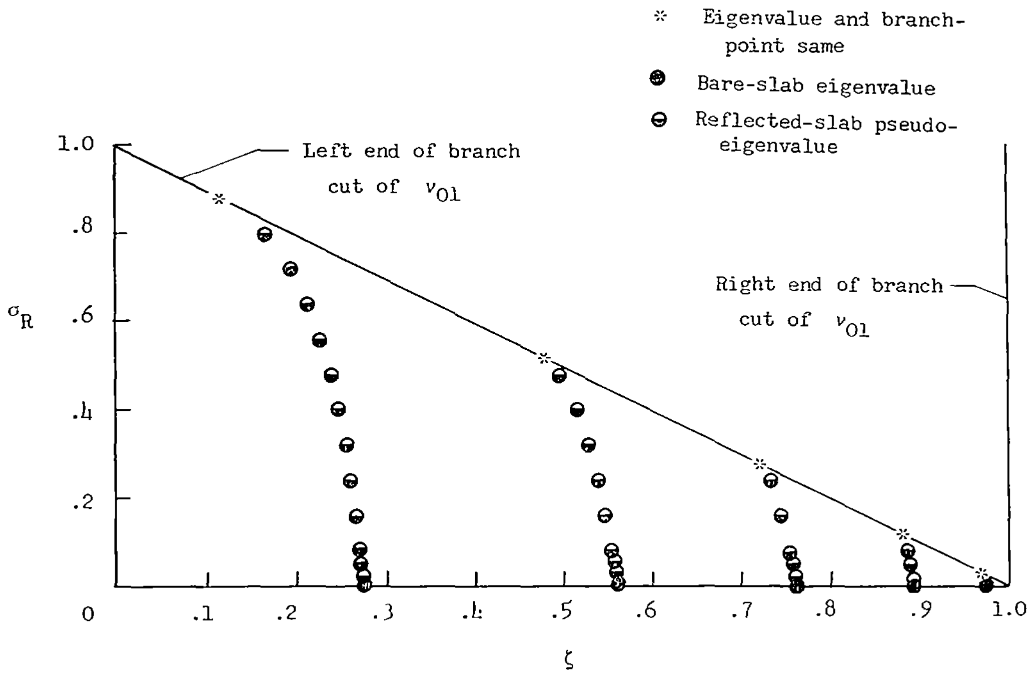

Figure 10.- Dependence of eigenvalues ζ_n on σ_R. $\sigma_D = \sigma_R - 1$; $A = 5$. Note that $c_2 < 1$ for this figure.

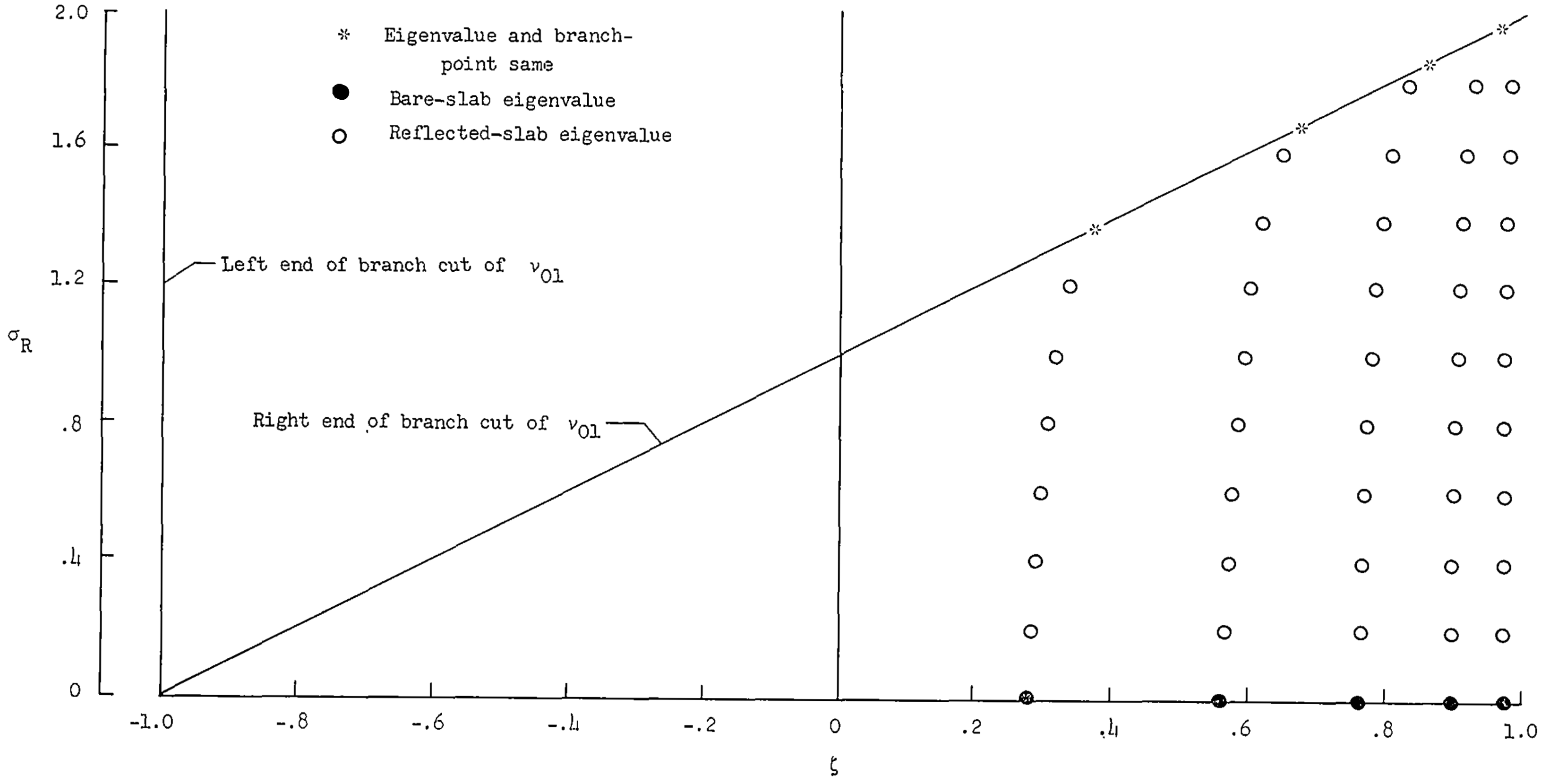

Figure 11.- Dependence of eigenvalues ζ_n on σ_R. $\sigma_D = 1$; $A = 5$.

Summary of Numerical Results

All numerical results indicate that real time eigenvalues $\{\zeta_n\}$ for material reflectors are finite in number and tend to eigenvalues previously obtained for a vacuum as $\sigma_R \rightarrow 0$, as do the pseudo-eigenvalues for $s < -\sigma_{min}$. If the set $\{\zeta_n\}$ is empty, the neutron density is necessarily decaying in time. Conversely, if the neutron density is stationary or increasing in time, the set $\{\zeta_n\}$ is not empty. One also expects that when $c_2 > 1$, a critical thickness should exist; that is, the largest eigenvalue ζ_0 must be present for large enough slab thicknesses for the given c_2. This effect can be seen from table I. For example, if $-\sigma_D = 0.8$, the eigenvalue ζ_0 for $A = 1$ is not present, whereas that for $A = 5$ would be, and represents a mode whose amplitude increases exponentially with time for $c_2 > 1/0.975$. That for $A = 20$ needs only $c_2 > 1/0.998$ in order to represent a time-increasing mode.

As pointed out at the beginning of this section, some speculations concerning the behavior of $\{\zeta_n\}$ for reflected slabs as a function of the slab half-thickness a can be made, that is, if c_m and σ_m are known what can be said about $\{\zeta_n\}$ as a function of a. The following conclusions are based on the observation that if ζ_0 at $\sigma_R = 0$ lies to the right of $-\sigma_D$, it appears to remain to the right of $-\sigma_D + \sigma_R$ as σ_R increases until $-\sigma_D + \sigma_R = 1$. (See figs. 7 and 9.) The dependence of $(\zeta_0)_{\sigma_R=0}$ on slab half-thickness is given in tables I to III and many more points are given in reference 12. First, if $-\sigma_D + \sigma_R \geqq 1$ (this inequality implies $c_2 < 1$), the set $\{\zeta_n\}$ is empty for all a. However, there may be pseudo-eigenvalues if $-\sigma_D > 0$. Next, if $-\sigma_D + \sigma_R < 1$, then two cases arise, depending on the value of σ_D

(1) When $-\sigma_D > 0$, then regardless of the value of c_2, one can find an a^* such that $a < a^*$ implies that the set $\{\zeta_n\}$ is empty, whereas $a > a^*$ implies that the set $\{\zeta_n\}$ is not empty. The number a^* is obtained from the bare-slab result $(\zeta_0)_{\sigma_R=0}$ as

$$\left[\zeta_0\left(c_2, \sigma_2, a^*\right)\right]_{\sigma_R=0} = -\sigma_D \tag{123}$$

(2) When $-\sigma_D \leqq 0$, the set $\{\zeta_n\}$ is never empty. Thus, given c_m, σ_m, a, and the bare-slab eigenvalues corresponding to c_2, σ_2, and a, one can say whether the set $\{\zeta_n\}$ is empty. Furthermore, the number of eigenvalues $\{\zeta_n\}$ will not exceed the number of bare-slab eigenvalues $\left\{(\zeta_n)_{\sigma_R=0}\right\}$ which are greater than $-\sigma_D$. Finally, the number of real reflected-slab eigenvalues and pseudo-eigenvalues does not exceed the number of bare-slab eigenvalues.

COMPARISON WITH OTHER ANALYTICAL SOLUTIONS

General Form

It has been shown by using Case's method that the solution of the initial-value problem of monoenergetic neutrons migrating in a finite slab $\left(\text{properties} \quad c_2, \sigma_2\right)$ with infinite reflectors $\left(\text{properties} \quad c_1, \sigma_1\right)$ can be written in the form

$$\Psi(x,\mu,t) = \Psi_u(x,\mu,t) + \sum_{s=s_n} \text{Residue}\Big[\psi(x,\mu,s)\Big]_{s_n} e^{s_n t}$$

$$+ \frac{1}{2\pi i}\int_{-\sigma_{min}}^{-\sigma_1(1-c_1)} \left\{\Big[\psi(x,\mu,s)\Big]^- - \Big[\psi(x,\mu,s)\Big]^+\right\} e^{st}\, ds$$

$$+ \frac{1}{2\pi i}\int_{-\sigma_{min}-i\infty}^{-\sigma_{min}+i\infty} \Big[\psi(x,\mu,s) - \psi_u(x,\mu,s)\Big] e^{st}\, ds \qquad \left(-\sigma_{min} < -\sigma_1\left(1 - c_1\right) < s_n\right) \qquad (124)$$

In this equation, t is the real time multiplied by the constant neutron speed, σ_{min} is the minimum of σ_1 and σ_2, and each ψ function is the sum of its definite parity parts $\psi_\pm$. Some terms of the solution (124) will not be present if $-\sigma_{min} \not< -\sigma_1\left(1 - c_1\right) \not< s_n$. That is, if $-\sigma_1\left(1 - c_1\right) < -\sigma_{min}$, then the branch-cut integral does not appear. Likewise, if all $s_n < -\sigma_1\left(1 - c_1\right)$, there are no residue terms. These discrete eigenvalue terms are characteristic of a finite slab whereas the branch-cut integral term is typical of a semi-infinite medium. The term $\Psi_u(x,\mu,t)$ describes the behavior of neutrons from the initial distribution $f(x,\mu)$ which have not suffered a scattering collision and its definite parity parts are given in equations (104) to (107). The discrete eigenvalue terms in equation (124) are given by equation (117) whereas the integrand of the branch-cut integral is given by equations (115) to (116). The definite parity parts of the last integrand are given by equation (103) and equations (70) to (73). The eigenvalues $\left\{s_n\right\}$ can be computed as was demonstrated in the previous section; thus, all terms in equation (124) can be calculated.

Special Cases

In all special cases of the present problem which have been solved using the Lehner-Wing technique (refs. 6 to 9), $c_1 = 0$. In these cases, there is no branch cut due to $\nu_{01}(s)$; therefore, the branch-cut integral is not present in equation (124). It was shown that as $c_1 \to 0$, the eigenvalues $\left\{s_n\right\}$ which are greater than $-\sigma_{min}$ approach those for a bare slab as do the pseudo-eigenvalues for $s < -\sigma_{min}$. The solution $\psi_\pm$ has the proper behavior as $c_1 \to 0$ since those terms of equations (65) and (66) which appear to blow up in such a limit actually cancel when the contour C' is collapsed on to the portion of the branch cut of $\Omega_m(z',s)$, $0 \leqq z' \leqq 1$. When the uncollided term is combined with the last integral, it is then seen that the solution (124) and the eigenvalues $\{s_n\}$ have the behavior required by the theorems of Lehner (ref. 8) and Hintz (ref. 9). The present problem reduces to those considered by Lehner and Hintz, respectively, when

$$\left.\begin{aligned} c_1 &= 0 \\ \sigma_1 &= \sigma_2 \end{aligned}\right\} \tag{125a}$$

and

$$\left.\begin{aligned} c_1 &= 0 \\ \sigma_1 &\neq \sigma_2 \end{aligned}\right\} \tag{125b}$$

In order to describe the same physical problem in the slab as that solved by Lehner and Wing (refs. 6 and 7), one must not only have

$$\left.\begin{aligned} c_1 &= 0 \\ \sigma_1 &= 0 \end{aligned}\right\} \tag{126}$$

but also

$$f(x,\mu) = 0 \qquad (x < -a, \quad \mu > 0 \quad \text{and} \quad x > a, \quad \mu < 0) \tag{127}$$

In other words, neutrons from the initial distribution outside the slab cannot impinge on the slab faces at times $t > 0$. Lehner and Wing solved the time-dependent problem with boundary conditions:

$$\Psi(\pm a,\mu,t) = 0 \qquad \left(\mu \lessgtr 0, \quad t > 0\right) \tag{128}$$

Restrictions (126) and (127) in the present solution make $I_{2\pm}(\mu,s)$ and therefore $A_{2\pm}(\mu,s)$ depend only on slab properties. Then, in looking for solutions inside the slab $(|x| < a)$, the inversion contour along $Re(s) = -\sigma_{min}$ can be deformed back to $Re(s) = -\sigma_2$, and one picks up a residue contribution from any pseudo-eigenvalue in the region and thus obtains the Lehner-Wing results. That is, the solution has the proper form and all bare-slab eigenvalues are recovered.

The analogous problem for $c_1 \neq 0$ in which the inversion contour can be deformed to the left of $\mathrm{Re}(s) = -\sigma_{min}$ for $|x| < a$ is obtained when $\sigma_2 > \sigma_1$ and $f_1(x,\mu) \equiv 0$; that is, if

$$f(x,\mu) \equiv 0 \qquad \left(|x| > a \text{ and } \sigma_2 > \sigma_1\right) \tag{129}$$

then all terms in $I_{2\pm}(\mu,s)$ which contain $s + \sigma_1$ factors in the exponentials are identically zero and this allows the contour along $\mathrm{Re}(s) = -\sigma_{min}$ to be deformed back to $\mathrm{Re}(s) = -\sigma_2$ when $|x| < a$. Such a deformation is not possible for $|x| > a$; for this range of x, one must stop at $\mathrm{Re}(s) = -\sigma_{min} = -\sigma_1$. If there are pseudo-eigenvalues in $-\sigma_2 < \mathrm{Re}(s) < -\sigma_1 = -\sigma_{min}$ (see, for example, fig. 9), they will appear in the solution for $|x| < a$ as residue terms which have the exponential time dependence. They are however not eigenvalues for the reflected slab since such terms do not appear for $|x| > a$. Erdmann (refs. 15 and 16) solved the time-dependent problem for two semi-infinite media where an isotropic pulse of neutrons was introduced at the interface, and found that the inversion contour for $x \in$ medium m could be deformed to the left as far as $\mathrm{Re}(s) = -\sigma_m$. In the present problem, such deformations can be made only when conditions (129) are satisfied. It appears that the contour $\mathrm{Re}(s) = -\sigma_{min}$ cannot be deformed to the left of $\mathrm{Re}(s) = -\sigma_2$, since the implicit equation which determines $A_{2\pm}(\mu,s)$ (see eq. (I3)) requires $\mathrm{Re}(s) \geqq -\sigma_2$. Apparently, $\mathrm{Re}(s) = -\sigma_2$ is the edge of a continuous spectrum in all cases for the reflected slab.

CONCLUDING REMARKS

The present solution has been shown to have the required properties in all special cases which have been solved previously by others using the Lehner-Wing technique. However, in all these rigorous solutions, there was no scattering outside the slab. It was seen that with infinite reflectors on the slab and neutrons anywhere outside the slab initially, it is possible for some neutrons which have spent their entire history in the reflector to impinge on the slab faces at later times. Such neutrons have a collision rate which is characteristic of reflector properties and this condition, in general, places a restriction on how the inversion contour can be deformed in the transform plane. Two cases have been illustrated in which a further deformation is possible for the solution inside the slab by eliminating neutrons outside the slab initially, which can later impinge on the slab faces. This condition is equivalent to a further restriction on the Hilbert space which has been used in some of the above-mentioned rigorous solutions. The exact eigenvalue condition has been obtained, and real time eigenvalues have been calculated for a number of combinations of material parameters. The largest eigenvalues have been shown to agree with the criticality results of others. Calculations also show that eigenvalues can disappear into the branch cut or continuum as material properties are varied and it was pointed out that all such disappearing eigenvalues correspond to exponentially time-decaying modes since the number of secondary neutrons per collision in the reflector was taken to be less than unity. It is expected (but has not been shown) that there is no drastic change in the shape of the solution when this situation occurs; one of the integrals in it probably has resonance-like terms caused perhaps by zeros of the eigenvalue condition on the next Riemann sheet. The assumption that the eigenvalues are real for arbitrary slab half-thicknesses has been made. This assumption has been shown to be true for thick slabs in this report and it has been proved rigorously by others for the above-mentioned special cases. On the basis of sample calculations, it is concluded that if one is given the material properties as well as the bare-slab eigenvalues corresponding to the slab properties, then he can conclude whether there are eigenvalues and the maximum number of them.

Perhaps the present results can serve as a guide for a rigorous Lehner-Wing type analysis of the reflected-slab problem. If the eigenvalues are all real, one might be able to prove it in such an analysis of the present problem.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., November 12, 1970.

APPENDIX A

SUMMARY OF ELEMENTARY SOLUTION PROPERTIES

In this appendix, the elementary solution properties derived by others (refs. 12 to to 16), following the lead of Case (refs. 2 and 11), are summarized. These solutions are obtained from equations (21) to (23) and are given by equations (24) to (26). Such solutions are complete and orthogonal in the following sense. A function, say $g(\mu)$, satisfying very weak restrictions (see, for example, appendix G of ref. 2) for $-1 \leqq \alpha \leqq \mu \leqq \beta \leqq 1$ can be expanded as follows:

(1) Full range ($\alpha = -1$; $\beta = 1$):

$$g(\mu) = \left[a_m \varphi_{\nu_{0m}}(\mu) + b_m \varphi_{-\nu_{0m}}(\mu)\right] \delta_m(s) + \int_{-1}^{1} A_m(\nu)\ \varphi_{m\nu}(\mu,s)\ d\nu \tag{A1}$$

where the notation $\delta_m(s)$ was defined by equation (30). The orthogonality relations used to determine the expansion coefficients in equation (A1) are

$$\int_{-1}^{1} \mu \varphi_{m\nu'}(\mu,s)\ d\mu \int_{-1}^{1} A_m(\nu)\ \varphi_{m\nu}(\mu,s)\ d\nu = A_m(\nu')\ \nu' \Omega_m^{+}(\nu',s)\ \Omega_m^{-}(\nu',s)$$

and for $s \in S_{mi}$

$$\left.\begin{aligned}
&\int_{-1}^{1} \mu \varphi_{m\nu'}(\mu,s)\ \varphi_{\pm\nu_{0m}}(\mu)\ d\mu = 0 \\
&\int_{-1}^{1} \mu \varphi_{\nu_{0m}}(\mu)\ \varphi_{-\nu_{0m}}(\mu)\ d\mu = 0 \\
&\int_{-1}^{1} \mu \varphi_{\pm\nu_{0m}}^{2}(\mu)\ d\mu = \frac{1}{2} c_m \sigma_m \nu_{0m}^{2} \Omega_m'\left(\pm\nu_{0m},s\right)
\end{aligned}\right\} \tag{A2}$$

where

$$\Omega_m'\left(\nu_{0m},s\right) = \frac{d}{dz} \Omega_m(z,s)\Big|_{z=\nu_{0m}} \tag{A3a}$$

and

$$\Omega_m^{\pm}(\nu,s) = \lambda_m(\nu,s) \pm \frac{i\pi c_m \sigma_m \nu}{2} \tag{A3b}$$

(2) Half range ($\alpha = 0$; $\beta = 1$):

$$g(\mu) = a_m \varphi_{\nu_{0m}}(\mu)\, \delta_m(s) + \int_0^1 A_m(\nu)\, \varphi_{m\nu}(\mu,s)\, d\nu \tag{A4}$$

Here the orthogonality relations for $s \in S_{mi}$ are

$$\left.\begin{aligned}
&\int_0^1 W_m(\mu)\, \varphi_{m\nu'}(\mu,s)\, d\mu \int_0^1 A_m(\nu)\, \varphi_{m\nu}(\mu,s)\, d\nu = A_m(\nu')\, W_m(\nu')\, \Omega_m^{+}(\nu',s)\, \Omega_m^{-}(\nu',s) \\
&\int_0^1 W_m(\mu)\, \varphi_{m\nu}(\mu,s)\, \varphi_{\nu_{0m}}(\mu)\, d\mu = 0 \\
&\int_0^1 W_m(\mu)\, \varphi_{m\nu}(\mu,s)\, \varphi_{-\nu_{0m}}(\mu)\, d\mu = \nu c_m \sigma_m \nu_{0m} X_m\left(-\nu_{0m},s\right) \varphi_{-\nu_{0m}}(\nu) \\
&\int_0^1 W_m(\mu)\, \varphi_{\nu_{0m}}(\mu)\, \varphi_{\pm\nu_{0m}}(\mu)\, d\mu = \mp\left(\frac{c_m \sigma_m \nu_{0m}}{2}\right)^2 X_m\left(\pm\nu_{0m},s\right) \\
&\int_0^1 W_m(\mu)\, \varphi_{\nu_{0m}}(\mu)\, \varphi_{m(-\nu)}(\mu,s)\, d\mu = \left(\frac{c_m \sigma_m}{2}\right)^2 \nu\nu_{0m} X_m(-\nu,s) \\
&\int_0^1 W_m(\mu)\, \varphi_{m\nu'}(\mu,s)\, \varphi_{m(-\nu)}(\mu,s)\, d\mu = \frac{c_m \sigma_m}{2} \nu'\left(\nu_{0m} + \nu'\right) X_m(-\nu,s)\, \varphi_{m(-\nu)}(\nu',s) \\
&\int_0^1 W_m(\mu)\, \varphi_{m\nu}(\mu,s)\, d\mu = \frac{1}{2} c_m \sigma_m \nu
\end{aligned}\right\} \tag{A5a}$$

where

$$W_m(\mu) = \frac{c_m \sigma_m \mu}{2\Omega_m(\infty,s)\, \left(\nu_{0m} + \mu\right) X_m(-\mu,s)} \qquad (0 \leqq \mu \leqq 1) \tag{A6a}$$

and

$$\frac{X_m^+(\mu,s)}{X_m^-(\mu,s)} \equiv \frac{\Omega_m^+(\mu,s)}{\Omega_m^-(\mu,s)} \qquad (0 \leqq \mu \leqq 1) \qquad \text{(A7a)}$$

with $X_m(z,s)$ given by

$$X_m(z,s) = \frac{1}{1 - z} \exp\left\{\frac{1}{2\pi i} \int_0^1 \log_e\left[\frac{\Omega_m^+(\nu,s)}{\Omega_m^-(\nu,s)}\right] \frac{d\nu}{\nu - z}\right\} \qquad \text{(A8a)}$$

Several other equivalent expressions for X_m, generally referred to as the X-identities, are given by

$$X_m(z,s) = \begin{cases} \dfrac{\Omega_m(z,s)}{\left(\nu_{0m}^2 - z^2\right)\Omega_m(\infty,s)\, X_m(-z,s)} \\[2ex] \dfrac{c_m \sigma_m}{2\Omega_m(\infty,s)} \displaystyle\int_0^1 \frac{\mu\, d\mu}{\left(\nu_{0m}^2 - \mu^2\right) X_m(-\mu,s)\, (\mu - z)} \end{cases} \qquad \text{(A9a)}$$

The orthogonality relations for $s \in S_{me}$ for the expansion (A4) are

$$\left.\begin{aligned} &\int_0^1 W_m(\mu)\, \varphi_{m\nu'}(\mu,s)\, d\mu \int_0^1 A_m(\nu)\, \varphi_{m\nu}(\mu,s)\, d\nu = A_m(\nu')\, W_m(\nu')\, \Omega_m^+(\nu',s)\, \Omega_m^-(\nu',s) \\ &\int_0^1 W_m(\mu)\, \varphi_{m\nu'}(\mu,s)\, \varphi_{m(-\nu)}(\mu,s)\, d\mu = \frac{1}{2} c_m \sigma_m \nu' X_{0m}(-\nu,s)\, \varphi_{m(-\nu)}(\nu',s) \\ &\int_0^1 W_m(\mu)\, \varphi_{m\nu}(\mu,s)\, d\mu = \frac{1}{2} c_m \sigma_m \nu \end{aligned}\right\} \qquad \text{(A5b)}$$

where

$$W_m(\mu) = \frac{c_m \sigma_m \mu}{2\Omega_m(\infty,s)\, X_{0m}(-\mu,s)} \qquad (0 \leqq \mu \leqq 1) \qquad \text{(A6b)}$$

and

$$\frac{X_{0m}^{+}(\mu,s)}{X_{0m}^{-}(\mu,s)} \equiv \frac{\Omega_m^{+}(\mu,s)}{\Omega_m^{-}(\mu,s)} \qquad (0 \leqq \mu \leqq 1) \tag{A7b}$$

with $X_{0m}(z,s)$ given by

$$X_{0m}(z,s) = \exp\left\{\frac{1}{2\pi i}\int_0^1 \log_e\left[\frac{\Omega_m^{+}(\nu,s)}{\Omega_m^{-}(\nu,s)}\right]\frac{d\nu}{\nu - z}\right\} \tag{A8b}$$

and the identities

$$X_{0m}(z,s) = \begin{cases} \dfrac{\Omega_m(z,s)}{\Omega_m(\infty,s)\, X_{0m}(-z,s)} \\ \\ 1 + \dfrac{c_m \sigma_m}{2\Omega_m(\infty,s)}\displaystyle\int_0^1 \frac{\mu\, d\mu}{X_{0m}(-\mu,s)\,(\mu - z)} \end{cases} \tag{A9b}$$

These half-range orthogonality relations and identities are obtained by extending the time-independent results of Kuščer, McCormick, and Summerfield (ref. 20).

A result, due to Kuščer and Zweifel (ref. 14), which is needed to continue solutions analytically follows from equations (A9a) and (A9b). For a fixed value of z, $X_m(z,s)$ does not become $X_{0m}(z,s)$ as s crosses C_m. However, it can be shown from the first line in equations (A9a) and (A9b) that

$$\left(\nu_{0m} - z\right)X_m(z,s)\Big|_{\substack{s \to C_m \\ s \in S_{mi}}} = X_{0m}(z,s)\Big|_{\substack{s \to C_m \\ s \in S_{me}}} \tag{A10}$$

By use of reference 14, the function $X_{0m}(z,s)$ is redefined to be continuous as $s \to C_m$ by equation (60). Such a function of the two complex variables z and s has the following analytical properties (ref. 14):

Fixed s: no singularity in z-plane cut along (0,1); one simple zero at $z = \nu_{0m}(s)$, $\mathrm{Re}\left(\nu_{0m}\right) \geqq 0$, only if $s \in S_{mi}$

Fixed z: no singularity in the s-plane cut along $\left(-\sigma_m, -\sigma_m\left(1 - c_m\right)\right)$; one simple zero at $s = -\sigma_m + c_m \sigma_m z \tanh^{-1}\frac{1}{z}$ for $\mathrm{Re}(z) > 0$

Note here that $X_{0m}(z,s)$ is a nonvanishing analytic function of z and s for $\mathrm{Re}(z) < 0$ and $s \notin \left(-\sigma_m, -\sigma_m\left(1 - c_m\right)\right)$, the branch cut of $\nu_{0m}(s)$.

APPENDIX B

DERIVATION OF $\psi_{mp\pm}(x,\mu,s)$

In this appendix, explicit forms of $\psi_{mp\pm}(x,\mu,s)$ are obtained. Consider, for medium m, the function $g_m(x,\mu;x_0)$ as

$$g_m(x,\mu;x_0) = \begin{cases} -D_m(x_0)\ \varphi_{-\nu_{0m}}(\mu)\ e^{(s+\sigma_m)(x-x_0)/\nu_{0m}}\ \delta_m(s) \\ -\int_{-1}^{0} C_m(x_0,\nu)\ \varphi_{m\nu}(\mu,s)\ e^{-(s+\sigma_m)(x-x_0)/\nu}\ d\nu & (x < x_0) \\ \\ C_m(x_0)\ \varphi_{\nu_{0m}}(\mu)\ e^{-(s+\sigma_m)(x-x_0)/\nu_{0m}}\ \delta_m(s) \\ +\int_{0}^{1} C_m(x_0,\nu)\ \varphi_{m\nu}(\mu,s)\ e^{-(s+\sigma_m)(x-x_0)/\nu}\ d\nu & (x > x_0) \end{cases} \tag{B1}$$

The expansion coefficients in equation (B1) are to be determined so that $g_m(x,\mu;x_0)$ satisfies equations (42) and (43); that is, on putting the expansion (B1) into equation (43), in the limit $x \rightarrow x_0$

$$\frac{f_m(x_0,\mu)}{\mu} = \left[C_m(x_0)\ \varphi_{\nu_{0m}}(\mu) + D_m(x_0)\ \varphi_{-\nu_{0m}}(\mu)\right]\delta_m(s) + \int_{-1}^{1} C_m(x_0,\nu)\ \varphi_{m\nu}(\mu,s)\ d\nu \tag{B2}$$

This equation is a full-range expansion (see eq. (A1)) of the function $f_m(x_0,\mu)/\mu$ and use of the orthogonality relations (A2) gives the coefficients as

$$C_m(x_0,\nu) = \frac{1}{\nu\Omega_m^{+}(\nu,s)\ \Omega_m^{-}(\nu,s)}\int_{-1}^{1} f_m(x_0,\mu)\ \varphi_{m\nu}(\mu,s)\ d\mu$$

and, if $s \in S_{mi}$

$$C_m(x_0) = \frac{2}{c_m\sigma_m\nu_{0m}^{2}\Omega_m'(\nu_{0m},s)}\int_{-1}^{1} f_m(x_0,\mu)\ \varphi_{\nu_{0m}}(\mu)\ d\mu \tag{B3a}$$

and

$$D_m(x_0) = \frac{2}{c_m \sigma_m \nu_{0m}^2 \Omega_m'(-\nu_{0m}, s)} \int_{-1}^{1} f_m(x_0, \mu)\, \varphi_{-\nu_{0m}}(\mu)\, d\mu \tag{B3b}$$

However, one needs expansion coefficients for $f_{m\pm}(x_0, \mu)/\mu$. It follows from equation tions (B3) that

$$C_{m\pm}(x_0, \nu) \equiv \frac{1}{2}\left[C_m(x_0, \nu) \mp C_m(-x_0, -\nu)\right]$$

and, if $s \in S_{mi}$,

$$C_{m\pm}(x_0, \nu_{0m}) \equiv \frac{1}{2}\left[C_m(x_0) \mp D_m(-x_0)\right] \tag{B4a}$$

and

$$C_{m\pm}(x_0, -\nu_{0m}) \equiv \frac{1}{2}\left[D_m(x_0) \mp C_m(-x_0)\right] \tag{B4b}$$

are the expansion coefficients of $f_{m\pm}(x_0, \mu)/\mu$; that is, equations (47).

In order to construct $\psi_{mp\pm}(x, \mu, s)$ according to equation (41), note that for $m = 2$

$$\int_{\text{medium 2}} (\ldots)\, dx_0 = \int_{-a}^{a} (\ldots)\, dx_0 = \int_{-a}^{x} (\ldots)\, dx_0 + \int_{x}^{a} (\ldots)\, dx_0 \tag{B5}$$

Upon using equation (B1), one obtains $\psi_{2p}(x, \mu, s)$ as

$$\begin{aligned}
\psi_{2p}(x, \mu, s) = {} & \left[\int_{-a}^{x} C_2(x_0)\, e^{(s+\sigma_2)x_0/\nu_{02}}\, dx_0\right] \psi_{\nu_{02}}(x, \mu, s)\, \delta_2(s) \\
& + \int_0^1 \left[\int_{-a}^{x} C_2(x_0, \nu)\, e^{(s+\sigma_2)x_0/\nu}\, dx_0\right] \psi_{2\nu}(x, \mu, s)\, d\nu \\
& - \left[\int_{x}^{a} D_2(x_0)\, e^{-(s+\sigma_2)x_0/\nu_{02}}\, dx_0\right] \psi_{-\nu_{02}}(x, \mu, s)\, \delta_2(s) \\
& - \int_0^1 \left[\int_{x}^{a} C_2(x_0, -\nu)\, e^{-(s+\sigma_2)x_0/\nu}\, dx_0\right] \psi_{2(-\nu)}(x, \mu, s)\, d\nu
\end{aligned} \tag{B6}$$

The definite parity particular solution $\psi_{2p\pm}(x,\mu,s)$ is then obtained by using equations (B6) and (B4) as

$$\psi_{2p\pm}(x,\mu,s) = \left\{\left[\int_{-a}^{x} C_{2\pm}(x_0,\nu_{02})\, e^{(s+\sigma_2)x_0/\nu_{02}}\, dx_0\right]\psi_{\nu_{02}}(x,\mu,s)\right.$$

$$\left.\pm\left[\int_{-a}^{-x} C_{2\pm}(x_0,\nu_{02})\, e^{(s+\sigma_2)x_0/\nu_{02}}\, dx_0\right]\psi_{-\nu_{02}}(x,\mu,s)\right\}\delta_2(s)$$

$$+\int_0^1\left[\int_{-a}^{x} C_{2\pm}(x_0,\nu)\, e^{(s+\sigma_2)x_0/\nu}\, dx_0\right]\psi_{2\nu}(x,\mu,s)\, d\nu$$

$$\pm\int_0^1\left[\int_{-a}^{-x} C_{2\pm}(x_0,\nu)\, e^{(s+\sigma_2)x_0/\nu}\, dx_0\right]\psi_{2(-\nu)}(x,\mu,s)\, d\nu \tag{B7}$$

That equation (B7) is a solution of equation (18) for $m = 2$ can be seen by direct substitution as follows. The $\psi_{m\nu}(x,\mu,s)$ in equation (B7) are solutions of equation (21), the homogeneous equation corresponding to equation (18). However, their coefficients in equation (B7) are functions of x so that some additional terms are obtained from the $\frac{\partial}{\partial x}$ operation. Thus,

$$\mu\left\{\left[C_{2\pm}(x,\nu_{02})\,\varphi_{\nu_{02}}(\mu) + C_{2\pm}(x,-\nu_{02})\,\varphi_{-\nu_{02}}(\mu)\right]\delta_2(s)\right.$$

$$\left.+\int_{-1}^{1} C_{2\pm}(x,\nu)\,\varphi_{2\nu}(\mu,s)\, d\nu\right\} = f_{2\pm}(x,\mu) \tag{B8}$$

which is an identity since according to equation (47), the $C_{2\pm}$ are the full-range expansion coefficients of $f_{2\pm}(x,\mu)/\mu$.

To get $\psi_{1p}(x,\mu,s)$ according to equation (41), first note that

$$\int_{\text{medium 1}}(\ldots)\, dx_0 = \int_{-\infty}^{a}(\ldots)\, dx_0 + \int_a^{\infty}(\ldots)\, dx_0$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{x}(\ldots)\, dx_0 + \int_x^{-a}(\ldots)\, dx_0 + \int_a^{\infty}(\ldots)\, dx_0 & (x < -a) \\ \displaystyle\int_{-\infty}^{-a}(\ldots)\, dx_0 + \int_a^{x}(\ldots)\, dx_0 + \int_x^{\infty}(\ldots)\, dx_0 & (x > a) \end{cases} \tag{B9}$$

By following the same procedure as before, one gets $\psi_{1p\pm}(x,\mu,s)$ as

$$\psi_{1p\pm}(x,\mu,s) = \left[\int_{-\infty}^{x} C_{1\pm}(x_0,\nu_{01})\, e^{(s+\sigma_1)x_0/\nu_{01}}\, dx_0\right] \psi_{\nu_{01}}(x,\mu,s)\, \delta_1(s)$$

$$+ \int_0^1 \left[\int_{-\infty}^{x} C_{1\pm}(x_0,\nu)\, e^{(s+\sigma_1)x_0/\nu}\, dx_0\right] \psi_{1\nu}(x,\mu,s)\, d\nu$$

$$+ \left[-\int_{x}^{-a} C_{1\pm}(x_0,-\nu_{01})\, e^{-(s+\sigma_1)x_0/\nu_{01}}\, dx_0\right.$$

$$\left.\pm \int_{-\infty}^{-a} C_{1\pm}(x_0,\nu_{01})\, e^{(s+\sigma_1)x_0/\nu_{01}}\, dx_0\right] \psi_{-\nu_{01}}(x,\mu,s)\, \delta_1(s)$$

$$+ \int_0^1 \left[-\int_{x}^{-a} C_{1\pm}(x_0,-\nu)\, e^{-(s+\sigma_1)x_0/\nu}\, dx_0\right.$$

$$\left.\pm \int_{-\infty}^{-a} C_{1\pm}(x_0,\nu)\, e^{(s+\sigma_1)x_0/\nu}\, dx_0\right] \psi_{1(-\nu)}(x,\mu,s)\, d\nu \qquad (x < -a) \qquad \text{(B10a)}$$

and

$$\psi_{1p\pm}(x,\mu,s) = \left[\int_{-\infty}^{-a} C_{1\pm}(x_0,\nu_{01})\, e^{(s+\sigma_1)x_0/\nu_{01}}\, dx_0\right.$$

$$\left.\mp \int_{-x}^{-a} C_{1\pm}(x_0,-\nu_{01})\, e^{-(s+\sigma_1)x_0/\nu_{01}}\, dx_0\right] \psi_{\nu_{01}}(x,\mu,s)\, \delta_1(s)$$

$$+ \int_0^1 \left[\int_{-\infty}^{-a} C_{1\pm}(x_0,\nu)\, e^{(s+\sigma_1)x_0/\nu}\, dx_0\right.$$

$$\left.\mp \int_{-x}^{-a} C_{1\pm}(x_0,-\nu)\, e^{-(s+\sigma_1)x_0/\nu}\, dx_0\right] \psi_{1\nu}(x,\mu,s)\, d\nu$$

$$\pm \left[\int_{-\infty}^{-x} C_{1\pm}(x_0,\nu_{01})\, e^{(s+\sigma_1)x_0/\nu}\, dx_0\right] \psi_{-\nu_{01}}(x,\mu,s)\, \delta_1(s)$$

$$\pm \int_0^1 \left[\int_{-\infty}^{-x} C_{1\pm}(x_0,\nu)\, e^{(s+\sigma_1)x_0/\nu}\, dx_0\right] \psi_{1(-\nu)}(x,\mu,s)\, d\nu \qquad (x > a) \qquad \text{(B10b)}$$

Again, it is easily shown by direct substitution that equations (B10a) and (B10b) are solutions of equation (18) for $m = 1$. By introducing the F functions of equation (46) and allowing x to take on negative and positive values, it follows that equations (B7), (B10a), and (B10b) can be written as equations (44) and (45).

Also note here that the $C_{m\pm}$ coefficients of equations (B4) have the property

$$\left.\begin{aligned} C_{m\pm}\left(-x_0,-\nu\right) &= \mp C_{m\pm}\left(x_0,\nu\right) \\ C_{m\pm}\left(-x_0,-\nu_{0m}\right) &= \mp C_{m\pm}\left(x_0,\nu_{0m}\right) \end{aligned}\right\} \tag{B11}$$

so that it then follows from equations (46) that

$$\left.\begin{aligned} F_{2\pm}(a,-\omega,s) &= \mp F_{2\pm}(a,\omega,s) \\ \tilde{F}_{\pm}(-a,-\omega,s) &= \mp \tilde{F}_{\pm}(-a,\omega,s) \end{aligned}\right\} \tag{B12}$$

APPENDIX C

TWO-MEDIA FULL-RANGE EXPANSIONS AND ORTHOGONALITY RELATIONS

In this appendix, some results of references 15, 16, and 20 are summarized and extended. In reference 15, it is shown that a function, for example, $h(\mu)$, satisfying very weak restrictions for μ on the interval $-1 \leqq \mu \leqq 1$ can be expanded as

$$h(\mu) = a_1\varphi_{\nu_{01}}(\mu)\ \delta_1(s) + b_2\varphi_{-\nu_{02}}(\mu)\ \delta_2(s) + \int_0^1 A_1(\nu)\ \varphi_{1\nu}(\mu,s)\ d\nu + \int_{-1}^0 A_2(\nu)\ \varphi_{2\nu}(\mu,s)\ d\nu \tag{C1}$$

This equation is a two-media full-range expansion of the function $h(\mu)$ and the expansion coefficients in it can be determined by using orthogonality relations which are easily determined from the time-independent ones of reference 20. For $\delta_1(s) = \delta_2(s) = 1$, that is, $s \in S_{1i} \cap S_{2i}$, these relations are

$$\left.\begin{aligned}
&\int_{-1}^1 W(\mu)\ \Phi_{\nu'}(\mu,s)\ d\mu \int_{-1}^1 A(\nu)\ \Phi_\nu(\mu,s)\ d\nu = A(\nu')\ W(\nu')\ \Omega^+(\nu',s)\ \Omega^-(\nu',s)\\
&\int_{-1}^1 W(\mu)\ \Phi_\nu(\mu,s)\ \varphi_{\nu_{01}}(\mu)\ d\mu = 0\\
&\int_{-1}^1 W(\mu)\ \Phi_\nu(\mu,s)\ \varphi_{-\nu_{02}}(\mu)\ d\mu = 0\\
&\int_{-1}^1 W(\mu)\ \varphi_{\nu_{01}}(\mu)\ \varphi_{-\nu_{02}}(\mu)\ d\mu = 0\\
&\int_{-1}^1 W(\mu)\ \Phi_\nu(\mu,s)\ \varphi_{-\nu_{01}}(\mu)\ d\mu = \nu c(\nu)\ \sigma(\nu)\ \nu_{01}\left(\nu_{02} - \nu_{01}\right)\chi\left(-\nu_{01},s\right)\ \varphi_{-\nu_{01}}(\nu)\\
&\int_{-1}^1 W(\mu)\ \Phi_\nu(\mu,s)\ \varphi_{\nu_{02}}(\mu)\ d\mu = \nu c(\nu)\ \sigma(\nu)\ \nu_{02}\left(\nu_{01} - \nu_{02}\right)\chi\left(\nu_{02},s\right)\ \varphi_{\nu_{02}}(\nu)\\
&\int_{-1}^1 W(\mu)\ \varphi_{\nu_{01}}(\mu)\ \varphi_{\pm\nu_{01}}(\mu)\ d\mu = -\left(\frac{c_1\sigma_1\nu_{01}}{2}\right)^2\left(\nu_{01} \pm \nu_{02}\right)\chi\left(\pm\nu_{01},s\right)
\end{aligned}\right\} \tag{C2}$$

Equations continued on next page

$$
\left.
\begin{aligned}
&\int_{-1}^{1} W(\mu)\ \varphi_{-\nu_{02}}(\mu)\ \varphi_{\pm\nu_{02}}(\mu)\ d\mu = \left(\frac{c_2\sigma_2\nu_{02}}{2}\right)^2 \left(\nu_{02} \mp \nu_{01}\right)\chi\left(\pm\nu_{02},s\right) \\
&\int_{-1}^{1} W(\mu)\ \varphi_{\nu_{01}}(\mu)\ \varphi_{\nu_{02}}(\mu)\ d\mu = -\frac{1}{2}c_1\sigma_1c_2\sigma_2\nu_{01}\nu_{02}^2\chi\left(\nu_{02},s\right) \\
&\int_{-1}^{1} W(\mu)\ \varphi_{-\nu_{02}}(\mu)\ \varphi_{-\nu_{01}}(\mu)\ d\mu = \frac{1}{2}c_1\sigma_1c_2\sigma_2\nu_{01}^2\nu_{02}\chi\left(-\nu_{01},s\right)
\end{aligned}
\right|
$$

where

$$
\left.
\begin{aligned}
&c(\nu),\sigma(\nu) = \begin{cases} c_1,\sigma_1 & (\nu > 0) \\ c_2,\sigma_2 & (\nu < 0) \end{cases} \\
&\Phi_\nu(\mu,s) = \begin{cases} \varphi_{1\nu}(\mu,s) & (\nu > 0) \\ \varphi_{2\nu}(\mu,s) & (\nu < 0) \end{cases} \\
&\Omega^\pm(\nu,s) = \begin{cases} \Omega_1^\pm(\nu,s) & (\nu > 0) \\ \Omega_2^\pm(\nu,s) & (\nu < 0) \end{cases} \\
&\chi(z,s) = X_1(z,s)\ X_2(-z,s) \\
&W(\nu) = \begin{cases} \left(\nu_{02} + \nu\right)X_2(-\nu,s)\ W_1(\nu) & (\nu > 0) \\ -\left(\nu_{01} - \nu\right)X_1(\nu,s)\ W_2(-\nu) & (\nu < 0) \end{cases}
\end{aligned}
\right\} \qquad \text{(C3)}
$$

All remaining quantities have been defined in appendix A.

Rather than write out explicit orthogonality relations for the other three regions of the transform plane in the present notation, one introduces a function which is continuous as $s \rightarrow C_m$. From the results of reference 14 quoted in appendix A, it is seen that one such function is given by equation (59) and can be written by using equation (60) as

$$X_0(z,s) = \begin{cases} \dfrac{(\nu_{02} - z)X_2(z,s)}{(\nu_{01} - z)X_1(z,s)} & (s \in S_{1i} \cap S_{2i}) \\[2ex] \dfrac{(\nu_{02} - z)X_2(z,s)}{X_{01}(z,s)} & (s \in S_{1e} \cap S_{2i}) \\[2ex] \dfrac{X_{02}(z,s)}{(\nu_{01} - z)X_1(z,s)} & (s \in S_{1i} \cap S_{2e}) \\[2ex] \dfrac{X_{02}(z,s)}{X_{01}(z,s)} & (s \in S_{1e} \cap S_{2e}) \end{cases} \tag{C4}$$

In terms of this function, $W(\nu)$ can be written as

$$W(\nu) = \begin{cases} \dfrac{c_1\sigma_1\nu}{2\Omega_1(\infty,s)} X_0(-\nu,s) & (\nu > 0) \\[2ex] \dfrac{c_2\sigma_2\nu}{2\Omega_2(\infty,s)} \dfrac{1}{X_0(\nu,s)} & (\nu < 0) \end{cases} \tag{C5}$$

The function $\chi(z,s)$ is expressed as

$$\chi(z,s) = \begin{cases} \dfrac{X_0(-z,s)}{(\nu_{02} + z)} \dfrac{\Omega_1(z,s)}{(\nu_{01} - z)\Omega_1(\infty,s)} & (\mathrm{Re}(z) > 0) \\[2ex] \dfrac{1}{(\nu_{01} - z)X_0(z,s)} \dfrac{\Omega_2(z,s)}{(\nu_{02} + z)\Omega_2(\infty,s)} & (\mathrm{Re}(z) < 0) \end{cases} \tag{C6}$$

In order to obtain a two-media expansion in the form of equation (C1), one generally has to switch some continuum solutions in one medium to those in the other. From the explicit form of $\varphi_{m\nu}(\mu,s)$ (eqs. (24)), it follows that

$$c_1\sigma_1\varphi_{2\nu}(\mu,s) - c_2\sigma_2\varphi_{1\nu}(\mu,s) = k\delta(\mu - \nu)$$

where

$$k \equiv s(c_1\sigma_1 - c_2\sigma_2) + \sigma_1\sigma_2(c_1 - c_2) \tag{C7}$$

It is seen that when the two media are the same, $k \equiv 0$. This quantity can be expressed in a number of different ways, and several that will be used are

$$k = \begin{cases} c_1\sigma_1\lambda_2(\nu,s) - c_2\sigma_2\lambda_1(\nu,s) \\ c_1\sigma_1\Omega_2\left(\nu_{01},s\right) \\ -c_2\sigma_2\Omega_1\left(\nu_{02},s\right) \end{cases} \tag{C8}$$

The orthogonality relations (eqs. (C2)) can now be written in terms of $X_0(z,s)$ and k as

$$\left.\begin{aligned}
&\int_{-1}^{1} W(\mu)\, \Phi_{\nu'}(\mu,s)\, d\mu \int_{-1}^{1} A(\nu)\, \Phi_\nu(\mu,s)\, d\nu = A(\nu')\, W(\nu')\, \Omega^+(\nu',s)\, \Omega^-(\nu',s) \\
&\int_{-1}^{1} W(\mu)\, \Phi_\nu(\mu,s)\, \varphi_{\nu_{01}}(\mu)\, d\mu = 0 \\
&\int_{-1}^{1} W(\mu)\, \Phi_\nu(\mu,s)\, \varphi_{-\nu_{02}}(\mu)\, d\mu = 0 \\
&\int_{-1}^{1} W(\mu)\, \varphi_{\nu_{01}}(\mu)\, \varphi_{-\nu_{02}}(\mu)\, d\mu = 0 \\
&\int_{-1}^{1} W(\mu)\, \Phi_\nu(\mu,s)\, \varphi_{-\nu_{01}}(\mu)\, d\mu = \frac{\nu c(\nu)\, \sigma(\nu)\, \nu_{01}k}{4\Omega_2(\infty,s)\, X_0\left(-\nu_{01},s\right)\left(\nu_{01}+\nu\right)} \\
&\int_{-1}^{1} W(\mu)\, \Phi_\nu(\mu,s)\, \varphi_{\nu_{02}}(\mu)\, d\mu = -\frac{\nu c(\nu)\, \sigma(\nu)\, \nu_{02}kX_0\left(-\nu_{02},s\right)}{4\Omega_1(\infty,s)\left(\nu_{02}-\nu\right)} \\
&\int_{-1}^{1} W(\mu)\, \varphi_{\nu_{01}}(\mu)\, \varphi_{\pm\nu_{01}}(\mu)\, d\mu = \begin{cases} \left(\frac{1}{2}c_1\sigma_1\nu_{01}\right)^2 \dfrac{\Omega_1'\left(\nu_{01},s\right)}{\Omega_1(\infty,s)} X_0\left(-\nu_{01},s\right) \\ \dfrac{c_1\sigma_1\nu_{01}k}{8\Omega_2(\infty,s)\, X_0\left(-\nu_{01},s\right)} \end{cases}
\end{aligned}\right\} \tag{C9}$$

Equations continued on next page

$$\int_{-1}^{1} W(\mu)\ \varphi_{-\nu_{02}}(\mu)\ \varphi_{\pm\nu_{02}}(\mu)\ d\mu = \begin{cases} \dfrac{c_2\sigma_2\nu_{02}kX_0\left(-\nu_{02},s\right)}{8\Omega_1(\infty,s)} \\[2ex] -\left(\dfrac{1}{2}c_2\sigma_2\nu_{02}\right)^2 \dfrac{\Omega_2'\left(\nu_{02},s\right)}{\Omega_2(\infty,s)}\ \dfrac{1}{X_0\left(-\nu_{02},s\right)} \end{cases}$$

$$\int_{-1}^{1} W(\mu)\ \varphi_{\nu_{01}}(\mu)\ \varphi_{\nu_{02}}(\mu)\ d\mu = \frac{c_1\sigma_1\nu_{01}\nu_{02}kX_0\left(-\nu_{02},s\right)}{4\Omega_1(\infty,s)\left(\nu_{01}-\nu_{02}\right)}$$

$$\int_{-1}^{1} W(\mu)\ \varphi_{-\nu_{02}}(\mu)\ \varphi_{-\nu_{01}}(\mu)\ d\mu = \frac{c_2\sigma_2\nu_{01}\nu_{02}k}{4\Omega_2(\infty,s)\left(\nu_{02}-\nu_{01}\right)}\ \frac{1}{X_0\left(-\nu_{01},s\right)}$$

These expressions appear to be more complicated than the corresponding ones in equations (C2); however, the orthogonality relations needed for all regions of the transform plane are given by equations (C9). That is, for $s \in S_{1e} \cap S_{2i}$, the proper orthogonality relations are the first, third, sixth, and eighth equations of equations (C9) with $X_0(z,s)$ given by equation (C4). Note here that $X_0(z,s)$ always appears in equations (C9) with $\mathrm{Re}(z) < 0$. It follows then from appendix A that for $\mathrm{Re}(z) < 0$, $X_0(z,s)$ is a nonvanishing analytic function of both z and s except for the branch cuts in the s-plane due to $\nu_{01}(s)$ and $\nu_{02}(s)$.

APPENDIX D

THE TWO-MEDIA FULL-RANGE EXPANSION FOR THIS PROBLEM

In this appendix, it is shown that application of the continuity condition (eq. (20)) results in a two-media full-range expansion of the type discussed in appendix C. For x = a, one readily obtains from equations (31) and (20) upon using the explicit forms of $\psi_{mc\pm}$ and $\psi_{mp\pm}$ given by equations (32), (39), (44), and (45) that

$$
\begin{aligned}
0 = {} & a_{2\pm}\left[\psi_{\nu_{02}}(a,\mu,s) \pm \psi_{-\nu_{02}}(a,\mu,s)\right] \delta_2(s) \\
& + \int_0^1 A_{2\pm}(\nu)\left[\psi_{2\nu}(a,\mu,s) \pm \psi_{2(-\nu)}(a,\mu,s)\right] d\nu \\
& \mp \left[a_{1\pm}\psi_{\nu_{01}}(a,\mu,s)\ \delta_1(s) + \int_0^1 A_{1\pm}(-\nu)\ \psi_{1\nu}(a,\mu,s)\ d\nu\right] \\
& + F_{2\pm}\left(a,\nu_{02},s\right)\ \psi_{\nu_{02}}(a,\mu,s)\ \delta_2(s) + \int_0^1 F_{2\pm}(a,\nu,s)\ \psi_{2\nu}(a,\mu,s)\ d\nu \\
& - F_{1\pm}\left(-a,\nu_{01},s\right)\left[\psi_{\nu_{01}}(a,\mu,s) \pm \psi_{-\nu_{01}}(a,\mu,s)\right] \delta_1(s) \\
& - \int_0^1 F_{1\pm}(-a,\nu,s)\left[\psi_{1\nu}(a,\mu,s) \pm \psi_{1(-\nu)}(a,\mu,s)\right] d\nu
\end{aligned}
\tag{D1}
$$

It was indicated in appendix C that according to Erdmann (ref. 15), the functions $\varphi_{\nu_{01}}(\mu)$, $\varphi_{-\nu_{02}}(\mu)$, $\varphi_{1\nu}(\mu,s)$ $(0 \leqq \nu \leqq 1)$ and $\varphi_{2\nu}(\mu,s)$ $(-1 \leqq \nu \leqq 0)$ form a complete orthogonal set of basis functions for the expansion of $h(\mu)$ $(-1 \leqq \mu \leqq 1)$ for $s \in S_{1i} \cap S_{2i}$. (See eq. (C1).) However, equation (D1) also contains terms in which $\varphi_{2\nu}(\mu,s)$ $(0 \leqq \nu \leqq 1)$ and $\varphi_{1\nu}(\mu,s)$ $(-1 \leqq \nu \leqq 0)$ appear. These continuum solutions must be replaced by corresponding continuum solutions for the other media. One uses the relationship (C7) to do this; that is,

$$
\left.
\begin{aligned}
\varphi_{2\nu}(\mu,s) &= \left[\frac{c_2\sigma_2}{c_1\sigma_1}\varphi_{1\nu}(\mu,s) + \frac{k}{c_1\sigma_1}\delta(\nu - \mu)\right] H(\nu) \\
\varphi_{1\nu}(\mu,s) &= \left[\frac{c_1\sigma_1}{c_2\sigma_2}\varphi_{2\nu}(\mu,s) - \frac{k}{c_2\sigma_2}\delta(\nu - \mu)\right] H(-\nu)
\end{aligned}
\right\}
\tag{D2}
$$

where

$$H(\nu) = \begin{cases} 1 & (\nu > 0) \\ 0 & (\nu < 0) \end{cases} \tag{D3}$$

When explicit forms of the elementary solutions and equations (D2) and (D3) are used in equation (D1), one obtains the two-media full-range expansion

$$\begin{aligned} h(\mu) = &\left[F_{1\pm}\left(-a,\nu_{01},s\right) \pm a_{1\pm}\right] e^{-(s+\sigma_1)a/\nu_{01}} \varphi_{\nu_{01}}(\mu)\ \delta_1(s) \\ &\mp a_{2\pm}\ e^{(s+\sigma_2)a/\nu_{02}} \varphi_{-\nu_{02}}(\mu)\ \delta_2(s) \\ &+ \int_0^1 \left\{\left[F_{1\pm}(-a,\nu,s) \pm A_{1\pm}(-\nu)\right] e^{-(s+\sigma_1)a/\nu}\right. \\ &\left. - \frac{c_2\sigma_2}{c_1\sigma_1}\left[F_{2\pm}(a,\nu,s) + A_{2\pm}(\nu)\right] e^{-(s+\sigma_2)a/\nu}\right\} \varphi_{1\nu}(\mu,s)\ d\nu \\ &\pm \int_{-1}^0 \left[\frac{c_1\sigma_1}{c_2\sigma_2} F_{1\pm}(-a,-\nu,s)\ e^{-(s+\sigma_1)a/\nu} - A_{2\pm}(-\nu)\ e^{-(s+\sigma_2)a/\nu}\right] \varphi_{2\nu}(\mu,s)\ d\nu \end{aligned} \tag{D4a}$$

where $h(\mu)$ is given by

$$\begin{aligned} h(\mu) = &\frac{k}{c_1\sigma_1} H(\mu) \left[F_{2\pm}(a,\mu,s) + A_{2\pm}(\mu)\right] e^{-(s+\sigma_2)a/\mu} \\ &\pm \frac{k}{c_2\sigma_2} H(-\mu)\ F_{1\pm}(-a,-\mu,s)\ e^{-(s+\sigma_1)a/\mu} \\ &+ \left[F_{2\pm}\left(a,\nu_{02},s\right) + a_{2\pm}\right] e^{-(s+\sigma_2)a/\nu_{02}} \varphi_{\nu_{02}}(\mu)\ \delta_2(s) \\ &\mp F_{1\pm}\left(-a,\nu_{01},s\right) e^{(s+\sigma_1)a/\nu_{01}} \varphi_{-\nu_{01}}(\mu)\ \delta_1(s) \end{aligned} \tag{D4b}$$

The orthogonality relations (eqs. (C9)) can be used on the expansion (D4) to obtain equations which determine the remaining unknown coefficients implicitly. However, it is convenient to introduce first the $E_{m\pm}$ coefficients given by equations (50). One then has, after some algebraic manipulation, equations (51) to (48).

APPENDIX E

CONTINUATION TO THE COMPLEX PLANE

In this appendix, the way in which equations (50) to (58) are extended to the complex plane $\mu \to z$ with s considered a parameter is outlined. As shown in reference 21, functions such as those introduced in equations (50) are extendable. The particular grouping of terms in equations (51) to (58) indicates some integrals and residues which go together.

The first functions to be considered are the $F_{m\pm}(x,\omega,s)$ functions given by equations (46). In equations (55) to (58), these functions appear with $Re(\omega) > 0$ so one considers the functions $L_{m\pm}(x,\nu,s)$ given by equation (69). When the explicit expressions of equations (46) and (47) are used, one can see that for $f_{m\pm}(x_0,\mu)$ extendable $\mu \to z$ without singularities in the finite z-plane, then $L_{m\pm}(x,\nu,s)$ can be extended to $L_{m\pm}(x,z,s)$ given by equations (67) and (68). As $z \to \nu \in (0,1)$, it can be seen that the limiting values of $L_{m\pm}$, namely $L_{m\pm}^{+}(x,\nu,s)$ and $L_{m\pm}^{-}(x,\nu,s)$, are identical. Thus, $L_{m\pm}$ does not contain the branch cut of $\Omega_m(z,s)$ as one might be led to expect from equation (67). There appear to be no other singularities of $L_{m\pm}$ in the finite z-plane, $Re(z) > 0$ and $Re(s) > \sigma_m$. It follows from equation (67) that

$$L_{m\pm}\left(x,\nu_{0m},s\right) = \frac{1}{2} c_m \sigma_m \nu_{0m} \Omega_m'\left(\nu_{0m},s\right) F_{m\pm}\left(x,\nu_{om},s\right) e^{-\left(s+\sigma_m\right)x/\nu_{0m}} \qquad \left(s \in S_{mi}\right) \tag{E1}$$

In order to extend the functions $I_{m\pm}(\nu)$ to the complex z-plane, one needs the identity

$$\frac{c_1\sigma_1\Omega_2^{+}(\nu,s)\ \Omega_2^{-}(\nu,s)}{c_2\sigma_2\Omega_1^{+}(\nu,s)\ \Omega_1^{-}(\nu,s)} = \frac{c_2\sigma_2}{c_1\sigma_1} + \frac{k}{c_1\sigma_1 c_2\sigma_2}\left[\frac{c_1\sigma_1\lambda_2(\nu,s) + c_2\sigma_2\lambda_1(\nu,s)}{\Omega_1^{+}(\nu,s)\ \Omega_1^{-}(\nu,s)}\right] \tag{E2}$$

which can be verified directly. On using this identity, one finds that $I_{m\pm}(\nu)$, given by equations (55) and (57), can be written, respectively, as equations (65) and (66). The restriction $Re(s) > -\sigma_{min}$ on these equations comes from the fact that $L_{m\pm}$ for both $m = 1$ and $m = 2$ occur in each $I_{m\pm}$. More will be said about this restriction later. The contours C' are given in figure 3. By letting $z = \nu_{02}$ in equation (65) and $z = \nu_{01}$ in equation (66), it can be seen that

$$\left.\begin{aligned} I_{2\pm}(\nu_{02},s) &\equiv J_{2\pm}(\nu_{02}) \\ I_{1\pm}(\nu_{01},s) &\equiv J_{1\pm}(\nu_{01}) \end{aligned}\right\} \tag{E3}$$

Thus, the inhomogeneous terms of equations (51) to (54) are seen to be extendable and related as shown in equations (E3). For $z \to \nu_{01}$ in $I_{2\pm}$ (eq. (65)) and $z \to \nu_{02}$ in $I_{1\pm}$ (eq. (66)), these functions might seem to be singular. However, upon examining the residues it is seen that this is not the case. Thus, the $I_{m\pm}(z,s)$ appear to be analytic in the finite z-plane, $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(s) > -\sigma_{min}$.

In equation (51), one now lets $\nu \to z$ and, for $\mathrm{Re}(s) > -\sigma_{min}$ and $\mathrm{Re}(z) > 0$ in the finite z-plane, finds that $E_{2\pm}(z,s)$ is given by the inhomogeneous term $I_{2\pm}(z)$, a term involving $a_{2\pm}$ if $s \in S_{2i}$ and an integral over $E_{2\pm}(\mu,s)$ $(0 \leqq \mu \leqq 1)$. A singularity occurs in the integrand when either $\Omega_2^+(\mu,s)$ or $\Omega_2^-(\mu,s)$ vanishes and this condition occurs for $s \in C_2$. However, for this case, it is seen that one obtains from equations (51) and (52) that $E_{2\pm}(\nu_{02},s)$ is related to $a_{2\pm}$. It appears that $E_{2\pm}(z,s)$ is analytic in the finite z-plane, $\mathrm{Re}(z) > 0$, $\mathrm{Re}(s) > -\sigma_{min}$ and can be written as equation (63). By following the same procedure with equation (52), one obtains an equation which is easily seen to be equation (63) evaluated at $z = \nu_{02}$; that is, $E_{2\pm}(\nu_{02},s)$ and $a_{2\pm}$ are related as

$$E_{2\pm}(\nu_{02},s) = \frac{1}{2} c_2 \sigma_2 \nu_{02} \Omega_2'(\nu_{02},s)\, a_{2\pm}\, e^{(s+\sigma_2)a/\nu_{02}} \qquad (s \in S_{2i}) \tag{E4}$$

In a similar manner, from equations (53) and (54) with $\nu \to z$ and from using equation (E4), one obtains equation (64) and again it follows that $E_{1\pm}(\nu_{01},s)$ and $a_{1\pm}$ are related as

$$E_{1\pm}(\nu_{01},s) = \frac{1}{2} c_1 \sigma_1 \nu_{01} \Omega_1'(\nu_{01},s)\, a_{1\pm}\, e^{-(s+\sigma_1)a/\nu_{01}} \qquad (s \in S_{1i}) \tag{E5}$$

It also appears that $E_{1\pm}(z,s)$ is analytic in the finite z-plane, $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(s) > -\sigma_{min}$.

The solutions $\psi_{mc\pm}$ and $\psi_{mp\pm}$ can now be written in terms of the $E_{m\pm}$ as shown in equations (70) to (73).

APPENDIX F

INVESTIGATION OF THE ASSOCIATED EIGENVALUE PROBLEM

In this appendix, the associated eigenvalue problem, that is, the problem for which $f(x,\mu) \equiv 0$, is considered. The inhomogeneous terms $I_{m\pm}$ (given by eqs. (65) to (67)) can be seen to be identically zero when $f(x,\mu)$ is zero everywhere. Solutions for $I_{m\pm} \equiv 0$ are denoted with a bar, that is, $\overline{\psi}_{m\pm}$. The unknown expansion coefficients for the associated eigenvalue problem $\overline{E}_{m\pm}$ are given by equations (63) and (64) with $I_{m\pm} = 0$. It is seen from such equations that $\overline{E}_{m\pm}$ can be determined only to within an arbitrary factor independent of z and that $\overline{E}_{1\pm}$ depends on $\overline{E}_{2\pm}$. Furthermore, the original normal-mode expansion coefficients for the eigenvalue problem are given by $\overline{E}_{m\pm}(\mu,s)$ $(0 \leqq \mu \leqq 1,\ m = 1,2)$, $\overline{E}_{1\pm}(\nu_{01},s)$ $(s \in S_{1i})$, and $\overline{E}_{2\pm}(\nu_{02},s)$ $(s \in S_{2i})$. Therefore, one must examine solutions of such equations as a function of the transform variable s for $z \to \mu$ with the contour C' collapsed onto the branch cut $(0,1)$ due to $\Omega_2(z',s)$, and for $z = \nu_{0m}$ when $s \in S_{mi}$. This procedure is followed for all s in some right-half s-plane and it is convenient to divide the plane into three regions: $s \in S_{2e}$, $s \in S_{2i}$, and $s \in C_2$.

When $s \in S_{2e}$, $\Omega_2(z',s)$ does not vanish within C' so that equation (63) with $I_{2\pm} = 0$ can be written as

$$\overline{E}_{2\pm}(\mu,s) = \pm\int_0^1 K(\mu,\nu)\, \overline{E}_{2\pm}(\nu,s)\, d\nu \qquad (s \in S_{2e},\ 0 \leqq \mu \leqq 1) \quad \text{(F1)}$$

where

$$K(\mu,\nu) = \frac{k}{2}\, \frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)}\, \frac{X_0(-\mu,s)\, X_0(-\nu,s)\, \nu}{\Omega_2^+(\nu,s)\, \Omega_2^-(\nu,s)\, (\nu+\mu)}\, e^{-2(s+\sigma_2)a/\nu} \qquad (0 \leqq \mu,\nu \leqq 1,\ \mu = \nu_{02}) \quad \text{(F2)}$$

When $s \in S_{2i}$, $\Omega_2(z',s)$ vanishes inside C' but not on the real interval $(0,1)$. As C' is collapsed on to $(0,1)$, a residue term appears so that equation (63) with $I_{2\pm} = 0$ takes the form

$$\overline{E}_{2\pm}(\mu,s) = \pm \frac{k}{c_2\sigma_2}\, \frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)}\, \frac{X_0(-\mu,s)\, X_0(-\nu_{02},s)}{\Omega_2'(\nu_{02},s)\, (\nu_{02}+\mu)}\, e^{-2(s+\sigma_2)a/\nu_{02}}\, \overline{E}_{2\pm}(\nu_{02},s)$$

$$\pm \int_0^1 K(\mu,\nu)\, \overline{E}_{2\pm}(\nu,s)\, d\nu \qquad (s \in S_{2i},\ 0 < \mu < 1) \quad \text{(F3)}$$

for $z = \mu$ where $K(\mu,\nu)$ is given by equation (F2). However, equation (63) with $I_{2\pm} = 0$ must also hold at $z = \nu_{02}$ and this requirement gives an additional constraint on solutions of equation (F3), namely,

$$\overline{E}_{2\pm}(\nu_{02},s) = \pm\frac{k}{c_2\sigma_2}\frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)}\frac{X_0^2(-\nu_{02},s)}{2\nu_{02}\Omega_2'(\nu_{02},s)}e^{-2(s+\sigma_2)a/\nu_{02}}\overline{E}_{2\pm}(\nu_{02},s)$$

$$\pm\int_0^1 K(\nu_{02},\nu)\overline{E}_{2\pm}(\nu,s)\,d\nu \qquad (s \in S_{2i}) \quad \text{(F4)}$$

When $s \in C_2$, the curve separating S_{2i} and S_{2e}, $\Omega_2^+(\nu,s)\,\Omega_2^-(\nu,s)$ vanishes for some ν on the interval (0,1); that is, ν_{02} is real and lies on (0,1). By setting $\nu_{02} = \eta$, one can put equation (63) with $I_{2\pm} = 0$ in the form

$$\overline{E}_{2\pm}(\mu,s) = \pm\frac{1}{2}\frac{k}{c_2\sigma_2}\frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)}\frac{X_0(-\mu,s)\,X_0(-\eta,s)}{\Omega_2'(\eta,s)\,(\eta+\mu)}e^{-2(s+\sigma_2)a/\eta}\overline{E}_{2\pm}(\eta,s)$$

$$\pm\frac{k}{2}\frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)}X_0(-\mu,s)\;\underset{\nu=\eta}{P}\int_0^1\frac{\overline{E}_{2\pm}(\nu,s)\,X_0(-\nu,s)}{\Omega_2^+(\nu,s)\,\Omega_2^-(\nu,s)}e^{-2(s+\sigma_2)a/\nu}\frac{\nu\,d\nu}{\nu+\mu}$$

$$(s \in C_2,\quad 0 \leqq \mu,\eta \leqq 1) \quad \text{(F5)}$$

Note that this equation would be obtained from either equation (F1) or equation (F3) for $s \to C_2$ from $s \in S_{2e}$ or $s \in S_{2i}$, respectively.

For arbitrary complex values of s, the kernel $K(\mu,\nu)$ which appears in equations (F1) and (F3) is not symmetric since

$$K(\mu,\nu) \neq [K(\nu,\mu)]^* \qquad (\mathrm{Im}(s) \neq 0) \quad \text{(F6)}$$

where the asterisk denotes complex conjugation. Note, however, that when $\mathrm{Im}(s) = 0$, the unknown functions $\overline{E}_{2\pm}(\mu,s)$ can be redefined so that a symmetric kernel is obtained. Solutions of equations (F1) and (F3) depend on the behavior of $K(\mu,\nu)$ generally through the quantity $B^2(s)$ given by

$$B^2(s) = \int_0^1\int_0^1 |K(\mu,\nu)|^2\,d\mu\,d\nu \quad \text{(F7)}$$

To study B^2, one introduces the nondimensional parameters ζ, σ_R, σ_D, and A given by equation (118) with $\zeta = \alpha + i\beta$. Note that α, β, σ_R, σ_D, and A are real whereas σ_R and A are nonnegative. In terms of these quantities, one has

$$\left|K(\mu,\nu)\right|^2 = \frac{\left\{\left[\alpha\left(\sigma_R - 1\right) - \sigma_D\right]^2 + \beta^2\left(\sigma_R - 1\right)^2\right\}\left[(\alpha - 1)^2 + \beta^2\right]}{4\left[\left(\alpha + \sigma_D - \sigma_R\right)^2 + \beta^2\right]} \frac{\left|X_0(-\mu,s)\right|^2 \left|X_0(-\nu,s)\right|^2 \nu^2}{\left|\frac{\Omega_2^+(\nu,s)}{c_2\sigma_2}\right|^2 \left|\frac{\Omega_2^-(\nu,s)}{c_2\sigma_2}\right|^2 (\nu + \mu)^2} e^{-4A\alpha/\nu} \tag{F8}$$

$$(0 \leqq \mu,\nu \leqq 1)$$

To make estimates of the function $\left|X_0(-\mu,s)\right|^2$, one uses an integral representation of the single-medium X-function given in reference 14, namely,

$$X_{0m}(-\mu,s) = \exp\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \log_e\left[\frac{\Omega_m(z',s)}{\Omega_m(\infty,s)}\right]\frac{dz'}{z' + \mu}\right\} \tag{F9}$$

Upon letting $z' = iy$ and using $\Omega_m(z',s) = \Omega_m(-z',s)$, one sees that equation (F9) becomes

$$X_{0m}(-\mu,s) = \exp\left\{\frac{\mu}{\pi}\int_0^{\infty} \log_e\left[\frac{\Omega_m(iy,s)}{\Omega_m(\infty,s)}\right]\frac{dy}{y^2 + \mu^2}\right\} \tag{F10}$$

which is real for s real. In terms of the quantities of equation (118), one finds

$$\left.\begin{aligned}
\left|X_{01}(-\mu,s)\right|^2 &= \exp\left\{\frac{2\mu}{\pi}\int_0^{\infty} \log_e\sqrt{\frac{\left[\alpha + \sigma_D - \sigma_R g(y)\right]^2 + \beta^2}{\left(\alpha + \sigma_D - \sigma_R\right)^2 + \beta^2}}\,\frac{dy}{y^2 + \mu^2}\right\} \\
\left|X_{02}(-\mu,s)\right|^2 &= \exp\left\{\frac{2\mu}{\pi}\int_0^{\infty} \log_e\sqrt{\frac{\left[\alpha - g(y)\right]^2 + \beta^2}{(\alpha - 1)^2 + \beta^2}}\,\frac{dy}{y^2 + \mu^2}\right\}
\end{aligned}\right\} \tag{F11}$$

where

$$g(y) = y \tan^{-1} \frac{1}{y} \tag{F12}$$

It follows from equation (F12) that $0 \leqq g(y) \leqq 1$ for $0 \leqq y \leqq \infty$ and that it is a monotonic increasing function of y on this interval. Furthermore, since

$$\frac{2\mu}{\pi}\int_0^\infty \frac{dy}{y^2 + \mu^2} = 1 \tag{F13}$$

the following bounds (perhaps rather loose) are obtained for $\left|X_{0m}(-\mu,s)\right|^2$:

$$\left.\begin{aligned} 1 &\leqq \left|X_{02}(-\mu,s)\right|^2 \leqq \sqrt{\frac{\alpha^2 + \beta^2}{(\alpha - 1)^2 + \beta^2}} \\ 1 &\leqq \left|X_{01}(-\mu,s)\right|^2 \leqq \sqrt{\frac{\left(\alpha + \sigma_D\right)^2 + \beta^2}{\left(\alpha + \sigma_D - \sigma_R\right)^2 + \beta^2}} \end{aligned}\right\} \tag{F14}$$

for $\alpha > \max\left(1, -\sigma_D + \sigma_R\right)$, and

$$\left.\begin{aligned} \sqrt{\frac{\beta^2}{(\alpha - 1)^2 + \beta^2}} &\leqq \left|X_{02}(-\mu,s)\right|^2 \leqq \max\left(1, \sqrt{\frac{\alpha^2 + \beta^2}{(\alpha - 1)^2 + \beta^2}}\right) \\ \sqrt{\frac{\beta^2}{\left(\alpha + \sigma_D - \sigma_R\right)^2 + \beta^2}} &\leqq \left|X_{01}(-\mu,s)\right|^2 \leqq \max\left(1, \sqrt{\frac{\left(\alpha + \sigma_D\right)^2 + \beta^2}{\left(\alpha + \sigma_D - \sigma_R\right)^2 + \beta^2}}\right) \end{aligned}\right\} \tag{F15}$$

for $\alpha < \max\left(1, -\sigma_D + \sigma_R\right)$ and $\beta \neq 0$. Note that in the ζ-plane, the points $\zeta = 1$ and $\zeta = -\sigma_D + \sigma_R$ are the right ends of the branch cuts of ν_{02} and ν_{01}, respectively, and those cuts lie on the real ζ-axis. The left ends are at $\zeta = 0$ and $\zeta = -\sigma_D$, respectively.

The functions $\left|\frac{\Omega_2^{\pm}(\nu,s)}{c_2\sigma_2}\right|^2$ are easily found to be

$$\left|\frac{\Omega_2^{\pm}(\nu,s)}{c_2\sigma_2}\right|^2 = \left(\alpha - \nu \tanh^{-1} \nu\right)^2 + \left(\beta \pm \frac{\pi\nu}{2}\right)^2 \tag{F16}$$

It was pointed out in the text that the curve C_2 (fig. 2) is given by

$$\alpha' = \frac{2\beta'}{\pi} \tanh^{-1} \frac{2\beta'}{\pi} \tag{F17}$$

The parametric form of this equation is

$$\left.\begin{aligned} |\beta'| &= \frac{\pi\nu}{2} && (0 \leqq \nu \leqq 1) \\ \alpha' &= \nu \tanh^{-1} \nu && (0 \leqq \nu \leqq 1) \end{aligned}\right\} \tag{F18}$$

It is seen from equation (F16) that $\left|\frac{\Omega_2^{\pm}(\nu,s)}{c_2\sigma_2}\right|^2$ are the squares of the distance (in the ζ-plane) from the point (α,β) to the points $(\alpha'(\nu), \mp\beta'(\nu))$, respectively, which lie on the curve C_2. Since these functions appear in the denominator of $|K(\mu,\nu)|^2$, the integral (F7) will not be bounded when α and β are related as in equation (F17). One defines $D_{min}(\alpha,\beta)$ as the minimum distance from the point (α,β) to the curve C_2 for $0 \leqq \nu \leqq 1$; that is,

$$D_{min}(\alpha,\beta) \equiv \min\left(\left|\frac{\Omega_2^{+}(\nu,s)}{c_2\sigma_2}\right|, \left|\frac{\Omega_2^{-}(\nu,s)}{c_2\sigma_2}\right|\right) \qquad (0 \leqq \nu \leqq 1) \tag{F19}$$

and $D_{min}(\alpha,\beta) \neq 0$ for $(\alpha,\beta) \notin C_2$. Therefore, from equations (F16) and (F19),

$$\frac{(c_2\sigma_2)^4}{\left|\Omega_2^{+}(\nu,s)\right|^2 \left|\Omega_2^{-}(\nu,s)\right|^2} \leqq \frac{1}{D_{min}^4(\alpha,\beta)} \qquad (0 \leqq \nu \leqq 1) \tag{F20}$$

Analytical bounds for this function are not as easy to determine. For $\beta = 0$, $\Omega_2^{+}(\nu,s) = \left[\Omega_2^{-}(\nu,s)\right]^*$ and

$$\left|\frac{c_2\sigma_2}{\Omega_2^{\pm}(\nu,s)}\right|^2_{\beta=0} = \frac{1}{\alpha^2}\, g\left(\frac{1}{\alpha}, \nu\right) \tag{F21}$$

where $g\left(\frac{1}{\alpha},\nu\right)$ has been investigated and tabulated by Case, De Hoffmann, and Placzek (ref. 25). They show that $g\left(\frac{1}{\alpha},\nu\right)\Big|_{max}$ occurs at $\nu = 0$ for $\alpha < \pi^2/8$ whereas for $\alpha > \pi^2/8$ it occurs for ν between 0 and 1. For α very large, they have $g_{max} \rightarrow 4\alpha^2/\pi^2$. The present geometric interpretation (ref. 26) is consistent with all these characteristics. The radius of curvature of the curve C_2 given by equation (F17) is $\pi^2/8$ at $(\alpha',\beta') = (0,0)$. For α' very large, $\beta' \rightarrow \pi/2$; thus, the minimum squared distance from $(\alpha,0)$ to (α',β') approaches $\pi^2/4$, in agreement with equation (F21) and $g_{max} \rightarrow 4\alpha^2/\pi^2$.

Note that the exponential factor $e^{-4\alpha A/\nu}$ in equation (F8) requires $\alpha > 0$ in order for $B^2(\alpha,\beta)$ to be bounded since both ν and A are nonnegative. On using estimates F(14), (F15), and (F20), one obtains from equation (F7) a bound for $B^2(\alpha,\beta)$ which is denoted as $B_{max}^2(\alpha,\beta)$:

$$B_{max}^2(\alpha,\beta) = \begin{cases} \dfrac{e^{-4\alpha A}}{4D_{min}^4(\alpha,\beta)} \dfrac{\left\{\left[\alpha(\sigma_R - 1) - \sigma_D\right]^2 + \beta^2(\sigma_R - 1)^2\right\}(\alpha^2 + \beta^2)}{(\alpha + \sigma_D - \sigma_R)^2 + \beta^2} & \left(\alpha > \max\left(1, -\sigma_D + \sigma_R\right)\right) \\ \dfrac{e^{-4\alpha A}}{4D_{min}^4(\alpha,\beta)} \left\{\max\left[\dfrac{\alpha^2 + \beta^2}{(\alpha - 1)^2 + \beta^2}, 1\right]\right\} \dfrac{\left\{\left[\alpha(\sigma_R - 1) - \sigma_D\right]^2 + \beta^2(\sigma_R - 1)^2\right\}\left[(\alpha - 1)^2 + \beta^2\right]}{\beta^2} & \left(\beta \neq 0;\ 0 < \alpha < \max\left(1, -\sigma_D + \sigma_R\right)\right) \end{cases} \tag{F22}$$

It can be seen that B_{max}^2 depends not only on α and β but also on the nondimensional material parameters σ_D and σ_R as well as the slab thickness parameter A. The estimate (F22) for B_{max}^2 is not bounded for the following regions in the s-plane:

$$\left.\begin{aligned} &\mathrm{Re}(s) < -\sigma_2(\alpha < 0) \\ &s \in C_2 \\ &s \in \text{Branch cuts of } \nu_{01}(s) \cup \nu_{02}(s) \end{aligned}\right\} \tag{F23}$$

These regions must be handled separately. Even for the general case, where s does not belong to any of the regions (F23), it appears difficult to say whether the eigenvalue problem has nontrivial solutions. One suspects that it has only trivial solutions for such regions since that is the result which has been found for certain special cases by others. Lehner and Wing (refs. 6 and 7) have shown this result for the bare slab, whereas Lehner (ref. 8) and Hintz (ref. 9) have obtained this result for the slab surrounded by pure absorbers. One shows that this result is also obtained for the special case $A \to \infty$, that is, a thick slab, as follows.

Since the slab-thickness parameter A appears only in the exponential term of equation (F22), it is seen that $B_{max}^2(\alpha,\beta)$ can be made as small as one likes as $A \to \infty$ if s does not belong to any of the regions given in regions (F23). For $|B_{max}(\alpha,\beta)| < 1$, the Neumann series solution of the inhomogeneous integral (eq. (F3)) converges to a unique solution. (See ref. 27, for example.) Fredholm's alternative theorem (ref. 27) then guarantees that the corresponding homogeneous equation, namely, equation (F1), has only the trivial solution. Thus, for $s \in S_{2e}$, the eigenvalue problem has only the trivial solution as $A \to \infty$. When $s \in S_{2i}$, the unique Neumann series solution of equation (F3) must satisfy the additional constraint (eq. (F4)). By using the condition $\Omega_2(\nu_{02},s) = 0$, one obtains

$$e^{-2(s+\sigma_2)a/\nu_{02}} = \left[\rho(s)\, e^{i\theta(s)}\right]^A \tag{F24}$$

where

$$\left.\begin{aligned} \rho^2(s) &= \frac{\left[\mathrm{Re}(\nu_{02}) - 1\right]^2 + \mathrm{Im}^2(\nu_{02})}{\left[\mathrm{Re}(\nu_{02}) + 1\right]^2 + \mathrm{Im}^2(\nu_{02})} \\ \theta(s) &= \tan^{-1}\left[\frac{\mathrm{Im}(\nu_{02})}{\mathrm{Re}(\nu_{02}) - 1}\right] - \tan^{-1}\left[\frac{\mathrm{Im}(\nu_{02})}{\mathrm{Re}(\nu_{02}) + 1}\right] \end{aligned}\right\} \tag{F25}$$

Now since $\mathrm{Re}(\nu_{02}) \geqq 0$, one has

$$\left[\rho(s)\right]^A \xrightarrow[A\to\infty]{} 0 \qquad \left(\mathrm{Re}(\nu_{02}) \neq 0\right) \tag{F26}$$

Therefore, the Neumann series solution is seen to converge to zero as $A \to \infty$ when $\mathrm{Re}(\nu_{02}) \neq 0$. Note that $\mathrm{Re}(\nu_{02}) = 0$ is the branch cut of $\nu_{02}(s)$ which is one of the regions given by region (F23) which must be considered separately. When $s \in C_2$, $\nu_{02} = \eta$, $0 \leqq \eta \leqq 1$ so that $\rho^2(s)$ of equation (F25) becomes

$$\rho^2(s) = \left(\frac{\eta - 1}{\eta + 1}\right)^2 \leqq 1 \tag{F27}$$

and $\rho = 1$ occurs only at $s = -\sigma_2$ (that is, $(\alpha,\beta) = (0,0)$). One uses equation (F27) in equation (F5) and, on taking the limit $A \to \infty$, finds that $\overline{E}_{2\pm}(\mu,s) \to 0$ for $s \in C_2$, $s \neq -\sigma_2$.

Summarizing the results then for $A \to \infty$ indicates that the eigenvalue problem has only the trivial solution for $\mathrm{Re}(s) > -\sigma_2$ unless s belongs to either the branch cut of $\nu_{01}(s)$ or $\nu_{02}(s)$. In order to determine what happens on these cuts, one must write equations (63) and (64) with $I_{m\pm} = 0$ in terms of the $X_m(-z,s)$ functions rather than in terms of the $X_0(-z,s)$ function. This will be done in appendix G. When A is not large, others (refs. 6 to 9) have shown for special cases of the present problem that if the eigenvalue problem has nontrivial discrete solutions, such conditions occur on the real s-axis. For the bare slab, it was shown in references 12 and 13 that these solutions lie on the branch cut of $\nu_0(s)$. In view of these results, it is assumed that the eigenvalue problem has nontrivial solutions for $\mathrm{Re}(s) > -\sigma_2$ only if s belongs to either the branch cut of $\nu_{01}(s)$ or $\nu_{02}(s)$.

APPENDIX G

SOLUTION OF THE ASSOCIATED EIGENVALUE PROBLEM FOR $s \in S_{2i}$

In this appendix, the solutions of equations (63) and (64) with $I_{m\pm} \equiv 0$ for s on the branch cuts of $\nu_{0m}(s)$ are examined. It is convenient to use coefficients related to the original expansion coefficients $\overline{A}_{m\pm}(\nu)$ and $\overline{a}_{m\pm}$ (the bar indicates that one is considering the associated eigenvalue problem). Recall that the $\overline{E}_{m\pm}$ are related to such coefficients by equations (50), (E4), and (E5). It was also noted in appendix F that the coefficients can be determined only to within an arbitrary factor independent of ν. By following references 12 and 13, one introduces coefficients $\overline{B}_{m\pm}$ as

$$\left.\begin{aligned} \overline{A}_{m\pm}(\nu) &= \overline{a}_{2\pm}\overline{B}_{m\pm}(\nu) \\ \overline{a}_{1\pm} &= \overline{a}_{2\pm}\overline{b}_{1\pm} \end{aligned}\right\} \tag{G1}$$

The estimate $B_{max}^2(\alpha,\beta)$ (eq. (F22)) was not bounded on the branch cuts of $\nu_{0m}(s)$. In that estimate the $X_0(-z,s)$ function was used so that the behavior for s inside, on, and outside the curve C_2 could be seen. To investigate what happens on the branch cuts of $\nu_{0m}(s)$, one should use the $X_m(z,s)$ functions (appendix A) which do not contain the branch cuts of ν_{0m}. Also, when $\nu_{01}(s)$ becomes pure imaginary (that is, on its branch cut), one cannot include the contribution (the pole at $z' = \nu_{01}$) in the integral over the contour C' of equation (64). Recall that the material properties c_m and σ_m determine where on the real s-axis the branch cuts of $\nu_{0m}(s)$ lie. The only restriction which has been made is that $c_1 < 1$ and this restriction alone does not specify the overlapping of the cuts. It does, however, guarantee that the branch cut of $\nu_{01}(s)$ lies entirely to the left of $s = 0$.

Consider $s \in S_{2i} \cap S_{1i}$ first. When the relationships (G1), (50), (E4), and (E5) are used in equations (F3) and (F4), one obtains after some algebra and use of the X-identities of appendix A, equations for $\overline{B}_{2\pm}(\mu)$ and the additional constraint, namely, equations (77) and (81). Recall that equations (F3) and (F4) were obtained from equations (63) with $I_{2\pm} \equiv 0$. Equations for $\overline{B}_{1\pm}(-\mu)$ and $\overline{b}_{1\pm}$ are obtained in a similar manner from equation (64) with $I_{1\pm} \equiv 0$ when the contour C' is collapsed on to the interval (0,1) of the branch cut of $\Omega_2(z',s)$. These equations are given as equations (78) and (79). The normal-mode expansion of the solution of the associated eigenvalue problem is given in terms of the $\overline{B}_{m\pm}$ coefficients by equation (76). Note that equation (81) is the exact eigenvalue condition since all material properties have been assumed to be known. It determines the values of s, $\{s_n\}$, for which the eigenvalue problem has nontrivial solutions. When s belongs to the branch cut of ν_{01}, equation (81) takes on different values above and below the ν_{01} cut. Therefore, it is concluded that the eigenvalue problem has only the trivial solution on the branch cut of ν_{01}. On that part of the branch cut of ν_{02} which is not also a part of the ν_{01} cut, equations (77) to (81) require that the limiting values of the coefficients above and below the ν_{02} cut be related as

$$\left.\begin{aligned} \left[\overline{B}_{m\pm}(\mu)\right]^{+} &= \pm\left[\overline{B}_{m\pm}(\mu)\right]^{-} \\ \left[\overline{b}_{1\pm}\right]^{+} &= \pm\left[\overline{b}_{1\pm}\right]^{-} \qquad \left(\mathrm{Re}\left(\nu_{02}\right) = \mathrm{Im}\left(\nu_{01}\right) = 0\right) \end{aligned}\right\} \tag{G2}$$

that is, where s is real and is given by $\max\left[-\sigma_2, -\sigma_1\left(1 - c_1\right)\right] < s < -\sigma_2\left(1 - c_2\right)$. It then follows from equations (76) and (G2) that the limiting values of $\overline{\Psi}_{\pm}(x,\mu,s)$ for the same region are given by equation (96). From the results of references 12 and 13 for the bare slab, it is expected that the eigenvalue problem has nontrivial solutions only at isolated points $\left\{s_n\right\}$ which lie on the branch cut of ν_{02} but not on the branch cut of ν_{01}.

In the limit $c_2\sigma_2 a \rightarrow \infty$ which was discussed in appendix F, one sees that equation (77) gives $\overline{B}_{2\pm}(\mu) \rightarrow 0$ whereas equation (81), the eigenvalue condition, becomes

$$0 = \frac{X_2\left(-\nu_{02},s\right)}{X_1\left(-\nu_{02},s\right)} \frac{e^{-\left(s+\sigma_2\right)a/\nu_{02}}}{\nu_{01} + \nu_{02}} \mp \frac{X_2\left(\nu_{02},s\right)}{X_1\left(\nu_{02},s\right)} \frac{e^{\left(s+\sigma_2\right)a/\nu_{02}}}{\nu_{01} - \nu_{02}}$$

$$\left(c_2\sigma_2 a \rightarrow \infty \quad \text{and} \quad \mathrm{Re}\left(\nu_{02}\right) = \mathrm{Im}\left(\nu_{01}\right) = 0\right) \tag{G3}$$

Equation (G3) is the "thick-slab" eigenvalue condition and for the region of the s-plane where it is valid, it can be seen that one obtains an even eigenvalue s_n if

$$\mathrm{Im}\left[\frac{X_2\left(\nu_{02},s\right)}{X_1\left(\nu_{02},s\right)} \frac{e^{\left(s_n+\sigma_2\right)a/\nu_{02}}}{\nu_{01} - \nu_{02}}\right] = 0 \tag{G4a}$$

and an odd eigenvalue s_n if

$$\mathrm{Re}\left[\frac{X_2\left(\nu_{02},s\right)}{X_1\left(\nu_{02},s\right)} \frac{e^{\left(s_n+\sigma_2\right)a/\nu_{02}}}{\nu_{01} - \nu_{02}}\right] = 0 \qquad \left(c_2\sigma_2 a \rightarrow \infty\right) \tag{G4b}$$

Note that equation (G3) has the same form as the zero-order approximation of the critical condition given in reference 2, except that here one has both even and odd parity solutions. Numerical solutions of the eigenvalue conditions are discussed in appendix J.

In the region of the s-plane, $s \in S_{1e} \cap S_{2i}$, one is specifically interested in the solution on the branch cut of $\nu_{02}(s)$ which lies to the left of $s = -\sigma_1$, that is, for s real and $-\sigma_2 < s \leqq -\sigma_1$. For such values of s, the solution (76) outside the slab is not bounded as $x \to \infty$, since

$$\psi_{1\nu}(x,\mu,s) = \varphi_{1\nu}(\mu,s)\ e^{-(s+\sigma_1)x/\nu} \qquad (0 \leqq \nu \leqq 1) \qquad \text{(G5)}$$

In addition, the restriction $\mathrm{Re}(s) > -\sigma_{min}$ on both inhomogeneous terms $I_{m\pm}$ (see eqs. (65) and (66)) also indicates that one cannot deform the inversion contour to the left of $\mathrm{Re}(s) = -\sigma_1$ in general. However, when one is looking for the solution inside the slab, $|x| \leqq a$, perhaps the inversion contour can be deformed to the left of $\mathrm{Re}(s) = -\sigma_1$ for special values of material properties and/or initial data. For $s \in S_{2i} \cap S_{1e}$, expansion coefficients for the solution inside the slab are obtained as equations (82) and (83). Note that equation (83) is exactly equation (81) with $X_{01}(z,s)$ replacing $(\nu_{01} - z)X_1(z,s)$. Recall from equation (A10) that these are the X-functions which are continuous as $s \to C_1$. Under the same replacement of $X_{01}(z,s)$ with $(\nu_{01} - z)X_1(z,s)$, equation (82) reduces to the equation from which equation (77) was obtained. Equation (83), which corresponds to the eigenvalue condition equation (81), determines the pseudo-eigenvalues, that is, the values of s, $-\sigma_2 < s < -\sigma_1$, where $\overline{\psi}_{2\pm}(x,\mu,s)$ has nontrivial solutions.

APPENDIX H

FORM OF $\psi_{\pm}(x,\mu,s)$ ON THE BRANCH CUTS OF $\nu_{0m}(s)$

In this appendix, the transformed solution $\psi_{\pm}(x,\mu,s)$ is put in a form where one can see how it behaves on the branch cuts of $\nu_{0m}(s)$. One expects that $\psi_{\pm}$ contain the branch cut of $\nu_{01}(s)$ since only one of the two discrete modes appears for $|x| > a$. Such branch cuts appeared in the half-space problems solved in references 14 to 16. One also expects that the branch cut of $\nu_{02}(s)$ does not appear in $\psi_{\pm}$ but instead one should find poles at $s = s_n$, the places where the associated eigenvalue problem has nontrivial solutions. This condition was found for the bare-slab problem analyzed in references 6, 7, 12, and 13.

It is not obvious from the equations which determine the expansion coefficients implicitly how one should group the terms to show what is expected. Consider first $\psi_{2\pm}(x,\mu,s)$. From equations (32) and (44),

$$\begin{aligned}\psi_{2\pm}(x,\mu,s) = {} & a_{2\pm}\left[\psi_{\nu_{02}}(x,\mu,s) \pm \psi_{-\nu_{02}}(x,\mu,s)\right] \\ & + \int_0^1 A_{2\pm}(\nu)\left[\psi_{2\nu}(x,\mu,s) \pm \psi_{2(-\nu)}(x,\mu,s)\right] d\nu \\ & + F_{2\pm}\left(x,\nu_{02},s\right)\psi_{\nu_{02}}(x,\mu,s) \pm F_{2\pm}\left(-x,\nu_{02},s\right)\psi_{-\nu_{02}}(x,\mu,s) \\ & + \int_0^1 \left[F_{2\pm}(x,\nu,s)\,\psi_{2\nu}(x,\mu,s) \pm F_{2\pm}(-x,\nu,s)\,\psi_{2(-\nu)}(x,\mu,s)\right] d\nu \end{aligned} \tag{H1}$$

Note that this equation can be obtained from equations (70) and (72). It is readily shown from the definition of the $F_{m\pm}$ functions (eqs. (46)) and the properties of the $C_{m\pm}$ (eqs. (B11)) that

$$\left.\begin{aligned} F_{2\pm}\left(x,\nu_{02},s\right) &= F_{2\pm}\left(a,\nu_{02},s\right) \pm F_{2\pm}\left(-x,-\nu_{02},s\right) \\ F_{2\pm}\left(x,-\nu_{02},s\right) &= F_{2\pm}\left(a,-\nu_{02},s\right) \pm F_{2\pm}\left(-x,\nu_{02},s\right) \end{aligned}\right\} \tag{H2}$$

It follows from equations (H2) that the two coefficients in equation (H1) can be written as

$$\left.\begin{aligned} F_{2\pm}(x,\nu_{02},s) &= \frac{1}{2}\left[F_{2\pm}(x,\nu_{02},s) \pm F_{2\pm}(-x,-\nu_{02},s)\right] + \frac{1}{2}F_{2\pm}(a,\nu_{02},s) \\ \pm F_{2\pm}(-x,\nu_{02},s) &= \frac{1}{2}\left[F_{2\pm}(x,-\nu_{02},s) \pm F_{2\pm}(-x,\nu_{02},s)\right] \pm \frac{1}{2}F_{2\pm}(a,\nu_{02},s) \end{aligned}\right\} \tag{H3}$$

where equation (B12) has been used to replace $F_{2\pm}(a,-\nu_{02},s)$. Equation (H1) then becomes

$$\begin{aligned} \psi_{2\pm}(x,\mu,s) &= \left[a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\right]\left[\psi_{\nu_{02}}(x,\mu,s) \pm \psi_{-\nu_{02}}(x,\mu,s)\right] \\ &\quad + \int_0^1 A_{2\pm}(\nu)\left[\psi_{2\nu}(x,\mu,s) \pm \psi_{2(-\nu)}(x,\mu,s)\right] d\nu \\ &\quad + \frac{1}{2}\left[F_{2\pm}(x,\nu_{02},s) \pm F_{2\pm}(-x,-\nu_{02},s)\right]\psi_{\nu_{02}}(x,\mu,s) \\ &\quad + \frac{1}{2}\left[F_{2\pm}(x,-\nu_{02},s) \pm F_{2\pm}(-x,\nu_{02},s)\right]\psi_{-\nu_{02}}(x,\mu,s) \\ &\quad + \int_0^1 \left[F_{2\pm}(x,\nu,s)\,\psi_{2\nu}(x,\mu,s) \pm F_{2\pm}(-x,\nu,s)\,\psi_{2(-\nu)}(x,\mu,s)\right] d\nu \end{aligned} \tag{H4}$$

When $s \in$ Branch cut of $\nu_{02}(s)$, then $\nu_{02} = i|\nu_{02}|$ for $\text{Im}(s) = 0^-$ and $\nu_{02} = -i|\nu_{02}|$ for $\text{Im}(s) = 0^+$. Therefore, on going from below to above the branch cut of ν_{02}, it is evident that the third and fourth terms in the right-hand side of equation (H4) simply interchange whereas those containing $F_{2\pm}(x,\nu,s)$ and $F_{2\pm}(-x,\nu,s)$ are unaffected since these functions do not depend on ν_{02}. The coefficient of $\left[a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\right]$ however changes sign for odd-parity solutions and the behavior of $A_{2\pm}(\nu)$ is not clear yet. By comparing equation (H4) with equation (76) for $|x| < a$, one suspects that $\left[a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\right]$ is the coefficient which excites the associated eigensolution $\overline{\psi}_{\pm}(x,\mu,s)$. This is the information needed to group terms in the implicit equations for the expansion coefficients.

Now examine the equations which determine the expansion coefficients. From equation (52) upon using the X-identities, the definition of the h_m functions (eqs. (80)), and the relationship between the $E_{m\pm}$ and the original expansion coefficients, the following equation is obtained:

$$0 = \left[a_{2\pm} + \frac{1}{2} F_{2\pm}\left(a, \nu_{02}, s\right)\right]\left[\frac{h_2\left(\nu_{02}\right)}{\nu_{01} + \nu_{02}} \pm \frac{h_2\left(-\nu_{02}\right)}{\nu_{01} - \nu_{02}}\right]$$

$$+ \frac{1}{2} F_{2\pm}\left(a, \nu_{02}, s\right)\left[\frac{h_2\left(\nu_{02}\right)}{\nu_{01} + \nu_{02}} \mp \frac{h_2\left(-\nu_{02}\right)}{\nu_{01} - \nu_{02}}\right] + \int_0^1 \left[A_{2\pm}(\mu) + F_{2\pm}(a, \mu, s)\right] h_2(\mu) \frac{d\mu}{\mu + \nu_{01}}$$

$$\pm \int_0^1 F_{1\pm}(-a, \mu, s)\, h_1(\mu) \frac{\mu + \nu_{01}}{\mu^2 - \nu_{02}^2}\, d\mu \pm F_{1\pm}\left(-a, \nu_{01}, s\right) h_1\left(\nu_{01}\right) \frac{2\nu_{01}}{\nu_{01}^2 - \nu_{02}^2} \tag{H5}$$

By following the same procedure with equation (51), one can obtain after making use of equation (H5):

$$A_{2\pm}(\nu) = \frac{c_1\sigma_1}{c_2\sigma_2} F_{1\pm}(-a, \nu, s)\, e^{\left(\sigma_1 - \sigma_2\right)a/\nu}$$

$$\pm \frac{k}{2} \frac{\Omega_2(\infty, s)}{\Omega_1(\infty, s)} \frac{\nu_{02}^2 - \nu^2}{\nu_{01}^2 - \nu^2} \frac{h_2(\nu)}{N_2(\nu)} \Bigg\{\left[a_{2\pm} + \frac{1}{2} F_{2\pm}\left(a, \nu_{02}, s\right)\right]\left[\frac{h_2\left(\nu_{02}\right)}{\nu + \nu_{02}} \pm \frac{h_2\left(-\nu_{02}\right)}{\nu - \nu_{02}}\right]$$

$$+ \frac{1}{2} F_{2\pm}\left(a, \nu_{02}, s\right)\left[\frac{h_2\left(\nu_{02}\right)}{\nu + \nu_{02}} \mp \frac{h_2\left(-\nu_{02}\right)}{\nu - \nu_{02}}\right] + \int_0^1 \left[A_{2\pm}(\mu) + F_{2\pm}(a, \mu, s)\right] h_2(\mu) \frac{d\mu}{\mu + \nu}$$

$$\mp \int_0^1 F_{1\pm}(-a, \mu, s)\, h_1(\mu) \frac{\nu_{01}^2 - \mu^2}{\nu_{02}^2 - \mu^2} \frac{2\varphi_{2\nu}(\mu, s)}{c_2\sigma_2\nu}\, d\nu\Bigg\} \tag{H6}$$

From equation (54), there is obtained:

$$\mp h_1\left(-\nu_{01}\right) a_{1\pm} = \left[a_{2\pm} + \frac{1}{2} F_{2\pm}\left(a, \nu_{02}, s\right)\right]\left[h_2\left(\nu_{02}\right) \pm h_2\left(-\nu_{02}\right)\right]$$

$$+ \frac{1}{2} F_{2\pm}\left(a, \nu_{02}, s\right)\left[h_2\left(\nu_{02}\right) \mp h_2\left(-\nu_{02}\right)\right]$$

$$+ \left(\nu_{02}^2 - \nu_{01}^2\right) \int_0^1 \left[A_{2\pm}(\mu) + F_{2\pm}(a, \mu, s)\right] h_2(\mu) \frac{d\mu}{\mu^2 - \nu_{01}^2}$$

$$\mp \left(\nu_{02}^2 - \nu_{01}^2\right) \int_0^1 F_{1\pm}(-a, \mu, s)\, h_1(\mu) \frac{d\mu}{\mu^2 - \nu_{02}^2}$$

$$+ F_{1\pm}\left(-a, \nu_{01}, s\right)\left[h_1\left(-\nu_{01}\right) \pm h_1\left(\nu_{01}\right)\right] \tag{H7}$$

Finally, from equation (53),

$$
\begin{aligned}
A_{1\pm}(-\nu) &\mp \frac{c_2\sigma_2}{c_1\sigma_1} A_{2\pm}(\nu)\, e^{(\sigma_1-\sigma_2)a/\nu} \\
&= \pm \left[\frac{k}{2}\,\frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)}\,\frac{h_1(\nu)}{N_1(\nu)}\right] \left\{\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]\left[\frac{h_2(\nu_{02})}{\nu - \nu_{02}} \pm \frac{h_2(-\nu_{02})}{\nu + \nu_{02}}\right]\right. \\
&\quad + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\left[\frac{h_2(\nu_{02})}{\nu - \nu_{02}} \mp \frac{h_2(-\nu_{02})}{\nu + \nu_{02}}\right] \\
&\quad + \int_0^1 \left[A_{2\pm}(\mu) + F_{2\pm}(a,\mu,s)\right] h_2(\mu)\, \frac{2\varphi_{1\nu}(\mu,s)}{c_1\sigma_1\nu}\, d\mu \\
&\quad \left. \mp \int_0^1 F_{1\pm}(-a,\mu,s)\, h_1(\mu)\, \frac{\nu_{01}^2 - \mu^2}{\nu_{02}^2 - \mu^2}\, \frac{d\mu}{\mu + \nu}\right\} \\
&\quad \mp \left[F_{1\pm}(-a,\nu,s) - \frac{c_2\sigma_2}{c_1\sigma_1} F_{2\pm}(a,\nu,s)\, e^{(\sigma_1-\sigma_2)a/\nu}\right]
\end{aligned}
\tag{H8}
$$

By following a procedure similar to that of reference 13, the expansion coefficients $A_{m\pm}(\mu)$ and $a_{1\pm}$ can be written in the form:

$$
\left.
\begin{aligned}
A_{m\pm}(\mu) &= \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right] A'_{m\pm}(\mu) + B_{m\pm}(\mu) \\
a_{1\pm} &= \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right] a'_{1\pm} + b_{1\pm}(\mu)
\end{aligned}
\right\}
\tag{H9}
$$

When equations (H9) are used in equations (H6) to (H8), it follows that

$$
\left.
\begin{aligned}
A'_{m\pm}(\mu) &= \overline{B}_{m\pm}(\mu) \\
a'_{1\pm} &= \overline{b}_{1\pm}
\end{aligned}
\right\}
\tag{H10}
$$

where $\overline{B}_{m\pm}(\mu)$ and $\overline{b}_{1\pm}$ are the expansion coefficients of the associated eigenvalue problem given by equations (77) to (79). The coefficients $B_{m\pm}(\mu)$ and $b_{1\pm}$ are found to be given by equations (85) to (87). The coefficient $a_{2\pm}$ is obtained from equation (H5) as

$$\left[a_{2\pm} + \frac{1}{2} F_{2\pm}\left(a,\nu_{02},s\right)\right]\left[\frac{h_2\left(\nu_{02}\right)}{\nu_{01} + \nu_{02}} \pm \frac{h_2\left(-\nu_{02}\right)}{\nu_{01} - \nu_{02}} + \int_0^1 \overline{B}_{2\pm}(\mu)\ h_2(\mu)\ \frac{d\mu}{\mu + \nu_{01}}\right]$$

$$= -\frac{1}{2} F_{2\pm}\left(a,\nu_{02},s\right)\left[\frac{h_2\left(\nu_{02}\right)}{\nu_{01} + \nu_{02}} \mp \frac{h_2\left(-\nu_{02}\right)}{\nu_{01} - \nu_{02}}\right] - \int_0^1 \left[B_{2\pm}(\mu) + F_{2\pm}(a,\mu,s)\right] h_2(\mu)\ \frac{d\mu}{\mu + \nu_{01}}$$

$$\mp \int_0^1 F_{1\pm}(-a,\mu,s)\ h_1(\mu)\ \frac{\mu + \nu_{01}}{\mu^2 - \nu_{02}^2}\ d\mu \mp F_{1\pm}\left(-a,\nu_{01},s\right)\ h_1\left(\nu_{01}\right)\ \frac{2\nu_{01}}{\nu_{01}^2 - \nu_{02}^2} \qquad \text{(H11)}$$

It can be seen from equation (81) that the coefficient of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}\left(a,\nu_{02},s\right)\right]$ in equation (H11) is the eigenvalue condition, and it will be zero at the places where the associated eigenvalue problem has nontrivial solutions. Equation (H11) is the same as equation (88). The solutions $\psi_{m\pm}(x,\mu,s)$ can now be written as equations (91) and (92).

APPENDIX I

BEHAVIOR OF $\psi_{\pm}(x,\mu,s)$ ON INVERSION CONTOURS

In this appendix, several points concerning the behavior of $\psi_{\pm}(x,\mu,s)$ on the integration contour of the inverse Laplace transformation and some parts of related deformed contours are discussed. First, one looks at the behavior of $\psi_{\pm}(x,\mu,s)$ as $|s| \to \infty$ with $Re(s) = \gamma$, a large finite positive number. It will be seen that $\psi_{\pm}(x,\mu,s)$ is not necessarily $0\left(\frac{1}{s}\right)$. Such parts of $\psi_{\pm}(x,\mu,s)$ are inverted separately and the resulting solutions are shown to satisfy the uncollided transport equation. Then one considers how $\psi_{\pm}(x,\mu,s)$ minus the uncollided term $\psi_{u\pm}(x,\mu,s)$ can be deformed around the poles and the branch cut of $\psi_{\pm}$.

It is of interest to examine the behavior of $\psi_{\pm}$ on the contour $Re(s) = \gamma$ as $|s| \to \infty$, where γ is finite (see fig. 4). For such cases, $s \in S_{1e} \cap S_{2e}$ and the solutions $\psi_{m\pm}$ can be seen from equations (32), (39), (44), and (45) to be

$$\psi_{2\pm}(x,\mu,s) = \int_0^1 \left[A_{2\pm}(\nu) + F_{2\pm}(x,\nu,s)\right] e^{-(s+\sigma_2)x/\nu} \varphi_{2\nu}(\mu,s)\, d\nu$$

$$\pm \int_0^1 \left[A_{2\pm}(\nu) + F_{2\pm}(-x,\nu,s)\right] e^{(s+\sigma_2)x/\nu} \varphi_{2\nu}(-\mu,s)\, d\nu \tag{I1}$$

and, for $x > a$,

$$\psi_{1\pm}(x,\mu,s) = \pm\int_0^1 \left[A_{1\pm}(-\nu) - \tilde{F}_{\pm}(-a,\nu,s) + F_{1\pm}(-x,-\nu,s)\right] e^{-(s+\sigma_1)x/\nu} \varphi_{1\nu}(\mu,s)\, d\nu$$

$$\pm \int_0^1 F_{1\pm}(-x,\nu,s)\, e^{(s+\sigma_1)x/\nu} \varphi_{1\nu}(-\mu,s)\, d\nu \tag{I2}$$

with an equation similar to equation (I2) for $x < -a$. One sees then that the coefficients $A_{1\pm}(-\nu)$, $A_{2\pm}(\nu)$, and the $F_{m\pm}$ functions are needed. The expansion coefficients are given implicitly in terms of the $F_{m\pm}$ as

$$A_{2\pm}(\mu) = \frac{\mu I_{2\pm}(\mu,s)}{N_2(\mu)} e^{-(s+\sigma_2)a/\mu}$$

$$\pm \frac{k}{c_2\sigma_2} \frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)} \frac{\mu X_0(-\mu,s)\, e^{-(s+\sigma_2)a/\mu}}{N_2(\mu)} \int_0^1 A_{2\pm}(\nu)\, X_0(-\nu,s)\, e^{-(s+\sigma_2)a/\nu} \varphi_{2\nu}(-\mu,s)\, d\nu \tag{I3}$$

and

$$A_{1\pm}(-\mu) = \frac{\mu I_{1\pm}(\mu,s)\, e^{(s+\sigma_1)a/\mu}}{N_1(\mu)} \pm \frac{c_1\sigma_1}{c_2\sigma_2} \frac{N_2(\mu)}{N_1(\mu)} e^{-(\sigma_2-\sigma_1)a/\mu} A_{2\pm}(\mu)$$

$$\mp \frac{\mu k\, e^{(s+\sigma_1)a/\mu}}{X_0(-\mu,s)\, N_1(\mu)} \int_0^1 \frac{A_{2\pm}(\nu)\, X\ (-\nu,s)\, e^{-(s+\sigma_2)a/\nu}}{c_2\sigma_2} \varphi_{2\nu}(\mu,s)\, d\nu \tag{I4}$$

where the $I_{m\pm}(\mu,s)$ are given by

$$I_{2\pm}(\mu,s) = \frac{c_2\sigma_2}{c_1\sigma_1\mu} N_1(\mu)\, F_{1\pm}(-a,\mu,s)\, e^{(s+\sigma_1)a/\mu}$$

$$\pm \frac{k}{c_2\sigma_2} \frac{\Omega_2(\infty,s)}{\Omega_1(\infty,s)} X_0(-\mu,s) \int_0^1 F_{2\pm}(a,\nu,s)\, e^{-(s+\sigma_2)a/\nu} X_0(-\nu,s)\, \varphi_{2\nu}(-\mu,s)\, d\nu$$

$$+ \frac{k}{c_1\sigma_1} X_0(-\mu,s) \int_0^1 \frac{F_{1\pm}(-a,\nu,s)\, e^{(s+\sigma_1)a/\nu}}{X_0(-\nu,s)} \varphi_{1\nu}(\mu,s)\, d\nu \tag{I5}$$

and

$$I_{1\pm}(\mu,s) = \mp \frac{N_1(\mu)}{\mu} F_{1\pm}(-a,\mu,s)\, e^{-(s+\sigma_1)a/\mu} \pm \frac{c_1\sigma_1}{c_2\sigma_2\mu} N_2(\mu)\, F_{2\pm}(a,\mu,s)\, e^{-(s+\sigma_2)a/\mu}$$

$$\mp \frac{k}{c_2\sigma_2} \frac{1}{X_0(-\mu,s)} \int_0^1 F_{2\pm}(a,\nu,s)\, e^{-(s+\sigma_2)a/\nu} X_0(-\nu,s)\, \varphi_{2\nu}(\mu,s)\, d\nu$$

$$- \frac{k}{c_1\sigma_1} \frac{\Omega_1(\infty,s)}{\Omega_2(\infty,s)} \frac{1}{X_0(-\mu,s)} \int_0^1 \frac{F_{1\pm}(-a,\nu,s)\, e^{(s+\sigma_1)a/\nu}}{X_0(-\nu,s)} \varphi_{1\nu}(-\mu,s)\, d\nu \tag{I6}$$

The behavior of various functions which appear in equations (I1) to (I6) as $|s| \to \infty$, $\mathrm{Re}(s) = \gamma$ is

$$\Omega_m(z,s), \quad \lambda_m(\nu,s), \quad k \to O(s)$$

$$X_0(z,s) \to O(1)$$

and for $0 \leqq \mu,\nu \leqq 1$,

$$\left.\begin{aligned} \varphi_{m\nu}(\mu,s) &\to O(s) \\ \varphi_{m\nu}(-\mu,s) &\to O(1) \end{aligned}\right\} \tag{I7}$$

The $F_{m\pm}$ functions appear with an exponential factor and the combined behavior in the same limit is

$$F_{m\pm}(x,\nu,s)\, e^{-(s+\sigma_m)x/\nu} \to \frac{1}{\nu(s+\sigma_m)} \int_{l_m}^{x} e^{-(s+\sigma_m)(x-x_0)/\nu} \left[f_{m\pm}(x_0,\nu) + O\left(\frac{1}{s}\right)\right] dx_0$$

$$\to O\left(\frac{1}{s}\right) \tag{I8}$$

On using equations (I7) and (I8) in equations (I5) and (I6), one finds that

$$\frac{\mu I_{2\pm}(\mu,s)}{N_2(\mu)} \to F_{1\pm}(-a,\mu,s)\, e^{(s+\sigma_1)a/\mu}$$

$$\to O\left(\frac{1}{s}\right) \tag{I9}$$

and

$$\frac{\mu I_{1\pm}(\mu,s)}{N_1(\mu)} \to \mp F_{1\pm}(-a,\mu,s)\, e^{-(s+\sigma_1)a/\mu} \pm F_{2\pm}(a,\mu,s)\, e^{-(s+\sigma_2)a/\mu}$$

$$\to O\left(\frac{1}{s}\right) \tag{I10}$$

The coefficient $A_{2\pm}(\mu)$ is obtained from the integral equation (I3). Since the kernel of this equation is also $O\left(\frac{1}{s}\right)$, the first term of the Neumann series solution will give the behavior of $A_{2\pm}(\mu)$ as $|s| \to \infty$. It follows then from equations (I3) and (I9) that

$$A_{2\pm}(\mu) - F_{1\pm}(-a,\mu,s)\, e^{-(\sigma_2-\sigma_1)a/\mu} \to O\left(\frac{1}{s}\right) \tag{I11}$$

Using equations (I10) and (I11) in equation (I4) yields

$$\left[A_{1\pm}(-\mu) \pm F_{1\pm}(-a,\mu,s)\right] - \pm F_{2\pm}(a,\mu,s)\, e^{-(\sigma_2-\sigma_1)a/\mu} \pm F_{1\pm}(-a,\mu,s)\, e^{-2(\sigma_2-\sigma_1)a/\mu} \to O\left(\frac{1}{s}\right) \tag{I12}$$

Equations (I11) to (I12) are used in equations (I1) and (I2) to get

$$\psi_{2\pm}(x,\mu,s) \to \begin{cases} (s+\sigma_2)\, e^{-(s+\sigma_2)x/\mu}\left[F_{2\pm}(x,\mu,s) + F_{1\pm}(-a,\mu,s)\, e^{-(\sigma_2-\sigma_1)a/\mu}\right] & (\mu > 0) \\ \pm(s+\sigma_2)\, e^{-(s+\sigma_2)x/\mu}\left[F_{2\pm}(-x,-\mu,s) + F_{1\pm}(-a,-\mu,s)\, e^{(\sigma_2-\sigma_1)a/\mu}\right] & (\mu < 0) \end{cases} \tag{I13}$$

and

$$\psi_{1\pm}(x,\mu,s) \to \begin{cases} \pm(s+\sigma_1)\, e^{(-s+\sigma_1)x/\mu}\Big[F_{1\pm}(-x,-\mu,s) - F_{1\pm}(-a,-\mu,s) \\ \quad \pm F_{2\pm}(a,\mu,s)\, e^{-(\sigma_2-\sigma_1)a/\mu} \pm F_{1\pm}(-a,\mu,s)\, e^{-2(\sigma_2-\sigma_1)a/\mu}\Big] & (\mu > 0) \\ \pm(s+\sigma_1)\, e^{-(s+\sigma_1)x/\mu}\, F_{1\pm}(-x,-\mu,s) & (\mu < 0) \end{cases} \tag{I14}$$

when $x > a$. For $x < -a$, $\psi_{1\pm}$ has a similar form. Upon using equation (I8) for the $F_{m\pm}$ functions, one finds that equations (I13) and (I14) can be written as equations (98) to (101) where the symmetry properties of $f_{m\pm}(x,\mu)$ have been used. It can be seen from equations (98) to (101) that $\psi_{m\pm}(x,\mu,s)$ is not necessarily $O\left(\frac{1}{s}\right)$. In fact, if $f_{m\pm}(x_0,\mu)$ contains $\delta(x - x_0)$, then $\psi_{\pm}(x,\mu,s)$ is $O(1)$ as $|s| \to \infty$, $\mathrm{Re}(s) = \gamma$. The parts of $\psi_{\pm}(x,\mu,s)$ which are not $O\left(\frac{1}{s}\right)$ can be inverted by inspection after a change of variables is made.

For $x > a$, $\mu > 0$, and <u>all s</u>, $\psi_{u\pm}(x,\mu,s)$ is defined as

$$\begin{aligned}\psi_{u\pm}(x,\mu,s) \equiv \frac{1}{\mu}\int_a^x e^{-(s+\sigma_1)(x-x_0)/\mu} f_{1\pm}(x_0,\mu)/dx_0 \\ + \frac{e^{-(\sigma_2-\sigma_1)(a-x)/\mu}}{\mu}\int_{-a}^{a} e^{-(s+\sigma_2)(x-x_0)/\mu} f_{2\pm}(x_0,\mu)\, dx_0 \\ + \frac{e^{-(\sigma_2-\sigma_1)2a/\mu}}{\mu}\int_{-\infty}^{-a} e^{-(s+\sigma_1)(x-x_0)/\mu} f_{1\pm}(x_0,\mu)\, dx_0\end{aligned} \tag{I15}$$

which gives the parts of equation (100) which are not $O\left(\frac{1}{s}\right)$ for $|s| \to \infty$, $\mathrm{Re}(s) = \gamma$. Now the change of variables

$$x - x_0 = \mu t \tag{I16}$$

where $t \geqq 0$ since $x \geqq x_0$ and $\mu > 0$ is made. Equation (I15) then becomes

$$\begin{aligned}\psi_{u\pm}(x,\mu,s) = \int_0^{(x-a)/\mu} e^{-(s+\sigma_1)t} f_{1\pm}(x - \mu t,\ \mu)\, dt \\ + \int_{(x-a)/\mu}^{(x+a)/\mu} e^{-(s+\sigma_2)t} e^{-(\sigma_2-\sigma_1)(a-x)/\mu} f_{2\pm}(x - \mu t,\ \mu)\, dt \\ + \int_{(x+a)/\mu}^{\infty} e^{-(s+\sigma_1)t} e^{-(\sigma_2-\sigma_1)2a/\mu} f_{1\pm}(x - \mu t,\ \mu)\, dt\end{aligned} \tag{I17}$$

which is easily seen to be equation (103) with $\Psi_{u\pm}(x,\mu,t)$ given by equation (106). For $\mu < 0$,

$$x_0 - x = |\mu| t \tag{I18}$$

is used. It is seen then that the results given as equations (103) to (107) follow.

Another point to be discussed in this appendix is the contribution to equation (114) from the contour C_ρ (see fig. 4) around the right-hand end of the branch cut of $\nu_{01}(s)$ as the radius ρ goes to zero. This branch point is located at $s = -\sigma_1(1 - c_1)$; thus,

$$s + \sigma_1(1 - c_1) \equiv \rho e^{i\varphi} \tag{I19}$$

Here $\nu_{01}(s) \rightarrow \infty$ as $\rho \rightarrow 0$ as

$$\nu_{01}^2 \xrightarrow[\rho \to 0]{} \frac{c_1\sigma_1}{3} \frac{1}{\rho e^{i\varphi}} \qquad (-\pi < \varphi < \pi) \tag{I20}$$

The branch cut has already been picked so that $\nu_{01}(s)$ is real when s is real and greater than $-\sigma_1(1 - c_1)$. The integral

$$\frac{1}{2\pi i} \int_{C_\rho} \psi_\pm(x,\mu,s)\ e^{st}\ ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho \left[\psi_\pm(x,\mu,s)\ e^{st}\right] e^{i\varphi}\ d\varphi \tag{I21}$$

with s given by equation (I19), is zero in the limit $\rho \rightarrow 0$ if

$$\lim_{\rho \to 0} \left[\rho\psi_\pm(x,\mu,s)\right] = 0 \tag{I22}$$

independent of φ. As pointed out in the text, the point $s = -\sigma_1(1 - c_1)$ may happen to satisfy the eigenvalue condition (eq. (81)). One assumes for the moment that it does not and shows later what changes are required if it does. The function $\Omega_1(\infty,s) \rightarrow 0$ as $\rho \rightarrow 0$ as

$$\Omega_1(\infty,s) \xrightarrow[\rho \to 0]{} \rho e^{i\varphi} \tag{I23}$$

so that

$$\nu_{01}^2\Omega_1(\infty,s)\Big|_{s=-\sigma_1(1-c_1)} = \frac{c_1\sigma_1}{3} \tag{I24}$$

At this branch point $s \in S_{1i} \cap S_{2i}$ so one needs to show the behavior of many functions given in the text as $\rho \to 0$. This behavior can be given in terms of the behavior of ν_{01} and $\Omega_1(\infty,s)$. In the relationships which follow, quantities which are functions of s will be given as $O(\nu_{01})$, $O(1/\nu_{01})$, $O(\Omega_1)$, $O(1)$, etc., as $s \to -\sigma_1(1 - c_1)$. For example,

$$\Omega_1(\infty,s)\, \nu_{01}^2 \to \text{Finite} \to O(1) \tag{27}$$

where the given equation number is that from which the relationship can be seen.

$$\left.\begin{aligned} & X_m(-\mu,s) \to O(1) \\ & X_m(\pm\nu_{02},s) \to O(1) \\ & X_m(\pm\nu_{01},s) \to O(1/\nu_{01}) \end{aligned}\right\} \tag{A8a}$$

$$k \to O(1) \tag{C7}$$

$$\left.\begin{aligned} & h_2(\omega) \to O(1) \qquad (\omega = \pm\nu_{02} \text{ and } \mu(0 \leqq \mu \leqq 1)) \\ & N_m(\mu) \to O(1) \\ & h_1(\mu) \to O(\Omega_1) \\ & h_1(\pm\nu_{01}) \to O(1/\nu_{01}) \end{aligned}\right\} \tag{80}$$

$$\overline{B}_{2\pm}(\mu) \to O(1) \tag{77}$$

$$\overline{b}_{1\pm} \to O(\nu_{01}) \tag{79}$$

$$\overline{B}_{1\pm}(-\mu) \to O(1) \tag{78}$$

$$\left.\begin{aligned} &\overline{\Psi}_{2\pm}(x,\mu,s) \to O(1) \\ &\overline{\Psi}_{1\pm}(x,\mu,s) \to O(\nu_{01}) \end{aligned}\right\} \tag{76}$$

$$\left.\begin{aligned} &F_{m\pm}(x,\mu,s) \to O(1) \\ &F_{2\pm}(x,\pm\nu_{02},s) \to O(1) \\ &F_{1\pm}(x,\pm\nu_{01},s) \to O(1/\nu_{01}) \end{aligned}\right\} \tag{46}$$

$$B_{2\pm}(\nu) \to O(1) \tag{85}$$

$$B_{1\pm}(-\nu) \to O(1) \tag{86}$$

$$\left.\begin{aligned} &\alpha_{1\pm} \to O(1) \\ &\alpha_{2\pm} \to O(1) \end{aligned}\right\} \tag{89}$$

$$\left.\begin{aligned} &\beta_{1\pm} \to O(1) \\ &\beta_{2\pm} \to O(1) \end{aligned}\right\} \tag{90}$$

$$\left.\begin{aligned} &\tilde{F}_{\pm}(-a,\nu_{01},s) \to O(1/\nu_{01}) \\ &\tilde{F}_{\pm}(-a,\nu,s) \to O(1) \end{aligned}\right\} \tag{46}$$

$$\left[b_{1\pm} - \tilde{F}_{\pm}\left(-a,\nu_{01},s\right)\right] \rightarrow O\left(\nu_{01}\right) \tag{87}$$

$$\left[a_{2\pm} + \frac{1}{2} F_{2\pm}\left(a,\nu_{02},s\right)\right] \rightarrow O(1) \tag{88}$$

$$\psi_{2\pm}(x,\mu,s) \rightarrow O(1) \tag{91}$$

$$\psi_{1\pm}(x,\mu,s) \rightarrow O\left(\nu_{01}\right) \tag{92}$$

From these last two relationships and equation (I24), it follows that

$$\rho\psi_{\pm}(x,\mu,s) \rightarrow \sqrt{\rho}\, O(1) \tag{I25}$$

so that equation (I22) is satisfied. Therefore, there is no contribution from the integral (I21) for the case when $s = -\sigma_1\left(1 - c_1\right)$ does not satisfy the eigenvalue condition (81).

If the point $s = -\sigma_1\left(1 - c_1\right)$ happens to satisfy the eigenvalue condition, the denominator of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}\left(a,\nu_{02},s\right)\right]$ which is equivalent to the eigenvalue condition (81) vanishes. It can be seen from equation (88) that the limiting form of this condition at the branch point is $\alpha_{1\pm} = 0$ and additional discussion about this condition is given in appendixes J and K. If one considers, for such cases, the function

$$\psi_{\pm}(x,\mu,s) - \left[a_{2\pm} + \frac{1}{2} F_{2\pm}\left(a,\nu_{02},s\right)\right] \overline{\psi}_{\pm}(x,\mu,s) \tag{I26}$$

instead of $\psi_{\pm}(x,\mu,s)$ as the integrand of the integral (I21), it follows that in the limit $\rho \rightarrow 0$, the contribution from such an integral vanishes. The part which has been subtracted from $\psi_{\pm}(x,\mu,s)$ in function (I26) is considered separately and would appear to have a pole because of the zero in the denominator of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}\left(a,\nu_{02},s\right)\right]$. Its contribution therefore does not vanish in the limit $\rho \rightarrow 0$; in fact, its contribution looks like a discrete residue term. However, the point is not isolated (remember that this is the branch point of ν_{01} at $s = -\sigma_1\left(1 - c_1\right)$) so it should be understood that its contribution is included in the branch-cut integral term of equation (124). The numerical results indicate that such points occur.

APPENDIX J

EQUATIONS AND PROCEDURES FOR COMPUTATION OF EIGENVALUES

The equations from which the time eigenvalues $\{s_n\}$ are determined for $s \epsilon S_{1i} \cap S_{2i}$ are equations (77) and (81). When $s \epsilon S_{1e} \cap S_{2i}$, the corresponding equations are equations (82) and (83) and they determine what has been called the pseudo-eigenvalues. These equations are solved numerically by using the procedure of references 12 and 13. As stated previously, these equations can be written in terms of the nondimensional quantities introduced in equations (118). By making the substitution

$$B_{\pm}(\mu) = \frac{\overline{B}_{2\pm}(\mu)\ h_2(\mu)}{\left(\nu_{01} + \mu\right)\nu_{02}} \sqrt{\frac{\Omega_1\left(\nu_{02},s\right)}{\Omega_1(\infty,s)}}\ (i)^{-\left(\frac{1\pm 1}{2}\right)} \tag{J1}$$

it follows that equation (77) can be written for ζ real and $\max\left(-\sigma_D + \sigma_R,\ 0\right) < \zeta < 1$ (that is, on that part of the branch cut of ν_{02} which is not also part of the branch cut of ν_{01}) as

$$B_{\pm}(\mu) = -g(\mu)\left[\frac{\mu g_{\pm} \pm \left|\nu_{02}\right| g_{\mp}}{\mu^2 + \left|\nu_{02}\right|^2} \pm \frac{1}{2}\int_0^1 B_{\pm}(\nu)\ \frac{d\nu}{\nu + \mu}\right] \tag{J2}$$

where

$$\left.\begin{aligned} g(\mu) &= \frac{\Omega_1\left(\nu_{02},\zeta\right)}{\Omega_1(\infty,\zeta)}\left[\frac{X_2(-\mu,\zeta)}{X_1(-\mu,\zeta)}\right]^2 \frac{\mu(1-\zeta)\ e^{-2\zeta A/\mu}}{\dfrac{\Omega_2^+(\mu,\zeta)\ \Omega_2^-(\mu,\zeta)}{c_2^2\sigma_2^2}}\ \frac{\mu^2 + \left|\nu_{02}\right|^2}{\left(\mu + \nu_{01}\right)^2} \\ g_{\pm} &= -\sqrt{\frac{\Omega_1\left(\nu_{02},\zeta\right)}{\Omega_1(\infty,\zeta)}}\ \begin{matrix}\mathrm{Im}\\ \mathrm{Re}\end{matrix}\left[\frac{X_2\left(\nu_{02},\zeta\right)}{X_1\left(\nu_{02},\zeta\right)}\ \frac{e^{\zeta A/\nu_{02}}}{\nu_{01} - \nu_{02}}\right] \end{aligned}\right\} \tag{J3}$$

Now one defines $\Delta_{\zeta\pm}$ as

$$\Delta_{\zeta\pm} \equiv -2g_{\pm} \mp \int_0^1 B_{\pm}(\nu)\ d\nu \tag{J4}$$

which is the eigenvalue condition (eq. (81)) if $\Delta_{\zeta\pm} = 0$.

Equations (J2) are reduced to two sets $(\pm)$ of N equations in the N unknowns $B_+(\mu_i)$ and $B_-(\mu_i)$ where $i = 1,\ldots,N$ (see, for example, ref. 28) given by

$$B_\pm(\mu_i) = -g(\mu_i)\left[\frac{\mu_i g_\pm \pm |\nu_{02}| g_\mp}{\mu_i^2 + |\nu_{02}|^2} \pm \frac{1}{2}\sum_{j=1}^{N} R_j \frac{B_\pm(\mu_j)}{\mu_j + \mu_i}\right] \tag{J5}$$

where R_j are the weighting functions for the numerical integration scheme which is used. Equation (J4) is written as

$$\Delta_{\zeta\pm} = -2g_\pm \mp \sum_{j=1}^{N} R_j B_\pm(\mu_j) \tag{J6}$$

Since a search must be made for values of ζ for which $\Delta_{\zeta\pm} = 0$, one must be able to compute all quantities which appear in equations (J3) for any value of ζ in the range given by equation (120)). These quantities are computed as follows:

$$\frac{\Omega_1(\nu_{02},\zeta)}{\Omega_1(\infty,\zeta)} = \frac{\zeta + \sigma_D - \zeta\sigma_R}{\zeta + \sigma_D - \sigma_R} \tag{J7}$$

$$\frac{\Omega_2^+(\mu,\zeta)\ \Omega_2^-(\mu,\zeta)}{c_2^2\sigma_2^2} = \left[\zeta - \frac{\mu}{2}\log_e\left(\frac{1 + \mu}{1 - \mu}\right)\right]^2 + \left(\frac{\pi\mu}{2}\right)^2 \tag{J8}$$

The functions ν_{0m} are determined by $\Omega_m(\nu_{0m},\zeta) = 0$ and they are computed numerically by using the Newton-Raphson iteration (ref. 28) on the nonlinear equations

$$|\nu_{02}| \tan^{-1}\frac{1}{|\nu_{02}|} = \zeta \tag{J9}$$

and

$$\nu_{01}\tanh^{-1}\frac{1}{\nu_{01}} = \frac{\zeta + \sigma_D}{\sigma_R} \tag{J10}$$

The X-functions are computed from equation (A8a); namely,

$$X_m(z,\zeta) = \frac{1}{1 - z} \exp\left\{\frac{1}{2\pi i} \int_0^1 \log_e\left[\frac{\Omega_m^+(\nu,\zeta)}{\Omega_m^-(\nu,\zeta)}\right] \frac{d\nu}{\nu - z}\right\} \tag{J11}$$

For ζ and z real where $z = -\mu$, $0 \leqq \mu \leqq 1$, one has from equation (J11) that

$$\frac{X_2(-\mu,\zeta)}{X_1(-\mu,\zeta)} = \exp\left\{\frac{1}{\pi} \int_0^1 \left[\theta_2(\zeta,\nu) - \theta_1\left(\zeta,\sigma_R,\sigma_D,\nu\right)\right] \frac{d\nu}{\nu + \mu}\right\} \tag{J12}$$

where

$$\left.\begin{aligned} \theta_2(\zeta,\nu) &= \tan^{-1}\left(\frac{\pi\nu/2}{\zeta - \nu \tanh^{-1} \nu}\right) \\ \theta_1\left(\zeta,\sigma_R,\sigma_D,\nu\right) &= \tan^{-1}\left(\frac{\sigma_R \pi\nu/2}{\zeta + \sigma_D - \sigma_R \nu \tanh^{-1} \nu}\right) \end{aligned}\right\} \tag{J13}$$

For ζ real and $z = \nu_{02}$, one calculates the real and imaginary parts of $\dfrac{X_2\left(\nu_{02},\zeta\right)}{X_1\left(\nu_{02},\zeta\right)}$ as

$$\left.\begin{aligned} \operatorname{Re}\left[\frac{X_2\left(\nu_{02},\zeta\right)}{X_1\left(\nu_{02},\zeta\right)}\right] &= e^{\Gamma_1/\pi} \cos \frac{\Gamma_2}{\pi} \\ \operatorname{Im}\left[\frac{X_2\left(\nu_{02},\zeta\right)}{X_1\left(\nu_{02},\zeta\right)}\right] &= e^{\Gamma_1/\pi} \sin \frac{\Gamma_2}{\pi} \end{aligned}\right\} \tag{J14}$$

where

$$\left.\begin{aligned} \Gamma_1 &= \int_0^1 \left[\theta_2(\zeta,\nu) - \theta_1\left(\zeta,\sigma_R,\sigma_D,\nu\right)\right] \frac{\nu \, d\nu}{\nu^2 + \left|\nu_{02}\right|^2} \\ \Gamma_2 &= \left|\nu_{02}\right| \int_0^1 \left[\theta_2(\zeta,\nu) - \theta_1\left(\zeta,\sigma_R,\sigma_D,\nu\right)\right] \frac{d\nu}{\nu^2 + \left|\nu_{02}\right|^2} \end{aligned}\right\} \tag{J15}$$

Integrals in equations (J12) and (J15) are computed as

$$\int_0^1 \left[\theta_2(\zeta,\nu) - \theta_1\left(\zeta,\sigma_R,\sigma_D,\nu\right)\right] f(\nu)\, d\nu = \sum_{i=1}^{M} R_i\left[\theta_2\left(\zeta,\nu_i\right) - \theta_1\left(\zeta,\sigma_R,\sigma_D,\nu_i\right)\right] f\left(\nu_i\right) \quad \text{(J16)}$$

where R_i are again the weighting functions for the numerical integration scheme.

In all numerical integrations, Gauss' method (ref. 28) was used. For integrations in equations (J5) and (J6), the interval (0,1) was split into four intervals

$$(0,1) = (0, 0.05) + (0.05, 0.1) + (0.1, 0.9) + (0.9, 1.0) \quad \text{(J17)}$$

and a 10-point Gaussian formula was used in each subinterval. For integrations in equations (J12) and (J15), the interval (0,1) was divided as

$$(0,1) = (0, 0.1) + (0.1, 0.9) + (0.9, 0.99) + (0.99, 0.999) + (0.999, 1.0) \quad \text{(J18)}$$

and in each of these subintervals a 10-point Gaussian formula was also used. The subdivision (J18) is the same as that used by Kowalska (ref. 29) and the X-functions calculated here agree with those she gives to all figures which she quotes except for the real and imaginary parts of $X_m\left(\nu_{02},\zeta\right)$. She apparently used Γ_2 instead of Γ_2/π in equations (J14) to obtain the numerical values for the real and imaginary parts given in part II of reference 29. Since her later published critical-slab results (ref. 24) agree with those of Mitsis (ref. 22) for a bare slab, it is expected that this oversight was corrected.

Conditions (82) and (83) which determine the pseudo-eigenvalues for $s \in S_{1e} \cap S_{2i}$ lead to very similar equations which will not be written down. In this region, the real s-axis corresponds to $0 \leqq \zeta \leqq -\sigma_D$ and such equations need be considered only if $-\sigma_D > 0$.

The procedure used to calculate the eigenvalues ζ_n is as follows. For fixed values of A, σ_R, and σ_D, one selects a number of ζ values in the interval given by equation (120). For each of these values, one obtains $|\nu_{02}|$ and ν_{01} from equations (J9) and (J10) by iteration (Newton-Raphson). Equations (J13) are evaluated at each of the 50 Gaussian integration points ν_i $(0 < \nu_i < 1)$. Next, the $\dfrac{X_2\left(-\mu_j,\zeta\right)}{X_1\left(-\mu_j,\zeta\right)}$ are calculated for each of the 40 Gaussian integration points, μ_j $(0 < \mu_j < 1)$ by using equation (J16) in equation (J19). The real and imaginary parts of $\frac{X_2(\nu_{02},\zeta)}{X_1(\nu_{02},\zeta)}$ are computed in the same way from equations (J14) to (J16). Now $g(\mu_j)$ can be computed from equation (J3) at each of the 40 points μ_j and all the coefficients in the two sets $(\pm)$ of N equations in the N unknowns $B_+(\mu_j)$ and $B_-(\mu_j)$ (eqs. (J5)) can be evaluated. These two sets of simultaneous equations are solved numerically for $B_\pm(\mu_j)$ which are then used to compute $\Delta_{\zeta\pm}$ from equation (J6) at the selected values of ζ. In this way, one locates the zeros of $\Delta_{\zeta\pm}$ approximately. A new set of ζ values, located about each approximate ζ_n, is selected and the process is repeated. For the present computations, the ζ_n were located to three figures. Discussion of the computed results is given in the text. The calculations were done on a Control Data 6600 computer system at the Langley Research Center.

In appendix G, the thick-slab eigenvalue condition was given as equation (G4). Note that $g_\pm$ quantities given by equations (J3) are, within a factor, exactly the quantities needed in equation (G4). Therefore, the thick-slab approximation eigenvalues are obtained from

$$g_\pm = 0 \tag{J19}$$

as would be expected from equation (J4).

The bare-slab eigenvalues are obtained when $\sigma_R = 0$ and it is easily shown that in this case, equations (J5) and (J6) no longer depend on σ_D; that is, for $\sigma_R = 0$, these equations do not contain σ_D.

It was noted in the text and in appendix I that the branch point of ν_{01} located at $s = -\sigma_1(1 - c_1)$ may happen to satisfy the eigenvalue condition which can be seen from equation (88) to be

$$\alpha_{1\pm} = 0 \tag{J20}$$

when $\nu_{01} \to \infty$. This point corresponds to $\zeta = -\sigma_D + \sigma_R$ and it can be shown that equation (J20) then determines values of $\zeta = \zeta_n$ which depend on neither σ_D nor σ_R; that is, if one uses $\zeta = -\sigma_D + \sigma_R$ to eliminate σ_D from the condition (J20), σ_R drops out of the equations. Equation (J20) determines the values of ζ at which eigenvalues disappear into the right end of the branch cut of ν_{01}. Also note that the limiting form of the pseudo-eigenvalue condition for $s = -\sigma_1$, which corresponds to $\zeta = -\sigma_D$, determines the values of ζ where the pseudo-eigenvalues disappear into the left end of the branch cut of ν_{01}. Such points, as well as those given by equation (J20), are labeled with an asterisk in figures 7 and 9 to 11.

APPENDIX K

REMARKS ON EIGENVALUE—BRANCH-POINT COINCIDENCE

In this appendix, a few remarks concerning the situation when the eigenvalues (or pseudo-eigenvalues) disappear into the branch cut of ν_{01} are made. This situation is somewhat analogous to that encountered by Hintz (ref. 9) for the slab surrounded by pure absorbers. He could not say whether a bare-slab eigenvalue (which does not depend on σ_D) that happened to coincide with $-\sigma_D$ belonged to the point spectrum or the continuous spectrum for his problem. In the present problem, the eigenvalues coincide with a branch point as they disappear into the branch cut of ν_{01}. A numerical study of the branch-cut integral in equation (124) has not been made nor has the eigenvalue condition on another Riemann sheet been investigated. It is suspected that there is no drastic change in the shape of the solution given by equation (124) when an eigenvalue disappears into the branch cut of ν_{01} and such studies would resolve this point. It was pointed out in appendix J that the condition (J20), which determines whether the point $s = -\sigma_1(1 - c_1)$, $(\zeta = -\sigma_D + \sigma_R)$ is a zero of the denominator of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\right]$ given by equation (88), depends neither on σ_D nor σ_R explicitly. In appendix I, it was indicated that the contribution from such points should be included in the branch-cut integral since it arises from the integration around the branch point. One understands then that such a contribution is included in equation (124) if $s = -\sigma_1(1 - c_1)$ happens to satisfy equation (J20). How such zeros of equation (J20) behave or appear in the solution after passing through the branch point as the material properties are varied has not been studied here.

If one considered the problem of a finite slab with symmetric reflectors of finite thickness, then what is happening at the places where the eigenvalues coincide with $\nu_{01} = \infty$ might be deduced. In such a problem, the solution probably does not contain the branch cut of ν_{01}, but instead has discrete eigenvalues along it. Even though there is another parameter in the problem, the reflector thickness, one might be able to carry out a numerical study of all the eigenvalues.

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