③ III

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APOLLO SPACECRAFT SYSTEMS ANALYSIS PROGRAM

CONSTRUCTION OF GENERAL PURPOSE PROGRAMS FOR THE NUMERICAL SOLUTION OF CERTAIN ELECTROMAGNETIC SCATTERING PROBLEMS

NAS 9-8166 TASK E-34D FEBRUARY 28, 1969

Prepared for
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MANNED SPACECRAFT CENTER
HOUSTON, TEXAS

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1. INTRODUCTION

This is Volume III in a series of four reports* aimed at providing a computable electromagnetic theory model for the lunar reflectivity problem. The problem is to construct the electromagnetic field reflected (scattered) by a very large target when a small portion of its surface is illuminated by an electromagnetic field of given description. Such a problem will be viewed in Volume IV as an asymptotic form of certain classical scattering problems discussed in Volume II in which the target is completely illuminated by the incident field and in which the surface bounding the target is allowed to be more general than a sphere or an ellipsoid. The present report is devoted to the construction of general purpose programs for the numerical solution of these classical problems.

Volumes I and II gave a mathematical demonstration of the applicability of a particular technique to classical problems involving spherical scatterers. The applicability of the technique in more general cases was seen to depend on the properties of a linear and continuous operator called the solvability operator. The point of departure for the construction of the algorithm is Theorem 13 of Volume II which states that the solvability operator preserves linear independence. This fact permits the adoption of an approach which may be stated heuristically as follows: attack the given scattering problem by the above method. If the problem is exactly solvable by the technique, then the algorithm gives a sequence which converges to a solution; if it is not exactly solvable, the algorithm gives a sequence which converges to "the closest thing to a solution". In any case, a number is given which measures the distance of the approximate solution to the exact solution. The algorithm is straightforward although it requires a large digital computer for its implementation. All the functions and numerical processes required are presently available as "off-the-shelf" subroutines although the usual care will have to be exercised in the selection and modification of these to suit the special

*Volume I = Reference 1; Volume II = Reference 2; Volume IV = Reference 3. Notations, conventions, and results of Volumes I and II will be freely used in the present report.

conditions at hand. Test cases may be easily devised for which the solutions are already available (References 4, 5, 6, and 7). The details pertinent to a particularly useful test case are given in Section 5.

2. THE METHOD

Sections 1.12, 3.5, and 4.2.2 of Volume II comprise the pertinent background for the method. The electromagnetic scattering problem is carefully stated in Section 1.12. If a scattering problem $\mathscr{P} \equiv \mathscr{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathscr{E}_0)$ is given involving the scattering surface $S(f, D_1, D_2, \hat{n})$, the following positive constants a, b ∋ are chosen

$$0 < a < \inf_{\omega \in \Omega} f(\omega) \leq \sup_{\omega \in \Omega} f(\omega) < b < \infty \tag{2-1}$$

These constants play the same roles as they did in Volume II in definitions involving the solvability operator. The tangential unit vectors $\hat{\xi}, \hat{\zeta}$ at points of S are available (Section 1.11 of Volume II). If T_0 is the associated solvability operator and $\mathscr{S}$ the boundary operator,

then

$$\mathscr{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathscr{E}_0) \text{ is } \underline{\text{exactly solvable}}$$

$$\leftrightarrow \quad \mathscr{S}\mathscr{E}_0 \in T_0[\mathscr{H}]$$

If P is exactly solvable, then (Section 3.5 of Volume II)

$$VT_0^{-1}(\mathscr{S}\mathscr{E}_0)$$

is the unique solution to P. A rigorous concept of approximate solvability will appear shortly.

The scattering problem $\mathscr{P}$ has been reduced to finding an element $\bar{v} \equiv (v_1, v_2, v_3, v_4)^T \in \mathscr{H}^4$ (if it exists) such that the equation

$$T_0 \bar{v} = \mathscr{S}\mathscr{E}_0 \tag{2-3}$$

is satisfied. Following the notation established in Section 4.2.2 of Volume II,

$$\bar{v} = \Sigma \left\{ b_\nu \tau_\nu \mid \nu \in \Lambda' \right\} = \Sigma \left\{ x_\nu (c_\nu \tau_\nu) \mid \nu \in \Lambda' \right\} \tag{2-4}$$

in which

$$x_\nu \equiv \left(\frac{b\nu}{x\nu}\right) \tag{2-5}$$

and

$$c_\nu \equiv \begin{cases} -\dfrac{k_2 h_n'(k_2 a)}{A_{mn}} \\ \dfrac{k_1 j_n'(k_1 b)}{A_{mn}} \end{cases} \tag{2-6}$$

In terms of (2-4), the equation (2-3) may be written

$$\Sigma\{x_\nu T_0(\chi_\nu) \mid \nu \in \Lambda'\} = \mathscr{J}\mathscr{E}_0, \tag{2-7}$$

where, as in Section 4.2.2 of Volume II,

$$\chi_\nu = c_\nu \tau_\nu \quad . \tag{2-8}$$

Hence the problem is to express the element $\mathscr{J}\mathscr{E}_0$ of $\mathscr{H}^4$ in terms of the linearly independent set $T_0(\chi_\nu)$ which is described in Table 3-1 of Volume II.

A systematic means of handling (2-3) numerically is available. For ease of presentation, denote

$$\xi_\nu = T_0(\chi_\nu) \tag{2-9}$$

and order the linearly independent set

$$\{\xi_\nu \mid \nu \in \Lambda'\}. \tag{2-10}$$

Henceforth the elements of this set will be written $\xi_1, \xi_2, \ldots,$ etc. Denote

$$\bar{g} \equiv \mathscr{J}\mathscr{E}_0. \tag{2-11}$$

Consider the approximate problem

Of all sequences

$$\tilde{x} \equiv \{ x_\nu \mid \nu \in \Lambda' \} \qquad (2\text{-}12a)$$

of constants in $\mathcal{C}$ such that

$$\| \tilde{x} \| = \left\{ \sum_{\nu=1}^{\infty} |x_\nu|^2 \right\}^{1/2} < \infty \qquad (2\text{-}12b)$$

find that one

$$\tilde{x}_0 \equiv \{ x_\nu^0 \mid \nu \in \Lambda' \} \qquad (2\text{-}12c)$$

for which the distance

$$\delta(\tilde{x}) \equiv \| \bar{g} - \sum_{\nu=1}^{\infty} x_\nu \mathcal{E}_\nu \| \qquad (2\text{-}12d)$$

is a minimum. (2-12)

Since $\{\mathcal{E}_\nu | \nu \in \Lambda'\}$ is linearly independent, the minimizing sequence (2-12c) will be uniquely determined. This problem is easily solved by the Gram-Schmidt orthogonalization procedure:

$$\left.\begin{aligned}
\psi_1 &= \xi_1 \\
\phi_1 &= \frac{\psi_1}{\|\psi_1\|} \\
&\vdots \\
\psi_{k+1} &= \xi_{k+1} - \sum_{i=1}^{k} (\xi_{k+1}, \phi_i)\, \phi_i \\
\phi_{k+1} &= \frac{\psi_{k+1}}{\|\psi_{k+1}\|} \\
&\vdots
\end{aligned}\right\} \qquad (2\text{-}13)$$

This process defines the elements of the infinite matrix

$$A = (\alpha_{ij}) \qquad (2\text{-}14)$$

of the transformation

$$\left.\begin{aligned}
\phi_1 &= \alpha_{11}\,\xi_1 \\
\phi_2 &= \alpha_{21}\,\xi_1 + \alpha_{22}\,\xi_2 \\
&\;\vdots \\
\phi_N &= \alpha_{N1}\,\xi_1 + \alpha_{N2}\,\xi_2 + \cdots + \alpha_{NN}\,\xi_N \\
&\;\vdots
\end{aligned}\right\} \qquad (2\text{-}15)$$

as inner products of the form (ξ_i, ξ_j). As is well-known, the result of the transformation

$$\{\xi_\nu \mid \nu \in \Lambda'\} \to \{\phi_\nu \mid \nu \in \Lambda'\}$$

given by (2-15) is an <u>orthonormal</u> set $\{\phi_\nu\}$ which spans the same linear manifold $T_0[\Lambda']$ as the set $\{\xi_\nu\}$. It is also well known that the element of $T_0[\Lambda']$ nearest to $\overline{g}$ is

$$\overline{g}' = \sum_{\nu=1}^{\infty} c_\nu \phi_\nu \qquad (2\text{-}16)$$

where

$$c_\nu = (\overline{g}, \phi_\nu). \qquad (2\text{-}17)$$

From this, it follows that the solution to the minimization problem (2-12) is exactly the solution to the equation

$$\sum_{\nu=1}^{\infty} x_\nu \xi_\nu = \sum_{\nu=1}^{\infty} c_\nu \phi_\nu \qquad (2\text{-}18)$$

which may be written*

$$\sum_{j=1}^{\infty} x_j \xi_j = \sum_{i=1}^{\infty} c_i \left(\sum_{j=1}^{\infty} \alpha_{ij}\, \xi_j \right) = \sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} c_i \alpha_{ij} \right) \xi_j \qquad (2\text{-}19)$$

---

*The rearrangement is formal, but this point is not important for numerical work, as will be seen below.

and hence, by the linear independence of the ξ_i,

$$x_j = \sum_{i=j}^{\infty} c_i \alpha_{ij} \quad , \; j = 1, 2, \ldots \tag{2-20}$$

Whenever the rearrangement of (2-19) is valid, (2-20) gives a rigorous solution to the minimization problem (2-18). The number δ_0 of

$$0 \leq \delta_0 \equiv \|\bar{g} - \bar{g}'\| \leq \delta(\tilde{\bar{x}}) \tag{2-21}$$

measures the degree to which $\mathcal{P}$ is solvable. If $\delta_0 = 0$, the problem is exactly solvable. Otherwise $\delta_0 > 0$, and it can be said that $\mathcal{P}$ is δ_0 - approximately solvable.

In practice, of course, one must be satisfied with truncating the infinite processes involved. The following inductive process explains the truncation criterion:

1) Choose $M \geq 1$.

2) Calculate

$$\bar{g}'_N = \sum_{\nu=1}^{M} c_\nu \phi_\nu \tag{2-22}$$

where

$$c_\nu = (\bar{g}, \phi_\nu) \tag{2-23}$$

3) Calculate

$$\delta'_M = \|\bar{g} - \bar{g}'_N\| \tag{2-24}$$

4) If

$$\delta'_M < \epsilon_0 \tag{2-25}$$

where ϵ_0 is acceptably small*, then calculate the elements of the sub-matrix A_M of A

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_M \end{bmatrix} = A_M \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_M \end{bmatrix} . \tag{2-26}$$

and proceed to the Step 5 below. Otherwise, increase M and go to Step 2. This will involve the calculation of some <u>more</u> c_ν in (2-23) since the sequence

$$\{ c_\nu \mid \nu = 1, 2, \ldots, M \} \tag{2-27}$$

is already available from the first pass. Calculate δ'_M for the new M etc.

5) Solve the equation

$$\sum_{j=1}^{M} x_j \, \xi_j = \sum_{i=1}^{M} c_i \phi_i = \sum_{i=1}^{M} c_i \left(\sum_{j=1}^{i} \alpha_{ij} \, \xi_j \right) = \sum_{j=1}^{M} \left(\sum_{i=j}^{M} c_i \, \alpha_{ij} \right) \tag{2-28}$$

to obtain

$$x_j = \sum_{i=j}^{M} c_i \, \alpha_{ij} \quad , \ j = 1, 2, \ldots, M. \tag{2-29}$$

The sequence

$$\tilde{x}'_M = \left\{ x_j \mid j = 1, 2, \ldots, M \right\} \tag{2-30}$$

determined by (2-29) may be taken to be an approximate solution to (2-3).

The next section summarizes the algorithm.

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*A feeling for "acceptably small" may be gained from a study of the solution to the classical problem with spherical boundary.

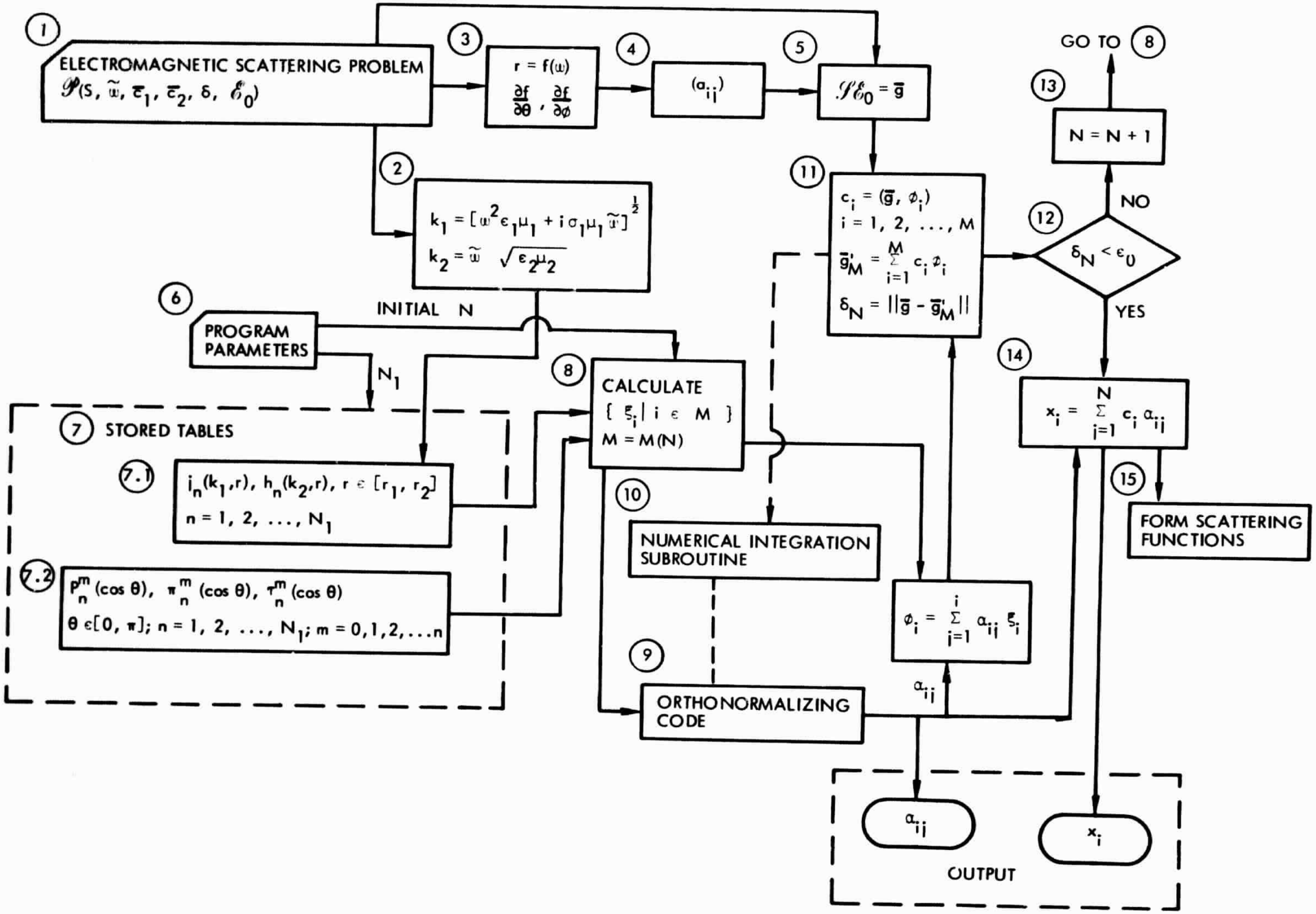

Figure 2-1. General Purpose Program

3. THE PROGRAM

Figure 2-1 gives a flow chart of a general-purpose program to solve the equation (2-3) associated with the scattering problem $\mathscr{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$. The program to be described makes use of stored tables of the functions required.

The description of the program will be given with reference to Figure 2-1. The numeration of the blocks corresponds to the suggested order of execution of the program. The discussion below is keyed to this numeration. References given pertain to section and equation numbers of Volume II. The notation used is, of course, precisely that of the last section.

(1.) (Sections 1.11 and 1.12). S is the starlike surface bounding the scatterer. Its description is given by the everywhere positive $C^1(\Omega)$ function

$$r = f(\omega).$$

All fields are taken to have time variation $e^{-i\tilde{\omega}t}$. The triple $\bar{c}_1 = (\epsilon_0, \mu_1, \sigma_1)$ gives the electromagnetic parameters for the domain D_1 interior to S and it is required that $\sigma_1 > 0$. The triple $\bar{c}_2 = (\epsilon_2, \mu_2, 0)$ gives the electromagnetic parameters for the domain D_2 exterior to S. δ is the "width" of the domain $D(S, \delta)$ in D_2 immediately adjacent to S and in which the incident field $\mathcal{E}_0$ is defined.

(2.) The constants k_i are the propagation constants.

(3.) If $r = f(\omega)$ is given numerically, then this block must form the functions

$$\left.\begin{aligned} f_\theta(\theta, \phi) &= \frac{\partial f}{\partial \theta} \\ f_\phi(\theta, \phi) &= \frac{\partial f}{\partial \phi} \end{aligned}\right\} \qquad (3\text{-}1)$$

numerically. It is of course desirable to specify $r = f(\omega)$ analytically so that (3-1) may also be computed as accurately as possible, preferably from analytically-given functions. For many classical problems, not only f but also f_θ and f_ϕ are available analytically.

4. The $a_{ij}(\omega)$ of the transformation

$$\left.\begin{aligned} \hat{r} &= a_{11}\hat{n} + a_{12}\hat{\xi} + a_{13}\hat{\zeta} \\ \hat{\theta} &= a_{21}\hat{n} + a_{22}\hat{\xi} + a_{23}\hat{\zeta} \\ \hat{\phi} &= a_{31}\hat{n} + a_{32}\hat{\xi} + a_{33}\hat{\zeta} \end{aligned}\right\} \tag{3-2}$$

between unit spherical base vectors and unit base vectors $(\hat{n}, \hat{\xi}, \hat{\zeta})$ defined on S, are found in terms of the functions f, f_θ, f_ϕ.

5. See Section 3. 1, Definition 5.

6. A list of the main program parameters and the criteria which affect their selection follows.

ϵ_0 The selection of this closeness-of-fit parameter should be made on the basis of study of the corresponding parameter for problems of the type $\mathscr{P}_0 = \mathscr{P}(S_c, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathscr{E}_0')$ where S_c is a sphere $r = c \ni$

$$c \in \left(\inf_{\omega\in\Omega} f(\omega), \sup_{\omega\in\Omega} f(\omega) \right)$$

and where $\mathscr{E}_0'$ is a plane polarized monochromatic wave. The solution to this problem is available (References 4, 5, and 6).

N This is the truncation index for the series

$$\left.\begin{aligned} \mathcal{E}_1 &= \Sigma\left\{\left(a_{\lambda 3}\,\mathcal{E}_\lambda^{1E} + a_{\lambda 4}\,\mathcal{E}_\lambda^{1H}\right) \,\middle|\, \lambda \in \Lambda_1 \cup \Lambda_2\right\} \\ \mathcal{E}_2 &= \Sigma\left\{\left(a_{\lambda 1}\,\mathcal{E}_\lambda^{2E} + a_{\lambda 2}\,\mathcal{E}_\lambda^{2H}\right) \,\middle|\, \lambda \in \Lambda_1 \cup \Lambda_2\right\} \end{aligned}\right\} \tag{3-3}$$

where the sequence

$$\left\{a_\nu \,\middle|\, \nu \in \Lambda'\right\} \tag{3-4}$$

is determined approximately by the finite sequence

$$\left\{\chi_\nu \,\middle|\, \nu \in \Lambda'_M\right\} \tag{3-5}$$

which is the output of the algorithm. The relationship between M and N will be discussed in 8 below. The initial choice of N should be made from a study of the corresponding quantity for $\mathcal{P}_0$.

N_1 Let N_0 be the lowest positive integer such that

$$\delta_{N_0} < \epsilon_0 \tag{3-6}$$

in 12 .

Let N_0' be the largest of these integers expected to be encountered for the contemplated application to classes of problems such as $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$. Then $N_1 \geq N_0'$. This number is the maximum index n expected to be required for j_n, h_n by other parts of the program.

r_1, r_2 As an example of how to select r_1, r_2, consider a class of scattering problems for the solution of which the program is to be used. To describe a typical situation, let $\mathcal{B}$ be an index set in the general sense and suppose for each $\beta \epsilon \mathscr{B}$, there is a function $f(\omega, \beta)$ on Ω to the positive reals, and such that $f(\omega, \beta) \epsilon C'(\Omega)$. The set $\mathscr{B}$ generates the family

$$\mathscr{S}(\mathscr{B}) = \left\{ S \mid \exists\ \beta \in \mathscr{B} \ni S = \{(r, \omega) \mid r = f(\omega, \beta)\} \right\} \tag{3-7}$$

of stars. Consider the class

$$\begin{aligned} \mathscr{A}(k_1, k_2, \mathscr{B}) = \{ \mathscr{P} \mid \mathscr{P} = \mathscr{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathscr{E}_0), \\ S \epsilon \mathscr{S}(\mathscr{B}), k_1 = k(\tilde{\omega}, \bar{c}_1), \\ k_2 = k(\tilde{\omega}, \bar{c}_2) \} \end{aligned} \tag{3-8}$$

of scattering problems with same k_1, k_2 and stars contained in the annulus

$$r_1 \leq r \leq r_2 \tag{3-9}$$

where

$$\left.\begin{aligned} r_1 &= \inf_{\beta \epsilon \mathscr{B}} \left(\inf_{\omega \epsilon \Omega} f(\omega) \right) \\ r_2 &= \sup_{\beta \epsilon \mathscr{B}} \left(\sup_{\omega \epsilon \Omega} f(\omega) \right). \end{aligned}\right\} \tag{3-10}$$

Other programming parameters are those associated with the numerical integration routine (10) and those associated with the tabulation and interpolation methods of (7).

(7.1) The computation of the function $j_n(k_1 r)$ and $h_n(k_2 r)$ is usually performed by forward recursion or backward recursion. Subroutines and check values are already in existence (References 8 and 9). Recall that

$k_1 \in \mathscr{C}_1$ and $k_2 \in \mathscr{R}_0$ (See Figure 1-1 of Volume II). Thus the arguments of $j_n(z)$ required will all fall on the line segment

$$\{ z \mid z = k_1 r, \quad r \in [r_1, r_2] \} \tag{3-11}$$

in the region $\mathscr{C}_1$, while the required arguments of $h_n(z)$ will all fall in the interval

$$[k_2 r_1, k_2 r_2] \tag{3-12}$$

of the positive reals. Hence, both $j_n(k_1 r)$ and $h_n(k_2 r)$ will be complex-valued.

7.2 The functions P_n^m, Π_n^m, and τ_n^m are generated recursively. Subroutines and tables are available for P_n^m (References 10, 11, and 12). The functions cos $m\phi$ and sin $m\phi$ will also be needed.

8 (See Table I of Volume I and Table 3-1 of Volume II). It is suggested that $\xi_i(\omega)$ be computed as required. Thus 8 would represent a block of subroutines. The following index sets will be useful (See Figure 1-4 of Volume II).

$$\left.\begin{aligned}
\mathscr{Y}_{1N} &\equiv \mathscr{Y}_1 - \{ \lambda \mid \lambda = (n, m, i) \in \mathscr{Y}_1, \quad n > N \} \\
\mathscr{Y}_{2N} &\equiv \mathscr{Y}_2 - \{ \lambda \mid \lambda = (n, m, i) \in \mathscr{Y}_2, \quad n > N \} \\
\Lambda_{1N} &\equiv \mathscr{Y}_{1N} \times \{1\} \\
\Lambda_{2N} &\equiv \mathscr{Y}_{2N} \times \{2\} \\
\Lambda_N &= \Lambda_{1N} \cup \Lambda_{2N} \quad .
\end{aligned}\right\} \tag{3-13}$$

which is closest to $\bar{g} = \mathcal{S}\mathcal{E}_0$. The distance is

$$\delta_N = \| \bar{g} - \bar{g}'_{M(N)} \| . \tag{3-16}$$

(12), (13) Unless δ_N is "acceptably small" there is no point in proceeding with the calculation of the solution sequence to be described below. If N gets so large that numerical errors inherent in the blocks (8) and (10) grow to unacceptable bounds and still the number δ_N is not less than ϵ_0, it must be concluded that the given scattering problem is one of the following:

a) Not exactly solvable

b) Not exactly solvable and not ϵ_0-approximately solvable

c) Not exactly solvable and ϵ_0-approximately solvable, but beyond the capabilities of the machine or the numerical analysis to implement by the method at least for the precision being attempted.

(14) After this sequence has been generated, it should be re-indexed in terms of the indices used in Table I of Volume I and Table 3-1 of Volume II. If the scattering problem $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1\, \bar{c}_2, \delta, \mathcal{E}_0)$ is already of the type $\mathcal{P}_0$ discussed in (6), then the coefficients x_ν should be transformed to a normalization which permits their comparison with tabulated values; for example, References 4-7. For these comparisons, it will also be desirable to form the far-field intensity functions and the scattering cross-section quantities whose formation in terms of x_ν is described in the next section.

The number of points in Λ_N is

$$\left[\frac{1}{2} N (N + 1) + N\right] + \left[\frac{1}{2} N (N + 1)\right]$$

$$= N (N + 1) + N = N (N + 2).$$

The index M of ⑧ is, therefore

$$M = M (N) \equiv 4 N (N + 2). \qquad (3\text{-}14)$$

This number gives the order of the triangular matrix $A_M = (\alpha_{ij})$ of complex constants to be computed in ⑨, etc.

⑨ Efficient schemes for calculating the coefficients α_{ij} required by the Gram Schmidt orthogonalization procedure (2. 15) are known as orthogonormalizing codes. These have been surveyed by Davis (References 13 and 14). Inspection of the matrix $A_M = (\alpha_{ij})$ will give qualitative answers to questions such as: "How close is the sequence ξ_ν to being orthogonal?" and "What is the best way to order the sequence ξ_ν?"

⑩ In the most general case, this will be a subroutine to perform numerical integration over the unit sphere. In certain special cases such as the one to be discussed in Section 5, numerical integration over the unit circle only is required. For some of these cases, it will be possible to give special purpose algorithms based on the particular symmetry of the problem.

⑪ The element $\bar{g}_M'$ is that element of the linear manifold spanned by

$$\{\xi_i \mid i \in \langle M \rangle\} \qquad (3\text{-}15)$$

4. FAR-FIELD INTENSITY FUNCTIONS

Suppose that the application of the algorithm described in the previous sections to the scattering problem $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$ has yielded the sequence $\{a_{\lambda j} | \lambda \in \Lambda_N\}$ for

$$\left.\begin{aligned} \mathcal{E}_1 &= \Sigma\left\{\left(a_{\lambda 3}\,\mathcal{E}_\lambda^{1E} + a_{\lambda 4}\,\mathcal{E}_\lambda^{1H}\right) \middle| \lambda \in \Lambda_N\right\} \\ \mathcal{E}_2 &= \Sigma\left\{\left(a_{\lambda 1}\,\mathcal{E}_\lambda^{2E} + a_{\lambda 2}\,\mathcal{E}_\lambda^{2H}\right) \middle| \lambda \in \Lambda_N\right\}. \end{aligned}\right\} \tag{4-1}$$

In particular, the outer field is $\mathcal{E}_2 = (\bar{E}_2, \bar{H}_2)$ where

$$\begin{aligned} \bar{E}_2 = &\left\{\Sigma\left[\dot{a}_{nm12}\frac{n(n+1)}{k_2 r} h_n P_n^m C_m + a_{nm22}\frac{n(n+1)}{k_2 r} h_n P_n^m S_n\right]\right\}\hat{r} \\ &+\left\{\Sigma\left[a_{nm11}\left(-m\,h_n \Pi_n^m S_m\right) + a_{nm21}\left(m\,h_n \Pi_n^m C_m\right)\right.\right. \\ &\quad\left.\left. + a_{nm12}\left(\mathcal{H}_n \tau_n^m C_m\right) + a_{nm22}\left(\mathcal{H}_n \tau_n^m S_m\right)\right]\right\}\hat{\theta} \\ &+\left\{\Sigma\left[a_{nm11}\left(-h_n \tau_n^m C_m\right) + a_{nm21}\left(-h_n \tau_n^m S_m\right)\right.\right. \\ &\quad\left.\left. + a_{nm12}\left(-m\,\mathcal{H}_n \Pi_n^m S_m\right) + a_{nm22}\left(m\,\mathcal{H}_n \Pi_n^m C_m\right)\right]\right\}\hat{\phi}\,. \end{aligned} \tag{4-2}$$

Figure 4-1 illustrates a classical scattering situation and makes use of conventions in the literature (e.g., Reference 15). In the most general case, the incident field is not polarized and the scatterer has no particular axis of symmetry. The energy of the incident field is scattered into all directions. If it is necessary to study the intensity per unit area of scattered energy on a spherical surface S_r of radius r with origin at the center O of the star S with respect to which the calculations were made, choose a plane $\phi = \phi_0$ through the z-axis. This plane cuts

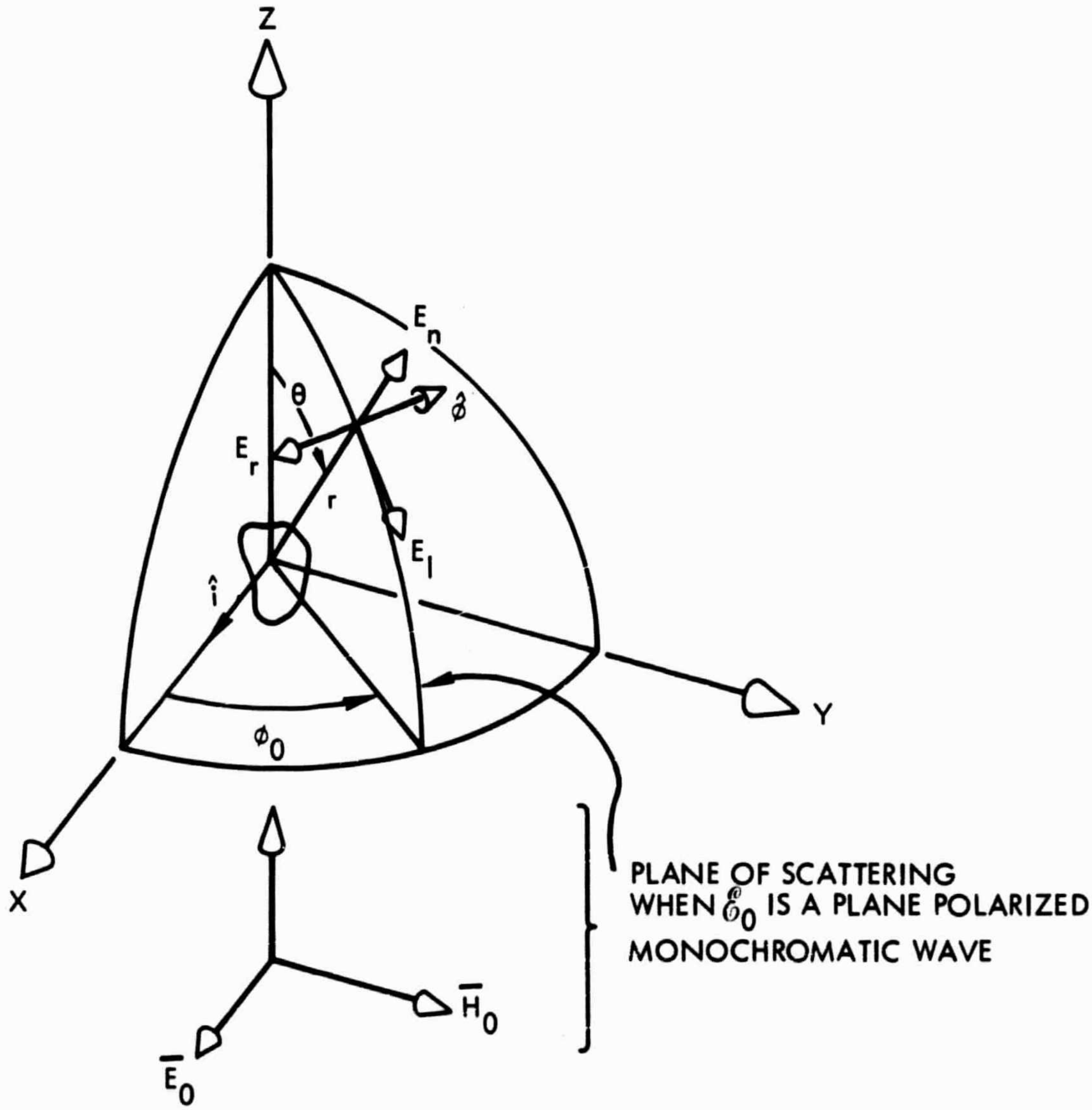

Figure 4-1. Formation of the Far-Field Scattering Functions

the sphere S_r on a circle. If an experimental apparatus is arranged so that an intensity measuring instrument with polarizers measures the intensity associated with

E_ℓ; the component of $\bar{E}_2$ parallel to the plane of polarization and in the direction $\hat{\theta}$

and

E_r, the component of $\bar{E}_2$ perpendicular to the plane of polarization and in the direction $(-\hat{\phi})$.

Then from (4-2),

$$E_\ell = \sum \Big[S_m \left(- m\, a_{nm11}\, h_n\, \Pi_n^m + a_{nm22}\, \mathscr{H}_n\, \tau_n^m \right) + C_m \left(m\, a_{nm21}\, h_n\, \Pi_n^m + a_{nm12}\, \mathscr{H}_n\, \tau_n^m \right) \Big] \tag{4-3}$$

and

$$E_r = -\sum \Big[C_m \left(m\, a_{nm22}\, \mathscr{H}_n\, \Pi_n^m - a_{nm11}\, h_n\, \tau_n^m \right) - S_m \left(a_{nm21}\, h_n\, \tau_n^m + m\, a_{nm12}\, \mathscr{H}_n\, \Pi_n^m \right) \Big] . \tag{4-4}$$

The asymptotic formulae

$$\left. \begin{aligned} hn\,(k_2 r) &\sim \frac{(-i)^n\, e^{ik_2 r}}{ik_2 r} \\ \mathscr{H}n\,(k_2 r) &\sim i\, \frac{(-i)^n\, e^{ik_2 r}}{ik_2 r} \end{aligned} \right\} r \to \infty \tag{4-5}$$

are easily derived from a well-known asymptotic expression for $H_\nu(z)$ (See the appendix). It follows that, for large r,

$$E_\ell \sim \frac{e^{ik_2 r}}{ik_2 r} \sum \left\{ (-i)^n \left[S_m \left(-m\, a_{nm11}\, \Pi_n^m + i\, a_{nm22}\, \tau_n^m \right) + C_m \left(m\, a_{nm21}\, \Pi_n^m + i\, a_{nm12}\, \tau_n^m \right) \right] \right\} \tag{4-6}$$

and

$$E_r \sim -\frac{e^{ik_2 r}}{ik_2 r} \sum \left\{ (-i)^n \left[C_m \left(i\, m\, a_{nm22}\, \Pi_n^m - a_{nm11}\, \tau_n^m \right) - S_m \left(a_{nm21}\, \tau_n^m + i\, m\, a_{nm12}\, \Pi_n^m \right) \right] \right\} . \tag{4-7}$$

The intensities associated with E_r and E_ℓ are

$$\left.\begin{aligned} \left|E_r\right|^2 &\sim \left(\frac{1}{k_2 r}\right)^2 \left| S_1 (\theta, \phi_0) \right|^2 \\ \left|E_\ell\right|^2 &\sim \left(\frac{1}{k_2 r}\right)^2 \left| S_2 (\theta, \phi_0) \right|^2 \end{aligned}\right\} \tag{4-8}$$

where

$$\left.\begin{aligned} S_2(\theta, \phi) &= \sum \left\{ (-i)^n \left[S_m \left(-m\, a_{nm11}\, \Pi_n^m + i\, a_{nm22}\, \tau_n^m \right) + C_m \left(m\, a_{nm21}\, \Pi_n^m + i\, a_{nm12}\, \tau_n^m \right) \right] \middle| (n, m) \epsilon\ \mathscr{Y}_N \right\} \\ S_1(\theta, \phi) &= -\sum \left\{ (-i)^n \left[C_m \left(i\, m\, a_{nm22}\, \Pi_n^m - a_{nm11}\, \tau_n^m \right) - S_m \left(a_{nm21}\, \tau_n^m + i\, m\, a_{nm12}\, \Pi_n^m \right) \right] \middle| (n, m) \epsilon\ \mathscr{Y}_N \right\} \end{aligned}\right\} \tag{4-9}$$

in which

$$\mathscr{Y}_N \equiv \mathscr{Y}_{1N} \cup \mathscr{Y}_{2N} . \tag{4-10}$$

The functions $S_1(\theta, \phi)$ and $S_2(\theta, \phi)$ are the ones observed by the instrument mentioned above. It is these functions which are of interest to most researchers dealing with classical scattering problems.

Another quantity of interest is the amount of energy radiated to infinity per unit time. This number has already been calculated in Section 1.10 of Volume II. The result will be stated here in the context of the notation of (3-3). If the scattering problem $\mathscr{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathscr{E}_0)$ has been solved exactly to obtain the sequence

$$\left\{ a_{\lambda j} \mid (\lambda, j) \in (\Lambda_1 \cup \Lambda_2) \times \langle 4 \rangle \right\} \tag{4-11}$$

for the solution fields

$$\left.\begin{aligned} \mathscr{E}_2 &= \sum \left\{ \left(a_{\lambda 1} \mathscr{E}_\lambda^{2E} + a_{\lambda 2} \mathscr{E}_\lambda^{2H} \right) \mid \lambda \in \Lambda_1 \cup \Lambda_2 \right\} \\ \mathscr{E}_1 &= \sum \left\{ \left(a_{\lambda 3} \mathscr{E}_\lambda^{1E} + a_{\lambda 4} \mathscr{E}_\lambda^{1H} \right) \mid \lambda \in \Lambda_1 \cup \Lambda_2 \right\}, \end{aligned}\right\} \tag{4-12}$$

then the energy radiated outwards by $\mathscr{E}_2$ per unit time through the spherical surface S_r of Figure 4-1 is

$$W_s(r) = \frac{1}{2} \operatorname{Re} \int_{S_r} \hat{r} \cdot (\bar{E}_2 \times \bar{H}_2^*) \, da \quad . \tag{4-13}$$

It was shown in Section 1.10 of Volume II that $W_s(r)$ is a constant W_s with respect to r and that its value is

$$W_s = \Pi \frac{1}{k_2} \sqrt{\frac{\epsilon_2}{\mu_2}} \sum \left\{ \frac{n(n+1)}{2n+1} \frac{(n+m)!}{(n-m)!} \left[|a_{\lambda 1}|^2 + |a_{\lambda 2}|^2 \right] \mid \lambda \in (n, m, i) \in (\Lambda_1 \cup \Lambda_2) \right\} \tag{4-14}$$

If the incident field is of the form

$$\mathscr{E}_0 = \begin{bmatrix} \bar{E}_0 \\ \\ \bar{H}_0 \end{bmatrix} = \begin{bmatrix} \hat{i}\, \bar{E}_0 \exp[ik_2\ r \cos\theta] \\ \\ \\ \dfrac{-i}{\tilde{\omega}\mu_2}\ \nabla \times \bar{E}_0 \end{bmatrix}, \tag{4-15}$$

the mean energy flow of the incident wave per unit area* is

$$S_0 = \frac{1}{2} E_0^{\ 2} \sqrt{\frac{\epsilon_2}{\mu_2}}. \tag{4-16}$$

The ratio

$$\left(\frac{W_s}{S_0}\right) \tag{4-17}$$

is known as the scattering cross section of the scattering target. The corresponding expressions specialize to those in Stratton (Reference 16, page 569) when S is a sphere, if proper account is taken of the varying notations for the coefficients $a_{\lambda j}$.

*Stratton (Reference 16, page 569)

5. A CLASS OF SPECIAL CASES

The purpose of this section is to display an interesting and useful three-parameter family $\mathscr{F}(\tilde{\omega}, \bar{c}_1, \bar{c}_2)$ of scattering problems $\mathscr{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathscr{E}_0)$ for fixed $\tilde{\omega}, \bar{c}_1, \bar{c}_2$. Every member of the family involves the incident field

$$\mathscr{E}_0 = \begin{bmatrix} \overline{E}_0 \\ \overline{H}_0 \end{bmatrix} = \begin{bmatrix} \hat{i} \quad \exp(i k_2 r \cos\theta) \\ \dfrac{-1}{\tilde{\omega}\mu_2} \nabla \times \overline{E}_0 \end{bmatrix} \tag{5-1}$$

(plane polarized monochromatic waves - Figure 4-1). The elements of $\mathscr{F}(\tilde{\omega}, \bar{c}_1, \bar{c}_2)$ involve stars S which are elements of a three-parameter family $\mathscr{U}$ of surfaces. The family $\mathscr{U}$ contains the spherical surfaces and also the spheroidal surfaces as special cases. The analytical representation $r = f$ for elements S of $\mathscr{U}$ varies smoothly with changes in the three parameters and all geometrical quantities involving f may be computed in closed form as smooth functions again involving these three parameters. Finally, the algorithm described in Sections 2 and 3 for any $\mathscr{P} \in \mathscr{F}(\tilde{\omega}, \bar{c}_1, \bar{c}_2)$ requires only the functions $j_n(k_1 f)$, $h_n(k_2 f)$, P_n^1, π_n^1, τ_n^1, i. e., those required for the computation of the classical scattering problem with spherical boundary. These features of the class $\mathscr{F}(\tilde{\omega}, \bar{c}_1, \bar{c}_2)$ recommend it as an ideal testing ground for general-purpose program embodying the ideas of Section 3.

The elements S of $\mathscr{U}$ are superellipses* of revolution

$$\left(\frac{x}{b}\right)^{\beta} + \left(\frac{z}{a}\right)^{\beta} = 1. \tag{5-2}$$

Figure 5-1 shows a family of superellipses for various values of the parameter β. When $\beta = 2$, the superellipse is an ellipse. As β goes

*The superellipse was introduced recently by the Danish writer and inventor Piet Hein, who noted its useful architectural properties (Ref. 17).

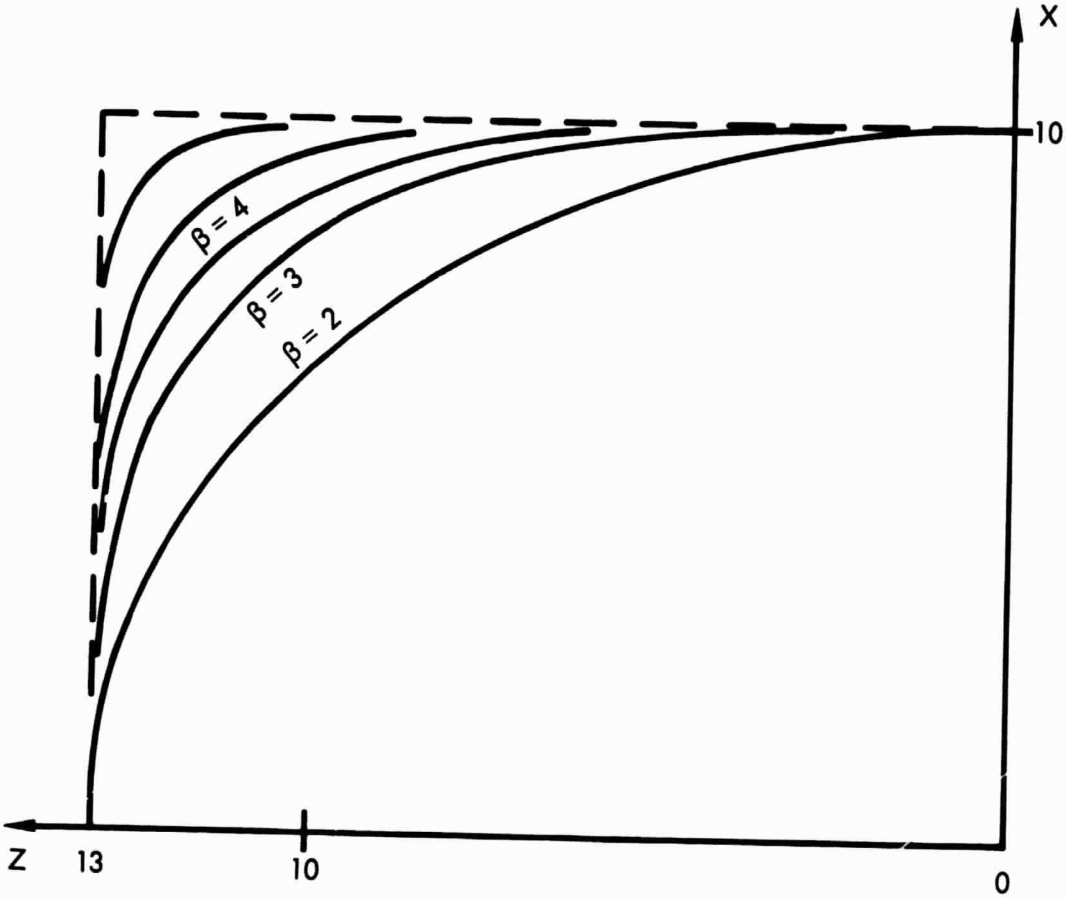

Figure 5-1. A Family of Superellipses for which a = 13, b = 10

from 2 to ∞, the curves approach closer and closer to the rectangle with the corner at $z = b$ and $x = a$. When $\beta = 2$ and $a = b$, the superellipse is a circle. The star S given by (5-2) may be described in the more usual manner by

$$r = f(\theta, \phi) \equiv g(\theta) = a \left[\frac{1}{(\cos\theta)^{\beta} + \left(\frac{a}{b}\right)^{\beta} (\sin\theta)^{\beta}} \right]^{1/\beta} . \tag{5-3}$$

The unit base vectors $\hat{n}$, $\hat{\xi}$, $\hat{\zeta}$ on S are given by

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = (a_{ij}(\theta)) \begin{bmatrix} \hat{n} \\ \hat{\xi} \\ \hat{\zeta} \end{bmatrix} \tag{5-4}$$

where

$$\left.\begin{aligned} & a_{11}(\theta) = a_{22}(\theta) \\ & a_{12}(\theta) = -a_{21}(\theta) \\ & a_{ij}(\theta) \equiv 0 \end{aligned}\right\} \text{for } (i, j) \in \{(3,1), (3,2), (1,3), (2,3)\} \tag{5-5}$$

and

$$\left.\begin{aligned} & a_{33} = 1 \\ & a_{11} = \frac{g}{\sqrt{g^2 + g_\theta^2}} \\ & a_{12} = \frac{g_\theta}{\sqrt{g^2 + g_\theta^2}} . \end{aligned}\right\} \tag{5-6}$$

Figure 5-2 illustrates the geometrical concepts. It is clear that once a, $\left(\frac{a}{b}\right)$, and β are given, a particular element of $\mathscr{U}$ is given by (5-3). For purposes of normalization of the elements of $\mathscr{F}(\tilde{\omega}, \bar{c}_1, \bar{c}_2)$, it is best to choose the dimensionless parameters

$$\left.\begin{aligned} \alpha_0 &\equiv k_2 a = \frac{2\pi}{\lambda} a \\ x &\equiv \frac{a}{b} \\ \beta &\in (2,\infty) \end{aligned}\right\} \qquad (5\text{-}7)$$

for specifying a particular problem of the family.

If $\mathscr{P}$ is any element of $\mathscr{F}(\tilde{\omega}, \bar{c}_1, \bar{c}_2)$, then a calculation very similar to that in Section 4. 2. 2 will reveal that the only $\xi_{\lambda j}$ required for the algorithm are

$$\{\xi_\nu \mid \nu = (\lambda, j) \in \Lambda_4\} \qquad (5\text{-}8)$$

where (See Section 4. 22 of Volume II in particular (4-46)).

$$\Lambda_4 = \{(n,1,2,1),\ (n,1,1,2),\ (n,2,1,3),\ (n,1,1,4)\}_{n=1}^{\infty} \qquad (5\text{-}9)$$

This index set is precisely that of the non-zero scattering coefficients for the solution of the classical scattering problem with S a sphere.

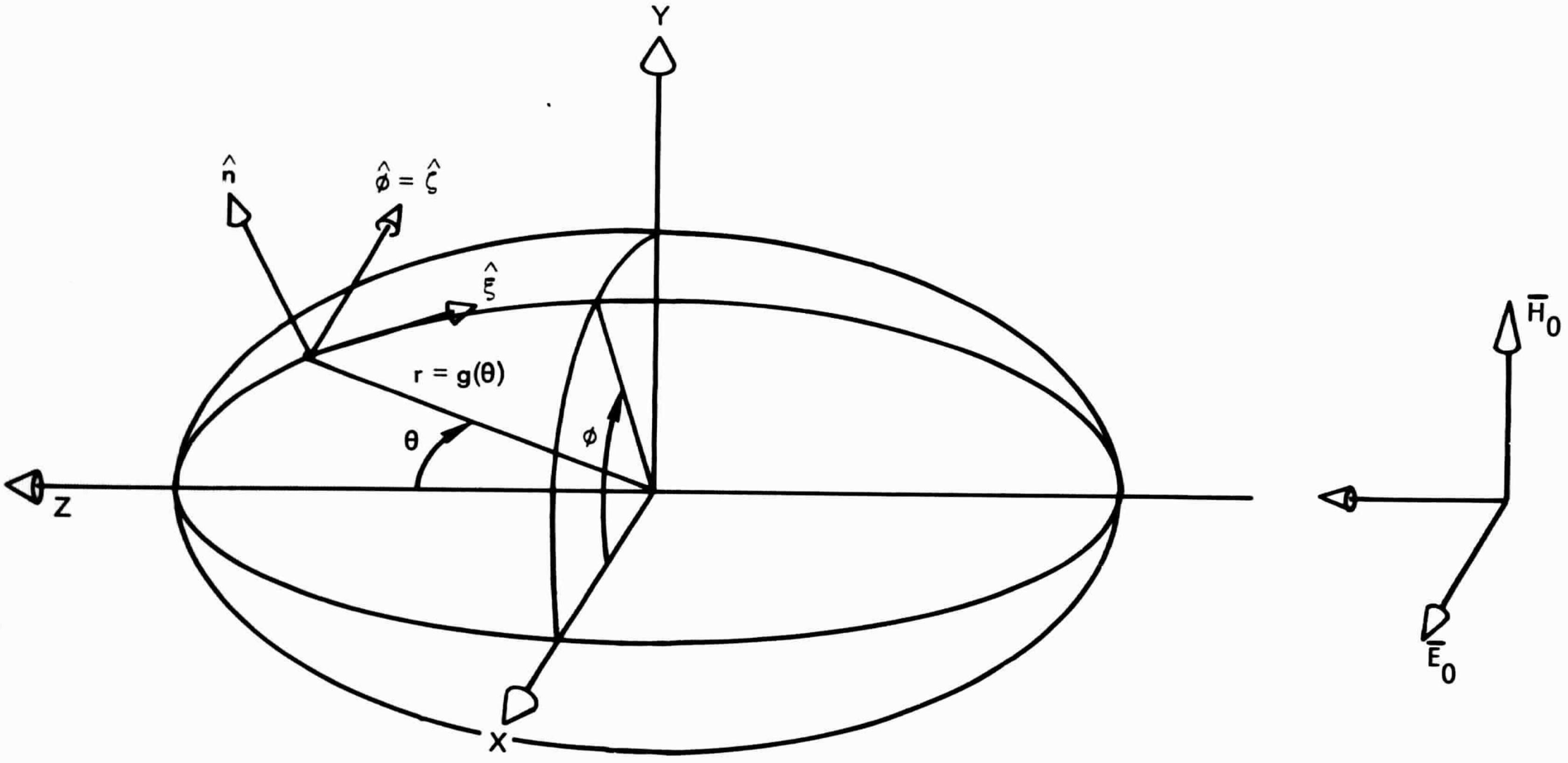

Figure 5-2. Plane Wave Symmetrically Incident Upon Superspheroid

APPENDIX

The formula

$$h_n(z) = i^{-n-1} z^{-1} e^{iz} \sum_{k=0}^{n} \frac{(n+k)!}{k!\Gamma(n+k)} (-2iz)^{-k} \qquad \text{(A-1)}$$

is found in NBS (Reference 18, p. 439). It may be derived from the asymptotic series (Reference 19, p. 197)

$$H_{\nu}^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \exp\left[i\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\right] \left[\sum_{k=0}^{m} \frac{\left(\frac{1}{2}-\nu\right)_k \left(\frac{1}{2}+\nu\right)_k}{k!} (2zi)^{-k} + O\left(|z|^{-m-1}\right)\right] \qquad \text{(A-2)}$$

for

$$\mathrm{Re}\,\nu > -\frac{1}{2} \text{ and } \quad -\pi + \delta \le \arg z \le 2\pi - \delta \qquad \text{(A-3)}$$

and where $(z)_k$ is Pochhammer's symbol

$$(z)_0 = 1 \quad (z)_k = \frac{\Gamma(z+k)}{\Gamma(z)} \qquad \text{(A-4)}$$

truncates at k=n where $\nu \equiv n + \frac{1}{2}$. It follows from (A-1) that

$$\left.\begin{array}{l} h_n(z) \sim (-i)^n \dfrac{e^{iz}}{iz} \\ |z| \to \infty \\ -\pi+\delta < \arg z \le 2\pi-\delta \,. \end{array}\right\} \qquad \text{(A-5)}$$

From this and

$$h_n'(z) = \frac{1}{2n+1} \left[n\, h_{n-1} - (n+1)\, h_{n+1}\right] \qquad \text{(A-6)}$$

it is easy to show that

$$h_n'(z) \sim (-i)^n \frac{e^{iz}}{z} \qquad \text{(A-7)}$$

and also

$$\mathscr{H}_n(z) \equiv \frac{1}{z} h_n(z) + h_n'(z) \sim i\,(-i)^n \frac{e^{iz}}{iz}\,. \qquad \text{(A-8)}$$

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