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APOLLO SPACECRAFT SYSTEMS ANALYSIS PROGRAM

A CLASS OF ELECTROMAGNETIC SCATTERING
PROBLEMS SOLVABLE BY DEBYE'S REPRESENTATION

NAS 9-8166

TASK E-34D

FEBRUARY 14, 1969

Prepared for
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MANNED SPACECRAFT CENTER
HOUSTON, TEXAS



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ABSTRACT

This report is the second in a series of four volumes devoted to certain classes of electromagnetic scattering problems. The general program of work is aimed at providing a tractable computer program for calculating the power scattered by a large, homogeneous body bounded by a smooth, undulated surface when it is illuminated by an incident field of finite extent. Such a model would provide the expected doppler spectrum for the velocity landing radar on the lunar module for certain deterministically rough lunar landing sites. The techniques used, however, are applicable to a very wide class of problems; therefore, it is desirable to maintain generality as long as possible. Thus, the first reports of the series may (deceptively) appear to be . . . "too abstract to be useful."

Volume I gave a construction by Debye's method of certain classes of electromagnetic fields with strongly convergent series expansions in terms of the well known $\bar{m}$, $\bar{n}$ fields of Stratton. This report (Volume II) is motivated by the question: Which scattering problems may be solved using these classes of electromagnetic fields? The question may be mathematically formulated as a problem involving a certain linear and continuous operator (called the solvability operator) in a suitable Hilbert space. The answer appears in terms of the dimension of the range of the solvability operator. The dimension of the range is denumerably infinite. This result suggests that a large class of scattering problems might be solvable by the method described here. It will be shown that any classical spherical scattering problem with non-zero inner conductivity, zero outer conductivity, and a complex refractive index greater than 1 in modulus is a special case of problems solvable by using Debye's construction. Volume III of the series will be devoted to the construction of solutions and demonstration of test cases. Volume IV will apply the ideas and programs to the lunar reflectivity problem mentioned above. The method proposed for constructing solutions is a straightforward application of the theoretical developments in the first two volumes.

CONTENTS

	Page
1. ELECTROMAGNETIC SCATTERING PROBLEMS	1-1
1.1 Introduction	1-1
1.2 Maxwell's Equations	1-2
1.3 Equation of Continuity	1-3
1.4 Divergence Equations	1-3
1.5 Constitutive Relations	1-4
1.6 Relaxation Time	1-5
1.7 Postulates for the Electromagnetic Field	1-6
1.8 Time-Harmonic Fields	1-7
1.9 The Classes $\mathcal{I}_{2a} \mathcal{I}'(\tilde{\omega}, \tilde{c}_2)$ and $\mathcal{I}_{1b} \mathcal{I}'(\tilde{\omega}, \tilde{c}_1)$	1-14
1.10 Energy Radiated to Infinity by the $\mathcal{I}_{2a} \mathcal{I}'(\tilde{\omega}, \tilde{c}_2)$ Fields	1-25
1.11 Stars	1-28
1.12 Statement of the Scattering Problem	1-34
2. UNIQUENESS	2-1
3. SOLVABILITY	3-1
3.1 The General Solvability Operator	3-1
3.2 The Solvability Operator	3-5
3.3 The Range of T_0 is Large	3-7
3.4 Action of T_0 in a Basis for $\mathcal{X}$	3-8
3.5 Solvability by Debye's Method	3-13
4. THE CLASSIC SPHERICAL PROBLEM IS SOLVABLE	4-1
4.1 Stratton's Treatment of the Classical Problem	4-1
4.1.1 Critique of the Derivation of the Vector Plane Wave Expansion	4-2
4.1.2 Critique of the Method Used to Fit Boundary Conditions	4-3
4.1.3 Critique of the Formally-Obtained Solution	4-7
4.2 Justification of the Formal Solution	4-7
4.2.1 Direct Proof of Plane Wave Expansion	4-7
4.2.2 Interpretation of the Classical Method in in the Light of the General Method	4-14

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CONTENTS (Continued)

	Page
4.2.3 The Scattering Coefficients are Well Defined	4-21
4.2.4 The Classical Problem is Solvable	4-24
APPENDIXES	
A PROPERTIES OF THE FIELDS $\bar{m}_\lambda^{(L)}, \bar{n}_\lambda^{(L)}$	A-1
B CALCULATION OF CERTAIN INTEGRALS	B-1
SYMBOLS	S-1
REFERENCES	R-1

TABLES

		Page
3-1	Action of Solvability Operator on a Complete Orthogonal Set	3-17
4-1	Plane Wave Scattered by Sphere	4-5
A-1	Orthogonality Properties of $\bar{m}_\lambda^{(1)}(\xi, \omega)$, $\bar{n}_\lambda^{(1)}(\xi, \omega)$	A-6
A-2	Orthogonality Properties of $\bar{m}_\lambda^{(2)}(\xi, \omega)$, $\bar{n}_\lambda^{(2)}(\xi, \omega)$	A-8

ILLUSTRATIONS

		Page
1-1	Location of Complex Propagation Constants	1-11
1-2	Geometrical Concepts Used in Section 1.9	1-15
1-3	Index Sets Used for Sequences $\bar{m}_{mni}$ and $\bar{n}_{mni}$	1-17
1-4	Schematic for Definitions	1-19
1-5	Geometrical Concepts Associated with Stars	1-33
1-6	Geometrical Concepts Associated with the Problem P ($S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \xi_0$)	1-35
1-7	Choice of Auxiliary Domains for a Given Scattering Problem	1-38
1-8	Situation of the Class of Outer Fields $\mathcal{I}_{2a} \mathcal{L}(\tilde{\omega}, \bar{c}_2)$ Within the Class of Fields Satisfying the Finite Energy Radiation Property	1-40
2-1	Geometrical Concepts Used in the Proof of the Uniqueness Theorem	2-3
2-2	Domains Used to Obtain the Energy Balance Integral Identity	2-5
2-3	Domains Used to Find the Cartesian Components of β	2-11
2-4	Domain of Holomorphy in the τ Plane of the Cartesian Components of $\bar{E}_2(\tau, \phi)$ for any $\phi \in (0, 2\pi)$	2-13

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PREFACE

One of the weak links in the mathematical models proposed for assessing the lunar module landing radar (LM LR) is the method of computing the power returned to the radar by the moon's surface. An attempt has been made to provide an analytical formulation of an electromagnetic theory model with a measure of the accuracy with which the model may be numerically implemented. In this model, the incident electromagnetic field, the shape of the lunar surface, and the electromagnetic parameters of the moon are given. It is necessary to compute the power return by invoking Maxwell's equations and the well-known boundary conditions at the lunar surface. This report is Volume II in a series of four reports. The first two volumes contain the results of a purely mathematical investigation on the applicability of a certain method to obtain the numerical solutions; the final two volumes deal with construction of algorithms, test cases, and suggested applications — among them the lunar reflectivity model.

In the lunar reflectivity problem, the LM landing radar is located at a given point above the lunar surface. The radar transmits an electromagnetic field of given description which strikes moon but does not illuminate all of the lunar surface. This incident field excites elements of charge and current which, in turn, act as sources of a secondary field, part of which is reflected (scattered) back to the radar. A part of the total power scattered is available at the radar. It is necessary to calculate the total power returned to the radar, and also the so-called power spectrum; i. e., the energy which at the radar unit crosses in unit time an element of unit area at a given orientation with respect to the lunar surface. The assumptions are as follows. The moon is assumed to be a linear homogenous isotropic medium of given electromagnetic parameters of which the conductivity must be positive. The conductivity of free space is assumed to be zero. The lunar surface is assumed to be a sufficiently smooth, though not necessary spherical, surface. The topography is assumed to be known at least in the area illuminated by the incident radar beam. It is also assumed that the radar unit does not itself rescatter the energy it receives from the scattered field nor does it affect the scattered field in any way. Subject to these assumptions, the answer must be in accord with electromagnetic theory. This may be regarded as being the mathematical theory

of Maxwell's partial differential equations, the solutions of which are required to behave in a prescribed manner at interfaces between media of differing electromagnetic constants and in regions of space far from the sources of the fields. Viewed in this context, the lunar reflectivity problem is a boundary value problem or more specifically, an electromagnetic scattering problem. In its general context, the problem has applications in radar meteorology, astronomy, and colloid chemistry. (See van de Hulst (Reference 1) and Kerker (Reference 2).)

One scattering problem already considered is that of the plane polarized monochromatic wave incident upon a spherical scatterer. The solution was given independently by several sources* although it is usually attributed to Mie (Reference 3). The solution was actually an algorithm, and one formidable enough that its systematic numerical exploration had to await the development of fast digital computers. Although, from the purely mathematical standpoint, Mie's solution is only formal, programs have been prepared in recent years to tabulate the solution. (See, for example, References 4, 5, and 6). Comparisons with experiments (for example, References 2 and 7) have fostered the conviction that the results are valid. Stratton's interpretation of the technique suggests the pattern for generalizations and for setting the theory on a rigorous mathematical foundation (Reference 8, pages 563-570). The general investigation to be described in this series of reports arose partly in response to this suggestion. In the course of this investigation, it will be shown (in the present report) that Mie's solution is mathematically valid, at least for an interesting range of scattering parameters.

The above method of solution calls for using series of fields $\bar{m}_{imn}$, $\bar{n}_{imn}$ (Reference 3, page 416) with constant coefficients to be determined so that the boundary conditions are satisfied. The fields $\bar{m}$, $\bar{n}$ are constructed by the application of suitable operators to appropriate classes of solutions which arise when Helmholtz's scalar equation is separated into spherical coordinates. These are the spherical Bessel and Hankel functions and the spherical harmonics -- all functions that have been extensively

* See N. Logan "Early History of the Mie Solution," Jour. Opt. Soc. Amer. 53, p. 342, 343.

studied and tabulated. The construction of the fields $\overline{m}$, $\overline{n}$ is due essentially to Peter Debye (Reference 9). (C. H. Wilcox in Reference 10, pages 167-171 gives a history of the subject.) A rigorous survey of the method with generalizations appropriate to this report is contained in a previous report (Reference 11).

The motivating question for the present mathematical investigation was: What classes of scattering problems may be solved by the use of the above method? To state the question precisely, it was necessary to define "scattering problem" precisely. This is presented in Section 1. Sections 2 and 3 deal with uniqueness and existence theorems, respectively. In this report, these theorems appear as "uniqueness and existence theorems relative to the above method." In Section 4, it will be shown that the class A of problems which may be solved by this method contains an interesting class of spherical scattering problems and hence the class A is not void. The answer to the motivating question in a more general case is related to the dimension of the range of a certain linear operator (the solvability operator) whose domain is a certain Hilbert space. The range contains a denumerably infinite, linearly independent set which may be readily exhibited.

The linearly independent set alluded to above may be orthogonalized by the application of the Gram-Schmidt process to form a set which spans the same linear manifold as the original set. This statement is the basis for a straightforward algorithm described in Volume III of this series, "Construction of General Programs for the Numerical Solution of Certain Electromagnetic Scattering Problems" (Reference 12). Applications to the lunar reflectivity study will be discussed in the fourth report of the series, "Electromagnetic Scattering of Waves of Finite Extent Incident Upon a Smooth, Undulated, Extended Surface" (Reference 13).

In describing the mathematical applicability investigation of the present report, it was necessary to rely on many details in the classical theory, the introduction of which would have necessitated innumerable digressions or a lengthy appendix. Volume I of the series was meant to include all material of this type and to give all the background and special results needed for this volume. Volume I is "Debye's Construction of Vector Wave Functions from Scalar Wave Functions" (Reference 11). Notations,

definitions and theorems established in Volume I will be used in this report (Volume II) freely and sometimes without further reference. A list of symbols is provided for reference. Volumes I, II, III, and IV, taken in that order, give a thorough and logically organized exposition of the investigation.

1. ELECTROMAGNETIC SCATTERING PROBLEMS

1.1 INTRODUCTION

Electromagnetic scattering problems are boundary value problems that involve Maxwell's equations. In general, a boundary value problem is said to be well-set if it defines one, and only one, solution, and if the solution "varies continuously with variations in the data." Reference to standard texts such as Stratton (Reference 8), Panofsky and Phillips (Reference 14), and Smythe (Reference 15), and to the literature on the subject shows that the type of scattering problem to be dealt with here is not rigorously stated anywhere and that well-setness propositions are not available. The present report is intended to provide what may be called "well-setness propositions relative to a given method." The problems involve time-harmonic, electromagnetic fields in linear homogeneous isotropic media, with the incident (driving) field being given in a medium of zero conductivity and the scatterer being bounded by a smooth, star-like surface, the interior of which has positive conductivity. It shall not be assumed that the scatterer is a perfect conductor, nor that the incident field is a plane-polarized wave of infinite extent. All fields will be assumed to be time-harmonic. In view of the purely mathematical nature of this investigation and of the disorganized state of affairs regarding the well-setness proposition, the statement of the problem cannot be referred to previous treatments, nor to the suggestive power of picturesque phrases such as those used above. The main purpose of Section 1 is to give a careful mathematical statement of the problem, and one which assumes no knowledge of physics or of electromagnetic theory. A secondary aim of Section 1 is to codify and simplify notations, without which most subsequent calculations would bristle with unnecessary symbols and the results would be overloaded with repetitive and lengthy locutions.

Terms like "Maxwell's equations," "electromagnetic field," "constitutive relations," "linear homogenous isotropic media," etc., are all defined in Sections 1.2 through 1.8. Section 1.9 summarizes important ideas from Volume I of this series (Reference 11). Briefly, the objective of Volume I was to show how to construct electromagnetic fields on domains

of the form

$$D_{1a} = \left\{ \bar{\mathbf{r}} \mid |\bar{\mathbf{r}}| < a \right\}$$

or

$$D_{2a} = \left\{ \bar{\mathbf{r}} \mid |\bar{\mathbf{r}}| > a \right\}$$

from pairs (v_1, v_2) of elements in $\mathcal{J}'^2(\Omega)$. The resulting fields were shown to possess strongly convergent series expansions involving the vector fields $\bar{\mathbf{m}}, \bar{\mathbf{n}}$, (Stratton, Reference 8, page 416). One important property of fields for D_{2a} constructed by this process is demonstrated in Section 1.10. Its physical interpretation is that the "energy carried to infinity by the outer field per unit time" is finite (and non-negative). The domains on which electromagnetic fields are to be considered are bounded by surfaces called stars, the definition and properties of which are given in Section 1.11. The results given may be generalized to a wider class of surfaces without requiring basic emmendations in proof. Such a generalization, however, requires attention to differential geometry and will be reserved for later investigations. The statement of the electromagnetic scattering problem is given in Section 1.12. Several papers dealing with the proposed construction of solutions for this problem are mentioned in Section 1.12. These usually assume the well-setness theorem, as well as representation theorems for the desired solutions and vary in the degree of mathematical justification provided.

1.2 MAXWELL'S EQUATIONS

The experimental researches of Coulomb, Ampere, and Faraday led Maxwell to the belief that all effects pertaining to macroscopic electric and magnetic phenomena could be explained as consequences of the equations

$$\left. \begin{aligned} \nabla \times \tilde{\mathbf{E}} + \dot{\tilde{\mathbf{B}}} &= \tilde{\mathbf{0}} \\ \nabla \times \tilde{\mathbf{H}} - \dot{\tilde{\mathbf{D}}} &= \tilde{\mathbf{J}} \end{aligned} \right\} \quad (1-1)$$

by suitable interpretation of the vector fields $\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\mathbf{D}}, \tilde{\mathbf{B}}, \tilde{\mathbf{J}}$ which satisfy Equation (1-1). It is said that an electromagnetic field exists in a domain

D of E^3 if $\tilde{E}, \tilde{H}$, etc. are each* $C^2(D)$ functions on D into the product space $\mathcal{V}^3$. From the books of Smythe (Reference 15), Panofsky and Phillips (Reference 14), and Stratton (Reference 8), it appears that much of electromagnetic field theory is merely an interpretation of mathematical consequences of Equation (1-1) and that the success of this program to date has justified Maxwell's belief in (1-1). The Equations (1-1) are called Maxwell's equations.

1.3 EQUATION OF CONTINUITY

According to the interpretation of electromagnetic theory, the field $\tilde{J}$ is related to a scalar function ρ by the equation

$$\nabla \cdot [\tilde{J}(\bar{r}, t)] + \frac{\partial}{\partial t} [\rho(\bar{r}, t)] = 0, \quad (\bar{r}, t) \in D \times \mathcal{T}. \quad (1-2)$$

The scalar function $\rho(\bar{r}, t)$ is called the free charge density, and the vector $\tilde{J}$ is called the current density. The equation (1-2) expresses the principle known as "conservation of charge" and is called the equation of continuity.

1.4 DIVERGENCE EQUATIONS

Maxwell's first equation yields

$$\nabla \cdot \dot{\tilde{B}} = \frac{\partial}{\partial t} [\nabla \cdot [\tilde{B}(\bar{r}, t)]] = 0,$$

and hence, at each $\bar{r} \in D$,

$$\nabla \cdot \tilde{B} = f(\bar{r}), \quad \text{independent of time.} \quad (1-3)$$

Similarly, the second Maxwell equation gives

$$\nabla \cdot [\dot{\tilde{D}}(\bar{r}, t) + \tilde{J}(\bar{r}, t)] = 0,$$

i. e.,

$$\frac{\partial}{\partial t} [\nabla \cdot [\tilde{D}(\bar{r}, t)]] + \nabla \cdot [\tilde{J}(\bar{r}, t)] = 0$$

* Sometimes $\mathcal{T} \equiv \mathcal{A}$, is denoted as the set of reals. (Refer to the list of symbols.)

But from the equation of continuity,

$$\frac{\partial}{\partial t} \left| \nabla \cdot [\tilde{D}(\bar{r}, t)] - \rho(\bar{r}, t) \right| = 0$$

and hence, at each $\bar{r} \in D$,

$$\nabla \cdot [\tilde{D}(\bar{r}, t)] = \rho(\bar{r}, t) + g(\bar{r}), \quad \forall (r, t) \in D \times \mathcal{T}. \quad (1-4)$$

The interchange of space and time differentiation operators is justified by the hypothesis that all fields occurring in (1-1) are in $C^2(D \times \mathcal{T}, \mathcal{C}^3)$. In (1-3) and (1-4) it is then postulated that $f(\bar{r}) \equiv g(\bar{r}) \equiv 0 \quad \forall \quad \bar{r} \in D$, so that these equations read

$$\begin{aligned} \nabla \cdot \tilde{B} &= 0 \\ \nabla \cdot \tilde{D} &= \rho. \end{aligned} \quad (1-5)$$

These are called the divergence equations.

1.5 CONSTITUTIVE RELATIONS

If D is a domain of E^3 occupied by matter, and an electromagnetic field $(\tilde{E}, \tilde{H}, \tilde{D}, \tilde{B}, \tilde{J})$ pervades D , the question is: How do the physical properties of the matter occupying D affect the electromagnetic field? For a large variety of materials, the answer is contained in the relations

$$\tilde{D} = \tilde{\epsilon} \tilde{E}, \quad \tilde{B} = \tilde{\mu} \tilde{H}, \quad \tilde{J} = \tilde{\sigma} \tilde{E} \quad (1-6)$$

where $\tilde{\epsilon}$, $\tilde{\mu}$, and $\tilde{\sigma}$ are matrices $(\epsilon_{ij}(\bar{r}))$, $(\mu_{ij}(\bar{r}))$, $(\sigma_{ij}(\bar{r}))$, respectively, of functions which may be experimentally measured or derived from quantum-mechanical theories. In any case, they are characteristic of the matter occupying D .

The relations (1-6) are called constitutive relations. A large variety of materials may be completely characterized for all purposes of electromagnetic theory by relations (1-6) of the form described. These materials are said to be linear.

* $C^n(\mathcal{X}, \mathcal{Y}) \equiv \{f \mid f: \mathcal{X} \rightarrow \mathcal{Y} \text{ and } f \in C^n(\mathcal{X})\}$.

If, in (1-6), all the elements of the matrices $[\epsilon_{ij}(\bar{r})]$, $[\mu_{ij}(\bar{r})]$, $[\sigma_{ij}(\bar{r})]$ are independent of $\bar{r}$ in D , then the material is said to be linear and homogeneous. If, in addition, the matrices are of the special form

$$\epsilon_{ij} = \epsilon \delta_{ij} , \quad \mu_{ij} = \mu \delta_{ij} , \quad \sigma_{ij} = \sigma \delta_{ij} , \quad (1-7)$$

where δ_{ij} is Kronecker's delta, then the material is said to be linear, homogeneous, and isotropic. Many materials are of this type. They are the only materials which will enter this discussion.

For the purposes of this document, then, the constitutive relations take the form

$$\tilde{D} = \epsilon \tilde{E} , \quad \tilde{B} = \mu \tilde{H} , \quad \tilde{J} = \sigma \tilde{E} \quad (1-8)$$

where ϵ , μ , and σ are real constants associated with the matter occupying D . Given ϵ , μ , σ , all mention is dropped of the matter occupying D and the statement is made that the field $(\tilde{E}, \tilde{H}, \tilde{D}, \tilde{B}, \tilde{J})$ pervading D is associated with ϵ , μ , σ , in D . It shall always be assumed

$$(\epsilon, \mu, \sigma) \in \mathcal{V} = \{(\epsilon, \mu, \sigma) | \epsilon > 0, \mu > 0, \sigma \geq 0\} . \quad (1-9)$$

The subsets will be defined

$$\mathcal{V}_2 = \{(\epsilon, \mu, \sigma) | \epsilon > 0, \mu > 0, \sigma = 0\} \quad (1-10)$$

and

$$\mathcal{V}_1 = \{(\epsilon, \mu, \sigma) | \epsilon > 0, \mu > 0, \sigma > 0\} . \quad (1-11)$$

The constants (ϵ, μ, σ) will be called the electromagnetic parameters associated with D . If ϵ , μ , σ are given for D , it can sometimes be said that the electromagnetic triple $\bar{c} = (\epsilon, \mu, \sigma)$ is associated with D .

1.6 RELAXATION TIME

If the domain D is occupied by a linear, homogeneous, isotropic medium with $\sigma > 0$, from the equation of continuity and the constitutive relations it follows that

$$\frac{\partial}{\partial t} \rho(\bar{r}, t) + \left(\frac{\sigma}{\epsilon}\right) \rho(\bar{r}, t) = 0, \quad \forall (\bar{r}, t) \in D \times \mathcal{J}. \quad (1-12)$$

The solution to this differential equation is

$$\rho(\bar{r}, t) = \rho(\bar{r}, 0) \exp \left[-\left(\frac{\sigma}{\epsilon}\right)t \right], \quad \forall (r, t) \in D \times \mathcal{T}. \quad (1-13)$$

This shows that if $\rho(\bar{r}, 0)$ is bounded in a linear, homogeneous, isotropic medium for which $\sigma > 0$, then

$$\lim_{t \rightarrow \infty} \rho(\bar{r}, t) = 0, \quad \forall \bar{r} \in D. \quad (1-14)$$

Since the quantity

$$\left(\frac{\epsilon}{\sigma}\right), \quad (1-15)$$

called the relaxation time of the medium, is usually small, the divergence equations for these media are usually written

$$\left. \begin{array}{l} \nabla \cdot \tilde{B} = 0 \\ \nabla \cdot \tilde{D} = 0 \end{array} \right\} (r, t) \in D \times \mathcal{T}. \quad (1-16)$$

If $\sigma = 0$, it is postulated that $\rho \equiv 0$.

1.7 POSTULATES FOR THE ELECTROMAGNETIC FIELD

The properties of the electromagnetic fields to be considered are summarized into the following two postulates.

Postulate 1.

An electromagnetic field pervading the domain D which is occupied by a linear, homogeneous, isotropic medium of electromagnetic parameters ϵ, μ, σ with

$$\bar{c} \equiv (\epsilon, \mu, \sigma) \in \mathcal{T} \quad (1-17)$$

satisfies the equations

$$\left. \begin{array}{ll} \nabla \times \tilde{E} + \dot{\tilde{B}} = 0 & \nabla \cdot \tilde{B} = 0 \\ \nabla \times \tilde{H} - \dot{\tilde{D}} = \tilde{J} & \nabla \cdot \tilde{D} = 0 \end{array} \right\} \text{ on } D \times \mathcal{T}. \quad (1-18)$$

and also

$$\tilde{D} = \epsilon \tilde{E}, \quad \tilde{B} = \mu \tilde{H}, \quad \tilde{J} = \sigma \tilde{E} \quad (1-19)$$

All fields $\mathcal{M} \equiv (\tilde{E}, \tilde{H}, \tilde{D}, \tilde{B}, \tilde{J}) \equiv (M_1, M_2, M_3, M_4, M_5)^T$ are composed of $M_i \in C^2(D \times \bar{J})$.

If D is occupied by free space, the above equations are used with* (See Stratton, Reference 8 pages 22 and 23)

$$\left. \begin{aligned} \epsilon_0 &= 8.854 \times 10^{-12} \frac{\text{coulomb}^2 \text{ sec}^2}{\text{kgm}_r \text{ m}^3} \\ \mu_0 &= 1.257 \times 10^{-6} \frac{\text{kgm}_r \text{ m}_r}{\text{coulomb}^2} \\ \sigma_0 &= 0. \end{aligned} \right\} \quad (1-20)$$

Postulate 2.

Given a domain D and an element $\bar{c} \in \mathcal{V}$, then any $\mathcal{M} \ni M_i \in C^2(D \times \bar{J}, \mathcal{C}^3)$ and $\mathcal{M}$ satisfies (1-19) and (1-20) is an electromagnetic field associated with $\bar{c}$ on D .

Definition 1.

Given $D \subset E^3$ and $\bar{c} \in \mathcal{V}$, denote by $\mathcal{M}(D, \bar{c})$ the class of all electromagnetic fields associated with $\bar{c}$ on D .

Henceforth, unless otherwise noted, the quantities $\tilde{\omega} \in \mathcal{M}_0$, $\bar{c} = (\epsilon, \mu, \sigma) \in \mathcal{V}$ will be assumed to be given, and the symbol D will stand for a domain in E^3 .

1.8 TIME-HARMONIC FIELDS

The following two theorems show that the time-independent part of time-harmonic electromagnetic fields are vector wave functions (Chapter VI of Volume I) and that, conversely, from vector wave functions, time-harmonic electromagnetic fields can be constructed. The proofs are

*For a discussion of units, see Stratton (Reference 8, pages 16-23).

trivial calculations with Maxwell's equations and the constitutive relations and will not be given.

Theorem 1

Given $\mathcal{F} \in \mathcal{C}^2(D, \bar{C}) \ni$

$$\mathcal{F} = \bar{\mathcal{F}} e^{-i\tilde{\omega}t} \text{ on } D \times \mathcal{J} \quad (1-21a)$$

where

$\bar{\mathcal{F}} = (\bar{E}, \bar{H}, \bar{D}, \bar{B}, \bar{J}) = (F_1, F_2, F_3, F_4, F_5)^T$ are independent of time, (1-21b)
then

$$F_i \in C^2(D, \mathcal{J}^3)$$

and

F_i each satisfy the system

$$\nabla \times \nabla \times \bar{F} - k^2 \bar{F} = 0, \quad \nabla \cdot \bar{F} = 0 \text{ on } D \quad (1-22)$$

where

$$k^2 = \tilde{\omega}^2 \epsilon_\mu + i\sigma\mu\tilde{\omega} \quad (1-23)$$

The fields F_i are related by

$$\left. \begin{aligned} \bar{E} &= \frac{1}{(\sigma - i\tilde{\omega}\epsilon)} \nabla \times \bar{H} \\ \bar{H} &= \frac{-i}{\tilde{\omega}\mu} \nabla \times \bar{E} \end{aligned} \right\} \quad (1-24)$$

and

$$\bar{D} = \epsilon \bar{E}, \quad \bar{B} = \mu \bar{H}, \quad \bar{J} = \sigma \bar{E}. \quad (1-25)$$

The converse is

Theorem 2

Given any $\bar{F}: D \rightarrow \mathcal{J}^3 \ni$

$$\bar{F} \in C^2(D)$$

and

$$\nabla \times \nabla \times \bar{F} - k^2 \bar{F} = 0, \quad \nabla \cdot \bar{F} = 0 \text{ on } D \quad (1-26)$$

where

$$k^2 = \tilde{\omega}^2 \epsilon \mu + i \sigma \mu \tilde{\omega}, \quad (1-27)$$

then the construction

either

$$\bar{\mathbf{E}} = \bar{\mathbf{F}}, \quad \bar{\mathbf{H}} = \frac{-i}{\tilde{\omega} \mu} \nabla \times \bar{\mathbf{F}}, \quad \bar{\mathbf{D}} = \epsilon \bar{\mathbf{E}}, \quad \bar{\mathbf{B}} = \mu \bar{\mathbf{H}}, \quad \bar{\mathbf{J}} = \sigma \bar{\mathbf{E}} \quad (1-28)$$

or

$$\bar{\mathbf{E}} = \frac{1}{(\sigma - i \tilde{\omega} \epsilon)} \nabla \times \bar{\mathbf{H}}, \quad \bar{\mathbf{H}} = \bar{\mathbf{E}}, \quad \bar{\mathbf{D}} = \epsilon \bar{\mathbf{E}}, \quad \bar{\mathbf{B}} = \mu \bar{\mathbf{H}}, \quad \bar{\mathbf{J}} = \sigma \bar{\mathbf{E}} \quad (1-29)$$

and

$$\mathcal{H}_1 = \begin{bmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \\ \bar{\mathbf{D}} \\ \bar{\mathbf{B}} \\ \bar{\mathbf{J}} \end{bmatrix} e^{-i \tilde{\omega} t} \quad \text{on } D \times \tilde{\mathcal{I}} \quad (1-30)$$

defines an element of $\mathcal{H}(D, \bar{c})$.

It is easy to show that

$$\mathcal{H}_i \in \mathcal{H}(D, \bar{c}), \quad i = 1, 2 \Rightarrow (\mathcal{H}_1 + \mathcal{H}_2) \in \mathcal{H}(D, \bar{c}).$$

It follows that if $\bar{\mathbf{F}}_1$ and $\bar{\mathbf{F}}_2$ are any two solutions of (1-28), then the construction

$$\left. \begin{aligned} \bar{\mathbf{E}} &= \bar{\mathbf{F}}_1 + \frac{1}{(\sigma - i \tilde{\omega} \epsilon)} \nabla \times \bar{\mathbf{F}}_2 \\ \bar{\mathbf{H}} &= \frac{-i}{\tilde{\omega} \mu} \nabla \times \bar{\mathbf{F}}_1 + \bar{\mathbf{F}}_2 \\ \bar{\mathbf{D}} &= \epsilon \bar{\mathbf{E}} \\ \bar{\mathbf{B}} &= \mu \bar{\mathbf{H}} \\ \bar{\mathbf{J}} &= \sigma \bar{\mathbf{E}} \end{aligned} \right\} \quad (1-31)$$

and

$$\mathcal{H}_3 = \begin{bmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \\ \bar{\mathbf{D}} \\ \bar{\mathbf{B}} \\ \bar{\mathbf{J}} \end{bmatrix} e^{-i \tilde{\omega} t} \quad (1-32)$$

defines an element $\mathcal{M}_3$ of $\mathcal{M}(D, \bar{c})$. Since $(\nabla \times \bar{F})_i$, $i = 1, 2$ are also solutions of (1-26), it follows by Theorem 1 that $\bar{E}$ and $\bar{H}$ are related by (1-24). $\mathcal{M}(D, \bar{c}, \tilde{\omega})$ denotes the subset of $\mathcal{M}(D, \bar{c})$, every element of which has the form (1). In (1-21b), the vector wave functions $\bar{E}, \bar{H}$, etc. associated with $\tilde{E}, \tilde{H}$, etc. are called the time-independent parts of the electromagnetic field $\mathcal{M}_1$. The vector wave functions $\bar{E}, \bar{H}, \dots$ are themselves often called electromagnetic fields, but this usage will not be followed, reserving the term electromagnetic field for the concept explained in Postulate 1.

The number k of (1-27) will now be uniquely defined by specifying

$$\text{Im}(k) \geq 0. \quad (1-33)$$

Generally,

$$[k(\tilde{\omega}, \bar{c})]^2 \in \mathcal{C}_0 \Rightarrow k(\tilde{\omega}, \bar{c}) \in \mathcal{C}_2,$$

where

$$\mathcal{C}_2 = \{z \mid \text{Re}(z) > 0, \quad 0 \leq \text{Im}(z) < \text{Re}(z)\}. \quad (1-34)$$

The number k is called the propagation constant. It is purely real if, and only if, $\sigma = 0$; i. e.,

$$\bar{c}_2 \in \mathcal{V}_2 \Leftrightarrow k(\tilde{\omega}, \bar{c}_2) \in \mathcal{R}_0 \quad (1-35)$$

Otherwise, in view of its definition,

$$\bar{c}_1 \in \mathcal{V}_1 \Leftrightarrow k(\tilde{\omega}, \bar{c}_1) \in \mathcal{C}_1 \quad (1-36)$$

where

$$\mathcal{C}_1 = \mathcal{C}_2 - \mathcal{R}_0. \quad (1-37)$$

These regions are illustrated in Figure 1-1; k_1 will be used for propagation constants in $\mathcal{C}_1$ and k_2 for propagation constants in $\mathcal{R}_0$.

The time-independent parts of $\mathcal{M}_1 \in \mathcal{M}(D, \bar{c}, \tilde{\omega})$ satisfy

$$\left. \begin{aligned} \nabla \times \bar{E} &= i\tilde{\omega}\mu \bar{H} \\ \nabla \times \bar{H} &= (\sigma - i\tilde{\omega}\epsilon) \bar{E} \end{aligned} \right\} \quad (1-38)$$

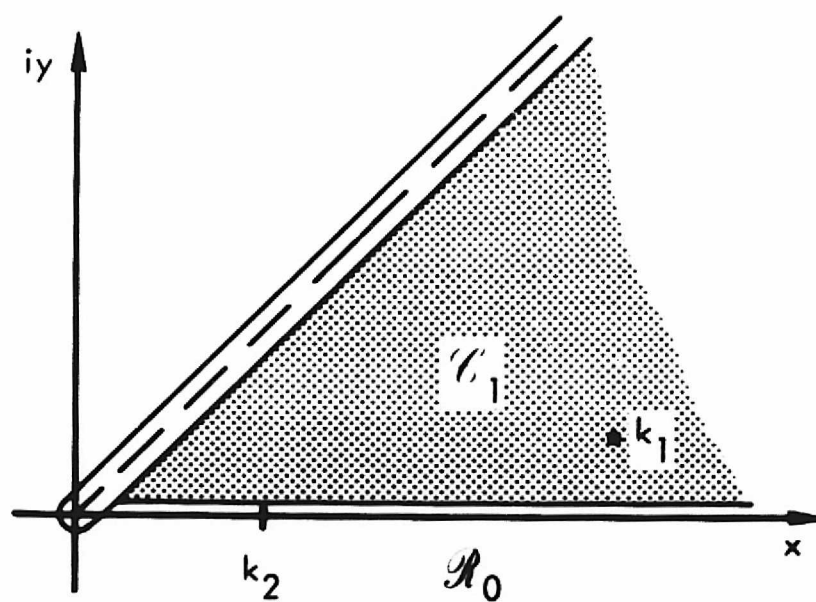
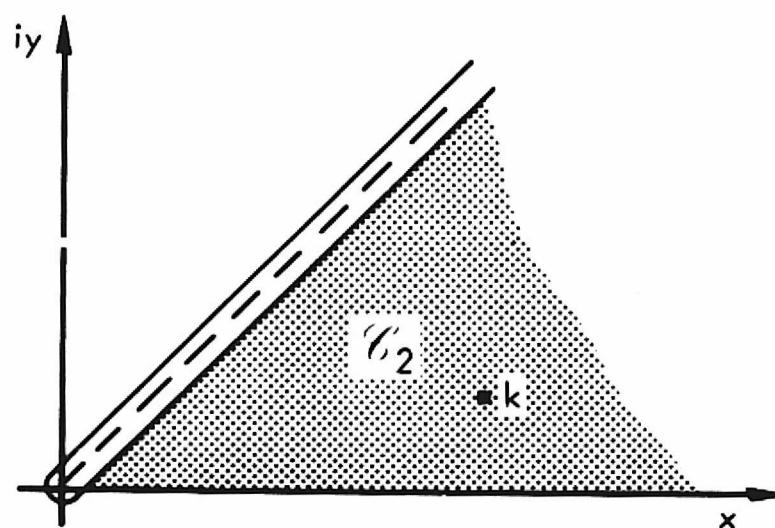
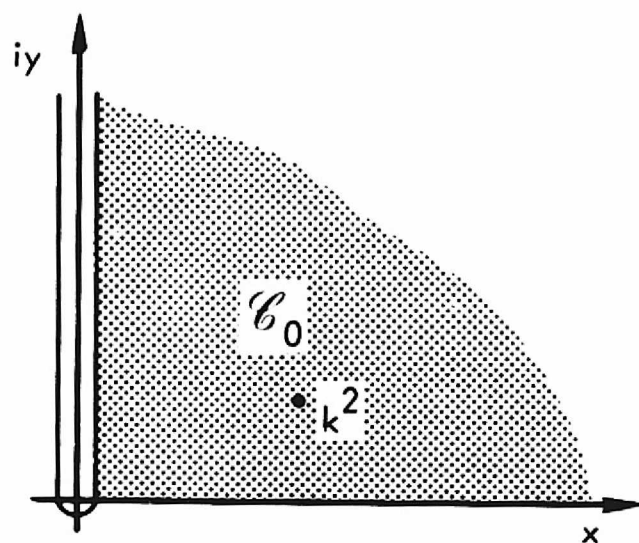


Figure 1-1. Location of Complex Propagation Constants

Vector fields $(\bar{E}, \bar{H})$ which satisfy these equations automatically satisfy the divergence equations. The following theorems give a very symmetrical characterization of time-harmonic fields. Again the proofs are trivial calculations which are omitted.

Theorem 3

Given $\tilde{\omega}$ and $\bar{c}$, let $k = k(\tilde{\omega}, \bar{c})$ and suppose $(\mathcal{A}, \mathcal{B}) \in [C^2(D, \mathcal{C}^3)]^2$ is any solution of the system

$$\left. \begin{aligned} \nabla \times \mathcal{A} &= k \mathcal{B} \\ \nabla \times \mathcal{B} &= k \mathcal{A} \end{aligned} \right\} \quad (1-39)$$

Then the relations

$$\left. \begin{aligned} \bar{E} &= (\tilde{\omega} \epsilon + i \sigma)^{-1/2} \exp(i \frac{\pi}{4}) \mathcal{A}, \quad \text{Im}(\tilde{\omega} \epsilon + i \sigma)^{1/2} \geq 0 \\ \bar{H} &= (\tilde{\omega} \mu)^{-1/2} \exp(-i \frac{\pi}{4}) \mathcal{B} \\ \bar{D} &= \epsilon \bar{E} \\ \bar{B} &= \mu \bar{H} \\ \bar{J} &= \sigma \bar{E} \end{aligned} \right\} \quad (1-40)$$

and

$$\mathcal{M}_1 = [\bar{E}, \bar{H}, \bar{D}, \bar{B}, \bar{J}]^T \exp(-i \tilde{\omega} t) \quad (1-41)$$

determine an element $\mathcal{M}_1$ of $\mathcal{M}(D, \bar{c}, \tilde{\omega})$.

The converse of Theorem 3 is Theorem 4.

Theorem 4

Given $\tilde{\omega}$ and $\bar{c}$, let $k = k(\tilde{\omega}, \bar{c})$ and suppose $(\bar{E}, \bar{H}, \dots)$ are the time independent parts of some element of $\mathcal{M}(D, \bar{c}, \tilde{\omega})$. Then the vector fields $\mathcal{A}$,

$\mathcal{B}$ defined by

$$\left. \begin{aligned} \overline{\mathbf{E}} &= (\tilde{\omega} \epsilon + i\sigma)^{-1/2} \exp(i \frac{\pi}{4}) \mathcal{A} \\ \overline{\mathbf{H}} &= (\tilde{\omega} \mu)^{-1/2} \exp(-i \frac{\pi}{4}) \mathcal{B} \end{aligned} \right\} \quad (1-42)$$

are $C^2(D, \mathcal{C}^3)$ solutions of

$$\left. \begin{aligned} \nabla \times \mathcal{A} &= k \mathcal{B} \\ \nabla \times \mathcal{B} &= k \mathcal{A} \end{aligned} \right\} \quad (1-43)$$

It is easy to show that if $(\overline{\mathbf{E}}, \overline{\mathbf{H}})$ are any two $C^2(D, \mathcal{C}^3)$ solution of (1-38), then both $\overline{\mathbf{E}}$ and $\overline{\mathbf{H}}$ satisfy (1-26). The pair $(\overline{\mathbf{E}}, \overline{\mathbf{H}})$ determines an element of $\mathcal{M}(D, \overline{c}, \tilde{\omega})$ by the process

$$\mathcal{G} \left[\begin{array}{c} \overline{\mathbf{E}} \\ \overline{\mathbf{H}} \end{array} \right] = \left[\begin{array}{c} \overline{\mathbf{E}} \\ \overline{\mathbf{H}} \\ \epsilon \overline{\mathbf{E}} \\ \mu \overline{\mathbf{H}} \\ \sigma \overline{\mathbf{E}} \end{array} \right] \exp(-i\tilde{\omega}t) . \quad (1-44)$$

In using this notation, it is understood that the ϵ, μ, σ and $\tilde{\omega}$ used in the process are precisely those used in the system (1-35) satisfied by $(\overline{\mathbf{E}}, \overline{\mathbf{H}})$.

Denote

$$\begin{aligned} \tilde{\mathcal{E}}(D, \overline{c}, \tilde{\omega}) &= \left\{ \mathcal{E} \mid \mathcal{E} = \left[\begin{array}{c} \overline{\mathbf{E}} \\ \overline{\mathbf{H}} \end{array} \right], \mathcal{E} \in \left[C^2(D, \mathcal{C}^3) \right]^2, \\ &\quad \mathcal{E} \text{ satisfies } \left\{ \begin{array}{l} \nabla \times \overline{\mathbf{E}} = i\tilde{\omega} \mu \overline{\mathbf{H}} \\ \nabla \times \overline{\mathbf{H}} = (\sigma - i\tilde{\omega} \epsilon) \overline{\mathbf{E}} \end{array} \right\} \right\} . \end{aligned} \quad (1-45)$$

It should be noted that

$$\mathcal{G}: \tilde{\mathcal{E}}(D, \overline{c}, \tilde{\omega}) \rightarrow \mathcal{M}(D, \overline{c}, \tilde{\omega}) . \quad (1-46)$$

1.9 THE CLASSES $\mathcal{L}_{2a} \mathcal{L}'(\tilde{\omega}, \bar{c}_2)$ and $\mathcal{L}_{1b} \mathcal{L}'(\tilde{\omega}, \bar{c}_1)$

In Volume I, Chapter V (Reference 11), it was shown how a class of solutions to

$$\nabla^2 u + k^2 u = 0 \quad (1-47)$$

can be constructed from the elements of $\mathcal{L}'^2(\Omega)$. In Volume I, Chapter VI (Reference 11), it was seen that when this class of scalar wave functions is passed through Debye's process

$$\begin{aligned} \bar{M} &= \nabla \times u \bar{r} \\ \bar{N} &= \frac{1}{k} \nabla \times \bar{M} , \end{aligned} \quad (1-48)$$

the resulting $\bar{M}$ and $\bar{N}$ vectors are solutions of the system

$$\nabla \times \nabla \times \bar{F} - k^2 \bar{F} = 0 , \quad \nabla \cdot \bar{F} = 0 . \quad (1-49)$$

Finally, in the last section of that report, it was shown how solutions of (1-47) can be used to generate time-harmonic electromagnetic fields, specifically, elements of $\mathcal{H}(D, \bar{c}, \tilde{\omega})$. It is thus evident that a class of elements of $\mathcal{H}(D, \bar{c}, \tilde{\omega})$ can be generated from $\mathcal{L}'^2(\Omega)$. The details of this process will be outlined in this section, which is the formal connecting link between Volume I (Reference 11) and the material of this volume.

The choice of notation and conventions was made with an eye towards the work in the sections to follow. Figure 1-2 illustrates the geometrical ideas involved. Fix a and b ,

$$0 < a < b < \infty . \quad (1-50)$$

Select

$$\left. \begin{aligned} \tilde{\omega} &> 0 \\ \bar{c}_1 &\in \mathcal{V}_1 \\ \bar{c}_2 &\in \mathcal{V}_2 \end{aligned} \right\} \quad (1-51)$$

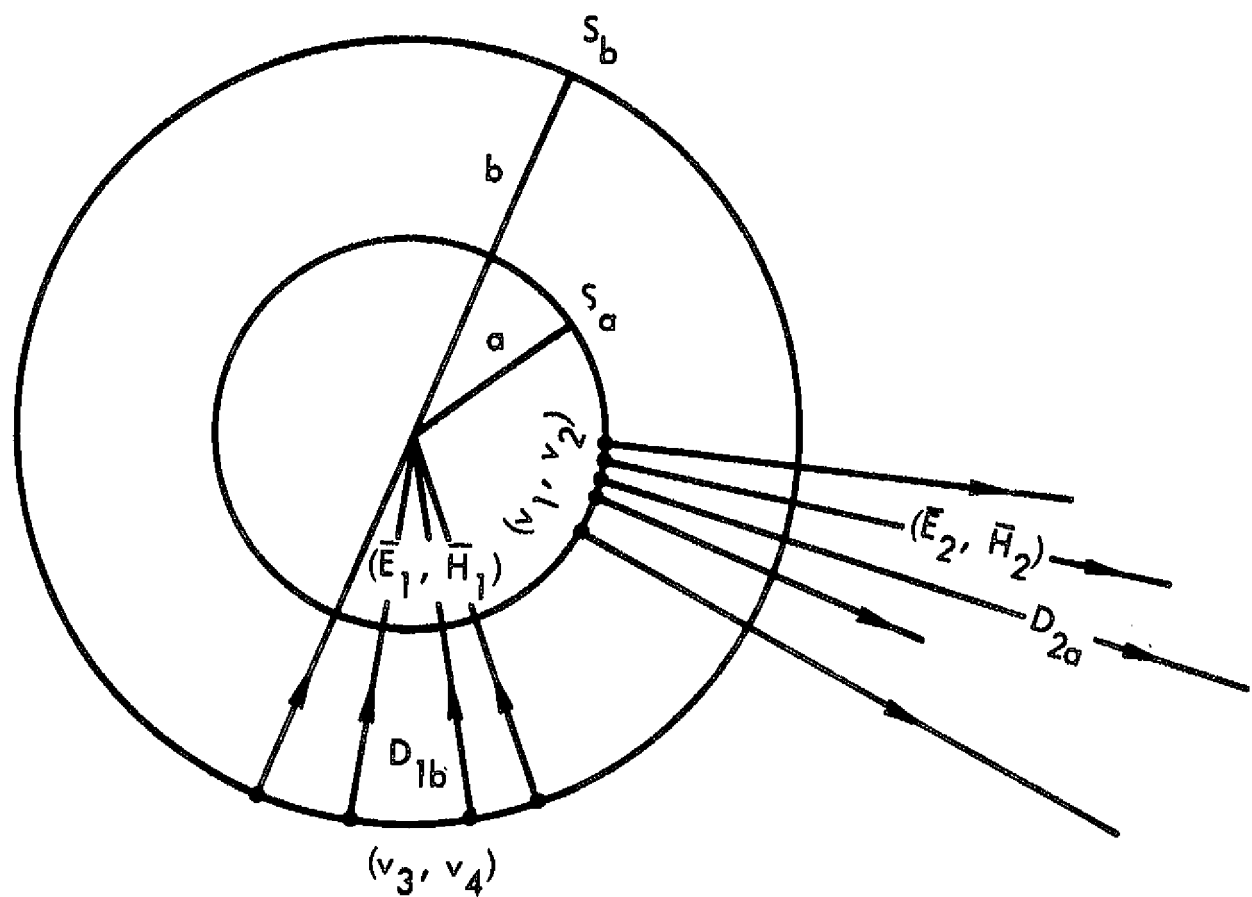


Figure 1-2. Geometrical Concepts Used in Section 1.9

and denote, as usual,

$$\left. \begin{aligned} k_1 &= k(\tilde{\omega}, \bar{c}_1) \\ k_2 &= k(\tilde{\omega}, \bar{c}_2) \end{aligned} \right\} \quad (1-52)$$

Operators

$$\mathcal{J}_{2a\bar{c}_2\tilde{\omega}} : [\mathcal{Y}^2(S_a)]^2 \rightarrow \mathcal{E}(D_{2a}, \bar{c}_2, \tilde{\omega})$$

and

$$\mathcal{J}_{1b\bar{c}_1\tilde{\omega}} : [\mathcal{Y}^2(S_b)]^2 \rightarrow \mathcal{E}(D_{1b}, \bar{c}_1, \tilde{\omega})$$

will be defined below such that the application of $\mathcal{J}$ to the elements of the ranges

$$\begin{aligned} \mathcal{J}_{2a\bar{c}_2\tilde{\omega}}[\mathcal{Y}^2(S_a)] &\equiv \mathcal{J}_{2a} \mathcal{Y}(\tilde{\omega}, \bar{c}_2) \subset \tilde{\mathcal{E}}(D_{2a}, \bar{c}_2, \tilde{\omega}) \\ \mathcal{J}_{1b\bar{c}_1\tilde{\omega}}[\mathcal{Y}^2(S_b)] &\equiv \mathcal{J}_{1b} \mathcal{Y}(\tilde{\omega}, \bar{c}_1) \subset \tilde{\mathcal{E}}(D_{1b}, \bar{c}_1, \tilde{\omega}) \end{aligned}$$

will yield elements in $\mathcal{E}(D_{2a}, \bar{c}_2, \tilde{\omega})$ and $\mathcal{E}(D_{1b}, \bar{c}_1, \tilde{\omega})$ respectively. Many important properties of these fields will follow immediately from their definition and from the appendix of Volume I (Reference 11).

The notation for much of what follows may be streamlined by the use of the index sets Figure 1-3).

$$\left. \begin{aligned} \mathcal{M}_0 &= \{(n, m) \mid n \in \langle \mathcal{N} \rangle, m \in \langle n \rangle\} \\ \mathcal{M}_1 &= \{(n, m) \mid n \in \langle \mathcal{N} \rangle, m \in \langle n \rangle\} \\ \mathcal{M}_2 &= \{(n, m) \mid n \in \langle \mathcal{N} \rangle, m \in \langle n \rangle\} \end{aligned} \right\} \quad (1-53)$$

and also

$$\left. \begin{aligned} \Lambda_1 &= \mathcal{M}_1 \times \{1\} \\ \Lambda_2 &= \mathcal{M}_2 \times \{2\} \\ \Lambda_0 &= \mathcal{M}_0 \times \{1, 2\} \end{aligned} \right\} \quad (1-54)$$

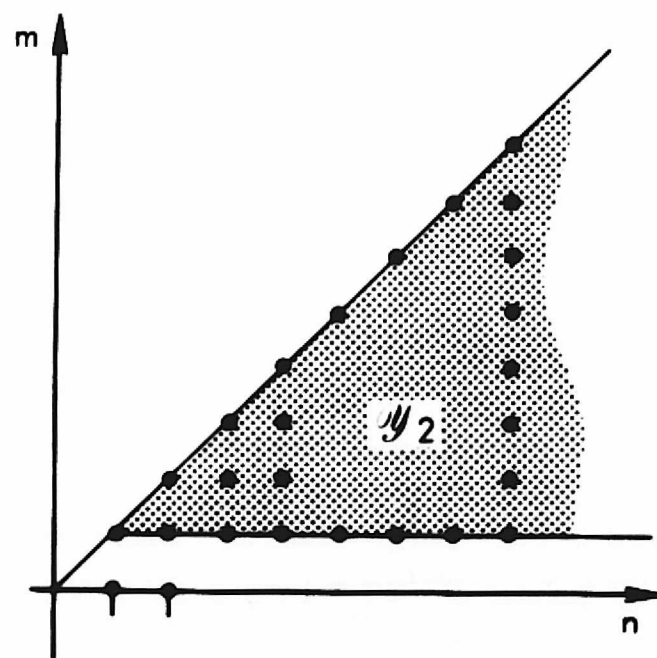
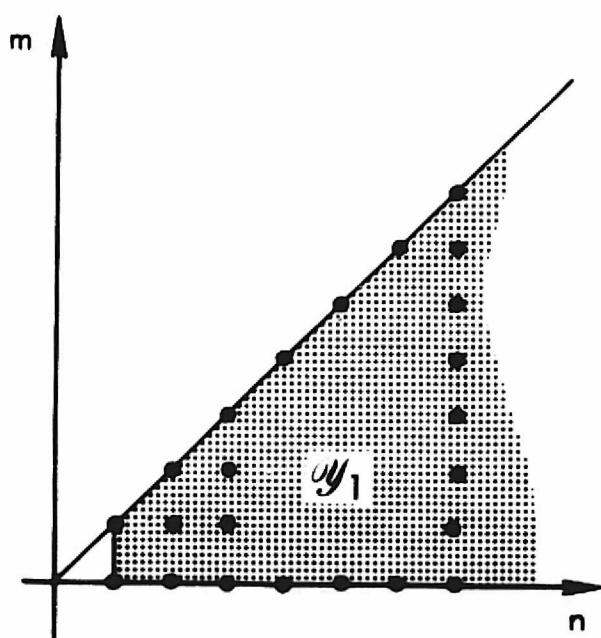
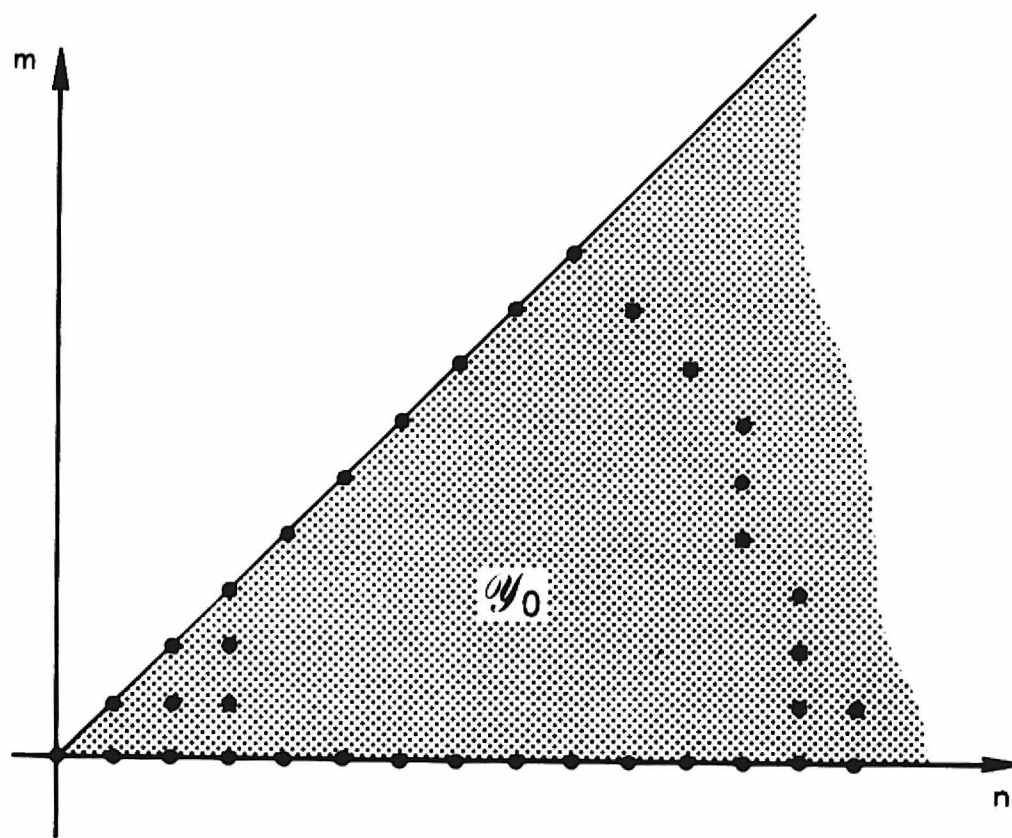


Figure 1-3. Index Sets Used for Sequences $\bar{m}_{mni}$ and $\bar{n}_{mni}$

Elements (n, m, j) of Λ_0 will be denoted λ , ν , or μ . With this notation, the sequence of normalized spherical harmonics described in Volume I, Chapter II, Section 3 (Reference 11) is

$$\Phi = \{\phi_\lambda \mid \lambda \in \Lambda_0\}. \quad (1-55)$$

For any element v in $\mathcal{L}(\Omega)$, the Fourier coefficients are

$$a_\lambda = (v, \phi_\lambda), \quad \lambda \in \Lambda_0 \quad (1-56)$$

and this sequence of coefficients satisfies

$$\|v\| \equiv \left\{ \int_{\Omega} |v|^2 d\omega \right\}^{1/2} = \Sigma \{ |a_\lambda|^2 \mid \lambda \in \Lambda_0 \}. \quad (1-57)$$

Also, the Laplace series

$$\Sigma \{ a_\lambda \phi_\lambda(\omega) \mid \lambda \in \Lambda_0 \}. \quad (1-58)$$

converges in mean square to $v(\omega)$.

The main results of Volume I, Chapter V (Reference 11) are summarized in Figure 1-4. Visualize (See Figure 1-2)

$$v_l = \Sigma \{ a_\lambda l \phi_\lambda \mid \lambda \in \Lambda_0 \} \quad (1-59)$$

as being elements of $\mathcal{L}^2(S_a)$ for $l = 1, 2$, and as elements of $\mathcal{L}^2(S_b)$ for $l = 3, 4$. The mappings

$$a_2 : \mathcal{L}^2(S_a) \rightarrow \mathcal{L}^2(S_a)$$

$$a_1 : \mathcal{L}^2(S_b) \rightarrow \mathcal{L}^2(S_b)$$

are of no concern here. The operator θ_{2a} satisfies the unicity property* and takes $\mathcal{L}^2(S_a)$ into a subset of $\mathcal{W}(D_{2a}, k_2)$. This is a set of solutions

*See Reference 11, Chapter V, Section 3

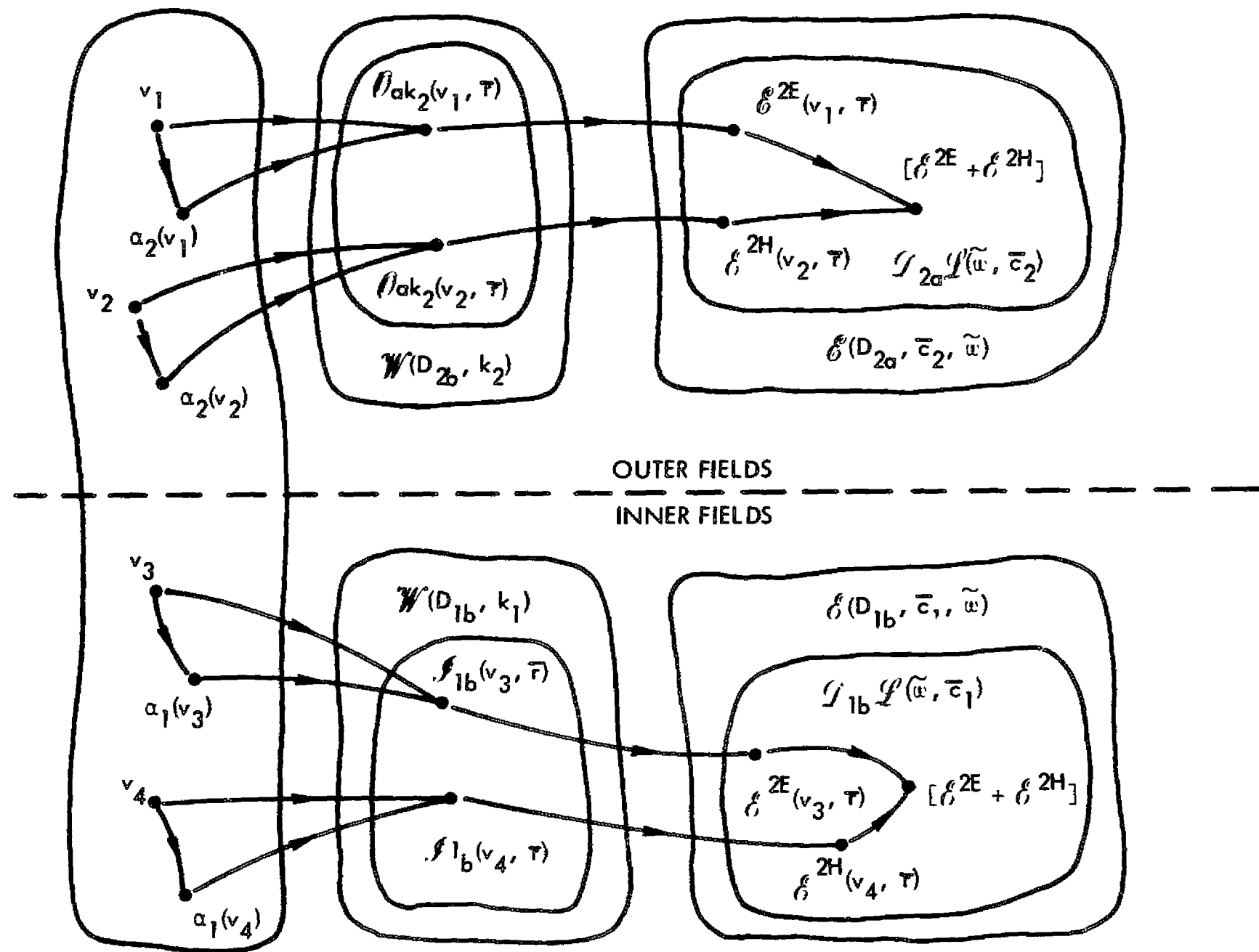


Figure 1-4. Schematic for Definitions

to

$$\nabla^2 u + k_2^2 u = 0$$

on D_{2a} . It was shown in Volume I, Chapter VI (Reference 11) that the vector fields

$$\left. \begin{aligned} \overline{M}^{(2)}(v, \overline{r}) &\equiv \nabla \times \left[\rho_{2a}(v, \overline{r}) \overline{r} \right] \\ \overline{N}^{(2)}(v, \overline{r}) &\equiv \frac{1}{k_2} \nabla \times \overline{M}^{(2)}(v, \overline{r}) \end{aligned} \right\} \quad (1-60)$$

each satisfy the system

$$\nabla \times \nabla \times \overline{F} - k_2^2 \overline{F} = 0, \quad \nabla \cdot \overline{F} = 0.$$

Therefore, from the last section, it follows that both

$$\mathcal{E}^{2E}(v_1, \overline{r}) = \begin{bmatrix} \overline{M}^{(2)}(v_1, \overline{r}) \\ -ik_2 \overline{N}^{(2)}(v_1, \overline{r}) \\ \overline{\omega}_{\mu 2} \end{bmatrix} \quad (1-61)$$

and

$$\mathcal{E}^{2H}(v_2, \overline{r}) = \begin{bmatrix} \overline{N}^{(2)}(v_2, \overline{r}) \\ -ik_2 \overline{M}^{(2)}(v_2, \overline{r}) \\ \overline{\omega}_{\mu 2} \end{bmatrix} \quad (1-62)$$

are elements of $\tilde{\mathcal{E}}(D_{2a}, \overline{c}_2, \tilde{\omega})$, and hence

$$\mathcal{E}^{2E}(v_1, \overline{r}) + \mathcal{E}^{2H}(v_2, \overline{r}) \quad (1-63)$$

must also be an element of $\tilde{\mathcal{E}}(D_{2a}, \overline{c}_2, \tilde{\omega})$.

A similar pattern exists for inner fields. The operator $\mathcal{I}_{1b}$ satisfies the unicity property* and takes $\mathcal{L}^2(S_b)$ into a subset of $\mathcal{W}(D_{1b}, k_1)$. This

*Reference 11, Chapter V, Section 5.

is a set of solutions to

$$\nabla^2 u + k_1^2 u = 0$$

on D_{1b} . By Volume I, Chapter VI (Reference 11), the vector fields

$$\left. \begin{aligned} \overline{M}^{(1)}(v, \overline{r}) &\equiv \nabla \times [\mathcal{J}_{1b}(v, \overline{r}) \overline{r}] \\ \overline{N}^{(1)}(v, \overline{r}) &\equiv \frac{1}{k_1} \nabla \times \overline{M}^{(1)}(v, \overline{r}) \end{aligned} \right\} \quad (1-64)$$

each satisfy the system

$$\nabla \times \nabla \times \overline{F} - k_1^2 \overline{F} = 0, \quad \nabla \cdot \overline{F} = 0.$$

It follows from the results of the last section that both

$$\mathcal{E}^{1E}(v_3, \overline{r}) \equiv \begin{bmatrix} \overline{M}^{(1)}(v_3, \overline{r}) \\ \frac{-ik_1}{\omega \mu_1} \overline{N}^{(1)}(v_3, \overline{r}) \end{bmatrix} \quad (1-65)$$

and

$$\mathcal{E}^{1H}(v_4, \overline{r}) \equiv \begin{bmatrix} \overline{N}^{(1)}(v_4, \overline{r}) \\ \frac{-ik_1}{\omega \mu_1} \overline{M}^{(1)}(v_4, \overline{r}) \end{bmatrix} \quad (1-66)$$

and hence also

$$\left[\mathcal{E}^{1E}(v_3, \overline{r}) + \mathcal{E}^{1H}(v_4, \overline{r}) \right] \quad (1-67)$$

are each elements of $\tilde{\mathcal{E}}(D_{1b}, \overline{c}_1, \tilde{\omega})$.

Figure 1-4 gives a schematic for the construction. With this background, the following definitions should not be unexpected.

Definition 2

Given $\tilde{\omega} > 0$ and $\bar{c}_2 \in \mathcal{V}_2$, let $k_2 \equiv k(\tilde{\omega}, \bar{c}_2)$. The outer Debye operator is defined as

$$\mathcal{I}_{2a\bar{c}_2\tilde{\omega}} : [\mathcal{I}'^2(\Omega)]^2 \rightarrow \tilde{\mathcal{O}}(D_{2a}, \bar{c}_2, \tilde{\omega})$$

and

$$\mathcal{I}_{2a\bar{c}_2\tilde{\omega}}(v_1, v_2) \equiv \mathcal{O}^{2E}(v_1, \bar{r}) + \mathcal{O}^{2H}(v_2, \bar{r}).$$

Definition 3.

Given $\tilde{\omega} > 0$ and $\bar{c}_1 \in \mathcal{V}_1$, let $k_1 \equiv k(\tilde{\omega}, \bar{c}_1)$. The inner Debye operator is defined as

$$\mathcal{I}_{1b\bar{c}_1\tilde{\omega}} : [\mathcal{I}^2(\Omega)]^2 \rightarrow \tilde{\mathcal{O}}(D_{1b}, \bar{c}_1, \tilde{\omega})$$

by

$$\mathcal{I}_{1b\bar{c}_1\tilde{\omega}}(v_3, v_4) \equiv \mathcal{O}^{1E}(v_3, \bar{r}) + \mathcal{O}^{1H}(v_4, \bar{r}).$$

The theorems which follow collect those results bearing on the elements of

$$\mathcal{I}_{1b} \mathcal{I}(\omega, \bar{c}_1) \equiv \mathcal{I}_{1b\bar{c}_1\tilde{\omega}} \left[[\mathcal{I}^2(\Omega)]^2 \right] \quad (1-68)$$

and

$$\mathcal{I}_{2a} \mathcal{I}(\omega, \bar{c}_2) \equiv \mathcal{I}_{2a\bar{c}_2\tilde{\omega}} \left[[\mathcal{I}^2(\Omega)]^2 \right] \quad (1-69)$$

which are immediate consequence of material already presented or found in the appendix. The following notation will be used:

$$\mathcal{G}_\lambda^{2E}(\bar{r}) = \mathcal{G}^{2E}\left(\left(\frac{k_2 h'_n(k_2 a)}{A_{mn}} \phi_\lambda\right), \bar{r}\right) \equiv \begin{bmatrix} \bar{m}_\lambda^{(2)} \\ \frac{-ik_2}{\omega\mu_2} \bar{n}_\lambda^{(2)} \end{bmatrix} \quad (1-70a)$$

$$\mathcal{G}_\lambda^{2H}(\bar{r}) = \mathcal{G}^{2H}\left(\left(\frac{k_2 h'_n(k_2 a)}{A_{mn}} \phi_\lambda\right), \bar{r}\right) \equiv \begin{bmatrix} \bar{n}_\lambda^{(2)} \\ \frac{-ik_2}{\omega\mu_2} \bar{m}_\lambda^{(2)} \end{bmatrix} \quad (1-70b)$$

$$\mathcal{G}_\lambda^{1E}(\bar{r}) = \mathcal{G}^{1E}\left(\left(\frac{k_1 j'_n(k_1 b)}{A_{mn}} \phi_\lambda\right), \bar{r}\right) \equiv \begin{bmatrix} \bar{m}_\lambda^{(1)} \\ \frac{-ik_1}{\omega\mu_1} \bar{n}_\lambda^{(1)} \end{bmatrix} \quad (1-71a)$$

$$\mathcal{G}_\lambda^{1H}(\bar{r}) = \mathcal{G}^{1H}\left(\left(\frac{k_1 j'_n(k_1 b)}{A_{mn}} \phi_\lambda\right), \bar{r}\right) \equiv \begin{bmatrix} \bar{n}_\lambda^{(1)} \\ \frac{-ik_1}{\omega\mu_1} \bar{m}_\lambda^{(1)} \end{bmatrix}, \quad (1-71b)$$

$$\lambda = (n, m, j) \in \Lambda_0 \quad (1-72)$$

where the nonzero constants A_{mn} and the vector fields $\bar{m}_\lambda$, $\bar{n}_\lambda$ have already been described in Volume I, Chapter VI (Reference 11).

Theorem 5

$$v_\ell = \sum \{ a_{\lambda\ell} \phi_\lambda \mid \lambda \in \Lambda_0 \}, \quad \ell = 1, 2 \quad (1-73a)$$

$$1) \quad \mathcal{G}_{2a\bar{c}_2} \tilde{\omega}(v_1, v_2) = \sum \left\{ \frac{A_{mn}}{k_2 h'_n(k_2 a)} \left(a_{\lambda 1} \mathcal{G}_\lambda^{2E} + a_{\lambda 2} \mathcal{G}_\lambda^{2H} \right) \mid \lambda \in \Lambda_1 \cup \Lambda_2 \right\} \quad (1-73b)$$

on $\bar{r} \in D_{2a}$.

2) The series on the RHS of (1-73b) converges subnormally on D_{2a} .

3) If $\mathcal{E}_\lambda$ is any one of the fields occurring under the summation signs in (1-73b), $\xi > a$, and

$$\mathcal{E}_\lambda(\xi, \omega) = \begin{bmatrix} E_\lambda(\xi, \omega) \\ H_\lambda(\xi, \omega) \end{bmatrix}$$

then

$$\begin{cases} E_\lambda(\xi, \omega) \neq \bar{0} \\ H_\lambda(\xi, \omega) \neq \bar{0} \end{cases}.$$

4) If $\exists \xi > a \ni$

$$\mathcal{D}_{2a\bar{c}_2} \tilde{\omega}(v_1, v_2) = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix} \quad \forall \omega \in \Omega,$$

then

$$a_{\lambda 1} = a_{\lambda 2} = 0 \quad \forall \lambda \in \Lambda_1 \cup \Lambda_2.$$

Theorem 6

$$v_\ell = \sum \{ a_{\lambda \ell} \phi_\lambda \mid \lambda \in \Lambda_0 \}, \quad \ell = 3, 4 \quad (1-74a)$$

$$\mathcal{I}_{1b\bar{c}_1} \tilde{\omega}(v_3, v_4) = \sum \left\{ \frac{A_{mn}}{k_{1j_n}(k_{1b})} \left(a_{\lambda 3} \mathcal{E}_\lambda^{1E} + a_{\lambda 4} \mathcal{E}_\lambda^{1M} \right) \right\}_{\lambda \in \Lambda_1 \cup \Lambda_2} \quad (1-74b)$$

on $\bar{r} \in D_{1b}$.

The series on the RHS of (1-75b) converge subnormally on D_{1b} .

If $\mathcal{E}_\lambda$ is any of the fields occurring under the summation signs in (1-74b), $\xi < b$, and

$$\mathcal{E}_\lambda(\xi, \omega) = \begin{bmatrix} E_\lambda(\xi, \omega) \\ \bar{M}_\lambda(\xi, \omega) \end{bmatrix}$$

then

$$\begin{cases} E_\lambda(\xi, \omega) \neq \tilde{0} \\ \bar{M}_\lambda(\xi, \omega) \neq \tilde{0} \end{cases}.$$

If $\exists$ positive $\xi < b$

$$\mathcal{D}_{1b\bar{c}_1\omega}(v_3, v_4) = \begin{bmatrix} \tilde{0} \\ \tilde{0} \end{bmatrix} \quad \forall \omega \in \Omega,$$

then

$$a_{\lambda 3} = a_{\lambda 4} = 0 \quad \forall \lambda \in \Lambda_1 \cup \Lambda_2$$

1.10 ENERGY RADIATED TO INFINITY BY $\mathcal{GD}_{2a}\mathcal{L}(\tilde{\omega}, c_2)$ FIELDS

The elements of the class $\mathcal{GD}_{2a}\mathcal{L}(\omega, \bar{c}_2)$ are time-harmonic electromagnetic fields on the outer domain $\mathcal{D}_{2a}$. Every electromagnetic field in this class may be expanded into subnormally convergent series of $\bar{m}_\lambda, \bar{n}_\lambda$ vectors. It will now be shown that every field of this class satisfies a very natural radiation condition; "the energy radiated to infinity per unit time is non-negative and finite". An exact expression will be developed for this quantity in terms of the Fourier coefficients of the underlying $\mathcal{L}^2(\Omega)$ functions. This expression will play an important part in the sequel.

Theorem 7

Given $c > 0, \tilde{\omega} > 0, \bar{c} = (\epsilon, \mu, \sigma) \in \mathcal{V}_2, k = k(\tilde{\omega}, \bar{c}_2)$ and

$$v_1 = \Sigma \{ a_\lambda \phi_\lambda \mid \lambda \in \Lambda_0 \}, \quad v_2 = \Sigma \{ b_\lambda \phi_\lambda \mid \lambda \in \Lambda_0 \} \quad (1-75a)$$

are any two elements of $\mathcal{L}^2(\Omega)$, and

$$\begin{bmatrix} \bar{E} \\ \bar{H} \end{bmatrix} = \mathcal{D}_{2c} \bar{c} \tilde{\omega} (v_1, v_2) . \quad (1-75b)$$

Then

$$\begin{aligned} & \operatorname{Re} \left\{ \int_{S_\xi} \hat{r} \cdot (\bar{E} \times \bar{H}) \, da \right\} \\ &= \frac{1}{k^2} \sqrt{\frac{\epsilon}{\mu}} \Sigma \left\{ \frac{n(n+1)}{|kh'_n(kc)|^2} \left[|a_\lambda|^2 + |b_\lambda|^2 \right] \mid \lambda \in \Lambda_1 \cup \Lambda_2 \right\} \quad (1-75c) \end{aligned}$$

Define

$$\left\{ \begin{array}{l} a_{\lambda} \equiv \frac{A_{mn}}{kh'_n(kc)} a_{\lambda} \\ \beta_{\lambda} \equiv \frac{A_{mn}}{kh'_n(kc)} b_{\lambda} \end{array} \right\} \lambda = (n, m, i) \in \Lambda_1 \cup \Lambda_2 \quad (1-76)$$

and $a_{\lambda} = \beta_{\lambda} = 0 \quad \lambda \in \Lambda_0^2 - (\Lambda_1 \cup \Lambda_2)$.

Then, in view of Appendix A, *

$$\begin{aligned} \bar{E} &= \sum \left\{ \left[a_{\lambda} \bar{m}_{\lambda}^{(2)} + \beta_{\lambda} \bar{n}_{\lambda}^{(2)} \right] \middle| \lambda \in \Lambda_0 \right\} \\ \bar{H} &= \frac{+ik}{\bar{\omega}_{\mu}} \sum \left\{ \left[a_{\lambda}^* \left[n_{\lambda}^{(2)} \right]^* + \beta_{\lambda}^* \left[\bar{m}_{\lambda}^{(2)} \right]^* \right] \middle| \lambda \in \Lambda_0 \right\}, \end{aligned} \quad (1-77)$$

and so

$$\begin{aligned} &\text{Re} \left\{ \int_{S_{\xi}} \hat{r} \cdot (\bar{E} \times \bar{H}^*) da \right\} \\ &= \text{Re} \left\{ \frac{ik}{\bar{\omega}_{\mu}} \sum \xi^2 \left[a_{\lambda} a_{\nu}^* A_{\lambda}^{\nu} + \beta_{\lambda} \beta_{\nu}^* B_{\lambda}^{\nu} + a_{\lambda} \beta_{\lambda}^* C_{\lambda}^{\nu} \right. \right. \\ &\quad \left. \left. + \beta_{\lambda} a_{\nu}^* D_{\lambda}^{\nu} \right] \middle| (\lambda, \nu) \in (\Lambda_0 \times \Lambda_0) \right\} \end{aligned} \quad (1-78)$$

where

$$\begin{aligned} A_{\lambda}^{\nu} &\equiv \int_{\Omega} \hat{r} \cdot (\bar{m}_{\lambda} \times \bar{n}_{\nu}^*) d\omega \\ B_{\lambda}^{\nu} &\equiv \int_{\Omega} \hat{r} \cdot (\bar{n}_{\lambda} \times \bar{m}_{\nu}^*) d\omega \\ C_{\lambda}^{\nu} &\equiv \int_{\Omega} \hat{r} \cdot (\bar{m}_{\lambda} \times \bar{m}_{\nu}^*) d\omega \\ D_{\lambda}^{\nu} &\equiv \int_{\Omega} \hat{r} \cdot (\bar{n}_{\lambda} \times \bar{n}_{\nu}^*) d\omega. \end{aligned} \quad (1-79)$$

* In this proof, $r = \xi$ is set in the pertinent definitions of $\bar{m}(r, \omega)$, $\bar{n}(r, \omega)$.

Now by straightforward calculations* using Volume I, Chapter VI, Table I (Reference 11) and certain identities of Volume I, Chapter II (Reference 11), and Appendix A of this volume, it can be shown that

$$C_{\lambda}^{\nu} = D_{\lambda}^{\nu} = 0 \quad \forall (\lambda, \nu) \in \Lambda_0^2. \quad (1-80)$$

and

$$\begin{cases} A_{\lambda}^{\nu} = \delta_{\lambda}^{\nu} E_{\lambda} \\ B_{\lambda}^{\nu} = \delta_{\lambda}^{\nu} F_{\lambda} \end{cases} \quad \forall (\lambda, \nu) \in \Lambda_0^2 \quad (1-81)$$

where*

$$E_{\lambda} \equiv \begin{cases} 0 & \lambda \in \Lambda_0 - (\Lambda_1 \cup \Lambda_2) \\ \pi \epsilon_m h_n \mathcal{H}_n^* B_n^m, & \lambda \in \Lambda_1 \cup \Lambda_2 \end{cases} \quad (1-82)$$

and

$$F_{\lambda} \equiv \begin{cases} 0 & \lambda \in \Lambda_0 - (\Lambda_1 \cup \Lambda_2) \\ -\pi \epsilon_m h_n^* \mathcal{H}_n B_n^m, & \lambda \in \Lambda_1 \cup \Lambda_2 \end{cases} \quad (1-83)$$

Then the RHS of (1-78) is

$$\begin{aligned} & \operatorname{Re} \left\{ \pi \frac{ik\xi^2}{\omega\mu} \sum \left\{ \left[\epsilon_m B_{mn} \left(h_n \mathcal{H}_n^* |a_{\lambda}|^2 - h_n^* \mathcal{H}_n |\beta_{\lambda}|^2 \right) \right] \middle| \lambda = (n, m, i) \in \Lambda_0 \right\} \right\} \\ & = \left\{ \pi \frac{k\xi^2}{\omega\mu} \sum \left\{ \left[\epsilon_m B_{mn} \left(\operatorname{Re}(ih_n H_n^*) \right) |a_{\lambda}|^2 + \operatorname{Re}(-ih_n^* H_n) |\beta_{\lambda}|^2 \right] \middle| \lambda = (n, m, i) \in \Lambda_0 \right\} \right\} . \end{aligned} \quad (1-84)$$

*These calculations are contained in Appendix B of this volume.

It can be easily shown that

$$\operatorname{Re}(ih_n \mathcal{H}_n^*) = \operatorname{Re}(-ih_n^* \mathcal{H}_n) = W \left| j_n, y_n \right| \quad (k) \equiv \left(\frac{1}{k\xi} \right)^2. \quad (1-85)$$

This result and

$$k = \tilde{\omega} \sqrt{\mu\epsilon} \quad (1-86)$$

yields the desired formula (1-75c).

It is important to realize that the RHS of (1-75c) is independent of ξ , and also that the summations range over precisely the same index set as in (1-73b). The quantity

$$\frac{1}{2} \operatorname{Re} \left[\int_{S_\xi} \tilde{\mathbf{r}} \cdot (\bar{\mathbf{E}} \times \bar{\mathbf{H}}^*) \, da \right] = \frac{1}{2} \left| \text{RHS of (1-75c)} \right|$$

may be interpreted (Stratton, Reference 8, pages 131-137) as the amount of energy per unit time carried outward through the spherical surface S_ξ . It follows that

$$\lim_{\xi \rightarrow \infty} \left\{ \frac{1}{2} \operatorname{Re} \left[\int_{S_\xi} \hat{\mathbf{r}} \cdot (\bar{\mathbf{E}} \times \bar{\mathbf{H}}^*) \, da \right] \right\} = \frac{1}{2} \left| \text{RHS of (1-75c)} \right| \quad (1-87)$$

is the amount of energy per unit time radiated to infinity by the electromagnetic field $\mathcal{G} \left[\bar{\mathbf{E}}_2, \bar{\mathbf{M}}_2 \right]^T$.

1.11 STARS

In this section, a class of surfaces that form boundaries for the domain D will be defined. Some of the subsequent developments will be valid for much more general surfaces than stars; however, such generalizations will be reserved for later investigations.

Definition 4

A star is a point set S in E^3 , for which there exists a point O called a center of S so that for any Cartesian coordinate system XYZ about

$O \ni f \in C^1(\Omega, \mathcal{R}_0)^*$ with the property,

$$S = \left\{ (x, y, z) \mid (x, y, z) \neq (0, 0, 0) \text{ and } \|(x, y, z)\| = f(\omega), \text{ where } \omega \equiv \frac{(x, y, z)}{\|(x, y, z)\|} \right\}.$$

The notation

$$\|(x, y, z)\| \equiv (x^2 + y^2 + z^2)^{1/2}. \quad (1-88)$$

has been used.

Theorem 8

A star is a closed, regular surface in the sense of Kellogg (Reference 16, pages 100-112).

Proof: The proof follows without difficulty from Kellogg's definitions.

In terms of the usual spherical coordinates in xyz

$$\left. \begin{aligned} x &= r \sin \theta \cos \varphi \\ y &= r \sin \theta \sin \varphi \\ z &= r \cos \theta \end{aligned} \right\} \quad (1-89)$$

the star S is represented by

$$r = f(\theta, \varphi) \equiv f(\omega),$$

where

$$\omega \equiv (\theta, \varphi) \text{ denotes a point on } \Omega.$$

The interior domain of the star S or, more simply, the interior of S is the domain

$$D_1 = \left\{ (r, \omega) \mid 0 \leq r < f(\omega) \right\}, \quad (1-90)$$

and the exterior domain of the star S or, more simply, the exterior of S is the domain

$$D_2 = \left\{ (r, \omega) \mid r > f(\omega) \right\}. \quad (1-91)$$

* $\mathcal{R}_0$ denotes the set of positive reals; Ω the set of points on the surface of the unit sphere; i. e., the set of all points distant 1 from O .

The space curves

$$\left. \begin{aligned} \Theta(\phi_0) &= \left\{ (f(\theta, \phi_0), \theta, \phi_0) \mid \theta \in [0, \pi] \right\}, \phi_0 \in [0, 2\pi] \\ \Phi(\theta_0) &= \left\{ (f(\theta_0, \phi), \theta_0, \phi) \mid \phi \in [0, 2\pi] \right\}, \theta_0 \in [0, \pi] \end{aligned} \right\} \quad (1-92)$$

form a system of (not necessarily orthogonal) curves on S and may be used to define a system of vectors tangential to S . The poles of S (under this coordinatization) are

$$\begin{aligned} P_1 &: (f(0, \phi), 0, \phi) \\ P_2 &: (f(\pi, \phi), \pi, \phi). \end{aligned} \quad (1-93)$$

For this coordinatization

$$\begin{aligned} S' &= S - \left\{ (P_1 \cup P_2) \right\} \\ \Omega' &= \Omega - \left\{ (0, \phi) \cup (\pi, \phi) \right\}. \end{aligned} \quad (1-94)$$

The position vector of S from O is

$$R = f(\omega) \hat{r}. \quad (1-95)$$

Then the vectors

$$\begin{aligned} \bar{R}_\theta &= f_\theta \hat{r} + f \hat{\theta} \\ \bar{R}_\phi &= f_\phi \hat{r} + f \hat{\phi} \end{aligned} \quad (1-96)$$

are defined on S' with respect to the unit base vectors $(\hat{r}, \hat{\theta}, \hat{\phi})$ of the coordinate system (1-89). The vectors $\bar{R}_\theta$ and $\bar{R}_\phi$ are tangential to S at points where they are defined. Moreover

$$\left. \begin{aligned} |\bar{R}_\theta|^2 &= f_\theta^2 + f^2 > 0 \\ |\bar{R}_\phi|^2 &= f_\phi^2 + f^2 > 0 \end{aligned} \right\} \text{ on } S'. \quad (1-97)$$

Also

$$\bar{R}_\theta \times \bar{R}_\phi = f^2 \hat{r} - f f_\theta \hat{\theta} + f f_\phi \hat{\phi} \quad (1-98)$$

and hence

$$|\bar{R}_\theta \times \bar{R}_\phi|^2 > 0 \text{ on } S'. \quad (1-99)$$

It follows that the vectors

$$\left. \begin{aligned} \hat{\xi}(\theta, \phi) &= \frac{1}{|\bar{R}_\theta|} \bar{R}_\theta \\ \hat{\zeta}(\theta, \phi) &= \frac{1}{|\bar{R}_\phi|} \bar{R}_\phi \\ \hat{n}(\theta, \phi) &= \hat{\xi}(\theta, \phi) \times \hat{\zeta}(\theta, \phi) \end{aligned} \right\} \quad (1-100)$$

are continuous functions of position on Ω' (or on S'). The relations (1-100) may be used to find the functions $a_{ij}(\theta, \phi)$ in the transformation

$$\left. \begin{aligned} \hat{r} &= a_{11}(\omega) \hat{n} + a_{12}(\omega) \hat{\xi} + a_{13}(\omega) \hat{\zeta} \\ \hat{\theta} &= a_{21}(\omega) \hat{n} + a_{22}(\omega) \hat{\xi} + a_{23}(\omega) \hat{\zeta} \\ \hat{\phi} &= a_{31}(\omega) \hat{n} + a_{32}(\omega) \hat{\xi} + a_{33}(\omega) \hat{\zeta} \end{aligned} \right\} \quad (1-101)$$

which is defined at every point of S' . The functions $a_{ij}(\omega)$ are all $C^0(\Omega')$.

The vectors $\hat{\xi}(\omega)$ and $\hat{\zeta}(\omega)$ are unit vectors which are tangent to S at the point $(f(\omega), \omega)$. Moreover $\hat{\xi} \times \hat{\zeta} \neq 0$ on S' so that any vector $\hat{t}$ which at $(f(\omega), \omega)$ is tangent to S may be written in a unique manner as

$$\hat{t} = a \hat{\xi}(\omega) + b \hat{\zeta}(\omega).$$

The vector $\hat{n}(\omega)$ is a unit vector which is normal to S at $(f(\omega), \omega)$ and is directed into the exterior of S . It is called the unit outward normal to S . The unit inward normal to S at $(f(\omega), \omega) \in S'$ is

$$\hat{n}_1(\omega) = -\hat{n}(\omega). \quad (1-102)$$

The unit outward normal is independent of the coordinatization $f(\omega)$ and XYZ used to define it; n may be defined (uniquely) even at the poles P_1, P_2 of S . But $\hat{\xi}$ and $\hat{\zeta}$ do depend on the particular coordinatization XYZ of S .

From (1-99) it is clear that

$$\hat{n}(\omega) \cdot \hat{r} > 0 \quad \forall \omega \in \Omega' . \quad (1-103)$$

This means that the line through the center O which pierces S at $(f(\omega), \omega)$ is not tangent to S at that point, no matter how the coordinates of S are defined. Equivalently, in (1-101)

$$a_{11}(\omega) > 0 \quad \forall \omega \in \Omega' . \quad (1-104)$$

Another useful property of stars is that $\exists \epsilon > 0$ such that the point sets

$$\left. \begin{aligned} S_1 &= \left\{ (r, \omega) \mid r = [f(\omega) - \epsilon] \right\} \\ S_2 &= \left\{ (r, \omega) \mid r = [f(\omega) + \epsilon] \right\} \end{aligned} \right\} \quad (1-105)$$

are also stars.

"Given the star $S(f, D_1, D_2, \hat{n}) \dots$ " means that S is a star about some center O; that about some spherical coordinate system with origin O, the star is given by $r = f(\omega)$; that D_1 is the domain interior to S and D_2 is the domain exterior to S; and that $\hat{n}$ is the unit outward normal at points of S. Figure 1-5 illustrates this terminology. The stated properties of stars are summarized:

Theorem 9

Given the star $S(f, D_1, D_2, \hat{n})$, then

$\hat{n}$ is a continuous vector-valued function on Ω (on S),

$\hat{r} \cdot \hat{n} \neq 0$ at points of S' ,

and the vectors

$$\left. \begin{aligned} \hat{s} &= \frac{1}{(f_\theta^2 + f^2)^{1/2}} (f_\theta \hat{r} + f \hat{\theta}) \\ \hat{t} &= \frac{1}{(f_\phi^2 + f^2)^{1/2}} (f_\phi \hat{r} + f \hat{\phi}) \end{aligned} \right\} \quad (1-106)$$

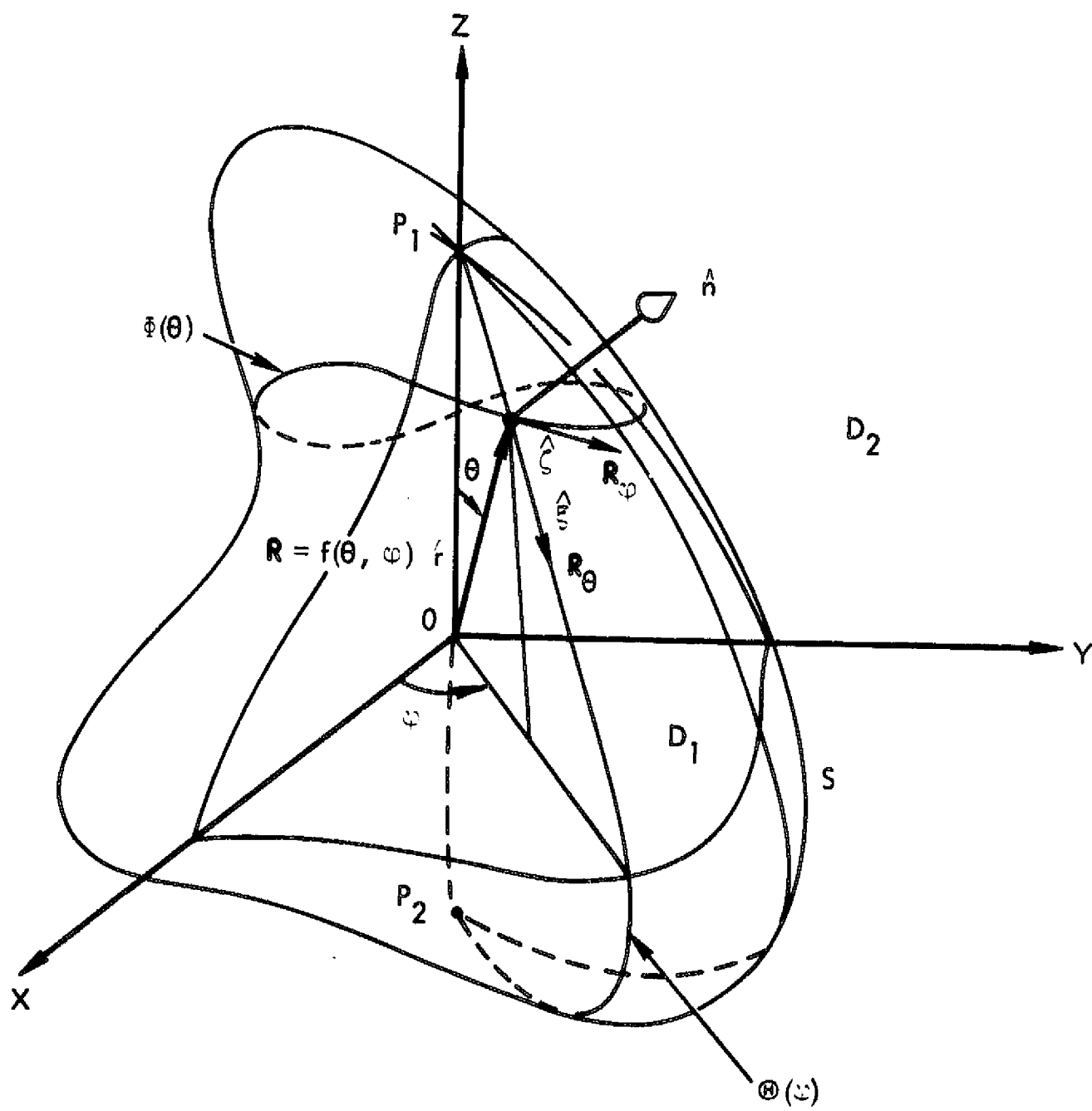


Figure 1-5. Geometrical Concepts Associated with Stars

form a set of unit tangent vectors to S on S' and are each continuous, vector-valued functions of position on S' (on Ω'). Moreover, $\hat{\xi} \times \hat{\xi} = 0$ on S' . Also, $\exists \epsilon > 0$ $\ni$

$$\left. \begin{aligned} S_1 &= \left\{ (r, \omega) \mid r = |f(\omega) - \epsilon| \right\} \\ S_2 &= \left\{ (r, \omega) \mid r = |f(\omega) + \epsilon| \right\} \end{aligned} \right\} \quad (1-107)$$

are stars whose centers are both at O .

1.12 STATEMENT OF THE SCATTERING PROBLEM

$\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \xi_0)$ (Figure 1-6)

A. Given

a) a star $S(f, D_1, D_2, \hat{n})$

and

b) an electromagnetic triple $\bar{c}_i \in \mathcal{V}_i$ associated with D_i , $i = 1, 2$,

and

c) a positive number δ , and the domain

$$D(S, \delta) = \left\{ (r, \omega) \mid f(\omega) < r < (f(\omega) + \delta) \right\} \subset D_2$$

such that $|f(\omega) + \delta|$ is a star.

d) $\mathcal{E}_0 \in \tilde{\mathcal{E}}(D(S, \delta), \bar{c}_2, \tilde{\omega})$

$\ni$

$$\mathcal{E}_{0S}(\omega) \equiv \lim_{\substack{r \rightarrow f(\omega) \\ (r, \omega) \in D_2}} \left[\mathcal{E}_0(r, \omega) \right]$$

exists and in which $\mathcal{E}_{0S}(\omega) \in [C^1(\Omega, \mathcal{C}^3)]^2$,

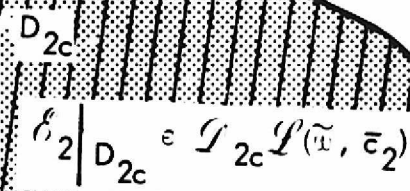
B. find

a pair $(\mathcal{E}_1, \mathcal{E}_2) \in [\tilde{\mathcal{E}}(D_1, \bar{c}_1, \tilde{\omega}) \times \tilde{\mathcal{E}}(D_2, \bar{c}_2, \omega)]$

C. such that

a) the limits

$$\mathcal{E}_{iS}(\omega) \equiv \lim_{\substack{r \rightarrow f(\omega) \\ (r, \omega) \in D_i}} \left[\mathcal{E}_i(r, \omega) \right], \quad i = 1, 2$$



1-35

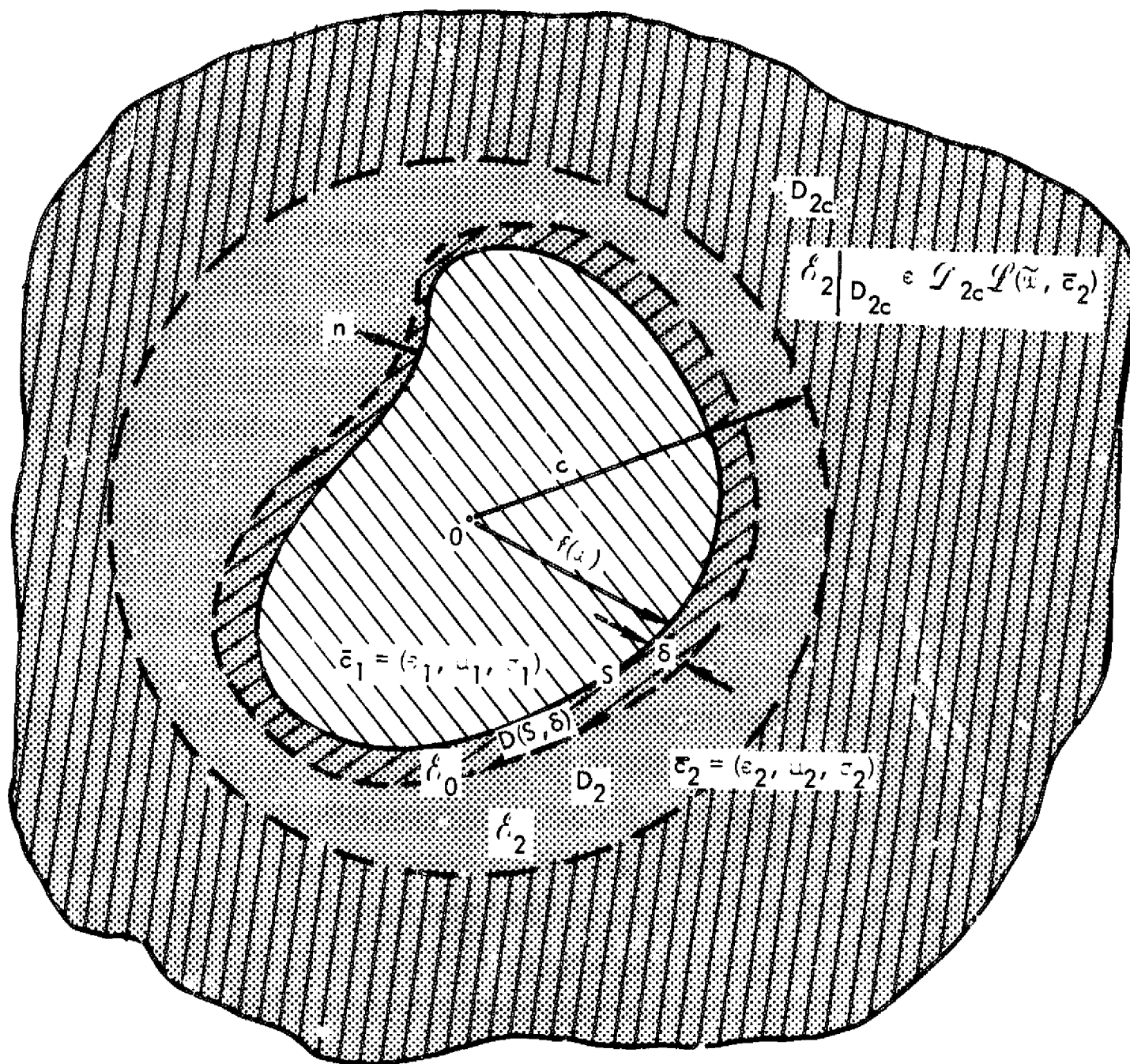


Figure 1-6. Geometrical Concepts Associated with the Problem $P(S, \tilde{\omega}, \bar{\epsilon}_1, \bar{\epsilon}_2, \delta_1, \xi_0)$

exist,

$$b) \quad \mathcal{E}_{iS} \in \left[C^1(\Omega, \mathcal{C}^3) \right]^2, \quad i = 1, 2$$

$$c) \text{ and } \hat{n}(\omega) \times \left[\mathcal{E}_{0S}(\omega) + \mathcal{E}_{2S}(\omega) \right] = \hat{n}(\omega) \times \mathcal{E}_{1S}(\omega) \quad \forall \omega \in \Omega$$

and

$$d) \quad \exists c > \sup_{\omega \in \Omega} f(\omega) \ni \mathcal{E}_2|_{D_{2c}} \in \mathcal{I}_{2c} \mathcal{I}'(\omega, \bar{c}_2) .$$

Remarks

1. A "solution" to the scattering problem $\mathcal{P}(S, \tilde{\omega}, \bar{c}, \bar{c}_2, \delta, \mathcal{E}_0)$ is now sharply defined as any pair $(\mathcal{E}_1, \mathcal{E}_2)$ described in (B) which satisfies all the conditions in (C).

2. When S is a sphere, $\delta = \infty$, and $\mathcal{E}_0$ is a plane monochromatic wave, i. e. one of the form

$$\mathcal{E}_0 \equiv \begin{bmatrix} \bar{E}_0 \\ \bar{H}_0 \end{bmatrix} = \begin{bmatrix} \hat{i} E_0 \exp(ik_2 z) \\ \hat{j} \frac{k_2}{\mu_2 \tilde{\omega}} E_0 \exp(ik_2 z) \end{bmatrix} \quad (1-108)$$

the problem $\mathcal{P}$ has been formally solved (References 3 and 9). Stratton's exposition of this problem (Reference 8, pages 563-570) will be discussed in Section 4. It contains the elements of the more general method to be proposed. The condition C(d) of $\mathcal{P}$ is no weaker or stronger than that used in the solution to the spherical problem. The boundary conditions are the usual ones arrived at by heuristic arguments (Stratton, Reference 8, pages 34-38 and 482-488). The radiation condition C(d) leads (rigorously) to the physically-nice property: the energy per unit time radiated to infinity is finite (Theorem 7).

3. The literature contains various constructive methods proposed for dealing with scattering problems with S more general than a sphere. Methods for perfect conductors have been given by Waterman (Reference 17), Andreason (Reference 18), and Schultz (References 19 and 20). Möglich's treatment (Reference 21) has many errors (Armijo (Reference 22) and has never been implemented numerically. Greenberg and Libelo

(Reference 23) and W. M. Watson (Reference 24) have proposed general methods. Stevenson's technique (Reference 25) reduces the problem to a series of problems in potential theory. In addition to these, there are various classical approximate methods which apply to special cases (van de Hulst, Reference 1). The method to be studied in this report deals with scatterers which have nonzero, positive, finite conductivity and are embedded in media of zero conductivity (perfect dielectrics). For these cases, only limited success, at best, has been reported in implementing the methods proposed. The ranges of applicability of various methods for a prolate conducting spheroid have been discussed by Mathur and Mueller (Reference 26).

4. The present investigation into the scattering problem is motivated by certain questions which should be reviewed in the light of the material heretofore presented. If a scattering problem $\mathcal{P}(S, \bar{c}_1, \bar{c}_2, \delta, \epsilon_0)$ has been given, with boundary the star

$$S(f, D_1, D_2, \hat{n}) .$$

Select real constants a, b so that (Figure 1-7)

$$\left. \begin{array}{l} 0 < a < \inf_{\omega \in \Omega} f(\omega) \\ \sup_{\omega \in \Omega} f(\omega) < b < \infty . \end{array} \right\} \quad (1-109)$$

Let these be used in Definitions 2 and 3 to generate the classes

$$\mathcal{I}_{1b} \mathcal{L}(\tilde{\omega}, \bar{c}_1)$$

and

$$\mathcal{I}_{2a} \mathcal{L}(\tilde{\omega}, \bar{c}_2)$$

which are subclasses of $\tilde{\mathcal{E}}(D_{1b}, \bar{c}_1, \tilde{\omega})$ and $\tilde{\mathcal{E}}(D_{2a}, \bar{c}_2, \tilde{\omega})$, respectively.

Consider the questions

Question 1. Among all pairs (ϵ_1, ϵ_2) in

$$\mathcal{I}_{1b} \mathcal{L}(\tilde{\omega}, \bar{c}_1) \times \mathcal{I}_{2a} \mathcal{L}(\tilde{\omega}, \bar{c}_2) , \quad (1-110)$$

do there exist some for which the boundary condition $C(c)$ in the statement of the scattering problem $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$ is satisfied?

Question 2. If $(\mathcal{E}_1^{(i)}, \mathcal{E}_2^{(i)})$ and $i = 1, 2$ are any two solutions of $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$, is the following true

$$\left\{ \begin{array}{l} \mathcal{E}_1^{(1)}(\bar{r}) \equiv \mathcal{E}_1^{(2)}(\bar{r}) \quad \forall \bar{r} \in D_1 \\ \mathcal{E}_2^{(1)}(\bar{r}) \equiv \mathcal{E}_2^{(2)}(\bar{r}) \quad \forall \bar{r} \in D_2 \end{array} \right\} \quad (1-111)$$

If the answer to Question 1 is "yes," then the pair

$$\left(\mathcal{E}_1|_{D_1}, \mathcal{E}_2|_{D_2} \right) \quad (1-112)$$

is clearly a solution to $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$. If, in addition, the answer to Question 2 is also "yes," then (1-112) is the unique solution to $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$. But note that "solution to $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$ " is now a phrase to be understood in the sharp and narrow sense of Remark 1.

To further emphasize the above point, consider Figure 1-8, which gives a schematic of the logical situation.

Denote

$$\mathcal{Q}(D_2, \bar{c}_2, \tilde{\omega}) = \left\{ \mathcal{E} \mid \mathcal{E} = (\bar{E}_2, \bar{H}_2) \in \tilde{\mathcal{E}}(D_2, \bar{c}_2, \tilde{\omega}), \right.$$

$$\left. \lim_{\xi \rightarrow \infty} \left\{ \frac{1}{2} \operatorname{Re} \left[\int_{S_\xi} \hat{r} \cdot (\bar{E}_2(\xi, \tilde{\omega}) \times [\bar{H}_2(\xi, \tilde{\omega})]^* da) \right] \right\} < \infty \right\} \quad (1-113)$$

Then, by Theorem 7,

$$\mathcal{I}_{2a} \mathcal{I}(\tilde{\omega}, \bar{c}_2) \subset \mathcal{Q}(D_2, \bar{c}_2, \tilde{\omega}) \quad (1-114)$$

and if c is the number which satisfies condition $C(d)$ then

$$\mathcal{I}_{2c} \mathcal{I}(\tilde{\omega}, \bar{c}_2) \subset \mathcal{Q}(D_2, \bar{c}_2, \tilde{\omega}) \quad (1-115)$$

But it has not been established that every element of $\mathcal{Q}(D_2, \bar{c}_2, \tilde{\omega})$ is also in $\mathcal{I}_{2a} \mathcal{I}(\tilde{\omega}, \bar{c}_2)$ or in $\mathcal{I}_{2c} \mathcal{I}(\tilde{\omega}, \bar{c}_2)$.

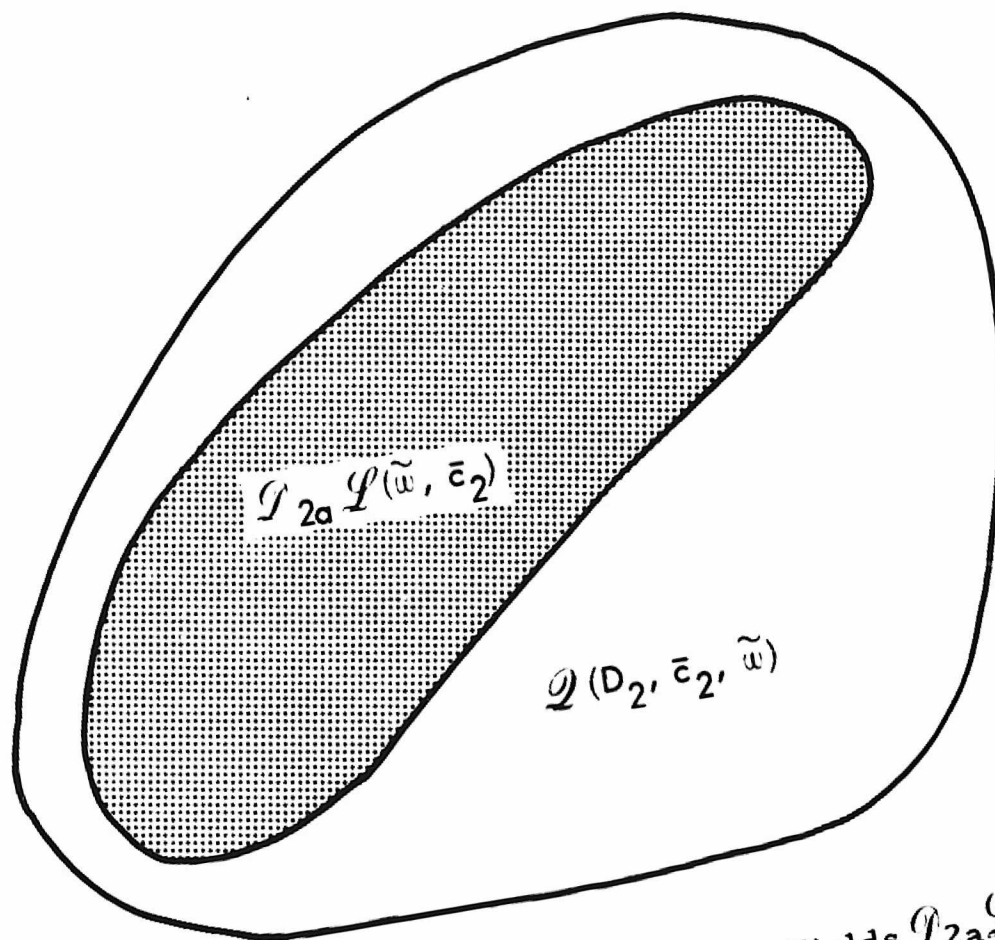


Figure 1-8. Situation of the Class of Outer Fields $\mathcal{I}_{2a} \mathcal{I}(\tilde{\omega}, \bar{\epsilon}_2)$ Within the Class of Fields Satisfying the Finite Energy Radiation Property

Question 3. Thus, is it conceivable that there is a pair $(\mathcal{E}_1, \mathcal{E}_2)$ in

$$\tilde{\mathcal{E}}(D_1, \bar{c}_1, \tilde{\omega}) \times \mathcal{Q}(D_2, \bar{c}_2, \tilde{\omega}) \quad (1-116)$$

such that

$$\mathcal{E}_2 \Big|_{D_{2c}} \in \mathcal{G}_{2c} \mathcal{P}(\tilde{\omega}, \bar{c}_2) \quad (1-117)$$

and yet $(\mathcal{E}_1, \mathcal{E}_2)$ satisfies all other requirements of (c). By definition, $(\mathcal{E}_1, \mathcal{E}_2)$ would not be a solution to $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$, and yet, on a broader and more natural interpretation of solution, it would have to be admitted as a solution. An investigation aimed at resolving this matter might lead to a restatement of the scattering problem in order to encompass a more general situation. If this could be done while retaining all the theorems thus far established, it would be a real contribution to the theory. In the meantime, it will be shown in the remainder of this volume and in those that follow, Volumes III and IV (References 12 and 13), that the above statement of the problem is both meaningful and useful.

The chapters that follow describe attempts to mathematize Questions 1 and 2. Section 2 gives an affirmative answer to Question 2. The purpose of Section 3 is to show how Question 1 may be posed as a problem involving a certain linear and continuous operator T (solvability operator) whose domain and codomain are the Hilbert space $(\mathcal{L}^2(\Omega))^4$. The mathematical properties of this operator T translate to qualitative features of the proposed construction which are independent of the means used to implement it. Section 3 gives a theorem on T which will be applied in Volume III to construct a straightforward algorithm for numerical construction of solutions to solvable or nearly solvable problems. Section 4 shows that there exists at least one class of such problems. This class of problems forms a useful class of scattering problems with spherical boundaries and incident field $\mathcal{E}_0$ given by (a) below. Thus, Section 4 serves the following purposes:

- a. Exhibit a problem amenable to the proposed method.
- b. Place the classical treatment on a firm mathematical foundation.
- c. Show that the classical problem may justifiably be used as a touchstone for perfecting general purpose programs to deal with more general problems.

2. UNIQUENESS

Theorem 10

Let the electromagnetic scattering problem $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$ be specified.

Suppose $(\bar{E}_1^{(i)}, \bar{H}_1^{(i)})(\bar{E}_2^{(i)}, \bar{H}_2^{(i)})$ is a solution to $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$ for $i = 1, 2$.

Then

$$\begin{bmatrix} \bar{E}_1^{(1)}(r) \\ \bar{H}_1^{(1)}(r) \\ \bar{E}_2^{(1)}(\bar{r}) \\ \bar{H}_2^{(1)}(\bar{r}) \end{bmatrix} = \begin{bmatrix} \bar{E}_1^{(2)}(\bar{r}) \\ \bar{H}_1^{(2)}(\bar{r}) \\ \bar{E}_2^{(2)}(\bar{r}) \\ \bar{H}_2^{(2)}(\bar{r}) \end{bmatrix} \quad \forall \bar{r} \in E^3. \quad (2-1)$$

Proof: Consider the difference field

$$\mathcal{F} = \begin{bmatrix} \bar{E}_1^{(1)} \\ \bar{H}_1^{(1)} \\ \bar{E}_2^{(1)} \\ \bar{M}_2^{(1)} \end{bmatrix} - \begin{bmatrix} \bar{E}_1^{(2)} \\ \bar{H}_1^{(2)} \\ \bar{E}_2^{(2)} \\ \bar{H}_2^{(2)} \end{bmatrix}. \quad (2-2)$$

A. It is easy to verify that the difference field is a solution to

$$\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_{00})$$

where $\mathcal{E}_{00} \equiv (\bar{E}_0, \bar{H}_0)$ is any "incident field" so that

$$\hat{n} \times \bar{E}_0 = \hat{n} \times \bar{H}_0 \equiv \tilde{O} \text{ on } S. \quad (2-3)$$

$\mathcal{E}_{00} \equiv (\bar{O}, \bar{O})$ is such a field. Fix $\mathcal{E}_{00}$, (2-3) holds.

B. Suppose, then, that $\mathcal{F}$ is any solution to $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \epsilon_{00})$.
Then

$$\left. \begin{aligned} \nabla \times \bar{E}_1 &= i\tilde{\omega} \mu_1 \bar{H}_1 \\ \nabla \times \bar{H}_1 &= (\sigma_1 - i\tilde{\omega} \epsilon_1) \bar{E}_1 \end{aligned} \right\} \quad (2-4)$$

and hence

$$\left\{ \begin{aligned} \nabla \times \bar{E}_1 &= i\tilde{\omega} \mu_1 \bar{H}_1 \\ (\nabla \times \bar{H}_1)^* &= (\sigma_1 + i\tilde{\omega} \epsilon_1) \bar{E}_1^* \end{aligned} \right\} \text{ on } D_1. \quad (2-5)$$

If these are used in conjunction with the identity

$$\nabla \cdot (\bar{E}_1 \times \bar{H}_1^*) = \bar{H}_1^* \cdot \nabla \times \bar{E}_1 - \bar{E}_1 \cdot \nabla \times \bar{H}_1^* = \bar{H}_1^* \cdot \nabla \times \bar{E}_1 - \bar{E}_1 \cdot (\nabla \times \bar{H}_1)^*,$$

the result is

$$\nabla \cdot (\bar{E}_1 \times \bar{H}_1^*) = i\tilde{\omega} \mu_1 |\bar{H}_1|^2 - (\sigma_1 + i\tilde{\omega} \epsilon_1) |\bar{E}_1|^2 \text{ on } D_1,$$

and by the divergence theorem

$$\int_S \hat{n} \cdot (\bar{E}_1 \times \bar{H}_1^*) da = -\sigma_1 \int_{R_1} |\bar{E}_1|^2 dv + i\tilde{\omega} \int_{R_1} (\mu_1 |\bar{H}_1|^2 - \epsilon_1 |\bar{E}_1|^2) dv, \quad (2-6)$$

where

$$R_1 \equiv D_1 \cup S. \quad (2-7)$$

Let c' be the number c of the condition $C(d)$ satisfied by $\mathcal{F}$. Choose $\xi > c'$, and define (see Figure 2-1)

$$\left. \begin{aligned} S_\xi &= \{(r, \omega) \mid r = \xi\} \\ D_{2\xi} &= \{(r, \omega) \mid f(\omega) < r < \xi\} \\ R_{2\xi} &= S \cup D_{2\xi} \cup S_\xi \end{aligned} \right\} \quad (2-8)$$

Apply the technique of the above paragraph to $\overline{E}_2, \overline{H}_2$ on $R_{2\xi}$ to obtain

$$\begin{aligned} & \int_{S_1} \hat{n}_1 \cdot (\overline{E}_2 \times \overline{H}_2^*) da + \int_{S_\xi} \hat{r} \cdot (\overline{E}_2 \times \overline{H}_2^*) da \\ &= -\sigma_2 \int_{R_{2\xi}} |\overline{E}_2|^2 dv + i\omega \int_{R_{2\xi}} (\mu_2 |\overline{H}_2|^2 - \epsilon_2 |\overline{E}_2|^2) dv. \end{aligned} \quad (2-9)$$

Next, choose $\epsilon \in (0, \delta)$, (Figure 2-2)

$$\left. \begin{aligned} S_1 &= \{(r, \omega) \mid r = [f(\omega) - \epsilon]\} \\ S_2 &= \{(r, \omega) \mid r = [f(\omega) + \epsilon]\} \end{aligned} \right\} \quad (2-10)$$

are both stars. This is possible by Theorem 11. Consider the stars

$$\begin{aligned} S_1 & \left([f(\omega) - \epsilon], D(1, 1, \epsilon), D(1, 2, \epsilon), \hat{n}_{01} \right) \\ S_2 & \left([f(\omega) + \epsilon], D(2, 1, \epsilon), D(2, 2, \epsilon), \hat{n}_{02} \right) \end{aligned} \quad (2-11)$$

and the associated domains

$$\left. \begin{aligned} D_{1\epsilon} &\equiv D_1 \cap D(1, 2, \epsilon) \\ D_{2\epsilon} &\equiv D(2, 1, \epsilon) \cap D_2 \end{aligned} \right\} \quad (2-12)$$

By definition of ϵ , $D_{2\epsilon} \subset D(S, \delta)$. The notation

$$R_{i\epsilon} \equiv S \cup D_{i\epsilon} \cup S_i, \quad i = 1, 2 \quad (2-13)$$

will be used.

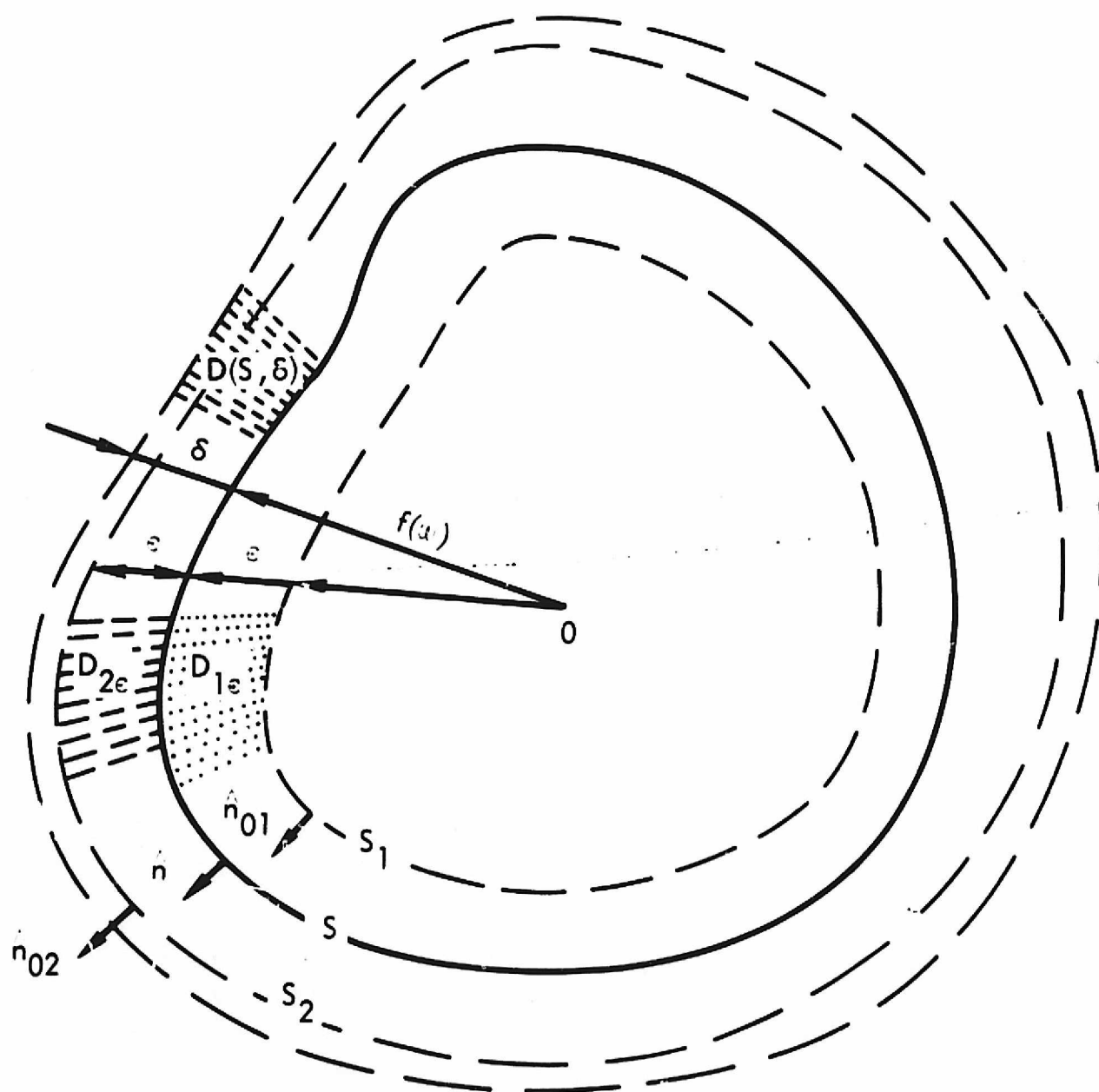


Figure 2-2. Domains Used to Obtain the Energy Balance Integral Identity

The technique used before is applied to $(\bar{\mathbf{E}}_1, \bar{\mathbf{H}}_1)$ on $R_{1\epsilon}$ to obtain

$$\begin{aligned}
 & - \int_{S_1} \hat{\mathbf{n}}_{01} \cdot (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_1^*) da + \int_S \hat{\mathbf{n}} \cdot (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2^*) da \\
 & = - \sigma_1 \int_{R_{1\epsilon}} |\bar{\mathbf{E}}_1|^2 dv + i\tilde{\omega} \int_{R_{1\epsilon}} (\mu_1 |\bar{\mathbf{H}}_1|^2 - \epsilon_1 |\bar{\mathbf{E}}_1|^2) dv. \quad (2-14)
 \end{aligned}$$

The technique is applied to $(\bar{\mathbf{E}}_0 + \bar{\mathbf{E}}_2, \bar{\mathbf{H}}_0 + \bar{\mathbf{H}}_2)$ on $R_{2\epsilon}$ to obtain

$$\begin{aligned}
 & \int_{S_2} \hat{\mathbf{n}}_{02} \cdot [(\bar{\mathbf{E}}_0 + \bar{\mathbf{E}}_2) \times (\bar{\mathbf{H}}_0 + \bar{\mathbf{H}}_2)^*] da + \int_{S_1} \hat{\mathbf{n}}_1 \cdot [(\bar{\mathbf{E}}_0 + \bar{\mathbf{E}}_2) \times (\bar{\mathbf{H}}_0 + \bar{\mathbf{H}}_2)^*] da \\
 & = - \sigma_2 \int_{R_{2\epsilon}} |\bar{\mathbf{E}}_0 + \bar{\mathbf{E}}_2|^2 da + i\tilde{\omega} \int_{R_{2\epsilon}} (\mu_2 |\bar{\mathbf{H}}_0 + \bar{\mathbf{H}}_2|^2 - \epsilon_2 |\bar{\mathbf{E}}_0 + \bar{\mathbf{E}}_2|^2) dv. \quad (2-15)
 \end{aligned}$$

(2-14) and (2-15) are added to obtain

$$\begin{aligned}
 & \int_{S_2} \hat{\mathbf{n}}_{02} \cdot [(\bar{\mathbf{E}}_0 + \bar{\mathbf{E}}_2) \times (\bar{\mathbf{H}}_0 + \bar{\mathbf{H}}_2)^*] da - \int_{S_1} \hat{\mathbf{n}}_{01} \cdot (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_1^*) da \\
 & + \int_S \left[\hat{\mathbf{n}} \cdot (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_1^*) - \hat{\mathbf{n}} \cdot [(\bar{\mathbf{E}}_0 + \bar{\mathbf{E}}_2) \times (\bar{\mathbf{H}}_0 + \bar{\mathbf{H}}_2)^*] \right] da \\
 & = - \left(\sigma_1 \int_{R_{1\epsilon}} |\bar{\mathbf{E}}_1|^2 dv + \sigma_2 \int_{R_{2\epsilon}} |\bar{\mathbf{E}}_0 + \bar{\mathbf{E}}_2|^2 dv \right) \\
 & + i\tilde{\omega} \left[\int_{R_{2\epsilon}} (\mu_2 |\bar{\mathbf{H}}_0 + \bar{\mathbf{H}}_2|^2 - \epsilon_2 |\bar{\mathbf{E}}_0 + \bar{\mathbf{E}}_2|^2) dv \right. \\
 & \left. + \int_{R_{1\epsilon}} (\mu_1 |\bar{\mathbf{H}}_1|^2 - \epsilon_1 |\bar{\mathbf{E}}_1|^2) dv \right]. \quad (2-16)
 \end{aligned}$$

The integrand of the surface integral over S may be dealt with by using vector identities and the boundary conditions at S :

$$\begin{aligned}
& \hat{n} \cdot (\bar{E}_1 \times \bar{H}_1) - \hat{n} \cdot \left[(\bar{E}_0 \times \bar{H}_0^*) + (\bar{E}_2 \times \bar{H}_2^*) + (\bar{E}_2 \times \bar{H}_0^*) + (\bar{E}_0 \times \bar{H}_2^*) \right] \\
&= \hat{n} \cdot (\bar{E}_1 \times \bar{H}_1^*) - \left\{ \hat{n} \cdot (\bar{E}_0 \times \bar{H}_0^*) + \hat{n} \cdot (\bar{E}_2 \times \bar{H}_2^*) + \hat{n} \cdot (\bar{E}_2 \times \bar{H}_0^*) \right. \\
&\quad \left. + \hat{n} \cdot (\bar{E}_0 \times \bar{H}_2^*) \right\} \\
&= \bar{H}_1^* \cdot \hat{n} \times \bar{E}_1 - \left[\bar{H}_0^* \cdot (\hat{n} \times \bar{E}_0) + \bar{H}_2^* \cdot (\hat{n} \times \bar{E}_2) + \bar{H}_0^* \cdot (\hat{n} \times \bar{E}_2) \right. \\
&\quad \left. + \bar{H}_2^* \cdot (\hat{n} \times \bar{E}_0) \right] \\
&= \bar{H}_1^* \cdot (\hat{n} \times \bar{E}_1) - \left[\bar{H}_0^* \cdot \left[(\hat{n} \times \bar{E}_0) + (\hat{n} \times \bar{E}_2) \right] + \bar{H}_2^* \cdot \left[(\hat{n} \times \bar{E}_2) \right. \right. \\
&\quad \left. \left. + (\hat{n} \times \bar{E}_0) \right] \right] \\
&= \bar{H}_1^* \cdot (\hat{n} \times \bar{E}_1) - \left[(\bar{H}_0^* + \bar{H}_2^*) \cdot \left[\hat{n} \times (\bar{E}_0 + \bar{E}_2) \right] \right] \\
&= \bar{H}_1^* \cdot (\hat{n} \times \bar{E}_1) - \left[(\bar{H}_0^* + \bar{H}_2^*) \cdot (\hat{n} \times \bar{E}_1) \right] \\
&= \left[\bar{H}_1^* - (\bar{H}_0^* + \bar{H}_2^*) \right] \cdot (\hat{n} \times \bar{E}_1) \\
&= \bar{E}_1 \cdot \left\{ \left[\hat{n} \times (\bar{H}_0 + \bar{H}_2) \right] - \hat{n} \times \bar{H}_1 \right\}^* = 0. \tag{2-17}
\end{aligned}$$

This shows that the integral over S in (2-16) is identically equal to zero. If ϵ goes to zero in (2-16), the (energy balance) identity

$$\int_S \hat{n}_1 \cdot \left[(\bar{E}_0 \times \bar{E}_2) \times (\bar{H}_0 \times \bar{H}_2)^* \right] da + \int_S \hat{n} \cdot (\bar{E}_1 \times \bar{H}_1^*) da = 0 \tag{2-18}$$

is obtained.

This identity may be written

$$\int_S \left[\hat{n}_1 \cdot (\bar{E}_0 \times \bar{H}_0^*) + \hat{n}_1 \cdot (\bar{E}_0 \times \bar{H}_2^*) + \hat{n}_1 \cdot (\bar{E}_2 \times \bar{H}_0^*) + \hat{n}_1 \cdot (\bar{E}_2 \times \bar{H}_2^*) \right] da \\ + \int_S \hat{n} \cdot (\bar{E}_1 \times \bar{H}_1^*) da = 0, \quad (2-19)$$

i. e.,

$$\left\{ \int_S \bar{H}_0^* \cdot (\hat{n}_1 \times \bar{E}_0) da + \int_S \bar{H}_2^* \cdot (\hat{n}_1 \times \bar{E}_0) da - \int_S \bar{E}_2 \cdot (\hat{n}_1 \times \bar{H}_0^*) da \right\} \\ + \int_S \hat{n}_1 \cdot (\bar{E}_2 \times \bar{H}_2^*) da + \int_S \hat{n} \cdot (\bar{E}_1 \times \bar{H}_1^*) da = 0. \quad (2-20)$$

But since for the present scattering problem

$$\hat{n} \times \bar{E}_0 = \hat{n} \times \bar{H}_0 = 0 \quad \text{on } S, \quad (2-21)$$

it follows from (2-20) that,

$$\int_S \hat{n}_1 \cdot (\bar{E}_2 \times \bar{H}_2^*) da + \int_S \hat{n} \cdot (\bar{E}_1 \times \bar{H}_1^*) da = 0, \quad (2-22)$$

a result which in view of (2-6) and (2-9), may be written

$$- \left(\sigma_1 \int_{R_1} |\bar{E}_1|^2 dv + \sigma_2 \int_{R_{2\xi}} |\bar{E}_2|^2 dv + \operatorname{Re} \left\{ \int_{S_\xi} \hat{r} \cdot (\bar{E}_2 \times \bar{H}_2^*) da \right\} \right) \\ + i \left(\tilde{\omega} \int_{R_1} (\mu_1 |\bar{H}_1|^2 - \epsilon_1 |\bar{E}_1|^2) dv + \tilde{\omega} \int_{R_{2\xi}} (\mu_2 |\bar{H}_2|^2 - \epsilon_2 |\bar{E}_2|^2) dv \right. \\ \left. - \operatorname{Im} \left\{ \int_{S_\xi} \hat{r} \cdot (\bar{E}_2 \times \bar{H}_2^*) da \right\} \right) = 0. \quad (2-23)$$

C. By Theorem 5,

$$\begin{aligned} & \operatorname{Re} \left\{ \int_{S_\xi} \hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_2^*) d\mathbf{a} \right\} \\ &= \frac{1}{k_2^2} \sqrt{\frac{\epsilon_2}{\mu_2}} \sum \left\{ \frac{n(n+1)}{|k_2 h'_n(k_2 c)|^2} \left[|a_\lambda|^2 + |b_\lambda|^2 \right] \middle| \lambda \in \Lambda_1 \cup \Lambda_2 \right\}. \end{aligned}$$

Denote

$$F(\xi) = \operatorname{Im} \left\{ \int_{S_\xi} \hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_2^*) d\mathbf{a} \right\}.$$

Then

$$\begin{aligned} & - \left(\sigma_1 \int_{R_1} |\bar{\mathbf{E}}_1|^2 d\mathbf{v} + \frac{1}{k_2^3} \sqrt{\frac{\epsilon_2}{\mu_2}} \sum \left\{ \frac{n(n+1)}{|h'_n(k_2 c)|^2} \left[|a_\lambda|^2 + |b_\lambda|^2 \right] \middle| \lambda \in \Lambda_1 \cup \Lambda_2 \right\} \right) \\ & + i \left(\tilde{\omega} \int_{R_1} (\mu_1 |\bar{\mathbf{H}}_1|^2 - \epsilon_1 |\bar{\mathbf{E}}_1|^2) d\mathbf{v} + \tilde{\omega} \int_{R_{2\xi}} (\mu_2 |\bar{\mathbf{H}}_2|^2 - \epsilon_2 |\bar{\mathbf{E}}_2|^2) d\mathbf{v} - F(\xi) \right) = 0. \end{aligned}$$

It follows that

$$\bar{\mathbf{E}}_1 = \bar{\mathbf{H}}_1 \equiv \bar{\mathbf{0}} \quad \text{on } R_1$$

and also that

$$a_\lambda = b_\lambda = 0 \quad \forall \lambda \in \Lambda_1 \cup \Lambda_2$$

$$\bar{\mathbf{E}}_2 = \bar{\mathbf{H}}_2 \equiv \bar{\mathbf{0}} \quad \text{on } D_c,$$

D. It remains to show that

$$\bar{\mathbf{E}}_2 = \bar{\mathbf{H}}_2 = \bar{\mathbf{0}} \quad \text{on } R_2 = S \cup D_2.$$

It has been shown that for each $\omega \in \Omega$, $\bar{E}_2(r, \omega) \equiv \bar{0}$ on $r \in (c', \infty)$. The necessary extension follows from the result

$$\bar{E}_2(r, \omega) = \frac{1}{4\pi} \int_S \left\{ \hat{n} \times (\nabla \times \bar{E}_2) \psi + (\hat{n} \times \bar{E}_2) \times \nabla \psi + (\hat{n} \cdot \bar{E}_2) \nabla \psi \right\} dS,$$

valid for $(r, \omega) \in D_2$, and where

$$\psi = \frac{e^{ik_2 \rho}}{\rho}$$

$$\nabla \psi = \hat{\rho} \left[\frac{ik_2 e^{ik_2 \rho}}{\rho} - \frac{e^{ik_2 \rho}}{\rho^2} \right] = (ik_2 - \frac{1}{\rho}) \psi \hat{\rho}.$$

This result is due to Wilcox (Reference 27, pages 116-120). In view of the fact that

$$\hat{n} \times \bar{E}_2 = \hat{n} \times (\nabla \times \bar{E}_2) = \tilde{0} \quad \text{on } S,$$

the above may be written

$$\bar{E}_2(r, \omega) = \frac{ik_2}{4\pi} \int_S (\hat{n} \cdot \bar{E}_2) \psi \hat{\rho} \, da - \frac{1}{4\pi} \int_S (\hat{n} \cdot \bar{E}_2) \frac{\psi}{\rho} \, da.$$

$\bar{E}_2(r, 0, \phi)$ may be considered as a function of r . The Cartesian components of $\hat{\rho}$ may be found from Figure 2-3:

$$\hat{\rho} = \frac{\left[(f(\omega) \sin \theta \cos \phi) \hat{i} + (f(\omega) \sin \theta \sin \phi) \hat{j} + (f(\omega) \cos \theta - r) \hat{k} \right]}{\left[[f(\omega)]^2 + r^2 - 2rf(\omega) \cos \theta \right]^{1/2}}$$

and the element of area da may be written

$$da = \frac{[f(\omega)]^2}{a_{11}(\omega)} d\omega.$$

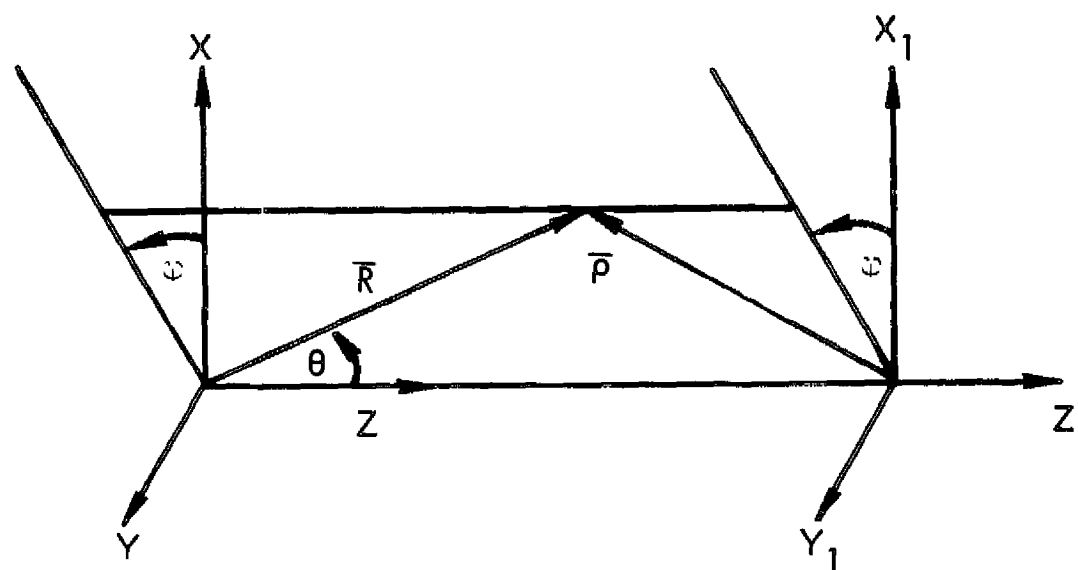


Figure 2-3. Domains Used to Find the Cartesian Components of $\hat{\rho}$

The Cartesian components of $\bar{E}_2(r, 0, \phi)$ are of the form

$$F(r) = \frac{ik_2}{4\pi} \int_{\Omega} g(\omega') \psi(r, f(\omega'), \omega') b_1(r, \omega') d\omega' \\ - \frac{1}{4\pi} \int_{\Omega} g(\omega') \frac{\psi(r, f(\omega'), \omega')}{\rho(r, f(\omega'), \omega')} b_2(r, \omega') d\omega'.$$

Write

$$f - 2rf \cos \theta + r^2 = (f - re^{i\theta})(f - re^{-i\theta}),$$

from which it follows that

$$\rho = \left| f - 2rf \cos \theta + r^2 \right|^{1/2} = 0$$

$\Leftrightarrow$

$$r = f(\theta, \phi) e^{\pm i\theta}.$$

This shows that the functions

$$\tilde{\psi}_j = \frac{\exp \left| ik_2 [f^2(\omega) - 2z f(\omega) \cos \theta + z^2]^{1/2} \right|}{\left([f^2(\omega) - 2z f(\omega) \cos \theta + z^2]^{1/2} \right)^j}, \quad j = 1, 2$$

are holomorphic everywhere on the z -plane except at

$$z = f(\omega) e^{\pm i\theta}.$$

By arguing as in the proof of Theorem 3 in the Appendix of Volume I (Reference 11) (or Hille, Reference 28, [page 182]) it can be shown that the function

$$\begin{aligned} \tilde{F}(z) = & \frac{ik_2}{4\pi} \int_{\Omega} g(\omega') \tilde{\psi}_1(z, f(\omega'), \omega') b_1(z, \omega') d\omega' \\ & - \frac{1}{4\pi} \int_{\Omega} g(\omega') \tilde{\psi}_2(z, f(\omega'), \omega') b_2(z, \omega') d\omega' \end{aligned}$$

is holomorphic on (Figure 2-4)

$$\mathcal{C} = \left\{ z \mid z = R e^{i\eta}, \quad R > \sup_{\phi \in [0, 2\pi]} f(\eta, \phi) \right\},$$

and it is clear that

$$\tilde{F}(z) = F(r) \quad \text{for } z = r \in (f(0, \phi), \infty).$$

The remainder of the proof follows by analytic continuation.

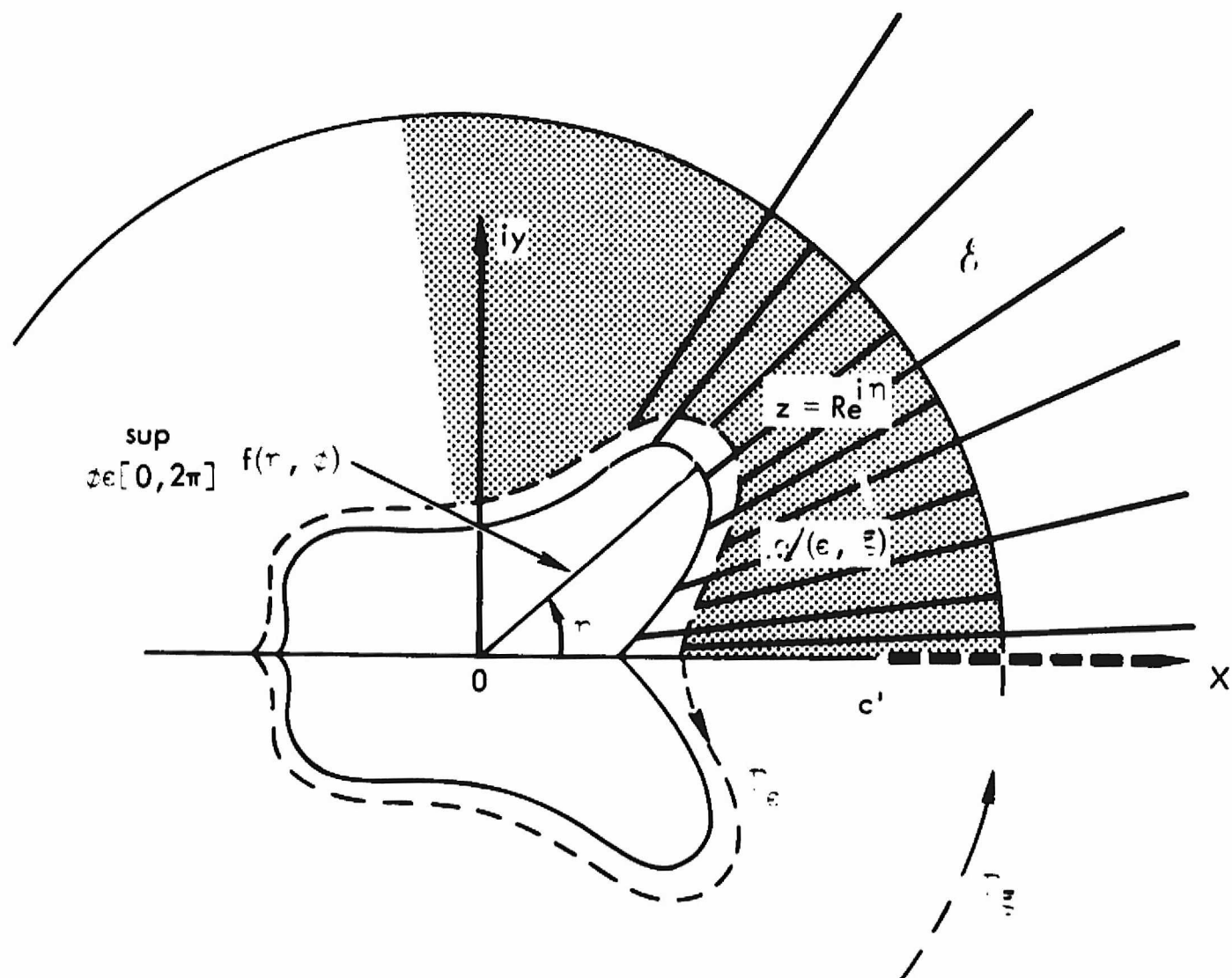


Figure 2-4. Domain of Holomorphy in the τ Plane of the Cartesian Components of $\bar{E}_2(\tau, \phi, \phi)$ for any $\phi \in (0, 2\pi)$

3. SOLVABILITY

3.1 THE GENERAL SOLVABILITY OPERATOR

The motivation and details for this section have already been provided in Section 1.12. Figure 1-8 should serve to review the definitions of basic geometrical concepts. The notations developed in Section 1.9 hold in toto. Figure 1-5 reviews the pertinent functions and spaces. It is assumed that a particular scattering problem $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{C}_0)$ has been given with the boundary of the star

$$S(f, D_1, D_2, \hat{n})$$

and that the radii a, b have already been selected. The coordinate system which gives rise to the representation of S by f and the choice of a, b will remain fixed. The unit tangent base vectors $\hat{\xi}$ and $\hat{\zeta}$ and the functions $a_{ji}(\omega)$ constructed in equations (1-101) and (1-102), respectively, from $f(\omega)$ are assumed to be available.

An operator is needed that takes pairs $(\bar{A}_1, \bar{A}_2)$ of vector-valued functions on the annulus

$$A_{ab} = D_{2a} \cap D_{1b},$$

which contains S , and gives the tangential components of $\bar{A}_1$ and $\bar{A}_2$ on S .

Definition 5.

The boundary operator $\mathcal{J}$ is the mapping

$$\mathcal{J}: [C^0(A_{ab}, \mathcal{C}^3)]^2 \rightarrow [C^0(\Omega, \mathcal{C})]^4$$

given by

$$\mathcal{J}(\bar{A}_1, \bar{A}_2) = \begin{bmatrix} (\bar{A}_1|_S) \cdot \hat{\xi} \\ (\bar{A}_1|_S) \cdot \hat{\zeta} \\ (\bar{A}_2|_S) \cdot \hat{\xi} \\ (\bar{A}_2|_S) \cdot \hat{\zeta} \end{bmatrix}$$

In terms of the operator $\mathcal{J}$, the boundary condition

$$\hat{n} \times [\mathcal{E}_{0s} + \mathcal{E}_{2s}] = \hat{n} \times \mathcal{E}_{1s}$$

in the statement of $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$ may be phrased as

$$\mathcal{J}[\mathcal{E}_1 - \mathcal{E}_2] = \mathcal{J}\mathcal{E}_0 \quad (3-1)$$

The space $\mathcal{L}^2(\Omega)$ is a Hilbert space $\mathcal{H}$ if the operations are defined in the usual way (Stone, Reference 29, pages 29-32). The space $\mathcal{H}^4 \equiv [\mathcal{L}^2(\Omega)]^4$ with operations

$$\bar{f}_i = (f_1^{(i)}, f_2^{(i)}, f_3^{(i)}, f_4^{(i)}) \in \mathcal{H}^4, \quad i = 1, 2$$

$\Rightarrow$

$$\bar{f}_1 + \bar{f}_2 = \bar{f} \equiv (f_1, f_2, f_3, f_4)$$

where

$$f_i \equiv f_i^{(1)} + f_i^{(2)}$$

$$\alpha \in \mathcal{C}, \quad \bar{f} \equiv (f_1, f_2, f_3, f_4) \in \mathcal{H}^4$$

$\Rightarrow$

$$\alpha \bar{f}_1 = (\alpha f_1, \alpha f_2, \alpha f_3, \alpha f_4)$$

and

$$(f_i, f_2) \equiv \sum_{i=1}^4 (f_i^{(1)}, f_i^{(2)}) \equiv \sum_{i=1}^4 \int_{\Omega} f_i^{(1)} f_i^{(2)*} d\omega$$

is also a Hilbert space. A complete orthonormal set for $\mathcal{H}^4$ is given by

$$\{\tau_\nu \mid \nu \in \Lambda_0'\} \quad (3-2)$$

where

$$\Lambda_0' \equiv \Lambda_0 \times (4) \quad (3-3)$$

and

$$\left. \begin{aligned} \tau_{\lambda 1} &\equiv \begin{bmatrix} \phi_\lambda & 0 & 0 & 0 \end{bmatrix} \\ \tau_{\lambda 2} &\equiv \begin{bmatrix} 0 & \phi_\lambda & 0 & 0 \end{bmatrix} \\ \tau_{\lambda 3} &\equiv \begin{bmatrix} 0 & 0 & \phi_\lambda & 0 \end{bmatrix} \\ \tau_{\lambda 4} &\equiv \begin{bmatrix} 0 & 0 & 0 & \phi_\lambda \end{bmatrix} \end{aligned} \right\} \quad (3-4)$$

Definition 6.

The general solvability operator is the mapping

$$T: \mathcal{H}^4 \rightarrow \mathcal{H}^4$$

defined by

$$T\bar{v} = \mathcal{J} \left| \mathcal{J}_{1b\bar{c}_1} \tilde{\omega}(v_3, v_4) - \mathcal{J}_{2a\bar{c}_2} \tilde{\omega}(v_1, v_2) \right|$$

where

$$\bar{v} \equiv (v_1, v_2, v_3, v_4).$$

Theorem 11.

T is linear and continuous on $\mathcal{H}^4$. The range of T is a proper subset of $\mathcal{H}^4$.

Proof: Linearity follows from the linearity of all operators involved in the definition. The proof of continuity depends on the fact that if

$$T\bar{v} \equiv \bar{g} \quad (3-5)$$

then $g_i \in C^0(\Omega)$. $\exists C$ independent of $\bar{v} \in \mathcal{H}^4 \ni$

$$\|T\bar{v}\| < C \|\bar{v}\| \quad \forall \bar{v} \in \mathcal{H}^4. \quad (3-6)$$

Since $g_i \in C^0(\Omega)$,

$$\|\bar{g}\| = \left\{ \sum_{i=1}^4 \int |g_i|^2 d\omega \right\}^{1/2} \leq \left\{ \sum_{i=1}^4 (4\pi \sup_{\omega \in \Omega} |g_i|^2) \right\}^{1/2} \leq 2\sqrt{\pi} \sum_{i=1}^4 \sup_{\omega \in \Omega} |g_i|, \quad (3-7)$$

$$\|T\bar{v}\| \leq 2\sqrt{\pi} \sum_{i=1}^4 \sup_{\omega \in \Omega} |g_i| = 2\sqrt{\pi} \left\{ \sup |(\bar{E}_1|_s) \cdot \hat{\xi} - (\bar{E}_2|_s) \cdot \hat{\xi}| + \sup |(\bar{E}_1|_s) \cdot \hat{\xi} - (\bar{E}_2|_s) \cdot \hat{\xi}| + \dots \text{etc.} \right\} \quad (3-8)$$

and

$$\begin{aligned} \sup_{\bar{r} \in S} |\bar{E}_1(\bar{r}) \cdot \hat{\xi} - \bar{E}_2(\bar{r}) \cdot \hat{\xi}| &\leq \sup_{\bar{r} \in S} |\bar{E}_1(\bar{r}) \cdot \hat{\xi}| + \sup_{\bar{r} \in S} |\bar{E}_2(\bar{r}) \cdot \hat{\xi}| \\ &\leq \sup_{\bar{r} \in S} |\bar{E}_1(\bar{r})| + \sup_{\bar{r} \in S} |\bar{E}_2(\bar{r})|. \end{aligned} \quad (3-9)$$

Consider, e.g.,

$$\bar{E}_2(\bar{r}) = \nabla \times [\theta(v_1, \bar{r}) \bar{r}] + \frac{1}{k_2} \nabla \times \nabla \times [\theta(v_2, \bar{r}) \bar{r}]. \quad (3-10)$$

$$\sup_{\bar{r} \in S} |\bar{E}_2(\bar{r})| = \sup_{\bar{r} \in S} |\nabla \times [\theta(v_1, \bar{r}) \bar{r}]| + \frac{1}{|k_2|} \sup_{\bar{r} \in S} |\nabla \times \nabla \times [\theta(v_2, \bar{r}) \bar{r}]|. \quad (3-11)$$

But

$$\nabla \times \theta \bar{r} = \theta \nabla \times \bar{r} + (\nabla \theta) \times \bar{r} = \left(\nabla \int [K_1 \alpha_2(v_1) - K_2 v_1] d\omega \right) \times \bar{r} \quad (3-12)$$

and

$$\nabla \int [K_1 \alpha_2(v_1) - K_2 v_1] d\omega = \int [(\nabla K_1) \alpha_2(v_1) - (\nabla K_2) v_1] d\omega. \quad (3-13)$$

Hence

$$\begin{aligned} \sup_{\bar{r} \in S} |\nabla \times [\theta \bar{r}]| &\leq \sup_{\bar{r} \in S} \left| \left(\int_{\Omega} [(\nabla K_1) \alpha_2(v_1) - (\nabla K_2) v_1] d\omega \right) \times \bar{r} \right| \\ &< b \left(\sup_{\bar{r} \in S} \left| \int (\nabla K_1) \alpha_2(v_1) d\omega \right| + \sup_{\bar{r} \in S} \left| \int (\nabla K_2) v_1 d\omega \right| \right). \end{aligned} \quad (3-14)$$

The Cauchy-Schwartz inequality is used to obtain

$$\sup_{\bar{r} \in S} |\nabla \times [\theta \bar{r}]| < b(M_1 \|\alpha_2(v_1)\| + M_2 \|v_1\|) \quad (3-15)$$

α_2 has already been shown to be bounded so

$$\sup_{\bar{r} \in S} |\nabla \times [\theta \bar{r}]| < C \|v_1\| \leq C \|\bar{v}\| \quad (3-16)$$

A similar pattern may be applied to bound all the terms on the RHS of (3-8).

3.2 THE SOLVABILITY OPERATOR

The closed linear manifold

$$\mathcal{X} = \overline{v} | \overline{v} = (v_1, v_2, v_3, v_4) \in \mathcal{K}^4, (v_i, \phi_\lambda) = 0 \forall \lambda \in \Lambda_0 - (\Lambda_1 \cup \Lambda_2) \text{ e} \quad (3-17)$$

is spanned by the orthonormal set

$$\{\tau_\nu | \nu \in \Lambda'\} \quad (3-18)$$

where

$$\Lambda' \equiv (\Lambda_1 \cup \Lambda_2) \times \langle 4 \rangle. \quad (3-19)$$

The zero element of $\mathcal{X}$ will be denoted 0_x .

Definition 7.

The solvability operator is the restriction of T to $\mathcal{X}$:

$$T_0 \equiv T|_{\mathcal{X}}.$$

In view of (1-73) and (1-74), it is clear that the range of T_0 is precisely the range of T .

Theorem 12.

- 1) T_0 is linear and continuous on $\mathcal{X}$.
- 2) The range of T_0 is a proper subset of $\mathcal{K}^4$.
- 3) T_0 takes $\mathcal{X}$ 1-1 onto its range.

Proof: (1) and (2) follow from Theorem 11. To prove (3), suppose the contrary. Then $\exists \overline{v}_i \in \mathcal{X}$, $i = 1, 2$ $\ni \overline{v}_1 \neq \overline{v}_2$ and $T\overline{v}_1 = T\overline{v}_2$.

Since T_0 is linear and since $\mathcal{X}$ is a linear space, the element

$$\overline{v} = \overline{v}_1 - \overline{v}_2 \equiv (v_1, v_2, v_3, v_4)$$

is not $0_x \in \mathcal{X}$ and yet

$$T(\overline{v}) = \mathcal{J}'(\overline{0}, \overline{0}).$$

In view of the definition of the solvability operator T_0 ,

$$\left(\mathcal{L}_{1b\bar{c}_2} \tilde{\omega}^{(v_3, v_4)}|_{D_1}, \mathcal{L}_{2a\bar{c}_2} \tilde{\omega}^{(v_1, v_2)}|_{D_2} \right)$$

is a solution to $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \frac{z}{\bar{c}_2}, \delta, \mathcal{G}_{00})$, where

$$\mathcal{G}_{00} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix} \quad (3-21)$$

It therefore follows from the proof of the Uniqueness Theorem that

$$\mathcal{L}_{1b\bar{c}_2} \tilde{\omega}^{(v_3, v_4)}|_{D_1} \equiv \begin{bmatrix} \bar{E}_1(\bar{r}) \\ \bar{H}_1(\bar{r}) \end{bmatrix} \equiv \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix} \quad \forall \bar{r} \in D_1 \quad (3-22)$$

and

$$\mathcal{L}_{2a\bar{c}_2} \tilde{\omega}^{(v_1, v_2)}|_{D_2} \equiv \begin{bmatrix} \bar{E}_1(\bar{r}) \\ \bar{H}_1(\bar{r}) \end{bmatrix} \equiv \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix} \quad \forall \bar{r} \in D_2. \quad (3-23)$$

In particular for $\eta, \xi \in$

$$\left\{ \begin{array}{l} 0 < \eta < a \\ b < \xi < \infty, \end{array} \right\} \quad (3-24)$$

by Theorem 5 and 6, for all $\omega \in \Omega$

$$\sum \left\{ \frac{A_{mn}}{k_1 j_n'(k_1 \eta)} (a_{\lambda 3} \mathcal{G}_{\lambda}^{1E}(\eta, \omega) + a_{\lambda 4} \mathcal{G}_{\lambda}^{2H}(\eta, \omega)) \mid \lambda \in \Lambda_1 \cup \Lambda_2 \right\} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix} \quad (3-25)$$

and

$$\sum \left\{ \frac{A_{mn}}{k_1 h_n'(k_2 \xi)} (a_{\lambda 1} \mathcal{G}_{\lambda}^{2E}(\xi, \omega) + a_{\lambda 2} \mathcal{G}_{\lambda}^{2H}(\xi, \omega)) \mid \lambda \in \Lambda_1 \cup \Lambda_2 \right\} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}, \quad (3-26)$$

and by the orthogonality properties of $\overline{m}^{(\ell)}$, $\overline{n}^{(\ell)}$ (Appendix A), it follows that

$$a_{\lambda i} = 0 \quad \forall (\lambda, i) \in (\Lambda_1 \cup \Lambda_2) \times \langle 4 \rangle. \quad (3-27)$$

and this implies that $\overline{v} = 0_x$, contrary to the original supposition.

3.3 THE RANGE OF T_0 IS LARGE

The theorem of the last section may be used to show that the range of T_0 contains a denumerably infinite linearly independent set; i.e., the linearly range of T_0 is large.

Theorem 13.

Given finite $N \geq 1$ and

$$\chi_i \in \mathcal{X}, \quad i \in \langle N \rangle$$

3

$$\{\chi_i \mid i \in \langle N \rangle\} \text{ is linearly independent,}$$

then

$$\{(T\chi_i) \mid i \in \langle N \rangle\} \text{ is linearly independent.}$$

Proof:

$$\sum_{i=1}^N b_i (T\chi_i) = \overline{0}$$

=

$$T\left(\sum_{i=1}^N b_i \chi_i\right) = \overline{0}$$

$\Rightarrow$ (by Theorem 10)

$$\sum_{i=1}^N b_i \chi_i = \overline{0}$$

$\Rightarrow$ (by linear independence)

$$b_i = 0 \quad \forall i \in \langle N \rangle.$$

3.4 ACTION OF T_0 ON A BASIS FOR $\mathcal{X}$.

Define the orthogonal set*

$$\chi_{\lambda i} = \begin{cases} -\frac{k_2 h'_n(k_2 a)}{A_{mn}} \tau_{\lambda i} & , (\lambda, i) \in (\Lambda_1 \cup \Lambda_2) \times \{1, 2\} \\ +\frac{k_1 j'_n(k_1 b)}{A_{mn}} \tau_{\lambda i} & , (\lambda, i) \in (\Lambda_1 \cup \Lambda_2) \times \{3, 4\} . \end{cases} \quad (3-28)$$

The set

$$\left\{ \chi_{\lambda i} \mid (\lambda, i) \in \Lambda' \right\}$$

spans the closed linear manifold $\mathcal{X}$; i. e., it is a basis for $\mathcal{X}$. The action of T_0 on this basis gives the sequence

$$\left\{ T_0 \chi_{\lambda i} \mid (\lambda, i) \in \Lambda' \right\} \quad (3-29)$$

of elements in $\mathcal{X}^4$. This sequence is described in more detail in Table 3-1 for S' given by

- A general star
- A surface of revolution with the Z-axis of the coordinates chosen to define T_0 taken along the axis of revolution
- A sphere.

The table will be of use in Section 4 and also in Volumes III and IV.

The calculation of the entries of Table 3-1 will be illustrated with one example, which will serve to review the definitions. Denote

$$c_\lambda \equiv \frac{k_2 h'_n(k_2 a)}{A_{mn}} , \quad \lambda = (n, m, i) \in \Lambda_0.$$

*Here $\lambda = (n, m, j) \in \Lambda_1 \cup \Lambda_2$.

Then

$$\begin{aligned}
 T_0 X_{\lambda 1} &= T_0 (-c_\lambda \tau_{\lambda 1}) = -c_\lambda T_0 [\phi_\lambda, 0, 0, 0] \\
 &= -c_\lambda \mathcal{I} \left[\mathcal{D}_{1b\bar{c}_1} \tilde{\omega}(0, 0) - \mathcal{I}_{2a\bar{c}_2} \tilde{\omega}(\phi_\lambda, 0) \right] \\
 &= +c_\lambda \mathcal{I} \mathcal{E}^{2E}(\phi_\lambda, \bar{r}) \\
 &= c_\lambda \mathcal{I} \begin{bmatrix} \bar{M}^{(2)}(\phi_\lambda, \bar{r}) \\ \frac{-ik_2}{\tilde{\omega} \mu_2} \bar{N}^{(2)}(\phi_\lambda, \bar{r}) \end{bmatrix}, \tag{3-30}
 \end{aligned}$$

where

$$\left\{ \begin{aligned} \bar{M}^{(2)}(\phi_\lambda, \bar{r}) &= \nabla \left[j_{ak_2}(\phi_\lambda, \bar{r}) \bar{r} \right] \\ \bar{N}^{(2)}(\phi_\lambda, \bar{r}) &= \frac{1}{k_2} \nabla \times \bar{M}^{(2)}(\phi_\lambda, \bar{r}). \end{aligned} \right\} \tag{3-31}$$

On the other hand, by Theorem 6 in Volume I, Chapter V, it follows that

$$j_{ak_2}(\phi_\lambda, \bar{r}) = \sum \left| \frac{h_n(k_2 r)}{k_2 h'_n(k_2 a)} a_\mu \phi_\mu \right|_{\mu = (n, m, i) \in \Lambda_0}$$

where

$$a_\mu = (\phi_\lambda, \phi_\lambda) = i_\nu^\mu,$$

and hence

$$\begin{aligned}
 j_{ak_2}(\phi_\lambda, \bar{r}) &= \frac{1}{k_2 h'_n(k_2 a)} h_n(k_2 r) \phi_\lambda \\
 &= \frac{1}{k_2 h'_n(k_2 a)} h_n(k_2 r) \left[A_{mn} P_n^m(\cos \theta) \Psi_{im}(\varphi) \right]. \tag{3-32}
 \end{aligned}$$

Thus

$$\nabla \times \left[\mathcal{O}_{ak_2}(\phi_\lambda, \bar{r}) \bar{r} \right] = \frac{A_{mn}}{k_2 h'_n(k_2 a)} \nabla \times \left[h_n(k_2 r) P_n^m(\cos \theta) \Psi_{im}(\varphi) \bar{r} \right] \quad (3-33)$$

i.e.,

$$\left\{ \begin{array}{l} \bar{M}^{(2)}(\phi_\lambda, \bar{r}) \equiv \frac{A_{mn}}{k_2 h'_n(k_2 a)} \bar{m}_\lambda^{(2)}(\bar{r}) \\ \bar{N}^{(2)}(\phi_\lambda, \bar{r}) \equiv \frac{A_{mn}}{k_2 h'_n(k_2 a)} \bar{n}_\lambda^{(2)}(\bar{r}) \end{array} \right\} \quad (3-34)$$

It follows that (see (1-70))

$$T_O \chi_{\lambda 1} = \mathcal{J}' \left[\begin{array}{c} \bar{m}_\lambda^{(2)}(\bar{r}) \\ \frac{-ik_2}{\omega \mu_2} \bar{n}_\lambda^{(2)}(\bar{r}) \end{array} \right] \equiv \mathcal{J}' \mathcal{E}_\lambda^{2E}(\bar{r}) \quad (3-35)$$

To give more details,

$$\left\{ \begin{array}{l} \bar{m}_\lambda^{(2)} = M_{2\lambda}^{(2)} \hat{\theta} + M_{3\lambda}^{(2)} \hat{\varphi} \\ \bar{n}_\lambda^{(2)} = N_{1\lambda}^{(2)} \hat{r} + N_{2\lambda}^{(2)} \hat{\theta} + N_{3\lambda}^{(2)} \hat{\varphi} \end{array} \right\} \quad (3-36)$$

where the coefficients of $\hat{r}$, $\hat{\theta}$, $\hat{\varphi}$ are given in Table I of Volume I (Reference 11). Using (1-101)

$$\left. \begin{array}{l} \hat{r} = a_{11} \hat{n} + a_{12} \hat{\xi} + a_{13} \hat{\varphi} \\ \hat{\theta} = a_{21} \hat{n} + a_{22} \hat{\xi} + a_{23} \hat{\varphi} \\ \hat{\varphi} = a_{31} \hat{n} + a_{32} \hat{\xi} + a_{33} \hat{\varphi} \end{array} \right\} \quad (3-37)$$

and the notation

$$N_{\lambda}^{(2)} = \begin{bmatrix} N_{1\lambda}^{(2)} \\ N_{2\lambda}^{(2)} \\ N_{3\lambda}^{(2)} \end{bmatrix} \quad (3-38)$$

$$M^{(2)} = \begin{bmatrix} M_{1\lambda}^{(2)} \\ M_{2\lambda}^{(2)} \\ M_{3\lambda}^{(2)} \end{bmatrix}, \quad M_{1\lambda}^{(1)} \equiv 0 \quad (3-39)$$

and also Bodewig's notation

$$(a_{ij}) = (a_{\cdot 1}, a_{\cdot 2}, a_{\cdot 3}), \quad (3-40)$$

(3-36) may be written as

$$\begin{aligned} \overline{m}_{\lambda}^{(2)} &= (M_{\lambda}^{(2)} \cdot a_{\cdot 1})^{\wedge}_{\hat{n}} + (M_{\lambda}^{(2)} \cdot a_{\cdot 2})^{\wedge}_{\hat{\xi}} + (M_{\lambda}^{(2)} \cdot a_{\cdot 3})^{\wedge}_{\hat{\xi}} \\ \overline{n}_{\lambda}^{(2)} &= (N_{\lambda}^{(2)} \cdot a_{\cdot 1})^{\wedge}_{\hat{n}} + (N_{\lambda}^{(2)} \cdot a_{\cdot 2})^{\wedge}_{\hat{\xi}} + (N_{\lambda}^{(2)} \cdot a_{\cdot 3})^{\wedge}_{\hat{\xi}} \end{aligned} \quad (3-41)$$

and so, by (3-35)

$$T_0 \chi_{\lambda 1} = \begin{bmatrix} M_{\lambda} \cdot a_{\cdot 2} \\ M_{\lambda} \cdot a_{\cdot 3} \\ \frac{-ik_2}{\tilde{\omega}_{\mu_2}} N_{\lambda} \cdot a_{\cdot 2} \\ \frac{-ik_2}{\tilde{\omega}_{\mu_2}} N_{\lambda} \cdot a_{\cdot 3} \end{bmatrix} \quad (3-42)$$

$$T_0^{\chi_{\lambda 1}} = \begin{bmatrix} M_{2\lambda} a_{22} + M_{3\lambda} a_{32} \\ M_{2\lambda} a_{23} + M_{3\lambda} a_{33} \\ \frac{-ik_2}{\tilde{\omega}_{\mu_2}} \left[N_{1\lambda} a_{12} + N_{2\lambda} a_{22} + N_{3\lambda} a_{32} \right] \\ \frac{-ik_2}{\tilde{\omega}_{\mu_2}} \left[N_{1\lambda} a_{12} + N_{2\lambda} a_{23} + N_{3\lambda} a_{33} \right] \end{bmatrix}, \quad (3-43)$$

which may be written

$$T_0^{\chi_{\lambda 1}} = \begin{bmatrix} M_{2\lambda}^{(2)} \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} + M_{3\lambda}^{(1)} \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \\ \frac{-ik_2}{\tilde{\omega}_{\mu_2}} \left[N_{1\lambda}^{(2)} \begin{bmatrix} a_{12} \\ a_{13} \end{bmatrix} + N_{2\lambda}^{(2)} \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} + N_{3\lambda}^{(2)} \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \right] \end{bmatrix}. \quad (3-44)$$

Then from Table 3-1 of Volume I (Reference 11),

$$T_0^{\chi_{lmn1}} = \begin{bmatrix} -mh_n \Pi_n^m S_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - h_n \tau_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \\ \frac{-ik_2}{\tilde{\omega}_{\mu_2}} \left[\frac{n(n+1)}{k_2 r} h_n P_n^m C_m \begin{bmatrix} a_{12} \\ a_{13} \end{bmatrix} + \mathcal{N}_n \tau_n^m C_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - m \mathcal{N}_n \Pi_n^m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \right] \end{bmatrix} \Bigg|_{r=f(\omega)} \quad (3-45)$$

For a surface of revolution with the axis of revolution lying about the Z-axis of coordinates, the matrix (a_{ij}) of (3-37) will have the form

$$(a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (3-46)$$

i. e.

$$a_{13} = a_{23} = a_{31} = a_{32} = 0 \quad (3-47a)$$

and

$$a_{33} = 1. \quad (3-47b)$$

This and (3-45) gives the element $T_0 \chi_{lmnl}$ of the table for S a surface of revolution.

Finally, for S a star, the matrix (a_{ij}) of (3-37) has the elements

$$a_{ij} = \delta_{ij} \quad (3-48)$$

This and the previously computed entry in the table gives the entry $T_0 \chi_{lmnl}$ for S' a sphere.

The remaining entries may be filled in by using Table 3-1 and the first row of Table 3-1 by formal substitutions.

3.5 SOLVABILITY BY DEBYE'S METHOD

Consider the mapping

$$v: \mathcal{X} \rightarrow \mathcal{X}^4 \quad (3-49)$$

given by

$$v(\bar{v}) = \left(\mathcal{I}_{1bc_1} \bar{\omega}(v_3, v_4) \Big|_{D_1}, \mathcal{I}_{2ac_2} \bar{\omega}(v_1, v_2) \Big|_{D_2} \right) \quad (3-50)$$

where

$$\bar{v} = (v_1, v_2, v_3, v_4) \in \mathcal{X}.$$

Suppose $\mathcal{J}\mathcal{E}_0$ falls in the range of T_0 ; then it is clear that

$$v T_0^{-1} (\mathcal{J}\mathcal{E}_0) \quad (3-51)$$

is the unique solution to $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$.

On the other hand, it is also possible that for any $\bar{y} \in T_0[\mathcal{X}] \ni (\delta, \mathcal{E}_0) \ni$

$$\mathcal{E}_0 \in \tilde{\mathcal{E}}(D(S, \delta), \bar{c}_2, \tilde{\omega})$$

and

$$\bar{y} = \mathcal{J}\mathcal{E}_0.$$

In this case,

$$\mathcal{J}\mathcal{E}_0 \in (\mathcal{X}^4 - T_0[\mathcal{X}]). \quad (3-52)$$

For some applications, the question:

$$\left. \begin{array}{l} \text{Given } \epsilon > 0 \text{ (acceptably small), does there exist} \\ \text{a } \bar{y} \in T_0[\mathcal{X}] \ni \|\bar{y} - \mathcal{J}\mathcal{E}_0\| < \epsilon ? \end{array} \right\} \quad (3-53)$$

is more meaningful.

At present, the above questions have not been mathematically resolved. In view of this unsatisfactory state of affairs, the question may be asked: Are there any scattering problems $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$ for which $\mathcal{J}\mathcal{E}_0 \in T_0[\mathcal{X}]$? The answer is yes, as will be shown in the next section.

At this point in the development of the theory, if $\mathcal{P}$ is not one of the problems demonstrated to be solvable in the next section, the following problem must be posed: Find that $\bar{y}' \in T_0[\mathcal{X}] \ni$

$$\|\bar{y}' - \mathcal{J}\mathcal{E}_0\| \leq \|\bar{y} - \mathcal{J}\mathcal{E}_0\| \quad \forall \bar{y} \in T_0[\mathcal{X}]. \quad (3-54)$$

Then, if

$$\|\bar{y}' - \mathcal{G}_0\| < \epsilon$$

for some acceptably small ϵ ,

$$V T_0^{-1}(\bar{y}')$$

is taken as the approximate solution. Numerical means for constructing $\bar{y}'$ with the property (3-54) will be the subject of Volume III.

Table 3-1. Action of Solvability Operator on a Complete Orthogonal Set

$\bar{v}$	$T_0 \bar{v}$	S a General Star	S a Surface of Revolution with axis of revolution along + Z-axis of coordinates	S a Sphere
x_{1mn1}	$\begin{bmatrix} \bar{m}_{1mn}^{(2)} \\ -\frac{ik_2}{\omega\mu_2} \bar{n}_{1mn}^{(2)} \end{bmatrix}$	$\begin{bmatrix} (-mh_n \bar{u}_n^m S_m) \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} + (-h_n \bar{r}_n^m C_m) \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \\ -\frac{ik_2}{\omega\mu_2} \left[\frac{n(n+1)}{k_2^2} h_n \bar{p}_n^m C_m \begin{bmatrix} a_{12} \\ a_{13} \end{bmatrix} + \bar{J}_n \bar{r}_n^m C_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - m \bar{J}_n \bar{u}_n^m S_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \right] \end{bmatrix}$	$\begin{bmatrix} -mh_n \bar{u}_n^m S_m a_{22} \\ -h_n \bar{r}_n^m C_m \\ -\frac{ik_2}{\omega\mu_2} \left[\frac{n(n+1)}{k_2^2} h_n \bar{p}_n^m C_m a_{12} + \bar{J}_n \bar{r}_n^m C_m a_{22} - m \bar{J}_n \bar{u}_n^m S_m \right] \end{bmatrix}$	$\begin{bmatrix} -mh_n \bar{u}_n^m S_m \\ -h_n \bar{r}_n^m C_m \\ -\frac{ik_2}{\omega\mu_2} \left[\bar{J}_n \bar{r}_n^m C_m - m \bar{J}_n \bar{u}_n^m S_m \right] \end{bmatrix}$
x_{2mn1}	$\begin{bmatrix} \bar{m}_{2mn}^{(2)} \\ -\frac{ik_2}{\omega\mu_2} \bar{n}_{2mn}^{(2)} \end{bmatrix}$	$\begin{bmatrix} (+mh_n \bar{u}_n^m C_m) \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} + (-h_n \bar{r}_n^m S_m) \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \\ -\frac{ik_2}{\omega\mu_2} \left[\frac{n(n+1)}{k_2^2} h_n \bar{p}_n^m S_m \begin{bmatrix} a_{12} \\ a_{13} \end{bmatrix} + \bar{J}_n \bar{r}_n^m S_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} + m \bar{J}_n \bar{u}_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \right] \end{bmatrix}$	$\begin{bmatrix} mh_n \bar{u}_n^m C_m a_{22} \\ -h_n \bar{r}_n^m S_m \\ -\frac{ik_2}{\omega\mu_2} \left[\frac{n(n+1)}{k_2^2} h_n \bar{p}_n^m S_m a_{12} + \bar{J}_n \bar{r}_n^m S_m a_{22} + m \bar{J}_n \bar{u}_n^m C_m \right] \end{bmatrix}$	$\begin{bmatrix} mh_n \bar{u}_n^m C_m \\ -h_n \bar{r}_n^m S_m \\ -\frac{ik_2}{\omega\mu_2} \left[\bar{J}_n \bar{r}_n^m S_m + m \bar{J}_n \bar{u}_n^m C_m \right] \end{bmatrix}$
x_{1mn2}	$\begin{bmatrix} \bar{n}_{1mn}^{(2)} \\ -\frac{ik_2}{\omega\mu_2} \bar{m}_{1mn}^{(2)} \end{bmatrix}$	$\begin{bmatrix} \left(\frac{n(n+1)}{k_2^2} h_n \bar{p}_n^m C_m \right) \begin{bmatrix} a_{12} \\ a_{13} \end{bmatrix} + \bar{J}_n \bar{r}_n^m C_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - m \bar{J}_n \bar{u}_n^m S_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \\ -\frac{ik_2}{\omega\mu_2} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mh_n \bar{u}_n^m S_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} - h_n \bar{r}_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \right] \end{bmatrix}$	$\begin{bmatrix} \frac{n(n+1)}{k_2^2} h_n \bar{p}_n^m C_m a_{12} + \bar{J}_n \bar{r}_n^m C_m a_{22} \\ -mh_n \bar{u}_n^m S_m a_{22} \\ -\frac{ik_2}{\omega\mu_2} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mh_n \bar{u}_n^m S_m a_{22} - h_n \bar{r}_n^m C_m \right] \end{bmatrix}$	$\begin{bmatrix} \bar{J}_n \bar{r}_n^m C_m \\ -m \bar{J}_n \bar{u}_n^m S_m \\ -\frac{ik_2}{\omega\mu_2} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mh_n \bar{u}_n^m S_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} - h_n \bar{r}_n^m C_m \right] \end{bmatrix}$
x_{2mn2}	$\begin{bmatrix} \bar{n}_{2mn}^{(2)} \\ -\frac{ik_2}{\omega\mu_2} \bar{m}_{2mn}^{(2)} \end{bmatrix}$	$\begin{bmatrix} \frac{n(n+1)}{k_2^2} h_n \bar{p}_n^m S_m \begin{bmatrix} a_{12} \\ a_{13} \end{bmatrix} + \bar{J}_n \bar{r}_n^m S_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} + m \bar{J}_n \bar{u}_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \\ -\frac{ik_2}{\omega\mu_2} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mh_n \bar{u}_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} - h_n \bar{r}_n^m S_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \right] \end{bmatrix}$	$\begin{bmatrix} \frac{n(n+1)}{k_2^2} h_n \bar{p}_n^m S_m a_{12} + \bar{J}_n \bar{r}_n^m S_m a_{22} \\ mh_n \bar{u}_n^m C_m a_{22} \\ -\frac{ik_2}{\omega\mu_2} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mh_n \bar{u}_n^m C_m a_{22} - h_n \bar{r}_n^m S_m \right] \end{bmatrix}$	$\begin{bmatrix} \bar{J}_n \bar{r}_n^m S_m \\ m \bar{J}_n \bar{u}_n^m C_m \\ -\frac{ik_2}{\omega\mu_2} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mh_n \bar{u}_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} - h_n \bar{r}_n^m S_m \right] \end{bmatrix}$
x_{1mn3}	$\begin{bmatrix} \bar{m}_{1mn}^{(1)} \\ -\frac{ik_1}{\omega\mu_1} \bar{n}_{1mn}^{(1)} \end{bmatrix}$	$\begin{bmatrix} -mj_n \bar{u}_n^m S_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - j_n \bar{r}_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \\ -\frac{ik_1}{\omega\mu_1} \left[\frac{n(n+1)}{k_1^2} j_n \bar{p}_n^m C_m \begin{bmatrix} a_{12} \\ a_{13} \end{bmatrix} + \bar{J}_n \bar{r}_n^m C_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - m \bar{J}_n \bar{u}_n^m S_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \right] \end{bmatrix}$	$\begin{bmatrix} -mj_n \bar{u}_n^m S_m a_{22} \\ -j_n \bar{r}_n^m C_m \\ -\frac{ik_1}{\omega\mu_1} \left[\frac{n(n+1)}{k_1^2} j_n \bar{p}_n^m C_m a_{12} + \bar{J}_n \bar{r}_n^m C_m a_{22} - m \bar{J}_n \bar{u}_n^m S_m \right] \end{bmatrix}$	$\begin{bmatrix} -mj_n \bar{u}_n^m S_m \\ -j_n \bar{r}_n^m C_m \\ -\frac{ik_1}{\omega\mu_1} \left[\bar{J}_n \bar{r}_n^m C_m - m \bar{J}_n \bar{u}_n^m S_m \right] \end{bmatrix}$
x_{2mn3}	$\begin{bmatrix} \bar{m}_{2mn}^{(1)} \\ -\frac{ik_1}{\omega\mu_1} \bar{n}_{2mn}^{(1)} \end{bmatrix}$	$\begin{bmatrix} mj_n \bar{u}_n^m C_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - j_n \bar{r}_n^m S_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \\ -\frac{ik_1}{\omega\mu_1} \left[\frac{n(n+1)}{k_1^2} j_n \bar{p}_n^m S_m \begin{bmatrix} a_{12} \\ a_{13} \end{bmatrix} + \bar{J}_n \bar{r}_n^m S_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} + m \bar{J}_n \bar{u}_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \right] \end{bmatrix}$	$\begin{bmatrix} mj_n \bar{u}_n^m C_m a_{22} \\ -j_n \bar{r}_n^m S_m \\ -\frac{ik_1}{\omega\mu_1} \left[\frac{n(n+1)}{k_1^2} j_n \bar{p}_n^m S_m a_{12} + \bar{J}_n \bar{r}_n^m S_m a_{22} + m \bar{J}_n \bar{u}_n^m C_m \right] \end{bmatrix}$	$\begin{bmatrix} mj_n \bar{u}_n^m C_m \\ -j_n \bar{r}_n^m S_m \\ -\frac{ik_1}{\omega\mu_1} \left[\bar{J}_n \bar{r}_n^m S_m + m \bar{J}_n \bar{u}_n^m C_m \right] \end{bmatrix}$
x_{1mn4}	$\begin{bmatrix} \bar{n}_{1mn}^{(1)} \\ -\frac{ik_1}{\omega\mu_1} \bar{m}_{1mn}^{(1)} \end{bmatrix}$	$\begin{bmatrix} \frac{n(n+1)}{k_1^2} j_n \bar{p}_n^m C_m \begin{bmatrix} a_{12} \\ a_{13} \end{bmatrix} + \bar{J}_n \bar{r}_n^m C_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - m \bar{J}_n \bar{u}_n^m S_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \\ -\frac{ik_1}{\omega\mu_1} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mj_n \bar{u}_n^m S_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} - j_n \bar{r}_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \right] \end{bmatrix}$	$\begin{bmatrix} \frac{n(n+1)}{k_1^2} j_n \bar{p}_n^m C_m a_{12} + \bar{J}_n \bar{r}_n^m C_m a_{22} \\ -mj_n \bar{u}_n^m S_m a_{22} \\ -\frac{ik_1}{\omega\mu_1} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mj_n \bar{u}_n^m S_m a_{22} - j_n \bar{r}_n^m C_m \right] \end{bmatrix}$	$\begin{bmatrix} \bar{J}_n \bar{r}_n^m C_m \\ -m \bar{J}_n \bar{u}_n^m S_m \\ -\frac{ik_1}{\omega\mu_1} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mj_n \bar{u}_n^m S_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} - j_n \bar{r}_n^m C_m \right] \end{bmatrix}$
x_{2mn4}	$\begin{bmatrix} \bar{n}_{2mn}^{(1)} \\ -\frac{ik_1}{\omega\mu_1} \bar{m}_{2mn}^{(1)} \end{bmatrix}$	$\begin{bmatrix} \frac{n(n+1)}{k_1^2} j_n \bar{p}_n^m S_m \begin{bmatrix} a_{12} \\ a_{13} \end{bmatrix} + \bar{J}_n \bar{r}_n^m S_m \begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} + m \bar{J}_n \bar{u}_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \\ -\frac{ik_1}{\omega\mu_1} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mj_n \bar{u}_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} - j_n \bar{r}_n^m S_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} \right] \end{bmatrix}$	$\begin{bmatrix} \frac{n(n+1)}{k_1^2} j_n \bar{p}_n^m S_m a_{12} + \bar{J}_n \bar{r}_n^m S_m a_{22} \\ mj_n \bar{u}_n^m C_m a_{22} \\ -\frac{ik_1}{\omega\mu_1} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mj_n \bar{u}_n^m C_m a_{22} - j_n \bar{r}_n^m S_m \right] \end{bmatrix}$	$\begin{bmatrix} \bar{J}_n \bar{r}_n^m S_m \\ m \bar{J}_n \bar{u}_n^m C_m \\ -\frac{ik_1}{\omega\mu_1} \left[\begin{bmatrix} a_{22} \\ a_{23} \end{bmatrix} - mj_n \bar{u}_n^m C_m \begin{bmatrix} a_{32} \\ a_{33} \end{bmatrix} - j_n \bar{r}_n^m S_m \right] \end{bmatrix}$

¹ The elements x_{lmn} are defined in Table 1 of Volume 1 (Reference 11).

² In the arguments of $\bar{m}^{(1)}(r, \omega)$, $\bar{n}^{(2)}(r, \omega)$, it is assumed that $r = f(\theta, \phi)$, where $f(\theta, \phi)$ gives the representation of the star S .

4. THE CLASSICAL SPHERICAL PROBLEM IS SOLVABLE

If $\tilde{\omega} > 0$, $\bar{c}_1 \in \mathcal{V}'_1$, $\bar{c}_2 \in \mathcal{V}'_2$ and

$$\left\{ \begin{array}{l} r = f(\omega) \equiv c \\ S \equiv S_c \\ \delta = \infty \\ \mathcal{E}_0 = \begin{cases} \hat{i} E_0 \exp[ik_2 r \cos \theta] \\ \frac{-i}{\tilde{\omega} \mu_2} \nabla \times \left\{ \hat{i} E_0 \exp[ik_2 r \cos \theta] \right\} \end{cases} \end{array} \right\} \quad (4-)$$

in the scattering problem $\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{E}_0)$, the resulting special case is the classical spherical scattering problem. In this chapter, consideration will be given to problems of this type for which the complex relative refractive index, defined as

$$m \equiv \frac{k_1}{k_2} = \frac{k(\tilde{\omega}, \bar{c}_1)}{k(\tilde{\omega}, \bar{c}_2)}, \quad (4-2)$$

is greater than 1 in modulus. It has already been pointed out that Stratton's treatment of this problem is formal and that it prompted the developments described in Sections 2 and 3. The purpose of this section is to demonstrate that the spherical scattering problem (4-1) is certainly solvable if $|m| > 1$. This means that the class of problems solvable by Debye's method is not void. Alternatively, the formulation $\mathcal{P}$ given in Section 1.12 of scattering problems is meaningful at least for S a sphere and the scattering parameters presently under consideration. These solvable spherical scattering problems may be used as a check for general purpose programs based on the general method.

4.1 STRATTON'S TREATMENT OF THE CLASSICAL PROBLEM

The statement of the scattering problem in Stratton is not altogether clear. In particular, the radiation condition to be satisfied is left open. Besides the assumption of existence and uniqueness, Stratton assumes not

only the expansibility of the desired solutions in the domains D_{1b} , D_{2b} but the convergence of these series on the boundary S as well, and strong enough convergence to justify the formulation of the boundary conditions in terms of these series. Also, Stratton's derivation of the series representation of the incident field $\mathcal{E}_0$ of (1) is purely formal. Finally, the solution obtained is a sequence of fractions, and it is not immediately evident that the denominators of these fractions are never zero. The nature of these objections will be discussed first, and then the objections will be removed.

4.1.1 Critique of the Derivation of the Vector Plane Wave Expansion

On page 419, Stratton (Reference 8) derives the formula

$$\hat{i} \exp[ik_2 r \cos \theta] = \sum_{n=1}^{\infty} \left[a_n \bar{m}_{21n}^{(0)} + b_n \bar{n}_{11n}^{(0)} \right] \quad (4-3a)$$

where

$$\begin{cases} a_n = i^n \frac{2n+1}{n(n+1)} \\ b_n = -i^{n+1} \frac{2n+1}{n(n+1)} \end{cases} \quad (4-3b)$$

and

$$\begin{cases} \bar{m}_{\lambda}^{(0)} \equiv \nabla \times [j_n(k_2 r) P_n^m(\cos \theta) \psi_{jm}(\varphi)] \\ \bar{n}_{\lambda}^{(0)} \equiv \frac{1}{k_2} \nabla \times m_{\lambda}^{(0)} \end{cases} \Bigg\}_{\lambda = (n, m, j) \in \Lambda_1 \cup \Lambda_2} \quad (4-3c)$$

The derivation of the formula is by the following pattern: Assume that the LHS is expansible as a series of vectors $m_{21n}^{(0)}$ and $n_{11n}^{(0)}$, then derive the coefficients a_n , b_n in the expansion by the use of the formula from Volume I, Chapter IV, (97) (Reference 11)

$$\exp[ik_2 r \cos \theta] = \sum_{n=0}^{\infty} i^n (2n+1) j_n(k_2 r) P_n(\cos \theta) \quad (4-4)$$

and the formulae of Theorem 2 in Volume I, Chapter VI, (Reference 11) giving the orthonormality property for $\bar{m}$, $\bar{n}$ fields. The trouble with this procedure is that the constants which result may give rise to a RHS of (4-3a) which, even if it converges, may not converge to the (desired) LHS from which it was constructed. A rigorous direct proof of formula (4-3a) is given in Section 4.2.

4.1.2 Critique of Method Used to Fit Boundary Conditions

Stratton's treatment of the classical problem is given as Table 4-1. The symmetry which results in this problem gives rise to a considerable simplification in the determination of the constant coefficients. However, it is not known ahead of time that the series (1b) and (1c) in Table 4-1 for the desired solution will converge sufficiently strongly on S_c , or for that matter anywhere else (to the desired solution), to justify the formulation of the boundary condition. It must, therefore, be granted that the sequence of coefficients obtained is still open to question. Rigor demands that with these coefficients, the series (1b) and (1c) must be shown to converge on S_c strongly enough to justify the rearrangements implicit in (2) and also that, with these coefficients, the series (1b) and (1c) converge to electromagnetic fields on D_{1c} and D_{2c} , respectively. These questions have thus far been ignored in view of the neatness of the final form and the authority of the source. They cannot be ignored in the present well-setness investigation.

The notation of Table 4-1 is almost that of Stratton. The correspondence is

Stratton pp. 564-565	This Report (Volume II)
$\bar{m}_{eln}^{(1)}$	$\bar{m}_{11n}^{(0)}$
$\bar{m}_{01n}^{(1)}$	$\bar{m}_{21n}^{(0)}$
$\bar{m}_{eln}^{(1)}$	$\bar{m}_{11n}^{(1)}$
$\bar{m}_{01n}^{(1)}$	$\bar{m}_{21n}^{(1)}$
$\bar{m}_{eln}^{(3)}$	$\bar{m}_{11n}^{(2)}$
$\bar{m}_{01n}^{(3)}$	$\bar{m}_{21n}^{(2)}$

The Debye representation of the fields is assumed to be

$$(1) \left\{ \begin{array}{l} (1a) \left\{ \begin{array}{l} \bar{E}_0 = \hat{E}_0 \exp(ik_2 z) = E_0 \sum_{n=1}^{\infty} D_n [\bar{m}_{21n}^{(0)} - i \bar{n}_{11n}^{(0)}] \\ \bar{H}_0 = \hat{H}_0 \frac{k_2}{\omega \mu_2} E_0 \exp(ik_2 z) = -\frac{k_2 E_0}{\omega \mu_2} \sum_{n=1}^{\infty} D_n [\bar{m}_{11n}^{(0)} + i \bar{n}_{21n}^{(0)}] \end{array} \right. \\ (1b) \left\{ \begin{array}{l} \bar{E}_1 = E_0 \sum_{n=1}^{\infty} D_n [a_n^t \bar{m}_{21n}^{(1)} - i b_n^t \bar{n}_{11n}^{(1)}] \\ \bar{H}_1 = -\frac{k_1}{\omega \mu_1} E_0 \sum_{n=1}^{\infty} D_n [b_n^t \bar{m}_{11n}^{(1)} + i a_n^t \bar{n}_{21n}^{(1)}] \end{array} \right. \\ (1c) \left\{ \begin{array}{l} \bar{E}_2 = E_0 \sum_{n=1}^{\infty} D_n [a_n^r \bar{m}_{21n}^{(2)} - i b_n^r \bar{n}_{11n}^{(2)}] \\ \bar{H}_2 = -\frac{k_2}{\omega \mu_2} E_0 \sum_{n=1}^{\infty} D_n [b_n^r \bar{m}_{11n}^{(2)} + i a_n^r \bar{n}_{21n}^{(2)}] \end{array} \right. \end{array} \right.$$

i. e., from Table 1 of Volume I:

$$(2) \left\{ \begin{array}{l} (2a) \left\{ \begin{array}{l} \bar{E}_0 = E_0 \sum_{n=1}^{\infty} D_n \left[-i \left| \frac{n(n+1)}{k_2^2 r} \right| j_{n0} P_n^1 C_1 \right] \quad \hat{r} + [j_{n0} \Pi_n^1 C_1 - i \int_{n0} \tau_n^1 C_1] \hat{\theta} + [-j_{n0} \tau_n^1 S_1 + i \int_{n0} \Pi_n^1 S_1] \hat{\phi} \\ \bar{H}_0 = -\frac{k_2 E_0}{\omega \mu_2} \sum_{n=1}^{\infty} D_n \left[i \left| \frac{n(n+1)}{k_2^2 r} \right| j_{n0} P_n^1 S_1 \right] \quad \hat{r} + [-j_{n0} \Pi_n^1 S_1 + i \int_{n0} \tau_n^1 S_1] \hat{\theta} + [-j_{n0} \tau_n^1 C_1 + i \int_{n0} \Pi_n^1 C_1] \hat{\phi} \end{array} \right. \\ (2b) \left\{ \begin{array}{l} \bar{E}_2 = E_0 \sum_{n=1}^{\infty} D_n \left[-i b_n^r \left| \frac{n(n+1)}{k_2^2 r} \right| h_n P_n^1 C_1 \right] \quad \hat{r} + [a_n^r \Pi_n^1 h_n C_1 - i b_n^r \mathcal{X}_n \tau_n^1 C_1] \hat{\theta} + [-a_n^r h_n \tau_n^1 S_1 + i b_n^r \mathcal{X}_n \Pi_n^1 S_1] \hat{\phi} \\ \bar{H}_2 = -\frac{k_2}{\omega \mu_2} E_0 \sum_{n=1}^{\infty} D_n \left[i a_n^r \left| \frac{n(n+1)}{k_2^2 r} \right| h_n P_n^1 S_1 \right] \quad \hat{r} + [-b_n^r h_n \tau_n^1 S_1 + i a_n^r \mathcal{X}_n \tau_n^1 S_1] \hat{\theta} + [-b_n^r h_n \tau_n^1 C_1 + i a_n^r \mathcal{X}_n \Pi_n^1 C_1] \hat{\phi} \end{array} \right. \\ (2c) \left\{ \begin{array}{l} \bar{E}_1 = E_0 \sum_{n=1}^{\infty} D_n \left[-i b_n^t \left| \frac{n(n+1)}{k_1^2 r} \right| j_n P_n^1 C_1 \right] \quad \hat{r} + [a_n^t j_n \Pi_n^1 C_1 - i b_n^t \int_n \tau_n^1 C_1] \hat{\theta} + [-a_n^t j_n \tau_n^1 S_1 + i b_n^t \int_n \Pi_n^1 S_1] \hat{\phi} \\ \bar{H}_1 = \frac{k_1}{\omega \mu_1} E_0 \sum_{n=1}^{\infty} D_n \left[i a_n^t \left| \frac{n(n+1)}{k_1^2 r} \right| j_n P_n^1 S_1 \right] \quad \hat{r} + [-b_n^t j_n \Pi_n^1 S_1 + i a_n^t \int_n \tau_n^1 S_1] \hat{\theta} + [-b_n^t j_n \tau_n^1 C_1 + i a_n^t \int_n \Pi_n^1 C_1] \hat{\phi} \end{array} \right. \end{array} \right.$$

The boundary conditions at $r = c$ lead to

$$\left\{ \begin{array}{l} \sum_{n=1}^{\infty} D_n [a_n^t j_n \Pi_n^1 C_1 - i b_n^t \int_n \tau_n^1 C_1 - a_n^r h_n \Pi_n^1 C_1 + i b_n^r \mathcal{X}_n \tau_n^1 C_1] \\ \sum_{n=1}^{\infty} D_n [j_{n0} \Pi_n^1 C_1 - i \int_{n0} \tau_n^1 C_1] \end{array} \right\} = \sum_{n=1}^{\infty} D_n [j_{n0} \Pi_n^1 C_1 - i \int_{n0} \tau_n^1 C_1]$$

The boundary conditions at $r = c$ lead to

$$(3) \begin{cases} (3a) \left\{ \begin{aligned} \sum_{n=1}^{\infty} D_n \left[a_n^t j_n \Pi_n^1 C_1 - i b_n^t \int_n \tau_n^1 C_1 - a_n^r h_n \Pi_n^1 C_1 + i b_n^r \mathcal{H}_n \tau_n^1 C_1 \right] &= \sum_{n=1}^{\infty} D_n \left[j_{n0} \Pi_n^1 C_1 - i \int_{n0} \tau_n^1 C_1 \right] \\ \sum_{n=1}^{\infty} D_n \left[-a_n^t j_n \tau_n^1 S_1 + i b_n^t \int_n \Pi_n^1 S_1 + a_n^r h_n \tau_n^1 S_1 - i b_n^r \mathcal{H}_n \Pi_n^1 S_1 \right] &= \sum_{n=1}^{\infty} D_n \left[-j_{n0} \tau_n^1 S_1 + i \int_{n0} \Pi_n^1 S_1 \right] \end{aligned} \right. \\ (3b) \left\{ \begin{aligned} \sum_{n=1}^{\infty} D_n \left[\frac{k_1}{\omega \mu_1} \left[-b_n^t j_n \Pi_n^1 S_1 + i a_n^t \int_n \tau_n^1 S_1 \right] + \frac{k_2}{\omega \mu_2} \left[b_n^r h_n \Pi_n^1 S_1 - i a_n^r \mathcal{H}_n \tau_n^1 S_1 \right] \right] &= \frac{k_2}{\omega \mu_2} \sum_{n=1}^{\infty} D_n \left[-j_{n0} \Pi_n^1 S_1 + i \int_{n0} \tau_n^1 S_1 \right] \\ \sum_{n=1}^{\infty} D_n \left[\frac{k_1}{\omega \mu_1} \left[-b_n^t j_n \tau_n^1 C_1 + i a_n^t \int_n \Pi_n^1 C_1 \right] + \frac{k_2}{\omega \mu_2} \left[b_n^r h_n \tau_n^1 C_1 - i a_n^r \mathcal{H}_n \Pi_n^1 C_1 \right] \right] &= \frac{k_2}{\omega \mu_2} \sum_{n=1}^{\infty} D_n \left[-j_{n0} \tau_n^1 C_1 + i \int_{n0} \Pi_n^1 C_1 \right] \end{aligned} \right. \end{cases}$$

The system (3) will be formally satisfied if there are constant $a_n^t, b_n^t, a_n^r, b_n^r$ such that

$$(4) \begin{cases} (4a) \left\{ \begin{aligned} a_n^t j_n(\alpha m) \tau_n^1 - i b_n^t \int_n(\alpha m) \tau_n^1 - a_n^r h_n(\alpha) \Pi_n^1 + i b_n^r \mathcal{H}_n(\alpha) \tau_n^1 &= j_n(\alpha) \Pi_n^1 - i \int_n(\alpha) \tau_n^1 \\ -a_n^t j_n(\alpha m) \tau_n^1 + i b_n^t \int_n(\alpha m) \Pi_n^1 + a_n^r h_n(\alpha) \tau_n^1 - i b_n^r \mathcal{H}_n(\alpha) \Pi_n^1 &= -j_n(\alpha) \tau_n^1 + i \int_n(\alpha) \Pi_n^1 \end{aligned} \right. \\ (4b) \left\{ \begin{aligned} -b_n^t m \mu_2 j_n(\alpha m) \Pi_n^1 + i a_n^t m \mu_2 \int_n(\alpha m) \tau_n^1 + b_n^r \mu_1 h_n(\alpha) \Pi_n^1 - i a_n^r \mu_1 \mathcal{H}_n(\alpha) \tau_n^1 &= \mu_1 \left[-j_n(\alpha) \Pi_n^1 + i \int_n(\alpha) \tau_n^1 \right] \\ -b_n^t m \mu_2 j_n(\alpha m) \tau_n^1 + i a_n^t m \mu_2 \int_n(\alpha m) \Pi_n^1 + b_n^r \mu_1 h_n(\alpha) \tau_n^1 - i a_n^r \mu_1 \mathcal{H}_n(\alpha) \Pi_n^1 &= \mu_1 \left[-j_n(\alpha) \tau_n^1 + i \int_n(\alpha) \Pi_n^1 \right] \end{aligned} \right. \end{cases} \quad \forall \alpha \in [0, \pi],$$

which is equivalent to

$$(5) \begin{cases} (5a) \left\{ \begin{aligned} \left| b_n^t \int_n(\alpha m) - b_n^r \mathcal{H}_n(\alpha) \right| \left[\left(\tau_n^1 \right)^2 - \left(\tau_n^1 \right)^2 \right] &= \int_n(\alpha) \left[\left(\tau_n^1 \right)^2 - \left(\tau_n^1 \right)^2 \right] \\ \left| a_n^t j_n(\alpha m) - a_n^r h_n(\alpha) \right| \left[\left(\tau_n^1 \right)^2 - \left(\tau_n^1 \right)^2 \right] &= j_n(\alpha) \left[\left(\tau_n^1 \right)^2 - \left(\tau_n^1 \right)^2 \right] \end{aligned} \right. \\ (5b) \left\{ \begin{aligned} \left| a_n^t m \mu_2 \int_n(\alpha m) - a_n^r \mu_1 h_n(\alpha) \right| \left[\left(\tau_n^1 \right)^2 - \left(\tau_n^1 \right)^2 \right] &= \mu_1 j_n(\alpha) \left[\left(\tau_n^1 \right)^2 - \left(\tau_n^1 \right)^2 \right] \\ \left| -b_n^t m \mu_2 j_n(\alpha m) + b_n^r \mu_1 h_n(\alpha) \right| \left[\left(\tau_n^1 \right)^2 - \left(\tau_n^1 \right)^2 \right] &= -\mu_1 j_n(\alpha) \left[\left(\tau_n^1 \right)^2 - \left(\tau_n^1 \right)^2 \right] \end{aligned} \right. \end{cases}$$

The system (5) is identically satisfied by the solutions of the systems

$$(6a) \left\{ \begin{aligned} a_n^t j_n(\alpha m) - a_n^r h_n(\alpha) &= j_n(\alpha) \\ a_n^t m \mu_2 \int_n(\alpha m) - a_n^r \mu_1 \mathcal{H}_n(\alpha) &= \mu_1 \int_n(\alpha) \end{aligned} \right\} \quad \text{and} \quad (6b) \left\{ \begin{aligned} b_n^t \int_n(\alpha m) - b_n^r \mathcal{H}_n(\alpha) &= \int_n(\alpha) \\ -b_n^t m \mu_2 j_n(\alpha m) + b_n^r \mu_1 h_n(\alpha) &= -\mu_1 j_n(\alpha) \end{aligned} \right\}$$

i. e., by

$$(7a) \left\{ \begin{aligned} a_n^r &= -\frac{\mu_1 j_n(\alpha m) \alpha \int_n(\alpha) - \mu_2 j_n(\alpha) \alpha m \int_n(\alpha m)}{\mu_1 j_n(\alpha m) \mathcal{H}_n(\alpha) - \mu_2 h_n(\alpha) m \int_n(\alpha m)} \\ a_n^t &= \frac{\mu_1 j_n(\alpha m) \alpha \mathcal{H}_n(\alpha) - \mu_1 h_n(\alpha) \alpha \int_n(\alpha)}{\mu_1 j_n(\alpha m) \alpha \mathcal{H}_n(\alpha) - \mu_2 h_n(\alpha) \alpha m \int_n(\alpha m)} \end{aligned} \right\} \quad \text{and} \quad (7b) \left\{ \begin{aligned} b_n^r &= -\frac{\mu_1 j_n(\alpha) \int_n(\alpha m) - \mu_2 m j_n(\alpha m) \int_n(\alpha)}{\mu_1 h_n(\alpha) \int_n(\alpha m) - \mu_2 m j_n(\alpha m) \mathcal{H}_n(\alpha)} \\ b_n^t &= \frac{\mu_1 h_n(\alpha) \int_n(\alpha) - \mu_1 j_n(\alpha) \mathcal{H}_n(\alpha)}{\mu_1 h_n(\alpha) \int_n(\alpha m) - \mu_2 m j_n(\alpha m) \mathcal{H}_n(\alpha)} \end{aligned} \right\}$$

The corresponding table for $\bar{n}_{\lambda}^{(l)}$ is obtained by formally replacing $\bar{m}$ by $\bar{n}$ in all entries above.

Also

Quantity	Stratton	This Report (Volume II)
radius of sphere	a	c
size ratio	$\rho = k_2 a$	$\alpha = k_2 c$
relative refractive index	$N = \frac{k_1}{k_2}$	$m = \frac{k_1}{k_2}$

Recall that (Figure 1-1)

$$k_2 \in \mathcal{R}_0$$

and

$$k_1 \in \mathcal{C}_1,$$

and hence

$$m \in \mathcal{C}_1.$$

4.1.3 Critique of the Formally-Obtained Solution

Note that in (7a) and (7b) on Table 4-1, it is not immediately evident that the denominators are always non-zero.

4.2 JUSTIFICATION OF FORMAL SOLUTION

This section removes all the objections raised in Section 4-1.

4.2.1 Direct Proof of Plane Wave Expansion

The proof of the plane wave expansion (4-3) is based on the formula (Volume I, Chapter IV, Section 7)

$$\exp[ik_2 r \cos \theta] = \sum_{n=0}^{\infty} i^n (2n+1) j_n(k_2 r) P_n(\cos \theta), \quad (4-5)$$

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which is a special case of the degenerate form of Gegenbauer's addition theorem (Theorem 7 in Volume I, Chapter IV (Reference 11) corresponding to $\nu = +1/2$ and of the formula given in Volume I, Chapter 4, Section 6 (104) (Reference 11)

$$z^{-1} \exp(iz\eta) = \frac{\cos z}{z} + i\eta \frac{\sin z}{z} - i(1 - \eta^2) \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} P'_n(\eta) j_n(z) \quad (4-6)$$

for

$$\begin{cases} z \equiv k_2 r \\ \eta \equiv \cos \theta \end{cases} \quad (4-7)$$

which follows from the above theorem by setting $\nu = -1/2$.

The convenient notation (4-7) will be followed. The convergence properties of the series in (4-6) are naturally those stated for the general theorem and are sufficient to guarantee rearrangements and term-by-term differentiation, the resulting series again converging subnormally on E^3 , etc. It will be evident from the proof that the statement of the theorem, though adequate for the present work, may be generalized.

Theorem 14:

For any real r and θ and for any $k_2 \in \mathcal{R}_0$,

$$\begin{aligned} \hat{i} \exp[ik_2 r \cos \theta] &= \sum_{n=1}^{\infty} D_n \left[\bar{m}_{21n}^{(0)} - i \bar{n}_{11n}^{(0)} \right] \\ &= \left[-i \sum_{n=1}^{\infty} D_n \frac{n(n+1)}{z} j_n P_n^1 C_1 \right]_{\hat{r}} \\ &\quad + \sum_{n=1}^{\infty} D_n \left[j_n \mathcal{M}_n^1 C_1 - i \frac{1}{z} [z j_n]' \tau_n^1 C_1 \right]_{\hat{\theta}} \\ &\quad + \sum_{n=1}^{\infty} D_n \left[-j_n \tau_n^1 S_1 + i \frac{1}{z} [z j_n]' \mathcal{M}_n^1 S_1 \right]_{\hat{\varphi}} \quad (4-8) \end{aligned}$$

in which

$$D_n = i^n \frac{2n+1}{n(n+1)} \quad (4-9)$$

and

$$z = k_2 r$$

The series on the RHS of (4-8) converge subnormally on E^3 .

Proof:

$$\text{Since } \hat{i} = \sin \theta \cos \varphi \hat{r} + \cos \theta \cos \varphi \hat{\theta} - \sin \varphi \hat{\phi}$$

and

$$\begin{aligned} \bar{m}_{21n} &= j_n \Pi_n^1 C_1 \hat{\theta} - j_n \tau_n^1 S_1 \hat{\phi} \\ \bar{n}_{11n} &= \frac{n(n+1)}{z} j_n P_n^1 C_1 \hat{r} + \frac{1}{z} [z j_n]' \tau_n^1 C_1 \hat{\theta} - \frac{1}{z} [z j_n]' \Pi_n^1 S_1 \hat{\phi} \end{aligned}$$

it suffices to show that

$$e^{iz\eta} \sin \theta \cos \theta = -i \sum_{n=1}^{\infty} D_n \frac{n(n+1)}{z} j_n P_n^1 C_1 \quad (4-10)$$

$$e^{iz\eta} \cos \theta \cos \varphi = \sum_{n=1}^{\infty} D_n \left[j_n \Pi_n^1 C_1 - i \frac{1}{z} [z j_n]' \tau_n^1 C_1 \right] \quad (4-11)$$

$$e^{iz\eta} (-\sin \varphi) = \sum_{n=1}^{\infty} D_n \left[-j_n \tau_n^1 S_1 + i \frac{1}{z} [z j_n]' \Pi_n^1 S_1 \right] \quad (4-12)$$

are identities.

To establish (4-10), it suffices to establish

$$\sin \theta e^{iz\eta} = -i \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \frac{n(n+1)}{z} j_n P_n^1, \quad (4-13)$$

which is

$$\begin{aligned}
 \sqrt{1-\eta^2} \sum_{n=0}^{\infty} i^n (2n+1) P_n(\eta) j_n(z) &= -i \frac{1}{z} \sum_{n=1}^{\infty} i^n (2n+1) P_n^1 j_n \\
 &= -i \frac{1}{z} \sum_{n=1}^{\infty} i^n (2n+1) \sqrt{1-\eta^2} \frac{dP_n}{d\eta} j_n \\
 &= -i \frac{1}{z} \sqrt{1-\eta^2} \frac{\partial}{\partial \eta} \left\{ \sum_{n=1}^{\infty} i^n (2n+1) P_n(\eta) j_n(z) \right\}
 \end{aligned}
 \tag{4-14}$$

and this is clearly the identity

$$\sqrt{1-\eta^2} e^{iz\eta} = -\frac{1}{z} \sqrt{1-\eta^2} \frac{\partial}{\partial \eta} \left\{ e^{iz\eta} - j_0(z) \right\} = e^{iz\eta}.
 \tag{4-15}$$

To show (4-11), note that

$$\Pi_n^1 = \frac{P_n^1}{\sin \theta} = P_n^1
 \tag{4-16}$$

and

$$\begin{aligned}
 \tau_n^1 &= \frac{\partial P_n^1}{\partial \theta} = -\sin \theta \frac{dP_n^1}{d\eta} = -\sqrt{1-\eta^2} \frac{d}{d\eta} \left\{ (1-\eta^2)^{\frac{1}{2}} \frac{dP_n}{d\eta} \right\} \\
 &= -(1-\eta^2)^{1/2} \left\{ (1-\eta^2)^{1/2} P_n'' - \eta (1-\eta^2)^{-1/2} P_n' \right\} \\
 &= \left\{ (1-\eta^2) P_n'' - \eta P_n' \right\} \\
 &= - \left\{ \left[(1-\eta^2) P_n'' - 2\eta P_n' \right] + \eta P_n' \right\} \\
 &= - \left\{ [-n(n+1) P_n] + \eta P_n' \right\} \\
 &= n(n+1) P_n - \eta P_n'.
 \end{aligned}
 \tag{4-17}$$

Using these identities, it is seen that (4-11) would follow from

$$e^{iz\eta} \cos \theta \cos \varphi = \sum_{n=1}^{\infty} D_n \left\{ j_n P_n^1 C_1 - i \frac{1}{z} [z j_n]' \tau_n^1 C_1 \right\} \quad (4-18)$$

i. e.

$$\begin{aligned} \eta \sum_{n=0}^{\infty} i^n (2n+1) j_n P_n &= \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left\{ j_n P_n^1 - i \frac{1}{z} [z j_n]' \tau_n^1 - i j_n' \tau_n^1 \right\} \\ &= \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left\{ j_n P_n^1 - i \left[\frac{1}{z} j_n + j_n' \right] \left[n(n+1) P_n - \eta P_n^1 \right] \right\} \end{aligned} \quad (4-19)$$

if it could be shown that

$$\begin{aligned} \eta \sum_{n=0}^{\infty} i^n (2n+1) j_n P_n &= \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left\{ -i \left[\frac{1}{z} j_n + j_n' \right] \left[n(n+1) P_n \right] \right. \\ &\quad \left. + \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left\{ j_n + i \left[\frac{1}{z} j_n + j_n' \right] \eta \right\} P_n^1 \right\} \\ &= \frac{-i}{z} \sum_{n=1}^{\infty} i^n (2n+1) j_n P_n - i \sum_{n=1}^{\infty} i^n (2n+1) j_n' P_n \\ &\quad + \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left\{ j_n + i \left[\frac{1}{z} j_n + j_n' \right] \eta \right\} P_n^1, \end{aligned} \quad (4-20)$$

and this is

$$\begin{aligned} \eta e^{iz\eta} &= \frac{-i}{z} \left\{ e^{iz\eta} - j_0(z) \right\} - i \frac{\partial}{\partial z} \left\{ e^{iz\eta} - j_0(z) \right\} \\ &\quad + \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} j_n P_n^1 + \frac{i}{z} \eta \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} j_n P_n^1 + i \eta \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} j_n' P_n^1 \\ &= \frac{-i}{z} \left\{ e^{iz\eta} - z^{-1} \sin z \right\} - i \frac{\partial}{\partial z} \left\{ e^{iz\eta} - z^{-1} \sin z \right\} \\ &\quad + (1 - i\eta z^{-1}) f(\eta, z) + i\eta f(\eta, z), \end{aligned} \quad (4-21)$$

in which

$$f(\eta, z) \equiv \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} P'_n(\eta) j_n(z). \quad (4-22)$$

Suppose $\eta \neq \pm 1$. Then (4-22) may be written (Volume I, Chapter IV (104))

$$\begin{aligned} f(\eta, z) &= \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} P'_n(\eta) j_n(z) \\ &= i(i-\eta^2)^{-1} z^{-1} e^{iz\eta} - i(1-\eta^2)^{-1} z^{-1} \cos z + \eta(1-\eta^2)^{-1} z^{-1} \sin z. \end{aligned} \quad (4-23)$$

This identity may be used to show

$$\begin{aligned} \eta e^{iz} &= -\frac{i}{z} \left[e^{iz\eta} - z^{-1} \sin z \right] - i \frac{\partial}{\partial z} \left[e^{iz\eta} - z^{-1} \sin z \right] \\ &\quad + (1 + i\eta z^{-1}) f(\eta, z) + i\eta f_z(\eta, z) \end{aligned} \quad (4-24)$$

by a straightforward algebraic exercise.

This shows that (4-21) and hence also (4-11) is an identity.

Finally, to show (4-12):

$$-S_1 e^{iz\eta} = \sum_{n=1}^{\infty} D_n \left[-j_n r_n^1 S_1 + i \frac{1}{z} [z j_n]' P_n^1 S_1 \right]. \quad (4-25)$$

It suffices to demonstrate

$$\begin{aligned} -\sum_{n=0}^{\infty} i^n (2n+1) P_n(\eta) j_n(z) &= \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left\{ -j_n [n(n+1) P_n - \eta P_n'] + i \left[\frac{1}{z} j_n + j_n' \right] P_n' \right\} \\ &= -\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} j_n n(n+1) P_n + \eta \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} j_n P_n' \\ &\quad + i \frac{1}{z} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} j_n P_n' + i \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} j_n' P_n' \end{aligned} \quad (4-26)$$

i. e.,

$$-e^{iz\eta} = -\left[e^{-iz\eta} - z^{-1} \sin z\right] + (\eta + i\frac{1}{z})f(\eta, z) + if_z(\eta, z). \quad (4-27)$$

That (4-27) is an identity follows from (4-23) by a straightforward exercise.

The identities (4-11) and (4-12) have been established for η complex and not equal to ± 1 . The remainder of the proof follows by analytic continuation and the convergence properties of the series in the degenerate Gegenbauer's addition theorem (Theorem 7, Volume I, Chapter IV (Reference 11)) from which (4-22) is derived, and by a well-known theorem on induced convergence (Hille, Reference 7, page 192).

To form the plane-wave expansion of $\hat{h}_0$ in (4-5) using the Theorem above and also Theorem 1, Volume I, Chapter VI (Reference 11)

$$\begin{aligned} \nabla \times \left[\hat{i} \exp(ik_2 r \cos \theta) \right] &= \nabla \times \left\{ \sum_{n=1}^{\infty} D_n \left[\bar{m}_{21n}^{(0)} - i \bar{n}_{11n}^{(0)} \right] \right\} \\ &= \sum_{n=1}^{\infty} D_n \left[\left(\nabla \times \bar{m}_{21n}^{(0)} \right) - i \left(\nabla \times \bar{n}_{11n}^{(0)} \right) \right] \\ &= \sum_{n=1}^{\infty} D_n \left[k_2 \bar{n}_{21n}^{(0)} - i k_2 \bar{m}_{11n}^{(0)} \right] \\ &= -ik_2 \sum_{n=1}^{\infty} D_n \left[\bar{m}_{11n}^{(0)} + i \bar{n}_{21n}^{(0)} \right]. \end{aligned} \quad (4-28)$$

Then

$$\bar{H}_0 = \frac{-i}{\bar{\omega}\mu_2} \nabla \times \left[\hat{i} \exp(ik_2 r \cos \theta) \right] = \frac{-k_2}{\bar{\omega}\mu_2} \sum_{n=1}^{\infty} D_n \left[\bar{m}_{11n}^{(0)} + i \bar{n}_{21n}^{(0)} \right]. \quad (4-29)$$

It follows that

$$\begin{aligned}
\epsilon_0 &= \begin{bmatrix} E_0 \sum D_n \left[\bar{m}_{21n}^{(0)} - i \bar{n}_{11n}^{(0)} \right] \\ \frac{-i}{\tilde{\omega}\mu_2} \sum D_n \left[\left(\nabla \times \bar{m}_{21n}^{(0)} \right) - i \left(\nabla \times \bar{n}_{11n}^{(0)} \right) \right] \end{bmatrix} \\
&= E_0 \sum_{n=1}^{\infty} D_n \left\{ \begin{bmatrix} \bar{m}_{21n}^{(0)} \\ \frac{-ik_2}{\tilde{\omega}\mu_2} \left[\frac{1}{k_2} \nabla \times \bar{m}_{21n}^{(0)} \right] \end{bmatrix} - i \begin{bmatrix} \bar{n}_{11n}^{(0)} \\ \frac{-ik_2}{\tilde{\omega}\mu_2} \left(\frac{1}{k_2} \nabla \times \bar{n}_{11n}^{(0)} \right) \end{bmatrix} \right\} \\
\epsilon_0 &= E_0 \sum_{n=1}^{\infty} D_n \left[\epsilon_{21n}^{0E} - i \epsilon_{11n}^{0H} \right].
\end{aligned}$$

A more useful form for the purposes below is obtained from (4-29):

$$\epsilon_0 = \begin{bmatrix} E_0 \sum D_n \left[\bar{m}_{21n}^{(0)} - i \bar{n}_{11n}^{(0)} \right] \\ \frac{-k_2}{\tilde{\omega}\mu_2} E_0 \sum D_n \left[\bar{m}_{11n}^{(0)} + i \bar{n}_{21n}^{(0)} \right] \end{bmatrix} \quad (4-30)$$

$$\begin{aligned}
&j_{n0} \pi_n^1 C_1 - i \int_{n0} \tau_n^1 C_1 \\
&-j_{n0} \tau_n^1 S_1 + i \int_{n0} \pi_n^1 S_1 \\
\mathcal{J}\epsilon_0 &= E_0 \sum_{n=1}^{\infty} D_n \begin{bmatrix} -j_{n0} \pi_n^1 S_1 + i \int_{n0} \tau_n^1 S_1 \\ \frac{-k_2}{\tilde{\omega}\mu_2} \left[-j_{n0} \tau_n^1 C_1 + i \int_{n0} \pi_n^1 C_1 \right] \end{bmatrix} \quad (4-31)
\end{aligned}$$

4.2.2 Interpretation of the Classical Method in the Light of the General Method

The approach taken in the following sections will be:

- 1) Assume that the problem is solvable
- 2) Define the constant coefficients conveniently
- 3) Prove that these constant coefficients are well-defined

- 4) Prove that the above sequence of coefficients of the problem really is solvable; i. e., that

$$\exists \bar{v} \in \mathcal{X} \ni T_0 \bar{v} = \mathcal{J} \mathcal{C}_0.$$

It turns out that the solution is exactly the classical solution. The above procedure shows that the classical solution arrived at by formal techniques is rigorously valid.

Since the process outlined in (1) through (4) above can be carried out for any $a, b \ni$

$$0 < a < \inf_{\omega \in \Omega} f(\omega) \leq \sup_{\omega \in \Omega} f(\omega) < b < \infty$$

(Figure 1-8), it follows, by the use of certain identity theorems that electromagnetic fields which comprise solutions to scattering problems of the form considered in this chapter are capable of being extended into domains which properly include the ones on which they were originally defined.* The details of these considerations will be reserved for further investigations not in this series of reports.

Assume that $\mathcal{P}(S_c, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \infty, \mathcal{C}_0)$ is solvable. Then, for some choice of $a, b \ni$

$$0 < a < c < b < \infty,$$

$$\begin{aligned} \exists \bar{v} \in \mathcal{X} \ni \\ T_0 \bar{v} = \mathcal{J} \mathcal{C}_0. \end{aligned} \quad (4-32)$$

To identify this $\bar{v}$ denote,

$$\bar{v} = (v_1, v_2, v_3, v_4)^T \quad (4-33a)$$

and suppose

$$v_j = \sum \{ b_{\lambda j} \phi_\lambda \mid \lambda \in \Lambda_1 \cup \Lambda_2 \}, \quad j \in \langle 4 \rangle. \quad (4-33b)$$

* A physical interpretation of this result amounts to an "Inverse of Huygens' Principle".

Denote

$$\begin{aligned} c_\lambda &\equiv -\frac{k_2 h' n(k_2 a)}{A_{mn}}, \quad \lambda = (n, m, i) \in (\Lambda_1 \cup \Lambda_2) \times \{1, 2\} \\ d_\lambda &\equiv \frac{k_2 j_n'(k_1 b)}{A_{mn}}, \quad \lambda = (n, m, i) \in (\Lambda_1 \cup \Lambda_2) \times \{3, 4\}, \end{aligned} \quad (4-34)$$

and define the sequence

$$|a_{\lambda j}| \mid (\lambda, j) \in \Lambda' \mid \quad (4-35)$$

by

$$b_{\lambda j} = \begin{cases} E_0 c_\lambda a_{\lambda j} & , \text{ for } (\lambda, j) \in (\Lambda_1 \cup \Lambda_2) \times \{1, 2\} \\ E_0 d_\lambda a_{\lambda j} & , \text{ for } (\lambda, j) \in (\Lambda_1 \cup \Lambda_2) \times \{3, 4\} . \end{cases} \quad (4-36)$$

Then the series (4-33b) may be written

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = E_0 \sum \left\{ \begin{bmatrix} \left(\frac{b_{\lambda 1}}{E_0 c_\lambda} \right) (c_\lambda \phi_\lambda) \\ \left(\frac{b_{\lambda 2}}{E_0 c_\lambda} \right) (c_\lambda \phi_\lambda) \\ \left(\frac{b_{\lambda 3}}{E_0 d_\lambda} \right) (d_\lambda \phi_\lambda) \\ \left(\frac{b_{\lambda 4}}{E_0 d_\lambda} \right) (d_\lambda \phi_\lambda) \end{bmatrix} \mid \lambda \in \Lambda_1 \cup \Lambda_2 \right\} \quad (4-37)$$

which is just

$$\bar{v} = E_0 \sum |a_{\lambda j} \chi_{\lambda j}| \mid (\lambda, j) \in \Lambda' \mid . \quad (4-38)$$

This means that (4-32) is

$$E_0 \sum |a_{\lambda j} (T \chi_{\lambda j})| \mid (\lambda, j) \in \Lambda' \mid = \mathcal{J} \mathcal{E}_0 . \quad (4-39)$$

From Table 3-1 and (4-31), this equation may be written

$$\begin{aligned}
 E_0 \mathcal{J} \sum & \left\{ a_{1mn1} \begin{bmatrix} \overline{m}_{1mn}^{(2)} \\ \frac{-ik_2}{\tilde{\omega}_{\mu 2}} \overline{n}_{1mn}^{(2)} \end{bmatrix} + a_{2mn1} \begin{bmatrix} \overline{m}_{2mn}^{(2)} \\ \frac{-ik_2}{\tilde{\omega}_{\mu 2}} \overline{n}_{2mn}^{(1)} \end{bmatrix} \right. \\
 & + a_{1mn2} \begin{bmatrix} \overline{n}_{1mn}^{(2)} \\ \frac{-ik_2}{\tilde{\omega}_{\mu 2}} \overline{m}_{1mn}^{(2)} \end{bmatrix} + a_{2mn2} \begin{bmatrix} \overline{n}_{2mn}^{(2)} \\ \frac{-ik_2}{\tilde{\omega}_{\mu 2}} \overline{m}_{2mn}^{(2)} \end{bmatrix} \\
 & + a_{1mn3} \begin{bmatrix} \overline{m}_{1mn}^{(1)} \\ \frac{-ik_1}{\tilde{\omega}_{\mu 1}} \overline{n}_{1mn}^{(1)} \end{bmatrix} + a_{2mn3} \begin{bmatrix} \overline{m}_{2mn}^{(1)} \\ \frac{-ik_1}{\tilde{\omega}_{\mu 1}} \overline{n}_{2mn}^{(1)} \end{bmatrix} \\
 & \left. + a_{1mn4} \begin{bmatrix} \overline{n}_{1mn}^{(1)} \\ \frac{-ik_1}{\tilde{\omega}_{\mu 1}} \overline{m}_{1mn}^{(1)} \end{bmatrix} + a_{2mn4} \begin{bmatrix} \overline{n}_{2mn}^{(1)} \\ \frac{-ik_2}{\tilde{\omega}_{\mu 2}} \overline{m}_{2mn}^{(1)} \end{bmatrix} \right\} (n, m) \in \mathcal{Y}_1 \cup \mathcal{Y}_2 \\
 & = E_0 \sum_{n=1}^{\infty} D_n \begin{bmatrix} (j_{n0} \Pi_n^1 - i \not{j}_{n0} \tau_n^1) C_1 \\ (-j_{n0} \tau_n^1 + i \not{j}_{n0} \Pi_n^1) S_1 \\ \frac{-k_2}{\tilde{\omega}_{\mu 2}} \begin{bmatrix} (-j_{n0} \Pi_n^1 + i \not{j}_{n0} \tau_n^1) S_1 \\ (-j_{n0} \tau_n^1 + i \not{j}_{n0} \Pi_n^1) C_1 \end{bmatrix} \end{bmatrix}
 \end{aligned}$$

(4-40)

However, by Table 3-1 and the orthogonality properties of the sequence $|C_m, S_m|$, the LHS of the above equation reduces to

$$E_0 \sum_{n=1}^{\infty} \left\{ \begin{aligned} & a_{21n1} \begin{bmatrix} h_n \Pi_n^1 C_1 \\ -h_n \tau_n^1 S_1 \\ \frac{-ik_2}{\omega \mu_2} \begin{bmatrix} \mathcal{H}_n \tau_n^1 S_1 \\ \mathcal{H}_n \Pi_n^1 C_1 \end{bmatrix} \end{bmatrix} + a_{11n2} \begin{bmatrix} \mathcal{H}_n \tau_n^1 C_1 \\ -\mathcal{H}_n \Pi_n^1 S_1 \\ \frac{-ik_2}{\tilde{\omega} \mu_2} \begin{bmatrix} -h_n \Pi_n^1 S_1 \\ -h_n \tau_n^1 C_1 \end{bmatrix} \end{bmatrix} \\ & + a_{21n3} \begin{bmatrix} j_n \Pi_n^1 C_1 \\ -j_n \tau_n^1 S_1 \\ \frac{-ik_1}{\omega \mu_1} \begin{bmatrix} \mathcal{J}_n \tau_n^1 S_1 \\ \mathcal{J}_n \Pi_n^1 C_1 \end{bmatrix} \end{bmatrix} + a_{11n4} \begin{bmatrix} \mathcal{J}_n \tau_n^1 C_1 \\ -\mathcal{J}_n \Pi_n^1 S_1 \\ \frac{-ik_1}{\omega \mu_1} \begin{bmatrix} -j_n \Pi_n^1 S_1 \\ -j_n \tau_n^1 C_1 \end{bmatrix} \end{bmatrix} \end{aligned} \right\} \quad (4-41)$$

Denote

$$\Lambda_3 \equiv \left\{ (n, 1, 2, 1), (n, 1, 1, 2), (n, 1, 2, 3), (n, 1, 1, 4) \right\} \subset (\Lambda_1 \cup \Lambda_2) \times \langle 4 \rangle$$

Then

$$a_{\lambda j} = 0 \quad \forall (\lambda, j) \in (\Lambda' - \Lambda_3).$$

Denote

$$a_{\lambda j} = D_n a_{\lambda j}, \quad (\lambda, j) \in \Lambda'.$$

Then (4-41) is

$$\sum_{n=1}^{\infty} D_n \begin{bmatrix} h_n \Pi_n^1 C_1 & \mathcal{H}_n \tau_n^1 C_1 & j_n \Pi_n^1 C_1 & \mathcal{J}_n \tau_n^1 C_1 \\ -h_n \tau_n^1 S_1 & -\mathcal{H}_n \Pi_n^1 S_1 & -j_n \tau_n^1 S_1 & -\mathcal{J}_n \Pi_n^1 S_1 \\ \frac{-ik_2}{\omega \mu_2} \mathcal{H}_n \tau_n^1 S_1 & \frac{ik_2}{\omega \mu_2} h_n \Pi_n^1 S_1 & \frac{-ik_1}{\omega \mu_1} \mathcal{J}_n \tau_n^1 S_1 & \frac{ik_1}{\omega \mu_1} j_n \Pi_n^1 S_1 \\ \frac{-ik_2}{\omega \mu_2} \mathcal{H}_n \Pi_n^1 C_1 & \frac{ik_2}{\omega \mu_2} h_n \tau_n^1 C_1 & \frac{-ik_1}{\omega \mu_1} \mathcal{J}_n \Pi_n^1 C_1 & \frac{ik_1}{\omega \mu_1} j_n \tau_n^1 C_1 \end{bmatrix} \begin{bmatrix} a_{21n1} \\ a_{11n2} \\ a_{21n3} \\ a_{11n4} \end{bmatrix} \quad (4-42)$$

From (4-41) and (4-40), it can be seen that (4-40) would be satisfied if constants $\{a_{\lambda j}\}$ could be found such that

$$\begin{bmatrix} h_n \Pi_n^1 & \mathcal{H}_n \tau_n^1 & j_n \Pi_n^1 & \mathcal{J}_n \tau_n^1 \\ -h_n \tau_n^1 & -\mathcal{H}_n \Pi_n^1 & -j_n \tau_n^1 & -\mathcal{J}_n \Pi_n^1 \\ \frac{-ik_2}{\omega \mu_2} \mathcal{H}_n \tau_n^1 & \frac{ik_2}{\omega \mu_2} h_n \Pi_n^1 & \frac{-ik_1}{\omega \mu_1} \mathcal{J}_n \tau_n^1 & \frac{ik_1}{\omega \mu_1} j_n \Pi_n^1 \\ \frac{-ik_2}{\omega \mu_2} h_n \tau_n^1 & \frac{ik_2}{\omega \mu_2} \mathcal{H}_n \Pi_n^1 & \frac{-ik_1}{\omega \mu_1} j_n \tau_n^1 & \frac{ik_1}{\omega \mu_1} \mathcal{J}_n \Pi_n^1 \end{bmatrix} \begin{bmatrix} a_{21n1} \\ a_{11n2} \\ a_{21n3} \\ a_{11n4} \end{bmatrix} = \begin{bmatrix} j_{n0} \Pi_n^{1-i} \mathcal{J}_{n0} \tau_n^1 \\ -j_{n0} \tau_n^{1+i} \mathcal{J}_{n0} \Pi_n^1 \\ \frac{-k_2}{\omega \mu_2} (-j_{n0} \Pi_n^{1+i} \mathcal{J}_{n0} \tau_n^1) \\ \frac{-k_2}{\omega \mu_2} (-j_{n0} \tau_n^{1+i} \mathcal{J}_{n0} \Pi_n^1) \end{bmatrix} \quad (4-43)$$

Denote

$$\begin{aligned} a_n^t &= A_{21n3} & -a_n^r &= a_{21n1} \\ -ib_n^t &= A_{11n4} & +ib_n^r &= a_{11n2} \end{aligned} \quad (4-44)$$

and multiply the third and fourth equations of (4-43) by $-\tilde{\omega}\mu_1\mu_2/k_2$ to obtain

$$\begin{bmatrix} -h_n \Pi_n^1 & i \mathcal{H}_n \tau_n^1 & j_n \Pi_n^1 & -i \int_n \tau_n^1 \\ +h_n \tau_n^1 & -i \mathcal{H}_n \Pi_n^1 & -j_n \tau_n^1 & +i \int_n \Pi_n^1 \\ -i\mu_1 \mathcal{H}_n \tau_n^1 & \mu_1 h_n \Pi_n^1 & +i\mu_2 m \int_n \tau_n^1 & -\mu_2 m j_n \Pi_n^1 \\ -i\mu_1 \mathcal{H}_n \Pi_n^1 & \mu_1 h_n \tau_n^1 & i\mu_2 m \int_n \Pi_n^1 & -m\mu_2 j_n \tau_n^1 \end{bmatrix} \begin{bmatrix} a_n^r \\ b_n^r \\ a_n^t \\ b_n^t \end{bmatrix} = \begin{bmatrix} (j_{n0} \Pi_n^1 - i \int_{n0} \tau_n^1) \\ (-j_{n0} \tau_n^1 + i \int_{n0} \Pi_n^1) \\ \mu_1 (-j_{n0} \Pi_n^1 + i \int_{n0} \tau_n^1) \\ \mu_1 (-j_{n0} \tau_n^1 + i \int_{n0} \Pi_n^1) \end{bmatrix} \quad (4-45)$$

which is exactly the system (4) of Table 4-1.

This shows that if $\{a_n^r, b_n^r, a_n^t, b_n^t\}$ is exactly the sequence given by (7) of Table 4-1, then the sequences $\{b_{\lambda j}\}$ given by

$$b_{\lambda j} = 0 \quad \forall (\lambda, j) \in (\Lambda' - \Lambda_3) \quad (4-46a)$$

and

$$\left\{ \begin{array}{l} b_{n,1,2,1} \equiv b_{21n1} = E_0 D_n \frac{k_2 h'_n(k_2 a)}{A_{1n}} a_n^r \\ b_{n,1,1,2} \equiv b_{11n2} = -E_0 D_n \frac{k_2 h'_n(k_2 a)}{A_{1n}} i b_n^r \\ b_{n,2,1,3} \equiv b_{21n3} = E_0 D_n \frac{k_1 j'_n(k_1 b)}{A_{1n}} a_n^t \\ b_{n,1,1,4} \equiv b_{11n4} = -i E_0 D_n \frac{k_1 j'_n(k_1 b)}{A_{1n}} b_n^t, \end{array} \right. \quad (4-46b)$$

where

$$D_n = i^n \frac{2n+1}{n(n+1)} \quad (4-46c)$$

and

$$A_{1n} = \sqrt{\frac{2n+1}{2\pi}} \sqrt{n(n+1)} \quad (4-46d)$$

(as before) define a $\bar{v}$ $\ni$ (4-32) is formally satisfied. It must now be shown that the sequence of fractions in (7) of Table 4-1 are always well-defined and also that the $\bar{v}$ is actually an element of $\mathcal{K}$.

4. 2. 3 The Scattering Coefficients Are Well-Defined

The determinant of the system (7a) in Table 4-1 is

$$\Delta_n^{(1)}(a, m, \mu_1, \mu_2) = -\mu_1 \mathcal{K}_n(a) j_n(am) + m\mu_2 h_n(a) \mathcal{J}_n(am) \quad (4-47)$$

and that of (7b) in Table 4-1 is

$$\Delta_n^{(2)}(a, m, \mu_1, \mu_2) = \mu_1 \int_n(am) h_n(a) - m\mu_2 j_n(am) H_n(a) \quad (4-48)$$

Theorem 15

For $n \in \langle \mathcal{N} \rangle$ and $a, \mu_1, \mu_2 \in \mathcal{R}_0$, $m \in \mathcal{C}_1$,

$$\Delta_n^{(l)}(a, m, \mu_1, \mu_2) \neq 0, \quad (l) = 1, 2.$$

Proof: Suppose

$$\Delta_n^{(1)}(a, m, \mu_1, \mu_2) = 0 \quad (4-49)$$

for some $n \in \langle \mathcal{N} \rangle$ and $(a, \mu_1, \mu_2, m) \in \mathcal{R}_0^3 \times \mathcal{C}_1$.

Then

$$\begin{aligned}
\mu_1 \mathcal{H}'_n(a) j_n(am) &= m \mu_2 h'_n(a) \mathcal{J}'_n(am) \\
\Rightarrow \\
\left(\frac{\mu_1}{\mu_2}\right) \frac{\mathcal{H}'_n(a)}{h'_n(a)} &= m \frac{\mathcal{J}'_n(am)}{j'_n(am)} \\
\Rightarrow \\
\left(\frac{\mu_1}{\mu_2}\right) \frac{\frac{1}{a} h'_n(a) + h''_n(a)}{h'_n(a)} &= m \frac{\frac{1}{ma} j'_n(am) + j''_n(am)}{j'_n(am)} \\
\Rightarrow \\
\left(\frac{\mu_1}{\mu_2}\right) \left\{ 1 + \left[a \frac{h''_n(a)}{h'_n(a)} \right] \right\} &= \left\{ 1 + \left[(am) \frac{j''_n(am)}{j'_n(am)} \right] \right\} . \tag{4-50}
\end{aligned}$$

But by Volume I, Chapter III (44) and (45)

$$\begin{aligned}
a \frac{h'_n(a)}{h'_n(a)} &= n - \tilde{\mathcal{H}}_{n+\frac{1}{2}}(a) \\
(am) \frac{j'_n(am)}{j'_n(am)} &= n - \tilde{\mathcal{J}}_{n+\frac{1}{2}}(am)
\end{aligned}$$

and hence (4-50) implies

$$\begin{aligned}
\left(\frac{\mu_1}{\mu_2}\right) \left\{ 1 + n - \tilde{\mathcal{H}}_{n+\frac{1}{2}}(a) \right\} &= 1 + n - \tilde{\mathcal{J}}_{n+\frac{1}{2}}(am) \\
\left(\frac{\mu_1}{\mu_2}\right) \operatorname{Im} \left[\tilde{\mathcal{H}}_{n+\frac{1}{2}}(a) \right] &= \operatorname{Im} \left[\tilde{\mathcal{J}}_{n+\frac{1}{2}}(am) \right] , \tag{4-51}
\end{aligned}$$

which contradicts the results of Theorems 8 and 9 in Volume I, Chapter III. This proves the theorem for $l = 1$.

For $l = 2$,

$$\begin{aligned} \Rightarrow \Delta_n^{(2)}(a, m, \mu_1, \mu_2) &= 0 \\ (4-52) \end{aligned}$$

$$\mu_1 h_n(a) \mathcal{J}_n(am) - m \mu_2 j_n(am) \mathcal{K}_n(a) = 0$$

$$\begin{aligned} \Rightarrow \left(\frac{\mu_1}{\mu_2} \right) \left[\frac{\mathcal{J}_n(am)}{(am) j_n(am)} \right] &= \left[\frac{\mathcal{K}_n(a)}{a h_n(a)} \right]. \\ (4-53) \end{aligned}$$

This may be written

$$\left(\frac{\mu_1}{\mu_2} \right) \left\{ \frac{1}{(am)^2} + \frac{1}{am} \frac{j_n(am)}{j_n(am)} \right\} = \frac{1}{a^2} + \frac{1}{a} \frac{h'_n(a)}{h_n(a)} \quad (4-54)$$

and again by Volume I, Chapter III, Section 3 (Reference 11) the above is

$$\left\{ \frac{1}{(z)^2} + \left[\frac{n}{(z)^2} - \frac{1}{(z)^2} \left(z \frac{J_\nu + 1(z)}{J_\nu(z)} \right) \right] \right\} = \frac{1}{x^2} + \left[\frac{n}{x} - \frac{1}{x^2} \mathcal{Y}'_\nu + \frac{1}{2}(x) \right] \quad (4-55)$$

where

$$\left. \begin{aligned} \mu &\equiv \frac{\mu_1}{\mu_2} \in \mathcal{R}_0 \\ a &\equiv x \in \mathcal{R}_0 \\ z &\equiv am \in \mathcal{R}_1 \\ \nu &\equiv n + \frac{1}{2} \end{aligned} \right\} \quad (4-56)$$

By the proof of Theorem 9 in Chapter III, Volume I (Reference 11) the LHS of (4-55) is

$$\mu \left\{ \frac{(n+1)}{z^2} - \frac{1}{z} \left[-2z \sum_{n=1}^{\infty} \frac{1}{z^2 - j_{\nu, n}^2} \right] \right\}.$$

Set

$$z^2 = \xi + i\eta. \quad (4-57)$$

Then

$$\operatorname{Im} \left\{ \text{LHS of (4-55)} \right\} = -\mu \left\{ \frac{n+1}{\xi^2 + \eta^2} \eta + 2\eta \sum_{n=1}^{\infty} \frac{1}{(\xi - j_{\nu, n})^2 + \eta^2} \right\} < 0 \quad (4-58)$$

since $\eta > 0$.

On the other hand, the RHS of (4-55) may be written

$$\begin{aligned} & \frac{1}{x^2} + \frac{1}{x^2} \left[x \frac{h'_n(x)}{h_n(x)} \right] \\ &= \frac{1}{x^2} + \frac{1}{x^2} \left[n - \tilde{\mathcal{H}}_{n+\frac{1}{2}}(x) \right] = (n+1) \frac{1}{x^2} - \frac{1}{x^2} \tilde{\mathcal{H}}_{\nu+\frac{1}{2}}(x) \end{aligned}$$

and hence

$$\operatorname{Im} \left\{ \text{RHS of (4-55)} \right\} = -\frac{1}{x^2} \operatorname{Im} \left\{ \tilde{\mathcal{H}}_{\nu}(x) \right\} = + \frac{2}{\pi |H_{\nu}(x)|^2 x} > 0. \quad (4-59)$$

Thus, (4-58) and (4-59) show that (4-55) is a contradiction.

4.2.4 The Classical Problem is Solvable

It remains only to show that the series of positive constants

$$\sum \left| b_{\lambda j} \right|^2 \mid \lambda \in \Lambda_1 \cup \Lambda_2 \mid, \quad j \in \langle 4 \rangle \quad (4-60)$$

with the sequences $\{b_{\lambda j}\}$ defined by (4-46), are convergent.

The series in (4-60) reduce to

$$\begin{aligned} \sum_{n=1}^{\infty} |b_{n, 1, 2, 1}|^2 &= E_0^2 k_2^2 \sum_{n=1}^{\infty} \left| \frac{D_n}{A_{1n}} h'_n(k_2 a) a_n^r \right|^2, \quad j=1 \\ \sum_{n=1}^{\infty} |b_{n, 1, 1, 2}|^2 &= E_0^2 k_2^2 \sum_{n=1}^{\infty} \left| \frac{D_n}{A_{1n}} h'_n(n_2 a) b_n^r \right|^2, \quad j=2 \\ \sum_{n=1}^{\infty} |b_{n, 2, 1, 3}|^2 &= E_0^2 |k_1|^2 \sum_{n=1}^{\infty} \left| \frac{D_n}{A_{1n}} j'_n(k_1 b) a_n^t \right|^2, \quad j=3 \\ \sum_{n=1}^{\infty} |b_{n, 1, 1, 4}|^2 &= E_0^2 |k_1|^2 \sum_{n=1}^{\infty} \left| \frac{D_n}{A_{1n}} j'_n(k_1 b) b_n^t \right|^2, \quad j=4, \end{aligned} \quad (4-61)$$

where a_n^r , a_n^t , b_n^r , b_n^t are given by (7) of Table 4-1. Note that

$$d_n \equiv \left| \frac{D_n}{A_{1n}} \right| = \frac{2n+1}{n(n+1)} \sqrt{\frac{2\pi}{(2n+1)n(n+1)}} = (2\pi)^{1/2} \frac{(2n+1)^{1/2}}{[n(n+1)]^{3/2}}, \quad (4-62)$$

and hence

$$\lim_{n \rightarrow \infty} d_n = 0. \quad (4-63)$$

Theorem 16.

For $\ell = 1$ and 2,

$$|m| > 1 \Rightarrow \lim_{n \rightarrow \infty} |\Delta_n^{(\ell)}(a, m, \mu_1, \mu_2)| = \infty.$$

Proof: Write

$$\begin{aligned} \Delta_n^{(1)} &= -\mu_1 \mathcal{H}'_n(a) j_n(am) + m \mu_2 h_n(a) \mathcal{J}'_n(am) \\ &= -\mu_1 \left[\frac{1}{a} h_n(a) + h'_n(a) \right] j_n(am) + m \mu_2 h_n(a) \left[\frac{1}{am} j_n(am) + j'_n(am) \right] \\ &= -\mu_1 \frac{1}{a} h_n(a) j_n(am) - \mu_1 h'_n(a) j_n(am) \\ &\quad + \frac{\mu_2}{a} h_n(a) j_n(am) + m \mu_2 h_n(a) j'_n(am) \\ &= \frac{1}{a} j_n(am) h_n(a) \left\{ (\mu_2 - \mu_1) + \mu_2 \left[(am) \frac{j'_n(am)}{j_n(am)} \right] - \mu_1 \left[a \frac{h'_n(a)}{h_n(a)} \right] \right\} \end{aligned}$$

and use the asymptotic estimates of Volume I, Chapter III, Section 4 (Reference 11) to obtain

$$\Delta_n^{(1)} \sim \frac{1}{a} \left[\frac{(am)^n}{(2n+1)!!} \right] \left[-i \frac{(2n-1)!!}{(a)^{n+1}} \right] \left\{ (\mu_2 - \mu_1) + \mu_2 \left[(am) \frac{n(am)^{n-1}}{(2n+1)!!} \cdot \frac{(2n+1)!!}{(am)^n} \right] \right. \\ \left. - \mu_1 \left[ai \frac{(n+1)(2n-1)!!}{a^{n+2}} \cdot \frac{a^{n+1}}{-i(2n-1)!!} \right] \right\}$$

$$= -i \frac{m^n}{\alpha} \frac{1}{2n+1} (\mu_2 - \mu_1) + \mu_2 n + \mu_1 (n+1)$$

$$= -i \frac{m^n}{\alpha} \frac{1}{2n+1} (\mu_2 - \mu_1) + n\mu_2 + \mu_1 n + \mu_1$$

$$= i \frac{m^n}{\alpha} \frac{1}{2n+1} \mu_2 + n (\mu_2 + \mu_1)$$

$$\left| \Delta_n^{(1)} \right| = \frac{1}{\alpha} \frac{|m|^n}{2n+1} \mu_2 + \frac{1}{\alpha} \frac{|m|^n}{2n+1} n (\mu_2 + \mu_1) \rightarrow \infty .$$

For $l = 2$, the calculation is

$$\begin{aligned} \Delta_n^{(2)} &= \mu_1 j_n(am) h_n(a) - m \mu_2 j_n(am) \mathcal{H}_n(a) \\ &= \mu_1 h_n(a) \left[\frac{1}{am} j_n(am) + j_n'(am) \right] - m \mu_2 j_n(am) \left[\frac{1}{a} h_n(a) + h_n'(a) \right] \\ &= \mu_1 \frac{1}{am} h_n(a) j_n(am) + \mu_1 h_n(a) j_n'(am) \\ &\quad - \frac{m \mu_2}{a} j_n(am) h_n(a) - m \mu_2 j_n(am) h_n'(a) \\ &= \frac{1}{a} j_n(am) h_n(a) \left\{ \left(\frac{\mu_1}{m} - m \mu_2 \right) + \mu_1 a \frac{j_n'(am)}{j_n(am)} - \mu_2 m \frac{h_n'(a)}{h_n(a)} \right\} \\ &= -i \frac{m^n}{\alpha} \frac{1}{2n+1} \left\{ \left(\frac{\mu_1}{m} - m \mu_2 \right) + \mu_1 a \frac{n(am)^{n-1}}{(2n+1)!!} \cdot \frac{(2n+1)!!}{(am)^n} \right. \\ &\quad \left. - \mu_2 m \frac{i(n+1)(2n-1)!!}{a^{n+2}} \frac{a^{n+1}}{-i(2n-1)!!} \right\} \end{aligned}$$

$$\begin{aligned}
&= -i \frac{m^n}{a^2} \frac{1}{2n+1} \left\{ \left(\frac{\mu_1}{m} - m\mu_2 \right) + \mu_1 a n \frac{1}{am} + \mu_2 m \frac{1}{a} \right\} \\
\Delta_n^{(2)} &= -i \frac{m^n}{a^2} \frac{1}{2n+1} \left\{ \frac{\mu_1}{m} (1+n) + \mu_2 m \left(\frac{1}{a} - 1 \right) \right\} \\
&= -i \frac{1}{a^2 (2n+1)} \left\{ \mu_1 (1+n) m^{n-1} + \left(\frac{1}{a} - 1 \right) m^{n+1} \right\} \\
\Delta_n^{(2)} &= \left| -i \frac{\mu_1}{a^2} \frac{n+1}{2n+1} m^{n-1} - i \left(\frac{1}{a} - 1 \right) \frac{1}{a^2} \frac{1}{2n+1} m^{n+1} \right| \\
&\cong \left| \frac{\mu_1}{a^2} \frac{n+1}{2n+1} |m|^{n-1} - \left| \frac{1}{a} - 1 \right| \frac{1}{a^2} \frac{1}{2n+1} |m|^{n+1} \right| \\
&= \left| \frac{\mu_1}{a^2} (n+1) - \left| \frac{1}{a} - 1 \right| \frac{1}{a^2} |m|^2 \right| \cdot |m|^{n-1} \rightarrow \infty .
\end{aligned}$$

Corollary 1.

$$|m| > 1$$

$\exists C$ independent of $n \in \langle N \rangle \Rightarrow$

$$0 < C < \Delta_n^{(\ell)}(a, m, \mu_1, \mu_2) \text{ for } \ell = 1, 2 .$$

Proof: Follows easily from Theorem 15 and 16.

Theorem 17.

The $\bar{v}$ defined by (4-33b) and (4-46) is an element of $\mathcal{K}$.

Proof: The proof uses Corollary 1 above and the asymptotic estimates of Volume I, Chapter III, Section 4 (Reference 11). For the C of Corollary 1,

$$\begin{aligned}
\left| a_n^r \right| &< \frac{1}{C} \left| \mu_1 j_n(am) a j_n'(a) - \mu_2 j_n(a) (am) j_n'(am) \right| \\
&\leq \frac{a}{C} \mu_1 \left| j_n(am) j_n'(a) \right| + \frac{\mu_2}{C} |am| \left| j_n(a) j_n'(am) \right| \\
&= \frac{a \mu_1}{C} \left| j_n(am) \left[\frac{1}{a} j_n(a) + j_n'(a) \right] \right| + \frac{\mu_2}{C} |am| \left| j_n(a) \left[\frac{1}{am} j_n(am) + j_n'(am) \right] \right| \\
&\leq \frac{\mu_1}{C} \left| j_n(am) j_n(a) \right| + \frac{a \mu_1}{C} \left| j_n(am) j_n'(a) \right| \\
&+ \frac{\mu_2}{C} \left| j_n(a) j_n(am) \right| + \frac{\mu_2}{C} |am| \left| j_n(a) j_n'(am) \right| \\
&= \frac{1}{C} (\mu_1 + \mu_2) \left| j_n(a) j_n(am) \right| + \frac{a \mu_1}{C} \left| \frac{(am)^n}{(2n+1)!!} \frac{n(a)^{n-1}}{(2n+1)!!} \right| \\
&\quad + \frac{\mu_2}{C} |am| \left| \frac{a^n}{(2n+1)!!} \cdot \frac{n(am)^{n-1}}{(2n+1)!!} \right| \\
&\equiv \mathcal{A}_n^r .
\end{aligned}$$

Also

$$\begin{aligned}
\mathcal{A}_n^r &\sim \frac{1}{C} (\mu_1 + \mu_2) \left| \frac{a^n}{(2n+1)!!} \frac{(am)^n}{(2n+1)!!} \right| + \frac{a \mu_1}{C} \frac{n |am|^{2n}}{a [(2n+1)!!]^2} \\
&\quad + \frac{\mu_2}{C} |am| \frac{1}{|am|} \frac{n |am|^{2n}}{(2n+1)!!^2} \\
&= \frac{1}{C} (\mu_1 + \mu_2) \left| \frac{1}{m} \frac{|am|^{2n}}{[(2n+1)!!]^2} + \frac{1}{C} \frac{n |am|^{2n}}{[(2n+1)!!]^2} (\mu_1 + \mu_2) \right| .
\end{aligned}$$

$$a = k_2 c$$

In view of

$$h'_n(k_2 a) \sim i \frac{(n+1)(2n-1)!!}{(k_2 a)^{n+2}}$$

and

$$am = k_2 c \frac{k_1}{k_2} = k_1 c, \quad a = k_2 c,$$

$$\begin{aligned} \left| h'_n(k_2 a) \sqrt{\nu_n} \right| &\sim \frac{1}{C} (\mu_1 + \mu_2) \frac{1}{|m|} \frac{k_1 c^{2n}}{[(2n+1)!!]^2} \frac{(n+1)(2n-1)!!}{|k_2 a|^{n+2}} \\ &\quad + \frac{1}{C} n \frac{|k_1 c|^{2n}}{[(2n+1)!!]^2} (\mu_1 + \mu_2) \frac{(n+1)(2n-1)!!}{|k_2 a|^{n+2}} \\ &= \frac{1}{C} (\mu_1 + \mu_2) \left\{ \frac{n+1}{(2n+1)(2n+1)!!} \left| \frac{(k_1 c)^2}{k_2 a} \right|^n \left| \frac{1}{k_2 a} \right|^2 \right. \\ &\quad \left. + \frac{n(n+1)}{(2n+1)(2n+1)!!} \left| \frac{(k_1 c)^2}{k_2 a} \right|^n \left| \frac{1}{k_2 a} \right|^2 \right\} \\ &= \frac{1}{C} (\mu_1 + \mu_2) \left| \frac{1}{k_2 a} \right|^2 \frac{(n+1)^2}{(2n+1)(2n+1)!!} \left| \frac{(k_1 c)^2}{k_2 a} \right|^n \end{aligned}$$

But

$$(2n+1)!! = \frac{2^n \Gamma(n + \frac{3}{2})}{\sqrt{\pi}}$$

$$\frac{1}{(2n+1)!!} = \frac{\sqrt{\pi}}{2^n \Gamma(n + \frac{3}{2})} < \frac{\sqrt{\pi}}{2^n (n+1)!},$$

and therefore

$$\left| h'_n(k_2 a) \mathcal{J}_n^\nu \right| = \frac{1}{C} (\mu_1 + \mu_2) \left| \frac{1}{k_2 a} \right|^2 \frac{n+1}{2n+1} \frac{\sqrt{\pi}}{n!} \left| \frac{1}{2} \frac{(k_1 c)^2}{k_2 a} \right|^n .$$

Therefore

$$\sum \left| h'_n(k_2 a) \mathcal{J}_n^\nu \right| < \infty ,$$

and hence also

$$\sum \left| d_n h'_n \mathcal{J}_n^\nu \right| < \infty .$$

This shows that

$$\sum \left| \frac{D_n}{A_{1n}} h'_n(k_2 a) a_n^\nu \right| < \infty$$

and hence also

$$\|v_1\|^2 = \sum \left\{ b_{\lambda 1} \left| \lambda \in \Lambda_1 \cup \Lambda_2 \right. \right\} = E_0^2 k_2^2 \sum_{n=1}^{\infty} \left| \frac{D_n}{A_{1n}} h'_n(k_2 a) a_n^\nu \right|^2 < \infty ,$$

i. e. , $v_1 \in \mathcal{L}^2(\Omega)$.

The calculations to show that

$$v_j \in \mathcal{L}^2(\Omega)$$

for $j = 2, 3, 4$ are very similar.

APPENDIX A

PROPERTIES OF THE FIELDS $\bar{m}_\lambda^{(\ell)}, \bar{n}_\lambda^{(\ell)}$

This appendix is a continuation of Volume I Chapter VI (Reference 11). It makes use of the notations introduced in Section 1.9. The objective is to provide material in support of Theorems 5, 6, and 7, which was not given in Volume I (Reference 11). The notational conventions of Volume I and of Section 1.9 will remain in force.

Recall that

$$\psi_{jm}(\phi) = \delta_1^j \cos m\phi + \delta_2^j \sin m\phi \quad (\text{A-1})$$

and

$$\left\{ \begin{array}{l} \bar{m}_{jmn}^{(1)} \equiv \nabla \times \left[\left(j_n(k_1 r) P_n^m(\cos \theta) \psi_{jm}(\phi) \right) \bar{r} \right] \\ \bar{n}_{jmn}^{(1)} \equiv \frac{1}{k_1} \nabla \times \bar{m}_{jmn}^{(1)} \\ \bar{m}_{jmn}^{(2)} \equiv \nabla \times \left[\left(h_n(k_2 r) P_n^m(\cos \theta) \psi_{jm}(\phi) \right) \bar{r} \right] \\ \bar{n}_{jmn}^{(2)} \equiv \frac{1}{k_2} \nabla \times \bar{m}_{jmn}^{(2)} \end{array} \right\}_{(n,m,j) \in \Lambda_0} \quad (\text{A-2})$$

The components of $\bar{m}_\lambda^{(\ell)}, \bar{n}_\lambda^{(\ell)}$ in the usual unit spherical base system $\hat{r}, \hat{\theta}, \hat{\phi}$ are given in Table I of Volume I, Chapter VI (Reference 11).

Lemma A-1

For $(n, m) \in \mathcal{M}_0$,

- 1) $\left\{ \begin{array}{l} j_n(k_1 r) \neq 0 \\ h_n(k_2 r) \neq 0 \end{array} \right\} r > 0$
- 2) $\left\{ \begin{array}{l} \tau_n^m(\eta) \neq 0 \quad \text{for } (n, m) \in \mathcal{M}_1 \\ \tau_n^m(\eta) \equiv 0 \quad \text{for } (n, m) \in \mathcal{M}_0 - \mathcal{M}_1 = \{(0, 0)\} \end{array} \right.$

$$3) \quad \Pi_n^m(\eta) \neq 0 \quad \text{for} \quad (n, m) \in \mathcal{H}_1$$

$$4) \quad P_n^m(\eta) \neq 0 \quad \text{for} \quad (n, m) \in \mathcal{H}_1$$

Proof: For (1), using (44), Volume I, Chapter III (Reference 11)

$$\tilde{J}_n(k_1 r) = \frac{1}{k_1 r} \frac{\partial}{\partial r} [r j_n(k_1 r)] = \frac{j_n(k_1 r)}{k_1 r} + j_n'(k_1 r) = 0 \quad (\text{A-3})$$

$\Rightarrow$

$$1 = - (k_1 r) \frac{j_n'(k_1 r)}{j_n(k_1 r)} = -n + \tilde{J}_{n+\frac{1}{2}}(k_1 r)$$

$\rightarrow$

$$0 = \text{Im} \left\{ \tilde{J}_{n+\frac{1}{2}}(k_1 r) \right\}. \quad (\text{A-4})$$

But by Theorem 9, Volume I, Chapter III (Reference 11)

$$\text{Im} \left\{ \tilde{J}_{n+\frac{1}{2}}(k_1 r) \right\} > 0 \quad \text{since} \quad k_1 \in \mathcal{C}_1 \Rightarrow (k_1 r) \in \mathcal{C}_1. \quad (\text{A-5})$$

Thus (A-4) is a contradiction and so (A-3) is false.

Similarly:

$$\mathcal{H}_n(k_2 r) = 0 \quad (\text{A-6})$$

$\rightarrow$

$$0 = \text{Im} \left\{ \tilde{\mathcal{H}}_{n+\frac{1}{2}}(k_2 r) \right\}. \quad (\text{A-7})$$

But by Theorem 8, Volume I, Chapter III (Reference 11),

$$\text{Im} \left\{ \tilde{\mathcal{H}}_{n+\frac{1}{2}}(k_2 r) \right\} < 0, \quad \text{since} \quad (k_2 r) \in \mathcal{H}_0. \quad (\text{A-8})$$

This shows that (A-7) is a contradiction and hence (A-6) is false.

To show (2), note that for $n = 0$, $m = 0$ also

and

$$\tau_0^0(\eta) = \frac{\partial P_0^0(\eta)}{\partial \theta} = \frac{\partial P_0(\eta)}{\partial \theta} = \frac{\partial 1}{\partial \theta} = 0; \quad (\text{A-9})$$

while, for $(n, m) \in \mathcal{H}_1$,

$$\Rightarrow \tau_n^m(\eta) = 0 \quad (\text{A-10})$$

$$\begin{aligned} \frac{\partial P_n^m(\eta)}{\partial \theta} &= \frac{d P_n^m(\eta)}{d \eta} \frac{d \eta}{d \theta} = \frac{d P_n^m(\eta)}{d \eta} [-\sqrt{1-\eta^2}] \\ &= -\sqrt{1-\eta^2} \frac{d}{d \eta} \left[(1-\eta^2)^{m/2} \frac{d^m P_n(\eta)}{d \eta^m} \right] \equiv 0 \end{aligned}$$

$\Rightarrow$

$$(1-\eta^2)^{m/2} \frac{d^m P_n(\eta)}{d \eta^m} \equiv \text{constant}$$

$\Rightarrow$

$$(1-\eta^2)^{m/2} \frac{d^m P_n(\eta)}{d \eta^m} \equiv 0$$

$\Rightarrow$

$$\frac{d^m P_n(\eta)}{d \eta^m} \equiv 0 \quad (\text{A-11})$$

$\Rightarrow$

$$P_n(\eta) = \sum_{i=0}^{m-1} a_i \eta^i, \quad 0 \leq m \leq n. \quad (\text{A-12})$$

This is a contradiction, since $P_n(\eta)$ is known to be a polynomial of order exactly n . This proves (2).

To show (3), note that

$$\begin{aligned} \Pi_n^m &= \frac{P_n^m(\eta)}{\sqrt{1-\eta^2}} = \frac{(1-\eta^2)^{m/2} \frac{d^m P_n(\eta)}{d \eta^m}}{(1-\eta^2)^{1/2}} = (1-\eta^2)^{(m-1)/2} \frac{d^m P_n(\eta)}{d \eta^m} = 0 \end{aligned} \quad (\text{A-13})$$

$\Rightarrow$

$$\frac{d^m P_n(\eta)}{d \eta^m} = 0$$

which implies (A-12), and this has already been noted to be a contradiction. Thus, (3) is proved.

To show (4), observe

$$P_n^m(\eta) = (\sqrt{1-\eta^2})^m \frac{d^m P_n(\eta)}{d\eta^m} = 0$$

$$\Rightarrow \frac{d^m P_n(\eta)}{d\eta^m} = 0,$$

a contradiction.

Theorem A-1

Let $\xi > 0$ and $\bar{r} \neq \bar{0}$. Then for $l = 1, 2$,

$$\left\{ \begin{array}{l} \left[\bar{m}_{lmn}^{(l)}(\xi, \omega) \right] \cdot \hat{\theta} \neq 0 \\ \left[\bar{m}_{lmn}^{(l)}(\xi, \omega) \right] \cdot \hat{\phi} \neq 0 \\ \left[\bar{n}_{lmn}^{(l)}(\xi, \omega) \right] \cdot \hat{r} \neq 0 \\ \left[\bar{n}_{lmn}^{(l)}(\xi, \omega) \right] \cdot \hat{\theta} \neq 0 \\ \left[\bar{n}_{lmn}^{(l)}(\xi, \omega) \right] \cdot \hat{\phi} \neq 0 \end{array} \right\} \forall (n, m) \in \mathcal{J}_1 - \{(0, 0)\} \equiv \mathcal{J}_1 \quad (A-14)$$

while

$$\left\{ \begin{array}{l} \bar{m}_{lmn}^{(l)}(\bar{r}) \equiv \bar{0} \\ \bar{n}_{lmn}^{(l)}(\bar{r}) \equiv \bar{0} \end{array} \right\} \forall \bar{r} \quad \text{for} \quad (n, m) \in \mathcal{J}_0 - \mathcal{J}_1 \equiv \{(0, 0)\}. \quad (A-15)$$

Also

$$\left\{ \begin{array}{l} \left[\bar{m}_{2mn}^{(l)}(\xi, \omega) \right] \cdot \hat{\theta} \neq 0 \\ \left[\bar{m}_{2mn}^{(l)}(\xi, \omega) \right] \cdot \hat{\phi} \neq 0 \\ \left[\bar{n}_{2mn}^{(l)}(\xi, \omega) \right] \cdot \hat{r} \neq 0 \\ \left[\bar{n}_{2mn}^{(l)}(\xi, \omega) \right] \cdot \hat{\theta} \neq 0 \\ \left[\bar{n}_{2mn}^{(l)}(\xi, \omega) \right] \cdot \hat{\phi} \neq 0 \end{array} \right\} \quad \forall (n, m) \in \mathbb{Z}_2 \quad (A-16)$$

while

$$\left\{ \begin{array}{l} \bar{m}_{2mn}^{(l)}(\bar{r}) \equiv 0 \\ \bar{n}_{2mn}^{(l)}(\bar{r}) \equiv 0 \end{array} \right\} \quad \forall \bar{r} \quad \text{for } (n, m) \in (\mathbb{Z}_0 - \mathbb{Z}_2) \quad (A-17)$$

Proof: The results (A-15) and (A-17) follow by inspection of Table I in Volume I, Chapter VI (Reference 11), while (A-14) and (A-16) are easily proved by contradiction using Lemma A-1.

One important property of the $\bar{m}_\lambda^{(l)}$, $\bar{n}_\lambda^{(l)}$ fields is their orthogonality over spheres. This property has already been noted in Volume I, Chapter VI (Reference 11). Tables A-1 and A-2 give a more detailed picture of these orthogonality properties:

$$\left\{ \begin{array}{l} \int_{\Omega} \left[\bar{m}_\lambda^{(l)}(\xi, \omega) \right] \cdot \left[\bar{m}_\nu^{(l)}(\xi, \omega) \right]^* d\omega = \delta_\lambda^\nu M_\lambda^{(l)}(\xi) \\ \int_{\Omega} \left[\bar{n}_\lambda^{(l)}(\xi, \omega) \right] \cdot \left[\bar{n}_\nu^{(l)}(\xi, \omega) \right]^* d\omega = \delta_\lambda^\nu N_\lambda^{(l)}(\xi) \\ \int_{\Omega} \left[\bar{m}_\lambda^{(l)}(\xi, \omega) \right] \cdot \left[\bar{n}_\nu^{(l)}(\xi, \omega) \right]^* d\omega = 0 \\ \int_{\Omega} \left[\bar{n}_\lambda^{(l)}(\xi, \omega) \right] \cdot \left[\bar{m}_\nu^{(l)}(\xi, \omega) \right]^* d\omega = 0 \end{array} \right\} \quad (\lambda, \nu) \in \Lambda_0^2 \quad (A-18a)$$

Table A-1. Orthogonality Properties of $\bar{m}_{\lambda}^{(1)}(\xi, \omega)$, $\bar{n}_{\lambda}^{(1)}(\xi, \omega)$

	$\bar{m}_{1m'n'}^{(1)}$	$\bar{m}_{2m'n'}^{(1)}$	$\bar{n}_{1m'n'}^{(1)}$	$\bar{n}_{2m'n'}^{(1)}$
$\bar{m}_{1mn}^{(1)}$	$M_{1mn}^{(1)}$	0	0	0
$\bar{m}_{2mn}^{(1)}$	0	$M_{2mn}^{(1)}$	0	0
$\bar{n}_{1mn}^{(1)}$	0	0	$N_{1mn}^{(1)}$	0
$\bar{n}_{2mn}^{(1)}$	0	0	0	$N_{2mn}^{(1)}$

$$M_{imn}^{(1)} = \begin{cases} 0 & , \lambda = (n, m, i) \in \Lambda_0 - (\Lambda_1 \cup \Lambda_2) \\ \pi \epsilon_m |j_n(k_1 \xi)|^2 B_{mn} & , \lambda = (n, m, i) \in (\Lambda_1 \cup \Lambda_2) . \end{cases}$$

$$N_{imn}^{(1)} = \begin{cases} 0 & , \lambda = (n, m, i) \in \Lambda_0 - (\Lambda_1 \cup \Lambda_2) \\ \pi \epsilon_m B_{mn} \left\{ n(n+1) + \left| \frac{j_n(k_1 \xi)}{k_1 \xi} \right|^2 + |j_n|^2 \right\} & , \lambda = (n, m, i) \in \Lambda_1 \cup \Lambda_2 . \end{cases}$$

where

$$\left\{ \begin{array}{l} M_{\lambda}^{(l)}(\xi) \neq 0 \\ N_{\lambda}^{(l)}(\xi) \neq 0 \end{array} \right\} \quad \text{for } \lambda \in (\Lambda_1 \cup \Lambda_2) \quad (\text{A-18b})$$

and

$$\left\{ \begin{array}{l} M_{\lambda}^{(l)}(\xi) \equiv 0 \\ N_{\lambda}^{(l)}(\xi) \equiv 0 \end{array} \right\} \quad \forall \lambda \in \Lambda_0 = (\Lambda_1 \cup \Lambda_2) \quad (\text{A-18c})$$

To establish these properties, the following formulas will be needed:

$$\left. \begin{aligned} \int_0^{\pi} P_n^m(\cos \theta) P_{n'}^m(\cos \theta) \sin \theta d\theta &= \delta_n^{n'} \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \\ \int_0^{\pi} \left[\tau_n^m \tau_{n'}^m + m^2 \Pi_n^m \Pi_{n'}^m \right] \sin \theta d\theta &= \delta_n^{n'} B_{mn} \end{aligned} \right\} \quad \begin{array}{l} (n, m) \in // 0 \\ n' \in (//) \end{array} \quad (\text{A-19})$$

$$\int_0^{\pi} \left[\tau_n^m \Pi_{n'}^m + \Pi_n^m \tau_{n'}^m \right] \sin \theta d\theta = 0, \quad \begin{array}{l} (n, m) \in // 2 \\ n' \in < // > \end{array} \quad (\text{A-20})$$

$$\int_0^{\pi} \left[\tau_n^m \Pi_{n'}^m + \Pi_n^m \tau_{n'}^m \right] \sin \theta d\theta = 0, \quad \begin{array}{l} (n, m) \in // 2 \\ n' \in < // > \end{array} \quad (\text{A-21})$$

Table A-2. Orthogonality Properties of $\bar{m}_\lambda^{(2)}(\xi, \omega)$, $\bar{n}_\lambda^{(2)}(\xi, \omega)$

	$\bar{m}_{1m'n'}^{(2)}$	$\bar{m}_{2m'n'}^{(2)}$	$\bar{n}_{1m'n'}^{(2)}$	$\bar{n}_{2m'n'}^{(2)}$
$\bar{m}_{1mn}^{(2)}$	$M_{1mn}^{(2)}$	0	0	0
$\bar{m}_{2mn}^{(2)}$	0	$M_{2mn}^{(2)}$	0	0
$\bar{n}_{1mn}^{(2)}$	0	0	$N_{1mn}^{(2)}$	0
$\bar{n}_{2mn}^{(2)}$	0	0	0	$N_{2mn}^{(2)}$

$$M_{imn}^{(2)} = \begin{cases} 0 & , \lambda \in \Lambda_0 - (\Lambda_1 \cup \Lambda_2) \\ \pi \epsilon_m |h_n(k_2 \xi)|^2 B_{mn} & , \lambda \in \Lambda_1 \cup \Lambda_2 \end{cases}$$

$$N_{imn}^{(2)} = \begin{cases} 0 & , \lambda \in \Lambda_0 - (\Lambda_1 \cup \Lambda_2) \\ \pi \epsilon_m B_{mn} \left\{ n(n+1) \left| \frac{h_n(k_2 \xi)}{k_2 \xi} \right|^2 + |W'_n|^2 \right\} & , \lambda \in \Lambda_1 \cup \Lambda_2 \end{cases}$$


where

$$B_{mn} = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} n(n+1) . \quad (A-22)$$

Of these, (A-19) is established in Whittaker and Watson (Reference 30 pages 324, 325), while (A-20) and (A-21) are proved below.

(A-19) may be written

$$\int_{-1}^1 P_n^m(\eta) P_{n'}^m(\eta) d\eta = \delta_n^m \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} . \quad (A-23)$$

To prove (A-21), observe that

$$\int_0^\pi \left[\tau_n^m \tau_{n'}^m + \Pi_n^m \tau_{n'}^m \right] \sin \theta d\theta = \int_0^\pi \left[\frac{\partial P_n^m}{\partial \theta} \frac{P_{n'}^m}{\sin \theta} + \frac{P_n^m}{\sin \theta} \frac{\partial P_{n'}^m}{\partial \theta} \right] \sin \theta d\theta .$$

Integrate by parts to obtain

$$\begin{aligned} \int_0^\pi \frac{\partial P_n^m}{\partial \theta} P_{n'}^m d\theta &= \int_{-1}^1 \left[\frac{d P_n^m}{d\eta} \left(\frac{d\eta}{d\theta} \right) \right] P_{n'}^m(\eta) \frac{-d\eta}{\sqrt{1-\eta^2}} \quad (A-24) \\ &= + \int_{-1}^1 \left[\frac{d P_n^m(x)}{dx} (-\sqrt{1-x^2}) \right] P_{n'}^m(x) \frac{-dx}{\sqrt{1-x^2}} \\ &= - \int_{-1}^1 \left(\frac{d P_n^m(x)}{dx} \right) P_{n'}^m(x) dx \\ &= - \left\{ P_n^m(x) P_{n'}^m(x) \right\}_{-1}^{+1} \\ &\quad - \int_{-1}^{+1} P_{n'}^m(x) \left(\frac{d P_n^m(x)}{dx} \right) dx \Bigg\} . \end{aligned}$$

Since

$$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}$$

it follows that

$$P_n^m(x) \Big|_{x=\pm 1} = 0, \quad \text{for } (n, m) \in \mathcal{H}_2$$

and hence

$$\begin{aligned} \int_0^\pi \frac{\partial P_n^m}{\partial \theta} P_{n'}^m d\theta &= - \int_{-1}^{+1} \left(\frac{d P_n^m(x)}{dx} \right) P_{n'}^m(x) dx \\ &= \int_{-1}^{+1} P_{n'}^m(x) \left(\frac{d P_n^m(x)}{dx} \right) dx. \end{aligned}$$

Hence

$$\begin{aligned} &- \int_{-1}^{+1} \left[\left(\frac{d P_n^m(x)}{dx} \right) P_{n'}^m(x) + P_{n'}^m(x) \left(\frac{d P_n^m(x)}{dx} \right) \right] dx \\ &= 0, \quad (n, m) \in \mathcal{H}_2. \end{aligned}$$

But by (A-24), this may be written

$$\int_0^\pi \left[\frac{\partial P_n^m}{\partial \theta} P_{n'}^m + \frac{\partial P_{n'}^m}{\partial \theta} P_n^m \right] d\theta = 0; \quad (n, m) \in \mathcal{H}_2.$$

This proves (A-21) for $(n, m) \in \mathcal{H}_2$.

To establish (A-20),

$$\tau_n^m = \frac{\partial P_n^m}{\partial \theta} = \frac{d P_n^m}{dx} \frac{d(\cos \theta)}{d\theta} = - \frac{d P_n^m}{dx} \sqrt{1-x^2} \quad (\text{A-25})$$

and

$$\Pi_n^m = \frac{P_n^m}{\sqrt{1-x^2}}. \quad (\text{A-26})$$

It follows that the LHS of (A-20) is

$$\begin{aligned} & - \int_{-1}^{+1} \left\{ \left[-\frac{d P_n^m}{dx} \sqrt{1-x^2} \right] \left[-\frac{d P_{n'}^m}{dx} \sqrt{1-x^2} \right] + m^2 \frac{P_n^m}{\sqrt{1-x^2}} \frac{P_{n'}^m}{\sqrt{1-x^2}} \right\} dx \\ & = \int_{-1}^{+1} \left\{ (1-x^2) \frac{d P_n^m}{dx} \cdot \frac{d P_{n'}^m}{dx} + m^2 \frac{P_n^m P_{n'}^m}{1-x^2} \right\} dx. \quad (\text{A-27}) \end{aligned}$$

Since $P_n^m(x)$ satisfies

$$\frac{d}{dx} \{ (1-x^2) f' \} + n(n+1) f = \frac{m^2}{1-x^2} f, \quad (\text{A-28})$$

it follows that (A-27) may be written

$$\begin{aligned} & \int_{-1}^{+1} \left\{ \left[(1-x^2) \frac{d P_n^m}{dx} \right] \frac{d P_{n'}^m}{dx} + P_{n'}^m \left[\frac{d}{dx} \left[(1-x^2) \frac{d P_n^m}{dx} \right] \right. \right. \\ & \quad \left. \left. + n(n+1) P_n^m \right] \right\} dx, \quad (\text{A-29}) \end{aligned}$$

and integration by parts and (A-19) yields

$$\begin{aligned} & \left[(1-x^2) \frac{d P_n^m}{dx} P_{n'}^m \right] \Big|_{-1}^{+1} - \int_{-1}^{+1} P_{n'}^m \frac{d}{dx} \left[(1-x^2) \frac{d P_n^m}{dx} \right] dx \\ & + \int_{-1}^{+1} P_{n'}^m \frac{d}{dx} \left[(1-x^2) \frac{d P_n^m}{dx} \right] dx \\ & + n(n+1) \int_{-1}^{+1} P_{n'}^m P_n^m dx \\ & = n(n+1) \delta_{n'}^n \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}. \end{aligned}$$

The formulas (A-18a), (A-18b), (A-18c) encompassing Tables A-1 and A-2 will now be established. Using a natural notation,

$$\begin{aligned}
 \left(\bar{m}_{1mn}^{(1)}, \bar{m}_{1\mu\nu}^{(1)} \right) &= \int_{\Omega} \left[-m j_n \Pi_n^m S_m \right] \left[-\mu j_\nu \Pi_\nu^\mu S_\mu \right]^* d\omega \\
 &\quad + \int_{\Omega} \left[-j_n \tau_n^m C_m \right] \left[-j_\nu \tau_\nu^\mu C_\mu \right]^* d\omega \\
 &= m \mu j_n j_\nu^* (S_m, S_\mu) \left(\Pi_n^m, \Pi_\nu^\mu \right) \\
 &\quad + j_n j_\nu^* (C_m, C_\mu) \left(\tau_n^m, \tau_\nu^\mu \right)
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 \left(\bar{m}_{1mn}^{(1)}, \bar{m}_{1\mu\nu}^{(1)} \right) &= \begin{cases} j_n j_\nu^* (C_m, C_\mu) \left(\tau_n^m, \tau_\nu^\mu \right) & \text{if } m \text{ or } \mu = 0 \\ m \mu j_n j_\nu^* (S_m, S_\mu) \left(\Pi_n^m, \Pi_\nu^\mu \right) \\ + j_n j_\nu^* (C_m, C_\mu) \left(\tau_n^m, \tau_\nu^\mu \right) & \text{if } (m, \mu) \in \mathcal{A}^2 \end{cases} \\
 &= \begin{cases} j_n j_\nu^* \left(\delta_m^\mu 2\pi \right) \left(\tau_n^m, \tau_\nu^\mu \right) & \text{if } m \text{ or } \mu = 0 \\ 0 & \text{if } m \neq \mu \\ j_n j_\nu^* \int_0^\pi \left[\tau_n^m \tau_\nu^m + m^2 \Pi_n^m \Pi_\nu^m \right] \sin \theta d\theta & m = \mu \end{cases} \\
 &= \begin{cases} j_n j_\nu^* 2\pi \left(\tau_n^0, \tau_\nu^0 \right) & \text{if } m \text{ or } \mu = 0 \\ 0 & \text{if } m \neq \mu \\ j_n j_\nu^* \pi \delta_n^\nu B_{mn} & \end{cases}
 \end{aligned}$$

This shows

$$\left(\bar{m}_{1mn}^{(1)}, \bar{m}_{1m'\nu'}^{(1)} \right) = \pi \delta_{mn}^{\mu\nu} \epsilon_m |j_n|^2 B_{mn} \quad \text{if } ((n, m), (n', m')) \in \mathcal{A}_1^2$$

$$\left(\bar{m}_{1mn}^{(1)}, \bar{m}_{1m'n'}^{(1)}\right) = 0 \quad \text{if} \quad (n, m) \text{ or } (n', m') \in \mathcal{M}_0 - \mathcal{M}_1.$$

Also

$$\begin{aligned} \left(\bar{m}_{1mn}^{(1)}, \bar{m}_{2\mu\nu}^{(1)}\right) &= \int_{\Omega} \left[-m j_n \Pi_n^m S_m\right] \left[+\mu j_\nu \Pi_\nu^\mu\right]^* d\omega \\ &+ \int_{\Omega} \left[-j_n \tau_n^m C_m\right] \left[-j_\nu \tau_\nu^\mu S_\mu\right]^* d\omega = 0, \end{aligned}$$

and clearly also

$$\left(\bar{m}_{2mn}^{(1)}, \bar{m}_{1\mu\nu}^{(1)}\right) = 0.$$

However, by the symmetry of Table 3-1, it can be seen that

$$\begin{aligned} \left(\bar{m}_{2mn}^{(1)}, \bar{m}_{2\mu\nu}^{(1)}\right) &= m \mu j_n j_\nu^* (C_m, C_\mu) \left(\Pi_n^m, \Pi_\nu^\mu\right) \\ &+ j_n j_\nu^* (S_m, S_\mu) \left(\tau_n^m, \tau_\nu^\mu\right) \\ &= \begin{cases} 0 & \text{if } m \text{ or } \mu = 0 \\ \delta_{mn}^{\mu\nu} \pi |j_n|^2 B_{mn} & \text{if } (n, m), (\nu, \mu) \in \mathcal{M}_2^2 \end{cases} \end{aligned}$$

i.e.,

$$\left(\bar{m}_{2mn}^{(1)}, \bar{m}_{2\mu\nu}^{(1)}\right) = \pi \delta_{mn}^{\mu\nu} |j_n|^2 B_{mn} \quad \text{if} \quad ((n, m), (\nu, \mu)) \in \mathcal{M}_2^2$$

and

$$\left(\bar{m}_{2mn}^{(1)}, \bar{m}_{2\mu\nu}^{(1)}\right) = 0 \quad \text{if} \quad \text{either } (n, m) \text{ or } (\nu, \mu) \text{ is in } \mathcal{M}_0 - \mathcal{M}_2$$

The results above may be summarized as

$$\left(\bar{m}_\lambda^{(1)}, \bar{m}_\nu^{(1)}\right) = \delta_\lambda^\nu M_\lambda^{(1)}$$

where

$$M_{\lambda}^{(1)} = 0 \quad \text{for} \quad \lambda \in \Lambda_0 - (\Lambda_1 \cup \Lambda_2)$$

and

$$M_{\lambda}^{(1)} = \pi \epsilon_m |j_n|^2 B_{mn} > 0 \quad \text{for} \quad \lambda = (n, m, i) \in (\Lambda_1 \cup \Lambda_2).$$

Next,

$$\begin{aligned} (\bar{n}_{1mn}^{(1)}, \bar{n}_{1\mu\nu}^{(1)}) &= \int_{\Omega} \left[\frac{n(n+1)}{k_1 \xi} j_n P_n^m C_m \right] \left[\frac{\nu(\nu+1)}{k_1 \xi} j_{\nu} P_{\nu}^{\mu} C_{\mu} \right]^* d\omega \\ &\quad + \int_{\Omega} \left[j_n \tau_n^m C_m \right] \left[j_{\nu} \tau_{\nu}^{\mu} C_{\mu} \right]^* d\omega \\ &\quad + \int_{\Omega} \left[-m j_n \Pi_n^m S_m \right] \left[-\mu j_{\nu} \Pi_{\nu}^{\mu} S_{\mu} \right]^* d\omega \\ &= \frac{n(n+1) \nu(\nu+1)}{|k_1 \xi|^2} j_n j_{\nu}^* (C_m, C_{\mu}) (P_n^m, P_{\nu}^{\mu}) \\ &\quad + j_n j_{\nu}^* (C_m, C_{\mu}) (\tau_n^m, \tau_{\nu}^{\mu}) \\ &\quad + \mu_m j_n j_{\nu}^* (S_m, S_{\mu}) (\Pi_n^m, \Pi_{\nu}^{\mu}) \\ &= \frac{n(n+1) \nu(\nu+1)}{|k_1 \xi|^2} j_n j_{\nu}^* (\delta_m^{\mu} \epsilon_m \pi) (P_n^m, P_{\nu}^{\mu}) \\ &\quad + \pi \delta_m^{\mu} j_n j_{\nu}^* \left[\epsilon_m (\tau_n^m, \tau_{\nu}^{\mu}) + \mu_m \delta_0^m (\Pi_n^m, \Pi_{\nu}^{\mu}) \right] \\ &= \begin{cases} 0 & \text{if } \mu \neq m \text{ or } n \neq \nu \\ \frac{[n(n+1)]^2}{|k_1 \xi|^2} |j_n|^2 \epsilon_m \pi \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} & \text{if } \mu = m \text{ and } n = \nu \\ + \pi |j_n|^2 \epsilon_m \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} n(n+1) & \end{cases} \\ &= \pi \delta_{mn}^{\mu\nu} \epsilon_m B_{mn} \left\{ n(n+1) \left| \frac{j_n(k_1 \xi)}{k_1 \xi} \right|^2 + |j_n|^2 \right\}. \end{aligned}$$

The above results may be summarized by

$$\left(\bar{n}_{1mn}^{(1)}, \bar{n}_{1\mu\nu}^{(1)} \right) = \delta_{mn}^{\mu\nu} N_{1mn}^{(1)} \quad \text{for } \left((n, m), (\nu, \mu) \right) \in \mathcal{M}_0^2$$

where

$$N_{1mn}^{(1)} = \begin{cases} 0 & \text{for } (n, m) \in \mathcal{M}_0 - \mathcal{M}_1 = \{(0, 0)\} \\ \pi \epsilon_m B_{mn} \left[n(n+1) \left| \frac{j_n(k_1 \xi)}{k_1 \xi} \right|^2 + |f_n|^2 \right] > 0 & \text{for } (n, m) \in \mathcal{M}_1 \end{cases}$$

It is clear by inspection of Table 3-1 that

$$\left(\bar{n}_{1nn}^{(1)}, \bar{n}_{2\mu\nu}^{(1)} \right) = \left(\bar{n}_{2mn}^{(1)}, \bar{n}_{1\mu\nu}^{(1)} \right) = 0 \quad \forall \left((m, n), (\mu, \nu) \right) \in \mathcal{M}_0^2.$$

Consider, however

$$\begin{aligned} \left(\bar{n}_{2mn}^{(1)}, \bar{n}_{2\mu\nu}^{(1)} \right) &= \frac{n(n+1) \nu(\nu+1)}{|k_1 \xi|^2} j_n j_\nu^* (S_m, S_\mu) \left(P_n^m, P_\nu^\mu \right) \\ &\quad + j_n j_\nu^* (S_m, S_\mu) \left(\tau_n^m, \tau_\nu^\mu \right) \\ &\quad + m_\mu j_n j_\nu^* (C_m, C_\mu) \left(\Pi_n^m, \Pi_\nu^\mu \right) \\ &= \begin{cases} 0 & \text{if } n, \nu, m \text{ or } \mu = 0 \\ \pi \delta_\mu^m \delta_n^\nu \left\{ \frac{[n(n+1)]^2}{|k_1 \xi|^2} |j_n|^2 \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \right. \\ &\quad \left. + |f_n|^2 \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} n(n+1) \right\} \end{cases} \end{aligned}$$

Hence

$$\left(\bar{n}_{2mn}^{(1)}, \bar{n}_{2\mu\nu}^{(1)} \right) = \delta_{mn}^{\mu\nu} N_{2mn}^{(1)}$$

where

$$N_{2mn}^{(1)} = \begin{cases} 0 & \text{for } (n, m) \in \mathcal{M}_0 - \mathcal{M}_2 \\ \pi \epsilon_m B_{mn} \left\{ n(n+1) \left| \frac{j_n(k_1 \xi)}{k_1 \xi} \right|^2 + |\psi_n|^2 \right\} & \text{for } (n, m) \in \mathcal{M}_2 \end{cases}$$

The results above may be summarized by

$$\left(\bar{n}_\lambda^{(1)}, \bar{n}_\nu^{(1)} \right) = \delta_\lambda^\nu N_\lambda^{(1)}$$

where

$$N_\lambda^{(1)} = \begin{cases} 0 & \text{for } \lambda = (n, m, i) \in \Lambda_0 - (\Lambda_1 \cup \Lambda_2) \\ \pi \epsilon_m B_{mn} \left\{ n(n+1) \left| \frac{j_n(k_1 \xi)}{k_1 \xi} \right|^2 + |\psi_n|^2 \right\} & \text{for } \lambda = (n, m, i) \in \Lambda_1 \cup \Lambda_2 \end{cases}$$

Next, it is clear that

$$\begin{aligned} \left(\bar{m}_{1mn}^{(1)}, \bar{n}_{1\mu\nu}^{(1)} \right) &= -m j_n \psi_\nu^* (S_m, C_\mu) \left(\Pi_n^m, \tau_\nu^\mu \right) \\ &+ \mu j_n \psi_\nu^* (C_m, S_\mu) \left(\tau_n^m, \Pi_\nu^\mu \right) = 0 \end{aligned}$$

However

$$\begin{aligned} \left(\bar{m}_{1mn}^{(1)}, \bar{n}_{2\mu\nu}^{(1)} \right) &= -m j_n \psi_\nu^* (S_m, S_\mu) \left(\Pi_n^m, \tau_\nu^\mu \right) \\ &- \mu j_n \psi_\nu^* (C_m, C_\mu) \left(\tau_n^m, \Pi_\nu^\mu \right) \end{aligned}$$

requires slightly more attention.

$$\left(\bar{m}_{1mn}^{(1)}, \bar{n}_{2\mu\nu}^{(1)} \right) = \begin{cases} 0 & \text{if } m = 0 \text{ or } \mu = 0 \\ -\pi \delta_m^\mu m j_n \int_0^\pi \left[\Pi_n^m \tau_\nu^m + \tau_n^m \Pi_\nu^m \right] \sin \theta d\theta = 0. \end{cases}$$

The result

$$\left(\bar{n}_{2mn}^{(1)}, \bar{n}_{2\mu\nu}^{(1)} \right) = 0$$

is clear by inspection,

while

$$\begin{aligned} \left(\bar{m}_{2mn}^{(1)}, \bar{n}_{1\mu\nu}^{(1)} \right) &= m j_n \int_0^\pi (C_m, C_\mu) \left(\Pi_n^m, \tau_\nu^\mu \right) \\ &\quad + \mu j_n \int_0^\pi (S_m, S_\mu) \left(\tau_n^m, \Pi_\nu^\mu \right) \\ &= \begin{cases} 0 & \text{if } m = 0 \text{ or } \mu = 0 \\ \pi \delta_m^\mu m j_n \int_0^\pi \left[\Pi_n^m \tau_\nu^m + \tau_n^m \Pi_\nu^m \right] \sin \theta d\theta = 0. \end{cases} \end{aligned}$$

From

$$\left(\bar{n}_{1mn}^{(1)}, \bar{m}_{1\mu\nu}^{(1)} \right) = \left(\bar{m}_{1\mu\nu}^{(1)}, \bar{n}_{1mn}^{(1)} \right)^*$$

it follows that

$$\left(\bar{n}_{1mn}^{(1)}, \bar{m}_{1\mu\nu}^{(1)} \right) = 0.$$

The results

$$\left(\bar{n}_{1mn}^{(1)}, \bar{m}_{2\mu\nu}^{(1)} \right) = \left(\bar{n}_{2mn}^{(1)}, \bar{m}_{1\mu\nu}^{(1)} \right) = \left(\bar{n}_{2mn}^{(1)}, \bar{m}_{2\mu\nu}^{(1)} \right) = 0$$

follow by similar reasoning from earlier results. This establishes Table A-1.

To establish Table A-2, note that from Table I of Volume I, Chapter IV (Reference 11), it is clear that the sequences

$$\left\{ \overline{m}_\lambda^{(2)} \mid \lambda \in \Lambda_0 \right\}$$

$$\left\{ \overline{n}_\lambda^{(2)} \mid \lambda \in \Lambda_0 \right\}$$

may be formally obtained from the sequences

$$\left\{ \overline{m}_\lambda^{(1)} \mid \lambda \in \Lambda_0 \right\}$$

$$\left\{ \overline{n}_\lambda^{(1)} \mid \lambda \in \Lambda_0 \right\}$$

by the formal replacements

$$k_1 \rightarrow k_2$$

$$j_n \rightarrow h_n$$

$$f_n \rightarrow \mathcal{K}'_n.$$

This establishes Table A-2.

APPENDIX B

CALCULATION OF CERTAIN INTEGRALS

The integrals*

$$A_{\lambda}^{\nu} = \int_{\Omega} \hat{\mathbf{r}} \cdot \left(\bar{\mathbf{m}}_{\lambda}^{(2)} \times \bar{\mathbf{n}}_{\nu}^{(2)*} \right) d\omega \quad (\text{B-1})$$

$$B_{\lambda}^{\nu} = \int_{\Omega} \hat{\mathbf{r}} \cdot \left(\bar{\mathbf{n}}_{\lambda}^{(2)} \times \bar{\mathbf{m}}_{\nu}^{(2)*} \right) d\omega \quad (\text{B-2})$$

$$C_{\lambda}^{\nu} = \int_{\Omega} \hat{\mathbf{r}} \cdot \left(\bar{\mathbf{m}}_{\lambda}^{(2)} \times \bar{\mathbf{m}}_{\nu}^{(2)*} \right) d\omega \quad (\text{B-3})$$

$$D_{\lambda}^{\nu} = \int_{\Omega} \hat{\mathbf{r}} \cdot \left(\bar{\mathbf{n}}_{\lambda}^{(2)} \times \bar{\mathbf{n}}_{\nu}^{(2)*} \right) d\omega \quad (\text{B-4})$$

were encountered in Section 1. 10. The purpose of this appendix is to record the calculations leading to the results (1-80), (1-81), (1-82) and (1-83) of Section 1. 10. The superscript (2) will be dropped.

Note that

$$-A_{\lambda}^{\nu*} = \int_{S_{\xi}} \hat{\mathbf{r}} \cdot \left(\bar{\mathbf{n}}_{\nu} \times \bar{\mathbf{m}}_{\lambda}^* \right) d\omega \equiv B_{\nu}^{\lambda} \quad (\text{B-5})$$

Hence it is necessary to calculate only (B-1), (B-3), and (B-4). For this purpose, use Table I of Volume I (Reference 11) and the relations

$$\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = - \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\theta}}$$

*The arguments of all $\bar{\mathbf{m}}$, $\bar{\mathbf{n}}$ fields are (ξ, ω) .

Consider

$$\begin{aligned}
 A_{lmn}^{1\mu\nu} &= \int_{S_\xi} \bar{r} \cdot (\bar{m}_{lmn} \times n_{l\mu\nu}^*) da \\
 &= m_\mu h_n \mathcal{H}_\nu^* (S_m, S_\mu) (\Pi_n^m, \Pi_\nu^\mu) + h_n \mathcal{H}_\nu^* (C_m, C_\mu) (\tau_n^m, \tau_\nu^\mu) \\
 &= \pi \delta_\mu^m h_n \mathcal{H}_\nu^* \left[\epsilon_m \tau_n^m \tau_\nu^\mu + m_\mu \Pi_n^m \Pi_\nu^\mu \right].
 \end{aligned}$$

This gives

$$\begin{aligned}
 A_{lmn}^{1\mu\nu} &= \begin{cases} \pi \delta_m^\mu 0 & \text{if } m=0 \text{ or } \mu=0 \text{ and } n=0 \text{ or } \nu=0 \\ \pi \delta_\mu^m \epsilon_m h_n \mathcal{H}_\nu^* (\tau_n^m, \tau_\nu^\mu) & \text{if } m=0 \text{ or } \mu=0 \\ \pi \delta_\mu^m h_n \mathcal{H}_\nu^* \int_0^\pi [\tau_n^m \tau_\nu^m + m^2 \Pi_n^m \Pi_\nu^m] \sin \theta d\theta, & m=\mu \end{cases} \\
 &= \begin{cases} 2\pi \delta_m^\mu \delta_n^\nu h_n \mathcal{H}_n^* B_{mn} & \text{if } m=0 \text{ or } \mu=0 \\ \pi \delta_m^\mu \delta_n^\nu h_n \mathcal{H}_n^* B_{mn} & , m=\mu \end{cases}
 \end{aligned}$$

i.e.,

$$A_{lmn}^{1\mu\nu} = \begin{cases} 0 & \text{if } ((n, m), (\nu, \mu)) \in \mathcal{H}_0^2 - \mathcal{H}_1^2 \\ \pi \delta_{mn}^{\mu\nu} \epsilon_m h_n \mathcal{H}_n^* B_{mn} & \text{if } ((n, m), (\nu, \mu)) \in \mathcal{H}_1^2 \end{cases}$$

Next, observe that

$$A_{lmn}^{2\mu\nu} = -m_\mu h_n \mathcal{H}_\nu^* (S_m, C_\mu) (\Pi_n^m, \Pi_\nu^\mu) + h_n \mathcal{H}_\nu^* (C_m, S_\mu) (\tau_n^m, \tau_\nu^\mu) = 0.$$

Also,

$$A_{2mn}^{1\mu\nu} = -m_\mu h_n \mathcal{H}_\nu^* (C_m, C_\mu) (\Pi_n^m, \Pi_\nu^\mu) + h_n \mathcal{H}_\nu^* (S_m, C_\mu) (\tau_n^m, \tau_\nu^\mu) = 0.$$

But

$$A_{2mn}^{2\mu\nu} = m_\mu h_n \mathcal{H}_\nu^* (C_m, C_\mu) (\Pi_n^m, \Pi_\nu^\mu) + h_n \mathcal{H}_\nu^* (S_m, S_\mu) (\tau_n^m, \tau_\nu^\mu)$$

$$= \begin{cases} 0 & \text{if } m = 0 \text{ or } \mu = 0 \\ \pi \delta_m^\mu h_n \mathcal{H}_\nu^* \int_0^\pi \left[\tau_n^m \tau_\nu^m + m^2 \Pi_n^m \Pi_\nu^m \right] \sin \theta d\theta & \end{cases}$$

i. e. ,

$$A_{2mn}^{2\mu\nu} = \begin{cases} 0 & \text{if } ((n, m), (\nu, \mu)) \in \mathcal{H}_0^2 - \mathcal{H}_2^2 \\ \pi \delta_m^\mu h_n \mathcal{H}_\nu^* \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} n(n+1) & , ((n, m), (\nu, \mu)) \in \mathcal{H}_2^2. \end{cases}$$

The above results may be summarized as

$$A_\lambda^\nu = \delta_\lambda^\nu E_\lambda \quad \forall (\lambda, \nu) \in \Lambda_0^2$$

where

$$E_\lambda = \begin{cases} 0, & \lambda \in \Lambda_0 - (\Lambda_1 \cup \Lambda_2) \\ \pi \epsilon_m h_n \mathcal{H}_n^* B_{mn}, & \lambda = (n, m, i) \in \Lambda_1 \cup \Lambda_2. \end{cases}$$

In view of (B-5), this shows also that

$$B_\nu^\lambda = - \left(\delta_\lambda^\nu E_\lambda \right)^* \quad \forall (\nu, \lambda) \in \Lambda_0^2$$

$$= \delta_\lambda^\nu F_\lambda,$$

where

$$F_\lambda = \begin{cases} 0 & , \lambda \in \Lambda_0 - (\Lambda_1 \cup \Lambda_2) \\ -\pi \epsilon_m h_n \mathcal{H}_n^* B_{mn} & , \lambda \in \Lambda_1 \cup \Lambda_2. \end{cases}$$

Next, to show that

$$C_{\lambda}^{\nu} = D_{\lambda}^{\nu} = 0 \quad \forall (\lambda, \nu) \in \Lambda_0^2.$$

First

$$\begin{aligned} C_{lmn}^{1\mu\nu} &= \int_{\Omega} \hat{r} \cdot (\bar{m}_{lmn} \times \bar{m}_{1\mu\nu}^*) d\omega \\ &= -m h_n h_{\nu}^* (S_m, C_{\mu}) (\Pi_n^m, \Pi_{\nu}^{\mu}) - \mu h_n h_{\nu}^* (C_m, S_{\mu}) (\tau_n^m, \tau_{\nu}^{\mu}) \\ &= 0. \end{aligned}$$

Also

$$\begin{aligned} C_{lmn}^{2\mu\nu} &= \int_{\Omega} \hat{r} \cdot (\bar{m}_{lmn} \times \bar{m}_{2\mu\nu}^*) d\omega \\ &= m h_n h_{\nu}^* (S_m, S_{\mu}) (\Pi_n^m, \tau_{\nu}^{\mu}) + \mu h_n h_{\nu}^* (C_m, C_{\mu}) (\tau_n^m, \Pi_{\nu}^{\mu}) \\ &= \begin{cases} 0 & \text{if } m = 0 \text{ or } \mu = 0 \\ \pi \delta_{\mu}^m m h_n h_{\nu}^* \int_0^{\pi} [\Pi_n^m \tau_{\nu}^m + \tau_n^m \Pi_{\nu}^m] \sin \theta d\theta = 0, & \end{cases} \end{aligned}$$

the latter result holding for

$$(m, \mu) \in \langle \mathcal{A} \rangle^2$$

$$(n, \nu) \in \langle \mathcal{A} \rangle^2,$$

and hence

$$C_{2\mu\nu}^{1mn} = \int_{\Omega} r \cdot (\bar{m}_{2\mu\nu} \times \bar{m}_{1mn}^*) d\omega = -C_{lmn}^{2\mu\nu*}.$$

Also by inspection,

$$C_{2mn}^{2\mu\nu} = 0 \quad \forall ((n, m), (\nu, \mu)) \in \mathcal{Y}_0^2.$$

The above results, taken together with theorem A-1 yield

$$C_\lambda^\nu = 0 \quad \forall (\lambda, \nu) \in \Lambda_0^2.$$

It remains only to examine D_λ^ν . First, by inspection

$$D_{1mn}^{1\mu\nu} = 0.$$

Next

$$\begin{aligned} D_{1mn}^{2\mu\nu} &= \int_{\Omega} \hat{r} \cdot (\bar{n}_{1mn} \times \bar{n}_{2\mu\nu}^*) d\omega \\ &= \mu \mathcal{H}_n \mathcal{H}_n^* (C_m, C_\mu) (\tau_n^m, \Pi_\nu^\mu) + m \mathcal{H}_n \mathcal{H}_\nu^* (S_m, S_\mu) (\Pi_n^m, \tau_\nu^\mu) \\ &= \begin{cases} 0 & \text{if } m \text{ or } \mu = 0 \\ m \delta_\mu^m \mathcal{H}_n \mathcal{H}_n^* \int_0^\pi [\tau_n^m \Pi_\nu^m + \Pi_n^m \tau_\nu^m] \sin \theta d\theta, & \end{cases} \end{aligned}$$

and by Appendix A,

$$\left. \begin{aligned} D_{1mn}^{2\mu\nu} &= 0 \\ \text{and hence also} \\ D_{2mn}^{1\mu\nu} &= 0 \end{aligned} \right\} \quad \forall ((n, m), (\nu, \mu)) \in \mathcal{Y}_0^2.$$

Finally, it is evident by inspection that

$$D_{2mn}^{2\mu\nu} = \int_{\Omega} \hat{r} \cdot (\bar{n}_{2mn} \times \bar{n}_{2\mu\nu}^*) d\omega = 0 \quad .$$

It has been shown that

$$D_{\lambda}^{\nu} = 0 \quad \forall (\lambda, \nu) \in \Lambda_0^2 \quad .$$

LIST OF SYMBOLS

In Order of Appearance

Symbols given in Volume I (Reference 11) will not be repeated here.

Section 1.

$(\tilde{E}, \tilde{H}, \tilde{B}, \tilde{D}, \tilde{J})$	General time-varying electromagnetic field
ρ	Charge density
ϵ, μ, σ	Electromagnetic constants
$\gamma, \gamma_2, \gamma_1$	Regimes of definition for electromagnetic constants $\bar{c} \equiv (\epsilon, \mu, \sigma)$
$\tilde{\omega}$	Time frequency of harmonically time-varying electromagnetic fields
ω	A point on the unit sphere Ω
$(\bar{E}, \bar{H}, \bar{B}, \bar{D}, \bar{J})$	Time-independent parts of an electromagnetic field of time-variation $\exp(-i\tilde{\omega}t)$
$C^n(X, Y) = \{f f: X \rightarrow Y, f \in C^n(X)\}$	
$\mathcal{M}(D, \bar{c})$	A class of electromagnetic fields on $D \subset E^3$ associated with $\bar{c} \in \gamma$ (See Definition 1)
k	Propagation constant
$\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$	Regions in the complex plane
$\mathcal{R}_0$	Set of strictly positive reals
$\mathcal{M}(D, \bar{c}, \tilde{\omega})$	Subset of $\mathcal{M}(D, \bar{c})$ consisting of fields of time variation $\exp(-i\tilde{\omega}t)$
$\tilde{\mathcal{E}}(D, \bar{c}, \tilde{\omega})$	A space of pairs $(\bar{E}, \bar{H})$ of related vector wave functions associated with $\bar{c} \in \gamma$ and $\tilde{\omega}$
$\mathcal{G}$	A mapping with domain $\tilde{\mathcal{E}}(D, \bar{c}, \tilde{\omega})$ and range $\mathcal{M}(D, \bar{c}, \tilde{\omega})$
$\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$	Index sets
$\Lambda_1, \Lambda_2, \Lambda_0$	Index sets
$\bar{M}^{(l)}(v, F)$	For $l = 2$ (See 1-60)
$\bar{N}^{(l)}(v, F)$	For $l = 1$ (See 1-64)

$$\mathcal{G}^{2E}(v_1, \bar{r}), \mathcal{G}^{2H}(v_2, \bar{r})$$

See (1-61) and (1-62). These are elements of $\mathcal{G}(D_{2a}, \bar{c}_2, \tilde{\omega})$

$$\mathcal{G}^{1E}(v_3, \bar{r}), \mathcal{G}^{1H}(v_4, \bar{r})$$

See (1-65) and (1-66). These are elements of $\mathcal{G}(D_{1b}, \bar{c}_1, \tilde{\omega})$

$$\left. \begin{array}{l} \mathcal{I}_{2a\bar{c}_2\tilde{\omega}}(v_1, v_2) \\ \mathcal{I}_{2a}\mathcal{I}'(\tilde{\omega}, \bar{c}_2) \end{array} \right\}$$

(See Definition 2)

$$\left. \begin{array}{l} \mathcal{I}_{1b\bar{c}_1\tilde{\omega}}(v_3, v_4) \\ \mathcal{I}_{1b}\mathcal{I}'(\tilde{\omega}, \bar{c}_1) \end{array} \right\}$$

(See Definition 3)

$$\mathcal{G}_\lambda^{(\ell)E}(\bar{r}), \mathcal{G}_\lambda^{(\ell)H}(\bar{r})$$

See (1-70) and (1-71)

$$S(f, D_1, D_2, \hat{n})$$

A star S given by $r = f(\omega)$ with interior domain D_1 , exterior domain D_2 , and unit outward normal vector $\hat{n}$ (See Section 1.11)

$$\hat{r}, \hat{\theta}, \hat{\phi}$$

Unit base vectors for the usual spherical coordinate system

$$\hat{\xi}, \hat{\zeta}$$

Unit vectors tangential to S

$$\mathcal{P}(S, \tilde{\omega}, \bar{c}_1, \bar{c}_2, \delta, \mathcal{G}_0)$$

An electromagnetic scattering problem involving a scatterer bounded by a star $S(f, D_1, D_2, \hat{n})$ with $\bar{c}_i$ an electromagnetic triple in $\mathcal{V}_i$ associated with the domain D_i , $i = 1, 2$; and with incident field $\mathcal{G}_0$ of time variation $e^{-i\tilde{\omega}t}$ given in a subdomain $D(S, \delta)$ of "width" δ in D_2 immediately adjacent to S . (See Figure 1-7.)

$$(\mathcal{G}_1, \mathcal{G}_2) = ((\bar{E}_1, \bar{H}_1), (\bar{E}_2, \bar{H}_2))$$

A solution to an electromagnetic scattering problem

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Section 3

$\mathcal{J}$	An operator which gives the tangential components on S of pairs of vector-valued functions on an annulus containing S
$\mathcal{H}$	The usual Hilbert space on $\mathcal{J}^2(\Omega)$
$\mathcal{H}^4$	The usual Hilbert space on $[\mathcal{J}^2(\Omega)]^4$
$\Phi \equiv \{\phi_\lambda \mid \lambda \in \Lambda_0\}$	The usual normalized complete orthogonal set of spherical harmonics
$\Lambda_0' \equiv \Lambda_0 \times \langle 4 \rangle$	An index set
$\{\tau_\lambda \mid \lambda \in \Lambda_0'\}$	A complete orthogonal set for $\mathcal{H}^4$
$T: \mathcal{H}^4 \rightarrow \mathcal{H}^4$	The general solvability operator (See Definition 6)
$\mathcal{X}$	A closed linear manifold in $\mathcal{H}^4$
$\Lambda' = (\Lambda_1 \cup \Lambda_2) \times \langle 4 \rangle$	An index set
$T_0 = T _{\mathcal{X}}$	The solvability operator
$\{\chi_{\lambda i} \mid (\lambda, i) \in \Lambda'\}$	An orthogonal set spanning $\mathcal{X}$
$V: \mathcal{X} \rightarrow \left(\tilde{\mathcal{C}}(D_{2a}, \bar{c}_2, \bar{\omega}) \times \tilde{\mathcal{C}}(D_{1b}, \bar{c}_1, \bar{\omega}) \right)$	An operator for forming <u>candidates</u> for solution to scattering problems

Section 4

m	Most often used as an index in $\lambda = (n, m, i) \in \Lambda_0$; also used to denote the relative refractive index $m = k_1/k_2$
$\hat{i}, \hat{j}, \hat{k}$	Unit base vectors for the usual Cartesian coordinate system
a	Size ratio of scattering theory

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$a_n^r, a_n^t, b_n^r, b_n^t$

Sequence of "scattering coefficients" for the classical scattering problems (See Table 4-1)

Λ_3

An index set

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