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Estimating the Parameters of the Chi-Square and Some Related Distributions Using Quantiles

I. Eisenberger

JET PROPULSION LABORATORY
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16. Abstract

A two-parameter family of distributions is generated by considering the random variable

$$Y_n = \sum_{i=1}^{n} x_i^2,$$

where the x_i are independent and distributed $N(0, \sigma^2)$. For $\sigma^2 = 1$, Y_n is said to have the chi-square distribution with n degrees of freedom. This report is concerned with the use of sample quantiles in solving the following estimation problems associated with this distribution:

(1) Estimating σ^2 when n is known.

(2) Estimating n when $\sigma^2 = 1$.

(3) Estimating the product $n\sigma^2$ when neither n nor σ^2 is known.

(4) Estimating both n and σ^2 when neither is known.

(5) Estimating σ for the special cases where $v_2 = Y_2$ and $v_3 = Y_3$.

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PASADENA, CALIFORNIA

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Preface

The work described in this report was performed by the Telecommunications Division of the Jet Propulsion Laboratory.

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(5) Estimating σ for the special cases where $\nu_2 = \sqrt{y_2}$ and $\nu_3 = \sqrt{y_3}$.

Monte Carlo methods are used in applying the various estimators, and the results are listed.

Estimating the Parameters of the Chi-Square and Some Related Distributions Using Quantiles

I. Introduction

This report presents additional results of the continuing investigation into the use of sample quantiles for data compression of space telemetry. Most of the previous results of this investigation are given in Refs. 1–4. Reference 1 deals with the problem of efficiently estimating the parameters of a normal distribution, using up to 20 quantiles, and also describes two goodness-of-fit tests, each using four quantiles. References 2–4 are concerned with hypothesis testing and estimation of the correlation coefficient of a bivariate normal distribution, using up to eight quantiles. In Ref. 5, estimators are given for the parameters of an extreme-value distribution, using up to 10 quantiles.

In this report, we consider initially the distribution of the sum of the squares of n independent, normally distributed random variables, each with zero mean and variance σ^2. For $\sigma^2 = 1$, this distribution is known as the chi-square (χ^2) distribution with n degrees of freedom (d.f.). For identification purposes, we will designate the generalized form as the χ^2 distribution with a scale factor (σ^2).

With respect to this distribution and assuming that n is known and the sample size is large, we first construct asymptotically optimum and unbiased quantile estimators of σ^2, using k quantiles, for $k = 1, 2, \cdots, 6$ and for $n = 1, 2, \cdots, 32, 34, \cdots, 40, 50, 60$. The efficiencies of the quantile estimators are also determined, for each value of n and k, relative to the maximum-likelihood (M.L.) estimators.

Estimating the power of white Gaussian noise is equivalent to estimating σ^2 for a χ^2 distribution with a scale factor. The problem of estimating σ^2 also arises in noncoherent communication analysis.

The χ^2 distribution has only one parameter, n. When n is unknown, it must be estimated in order for the distribution to be completely specified. Accordingly, quantile estimators of n are constructed using up to six quantiles. For an assumed sample size of $m = 100$, the probability that the quantile estimate of n will equal n is compared with the same probability when the M.L. estimator is used, for $n = 1, 2, \cdots, 5, 10, 15, \cdots, 70$. The probability that each quantile estimate of n will differ from n by at most one is also given.

The estimation of the length of noise bursts is equivalent to estimating the d.f. of a χ^2 distribution.

For the case where both n and σ^2 are unknown, quantile estimators of the mean, $n\sigma^2$, using from two to six quantiles, are constructed from the optimum estimators of σ^2 for $n = 20$. With this estimate, $\widehat{n\sigma^2}$, and the same quantiles used in estimating it, an unusual type of quantile estimator is constructed, using four and six quantiles, in order to estimate n and σ^2 individually.

Two distributions which are related to the χ^2 distribution with a scale factor are of special practical interest. These are the square roots of the χ^2 distributions with a scale factor and two and three d.f. Asymptotically optimum and unbiased estimators of σ are constructed for each case, using up to six quantiles. The efficiencies of these estimators are also determined relative to the M.L. estimators.

Finally, a Monte Carlo study was made for each type of estimator, and the results were listed.

II. Review of Quantiles

To define a quantile, consider m independent values $x_1, x_2, \cdots, x_m$ taken from a distribution of a continuous type with distribution function $G(x)$ and density function $g(x)$. The pth quantile, or the quantile of order p of the distribution or population, denoted by ζ_p, is defined as the root of the equation $G(\zeta_p) = p$; that is,

$$p = \int_{-\infty}^{\zeta_p} dG(x) = \int_{-\infty}^{\zeta_p} g(x)\, dx$$

The corresponding sample quantile z_p is defined as follows: If the sample values are arranged in non-decreasing order of magnitude

$$x_{(1)} \leqq x_{(2)} \leqq \cdots \leqq x_{(m)}$$

then $x_{(i)}$ is called the ith-order statistic and

$$z_p = x_{([mp]+1)}$$

where $[mp]$ is the greatest integer $\leqq mp$.

If $g(x)$ is differentiable in some neighborhood of each quantile value considered, it has been shown (Ref. 6) that the joint distribution of any number of quantiles is asymptotically normal as $n \to \infty$ and that, asymptotically,

$$E(z_p) = \zeta_p$$

$$\operatorname{var}(z_p) = \frac{p(1-p)}{mg^2(\zeta_p)}$$

$$\rho_{12} = \left[\frac{p_1(1-p_2)}{p_2(1-p_1)}\right]^{1/2}$$

where ρ_{12} is the correlation between z_{p_1} and z_{p_2} and $p_1 < p_2$.

Throughout this report, we will denote by $k_n(x)$ and $K_n(x)$ the density function and distribution function, respectively, of the χ^2 distribution with n d.f., given by

$$k_n(x) = \frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)} x^{n/2-1} e^{-x/2}, x \geqq 0$$

$$K_n(x) = \int_0^x k_n(t)\, dt$$

The density function of the χ^2 distribution with a scale factor is easily derived, and is given by

$$\frac{1}{\sigma^2} k_n\left(\frac{x}{\sigma^2}\right) = \frac{1}{2^{n/2}\,\sigma^n\,\Gamma\left(\frac{n}{2}\right)} x^{n/2-1} e^{-x/2\sigma^2}, x \geqq 0$$

Thus, one has

$$p = \frac{1}{\sigma^2}\int_0^{\zeta_p^*} k_n\left(\frac{x}{\sigma^2}\right) dx = \int_0^{\zeta_p^*/\sigma^2} k_n(x)\, dx = \int_0^{\zeta_p} k_n(x)\, dx$$

Hence, one sees that, asymptotically, if z_p is a sample quantile of order p from a χ^2 distribution with a scale factor, then

$$E(z_p) = \zeta_p^* = \sigma^2\,\zeta_p$$

and, since

$$\frac{1}{\sigma^2} k_n \left(\frac{\zeta_p^*}{\sigma^2} \right) = \frac{1}{\sigma^2} k_n (\zeta_p)$$

$$\operatorname{var}(z_p) = \frac{\sigma^4 p (1-p)}{m k_n^2 (\zeta_p)}$$

so that the moments of the sample quantiles of a χ^2 distribution with a scale factor are expressable in terms of the χ^2 distribution.

When $t\,(>1)$ quantiles are being considered, the sample quantiles will be denoted as z_i of order $p_i, i = 1, 2, \cdots, t$ and $p_i < p_j$ for $i < j$. The corresponding population quantile of the χ^2 distribution will be denoted by ζ_i.

III. Estimating the Parameter σ^2 of a χ^2 Distribution With a Scale Factor, Using Quantiles

The quantile estimators of σ^2 will be constructed as linear combinations of k quantiles of the form

$$\hat{\sigma}^2 = \sum_{i=1}^{k} \beta_i z_i \tag{1}$$

where the β_i and the orders of the quantiles will be chosen so as to satisfy simultaneously the following two conditions:

(1) The expected value of $\hat{\sigma}^2\,(E(\hat{\sigma}^2)) = \sigma^2$.

(2) $\operatorname{Var}(\hat{\sigma}^2) =$ minimum.

Beginning with one quantile, one has, for a given value of n and a fixed value of p,

$$\hat{\sigma}^2 = \beta z_p, \qquad E(\hat{\sigma}^2) = \beta \zeta_p \sigma^2$$

It is evident that in order to satisfy the first condition, we must put $\beta = 1/\zeta_p$. The variance of $\hat{\sigma}^2$ is given by

$$\operatorname{var}(\hat{\sigma}^2) = \frac{\sigma^4 p (1-p)}{m \zeta_p^2 k_n^2 (\zeta_p)}$$

so that the second condition will be satisfied if the order of the quantile is chosen to minimize $\operatorname{var}(\hat{\sigma}^2)$. It is more convenient, however, to adopt the equivalent procedure of minimizing $\operatorname{var}(\hat{\sigma}^2)$ with respect to ζ_p. Accordingly, one has

$$\frac{\partial p}{\partial \zeta_p} = k_n (\zeta_p)$$

$$\frac{\partial k_n (\zeta_p)}{\partial \zeta_p} = \frac{k_n (\zeta_p)}{2 \zeta_p} (n - 2 - \zeta_p)$$

Then, setting $\partial \operatorname{var}(\hat{\sigma}^2)/\partial \zeta_p = 0$ results in

$$\zeta_p k_n (\zeta_p)(1 - 2p) - p(1-p)(n - \zeta_p) = 0 \tag{2}$$

Solving Eq. (2) for p gives the optimum order of the single sample quantile used in estimating σ^2.

For $k > 1$, from Eq. (1) one has

$$E(\sigma^2) = \sigma^2 \sum_{i=1}^{k} \beta_i \zeta_i$$

and σ^2 will be unbiased if we put

$$\sum_{i=1}^{k} \beta_i \zeta_i = 1$$

The minimization of $\operatorname{var}(\hat{\sigma}^2)$ will be accomplished in two steps. Since β_k can now be expressed as

$$\beta_k = \frac{\left(1 - \sum_{i=1}^{k-1} \beta_i \zeta_i\right)}{\zeta_k} \tag{3}$$

we first determine the values of the remaining β_i which will minimize $\operatorname{var}(\hat{\sigma}^2)$ for a fixed set of the p_i.

Let σ_i^2 denote the variance of z_i and σ_{ij} the covariance between z_i and z_j. Then one has

$$\operatorname{var}(\hat{\sigma}^2) = \sum_{i=1}^{k} \beta_i \sigma_i^2 + 2 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \beta_i \beta_j \sigma_{ij} \tag{4}$$

Substituting in Eq. (4) the value of β_k given in Eq. (3) and then setting the partial derivatives of $\operatorname{var}(\hat{\sigma}^2)$ with respect to β_i equal to zero for $i = 1, 2, \cdots, k-1$ results in $k-1$ linear equations of the form

$$\sum_{j=1}^{k-1} A_{ij} \beta_j = b_i, \qquad i = 1, 2, \cdots, k-1$$

where

$$A_{ii} = \zeta_k \sigma_i^2 + \zeta_i^2 \sigma_k^2/\zeta_k - 2\zeta_i \sigma_{ik}, \qquad i = 1, 2, \cdots, k-1$$

$$A_{ij} = \zeta_i \zeta_j \sigma_k^2/\zeta_k + \zeta_k \sigma_{ij} - \zeta_j \sigma_{ik} - \zeta_i \sigma_{jk}, \qquad i = 1, 2, \cdots, k-2, \qquad j = i+1, \cdots, k-1$$

$$A_{ji} = A_{ij}$$

$$b_i = \zeta_i \sigma_k^2/\zeta_k - \sigma_{ik}, \qquad i = 1, 2, \cdots, k-1$$

The values of the β_i obtained from the above set of $k-1$ linear equations and Eq. (3) are those which minimize var $(\hat{\sigma}^2)$ for the fixed set of p_i. Then, by varying the p_i and determining the values of the β_i for each variation by the above procedure, the set of p_i and β_i are obtained for which var $(\hat{\sigma}^2)$ is an absolute minimum and $E(\hat{\sigma}^2) = \sigma^2$.

Using all the m sample values, the M.L. estimator of σ^2 and its variance for each n are given by

$$\tilde{\sigma}^2 = \frac{1}{nm} \sum_{i=1}^{m} x_i$$

$$\text{var}(\tilde{\sigma}^2) = \frac{2\sigma^4}{nm}$$

The efficiency of $\hat{\sigma}^2$ is defined for all values of k and n as

$$\text{eff}(\hat{\sigma}^2) = \frac{2\sigma^4}{nm \, \text{var}(\hat{\sigma}^2)}$$

Tables 1–5 give the optimum orders of the quantiles and the coefficients to be used in the estimators

$$\hat{\sigma}^2 = \sum_{i=1}^{k} \beta_i z_i$$

for $k = 1(1)6$ and $n = 1(1)32(2)40(10)60$. The efficiencies are also given.

For intermediate values of n, interpolation should give good results. For $60 < n \lessapprox 70$, extrapolation should also give good results. For $n > 70$, since the χ^2 distribution approaches normality as $n \to \infty$, one can use the estimators of the mean of a normal distribution given in Ref. 1 (p. 104) for $k = 1, 2, 4, \cdots, 20$. For $k = 3, 5, \cdots, 9$, the estimators given in Ref. 7 (p. 279) can be used. It should be noted, however, that since the mean of a χ^2 distribution with a scale factor is $n\sigma^2$, the estimators given in Refs. 1 and 7 should be divided by n to obtain the estimate of σ^2.

For $n = 2$, the density function of a χ^2 distribution with a scale factor is

$$\frac{1}{\sigma^2} k_2 \left(\frac{x}{\sigma^2} \right) = \frac{1}{2\sigma^2} e^{-x/2\sigma^2}$$

which is the density of an exponential distribution of the form

$$\lambda e^{-\lambda t}$$

where, in this case, $\lambda = 1/2\sigma^2$. Now, if we denote by $\bar{\zeta}_i$ and z_i, $i = 1, 2, \cdots, k$, the population and sample quantiles, respectively, of an exponential distribution with an unknown parameter λ, of the same orders as given in Tables 1–5 for $n = 2$, then if ζ_i again denotes the corresponding population quantile of the χ^2 distribution with two d.f., one has

$$\zeta_i = 2\lambda \bar{\zeta}_i \qquad i = 1, 2, \cdots, k$$

and if we let

$$\alpha = \sum_{i=1}^{k} \beta_i z_i$$

then

$$E(\alpha) = \sum_{i=1}^{k} \beta_i \bar{\zeta}_i = \sum_{i=1}^{k} \frac{\beta_i \zeta_i}{2\lambda} = \frac{1}{2\lambda}$$

Thus, multiplying by two the estimators of σ^2 for $n = 2$ gives the optimum, unbiased k-quantile estimator of $1/\lambda$ of the exponential distribution for each value of k.

Table 1. Optimum orders p_i, coefficients β_i, and efficiencies of quantile estimators of σ^2 for a χ^2 distribution with a scale factor and n d.f. of the form

$$\hat{\sigma}^2 = \sum_{i=1}^{k} \beta_i z_i, \qquad \text{for } k = 1, 2$$

n	$k = 1$			$k = 2$				
	p_1	β_1	Eff $(\hat{\sigma}^2)$	p_1	p_2	β_1	β_2	Eff $(\hat{\sigma}^2)$
1	0.8617	0.4552	0.6522	0.7458	0.9540	0.4150	0.1157	0.8244
2	0.7968	0.3138	0.6476	0.6385	0.9265	0.2616	0.08955	0.8202
3	0.7575	0.2391	0.6451	0.5781	0.9075	0.1878	0.07344	0.8179
4	0.7306	0.1931	0.6435	0.5387	0.8939	0.1455	0.06223	0.8164
5	0.7107	0.1619	0.6424	0.5107	0.8827	0.1180	0.05423	0.8154
6	0.6951	0.1394	0.6416	0.4903	0.8745	0.09909	0.04788	0.8146
7	0.6826	0.1223	0.6410	0.4730	0.8665	0.08511	0.04316	0.8141
8	0.6722	0.1090	0.6405	0.4601	0.8604	0.07452	0.03917	0.8136
9	0.6634	0.09829	0.6401	0.4476	0.8547	0.06619	0.03600	0.8132
10	0.6558	0.08950	0.6398	0.4383	0.8494	0.05944	0.03330	0.8129
11	0.6492	0.08215	0.6396	0.4297	0.8454	0.05396	0.03094	0.8127
12	0.6434	0.07592	0.6394	0.4235	0.8419	0.04938	0.02885	0.8125
13	0.6382	0.07056	0.6391	0.4168	0.8383	0.04546	0.02711	0.8123
14	0.6335	0.06591	0.6390	0.4106	0.8351	0.04212	0.02557	0.8121
15	0.6292	0.06185	0.6388	0.4062	0.8323	0.03922	0.02416	0.8120
16	0.6254	0.05824	0.6387	0.4013	0.8294	0.03667	0.02294	0.8119
17	0.6219	0.05503	0.6386	0.3973	0.8273	0.03445	0.02180	0.8118
18	0.6186	0.05216	0.6385	0.3935	0.8250	0.03247	0.02079	0.8117
19	0.6156	0.04958	0.6384	0.3898	0.8227	0.03068	0.01989	0.8116
20	0.6129	0.04724	0.6383	0.3869	0.8209	0.02910	0.01904	0.8115
21	0.6102	0.04511	0.6382	0.3841	0.8200	0.02769	0.01822	0.8114
22	0.6078	0.04316	0.6382	0.3814	0.8176	0.02636	0.01755	0.8113
23	0.6056	0.04137	0.6381	0.3781	0.8157	0.02515	0.01692	0.8113
24	0.6034	0.03973	0.6381	0.3755	0.8139	0.02405	0.01633	0.8112
25	0.6014	0.03821	0.6380	0.3735	0.8126	0.02306	0.01575	0.8112
26	0.5995	0.03681	0.6379	0.3710	0.8111	0.02213	0.01524	0.8111
27	0.5978	0.03550	0.6379	0.3697	0.8101	0.02128	0.01473	0.8111
28	0.5961	0.03428	0.6378	0.3673	0.8085	0.02048	0.01429	0.8110
29	0.5945	0.03315	0.6378	0.3654	0.8074	0.01974	0.01385	0.8110
30	0.5929	0.03208	0.6378	0.3647	0.8066	0.01907	0.01343	0.8110
31	0.5915	0.03108	0.6377	0.3624	0.8053	0.01842	0.01306	0.8109
32	0.5901	0.03015	0.6377	0.3611	0.8044	0.01782	0.01269	0.8109
34	0.5874	0.02843	0.6376	0.3576	0.8022	0.01672	0.01204	0.8108
36	0.5851	0.02690	0.6376	0.3557	0.8006	0.01576	0.01144	0.8108
38	0.5828	0.02553	0.6375	0.3533	0.7993	0.01490	0.01089	0.8107
40	0.5808	0.02429	0.6375	0.3504	0.7972	0.01411	0.01042	0.8107
50	0.5725	0.01954	0.6373	0.3414	0.7904	0.01118	0.008525	0.8105
60	0.5663	0.01635	0.6372	0.3351	0.7857	0.009246	0.007213	0.8104

Table 2. Optimum orders p_i, coefficients β_i, and efficiencies of quantile estimators of σ^2 for a χ^2 distribution with a scale factor and n d.f.,

$$\hat{\sigma}^2 = \sum_{i=1}^{3} \beta_i z_i$$

n	p_1	p_2	p_3	β_1	β_2	β_3	Eff $(\hat{\sigma}^2)$
1	0.6599	0.8904	0.9793	0.3808	0.1557	0.04750	0.8943
2	0.5300	0.8305	0.9656	0.2240	0.1131	0.03861	0.8911
3	0.4581	0.7895	0.9550	0.1534	0.08928	0.03311	0.8892
4	0.4168	0.7629	0.9475	0.1150	0.07306	0.02863	0.8880
5	0.3872	0.7428	0.9415	0.09123	0.06189	0.02529	0.8871
6	0.3639	0.7254	0.9362	0.07501	0.05390	0.02278	0.8865
7	0.3461	0.7110	0.9315	0.06339	0.04767	0.02082	0.8860
8	0.3326	0.6997	0.9276	0.05477	0.04269	0.01911	0.8857
9	0.3213	0.6900	0.9243	0.04811	0.03866	0.01768	0.8854
10	0.3118	0.6808	0.9211	0.04278	0.03534	0.01649	0.8851
11	0.3038	0.6742	0.9187	0.03853	0.03252	0.01539	0.8849
12	0.2962	0.6669	0.9162	0.03494	0.03017	0.01449	0.8847
13	0.2906	0.6611	0.9139	0.03197	0.02808	0.01369	0.8846
14	0.2852	0.6558	0.9119	0.02945	0.02628	0.01296	0.8844
15	0.2799	0.6510	0.9102	0.02727	0.02472	0.01231	0.8843
16	0.2762	0.6466	0.9083	0.02539	0.02328	0.01173	0.8842
17	0.2719	0.6426	0.9069	0.02373	0.02205	0.01119	0.8841
18	0.2683	0.6389	0.9054	0.02227	0.02092	0.01071	0.8840
19	0.2654	0.6355	0.9039	0.02097	0.01989	0.01028	0.8840
20	0.2625	0.6323	0.9026	0.01981	0.01897	0.009875	0.8839
21	0.2597	0.6294	0.9014	0.01877	0.01813	0.009501	0.8838
22	0.2568	0.6266	0.9004	0.01782	0.01738	0.009152	0.8838
23	0.2546	0.6240	0.8992	0.01697	0.01666	0.008834	0.8837
24	0.2523	0.6215	0.8982	0.01619	0.01602	0.008537	0.8837
25	0.2506	0.6192	0.8971	0.01548	0.01540	0.008266	0.8836
26	0.2483	0.6170	0.8963	0.01481	0.01485	0.008002	0.8836
27	0.2466	0.6150	0.8954	0.01421	0.01433	0.007760	0.8836
28	0.2448	0.6130	0.8946	0.01365	0.01385	0.007532	0.8835
29	0.2435	0.6111	0.8937	0.01314	0.01339	0.007323	0.8835
30	0.2417	0.6093	0.8930	0.01265	0.01297	0.007117	0.8835
31	0.2404	0.6077	0.8922	0.01221	0.01257	0.006928	0.8834
32	0.2391	0.6060	0.8915	0.01779	0.01219	0.006748	0.8834
34	0.2363	0.6030	0.8902	0.01103	0.01151	0.006413	0.8834
36	0.2340	0.6002	0.8890	0.01036	0.01090	0.006112	0.8833
38	0.2320	0.5977	0.8874	0.009761	0.01034	0.005841	0.8833
40	0.2300	0.5953	0.8867	0.009227	0.009848	0.005591	0.8832
50	0.2218	0.5852	0.8821	0.007227	0.007941	0.004619	0.8831
60	0.2166	0.5783	0.8787	0.005937	0.006647	0.003934	0.8830

Table 3. Optimum orders p_i, coefficients β_i, and efficiencies of quantile estimators of σ^2 for a χ^2 distribution with a scale factor and n d.f.,

$$\hat{\sigma}^2 = \sum_{i=1}^{4} \beta_i \mathbf{z}_i$$

n	p_1	p_2	p_3	p_4	β_1	β_2	β_3	β_4	Eff ($\hat{\sigma}^2$)
1	0.5972	0.8308	0.9429	0.9890	0.3530	0.1697	0.07626	0.02417	0.9294
2	0.4512	0.7418	0.9067	0.9810	0.1954	0.1181	0.05977	0.02046	0.9269
3	0.3763	0.6869	0.8814	0.9750	0.1290	0.08970	0.04908	0.01775	0.9254
4	0.3338	0.6503	0.8636	0.9705	0.09406	0.07182	0.04145	0.01559	0.9245
5	0.3031	0.6219	0.8487	0.9664	0.07290	0.05983	0.03606	0.01405	0.9238
6	0.2828	0.6021	0.8374	0.9633	0.05919	0.05103	0.03178	0.01271	0.9233
7	0.2661	0.5845	0.8268	0.9602	0.04941	0.04445	0.02855	0.01173	0.9229
8	0.2521	0.5695	0.8184	0.9579	0.04215	0.03948	0.02595	0.01082	0.9226
9	0.2412	0.5572	0.8104	0.9555	0.03666	0.03538	0.02378	0.01012	0.9224
10	0.2320	0.5463	0.8040	0.9535	0.03233	0.03210	0.02195	0.009475	0.9222
11	0.2249	0.5373	0.7985	0.9518	0.02889	0.02933	0.02037	0.008893	0.9220
12	0.2180	0.5291	0.7932	0.9501	0.02605	0.02701	0.01901	0.008415	0.9219
13	0.2136	0.5232	0.7891	0.9488	0.02366	0.02497	0.01779	0.007944	0.9217
14	0.2085	0.5170	0.7849	0.9474	0.02178	0.02326	0.01674	0.007556	0.9216
15	0.2048	0.5120	0.7812	0.9462	0.02011	0.02173	0.01579	0.007197	0.9215
16	0.2005	0.5066	0.7775	0.9450	0.01863	0.02041	0.01498	0.006881	0.9214
17	0.1975	0.5022	0.7746	0.9440	0.01737	0.01923	0.01422	0.006581	0.9214
18	0.1944	0.4981	0.7716	0.9429	0.01625	0.01817	0.01355	0.006315	0.9213
19	0.1907	0.4929	0.7683	0.9419	0.01521	0.01725	0.01297	0.006080	0.9212
20	0.1882	0.4898	0.7658	0.9410	0.01434	0.01639	0.01240	0.005853	0.9212
21	0.1857	0.4863	0.7635	0.9402	0.01355	0.01563	0.01189	0.005640	0.9211
22	0.1836	0.4830	0.7610	0.9394	0.01283	0.01491	0.01143	0.005451	0.9211
23	0.1815	0.4804	0.7591	0.9387	0.01219	0.01427	0.01098	0.005267	0.9210
24	0.1794	0.4773	0.7569	0.9380	0.01160	0.01368	0.01059	0.005097	0.9210
25	0.1778	0.4748	0.7553	0.9373	0.01107	0.01314	0.01021	0.004938	0.9210
26	0.1760	0.4720	0.7530	0.9365	0.01057	0.01262	0.009873	0.004800	0.9209
27	0.1743	0.4697	0.7512	0.9360	0.01012	0.01216	0.009550	0.004659	0.9209
28	0.1730	0.4676	0.7500	0.9355	0.009706	0.01173	0.009243	0.004521	0.9209
29	0.1720	0.4660	0.7485	0.9349	0.009337	0.01132	0.008951	0.004398	0.9209
30	0.1706	0.4640	0.7470	0.9344	0.008979	0.01094	0.008685	0.004280	0.9208
31	0.1692	0.4620	0.7453	0.9338	0.008643	0.01058	0.008437	0.004175	0.9208
32	0.1682	0.4601	0.7441	0.9333	0.008335	0.01025	0.008199	0.004071	0.9208
34	0.1656	0.4564	0.7415	0.9324	0.007767	0.009652	0.007769	0.003877	0.9207
36	0.1640	0.4538	0.7392	0.9315	0.007288	0.009105	0.007372	0.003702	0.9207
38	0.1623	0.4513	0.7374	0.9309	0.006859	0.008628	0.007015	0.003534	0.9207
40	0.1606	0.4485	0.7352	0.9301	0.006468	0.008192	0.006697	0.003392	0.9206
50	0.1543	0.4383	0.7270	0.9269	0.005035	0.006535	0.005452	0.002818	0.9205
60	0.1500	0.4316	0.7213	0.9247	0.004118	0.005433	0.004594	0.002408	0.9205

Table 4. **Optimum orders p_i, coefficients β_i, and efficiencies of quantile estimators of σ^2 for a χ^2 distribution with a scale factor and n d.f.,**

$$\hat{\sigma}^2 = \sum_{i=1}^{5} \beta_i z_i$$

n	p_1	p_2	p_3	p_4	p_5	β_1	β_2	β_3	β_4	β_5	Eff $(\hat{\sigma}^2)$
1	0.5497	0.7793	0.9034	0.9669	0.9936	0.3292	0.1735	0.09233	0.04294	0.01380	0.9496
2	0.3935	0.6671	0.8436	0.9433	0.9885	0.1731	0.1160	0.06998	0.03548	0.01215	0.9476
3	0.3177	0.6006	0.8032	0.9262	0.9845	0.1103	0.08537	0.05601	0.03006	0.01076	0.9464
4	0.2734	0.5562	0.7740	0.9131	0.9813	0.07841	0.06690	0.04660	0.02605	0.009658	0.9456
5	0.2447	0.5255	0.7528	0.9030	0.9788	0.05982	0.05477	0.03973	0.02289	0.008724	0.9451
6	0.2244	0.5015	0.7359	0.8945	0.9766	0.04776	0.04622	0.03463	0.02047	0.007978	0.9447
7	0.2088	0.4834	0.7220	0.8877	0.9748	0.03949	0.03989	0.03068	0.01851	0.007327	0.9444
8	0.1966	0.4666	0.7088	0.8811	0.9730	0.03334	0.03494	0.02764	0.01699	0.006842	0.9441
9	0.1871	0.4547	0.6995	0.8759	0.9715	0.02884	0.03114	0.02502	0.01561	0.006388	0.9439
10	0.1789	0.4431	0.6901	0.8708	0.9701	0.02525	0.02801	0.02292	0.01451	0.006010	0.9437
11	0.1726	0.4342	0.6825	0.8665	0.9688	0.02247	0.02543	0.02110	0.01354	0.005672	0.9436
12	0.1671	0.4261	0.6757	0.8628	0.9677	0.02018	0.02327	0.01957	0.01268	0.005369	0.9435
13	0.1623	0.4193	0.6703	0.8595	0.9668	0.01830	0.02147	0.01823	0.01191	0.005087	0.9434
14	0.1576	0.4130	0.6648	0.8567	0.9659	0.01669	0.01992	0.01709	0.01125	0.004834	0.9433
15	0.1547	0.4077	0.6604	0.8540	0.9651	0.01537	0.01855	0.01605	0.01604	0.004610	0.9432
16	0.1517	0.4028	0.6558	0.8516	0.9643	0.01422	0.01734	0.01515	0.01011	0.004407	0.9432
17	0.1492	0.3988	0.6522	0.8491	0.9636	0.01324	0.01629	0.01432	0.009623	0.004227	0.9431
18	0.1461	0.3942	0.6483	0.8469	0.9629	0.01233	0.01537	0.01360	0.009193	0.004057	0.9430
19	0.1435	0.3905	0.6447	0.8447	0.9622	0.01155	0.01453	0.01294	0.008800	0.003911	0.9430
20	0.1414	0.3869	0.6413	0.8426	0.9617	0.01086	0.01378	0.01234	0.008447	0.003767	0.9429
21	0.1391	0.3835	0.6382	0.8408	0.9610	0.01024	0.01310	0.01180	0.008112	0.003641	0.9429
22	0.1378	0.3801	0.6352	0.8391	0.9605	0.009687	0.01246	0.01131	0.007809	0.003516	0.9429
23	0.1355	0.3769	0.6320	0.8372	0.9599	0.009167	0.01190	0.01086	0.007535	0.003408	0.9428
24	0.1340	0.3743	0.6294	0.8356	0.9594	0.008717	0.01138	0.01044	0.007273	0.003306	0.9428
25	0.1325	0.3718	0.6269	0.8340	0.9588	0.008302	0.01091	0.01004	0.007030	0.003207	0.9428
26	0.1309	0.3693	0.6246	0.8323	0.9584	0.007919	0.01047	0.009674	0.006805	0.003119	0.9427
27	0.1296	0.3669	0.6224	0.8313	0.9580	0.007571	0.01007	0.009350	0.006587	0.003026	0.9427
28	0.1284	0.3646	0.6199	0.8298	0.9576	0.007248	0.009681	0.009038	0.006397	0.002945	0.9427
29	0.1270	0.3628	0.6184	0.8288	0.9572	0.006953	0.009346	0.008740	0.006202	0.002865	0.9427
30	0.1260	0.3605	0.6160	0.8272	0.9567	0.006677	0.009008	0.008464	0.006033	0.002799	0.9427
31	0.1250	0.3588	0.6147	0.8266	0.9564	0.006425	0.008713	0.008213	0.005855	0.002723	0.9426
32	0.1242	0.3576	0.6135	0.8257	0.9562	0.006199	0.008433	0.007964	0.005691	0.002652	0.9426
34	0.1223	0.3542	0.6102	0.8237	0.9555	0.005770	0.007916	0.007521	0.005402	0.002529	0.9426
36	0.1206	0.3505	0.6064	0.8212	0.9547	0.005387	0.007443	0.007125	0.005154	0.002427	0.9426
38	0.1190	0.3478	0.6037	0.8194	0.9542	0.005057	0.007034	0.006766	0.004916	0.002325	0.9425
40	0.1174	0.3451	0.6012	0.8179	0.9536	0.004758	0.006670	0.006446	0.004699	0.002232	0.9425
50	0.1119	0.3351	0.5912	0.8113	0.9513	0.003676	0.005286	0.005200	0.003851	0.001862	0.9424
60	0.1080	0.3281	0.5839	0.8064	0.9496	0.002986	0.004372	0.004357	0.003266	0.001599	0.9423

Table 5. Optimum orders p_i, coefficients β_i, and efficiencies of quantile estimators of σ^2 for a χ^2 distribution with a scale factor and n d.f.,

$$\hat{\sigma}^2 = \sum_{i=1}^{6} \beta_i \mathbf{z}_i$$

n	p_1	p_2	p_3	p_4	p_5	p_6	β_1	β_2	β_3	β_4	β_5	β_6	Eff $(\hat{\sigma}^2)$
1	0.5116	0.7326	0.8629	0.9386	0.9787	0.9958	0.3073	0.1726	0.1021	0.05680	0.02706	0.008220	0.9622
2	0.3478	0.6042	0.7828	0.8978	0.9631	0.9925	0.1554	0.1114	0.07462	0.04509	0.02282	0.007811	0.9606
3	0.2732	0.5319	0.7309	0.8684	0.9508	0.9897	0.09639	0.07966	0.05783	0.03718	0.01981	0.007055	0.9596
4	0.2275	0.4799	0.6927	0.8462	0.9414	0.9875	0.06648	0.06154	0.04762	0.03178	0.01743	0.006398	0.9590
5	0.2026	0.4506	0.6677	0.8306	0.9344	0.9858	0.05029	0.04949	0.03977	0.02742	0.01542	0.005782	0.9585
6	0.1837	0.4264	0.6485	0.8184	0.9287	0.9844	0.03972	0.04142	0.03430	0.02411	0.01381	0.005274	0.9582
7	0.1712	0.4093	0.6327	0.8080	0.9239	0.9831	0.03269	0.03534	0.03001	0.02155	0.01253	0.004859	0.9579
8	0.1614	0.3941	0.6192	0.7994	0.9198	0.9820	0.02750	0.03074	0.02676	0.01950	0.01147	0.004507	0.9577
9	0.1520	0.3806	0.6067	0.7906	0.9155	0.9809	0.02358	0.02717	0.02411	0.01786	0.01066	0.004247	0.9576
10	0.1454	0.3712	0.5981	0.7846	0.9126	0.9801	0.02066	0.02438	0.02191	0.01640	0.009883	0.003964	0.9574
11	0.1389	0.3606	0.5880	0.7773	0.9087	0.9791	0.01820	0.02200	0.02011	0.01523	0.009307	0.003780	0.9573
12	0.1333	0.3525	0.5804	0.7725	0.9063	0.9784	0.01626	0.02010	0.01860	0.01420	0.008719	0.003568	0.9572
13	0.1290	0.3449	0.5728	0.7667	0.9033	0.9775	0.01466	0.01842	0.01726	0.01332	0.008256	0.003415	0.9571
14	0.1247	0.3377	0.5658	0.7615	0.9007	0.9769	0.01329	0.01701	0.01611	0.01255	0.007845	0.003263	0.9571
15	0.1214	0.3316	0.5586	0.7563	0.8978	0.9761	0.01217	0.01573	0.01509	0.01188	0.007489	0.003141	0.9570
16	0.1182	0.3265	0.5533	0.7520	0.8956	0.9754	0.01120	0.01468	0.01418	0.01125	0.007145	0.003021	0.9569
17	0.1154	0.3209	0.5470	0.7477	0.8933	0.9747	0.01035	0.01371	0.01340	0.01072	0.006845	0.002914	0.9569
18	0.1126	0.3155	0.5417	0.7436	0.8912	0.9742	0.009590	0.01287	0.01270	0.01023	0.006573	0.002806	0.9568
19	0.1102	0.3110	0.5367	0.7399	0.8891	0.9736	0.008940	0.01212	0.01206	0.009775	0.006322	0.002714	0.9568
20	0.1081	0.3072	0.5327	0.7365	0.8875	0.9731	0.008377	0.01146	0.01147	0.009361	0.006083	0.002623	0.9568
21	0.1057	0.3029	0.5288	0.7337	0.8860	0.9727	0.007848	0.01087	0.01097	0.008979	0.005845	0.002536	0.9567
22	0.1036	0.2994	0.5251	0.7312	0.8847	0.9723	0.007385	0.01034	0.01050	0.008630	0.005643	0.002452	0.9567
23	0.1018	0.2959	0.5210	0.7280	0.8830	0.9718	0.006972	0.009834	0.01005	0.008317	0.005464	0.002382	0.9567
24	0.1000	0.2930	0.5177	0.7254	0.8817	0.9714	0.006602	0.009390	0.009640	0.008021	0.005285	0.002313	0.9566
25	0.09923	0.2913	0.5162	0.7238	0.8805	0.9710	0.006298	0.008998	0.009257	0.007708	0.005106	0.002246	0.9566
26	0.09866	0.2902	0.5148	0.7223	0.8797	0.9708	0.006030	0.008634	0.008895	0.007427	0.004936	0.002177	0.9566
27	0.09771	0.2879	0.5117	0.7202	0.8785	0.9704	0.005760	0.008272	0.008570	0.007189	0.004788	0.002122	0.9566
28	0.09670	0.2862	0.5094	0.7181	0.8773	0.9701	0.005514	0.007952	0.008259	0.006960	0.004652	0.002067	0.9566
29	0.09594	0.2845	0.5076	0.7166	0.8763	0.9698	0.005290	0.007658	0.007978	0.006734	0.004519	0.002013	0.9565
30	0.09514	0.2828	0.5059	0.7152	0.8756	0.9695	0.005080	0.007385	0.007718	0.006530	0.004389	0.001959	0.9565
31	0.09429	0.2816	0.5043	0.7138	0.8750	0.9693	0.004888	0.007135	0.007469	0.006339	0.004266	0.001908	0.9565
32	0.09367	0.2803	0.5032	0.7130	0.8744	0.9691	0.004712	0.006903	0.007242	0.006149	0.004146	0.001858	0.9565
34	0.09258	0.2783	0.5007	0.7107	0.8730	0.9687	0.004397	0.006474	0.006813	0.005808	0.003933	0.001772	0.9565
36	0.09133	0.2762	0.4983	0.7086	0.8716	0.9682	0.004114	0.006098	0.006436	0.005501	0.003741	0.001694	0.9564
38	0.09044	0.2740	0.4961	0.7068	0.8707	0.9680	0.003866	0.005755	0.006104	0.005232	0.003566	0.001618	0.9564
40	0.08940	0.2723	0.4939	0.7053	0.8697	0.9676	0.003643	0.005453	0.005817	0.004987	0.003406	0.001551	0.9564
50	0.08550	0.2642	0.4849	0.6976	0.8651	0.9661	0.002816	0.004304	0.004649	0.004041	0.002795	0.001292	0.9563
60	0.08201	0.2572	0.4763	0.6904	0.8609	0.9648	0.002273	0.003539	0.003873	0.003410	0.002385	0.001115	0.9563

IV. Estimating the Mean of a χ^2 Distribution Using Quantiles

Estimating the mean of a χ^2 distribution is equivalent to estimating the number of d.f., since $E(x) = n$. The quantile estimators of n will be constructed on an entirely different basis than was used for estimating σ^2 in the preceding section. If one looks at the entries in Table 7 of Ref. 8, which gives the values of the probability integral of the χ^2 distribution for $n = 1(1)30(2)70$, one observes that the median, the quantile of order 0.5, falls between $n - 1$ and n for all values of n listed. Thus, a simple one-quantile estimate of n can be obtained by determining the sample median, and if it falls between $l - 1$ and l, take l as $\hat{n}$. More formally, the estimator would be

$$\hat{n} = [z(0.5)] + 1 = [n^*] + 1$$

where $[x]$ denotes the greatest integer $\leqq x$.

It is useful to establish a criterion for deciding how "good" one estimator is as compared with another; we will therefore use as a criterion the probability that $\hat{n} = n$. In order to compute this probability, we first determine the mean and variance of n^*, and then we have

$$\text{prob}\,(\hat{n} = n) = \text{prob}\,(n - 1 \lesseqgtr n^* \lesseqgtr n)$$

which can easily be computed by assuming the asymptotic normality of n^*.

The moments of $\hat{n}$ are computed as follows:

$$E(\hat{n}) = \sum_{j=1}^{\infty} j\,\text{prob}\,(\hat{n} = j) = \sum_{j=1}^{\infty} j\,\text{prob}\,(j - 1 \lesseqgtr n^* < j)$$

$$E(\hat{n}^2) = \sum_{j=1}^{\infty} j^2\,\text{prob}\,(\hat{n} = j) = \sum_{j=1}^{\infty} j^2\,\text{prob}\,(j - 1 \lesseqgtr n^* \lesseqgtr j)$$

$$\text{var}\,(\hat{n}) = E(\hat{n}^2) - [E(\hat{n})]^2$$

Because of the assumption that n^* is normal, if we choose that quantile which, for a given value of n, $E(n^*) = n - 0.5$, we are likely to get a better estimator than that using the median. However, this reasoning is complicated by the fact that the optimum order of the quantile for one value of n is not necessarily optimum for another value, and, in the absence of any knowledge concerning the value of n, we must specify the order to be used whatever n happens to be. Nevertheless, it turns out that by using $p = 0.5085$, the order for which $E(n^*) = n - 0.5$ (from which it follows that $E(\hat{n}) = n$) for $n = 30$, we obtain results that are slightly better for all values of n than those obtained by using $p = 0.5$. Table 6 gives the results for $p = 0.5085$ and $n = 1(1)5(5)70$. Included in the table is the probability that $\hat{n}$ will differ from n by at most one. Column 6 gives the prob $(\hat{n} = n)$ for $p = 0.5$. It should be emphasized that in all cases a sample size of $m = 100$ was assumed and that the probabilities in the table apply only for this sample size. It can be seen from the table that for no value of n does $E(n)$ differ substantially from n.

In general, the two most common statistics used for estimating the mean of a distribution are the sample mean $\bar{x}$ and the M.L. equation. Sometimes the estimators from these two statistics coincide, but in this case they do not. In order to compare the quantile estimators with those obtained from $\bar{x}$ and the M.L. equation, the following procedures were adopted.

In the case of $\bar{x}$, we took as the estimate of n

$$\tilde{n} = \text{the nearest integer to } \bar{x}$$

Then, because of the large sample size $(m = 100)$, we assumed that $\bar{x}$ is normal, with mean n and variance $2n/m$ and computed prob $(\tilde{n} = n) = \text{prob}\,(n - 0.5 \lesseqgtr \bar{x} \lesseqgtr n + 0.5)$ and $\text{prob}\,(n - 1 \lesseqgtr \tilde{n} \lesseqgtr n + 1) = \text{prob}\,(n - 1.5 \lesseqgtr \bar{x} \lesseqgtr n + 1.5)$.

To compute the probability that the M.L. estimator of n is equal to n, one has the following:

$$L_n = \frac{1}{2^{mn/2}\,\Gamma\left(\frac{n}{2}\right)^m} \prod_{i=1}^{m} x_i^{(n/2)-1} \exp\left(-\sum_{i=1}^{m} x_{i/2}\right)$$

$$\ln L_n = -\frac{nm}{2}\ln 2 - m \ln \Gamma\left(\frac{n}{2}\right) + \left(\frac{n-2}{2}\right)\sum_{i=1}^{m} \ln x_i - \sum_{i=1}^{m} x_{i/2}$$

The M.L. estimator of n is taken to be that positive integer for which $\ln L_n$ is a maximum. Denoting the M.L. estimate of n by $\hat{\hat{n}}$, the prob $(\hat{\hat{n}} = n)$ is equivalent to the probability that $\ln L_n$ is greater than both $\ln L_{n-1}$ and $\ln L_{n+1}$. This is the same as

$$\text{prob}\left(m \ln 2 + 2m\left[\ln \Gamma\left(\frac{n}{2}\right) - \ln \Gamma\left(\frac{n-1}{2}\right)\right] < \sum_{i=1}^{m} \ln x_i < m \ln 2 + 2m\left[\ln \Gamma\left(\frac{n+1}{2}\right) - \ln \Gamma\left(\frac{n}{2}\right)\right]\right)$$

Table 6. Probabilities associated with estimating the mean (d.f.) of a χ^2 distribution using one quantile, the sample mean, and the M.L. estimator

n	$p = 0.5085$				$p = 0.5$	$\bar{x}$		M.L.
	$E(n^*)$	$E(\hat{n})$	Prob $(\hat{n} = n)$	Prob $(n-1 < \hat{n} < n+1)$	Prob $(\hat{n} = n)$	Prob $(\tilde{n} = n)$	Prob $(n-1 < \tilde{n} < n+1)$	Prob $(\hat{\hat{n}} = n)$
1	0.4733	1.0	1.0	1.0	1.0	0.9998	1.0	1.0
2	1.4207	1.9829	0.9785	1.0	0.9722	0.9876	1.0	0.9956
3	2.4117	2.9512	0.9222	1.0	0.9070	0.9588	1.0	0.9810
4	3.4115	3.9330	0.8648	1.0	0.8462	0.9229	1.0	0.9540
5	4.4141	4.9253	0.8136	1.0	0.7945	0.8862	1.0	0.9199
10	9.4341	9.9345	0.6401	0.9940	0.6264	0.7364	0.9992	0.7610
15	14.4534	14.9534	0.5420	0.9740	0.5325	0.6387	0.9938	0.6552
20	19.4705	19.9705	0.4778	0.9451	0.4710	0.5708	0.9823	0.5827
25	24.4859	24.9859	0.4318	0.9131	0.4267	0.5205	0.9661	0.5295
30	29.5	30.0	0.3968	0.8811	0.3929	0.4814	0.9472	0.4885
35	34.5131	35.0131	0.3691	0.8505	0.3661	0.4499	0.9270	0.4557
40	39.5253	40.0253	0.3464	0.8219	0.3440	0.4239	0.9065	0.4287
45	44.5368	45.0368	0.3275	0.7952	0.3256	0.4018	0.8862	0.4059
50	49.5476	50.0476	0.3113	0.7706	0.3098	0.3829	0.8664	0.3865
55	54.5580	55.0580	0.2973	0.7478	0.2961	0.3664	0.8473	0.3696
60	59.5680	60.0680	0.2850	0.7267	0.2841	0.3519	0.8291	0.3547
65	64.5775	65.0775	0.2741	0.7071	0.2733	0.3390	0.81169	0.3415
70	69.5867	70.0867	0.2644	0.6889	0.2638	0.3274	0.7951	0.3296

$\hat{n} = [z(p)] + 1 = [n^*] + 1$
$\tilde{n}$ = nearest integer to $\bar{x}$
$\hat{\hat{n}}$ = M.L. estimate
$m = 100$

Assuming that

$$\sum_{i=1}^{m} \ln x_i$$

is normal, the mean and variance of $\ln x_i$ were computed and used to find the prob $(\hat{\hat{n}} = n)$. The results for these estimators are also given in Table 6, in which a probability of one means that the actual probability was >0.9999.

The ideal estimator of n using more than one quantile for, say, $n = n_1$ would be of the form

$$\hat{n}_1 = [\sum_{i=1}^{k} \beta_i z_i] + 1 = [n_1^*] + 1, \qquad k > 1$$

where the β_i and p_i were selected so as to maximize prob $(n_1 = n_1)$. However, if no restraint were placed on the value of $E(n_1^*)$, this method of choosing the β_i and p_i would give very poor results even for values of n close to n_1. In order for an estimator to be of practical value over the entire range of values of n, it is essential that $E(n^*)$ should lie between $n - 1$ and n for all values of n, preferably as close to $n - 0.5$ as possible. This can best be accomplished by first choosing values of β_i and p_i which meet this condition and then optimizing for some intermediate value of $n = n_0$ under the constraint that $E(n_0^*) = n_0 - 0.5$. Good results were obtained with this procedure when an estimator was desired which was to be used for all possible values of n. The choice of $n_0 = 30$ was dictated by the fact that prob $(\hat{n} = n)$, even when only one quantile is used, is quite large for small values of n, but is small for relatively large values of n, so that it seems more important to try to increase this probability for large values of n even if by doing so it is degraded somewhat for small values.

In some practical situations, it may be known that $a \lessapprox n \lessapprox b$. In this case, optimization for some value of n in the interval (a, b) will give a better estimate than the one designed for all values. Therefore, estimators which have been optimized for values of n other than $n = 30$ will also be given.

A simple estimator of n using two quantiles, given by

$$\hat{n} = [0.5\,(z\,(0.25) + z\,(0.75))] + 1 = [n^*] + 1$$

meets the requirement $n - 1 < n^* < n$ for all values of n and is a better estimator than the optimum one-quantile estimator for $n > 4$. However, optimizing with respect to

p_1 and p_2 and keeping $\beta_1 = \beta_2 = 0.5$ for $n = 30$ results in the estimator

$$\hat{n}_{30} = [0.5\,(z\,(0.2643) + z\,(0.7280))] + 1 = [n^*_{30}] + 1$$

which is a better estimator than $\hat{n}$ for all values of n and better than the one-quantile estimator for $n > 2$. Table 7 lists the pertinent details associated with $\hat{n}_{30}$.

Optimization with respect to the β_i as well as the p_i for $n = 30$ results in prob $(\hat{n} = n) = 0.44290$ for $n = 30$ as compared to 0.44288 for the same probability when $\hat{n}_{30}$ is used. The improvement is so insignificant that it is not worthwhile considering.

Optimization for $n = 10$ gives the following:

$$p_1 = 0.2579 \qquad \beta_1 = 0.4827$$

$$p_2 = 0.7268 \qquad \beta_2 = 0.5096$$

$$\text{prob}\,(\hat{n}_{10} = n) = 0.7016 \text{ for } n = 10$$

Optimization for $n = 4$ also results in

$$p_1 = 0.2384 \qquad \beta_1 = 0.4754$$

$$p_2 = 0.7270 \qquad \beta_2 = 0.5087$$

$$\text{prob}\,(\hat{n}_4 = n) = 0.9177 \text{ for } n = 4$$

By comparing the above probabilities with the corresponding ones given in Table 7, it can be seen that $\hat{n}_4$ and $\hat{n}_{10}$ should be used only for values of n close to 4 and 10, respectively, when two quantiles are used to estimate n.

To estimate n using three quantiles, we considered an estimator of the form

$$\hat{n} = [\beta_1\,(z_1 + z_3) + \beta_2 z_2] + 1$$

and, to start with, put

$$\beta_1 = 0.25 \qquad p_1 = 0.25$$

$$\beta_2 = 0.5 \qquad p_2 = 0.5$$

$$p_3 = 0.75$$

Table 7. Probabilities associated with estimating the mean (d.f.) of a χ^2 distribution using two quantiles

n	$E\,(n^*)$	$E\,(\hat{n})$	Prob $(\hat{n} = n)$	Prob $(n-1 \leq \hat{n} \leq n+1)$
1	0.6602	1.0024	0.9976	1.0
2	1.6087	2.0209	0.9775	1.0
3	2.5588	3.0395	0.9427	1.0
4	3.5761	4.0493	0.9013	1.0
5	4.5678	5.0529	0.8598	1.0
10	9.5441	10.0432	0.6978	0.9980
15	14.5295	15.0294	0.5975	0.9880
20	19.5181	20.0181	0.5298	0.9697
25	24.5085	25.0085	0.4806	0.9468
30	29.50	30.00	0.4429	0.9218
35	34.4923	34.9923	0.4127	0.8966
40	39.4851	39.9851	0.3880	0.8719
45	44.4785	44.9785	0.3672	0.8482
50	49.4722	49.9722	0.3494	0.8257
55	54.4663	54.9663	0.3339	0.8045
60	59.4607	59.9601	0.3203	0.7846
65	64.4553	64.9553	0.3083	0.7658
70	69.4501	69.9501	0.2975	0.7481

$m = 100$
$\hat{n} = [0.5\,(z\,(0.2643) + z\,(0.7280))] + 1 = [n^*] + 1$

For these values, $\hat{n}$ is an excellent estimator for $n \leq 10$ and, in fact, is a better estimator than the optimum two-quantile estimator for all values of n. Keeping β_1 and β_2 fixed and optimizing with respect to the p_i for $n = 30$ gives the estimator

$$\hat{n}_{30} = [0.25\,(z\,(0.1327) + z\,(0.8575)) + 0.5\,z\,(0.4846)] + 1$$

which is better than $\hat{n}$ for $n > 10$. Optimization for $n = 4$ and $n = 10$ gives the following:

$$\hat{n}_4 = [0.25\,(z\,(0.08524) + z\,(0.8585)) + 0.5\,z\,(0.4537)] + 1$$

$$\hat{n}_{10} = [0.25\,(z\,(0.1207) + z\,(0.8577)) + 0.5\,z\,(0.4726)] + 1$$

Optimization with respect to the β_i as well as the p_i produces no significant improvement, even for the values of n for which the optimization was carried out. Hence, Table 8 gives prob $(n = n)$ and prob $(n - 1 \leq \hat{n} \leq n + 1)$ only for the above four estimators.

For the estimator of n using four quantiles, given by

$$\hat{n} = [0.25\,(z\,(0.2) + z\,(0.4) + z\,(0.6) + z\,(0.8))] + 1 = [n^*] + 1$$

Table 8. Probabilities associated with estimating the mean (d.f.) of a χ^2 distribution using three quantiles

n	Prob $(\hat{n} = n)$	Prob $(n-1 \leq \hat{n} \leq n+1)$	Prob $(\hat{n}_{30} = n)$	Prob $(n-1 \leq \hat{n}_{30} \leq n+1)$	Prob $(\hat{n}_{10} = 10)$	Prob $(n-1 \leq \hat{n}_{10} \leq n+1)$	Prob $(\hat{n}_4 = n)$	Prob $(n-1 \leq \hat{n}_4 \leq n+1)$
1	1.0	1.0	0.9752	1.0	0.9820	1.0	0.9893	1.0
2	0.9931	1.0	0.9364	1.0	0.9559	1.0	0.9792	1.0
3	0.9639	1.0	0.9028	1.0	0.9311	1.0	0.9642	1.0
4	0.9220	1.0	0.8711	1.0	0.9030	1.0	0.9333	1.0
5	0.8785	1.0	0.8401	1.0	0.8719	1.0	0.8902	1.0
10	0.7097	0.9985	0.7068	0.9984	0.7234	0.9989	0.6719	0.9975
15	0.6067	0.9896	0.6137	0.9907	0.6181	0.9913		
20	0.5377	0.9726	0.5475	0.9758	0.5446	0.9789		
25	0.4877	0.9507	0.4981	0.9560	0.4907	0.9524		
30	0.4494	0.9267	0.4595	0.9337	0.4493	0.9268		
35	0.4189	0.9021	0.4285	0.9104				
40	0.3938	0.8780	0.4028	0.8871				
45	0.3727	0.8548	0.3811	0.8644				
50	0.3547	0.8327	0.3625	0.8426				
55	0.3390	0.8118	0.3464	0.8218				
60	0.3253	0.7921	0.3322	0.8020				
65	0.3131	0.7735	0.3195	0.7833				
70	0.3022	0.7560	0.3082	0.7657				

$\hat{n} = [0.25\,(z\,(0.25) + z\,(0.75)) + 0.5\,z\,(0.5)] + 1$
$\hat{n}_{30} = [0.25\,(z\,(0.1327) + z\,(0.8575)) + 0.5\,z\,(0.4846)] + 1$
$\hat{n}_{10} = [0.25\,(z\,(0.1207) + z\,(0.8577)) + 0.5\,z\,(0.4726)] + 1$
$\hat{n}_4 = [0.25\,(z\,(0.08524) + z\,(0.8585)) + 0.5\,z\,(0.4537)] + 1$
$m = 100$

$E\,(n^*)$ is approximately $n = 0.4$ for all values of $n > 1$. Holding the β_i fixed at 0.25 and optimizing with respect to the p_i for $n = 30$, 10, and 4 gives the estimators

$$\hat{n}_{30} = [0.25\,(z\,(0.1277) + z\,(0.3618) + z\,(0.6069) + z\,(0.8590))] + 1$$

$$\hat{n}_{10} = [0.25\,(z\,(0.1139) + z\,(0.3460) + z\,(0.5976) + z\,(0.8581))] + 1$$

$$\hat{n}_4 = [0.25\,(z\,(0.07487) + z\,(0.3167) + z\,(0.5847) + z\,(0.8580))] + 1$$

Table 9 lists prob $(\hat{n} = n)$ and prob $(n - 1 \leq \hat{n} \leq n + 1)$ for the above four estimators.

For the four-quantile estimator of the form

$$\hat{n}' = [\beta_1\,(z_1 + z_4) + \beta_2\,(z_2 + z_3)] + 1$$

optimization with respect to both β_i and p_i for $n = 30$, 10, and 4 gives the following:

$$\hat{n}'_{30} = [0.1944\,(z\,(0.0962) + z\,(0.8875)) + 0.3072\,(z\,(0.3288) + z\,(0.6289))] + 1$$

$$\text{prob}\,(\hat{n}'_{30} = n) = 0.4692 \qquad \text{for } n = 30$$

$$\hat{n}'_{10} = [0.1968\,(z\,(0.0868) + z\,(0.8857)) + 0.3079\,(z\,(0.3082) + z\,(0.6166))] + 1$$

$$\text{prob}\,(\hat{n}'_{10} = n) = 0.7349 \qquad \text{for } n = 10$$

$$\hat{n}'_4 = [0.1871\,(z\,(0.0381) + z\,(0.8927)) + 0.3171\,(z\,(0.2733) + z\,(0.6130))] + 1$$

$$\text{prob}\,(\hat{n}'_4 = n) = 0.9407 \qquad \text{for } n = 4$$

To estimate n using five quantiles, we first form the estimator

$$\hat{n} = [\beta_1\,(z_1 + z_2 + z_4 + z_5) + \beta_3 z_3] + 1$$

where $p_j = j/6$, $j = 1, 2, \cdots, 5$ and $\beta_1 = 0.125$, $\beta_3 = 0.5$.

Fixing β_1 and β_3 and optimizing with respect to the p_i for $n = 30$ and $n = 10$ gives estimators which are not as good as the comparable ones using four quantiles. Optimization for $n = 4$ does give results that compare favorably with the estimator using four quantiles. Fixing the p_i and optimizing with respect to the β_i for $n = 30$ gives the following estimator, which is better than $\hat{n}_{30}$ using four quantiles for $n \geq 10$:

$$\hat{n}_{30} = [0.208\,(z_1 + z_2 + z_4 + z_5) + 0.163z\,(3)] + 1$$

Table 9. Probabilities associated with estimating the mean (d.f.) of a χ^2 distribution using four quantiles

n	Prob $(\hat{n} = n)$	Prob $(n-1 \lessgtr \hat{n} \lessgtr n+1)$	Prob $(\hat{n}_{30} = n)$	Prob $(n-1 \lessgtr \hat{n}_{30} \lessgtr n+1)$	Prob $(\hat{n}_{10} = n)$	Prob $(n-1 \lessgtr \hat{n}_{10} \lessgtr n+1)$	Prob $(\hat{n}_4 = n)$	Prob $(n-1 \lessgtr \hat{n}_4 \lessgtr n+1)$
1	0.9980	1.0	0.9542	1.0	0.9674	1.0	0.9807	1.0
2	0.9774	1.0	0.9141	1.0	0.9426	1.0	0.9749	1.0
3	0.9435	1.0	0.8837	1.0	0.9223	1.0	0.9654	1.0
4	0.9047	1.0	0.8566	1.0	0.8996	1.0	0.9388	1.0
5	0.8660	1.0	0.8302	1.0	0.8732	1.0	0.8964	1.0
10	0.7121	0.9986	0.7106	0.9986	0.7333	0.9991	0.6580	0.9973
15	0.6142	0.9908	0.6215	0.9918	0.6270	0.9925		
20	0.5471	0.9757	0.5563	0.9784	0.5514	0.9771		
25	0.4978	0.9559	0.5068	0.9602	0.4955	0.9550		
30	0.4597	0.9338	0.4680	0.9392				
35	0.4291	0.9109	0.4365	0.9169				
40	0.4039	0.8882	0.4104	0.8944				
45	0.3827	0.8661	0.3883	0.8722				
50	0.3645	0.8449	0.3693	0.8508				
55	0.3486	0.8247	0.3528	0.8302				
60	0.3347	0.8056	0.3383	0.8106				
65	0.3223	0.7875	0.3253	0.7920				
70	0.3111	0.7703	0.3137	0.7744				

$\hat{n} = [0.25\,(z\,(0.2) + z\,(0.4) + z\,(0.6) + z\,(0.8))] + 1$
$\hat{n}_{30} = [0.25\,(z\,(0.1277) + z\,(0.3618) + z\,(0.6069) + z\,(0.8590))] + 1$
$\hat{n}_{10} = [0.25\,(z\,(0.1139) + z\,(0.3460) + z\,(0.5976) + z\,(0.8581))] + 1$
$\hat{n}_4 = [0.25\,(z\,(0.07487) + z\,(0.3167) + z\,(0.5847) + z\,(0.8580))] + 1$
$m = 100$

For $\beta_1 = 0.208$, $\beta_2 = 0.163$, and optimizing with respect to the p_i for $n = 30$ gives the following:

$$\hat{n}'_{30} = [0.208\,(z\,(0.1101) + z\,(0.3075) + z\,(0.6745) + z\,(0.8844)) + 0.163 z\,(0.4894)] + 1$$

$$\text{prob}\,(\hat{n}'_{30} = n) = 0.4724 \qquad \text{for } n = 30$$

Table 10 presents prob $(\hat{n} = n)$ and prob $(n-1 \lessgtr \hat{n} \lessgtr n+1)$ for $\hat{n}$, $\hat{n}_{30}$, $\hat{n}_4$.

The six-quantile estimator of n, given by

$$n = \left[\frac{1}{6}\sum_{i=1}^{6} z_i\right] + 1$$

was first constructed, where $p_j = j/7$, $j = 1, 2, \cdots, 6$.

Then, optimizing with respect to the p_i for $n = 30, 10, 4$ gave

$$\hat{n}_{30} = \left[\frac{1}{6}\,(z\,(0.08242) + z\,(0.2366) + z\,(0.3978) + z\,(0.5632) + z\,(0.7326) + z\,(0.9052))\right] + 1$$

$$\hat{n}_{10} = \left[\frac{1}{6}\,(z\,(0.06856) + z\,(0.2169) + z\,(0.3802) + z\,(0.5507) + z\,(0.7259) + z\,(0.9044))\right] + 1$$

$$\hat{n}_4 = \left[\frac{1}{6}\,(z\,(0.02699) + z\,(0.1740) + z\,(0.3508) + z\,(0.5333) + z\,(0.7182) + z\,(0.9043))\right] + 1$$

Table 11 gives prob $(\hat{n} = n)$ and prob $(n-1 \lessgtr \hat{n} \lessgtr n+1)$ for the above four estimators.

A comparison between the probabilities associated with the quantile estimators given in Tables 6–11 and those associated with the M.L. estimator indicates that very little can be gained by using more than six quantiles. More important, perhaps, is the decision as to which estimator to use. The tables were designed to aid the user as much as possible in making this decision. Of particular significance, we believe, is the high probability, even for large values of n and a small number of quantiles, that the estimate will differ from the true value of n by no more than one. One should also bear in mind that a sample size of 100, as was assumed in the above analysis, is by no means large when quantiles are being considered in

Table 10. Probabilities associated with estimating the mean (d.f.) of a χ^2 distribution using five quantiles

n	Prob $(\hat{n} = n)$	Prob $(n - 1 \leq \hat{n} \leq n + 1)$	Prob $(\hat{n}_{30} = n)$	Prob $(n - 1 \leq \hat{n}_{30} \leq n + 1)$	Prob $(\hat{n}_4 = n)$	Prob $(n - 1 \leq \hat{n}_4 \leq n + 1)$
1	0.9999	1.0	0.9926	1.0	0.9704	1.0
2	0.9912	1.0	0.9628	1.0	0.9688	1.0
3	0.9624	1.0	0.9259	1.0	0.9646	1.0
4	0.9223	1.0	0.8905	1.0	0.9414	1.0
5	0.8804	1.0	0.8563	1.0	0.8968	1.0
10	0.7146	0.9987	0.7171	0.9988	0.6157	0.9960
15	0.6120	0.9904	0.6284	0.9921		
20	0.5429	0.9743	0.5575	0.9788		
25	0.4927	0.9533	0.5077	0.9608		
30	0.4542	0.9301	0.4691	0.9399		
35	0.4235	0.9061	0.4375	0.9177		
40	0.3982	0.8825	0.4113	0.8953		
45	0.3769	0.8597	0.3890	0.8728		
50	0.3588	0.8379	0.3699	0.8515		
55	0.3430	0.8172	0.3534	0.8310		
60	0.3291	0.7976	0.3392	0.8111		
65	0.3168	0.7792	0.3257	0.7926		
70	0.3058	0.7617	0.3141	0.7749		

$\hat{n} = [0.125\,(z\,(0.1667) + z\,(0.3333) + z\,(0.6667) + z\,(0.8333)) + 0.5\,z\,(0.5)] + 1$

$\hat{n}_{30} = [0.208\,(z\,(0.1667) + z\,(0.3333) + z\,(0.6667) + z\,(0.8333)) + 0.163\,z\,(0.5)] + 1$

$\hat{n}_4 = [0.125\,(z\,(0.01545) + z\,(0.1198) + z\,(0.7871) + z\,(0.9281)) + 0.5\,z\,(0.4428)] + 1$

$m = 100$

Table 11. Probabilities associated with estimating the mean (d.f.) of a χ^2 distribution using six quantiles

n	Prob $(\hat{n} = n)$	Prob $(n - 1 \leq \hat{n} \leq n + 1)$	Prob $(\hat{n}_{30} = n)$	Prob $(n - 1 \leq \hat{n}_{30} \leq n + 1)$	Prob $(\hat{n}_{10} = n)$	Prob $(n - 1 \leq \hat{n}_{10} \leq n + 1)$	Prob $(\hat{n}_4 = n)$	Prob $(n - 1 \leq \hat{n}_4 \leq n + 1)$
1	0.9881	1.0	0.9020	1.0	0.9314	1.0	0.9594	1.0
2	0.9481	1.0	0.8665	1.0	0.9146	1.0	0.9653	1.0
3	0.9068	1.0	0.8434	1.0	0.9029	1.0	0.9655	1.0
4	0.8686	1.0	0.8241	1.0	0.8887	1.0	0.9453	1.0
5	0.8335	1.0	0.8052	1.0	0.8697	1.0	0.9016	1.0
10	0.6971	0.9982	0.7085	0.9986	0.7433	0.9993	0.6098	0.9963
15	0.6077	0.9900	0.6269	0.9925	0.6345	0.9934		
20	0.5448	0.9752	0.5639	0.9805	0.5545	0.9782		
25	0.4978	0.9561	0.5952	0.9638	0.4950	0.9553		
30	0.4611	0.9349	0.4758	0.9440	0.4492	0.9279		
35	0.4314	0.9129	0.4439	0.9226				
40	0.4068	0.8910	0.4173	0.9007				
45	0.3895	0.8697	0.3947	0.8689				
50	0.3679	0.8492	0.3752	0.8577				
55	0.3523	0.8296	0.3582	0.8372				
60	0.3384	0.8109	0.3432	0.8175				
65	0.3261	0.7932	0.3299	0.7988				
70	0.3150	0.7765	0.3180	0.7810				

$$\hat{n} = \left[\frac{1}{6}\,(z\,(0.1429) + z\,(0.2857) + z\,(0.4286) + z\,(0.5714) + z\,(0.7143) + z\,(0.8571))\right] + 1$$

$$\hat{n}_{30} = \left[\frac{1}{6}\,(z\,(0.08242) + z\,(0.2366) + z\,(0.3978) + z\,(0.5632) + z\,(0.7326) + z\,(0.9052))\right] + 1$$

$$\hat{n}_{10} = \left[\frac{1}{6}\,(z\,(0.06856) + z\,(0.2169) + z\,(0.3802) + z\,(0.5506) + z\,(0.7259) + z\,(0.9044))\right] + 1$$

$$\hat{n}_4 = \left[\frac{1}{6}\,(z\,(0.02699) + z\,(0.1740) + z\,(0.3508) + z\,(0.5333) + z\,(0.7182) + z\,(0.9043))\right] + 1$$

$m = 100$

order to effect data compression of space telemetry, and the larger the sample size, the less becomes the probability of error.

V. Estimating n and σ^2 Using Quantiles When Neither Is Known

In Section III, we constructed quantile estimators for σ^2 when n is assumed to be known, and in Section IV, we described quantile estimators for n when, in effect, σ^2 is assumed to be equal to one. But suppose we know neither of these parameters and would like to estimate them using quantiles. To accomplish this, we will first construct k-quantile estimators of the mean $n\sigma^2$ for $k = 2(1)6$. Then, for $k = 4$ and 6, we will describe estimators of n using the same quantiles as were used to estimate $n\sigma^2$, and then take $\hat{\sigma}^2 = \widehat{n\sigma^2}/\hat{n}$ as the estimate for σ^2.

For a given value of n, say $n = n_1$, if we multiply by n_1 the coefficients of the k-quantile estimator of σ^2 for $n = n_1$ given in Section III, we then have the optimum, unbiased estimator of $n_1\sigma^2$, the mean of a χ^2 distribution with a scale factor and n_1 d.f. However, n is unknown, and since we must use the same estimator no matter what n happens to be, the question then becomes whether we can find a value of n_1 such that the estimator obtained in this way will be adequate for all or most of the values of n, in the sense that the bias will be small and the efficiency remain high for $n \neq n_1$. Estimators of $n\sigma^2$ were constructed using this method by letting $n_1 = 20$, and are given by

$$\widehat{n\sigma^2} = 0.5820z\,(0.3869) + 0.3808z\,(0.8209)$$

$$\widehat{n\sigma^2} = 0.3963z\,(0.2625) + 0.3793z\,(0.6323) + 0.1975z\,(0.9026)$$

$$\widehat{n\sigma^2} = 0.2868z\,(0.1882) + 0.3278z\,(0.4898) + 0.2479z\,(0.7685) + 0.1171z\,(0.9410)$$

$$\widehat{n\sigma^2} = 0.2172z\,(0.1414) + 0.2755z\,(0.3869) + 0.2468z\,(0.6413) + 0.1689z\,(0.8426) + 0.07534z\,(0.9617)$$

$$\widehat{n\sigma^2} = 0.1675z\,(0.1081) + 0.2291z\,(0.3072) + 0.2294z\,(0.5327) + 0.1872z\,(0.7365) + 0.1217z\,(0.8875) + 0.05246z\,(0.9731)$$

Table 12 presents the normalized expected values and efficiencies of the estimators. The efficiencies are given relative to the sample mean, which has a variance of $2n\sigma^4/m$. It can be seen that the bias of the estimators is quite small for $n > 2$, and for $k = 6$ the bias is not excessive even for $n = 1$ and 2. The high efficiencies shown for $n = 1$ and 2 are not too meaningful because they are due in part to the nature of the bias of the estimators.

Now, given that we have used, say, the above six-quantile estimator of $n\sigma^2$, we would have immediately available to us the values of six sample quantiles z_i, $i = 1, \cdots, 6$, of specified orders p_i and the calculated estimate $\widehat{n\sigma^2}$. If we define

$$z'_i = \frac{z_i}{\widehat{n\sigma^2}}, \qquad i = 1, \cdots, 6$$

each z'_i can be considered as being approximately equal to the sample quantile of order p_i of a set of sample values taken from a population with the distribution associated with the random variable $y = x/n$, where x has the χ^2 distribution with n d.f. The density function associated with this distribution is given by

$$nk_n(ny) = \frac{\left(\frac{n}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} y^{n/2-1} e^{-ny/2}$$

The problem now is to estimate n using the z'_i, the values of which are assumed to be independent of σ^2. A somewhat unusual method will now be proposed for doing this.

Defining S_n as

$$S_n = z'_4 + z'_5 + z'_6 - z'_1 - z'_2 - z'_3$$

we calculate intervals (a_n, b_n) for $n = 1(1)60$, where

$$a_n = \frac{(E(S_n) + E(S_{n+1}))}{2} \qquad \text{for } n < 60$$

$$b_n = \frac{(E(S_n) + E(S_{n-1}))}{2} \qquad \text{for } n > 1$$

$$a_{60} = 0$$

$$b_1 = \infty$$

The estimate of n can now be taken as $\hat{n} = t$ if S_n falls in the interval (a_t, b_t) for $t < 60$, and if $S_n < b_{60}$, $n \geqq 60$. In order to estimate n in this manner, a table of intervals (a_n, b_n) is required and will be given. But in order to

Table 12. The normalized expected values and efficiencies of k-quantile estimators of the mean, $n\sigma^2$, of the χ^2 distribution with a scale factor and n d.f., for $k = 2, 3, 4, 5$, and 6

n	$k = 2$		$k = 3$		$k = 4$		$k = 5$		$k = 6$	
	$E(\widehat{n\sigma^2})/n\sigma^2$	Eff $(\widehat{n\sigma^2})$	$E(\widehat{n\sigma^2})/n\sigma^2$	Eff $(\widehat{n\sigma^2})$	$E(\widehat{n\sigma^2})/n\sigma^2$	Eff $(\widehat{n\sigma^2})$	$E(\widehat{n\sigma^2})/n\sigma^2$	Eff $(\widehat{n\sigma^2})$	$E(\widehat{n\sigma^2})/n\sigma^2$	Eff $(\widehat{n\sigma^2})$
1	0.8363	0.9977	0.8949	1.0023	0.9267	1.0057	0.9462	1.0050	0.9575	1.0060
2	0.9397	0.8634	0.9601	0.9242	0.9717	0.9492	0.9791	0.9641	0.9829	0.9745
3	0.9731	0.8289	0.9821	0.8978	0.9871	0.9313	0.9905	0.9507	0.9919	0.9638
4	0.9877	0.8157	0.9918	0.8875	0.9941	0.9239	0.9957	0.9449	0.9961	0.9591
5	0.9949	0.8097	0.9967	0.8827	0.9977	0.9204	0.9984	0.9422	0.9983	0.9568
10	1.0031	0.8055	1.0022	0.8792	1.0017	0.9177	1.0014	0.9401	1.0009	0.9547
15	1.0021	0.8083	1.0014	0.8814	1.0011	0.9193	1.0009	0.9414	1.0006	0.9557
20	1.0	0.8114	1.0	0.8839	1.0	0.9212	1.0	0.9429	1.0	0.9568
25	0.9981	0.8141	0.9986	0.8861	0.9989	0.9228	0.9992	0.9442	0.9994	0.9578
30	0.9964	0.8163	0.9974	0.8879	0.9980	0.9242	0.9984	0.9454	0.9988	0.9587
35	0.9948	0.8183	0.9962	0.8895	0.9972	0.9254	0.9978	0.9464	0.9983	0.9595
40	0.9935	0.8200	0.9953	0.8909	0.9964	0.9265	0.9972	0.9472	0.9978	0.9601
45	0.9923	0.8215	0.9944	0.8921	0.9958	0.9274	0.9967	0.9480	0.9974	0.9607
50	0.9912	0.8228	0.9936	0.8931	0.9952	0.9282	0.9962	0.9486	0.9971	0.9612
60	0.9894	0.8251	0.9923	0.8950	0.9942	0.9296	0.9954	0.9497	0.9965	0.9621

eliminate the need for consulting a table, the following procedure can be adopted. Define n_1^*, n_2^*, n_3^* as

$$n_1^* = 60.8422 - 37.7767S_n + 9.5807S_n^2 - 1.1090S_n^3 + 0.04811S_n^4$$

$$n_2^* = 139.0614 - 127.7894S_n + 44.0720S_n^2 - 5.3841S_n^3$$

$$n_3^* = 325.8605 - 462.3892S_n + 244.8333S_n^2 - 45.6635S_n^3$$

If $2.6135 \leqq S_n < 6.9285$, compute n_1^*. If $1.6555 \leqq S_n < 2.6135$, compute n_2^*. If $1.0341 \leqq S_n < 1.6555$, compute n_3^*. Then $\hat{n}$ = nearest integer to n_j^* for the appropriate value of j.

If $S_n \leqq 6.9285$, $\hat{n} = 1$. n_1^* is a quartic such that $n_1^*(E(S_n)) = n$ for $n = 1, 2, 4, 7, 10$; n_2^* and n_3^* are cubics such that $n_2^*(E(S_n)) = n$, for $n = 10, 13, 17, 20$ and $n_3^*(E(S_n)) = n$, for $n = 20, 30, 40, 50$.

When the same calculations were performed for estimating n using four quantiles and defining S_n as

$$S_n = z_3' + z_4' - z_1' - z_2'$$

the following results were obtained:

$$n_1^* = 62.0627 - 66.5100S_n + 29.1473S_n^2 + 5.8457S_n^3 - 0.4407S_n^4$$

$$n_2^* = 139.7787 - 219.1982S_n + 128.9406S_n^2 - 26.8681S_n^3$$

$$n_3^* = 326.9103 - 790.1756S_n + 712.5095S_n^2 - 226.3097S_n^3$$

If $1.5309 \leqq S_n < 3.9452$, compute n_1^*. If $0.9720 \leqq S_n < 1.5309$, compute n_2^*. If $0.6097 \leqq S_n < 0.9720$, compute n_3^*. Then $\hat{n}$ is defined as in the six-quantile case except that here, if $S_n \geqq 3.9452$, $\hat{n} = 1$.

Table 13 lists the a_n and b_n for estimating n using four and six quantiles. Table 14 gives the values of n^* when the appropriate n_j^* is evaluated at $E(S_n)$.

Finally, given the estimates $\widehat{n\sigma^2}$ and $\hat{n}$ from the same set of quantiles, one can take as the estimate of σ^2

$$\hat{\sigma}^2 = \frac{\widehat{n\sigma^2}}{\hat{n}}$$

VI. Estimating the Parameters of Two Distributions Related to the χ^2, Using Quantiles

If the horizontal and vertical deviations x_1 and x_2 of a shot from the center of the target are independent and normal $(0, \sigma^2)$, the distance $x = \sqrt{x_1^2 + x_2^2}$ from the center will have the density function

$$\frac{2x}{\sigma^2} k_2\left(\frac{x^2}{\sigma^2}\right) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}$$

Thus, x is the square root of a random variable that has the χ^2 distribution with a scale factor and two d.f.

Table 13. Values of a_n and b_n to be used in estimating n, the number of d.f. of a χ^2 distribution with a scale factor, using k quantiles, for $k = 4, 6$, when n and σ^2 are unknown

n	$k = 4$		$k = 6$		n	$k = 4$		$k = 6$	
	a_n	b_n	a_n	b_n		a_n	b_n	a_n	b_n
1	3.9452	∞	6.9285	∞	31	0.8443	0.8586	1.4373	1.4617
2	3.1070	3.9452	5.3685	6.9285	32	0.8308	0.8443	1.4142	1.4373
3	2.6285	3.1070	4.5191	5.3685	33	0.8178	0.8308	1.3922	1.4142
4	2.3139	2.6285	3.9683	4.5191	34	0.8055	0.8178	1.3711	1.3922
5	2.0879	2.3139	3.5752	3.9683	35	0.7936	0.8055	1.3508	1.3711
6	1.9157	2.0879	3.2769	3.5752	36	0.7823	0.7936	1.3316	1.3508
7	1.7790	1.9157	3.0399	3.2769	37	0.7714	0.7823	1.3130	1.3316
8	1.6673	1.7790	2.8473	3.0399	38	0.7610	0.7714	1.2952	1.3130
9	1.5738	1.6673	2.6872	2.8473	39	0.7510	0.7610	1.2781	1.2952
10	1.4941	1.5738	2.5502	2.6872	40	0.7413	0.7510	1.2616	1.2781
11	1.4251	1.4941	2.4316	2.5502	41	0.7321	0.7413	1.2458	1.2616
12	1.3646	1.4251	2.3278	2.4316	42	0.7231	0.7321	1.2305	1.2458
13	1.3111	1.3646	2.2360	2.3278	43	0.7145	0.7231	1.2158	1.2305
14	1.2633	1.3111	2.1540	2.3360	44	0.7061	0.7145	1.2016	1.2158
15	1.2202	1.2633	2.0502	2.1540	45	0.6981	0.7061	1.1879	1.2016
16	1.1812	1.2202	2.0134	2.0802	46	0.6903	0.6981	1.1746	1.1879
17	1.1456	1.1812	1.9524	2.0134	47	0.6827	0.6903	1.1617	1.1746
18	1.1132	1.1456	1.8966	1.9524	48	0.6754	0.6827	1.1493	1.1617
19	1.0831	1.1132	1.8452	1.8966	49	0.6684	0.6754	1.1372	1.1493
20	1.0551	1.0831	1.7976	1.8452	50	0.6615	0.6684	1.1254	1.1372
21	1.0294	1.0551	1.7536	1.7976	51	0.6541	0.6615	1.1141	1.1254
22	1.0053	1.0294	1.7124	1.7536	52	0.6476	0.6541	1.1031	1.1141
23	0.9829	1.0053	1.6741	1.7124	53	0.6421	0.6476	1.0924	1.1031
24	0.9618	0.9829	1.6381	1.6741	54	0.6360	0.6421	1.0820	1.0924
25	0.9421	0.9618	1.6044	1.6381	55	0.6300	0.6360	1.0719	1.0820
26	0.9235	0.9421	1.5725	1.6044	56	0.6243	0.6301	1.0621	1.0719
27	0.9059	0.9235	1.5426	1.5725	57	0.6187	0.6243	1.0525	1.0621
28	0.8893	0.9059	1.5141	1.5426	58	0.6150	0.6187	1.0431	1.0525
29	0.8735	0.8893	1.4872	1.5141	59	0.6097	0.6150	1.0341	1.0431
30	0.8586	0.8735	1.4617	1.4872	≥ 60	0	0.6097	0	1.0341

Table 14. n^* evaluated at $E(S_n)$ for $n = 1, 2, \cdots, 60$ and $k = 4, 6$

n	$k = 4$	$k = 6$	n	$k = 4$	$k = 6$
	$n^*(E(S_n))$	$n^*(E(S_n))$		$n^*(E(S_n))$	$n^*(E(S_n))$
1	1.0	1.0	31	30.9801	30.9796
2	2.0	2.0	32	31.9657	31.9653
3	3.0824	3.0923	33	32.9567	32.9571
4	4.0	4.0	34	33.9536	33.9505
5	4.9603	4.9573	35	34.9553	34.9554
6	5.9700	5.9683	36	35.9595	35.9583
7	7.0	7.0	37	36.9674	36.9650
8	8.0249	8.0263	38	37.9773	37.9775
9	9.0283	9.0295	39	38.9883	38.9864
10	10.0	10.0	40	40.0	40.0
11	11.0224	11.0226	41	41.0121	41.0088
12	12.0135	12.0138	42	42.0218	42.0224
13	13.0	13.0	43	43.0304	43.0316
14	13.9914	13.9905	44	44.0376	44.0331
15	14.9899	14.9897	45	45.0406	45.0383
16	15.9937	15.9931	46	46.0406	46.0378
17	17.0	17.0	47	47.0379	47.0365
18	18.0051	18.0051	48	48.0302	48.0246
19	19.0059	19.0067	49	49.0169	49.0157
20	20.0	20.0	50	50.0	50.0
21	20.9843	20.9841	51	50.9776	50.9753
22	21.9574	21.9571	52	52.1751	51.9484
23	22.9177	22.9185	53	52.9156	52.9089
24	23.8639	23.8645	54	53.8741	53.8730
25	25.1696	25.1707	55	54.8283	54.8211
26	26.1328	26.1324	56	55.7752	55.7703
27	27.0945	27.0947	57	56.7166	56.7098
28	28.0585	28.0558	58	57.6498	57.6482
29	29.0264	29.0264	59	58.5767	58.5744
30	30.0	30.0	60	59.4963	59.4869

$\hat{n}$ = nearest integer to n^*

The mean and variance of this distribution are

$$E(x) = \sigma\sqrt{\frac{\pi}{2}} = 1.2533\sigma$$

$$\text{var}(x) = \left(2 - \frac{\pi}{2}\right)\sigma^2 = 0.4292\sigma^2$$

and the mode is at $x = \sigma$.

The M.L. estimator $\tilde{\sigma}$ of σ and its asymptotic variance are given by

$$\tilde{\sigma} = \sqrt{\frac{1}{2m}\sum_{i=1}^{m} x_i^2}$$

$$\text{var}(\tilde{\sigma}) = \frac{\sigma^2}{4m}$$

The unbiased estimator σ^*, obtained by using the sample mean, and its variance are given by

$$\sigma^* = \bar{x}\sqrt{\frac{2}{\pi}}$$

$$\text{var}(\sigma^*) = 0.2732\frac{\sigma^2}{m}$$

so that the efficiency of σ^* relative to $\tilde{\sigma}$ is 0.9149.

Optimum asymptotically unbiased k-quantile estimators of σ of the form

$$\hat{\sigma} = \sum_{i=1}^{k} \beta_i z_i$$

were constructed for $k = 1, \cdots, 6$ by the same method used in Section III to obtain estimators for σ^2. The details will therefore be omitted. Table 15 lists the orders of the quantiles, the coefficients, and the spacings ζ_i, which represent $E(z_n)/\sigma$. It can be seen from the table that for $k > 3$, the efficiencies of the quantile estimators are higher than that of σ^*.

Table 15. Optimum orders p_i, the corresponding coefficients β_i and spacings ζ_i, and efficiencies of quantile estimators of the parameter σ of the square root of a χ^2 distribution with a scale factor and two d.f.

k	1	2	3	4	5	6
p_1	0.7968	0.6378	0.5291	0.4512	0.3942	0.3476
β_1	0.5602	0.3696	0.2746	0.2140	0.1736	0.1436
ζ_1	1.7852	1.4252	1.2272	1.0955	1.0012	0.9243
p_2		0.9263	0.8295	0.7418	0.6682	0.6039
β_2		0.2072	0.2134	0.1944	0.1720	0.1516
ζ_2		2.2840	1.8810	1.6455	1.4854	1.361
p_3			0.9653	0.9067	0.8441	0.7826
β_3			0.1009	0.1302	0.1347	0.1305
ζ_3			2.5930	2.1780	1.9279	1.747
p_4				0.9810	0.9437	0.8978
β_4				0.05762	0.08468	0.09643
ζ_4				2.8160	2.3987	2.136
p_5					0.9886	0.9631
β_5					0.03609	0.05858
ζ_5					2.9905	2.569
p_6						0.9925
β_6						0.02443
ζ_6						3.128
Eff	0.6476	0.8202	0.8910	0.9269	0.9476	0.9606

$\hat{\sigma} = \sum_{i=1}^{k} \beta_i z_i, \quad k = 1, \cdots, 6$

If the components x_1, x_2, x_3 of the velocity of a molecule with respect to a system of rectangular coordinates are independent and normal $(0, \sigma^2)$, the velocity

$$x = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

will have the density function

$$\frac{2x}{\sigma^2} k_3 \left(\frac{x^2}{\sigma^2}\right) = \sqrt{\frac{2}{\pi}} \frac{x^2}{\sigma^3} e^{-x^2/2\sigma^2}$$

with a mean and variance given by

$$E(x) = 2\sigma \sqrt{\frac{2}{\pi}} = 1.5958\sigma$$

$$\text{var}(x) = \left(3 - \frac{8}{\pi}\right) \sigma^2 = 0.4535\sigma^2$$

with the mode at $x = \sigma\sqrt{2}$.

The M.L. estimator $\tilde{\sigma}$ and the estimator σ^* obtained from the sample mean, and their variances, are given by

$$\tilde{\sigma} = \sqrt{\frac{1}{3m} \sum_{i=1}^{m} x_i^2}$$

$$\text{var}(\tilde{\sigma}) = \frac{\sigma^2}{6m}$$

$$\sigma^* = \bar{x} \sqrt{\frac{\pi}{8}}$$

$$\text{var}(\sigma^*) = 0.1781 \frac{\sigma^2}{m}$$

so that eff $(\sigma^*) = 0.9358$.

Optimum k-quantile estimators of σ were constructed of the form

$$\hat{\sigma} = \sum_{i=1}^{k} \beta_i z_i$$

for $k = 1, \cdots, 6$. Like Table 15, Table 16 lists the orders of the quantiles p_i, the coefficients β_i of the estimators, and the optimum spacings. For $k > 4$, the efficiency of the quantile estimators exceeds that of σ^*.

Table 16. Optimum orders p_i, the corresponding coefficients β_i and spacings ζ_i, and efficiencies of quantile estimators of the parameter σ of the square root of a χ^2 distribution with a scale factor and three d.f.

k	1	2	3	4	5	6
p_1	0.7576	0.5778	0.4589	0.3775	0.3177	0.2730
β_1	0.4890	0.3131	0.2252	0.1713	0.1352	0.1101
ζ_1	2.0451	1.6756	1.4676	1.3287	1.2247	1.1442
p_2		0.9076	0.7899	0.6870	0.6008	0.5309
β_2		0.1874	0.1897	0.1690	0.1467	0.1269
ζ_2		2.5360	2.127	1.887	1.718	1.592
p_3			0.9552	0.8814	0.8035	0.7307
β_3			0.09369	0.1187	0.1212	0.1150
ζ_3			2.839	2.421	2.169	1.982
p_4				0.9750	0.9264	0.8685
β_4				0.05434	0.07904	0.08827
ζ_4				3.057	2.636	2.371
p_5					0.9846	0.9509
β_5					0.03460	0.05544
ζ_5					3.226	2.803
p_6						0.9898
β_6						0.02364
ζ_6						3.361
Eff	0.6451	0.8179	0.8892	0.9254	0.9464	0.9596

$\hat{\sigma} = \sum_{i=1}^{k} \beta_i z_i, \quad k = 1, \cdots, 6$

VII. Estimating the Parameters Using Real Data

In order to generate m independent sample values that can be assumed to be taken from a population which has a χ^2 distribution with a scale factor and n d.f., the following procedure was adopted. First, nm independent random numbers $p_1, p_2, \cdots, p_{nm}$, each uniformly distributed over $(0, 1)$, were generated. Then the pairwise transformations

$$y_j = \sigma \cos(2\pi p_j) \sqrt{-2 \ln p_{j+1}}$$

$$j = 1, 3, 5, \cdots, nm - 1$$

$$y_{j+1} = \sigma \sin(2\pi p_j) \sqrt{-2 \ln p_{N=1}}$$

were performed, and it can be shown that the y_i are independent and $N(0, \sigma^2)$. Finally, the transformation

$$x_j = \sum_{i=n(j-1)+1}^{nj} y_i^2, \qquad j = 1, 2, \cdots, m$$

provides the m sample values with the required distribution.

For $n = 10$, $\sigma^2 = 1$, and $m = 100$, 25 sets of sample values were generated. For each set, the optimum k quantiles were determined for $k = 1, 2, \cdots, 6$ and used to obtain estimates of σ^2. The M.L. estimates were also computed. Denoting by $\bar{\hat{\sigma}}_k^2$ the average of the 25 estimates of σ^2 using k quantiles, and by $\bar{\tilde{\sigma}}^2$ the average of the 25 M.L. estimates, the following results were obtained:

$$\bar{\tilde{\sigma}} = 1.0143\,(0.0389) \qquad \bar{\hat{\sigma}}_4^2 = 1.0153\,(0.0409)$$

$$\bar{\hat{\sigma}}_1^2 = 1.0159\,(0.0463) \qquad \bar{\hat{\sigma}}_5^2 = 1.0206\,(0.0411)$$

$$\bar{\hat{\sigma}}_2^2 = 1.0063\,(0.0476) \qquad \bar{\hat{\sigma}}_6^2 = 1.0236\,(0.0398)$$

$$\bar{\hat{\sigma}}_3^2 = 1.0236\,(0.0459)$$

where the numbers in parentheses denote the sample standard deviation of the estimates, given by

$$\sqrt{\sum_{i=1}^{25} \frac{(x_i - \bar{x})^2}{24}}$$

For $n = 10$ and $m = 100$, 25 sets of sample values were generated for a χ^2 distribution. For each set, the M.L. estimate $\tilde{n}$, $n^* = \bar{x}$ and $\hat{n}_k; k = 1, \cdots, 6$, the estimates using k quantiles were determined. The results in terms of the number of times a particular estimate was obtained are as follows:

Estimate	$\tilde{n}$	n^*	$\hat{n}_1$	$\hat{n}_2$	$\hat{n}_3$	$\hat{n}_4$	$\hat{n}_5$	$\hat{n}_6$
9		3	5	2		2	3	1
10	22	19	17	18	21	19	18	20
11	3	3	3	5	4	4	4	4

For $n = 20$, the following results were obtained:

Estimate	$\tilde{n}$	n^*	$\hat{n}_1$	$\hat{n}_2$	$\hat{n}_3$	$\hat{n}_4$	$\hat{n}_5$	$\hat{n}_6$
18			2	1	1			
19	5	5	5	5	6	10	6	6
20	15	14	13	16	11	10	16	15
21	5	6	5	3	7	5	3	4

When both σ^2 and n are unknown, estimates can be obtained using all the sample values by the method of moments. One equates the first two sample moments, $\bar{x}$ and s^2, with the corresponding population moments and solves the two equations for n and σ^2. Thus, solving $\bar{x} = n\sigma^2$ and $s^2 = 2n\sigma^4$ gives $\tilde{\sigma}^2 = s^2/2\bar{x}$, $\tilde{n} = 2\bar{x}^2/s^2$.

For $\sigma^2 = 1$, $m = 100$, and $n = 10, 20, 30$, fifty sets of sample values were generated for each value of n. The estimators given in Section V using four and six quantiles were used to determine $\hat{n}_4$, $\hat{\sigma}_4^2$, $\hat{n}_6$, and $\hat{\sigma}_6^2$ for each set of sample values. The estimates $\tilde{n}$ and $\tilde{\sigma}^2$ were also computed. The averages of the estimates as well as their sample standard deviations are:

For $n = 10$,

$$\bar{\tilde{n}} = 10.46\,(1.515) \qquad \bar{\tilde{\sigma}}^2 = 0.9814\,(0.1432)$$

$$\bar{\hat{n}}_4 = 10.57\,(1.893) \qquad \bar{\hat{\sigma}}_4^2 = 0.9728\,(0.1746)$$

$$\bar{\hat{n}}_6 = 10.30\,(1.787) \qquad \bar{\hat{\sigma}}_6^2 = 1.0025\,(0.1715)$$

For $n = 20$,

$$\bar{\tilde{n}} = 20.20\,(2.703) \qquad \bar{\tilde{\sigma}}^2 = 1.0144\,(0.1440)$$

$$\bar{\hat{n}}_4 = 19.44\,(3.176) \qquad \bar{\hat{\sigma}}_4^2 = 1.0478\,(0.1636)$$

$$\bar{\hat{n}}_6 = 19.62\,(3.036) \qquad \bar{\hat{\sigma}}_6^2 = 1.0499\,(0.1813)$$

For $n = 30$,

$$\bar{\tilde{n}} = 29.78\,(4.358) \qquad \bar{\tilde{\sigma}}^2 = 1.0258\,(0.1501)$$

$$\bar{\hat{n}}_4 = 30.34\,(4.737) \qquad \bar{\hat{\sigma}}_4^2 = 1.0361\,(0.1630)$$

$$\bar{\hat{n}}_6 = 30.16\,(5.270) \qquad \bar{\hat{\sigma}}_6^2 = 1.0189\,(0.1908)$$

Sets of sample values were generated in order to estimate the mean $n\sigma^2$ of a χ^2 distribution with a scale factor when n and σ^2 are unknown, using the sample mean and the k-quantile estimators given in Section V, for $k = 2, \cdots, 6$; $m = 100$ and $\sigma^2 = 1$ in all cases.

Let m_1 denote the number of sets of m sample values generated for a given value of n. Let x denote the average of m_1 estimates of $n\sigma^2$ using all the sample values, and $\overline{\widehat{n\sigma_k^2}}$ the average of m_1 estimates using k quantiles. Then, for $n = 6$ and $m_1 = 25$,

$$\bar{\bar{x}} = 6.046\,(0.3300) \qquad \overline{\widehat{n\sigma_4^2}} = 6.051\,(0.3469)$$

$$\overline{\widehat{n\sigma_2^2}} = 5.984\,(0.4181) \qquad \overline{\widehat{n\sigma_5^2}} = 6.061\,(0.3415)$$

$$\overline{\widehat{n\sigma_3^2}} = 6.11\,(0.3217) \qquad \overline{\widehat{n\sigma_6^2}} = 6.052\,(0.3544)$$

where the numbers in parentheses again denote the sample standard deviation of the estimates.

For $n = 10$ and $m_1 = 85$,

$$\bar{\bar{x}} = 10.071\,(0.4705) \qquad \overline{\widehat{n\sigma_4^2}} = 10.108\,(0.5011)$$

$$\overline{\widehat{n\sigma_2^2}} = 10.162\,(0.5710) \qquad \overline{\widehat{n\sigma_5^2}} = 10.136\,(0.5113)$$

$$\overline{\widehat{n\sigma_3^2}} = 10.155\,(0.5174) \qquad \overline{\widehat{n\sigma_6^2}} = 10.087\,(0.4850)$$

For $n = 20$, and $m_1 = 110$,

$$\bar{\bar{x}} = 20.109\,(0.5861) \qquad \overline{\widehat{n\sigma_4^2}} = 20.089\,(0.5912)$$

$$\overline{\widehat{n\sigma_2^2}} = 20.114\,(0.6577) \qquad \overline{\widehat{n\sigma_5^2}} = 20.158\,(0.6296)$$

$$\overline{\widehat{n\sigma_3^2}} = 20.168\,(0.6157) \qquad \overline{\widehat{n\sigma_6^2}} = 20.108\,(0.5955)$$

For $n = 30$ and $m_1 = 80$,

$$\bar{\bar{x}} = 29.978\,(0.6984) \qquad \overline{\widehat{n\sigma_4^2}} = 29.865\,(0.7436)$$

$$\overline{\widehat{n\sigma_2^2}} = 29.868\,(0.8022) \qquad \overline{\widehat{n\sigma_5^2}} = 29.954\,(0.6910)$$

$$\overline{\widehat{n\sigma_3^2}} = 29.952\,(0.7415) \qquad \overline{\widehat{n\sigma_6^2}} = 29.904\,(0.7296)$$

For $\sigma^2 = 1$ and $m = 100$, 25 sets of sample values were generated from the square root of a χ^2 distribution with a scale factor and two d.f. The M.L. estimate $\tilde{\sigma}$, the estimate σ^* from the sample mean, and the k-quantile estimates σ_k, $k = 1, \cdots, 6$ were determined for each set. The averages and the sample deviations of the estimates are as follows:

$$\bar{\tilde{\sigma}} = 0.9937\,(0.0494) \qquad \bar{\hat{\sigma}}_3 = 0.9905\,(0.0523)$$

$$\bar{\sigma}^* = 0.9894\,(0.0520) \qquad \bar{\hat{\sigma}}_4 = 1.0014\,(0.0533)$$

$$\bar{\hat{\sigma}}_1 = 1.0075\,(0.0663) \qquad \bar{\hat{\sigma}}_5 = 0.9912\,(0.0503)$$

$$\bar{\hat{\sigma}}_2 = 0.9930\,(0.0501) \qquad \bar{\hat{\sigma}}_6 = 0.9982\,(0.0537)$$

The same statistics were determined for the square root of a χ^2 distribution with a scale factor and three d.f.:

$$\bar{\tilde{\sigma}} = 1.0028\,(0.0501) \qquad \bar{\hat{\sigma}}_3 = 0.9985\,(0.0548)$$

$$\bar{\sigma}^* = 1.005\,(0.0494) \qquad \bar{\hat{\sigma}}_4 = 1.0059\,(0.0534)$$

$$\bar{\hat{\sigma}}_1 = 0.9985\,(0.0704) \qquad \bar{\hat{\sigma}}_5 = 1.0017\,(0.0499)$$

$$\bar{\hat{\sigma}}_2 = 1.0010\,(0.0563) \qquad \bar{\hat{\sigma}}_6 = 1.0033\,(0.0524)$$

VIII. Conclusion

It can be seen that the efficiencies of the k-quantile estimators of σ^2 given in Tables 1–5 are decreasing functions of n. Since the χ^2 distribution approaches the normal as $n \to \infty$, the limiting value of these efficiencies is the efficiency of the corresponding optimum k-quantile estimator of the mean of a normal distribution, as given in Refs. 1 and 7.

Although prob $(\hat{n} = n)$ and prob $(n - 1 \lesssim \hat{n} \lesssim n + 1)$ for the estimators $\hat{n}$ as defined in Tables 6–11 decrease as n increases, the rate of decrease for large n also decreases, so that even for $n > 70$, $\hat{n}$ should give excellent results when compared to the M.L. estimator and the estimate obtained from the sample mean.

The results of the Monte Carlo study with respect to the estimators of n and σ^2 when neither are known show that, by comparison with the estimates obtained from all the sample values, the quantile estimators can be considered good estimators. However, the large sample standard deviation of the estimates again points up the importance of a large sample size in many areas of parameter estimation.

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