

Local Behavior of Autonomous Neutral Functional Differential Equations*

by

Jack K. Hale

**CASE FILE
COPY**

A neutral functional differential equation is a model for an hereditary system which depends upon the present and past values of the state and the derivative of the state of the system as a function of time. In this generality, it does not seem possible to develop a general qualitative theory. In the past few years, my colleagues and I have introduced a special class of these systems which is simple enough to have many interesting mathematical properties and yet sufficiently general to include retarded functional differential equations, difference equations and several important applications. Considered separately, many of the results which have been obtained appear at first glance to be special. In this paper, we formulate some basic problems for a more restricted class of neutral functional differential equations in an abstract manner and then indicate the manner in which the alluded to special results contribute to a general qualitative theory in the neighborhood of an equilibrium point.

*

This research was supported in part by the National Aeronautics and Space Administration under Grant No. NGL 40-002-015, in part by the Air Force Office of Scientific Research under Grant No. AF-AFOSR 67-0693D, and in part by the United States Army - Durham under Grant No. DA-31-124-ARO-D-270.

Suppose $r \geq 0$ is a given real number, E^n is an n -dimension linear vector space with norm $|\cdot|$ and $C = C([-r, 0], E^n)$ is the space of continuous functions mapping the interval $[-r, 0]$ into E^n with $|\varphi| = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|$ for $\varphi \in C$. Let $\mathcal{B}(C, E^n)$ be the space of bounded linear operators with $\|B\| = \inf\{k: |B\varphi| \leq k|\varphi|\}$ for any $B \in \mathcal{B}(C, E^n)$. Let $\mathcal{G} \subset \mathcal{B}(C, E^n)$ be defined by

$$\mathcal{G} = \{G \in \mathcal{B}(C, E^n): G(\varphi) = \varphi(0) - g(\varphi), g(\varphi) = \int_{-r}^0 [d\mu(\theta)]\varphi(\theta),$$

μ an $n \times n$ matrix function of bounded variation and nonatomic at zero}

where we say an $n \times n$ matrix function μ of bounded variation on $[-r, 0]$ is nonatomic at zero if there is a continuous nonnegative scalar function

$\gamma(s)$, $s \geq 0$, $\gamma(0) = 0$ such that $|\int_{-s}^0 [d\mu(\theta)]\varphi(\theta)| \leq \gamma(s)|\varphi|$ for all $s \geq 0$, $\varphi \in C$. Let $\bar{\Omega}$ be the closure of an open bounded subset Ω of C ,

$$\mathcal{F} = \{f: \Omega \rightarrow E^n, f, f' \text{ continuous and bounded in } \bar{\Omega}\}$$

and let $\|f\| = \sup_{\varphi \in \bar{\Omega}} \{|\varphi(0)| + \|f'(\varphi)\|\}$ for $f \in \mathcal{F}$. If $A \geq 0$ and

$x: [\sigma-r, \sigma+A] \rightarrow E^n$, then for any $t \in [\sigma, \sigma+A]$ we let $x_t: [-r, 0] \rightarrow E^n$ be defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$.

For any pair $(G, f) \in \mathcal{G} \times \mathcal{F}$, we define a neutral functional differential equation and designate it as NFDE(G, f) by a relation

$$(1) \quad \frac{d}{dt}G(x_t) = f(x_t).$$

For any φ in $\bar{\Omega}$, a solution $x(\varphi)$ of (1) through φ is a continuous function $x: [-r, A) \rightarrow E^n$ for some $A > 0$ such that $x_t(\varphi) \in \Omega$, $t \in [0, A)$, $G(x_t)$ is continuously differentiable on $(0, A)$ and satisfies (1) on $(0, A)$. For the purpose of this paper, we assume that all solutions remain in $\bar{\Omega}$ for all $t \geq 0$ for each NFDE(G, f) in $\mathcal{G} \times \mathcal{F}$. Of course, this implies that in our definition of $\mathcal{G}$ and $\mathcal{F}$ we have implicitly put limitations which will ensure this latter property.

For any given NFDE(G, f) and a solution $x(\varphi, G, f)$ through φ , we can define a mapping

$$(2) \quad T_{G, f}(t): \bar{\Omega} \rightarrow \bar{\Omega}, \quad T_{G, f}(t)\varphi = x_t(\varphi, G, f)$$

for $t \geq 0$. The following properties are consequences of the same type of arguments used in [1, 2]:

$$(3) \quad T_{G, f}(t)\varphi \text{ is continuously differentiable in } (G, f, t, \varphi) \in \mathcal{G} \times \mathcal{F} \times [0, \infty) \times \Omega.$$

$$(4) \quad T_{G, f}(0) = I, \text{ the identity}$$

$$(5) \quad T_{G, f}(t+s)\varphi = T_{G, f}(t)T_{G, f}(s)\varphi, \quad t \geq 0, \quad s \geq 0.$$

In particular, $T_{G,f}$ is a dynamical system.

By a linear neutral functional differential equation, we mean a NFDE(G, L) with $(G, L) \in \mathcal{G} \times \mathcal{B}(C, E^n)$. In this case, the family of operators $\{T_{G,L}(t)\}$ form a strongly continuous semigroup of linear operators on C .

Our fundamental goal is to obtain as much information as possible about the properties of solutions of a NFDE(G, f) and to discover in what sense these properties are insensitive to small changes in (G, f) in the topology of $\mathcal{G} \times \mathcal{F}$. One can hope to eventually develop a qualitative theory in the spirit of the recent generic theory for ordinary differential equations. Unfortunately, at the present time we must limit our considerations to the neighborhood of an equilibrium point. Even for this theory, more restrictions are imposed on the equation. We indicate below the nature of the theory and point out occasionally the implications of the hypotheses for some nonlocal problems in NFDE's.

Before embarking on details, it is worthwhile to mention that there is nothing magic about the space C as the underlying space for the definition of a NFDE. Any of the Sobolev spaces $W_p^{(k)}$ are perfectly legitimate and, in fact, as the global theory evolves, it certainly will be advantageous to have a norm which is differentiable.

For any $G \in \mathcal{G}$, let

$$(6) \quad a_G = \inf\{a: \text{there is a constant } K = K(a) \text{ with the solution operator } T_G^0 \text{ of } G(x_t) = 0 \text{ satisfying } \|T_G^0(t)\| \leq Ke^{at}, t \geq 0\}$$

that is, a_G is the type of the semigroup of linear operators $T_G^0(t)$ acting on the Banach space $\{\varphi \in C: G(\varphi) = 0\}$. Following [3], the operator G is said to be stable if $a_G < 0$.

In terms of a_G , a fundamental property which is a consequence of the proof in [12] is that, for any $(G, f) \in \mathcal{G} \times \mathcal{F}$, there is a continuous linear operator $\psi: C \rightarrow C$ such that

$$(7) \quad T_{G,f}(t) = T_G^0(t)\psi + T_{G,f}^1(t)\psi$$

where $T_G^0(t)$ is the semigroup of operators given in the definition (6) and $T_{G,f}^1(t)$ is completely continuous for $t \geq r$. In particular, if G is stable, then for any α , $a_D < -\alpha < 0$, there is a constant $K = K(\alpha)$ such that

$$(8) \quad \|T_G^0(t)\| \leq Ke^{-\alpha t}, \quad t \geq 0.$$

Consequently, $T_{G,f}(t)$ is a contraction operator plus a completely continuous operator for t sufficiently large and, in particular, has the fixed point property. Also, the representation (7) and relation (8) show that $T_{G,f}(t)$ has a smoothing property in the sense that the image of a bounded set under $T_{G,f}(t)$ gets closer and closer to a compact set as t gets large. This property has recently been exploited in a global theory of dissipative processes (see [4]).

Another basic implication of the hypothesis that G is stable is that the union of orbits with initial values in a compact set being

bounded implies this union is precompact. See the above mentioned paper [4] for an application to dissipative processes and see [3] for a rather complete theory of stability using Liapunov functionals.

If $(G, L) \in \mathcal{G} \times \mathcal{B}(C, E^n)$, then the characteristic values of NFDE (G, L) are defined to be the roots λ of the characteristic equation

$$(9) \quad \det \Delta(\lambda, G, L) \stackrel{\text{def}}{=} \det[\lambda G(e^{\lambda \cdot} I) - L(e^{\lambda \cdot} I)] = 0.$$

It is shown in [5] that there are only a finite number of roots of (9) with real parts $\geq a$ if $a > a_G$. If $a > a_G$ and $\Lambda(a, G, L) = \{\lambda: \det \Delta(\lambda, G, L) = 0, \operatorname{Re} \lambda \geq a\}$, then it is a consequence of [6] that there are only a finite number of solutions of NFDE (G, L) of the form $p(t)e^{\lambda t}$, $\lambda \in \Lambda(a, G, L)$, $p(t)$ a polynomial. Furthermore, the initial values of these solutions form a finite dimensional subspace of C which is invariant under $T_{G, L}(t)$ and there is a complementary subspace of C also invariant under $T_{G, L}(t)$. If we designate the corresponding projection operators by $\pi(G, L, a)$ and $\pi^\perp(G, L, a) \stackrel{\text{def}}{=} I - \pi(G, L, a)$, then it is shown in [5] that there is a constant $K = K(a, G, L)$ such that

$$(10) \quad \|T_{G, L}(t)\pi^\perp(G, L, a)\| \leq Ke^{at}, \quad t \geq 0.$$

Furthermore, $T_{G, L}(t)\pi(G, L, a)$ is equivalent to a linear ordinary differential equation with constant coefficients with the spectrum of the coefficient matrix being $\Lambda(a, G, L)$. We shall refer to $\pi(G, L, a)$, $\pi^\perp(G, L, a)$ as the projections associated with the decomposition of C

by $\Lambda(a, G, L)$.

The estimate (10) was obtained using rather complicated arguments in Laplace transforms. Using the representation (7) for $T_{G,L}(t)$, one can also obtain this estimate from some general results in functional analysis dealing with completely continuous perturbations of bounded operators. It is instructive to investigate this approach for two reasons:

- 1) The ideas lead to an extension of Floquet theory for periodic systems which will not be exploited here;
- 2) Later remarks will indicate the need for a more detailed functional analytic investigation of completely continuous perturbations of bounded operators.

Suppose A is a bounded linear operator and B is a completely continuous linear operator mapping some Banach space X into itself. A complex number λ is called a normal eigenvalue of A if its generalized eigenspace X_λ is finite dimensional and $X = X_\lambda \oplus Y_\lambda$ where Y_λ is also invariant under A . A point λ is called a normal point of A if λ is either in the resolvent set of A or λ is a normal eigenvalue. Let $\tilde{\rho}(A)$ be the set of normal points of A . If C is any connected component of $\tilde{\rho}(A)$ and C contains at least one point of the resolvent set of $A+B$, then C is a connected component of $\tilde{\rho}(A+B)$ (see [7, p. 22]). To apply this to our problem, note that the spectral radius of $T_G^0(r)$ is $\exp(ra_G)$ and, therefore, $\exp \lambda r$ is in $\tilde{\rho}(T_G^0(r))$ for any λ with $\operatorname{Re} \lambda \geq a_1 > a_G$. Since $T_{G,L}(t)$ is a strongly continuous semigroup, there is certainly a λ with $\operatorname{Re} \lambda \geq a_1$ such that $\exp \lambda r$ is in the resolvent set of $T_{G,L}(r)$.

Since $T_{G,L}^1(r)$ is completely continuous, any λ with $\operatorname{Re} \lambda \geq a_1$ is in $\tilde{\rho}(T_{G,L}(r))$. Consequently, the spectral radius of $T_{G,L}(r)\pi^1(Q,L,a_1)$ is $\leq \exp a_1 r$ and estimate (10) follows by a standard procedure.

From the preceding remarks, it looks as if we should give up Laplace transforms and use only functional analysis. However, recall that we wish to study properties of the equations which are insensitive to small changes in G,L . In particular, we need the following result: For any fixed $a > a_D$, there is a neighborhood $U_{G,L}$ of (G,L) in $\mathcal{G} \times \mathcal{B}(C, E^n)$ and a positive number $K = K(a, U_{G,L})$ such that

$$(11) \quad \|T_{\bar{G}, \bar{L}}(t)\pi^1(\bar{G}, \bar{L}, a)\| \leq Ke^{at}, \quad t \geq 0, \quad \text{for all } (\bar{G}, \bar{L}) \in U_{G,L}.$$

The functional analytic approach does not seem to suffice for such general estimates and one must get more involved with the details of the spectrum of each operator. Estimate (11) could be obtained as a consequence of the method of integral manifolds to be mentioned later, but we prefer to proceed in a more detailed manner to obtain some results of independent interest. These propositions are stated as lemmas with only brief indications of the proofs.

Lemma 1. For any $\varepsilon > 0$, there exists an open neighborhood $U_G(\varepsilon)$ in $\mathcal{G}$ such that $a_{\bar{G}} < a_G + \varepsilon$ for every $\bar{G} \in U_G(\varepsilon)$. Furthermore, for any $a > a_G + \varepsilon$, there is a constant $K = K(a, U_G(\varepsilon), \varepsilon)$ such that

$$\|T_{\bar{G}}^0(t)\| \leq Ke^{at}, \quad t \geq 0,$$

for every $\bar{G} \in U_G(\mathcal{E})$.

Proof: To find $a_{\bar{G}}$, we must estimate the solutions of $\bar{G}(x_t) = 0$ for $t \geq 0$. Writing this as $G(x_t) = (G - \bar{G})(x_t)$ and using the same type of argument as used in the proof of Lemma 3.4 of [3], one obtains the result.

Lemma 2. For any $a > a_G$, there is an open neighborhood U_G of G in $\mathcal{G}$ and a $\delta = \delta(a, U_G) > 0$ such that $|\det \bar{G}(e^{\lambda \cdot} I)| \geq \delta$ on $\text{Re } \lambda = a$ for all $\bar{G} \in U_G$.

Proof: Choose $\mathcal{E} > 0$ such that $a_G + \mathcal{E} < a$ and a neighborhood $U_G(\mathcal{E})$ as in Lemma 1. Then Lemma 3.4 of [5] implies that all roots of $\det \bar{G}(e^{\lambda \cdot} I) = 0$ are to the left of the line $\text{Re } \lambda = a_G < a + \mathcal{E}$ and there is a $\delta(a, \bar{G}) > 0$ such that $|\det \bar{G}(e^{\lambda \cdot} I)| \geq \delta(a, \bar{G})$ on $\text{Re } \lambda = a$ for each $\bar{G} \in U_G(\mathcal{E})$.

If no neighborhood and δ exists as stated in the lemma, then there are sequences $G_k \in U_G(\mathcal{E})$, $G_k \rightarrow G$ as $k \rightarrow \infty$ and $\lambda_k = \lambda_k(G_k)$, $\text{Re } \lambda_k = a$, such that $|\det G_k(e^{\lambda_k \cdot} I)| \leq 1/k$, $k = 1, 2, \dots$. The proof now proceeds as in the proof of Lemma 3.4 of [5] using the appropriate generalization of Lemma 3.2 of [5].

Lemma 3. For any $a > a_D$, there is an open neighborhood $U_{G,L}$ of (G, L) in $\mathcal{G} \times \mathcal{B}(C, E^n)$ such that for each $(\bar{G}, \bar{L}) \in U_{G,L}$ the characteristic equation

$$\det \Delta(\lambda, \bar{G}, \bar{L}) = 0$$

has the same finite number of characteristic roots with $\operatorname{Re} \lambda \geq a$. Furthermore, if $\pi(\bar{G}, \bar{L}, a)$, $\pi^\perp(\bar{G}, \bar{L}, a)$ are the projection operators associated with the decomposition of C by $\Lambda(a, \bar{G}, \bar{L})$, then there is a constant $K = K(a, U_{G, L})$ such that (11) holds for every $(\bar{G}, \bar{L}) \in U_{G, L}$. Furthermore, $\pi(\bar{G}, \bar{L}, a)$ is continuously differentiable in $\bar{G}, \bar{L}$.

Proof: The remark about the number of roots is proved in the same manner as in the proof of Lemma 3.5 of [5]. The existence of the projection operator is a consequence of the general theory in [6]. The continuity and differentiability of the projection follows from the contour integral representation of the projection (see, e.g. [7, pp. 14-15]). To obtain the estimate (11), repeat the proof of Theorem 4.1 in [5].

As a first application of the previous results, let us consider as in [5] the saddle point property. Suppose G is stable, no roots of the characteristic equation (9) lie on the imaginary axis and $\pi(G, L, 0)$, $\pi^\perp(G, L, 0)$ are the projection operators associated with the decomposition of C by $\Lambda(0, G, L)$. Since G is stable, there is an $a_0 < 0$ such that $\pi(G, L, a) = \pi(G, L, 0)$ for $a_0 \leq a \leq 0$. We know that $T_{G, L}(t)\pi(G, L, 0)$ is equivalent to a linear ordinary differential equation whose coefficient matrix has spectrum with positive real parts and $\|T_{G, L}(t)\pi^\perp(G, L, 0)\| \leq K \exp at$ for $t \geq 0$ and some $K > 0$, $a < 0$. Therefore, the solutions of the NFDE(G, L) in C is similar to a saddle point as shown in the accompanying figure.

Figure 1

From Lemma 3, there is a neighborhood U_G of G in $\mathcal{G}$ such that the same behavior is preserved for $(\bar{G}, L)$ with $\bar{G} \in U_G$ and $\pi(G, L, 0)$ replaced by $\pi(\bar{G}, L, 0)$.

In what sense is this qualitative picture preserved for $(\bar{G}, f)$ in a neighborhood of (G, L) ? We say the saddle point property of (G, L) is preserved if (G, L) is as described above and there are an open neighborhood $U_{G,L}$ of (G, L) in $\mathcal{G} \times \mathcal{F}$ and a neighborhood W of $\varphi = 0$ such that the following properties hold:

(i) In W , there is a unique equilibrium point $e(\bar{G}, f)$ which is continuously differentiable in $(\bar{G}, f)$ for each $(\bar{G}, f) \in U_{G,L}$.

(ii) The stable and unstable manifolds of $e(\bar{G}, f)$ in W are continuously differentiable in $(\bar{G}, f) \in U_{G,L}$ and are diffeomorphic to those of (G, L) in W .

(iii) Any solution of the NFDE $(\bar{G}, f)$ with initial value in W which is not on the stable manifold must leave W in a finite time for each $(\bar{G}, f) \in U_{G,L}$.

Theorem 1. The saddle point property of (G, L) is preserved.

Proof: Suppose $\bar{G} \in U_G$, the neighborhood given by Lemma 3. For any $f \in \mathcal{F}$, consider the NFDE $(\bar{G}, f)$ written as

$$(12) \quad \begin{aligned} \frac{d}{dt} \bar{G}(x_t) &= L(x_t) + h(x_t) \\ h(\varphi) &\stackrel{\text{def}}{=} f(\varphi) - L(\varphi) \end{aligned}$$

The variation of constants formula implies the initial value problem for (12) is equivalent to

$$(13) \quad x_t = T_{\bar{D}, L}(t)\varphi + \int_0^t T_{\bar{D}, L}(t-s)X_0 h(x_s)ds$$

where X_0 is the $n \times n$ matrix function on $[-r, 0]$ with $X_0(\theta) = 0$ for $-r \leq \theta < 0$; $X_0(0) = I$ (see [6, p. 13]). With the known estimates of $T_{\bar{D}, L}(t)$ on $\pi(\bar{D}, L, 0)$, $\pi^\perp(\bar{D}, L, 0)$ given by Lemma 3, one can repeat the proof of a similar theorem for retarded equations given in [8] (see, also [9]).

In general, the solution operator $T_{G, f}(t)$ of a NFDE (D, f) need not be one-to-one. On the other hand, if the function μ of bounded variation used in the definition of G has a jump at $-r$ given by a non-singular matrix B , then it is shown in [2] that $T_{G, f}(t)$ is a homeomorphism. Consequently, on an open dense set V in $\mathcal{G} \times \mathcal{F}$, one can assume $T_{G, f}(t)$ is a homeomorphism. We can thus introduce on $\dot{V}$ the concept of

trajectory equivalence of NFDE's as in ordinary differential equations. The details have not been worked out, but there should be no essential difficulty in proving the following result: If $(G, L) \in V$ and (G, L) has the saddle point property, then there are a neighborhood W of $\varphi = 0$ and a neighborhood $U \subset V$ of (G, L) such that $(\bar{G}, f)$ is equivalent to (G, L) in W for each $(\bar{G}, f) \in U$.

An important subclass of NFDE's are the retarded functional differential equations; namely, those NFDE(G, f) for which $G(\varphi) = \varphi(0)$. In this case, one can state conditions on (G, f) to ensure that $T_{G, f}(t)$ is one-to-one (see [2]), but this map will never be a homeomorphism if $r > 0$. Therefore, even the definition of trajectory equivalence seems to be a challenging problem.

Now let us turn to a more difficult question. Suppose we are given a NFDE(G, f) with G stable, $f(0) = 0$, $f = L + f_1$, $L = f'(0)$, p roots of the characteristic equation (9) have zero real parts and the remaining roots have negative real parts. Knowing the behavior of solutions of the NFDE(G, f), is it possible to discuss the behavior of solutions of the NFDE($\bar{G}, \bar{f}$) for all $(\bar{G}, \bar{f})$ in a neighborhood of (G, f) ? A partial answer to this question is given below.

Since G is stable, there is a $d < 0$ such that no characteristic roots of (9) lie in the strip $d \leq \operatorname{Re} \lambda < 0$. Choose a real number a such that $d < a < 0$. Then Lemma 3 implies there are a neighborhood U_G of G in $\mathcal{G}$, projection operators $\pi(\bar{G}, L, a)$, $\pi^\perp(\bar{G}, L, a)$ associated with the decomposition of C by $N(a, \bar{G}, L)$ and a constant $K = K(a, U_G)$ such that

(11) is satisfied for each $(\bar{G}, L) \in U_G \times \{L\}$. Furthermore, the dimension of the projection operators $\pi(\bar{G}, L, a)$ is p for every $\bar{G} \in U_G$. For any $\bar{f} \in \mathcal{F}$, write the NFDE $(\bar{G}, \bar{f})$ as

$$(14) \quad \begin{aligned} \frac{d}{dt} \bar{G}(x_t) &= L(x_t) + h_1(x_t), \\ h_1(\varphi) &= [\bar{f}(\varphi) - f(\varphi)] + [f(\varphi) - L(\varphi)]. \end{aligned}$$

As before, the variation of constants formula for (14) is

$$(15) \quad x_t(\varphi) = T_{\bar{G}, L}^-(t)\varphi + \int_0^t T_{\bar{G}, L}^-(t-s)X_0 h_1(x_s)ds.$$

Using this relation and the known estimates on $T_{\bar{G}, L}^-(t)$, one can apply the same reasoning as in [10] for retarded functional differential equations to obtain the following results.

Theorem 2. Under the above hypotheses on (G, f) , there are an open neighborhood $U_{G, f}$ of (G, f) in $\mathcal{G} \times \mathcal{F}$, open neighborhoods V, W of $\varphi = 0$, positive constants K, α and a diffeomorphism $h_{\bar{G}, \bar{f}}: \pi(\bar{G}, L, a)C \cap V \rightarrow W$ such that the set

$$M_{\bar{G}, \bar{f}} = \{\varphi \in C: \varphi = h_{\bar{G}, \bar{f}}\psi, \psi \in \pi(\bar{G}, L, a)C\}$$

is a local integral manifold of the NFDE $(\bar{G}, \bar{f})$ for $(\bar{G}, \bar{f}) \in U_{G, f}$. The solutions of the NFDE $(\bar{D}, \bar{f})$ on $M_{\bar{G}, \bar{f}}$ is equivalent to an ordinary

differential equation and $h_{\bar{G}, \bar{F}}$ is differentiable in $(\bar{G}, \bar{F}) \in U_{G, f}$. Finally, any solution $x(\varphi)$ of the NFDE $(\bar{G}, \bar{F})$ satisfies

$$|x_t(\varphi) - h_{\bar{G}, \bar{F}}(\pi(\bar{G}, L, a)x_t(\varphi))| \leq Ke^{-\alpha t}|\varphi|$$

as long as $x_t(\varphi) \in W$, $\pi(\bar{G}, L, a)x_t(\varphi) \in V$.

Corollary 1. With $h_{\bar{G}, \bar{F}}$ as in Theorem 2, any solution $x(\varphi)$ satisfying $x_t(\varphi) \in W$, $\pi(\bar{G}, L, a)x_t(\varphi) \in V$ for $t \leq 0$ must lie on $M_{\bar{G}, \bar{F}}$.

Following the ideas used in the proof of Theorem 3.3 in [11] and Theorems 4.3 and 7.1 of [10] one can obtain

Corollary 2. Suppose $U_{G, f}$ is the neighborhood of Theorem 2, $(\bar{G}, \bar{F}) \in U_{G, f}$ and there is a unique equilibrium point (say $\varphi = 0$) of the NFDE $(\bar{G}, \bar{F})$.

If the zero solution of the ordinary differential equation on $M_{\bar{G}, \bar{F}}$ is uniformly asymptotically stable (unstable), then the zero solution of the NFDE $(\bar{G}, \bar{F})$ is uniformly asymptotically stable (unstable).

Corollary 3. Let $U_{G, f}, V, W$ be the neighborhoods of Theorem 2 and suppose $(\bar{G}, \bar{F}) \in U_{G, f}$. If $x(\varphi)$ is a solution of the NFDE $(\bar{G}, \bar{F})$ with $\pi(\bar{G}, L, a)x_t(\varphi) \in V$, $x_t(\varphi) \in W$ for $t \geq 0$, then the ω -limit set of φ belongs to $M_{\bar{G}, \bar{F}}$ and is an invariant set. Furthermore, if there is a neighborhood on $M_{\bar{G}, \bar{F}}$ which is positively invariant under the corresponding ordinary differential equation, then there is a neighborhood W' of $\varphi = 0$ such that the ω -limit set of every solution of the NFDE $(\bar{G}, \bar{F})$ with initial value $\varphi \in W'$ belongs to $M_{\bar{G}, \bar{F}}$.

As an application of Corollary 3 to a bifurcation problem, consider the system

$$\frac{d}{dt}G(x_t, \varepsilon) = f(x_t, \varepsilon)$$

where ε is a small real parameter and $G(\varphi, \varepsilon)$, $f(\varphi, \varepsilon)$ are very smooth functions of φ, ε for $\varphi \in \mathbb{C}$, $|\varepsilon| \leq \varepsilon_0$, $f(0, \varepsilon) = 0$. Suppose $L(\cdot, \varepsilon)$ is the derivative of $f(\varphi, \varepsilon)$ with respect to φ evaluated at $\varphi = 0$. Suppose the characteristic equation $\det \Delta(\lambda, G(\cdot, \varepsilon), L(\cdot, \varepsilon)) = 0$ has a pair of simple roots $\nu(\varepsilon) \pm i\mu(\varepsilon)$, $\varepsilon\nu(\varepsilon) > 0$ for $\varepsilon \neq 0$, $\nu(0) = 0$, $\mu(\varepsilon) > 0$, $|\varepsilon| \leq \varepsilon_0$, the remaining roots have negative real parts and there is a $\delta > 0$ such that $\operatorname{Re} \lambda_{G(\cdot, \varepsilon)} \leq -\delta < 0$, $|\varepsilon| \leq \varepsilon_0$. If we further assume that the zero solution of the NFDE($G(\cdot, 0), f(\cdot, 0)$) is uniformly asymptotically stable, then Corollary 3 and a Poincaré - Bendixson argument imply there is at least one nontrivial periodic solution of the NFDE($G(\cdot, \varepsilon), f(\cdot, \varepsilon)$) for $0 < \varepsilon \leq \varepsilon_1$, ε_1 sufficiently small.

Of course, to determine when the zero solution of a system with roots on the imaginary axis is uniformly asymptotically stable is an extremely difficult problem even for ordinary differential equations. Some success for NFDE's has recently been obtained in [11] for all roots on the imaginary axis being zero and for the general case by A. Hausrath in a Ph.D. thesis at Brown. Basic to this investigation is Corollary 2. This corollary implies that it is sufficient to discuss the ordinary differential equation on $M_{\overline{G}, \overline{F}}$. Therefore, one can hope to approximate the manifold $M_{\overline{G}, \overline{F}}$ to a sufficient accuracy so that a decision can be made. This is the approach that was taken in the above papers and the final sufficient conditions are expressed in terms of computable quantities.

References

- [1] Hale, J. K. and M. A. Cruz, Existence, uniqueness and continuous dependence in hereditary systems. *Annali Mat. Pura Appl.* 85(1970), 63-82.
- [2] Hale, J. K., Forward and backward continuation for neutral functional differential equations. *J. Differential Equations* 9(1971), 168-181.
- [3] Cruz, M. A. and J. K. Hale, Stability of functional differential equations of neutral type. *J. Differential Equations* 7(1970), 334-355.
- [4] Hale, J. K., LaSalle, J. P. and M. Slemrod, Theory of a general class of dissipative processes. *J. Math. Ana. Appl.* To appear.
- [5] Cruz, M. A. and J. K. Hale, Exponential estimates and the saddle point property for neutral functional differential equations. *J. Math. Ana. Appl.* 34(1971).
- [6] Hale, J. K. and K. R. Meyer, A class of functional equations of neutral type. *Mem. Amer. Math. Soc.* 76(1967), 65 pages.
- [7] Gohberg, I. C. and M. G. Krein, Introduction to the theory of linear nonself-adjoint operators. Vol. 18, *Trans. Math. Monographs. Am. Math. Soc.* 1969.
- [8] Hale, J. K. and C. Perelló, The neighborhood of a singular point for functional differential equations. *Contr. to Differential Eqns.* 2(1964), 351-375.
- [9] Hale, J. K., Functional Differential Equations, *Applied Math. Sciences*, Vol. 3, Springer-Verlag, 1971.
- [10] Chafee, N., A bifurcation problem for a functional differential equation of finitely retarded type. *J. Math. Ana. Appl.* 34(1971).
- [11] Hale, J. K., Critical cases for neutral functional differential equations. *J. Differential Eqns.* 1971.
- [12] Hale, J. K., A class of neutral equations with the fixed point property. *Proc. Nat. Acad. Sci. USA*, 67(1970), 136-137.

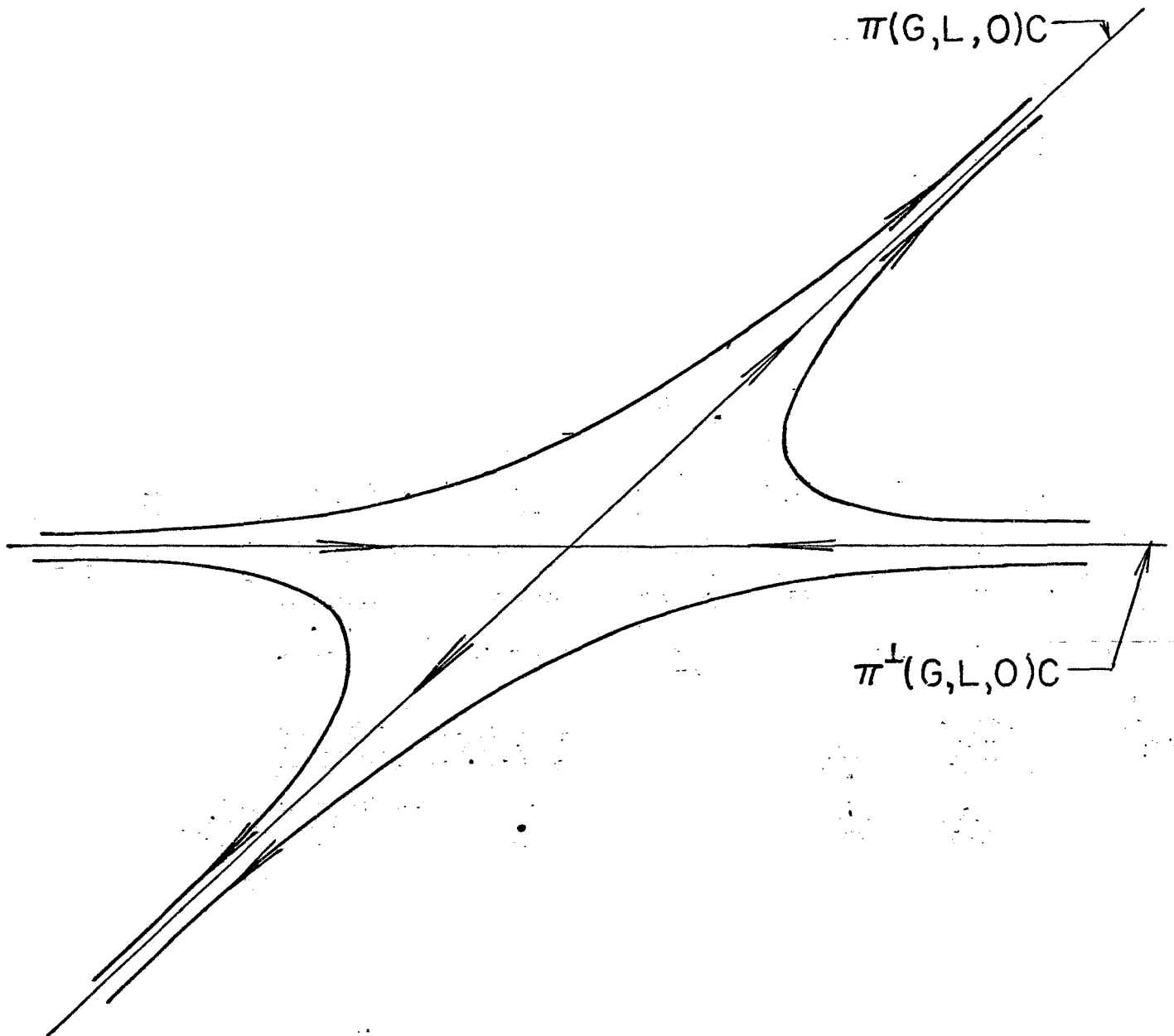


FIG. I