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ABSTRACT

The current collection characteristics of moving cylindrical and spherical electrostatic probes are evaluated for velocities encountered in earth satellites and planetary probes. Simple asymptotic current formulas are developed for large values of the Mach number and potential. For Mach numbers greater than 2.5, the orbital motion limited current is given within 1% by

$$i = N e A w \left(1 + \frac{kT}{mw^2} + \frac{2V}{mw^2}\right)^{\frac{d-1}{2}}$$

where d = 1, 2, 3 for a plane, cylinder, or sphere respectively. A data reduction technique for determining ion densities from the accelerated ion current is discussed. The current expressions are given in terms of standard functions and recurrence relations, therefore allowing straightforward computer programming of all the probe current formulas.

CONTENTS

INTRODUCTION

Cylindrical and spherical electrostatic probes have been widely employed on rockets and satellites for charged particle temperature and density measurements.[1-3] In the future these devices may be used on planetary missions. A common feature of the planetary and earth electrostatic probes is the rapid vehicle velocity relative to the plasma thermal velocity. This results in high ion Mach numbers and non-negligible electron Mach numbers.

The formula for the current to a moving cylinder has been obtained by Kanal[4]; however the series expansions derived in reference 4 are not valid for high Mach numbers. A number of authors[5-7] have investigated the current to a moving sphere, however only Kanal[7] has obtained the current expression for an arbitrary sized sheath.

We reexamine the expression for the current to a moving cylindrical probe (eqs. 15 and 27 of reference 4) in order to derive expansions valid at high Mach numbers. We include the expressions for the current to a moving sphere so that the asymptotic current expressions may be compared with those of the cylinder.

The key assumptions used in deriving the probe currents are that: 1) the probes are surrounded by a sheath of ideal geometry (cylinder or sphere); 2) no collisions take place within the sheath; and 3) the particle distribution outside the sheath is Maxwellian with a superimposed drift velocity. The sheath symmetry becomes distorted at high Mach numbers; however most of the collected current then originates in a small region about the velocity vector where sheath symmetry is approximately maintained. The effects due to the finite length of the cylinder probe will be neglected in the present treatment.

BASIC CURRENT EQUATIONS

The derivation from first principles of the integral expression for the current to a moving sphere or cylinder has been treated in the literature. Both the sphere and cylinder probes may be treated by the method of Kanal.[4] We express the current formulas in normalized form,

$$I = i/i_{random}, \tag{1}$$

where i is the actual current and i_{random} is the random current to an uncharged probe of area A due to particles of mass m with number density N and at temperature T,

$$i_{random} = \left(\frac{kT}{2\pi m}\right)^{1/2} NAe. \tag{2}$$

The normalized accelerated particle current is given by the formula,

$$\begin{aligned} I_{acc} = & \int_{\gamma V^{1/2}}^{\infty} sds\,(s^2 + V)^{\frac{d-1}{2}}\, I^{(d)}(s, M) \\ & + \left(\frac{a}{r}\right)^{d-1} \int_0^{\gamma V^{1/2}} sds\, s^{d-1}\, I^{(d)}(s, M), \end{aligned} \tag{3}$$

where d = 2, 3 for cylinder and sphere respectively, and

$$I^{(2)}(s, M) = \frac{4}{\pi^{1/2}}\, e^{-(M^2+s^2)}\, I_0(2Ms), \tag{4}$$

$$I^{(3)}(s, M) = \frac{e^{-(s-M)^2} - e^{-(s+M)^2}}{2Ms}. \tag{5}$$

The symbols are defined by:

$M = (m w^2 \sin^2 \theta / 2kT)^{1/2}$ = Mach number,

θ = angle between probe axis and velocity vector for cylinder; $\theta = \pi/2$ for sphere,

w = probe speed relative to the stationary plasma,

$\gamma = 1/(a^2/r^2 - 1)^{1/2}$,

a = sheath radius,

r = probe radius,

$V = \left|\frac{e\phi}{kT}\right|$ = ratio of potential to thermal energy.

The normalized retarded particle current is:

$$I_{ret} = \int_{V^{1/2}}^{\infty} s ds \, (s^2 - V)^{\frac{d-1}{2}} \, I^{(d)}(s, M). \qquad (6)$$

In formula (4), I_0 is the modified Bessel function of order zero.[8] The integrals for the sphere may be readily performed, and yield exponentials and error functions. The integrals for the cylinder can at best only be reduced to the form of a power series whose coefficients are known functions. Kanal[4] obtained a single power series in powers of $M/V^{1/2}$ with coefficients depending on Bessel functions and incomplete gamma functions for I_{acc} and modified Bessel functions for I_{ret}.

We derive a power series representation for the cylinder current in power of M^2 and V from which asymptotic formulas valid for high M or V can be obtained by summing over M or V. The summation yields series in confluent hypergeometric functions.[8]

The evaluation of the cylinder current reduces to the problem of performing the integral,

$$I(x, y) = \frac{4}{\pi^{1/2}} \int_{x^{1/2}}^{\infty} s\,ds\,(s^2 + y)^{1/2}\, e^{-(s^2 + M^2)}\, I_0(2Ms) \tag{7}$$

The dependence of I (x, y) on M^2 is simplified by performing a Laplace transform in M^2:

$$\mathcal{L}\, I(x, y) = \int_0^{\infty} dM^2\, e^{-pM^2}\, I(x, y). \tag{8}$$

Using the Laplace transform of the modified Bessel function[9], we obtain the result:

$$\mathcal{L}\, I(x, y) = \frac{2}{\pi^{1/2}} \frac{1}{1 + p} \int_x^{\infty} ds\,(s + y)^{1/2}\, e^{-ps/(1+p)}. \tag{9}$$

This is in the form of an integral representation of the confluent hypergeometric function[8], $\psi(a, c; z)$. Therefore using the relation,

$$\psi(a, c; z) = z^{1-c}\, \psi(a - c + 1, 2 - c; z), \tag{10}$$

we find:

$$\mathcal{L}\,I(x,y) = \frac{2}{\pi^{1/2}}\,\frac{1}{1+p}\,e^{-px/(1+p)}\left(\frac{1+p}{p}\right)^{3/2}\psi\left(-\frac{1}{2},-\frac{1}{2};\frac{p(x+y)}{1+p}\right). \tag{11}$$

To facilitate the separation of variables and taking the inverse Laplace transform, we partition I into two parts,

$$I(x,y) = I_1(y) + I_2(x,y) \tag{12}$$

$$\mathcal{L}\,I(x,y) = \mathcal{L}\,I_1(y) + \mathcal{L}\,I_2(x,y),$$

according to the relation between ψ and ϕ,

$$\psi(a,c;z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,\phi(a,c;z) + \frac{\Gamma(c-1)}{\Gamma(a)}\,z^{1-c}\,\phi(a-c+1,2-c;z), \tag{13}$$

$$\phi(a,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n\,z^n}{(c)_n\,n!}. \tag{14}$$

The result of this partition is:

$$\mathcal{L}\,I_1(y) = \frac{1}{1+p}\left(\frac{1+p}{p}\right)^{3/2} e^{py/(1+p)}, \tag{15}$$

$$\mathcal{L}\,I_2(x,y) = -\frac{1}{1+p}\,(x+y)^{3/2}\,e^{-py/(1+p)}\,\phi\left(1,\frac{5}{2};\frac{p(x+y)}{1+p}\right)\Big/\Gamma\left(\frac{5}{2}\right). \tag{16}$$

In performing the inverse Laplace transform on Equations (15) and (16), the path of integration reduces to a circle of radius greater than unity and centered at p = -1. For I_1 (y) a branch cut extends from p = -1 to p = 0. Both integrals, I_1 and I_2, are evaluated with the change of variable p = -1 + q, and an expansion of the integrands in powers of q. Evaluating the residue at q = 0 we find:

$$I_1(y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-M^2)^n\, y^m\, \Gamma(n+m-1/2)}{n!\, m!\, \Gamma(m-1/2)\, \Gamma(n+1)}, \tag{17}$$

$$I_2(x,y) = -(x+y)^{3/2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-M^2)^n (-x)^m (x+y)^r\, \Gamma(n+m+r+1)}{n!\, m!\, \Gamma\left(r+\frac{5}{2}\right) \Gamma(m+r+1)\, \Gamma(n+1)} \tag{18a}$$

$$I_2(x,y) = -(x+y)^{3/2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-x)^m (x+y)^n\, \phi(n+m+1,\, 1;\, -M^2)}{m!\, \Gamma\left(n+\frac{5}{2}\right)}. \tag{18b}$$

These basic formal power series will be rearranged later as series in confluent hypergeometric functions (in the manner of Equation 18b) which can either be programmed on a computer or expressed in analytic, asymptotic form.

Finally we give the connection between the basic formulas, Equations 17 and 18, and the normalized cylinder currents.

$$I_{\substack{acc\\ cylinder}} = I_1(V) + I_2(\gamma^2 V,\, V) - \frac{a}{r} I_2(\gamma^2 V,\, 0) \tag{19}$$

$$I_{\substack{ret\\ cylinder}} = I_1(-V). \tag{20}$$

We note that for sufficiently small values of M and V, the power series representation of Equations 17 and 18 are rapidly convergent and would therefore be appropriate for computer evaluation.

Two limiting forms of the accelerated current are commonly employed: 1) the sheath area limited case when the potential is small and the sheath collapses to the probe, $a/r \rightarrow 1$ and $\gamma \rightarrow \infty$; and 2) the orbital motion limited case when the sheath is much larger than the probe, $a/r \gg 1$, $\gamma \rightarrow r/a$. The normalized current expressions for these two ideal cases are:

$$I_{\substack{sal \\ cylinder}} = \frac{a}{r} I_1 (0), \tag{21}$$

$$I_{\substack{oml \\ cylinder}} = I_1 (V) + I_2 (0, V). \tag{22}$$

The expressions for the normalized current to a sphere are obtained by integrating Equations 3 and 6:

$$\begin{aligned} I_{\substack{acc \\ sphere}} = \left(\frac{a}{r}\right)^2 & \left\{ \frac{\pi^{1/2}}{2} \frac{\left(M^2 + \frac{1}{2}\right)}{2M} \, 2 \, \mathrm{erf}\,(M) \right. \\ & \left. + \frac{\pi^{1/2}}{2} \frac{V}{2M} \left[\mathrm{erf}\,(M - \gamma V^{1/2}) + \mathrm{erf}\,(M + \gamma V^{1/2})\right] + \frac{1}{2} e^{-M^2} \right\} \\ & - \left(\frac{a^2}{r^2} - 1\right) \left\{ \frac{\pi^{1/2}}{2} \frac{\left(M^2 + \frac{1}{2} + V\right)}{2M} \left[\mathrm{erf}\,(M - \gamma V^{1/2}) + \mathrm{erf}\,(M + \gamma V^{1/2})\right] \right. \\ & \left. + \frac{M + \gamma V^{1/2}}{4M} e^{-(M - \gamma V^{1/2})^2} + \frac{M - \gamma V^{1/2}}{4M} e^{-(M + \gamma V^{1/2})^2} \right\}, \end{aligned} \tag{23}$$

$$I_{\substack{ret \\ sphere}} = \frac{\pi^{1/2}}{2} \frac{\left(M^2 + \frac{1}{2} - V\right)}{2M} \left[\mathrm{erf}\,(M - V^{1/2}) + \mathrm{erf}\,(M + V^{1/2})\right]$$

$$+\frac{M + V^{1/2}}{4M}\, e^{-(M - V^{1/2})^2}$$

$$+\frac{M - V^{1/2}}{4M}\, e^{-(M + V^{1/2})^2}\,. \tag{24}$$

For the two idealized cases the accelerated current reduces to:

$$I_{\substack{sal\\ sphere}} = \frac{a^2}{r^2}\left\{\frac{\pi^{1/2}}{2}\,\frac{M^2 + \frac{1}{2}}{M}\,\mathrm{erf}\,(M) + \frac{1}{2}e^{-M^2}\right\}, \tag{25}$$

$$I_{\substack{oml\\ sphere}} = \frac{\pi^{1/2}}{2}\,\frac{M^2 + \frac{1}{2} + V}{M}\,\mathrm{erf}\,(M) + \frac{1}{2}e^{-M^2}\,. \tag{26}$$

Equation 23 has been given by Kanal[7] and Equation 25 by Sagalyn.[6]

SHEATH AREA LIMITED CURRENT

The sheath area limited current is determined by the condition that all particles that enter the sheath are collected; it is obtained from the general accelerated current by taking the limit as $a/r \to 1$ and multiplying the result by $(a/r)^{d-1}$.

For the cylinder, the sheath area limited current is given in closed form:

$$I_{\substack{sal\\ cylinder}} = \frac{a}{r}\,\varphi\left(-\frac{1}{2},\ 1;\ -M^2\right). \tag{27}$$

Using the contiguous functions, the expression for the derivative of ϕ and the confluent hypergeometric representation of the modified Bessel function, one can demonstrate the equivalence of this result with Equation 22 of Reference 4.

$$I_{\substack{sal\\ cylinder}} = \frac{a}{r} e^{-M^2/2} \left[(1 + M^2)\, I_0 \left(\frac{M^2}{2}\right) + M^2\, I_1 \left(\frac{M^2}{2}\right) \right]$$

A comparison of these two representations demonstrates the usefulness of the confluent hypergeometric functions; only one function need be evaluated instead of an exponential and two Bessel functions and the asymptotic form for large M is easily obtained from ϕ. For small M, the Taylor series representation of ϕ provides a convergent calculation of I_{sal}.

For large M, the asymptotic series is derived from the following exact series which is obtained from the integral representation of ϕ:

$$\phi(a, c; -z) = \sum_{n=0}^{\infty} \frac{(a - c + 1)_n\, \Gamma(c)\, \gamma(a + n, z)}{n!\, \Gamma(c - a)\, \Gamma(a)\, z^{a+n}} \tag{28a}$$

a − c ≠ negative integer,

$$\phi(a, c; z) = \sum_{n=0}^{\infty} \frac{(1 - a)_n\, \Gamma(c)\, \gamma(c - a + n, z)\, e^z}{n!\, \Gamma(a)\, \Gamma(c - a)\, z^{c-a+n}} \tag{28b}$$

a ≠ negative integer,

where the incomplete gamma function may be written as

$$\gamma(a, x) = a^{-1}\, x^a\, \phi(a,\ a + 1;\ -x)$$

For sufficiently large z, the following equation holds;

$$\Gamma(a) - \gamma(a, z) \sim z^{a-1}\, e^{-z},$$

Therefore we have the asymptotic expansion:

$$I_{\substack{sal\\ cylinder}} \underset{(M\ large)}{\cong} \frac{a}{r}\frac{2}{\pi^{1/2}} M \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n \left(-\frac{1}{2}\right)_n}{n!} \left(\frac{1}{M^2}\right)^n . \tag{29}$$

Writing out the first few terms we have,

$$I_{\substack{sal\\ cylinder}} \underset{(M\ large)}{\cong} \frac{a}{r}\frac{2}{\pi^{1/2}} \left\{1 + \frac{1}{4M^2} + \frac{1}{32M^4} + \cdots\right\} . \tag{30}$$

Retaining only the first term and multiplying by the random current, we find that the actual current is equal to

$$New \frac{a}{r}\frac{A\,|\sin\theta|}{\pi} ,$$

which is the product of the number density, charge, relative velocity, and the cylinder probe area projected normal to the velocity,

$$\frac{a}{r} A\,|\sin\theta|/\pi = 2\,a\,\ell\,|\sin\theta| .$$

The sphere sheath area limited current is given by Equation 25, the asymptotic form for large Mach number, M, is:

$$I_{\substack{sal\\ sphere}} \underset{(M\ large)}{\cong} \frac{\pi^{1/2}}{2} M \left(1 + \frac{1}{2M^2}\right) , \tag{31}$$

where we have neglected terms of order e^{-M^2}.

RETARDED CURRENT

The retarded current is independent of the sheath size and depends only on the potential across the sheath. The cylinder retarded current is given by the double series (Equation 20):

$$I_{\substack{ret\\cylinder}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-M^2)^n (-V)^m \Gamma\left(n+m-\frac{1}{2}\right)}{n!\, m!\, \Gamma\left(m-\frac{1}{2}\right) \Gamma(n+1)}. \tag{32}$$

For small M and V this series provides a convergent means of calculating the current.

For small M and arbitrary V, we sum the series in V giving:

$$I_{\substack{ret\\cylinder}} = \sum_{n=0}^{\infty} \binom{n-\frac{3}{2}}{n} \frac{(-M^2)^n}{n!} \phi\left(n-\frac{1}{2}, -\frac{1}{2}; -V\right). \tag{33}$$

The coefficients of the powers of M^2 (the ϕ) are polynomials in V multiplied by e^{-V}. Knowing the first two functions:

$$\begin{aligned} \phi\left(-\frac{1}{2}, -\frac{1}{2}, -V\right) &= e^{-V} \\ \phi\left(\frac{1}{2}, -\frac{1}{2}, -V\right) &= (1+2V)\, e^{-V}, \end{aligned} \tag{34}$$

the general terms can be found from the recurrence relation:

$$\begin{aligned} \phi\left(n+\frac{3}{2}, -\frac{1}{2}; -V\right) = \Bigg[&\left(2n+\frac{3}{2}-V\right) \phi\left(n+\frac{1}{2}, -\frac{1}{2}; -V\right) \\ &- (n+1)\, \phi\left(n-\frac{1}{2}, -\frac{1}{2}; -V\right)\Bigg] \Big/ n+\frac{1}{2}. \end{aligned} \tag{35}$$

This provides a useful computer scheme for evaluating Equation 33. Writing out the first few terms, we have

$$I_{\substack{ret \\ cylinder}} = e^{-V} \left\{ 1 + \frac{1 + 2V}{2} M^2 - \frac{(1 + 4V - 4V^2)}{16} M^4 + \cdots \right\} . \tag{36}$$

The series given by Equations 33 and 36 would adequately treat the retarded electron current for earth or planetary probes.

For higher Mach numbers we perform the sum over M^2 to obtain:

$$I_{\substack{ret \\ cylinder}} = \sum_{n=0}^{\infty} \frac{(-V)^m}{m!} \phi \left(m - \frac{1}{2}, 1; -M^2 \right). \tag{37}$$

The first term in this sum is proportional to the sheath area limited current; the general terms are determined from the first two ϕ and the recurrence relation:

$$\begin{aligned} \phi\left(-\frac{1}{2}, 1; -M^2\right) &= e^{-M^2/2} \left[(1 + M^2)\, I_0 \left(\frac{M^2}{2}\right) + M^2\, I_1 \left(\frac{M^2}{2}\right) \right] \\ \phi\left(\frac{1}{2}, 1; -M^2\right) &= e^{-M^2/2}\, I_0 \left(\frac{M^2}{2}\right) \\ \phi\left(m + \frac{3}{2}, 1; -M^2\right) &= \left[(2m - M^2)\, \phi\left(m + \frac{1}{2}, 1; -M^2\right) \right. \\ &\quad \left. + \left(\frac{1}{2} - m\right) \phi\left(m - \frac{1}{2}, 1; -M^2\right) \right] \Big/ m + \frac{1}{2} . \end{aligned} \tag{38}$$

Using the asymptotic form of ϕ for large M, this series becomes,

$$I_{\substack{ret \\ cylinder}} \cong \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{-V}{M^2}\right)^m \left(\frac{1}{M^2}\right)^{n-1/2} \left(m - \frac{1}{2}\right)_n \left(m - \frac{1}{2}\right)_n}{n!\, m!\, \Gamma\left(\frac{3}{2} - m\right)} . \tag{39}$$

The summation over the index m converges for $M^2 > V$ and gives a series in $1/M^2$:

$$I_{\substack{ret \\ cylinder}} \cong \sum_{n=0}^{\infty} C_n$$

$$C_0 = \frac{2}{\pi^{1/2}} (M^2 - V)^{1/2} \tag{40}$$

$$C_{n+1} = \frac{1}{n+1} \frac{\partial}{\partial M^2} M^2 \frac{\partial}{\partial M^2} C_n .$$

The first few terms of the asymptotic expansion are given as:

$$\underset{\substack{M^2 > V}}{I_{\substack{ret \\ cylinder}} \cong \frac{2}{\pi^{1/2}}} M \left\{ \left(1 - \frac{V}{M^2}\right)^{1/2} + \frac{1}{2M^2} \left(\left(1 - \frac{V}{M^2}\right)^{-1/2} - \frac{1}{2}\left(1 - \frac{V}{M^2}\right)^{-3/2} \right) \right.$$

$$- \frac{1}{4M^2} \left(\left(1 - \frac{V}{M^2}\right)^{-3/2} - 3\left(1 - \frac{V}{M^2}\right)^{-5/2} \right. \tag{41}$$

$$\left. \left. + \frac{15}{8} \left(1 - \frac{V}{M^2}\right)^{-7/2} \right) + \cdots \right\} .$$

The retarded current to a sphere is given by Equation 24; the asymptotic form for large M is:

$$I_{\substack{ret \\ sphere}} \underset{M\ large}{\cong} \frac{\pi^{1/2}}{2} M \left(1 + \frac{1}{2M^2} - \frac{V}{M^2}\right) . \tag{42}$$

ORBITAL MOTION LIMITED CURRENT

The orbital motion limited current is independent of the sheath size and depends only on the potential across the sheath. The current to a cylinder (Equation 22) is given by the double sum:

$$I_{\substack{oml \\ cylinder}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-M^2)^n V^m}{n!\, m!} \left\{ \frac{\Gamma\left(n+m-\frac{1}{2}\right)}{\Gamma\left(m-\frac{1}{2}\right)\Gamma(n+1)} - \frac{V^{3/2}\,\Gamma(n+m+1)}{\Gamma\left(m+\frac{5}{2}\right)\Gamma(n+1)} \right\} \tag{43}$$

The formula suitable for low Mach numbers is obtained by summing over V:

$$I_{\substack{oml \\ cylinder}} = \frac{2}{\pi^{1/2}} \sum_{n=0}^{\infty} \binom{1/2}{n} M^{2n}\, \psi\left(n-\frac{1}{2}, -\frac{1}{2}; V\right) \tag{44}$$

From the recursion relations among the ψ and the fact that,

$$\psi\left(\frac{1}{2}, \frac{1}{2}; V\right) = \pi^{1/2} e^V \operatorname{erfc} V^{1/2}, \tag{45}$$

we can determine all the coefficients of M^2:

$$\psi\left(-\frac{1}{2}, -\frac{1}{2}; V\right) = \frac{\pi^{1/2}}{2} e^V \operatorname{erfc} V^{1/2} + V^{1/2}, \tag{46}$$

$$\psi\left(\frac{1}{2}, -\frac{1}{2}; V\right) = \frac{\pi^{1/2}}{2} e^V (1-2V) \operatorname{erfc} V^{1/2} + V^{1/2}, \tag{47}$$

$$\psi\left(n+2-\frac{1}{2}, -\frac{1}{2}; V\right) = \left[\left(2n+\frac{3}{2}+V\right)\psi\left(n+1-\frac{1}{2}, -\frac{1}{2}; V\right) - \psi\left(n-\frac{1}{2}, -\frac{1}{2}; V\right)\right] \Big/ \left(n+\frac{1}{2}\right)(n+2) \tag{48}$$

Use of the recurrence relation (48) in Equation 44 provides a convenient means of evaluating the orbital motion limited current to a cylinder for $M < 4$. Writing out the first few terms we find:

$$I_{\substack{oml \\ cylinder \\ M\ small}} = \frac{2}{\sqrt{\pi}}\sqrt{V} + e^V \operatorname{erfc}(\sqrt{V}) + \frac{M^2}{2}\left[\frac{2}{\sqrt{\pi}}\sqrt{V} + (1-2V)\, e^V \operatorname{erfc}(\sqrt{V})\right] - \frac{M^4}{8}\left[\frac{2}{\sqrt{\pi}}\left(V+\frac{1}{2}\right)\sqrt{V} + \left(\frac{1}{2}-2V-2V^2\right) e^V \operatorname{erfc}(\sqrt{V})\right] + \cdots \tag{49}$$

Using the exact representation for ψ:

$$\psi(a, c; z) = \sum_{n=0}^{\infty} \frac{(1+a-c)_n}{n!\,\Gamma(a)} \left[\frac{\gamma(a+n, z)}{(-z)^n z^a} + z^{1-c} (-z)^n \Gamma(c-1-n, z) \right], \tag{50}$$

we obtain an asymptotic expansion for ψ at large potentials:

$$\psi\left(n - \frac{1}{2}, -\frac{1}{2}; V\right) \cong V^{1/2-n} \left[\sum_{m=0}^{R} \frac{(n+1)_m \left(n - \frac{1}{2}\right)_m (-)^m}{m!\, V^m} + 0\,(V^{-R-1}) \right] \tag{51}$$

Therefore we obtain an expression valid for $V \gg 1$:

$$I_{\substack{oml\\ cylinder}} \cong \frac{2}{\sqrt{\pi}} \sqrt{V} \sum_{m=0}^{R} \left(-\frac{1}{V}\right)^m \frac{1}{m!} \sum_{n=0}^{\infty} \left(\frac{M^2}{V}\right)^n \binom{1/2}{n} (n+1)_m \left(n - \frac{1}{2}\right)_m$$

When $V > M^2$ the series in n converges and is given by:

$$I_{\substack{oml\\ cylinder}} \cong \sum_{n=0}^{R} c_n,\; c_0 = \frac{2}{\sqrt{\pi}} \sqrt{V + M^2},\; c_{n+1} = \left(1 + \frac{M^2}{n+1} \frac{\partial}{\partial M^2}\right) \frac{\partial c_n}{\partial V} \tag{52a}$$

$$I_{\substack{oml\\ cylinder\\ V > M^2}} \cong \frac{2}{\sqrt{\pi}} V^{1/2} \left\{ \left(1 + \frac{M^2}{V}\right)^{1/2} + \frac{1}{2V} \left[\left(1 + \frac{M^2}{V}\right)^{-1/2} - \frac{M^2}{2V} \left(1 + \frac{M^2}{V}\right)^{-3/2} \right] \right.$$

$$- \frac{1}{4V^2} \left[\left(1 + \frac{M^2}{V}\right)^{-3/2} - \frac{9M^2}{4V} \left(1 + \frac{M^2}{V}\right)^{-5/2} \right. \tag{52b}$$

$$\left. \left. + \frac{15 M^4}{8 V^2} \left(1 + \frac{M^2}{V}\right)^{-7/2} \right] + \cdots \right\}$$

We obtain an expression useful for high Mach numbers by summing over M :

$$I_{\substack{oml \\ cylinder}} = \sum_{m=0}^{\infty} \frac{V^m}{m!} \left\{ \phi\left(m - \frac{1}{2}, 1; -M^2\right) - \frac{V^{3/2}\,\Gamma(m+1)}{\Gamma\left(m + \frac{5}{2}\right)} \phi(m+1, 1; -M^2) \right\} \quad (53)$$

The second set of confluent hypergeometric functions, $\phi(m + 1, 1; - M^2)$, consist of Laguerre polynomials in M^2 multiplied by e^{-M^2}, therefore we may neglect them in the asymptotic expansion. Consequently the orbital motion limited current for large M is identical with the asymptotic retarded cylinder current with the substitution $V \to - V$:

$$I_{\substack{oml \\ cylinder}} \underset{M\ large}{\cong} \frac{2}{\sqrt{\pi}} M \left\{ \left(1 + \frac{V}{M^2}\right)^{1/2} + \frac{1}{2M^2}\left[\left(1 + \frac{V}{M^2}\right)^{-1/2} - \frac{\frac{1}{2}}{\left(1 + \frac{V}{M^2}\right)^{3/2}}\right] \right.$$

$$\left. - \frac{1}{4M^4}\left[\left(1 + \frac{V}{M^2}\right)^{-3/2} - 3\left(1 + \frac{V}{M^2}\right)^{-5/2} + \frac{15}{8}\left(1 + \frac{V}{M^2}\right)^{-7/2}\right] + \cdots \right\} \quad (54)$$

It is interesting to note that up to the second term the expansion for I_{oml}, when $M^2 < V$ and $M^2 > V$, Equations 52 and 53, are identical.

Taking the lowest order terms we find that Equation 54 can be approximated by the single square root expression:

$$I_{\substack{oml \\ cylinder}} \underset{M^2 > V}{\cong} \frac{2}{\sqrt{\pi}} \left(\frac{1}{2} + M^2 + V\right)^{1/2} \quad (55)$$

In Figure 1, there are displayed contours of the relative error between approximation 55 and I_{oml} computed either from Equation 44 (for small M) or

Equation 54 (for large M). It is seen that the relative error is less than 1% for $M > 2.5$ and all values of the potential.

From Equation 52, the asymptotic expansion for $M^2 < V$, we obtain the approximation:

$$I_{\substack{oml \\ cylinder}} \cong \frac{2}{\sqrt{\pi}} (1 + V + M^2)^{1/2} \qquad (56)$$

In Figure 2, there are displayed contours of the percent relative difference between approximation (56) and I_{oml} as computed by Equations 44 and 54. For potentials $V > 5.5$ and any value of M, the relative error is less than 1%. We see from Figures 1 and 2 that approximation (55) is most appropriate for fixed M when the potential is varied, and that approximation (56) is most appropriate for fixed V when M is varied. It is pointed out that Equations 55 and 56 are the generalizations to be taken for the Mott-Smith and Langmuir small cylinder current expressions[10]:

$$I_{\substack{oml \\ cylinder}} \underset{M=0}{\cong} \frac{2}{\sqrt{\pi}} \sqrt{1 + V}, \qquad (57)$$

Mott-Smith and Langmuir

$$I_{\substack{oml \\ cylinder}} \underset{M>>1}{\cong} \frac{2}{\sqrt{\pi}} \sqrt{V + M^2}, \qquad (58)$$

Mott-Smith and Langmuir

Where Equation 56 is the generalization of Equation (57) and Equation (55) is the generalization of Equation (58). It is interesting that the thermal effects are added to the Mott-Smith and Langmuir formula (58) by adding the term, 1/2, under the square root.

The orbital motion limited current to a sphere is given by Equation (26); the asymptotic form is simply:

$$I_{\substack{oml\\ sphere}} \underset{M \gg 1}{\cong} \frac{\sqrt{\pi}}{2} M \left(1 + \frac{1}{2M^2} + \frac{V}{M^2}\right), \tag{59}$$

neglecting terms of order e^{-M^2}.

Thus we find that the approximation for large M^2 for the actual orbital motion limited current to a sphere and cylinder (and plane) are similar in form,

$$I_{oml} \underset{M \gg 1}{\cong} \text{New } A \left(1 + \frac{kT}{mw^2} + \frac{2V}{mw^2}\right)^{\frac{d-1}{2}}, \tag{60}$$

Where A is the probe area projected perpendicular to the velocity vector and d = 1, 2, 3 for a plane, cylinder, and sphere respectively.

ACCELERATED CURRENT

The accelerated cylinder current as given by Equation 19 is the sum of three terms, one double sum and two triple sums. We will not write down the formula as it is contained in Equations 17-19. For small values of V and M this power series would be suitable for computer evaluation. For small M and arbitrary V we may express the accelerated current as a power series in M^2:

$$I_{\substack{acc\\ cylinder}} = \frac{2}{\sqrt{\pi}} e^{-M^2} \sum_{n=0}^{\infty} \frac{(M^2)^n}{(n!)^2} (f_n + g_n), \tag{61}$$

where the initial values of f_n and g_n and their recurrence relations are determined most readily from the integral representation, Equation (3):

$$f_{n+2} = \left(n + \frac{5}{2} - V\right) f_{n+1} + V(n+1) f_n + \gamma^2 \left((1+\gamma^2) V\right)^{3/2} e^{-\gamma^2 V} (\gamma^2 V)^n \tag{62}$$

$$f_0 = \sqrt{(1+\gamma^2)V}\; e^{-\gamma^2 V} + \frac{\sqrt{\pi}}{2} e^V \operatorname{erfc}\left(\sqrt{(1+\gamma^2)V}\right). \tag{63}$$

$$f_1 = \left(\frac{3}{2} + V\gamma^2\right) \sqrt{(1+\gamma^2)V}\; e^{-\gamma^2 V} + \left(\frac{3}{2} - V\right) \frac{\sqrt{\pi}}{2} e^V \operatorname{erfc}\left(\sqrt{(1+\gamma^2)V}\right). \tag{64}$$

$$g_{n+1} = \left(n + \frac{3}{2}\right) g_n - \frac{a}{r} (\gamma^2 V)^{n+3/2}\, e^{-\gamma^2 V}, \tag{65}$$

$$g_0 = \frac{a}{r}\left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(\gamma\sqrt{V}) - \gamma\sqrt{V}\, e^{-\gamma^2 V}\right] \tag{66}$$

This scheme is recommended for computer evaluation of I_{acc} for small M, since it involves only multiplication and addition after the initial terms f_0, f_1, g_0 have been evaluated. We write out explicitly the first two terms in the expansion:

$$\underset{\text{cylinder}}{I_{acc}} \underset{M \ll 1}{=} e^{-M^2} \left\{ e^V \operatorname{erfc}\left(\sqrt{(1+\gamma^2)V}\right) + \frac{a}{r} \operatorname{erf}\left(\sqrt{\gamma^2 V}\right) \right.$$
$$\left. + M^2 \left[\left(\frac{3}{2} - V\right) e^V \operatorname{erfc}\sqrt{(1+\gamma^2)V} + \frac{3}{2}\frac{a}{r} \operatorname{erf}\left(\sqrt{\gamma^2 V}\right)\right] \cdots \right\} \tag{67}$$

For large M, Equation 18b) together with the asymptotic form Equation 28b), show that $I_2 \to 0$. Therefore the accelerated cylindrical current for large M is I_1 (v) which is given by Equation (54) or (38) with $V \to -V$.

$$\underset{\text{cylinder}}{I_{acc}} \cong I_1(V) = I_{om1} \quad \text{for } M^2 \gg 1 \tag{68}$$

The accelerated current for a sphere is given by Equation (23); for large M we have,

$$\underset{\text{sphere}}{I_{acc}} \cong \underset{\text{sphere}}{I_{om1}} = \frac{\sqrt{\pi}}{2} M \left(1 + \frac{1}{2M^2} + \frac{V}{M^2}\right) \quad M^2 \gg 1 \tag{69}$$

Therefore it is a property of probes with simple geometry that the accelerated current is given by the orbital motion limited current at large Mach numbers.

ION DENSITY DETERMINATION

We consider a method of determining the ion densities from the volt-ampere characteristics of a moving probe. We treat only the case when the ions are accelerated and the electron current is negligible. Unless the probe is driven to very high negative potentials with respect to the plasma, it will in practice be difficult to obtain more than two averaged parameters from the experimental accelerated ion current. However let us consider the general technique of fitting the accelerated ion current to a polynomial in the potential regardless of the information content of the data. One advantage of the polynomial method is that the number of terms in the fit is determined only by the amount of information contained in the data, independent of the number of ion species present. For sufficiently high Mach number $M^2 > V$ or $(mw^2 \sin^2 \theta)/2 > e\phi$, the orbital motion limited current is the polynomial:

$$i = e\, 2\, r\, \ell\, w\, |\sin \theta| \sum_{n=0}^{p} a_n V^n, \tag{70}$$

$$a_n = \sum_{\ell=0}^{R} \left(\frac{2kT}{w^2 \sin^2 \theta}\right)^{n+\ell} \binom{1/2}{n} \frac{\left(n - \frac{1}{2}\right)_\ell \left(n - \frac{1}{2}\right)_\ell}{\ell!} m^{(n+\ell)}, \tag{71}$$

$$m^{(s)} = \sum_a N_a / (m_a)^s , \tag{72}$$

where N_a is the number density and m_a the mass of species a. The coefficients of the powers of V are themselves expansions in the weighted inverse powers of the mass, $m^{(s)}$. The highest power of V, V^P, depends on the potential range over which the data is taken, and can be estimated by the requirement that $\binom{1/2}{P} (V/M^2)^P$ be not much less than unity. The termination of the series for a_n in powers of $1/M^2$ is determined by the value of the smallest Mach number. It is assumed that the area of the probe and $w \sin \theta$ are known, and that T can be determined (possibly from the electron temperature). Then by determining the coefficients, a_n, from a least square polynomial fit to the data one obtains a set of the weighted inverse powers of the mass, N, $m^{(1)}$, $m^{(2)}$, . . . by solving the set of equations, Equation 71, starting with a_p. If the ion species are known and a sufficient number of the $m^{(s)}$ are evaluated, then the mass concentrations can be found. If two known species are present, then only a knowledge of the first two quantities a_0 and a_1 are necessary to determine the concentrations.

The orbital motion limited current to a sphere at sufficiently high Mach number is:

$$i = e \pi r^2 w (a_0 + a_1 V), \tag{73}$$

$$a_0 = N + \frac{kT}{w^2} m^{(1)} \tag{74}$$

$$a_1 = \frac{2kT}{w^2} m^{(1)} . \tag{75}$$

Therefore the polynomial fit for the sphere is nearly the same as that for the cylinder, however with truncation of the series in V and $1/M^2$.

An alternate method of fitting the accelerated ion current proceeds on the assumption that the species as well as w sin θ and T are known. Then using the asymptotic expansion for the orbital motion limited current, Equations 54 or 52, the current is a linear function of the number densities, N_a, and may be statistically fitted to the data points,

$$i = \sum_a f_a (V) N_a , \tag{76}$$

where the coefficients f_a (V) depend on the species (mass), potential, temperature, and relative velocity w. This technique works best when there are primarily two species present. Although it is not unique, this technique can also be used to guess the masses of two species by examining the sign of the number densities, N_a computed for an assumed mass pair and then comparing the net density $N = N_1 + N_2$ with the electron density. Additional information to aid in the ion density determination is contained in the altitude variation of the ion species.

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LIST OF FIGURES

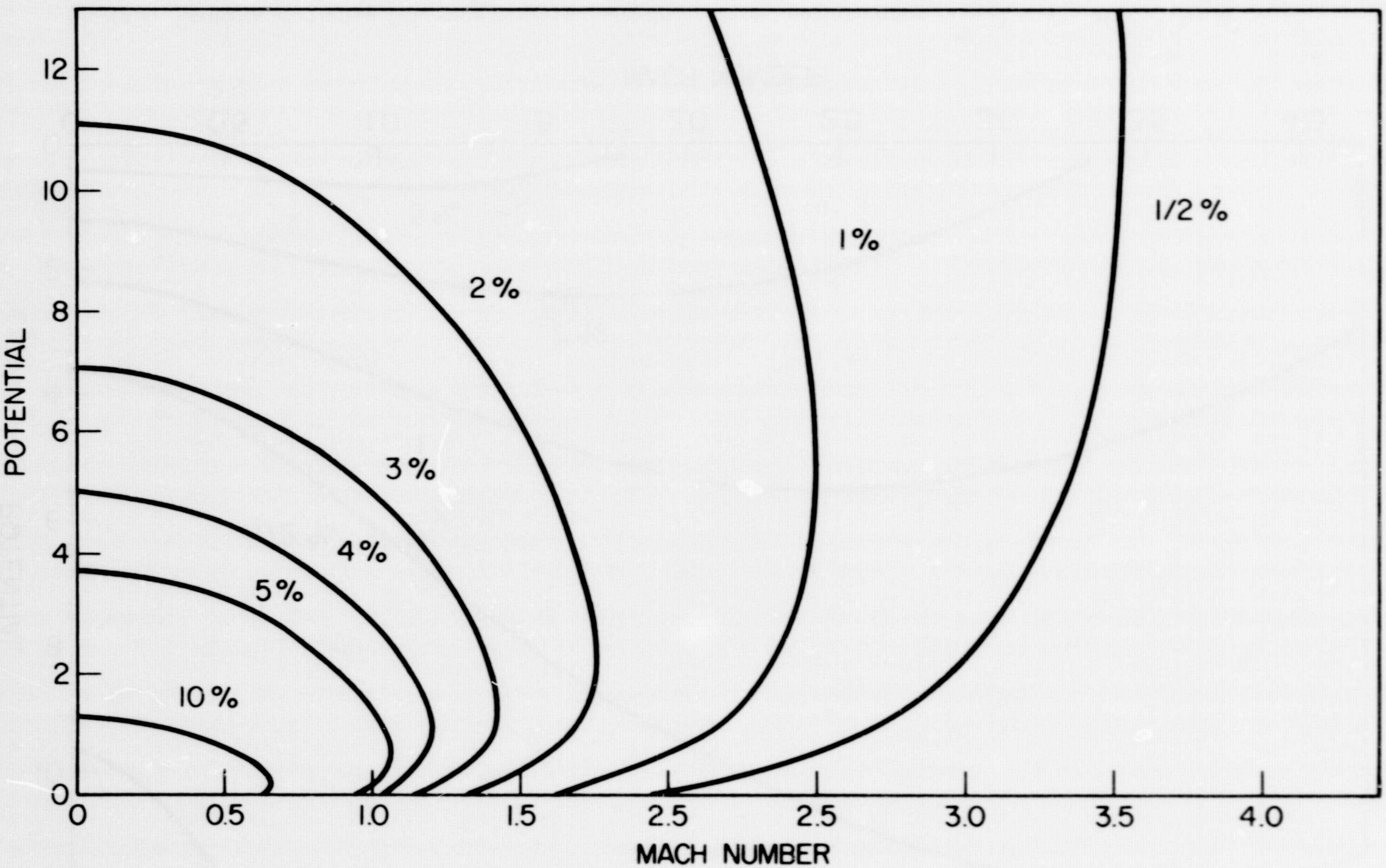

Figure 1. Contours of relative difference between the cylinder current approximation, $\frac{2}{\sqrt{\pi}} M (1 + 1/2\, M^2 + V/M^2)^{1/2}$, for $M^2 > V$, and the orbital motion limited current.

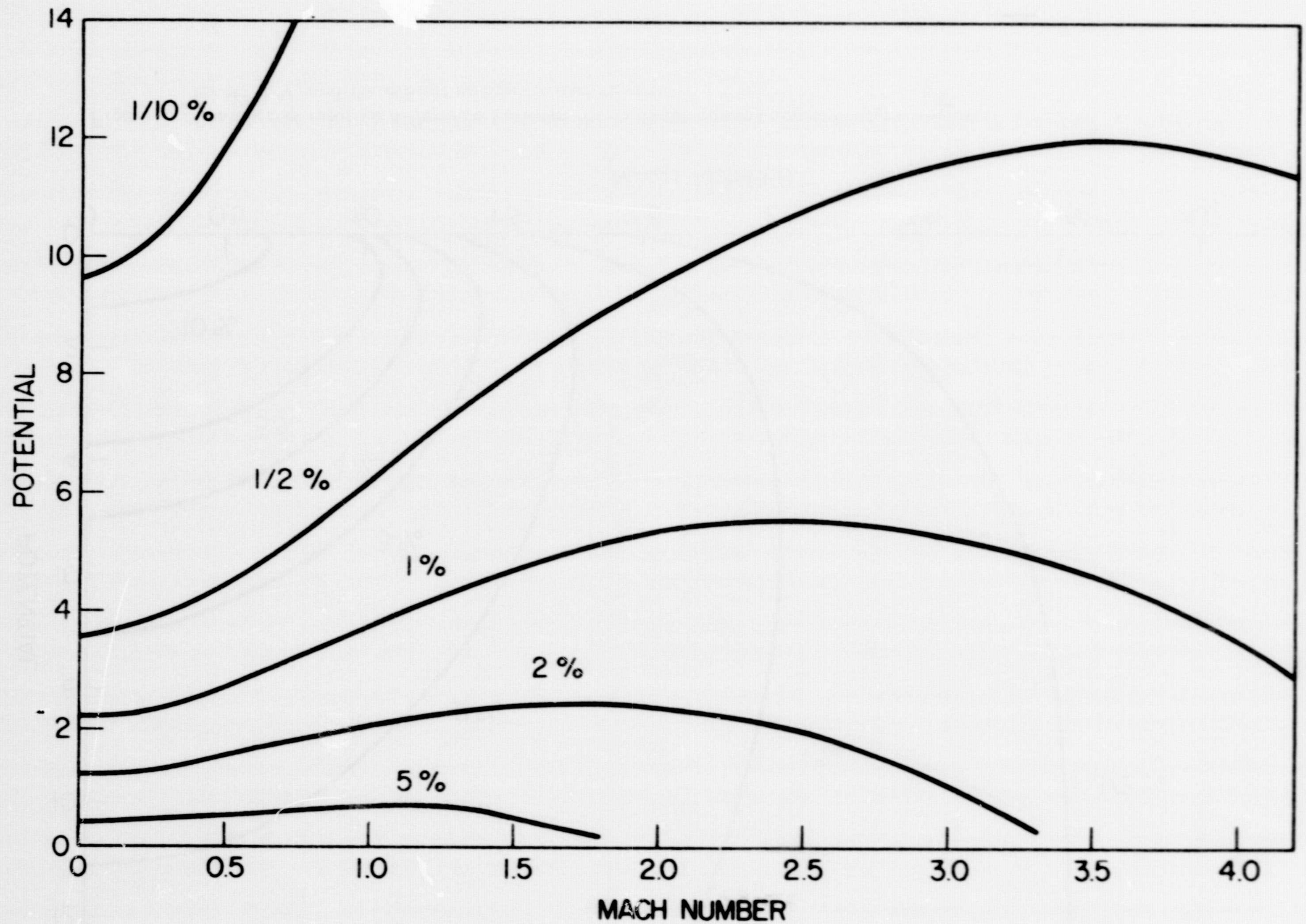

Figure 2. Contours of relative difference between the cylinder current approximation, $\frac{2}{\sqrt{\pi}}(1 + V + M^2)^{1/2}$, for $M^2 < V$, and the orbital motion limited current.