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IDENTIFICATION OF SYSTEM PARAMETERS FROM INPUT-OUTPUT DATA WITH APPLICATION TO AIR VEHICLES

by Dallas G. Denery

Ames Research Center
Moffett Field, Calif. 94035

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16. Abstract

A new algorithm is developed for estimating system parameters from input-output data. If the noise or uncertainty in the system is small, the algorithm does not require a prior estimate of the unknown parameters and if the noise has a zero mean, the final parameter estimates will not be biased. A method for reducing the computations required to obtain the parameter estimates is also presented.

A general canonical realization is developed for multi-input, multioutput, constant-coefficient, linear equations. If the unknown system is modeled in its canonical form, the unknown parameters are uniquely identifiable.

An analogy is established between a parameter estimation procedure developed by Shinbrot and the concept of linear observers developed by Luenberger. It is shown that observers of lower order can be designed quite easily using an extension of Shinbrot's method.

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SYMBOLS

All vectors are denoted by lower case letters.

All matrices are denoted by capital Roman letters.

All elements in the matrices are denoted by the corresponding lower case Roman letters.

Small Roman

$c_{(i)}$	ith row of the matrix C
e	base of the natural system of logarithms
l	constant parameter vector
m	number of measurements
n	system order
p	number of inputs
p_i	number of linearly independent rows of O_b that involve a multiplication by the ith row of the observation matrix
$p_{(i)}$	ith row of the matrix P
q	attitude rate; the quotient of n/m; the number of unknown parameters
$q^{(i)}$	ith column of the matrix Q
r	remainder of n/m
s_i	complex number
t	time
t_f	final time
t_i	discretized time
u	forward velocity; input vector
v,w	zero mean noise
x	state vector
y	output vector

y_e difference between y and y_t

y_t output vector in absence of uncertainty

z state vector

$(\ )_A$ linearized approximation of $(\ )$

$(\ )_i$ ith component of the vector $(\ )$

$(\ )_{i,j}$ element in the ith row and jth column on the matrix denoted by capital $(\ )$

$(\ )^i$ ith vector in a sequence of vectors

$(\ )_{(i)}$ ith row of the matrix denoted by capital $(\ )$

$(\ )^{(i)}$ ith column of the matrix denoted by capital $(\ )$

$(\ )_N$ nominal or initial estimate of $(\ )$

$(\ )_o$ trim condition; initial conditions

$(\ )^T$ transpose of the vector $(\ )$

Capital Roman

Any capital letters that appear in the text and which are not defined here are constant parameter matrices.

A time varying matrix of sensitivity functions

A_E portion of the matrix of sensitivity functions not correlated with the known system input

A_t difference between A and A_E

$E\{\ \}$ expected value of $\{\ \}$

I identity matrix

I_y inertia about the pitch axis

J least squares functional

$L_\alpha, L_q, L_{\delta_e}$ partial derivatives of lift with respect to α, q, and δ_e, respectively

$M_\alpha, M_q, M_{\delta_e}$	partial derivatives of moment with respect to α, q, and δ_e, respectively
O_b	observability matrix (see section 3.2)
P	covariance matrix; matrix used in constructing the canonical transformation (see section 3.2)
Q	inverse of P
T	thrust
$(\ )^i$	matrix ( ) multiplied by itself i times

Small Greek

α	angle of attack
γ	vector of unknown parameters
$\delta(\ )$	perturbation of ( ); unit impulse function
δ_e	elevator deflection
δ_{ij}	Kronecker delta
ε	error
ξ^i	ith vector in a sequence of vectors used to generate the sensitivity functions
θ	vector $\left[\xi^{1T} \,\vdots\, \xi^{2T} \,\vdots\, \cdots\right]$; attitude
ν	residuals
ρ^i	sequence of vectors
τ	dummy variable in the convolution of two functions
$(\ )_\varepsilon$	error in ( )
$(\ )^i_j$	jth component of the vector $(\ )^i$
$(\ )_N$	nominal or initial estimate of ( )

Subscripts and Superscripts

$$s_i = 1 + \sum_{j=1}^{i-1} p_j \qquad\qquad l_i = \sum_{j=1}^{i} p_j$$

IDENTIFICATION OF SYSTEM PARAMETERS FROM

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Ames Research Center

SUMMARY

This report is concerned with measurements of the input and output to a dynamic system in order to estimate the parameters in the differential equations that describe the input-output behavior. Two general methods can be used to estimate these parameters: the equations of motion method and the response curve fitting method. The equations of motion method is characterized by a single step solution that does not require prior estimation of the unknown parameters. However, unbiased noise in measurements of the system response causes a bias in the estimated parameters. The response curve fitting method is characterized by iterative solution techniques that require prior estimation of the unknown parameters and provides an unbiased estimate. The algorithm presented here uses the best features of both methods. If the system noise is small, the algorithm does not require a prior estimate of the unknown parameters, and if the noise has a zero mean, the final parameter estimates will not be biased. The algorithm is applied to simulated and flight data.

A feature of this report is the development of a canonical form for multioutput systems. When the unknown system is modeled in this canonical form, an identifiable set of parameters is defined and can be estimated by the combined algorithm. Although other canonical forms for multivariable systems are available, the parameters in those forms cannot be estimated directly by the combined algorithm.

In order to use the combined algorithm, the sensitivity functions for the system parameters and initial conditions must be computed. For constant coefficient linear systems all possible sensitivity functions can be obtained by linear combinations of the solution to only $(p + 2)$ differential equations of order n, where p is the number of independent inputs to the system, and n is the minimal order realization for the system. This is a smaller number of differential equation solutions than was previously thought to be necessary for the generation of the sensitivity functions.

An analogy is established between the equations of motion theory developed by Shinbrot and the concept of a state observer as formulated and discussed by Luenberger and Bryson. It is shown that observers of reduced order can be designed quite easily using the equations of motion theory.

I INTRODUCTION

The equations of motion for a flight vehicle describe its response to external disturbances and control inputs. They are based on Newton's laws as formulated by Euler (ref. 1). The forces in these equations are primarily aerodynamic, gravitational, and propulsive. These forces are functions of the vehicle state variables (position, velocity, angular orientation, and rate of change of angular orientation) and of the vehicle's control variables. If the deviations in the state and control variables from an equilibrium state are small, the vehicle's response can often be well approximated by a set of constant-coefficient linear equations. The coefficients in these constant-coefficient, linear, differential equations are called the stability derivatives. The stability derivatives can be estimated from aerodynamic theory or from wind-tunnel tests or both. The linearized equations of motion can then be used to predict small perturbations of the vehicle response about steady-state flight prior to flight. Needless to say there are often significant differences between the vehicle's predicted and actual response. These discrepancies can usually be attributed to errors in the estimates of the stability derivatives, and motivate the use of the flight data to improve the estimates of these stability derivatives.

1.1 IDENTIFICATION TECHNIQUES

The use of flight measurements to improve the estimates of the stability derivatives has been an area of research throughout the history of aviation (ref. 2). The first work in this area appears to have occurred in the years 1922 - 1925. During this time the National Advisory Committee for Aeronautics demonstrated the possibility of determining natural frequencies, damping ratios, time constants, and steady-state gains from flight data (refs. 3, 4). The techniques developed during this period were used with little change for the next 20 years. However, after World War II, many contributions were made to the analysis of flight data. Most of the methods used today have, as a basis, the results obtained during the years 1947 through 1953.

It was during this period that frequency response methods were first applied to the analysis of flight data. These methods included the analysis of steady-state oscillatory responses as well as the analysis of transient responses. In a particular application of the steady-state oscillation method (ref. 5) (taken from ref. 2), the elevator of an airplane was oscillated by means of an autopilot at a series of frequencies (0.5 to approximately 1.5 Hz) and the response of the airplane was measured. From these data a frequency response curve was established. Although the procedure worked satisfactorily, it required considerable flight time. Attention was therefore directed to the analysis of data from transient responses such as the response to a pulse in elevator deflection. The frequency response of the vehicle was obtained by taking the Fourier transforms of the input and response measurements and forming their ratio at discrete frequencies (ref. 6). This procedure reduced the required flight time to a small fraction of that necessary for steady-state oscillation tests. An inherent difficulty in any frequency response method, however, is that a frequency response curve is obtained instead of the parameters in the equations of motion. Methods were developed for curve fitting a transfer function of the assumed form to the measured frequency response curve in order to obtain an estimate of the parameters. Some of these methods are discussed in references 7 and 8.

In addition to frequency response methods, several other parameter estimation procedures evolved during this period which could be used to estimate the coefficients in the assumed equations of motion directly. Milliken credits Seckel with having categorized these methods as being either equations of motion methods or as response curve fitting methods (ref. 2).

The equations of motion methods are formulated by substituting measurements of the system variables (states and control positions) and their derivatives in the assumed equations of motion for the system. The resulting equations at any discrete time are then algebraic in the unknown parameters. In many cases these algebraic equations are linear and the parameters can be estimated by the solution of a set of linear equations. Shinbrot generalized this concept by considering integral transforms of the assumed equations of motion for the system, and substituting integral transforms of the measurements into these equations (ref. 8). The net result is still a set of equations which are algebraic in the unknown coefficients. Shinbrot showed that the curve fitting methods used to obtain coefficients from frequency response curves could be considered as equations of motion methods, within this generalized interpretation. It is shown in this report that the construction of linear observers can also be considered as an application of the generalized equations of motion theory. This latter material is not directly related to the rest of the thesis but is included in appendix B as a matter of interest.

In the application of an equations of motion method, there are typically more equations than unknown parameters. A least squares error criterion is therefore used to estimate the parameters. It was realized that this was not a correct application of the principle of least squares if noise was present in the measurements (ref. 9). In fact, this procedure will cause a bias in the parameter estimates even though the noise in the measurements has zero mean (refs. 10, 11). By a bias, we mean that the expected value of the error in the parameter estimates is not zero and does not go to zero with increasing amounts of data.

The response curve fitting methods were developed in order to apply the principle of least squares correctly (refs. 7, 9, 12). In these methods, the measured input is used to drive a model of the vehicle. The unknown parameters in the model are then adjusted until the model response agrees with the measured response in a least squares sense. It has been shown that the response curve fitting methods do not cause a bias to first order in the parameter estimates if there is no noise in the measured input and if the noise in the measurements of the output has zero mean (refs. 10, 13). Because the model response is a nonlinear function of the unknown parameters, an iterative estimation procedure is usually required. Shinbrot proposed several such algorithms for estimating the unknown parameters. These included a gradient procedure, a quasi-linearization procedure (referred to as a Taylor series method), and a relaxation procedure. To illustrate the feasibility of these various techniques, he applied them to some artificial data. However, because there were no high-speed digital computers at that time, the general feeling appeared to be that these methods were not practical (ref. 2).

Frequency response methods and equations of motion methods were used almost exclusively for analyzing flight data during the next 15 years. The only response curve fitting method used to any extent was the analog matching technique. In this technique, the equations of motion for an airplane are programmed on an analog computer and the unknown parameters are adjusted manually until the model response agrees with the flight measurements (ref. 14). The idea of using the digital computer to implement the powerful techniques pioneered by Shinbrot and Greenberg for systematically adjusting the parameters was not investigated until around 1966. At this time Bellman independently formulated a response curve fitting method with emphasis on digital computer implementation (ref. 15). In 1968, Cornell Aeronautical Laboratories applied this technique to some preliminary flight data (ref. 16) and in 1969, Lawrence Taylor of the Flight Research Center independently applied a similar method to analyze routine flight records (ref. 17). Taylor also presented some comparisons between parameter estimates obtained using the response curve fitting method (referred to as an approximated Newton Raphson Procedure) and the more conventional equations of motion methods. The results clearly indicated that the response curve fitting method improved parameter estimates and that with the digital computer these methods are indeed practical.

Parallel developments in identifying parameters have occurred in fields other than the field of aviation. The method of maximum likelihood estimation is one approach which has achieved wide acclaim in the fields of econometrics and statistics. Cramer has stated "From a theoretical point of view, the most important general method of estimation so far known is the method of maximum likelihood" (ref. 18). The response curve fitting methods developed within the field of aviation can be considered as maximum likelihood estimates if it is assumed that the noise in the measurements is gaussian and white, and that there are no unmeasured disturbing forces. If there are unmeasured random disturbing forces in the system, the response curve fitting methods must be modified slightly in order to obtain maximum likelihood estimates. The basic idea is that instead of modeling the unknown system by its equations of motion, it should be modeled by its optimal filter (refs. 19, 20). This idea has not yet been applied to the analysis of flight data, but may provide an improvement over the conventional response curve fitting methods if the unmeasured disturbances are substantial.

Several identification procedures are surveyed in greater detail in Chapter 2. The different techniques are illustrated by using the longitudinal equations of motion for a conventional aircraft as an example. The purpose of this chapter is to illustrate the differences between the equations of motion methods and the response curve fitting methods. The material in Chapter 2 forms the foundation on which the material in this thesis is developed.

1.2 A NEW COMBINED IDENTIFICATION ALGORITHM

From the previous discussion, it is evident that response curve fitting methods are usually superior to the equations of motion methods for estimating the coefficients in the equations of motion for an airplane. Nevertheless, equations of motion methods are useful in obtaining initial estimates of the unknown parameters which can then be used to start a response curve fitting algorithm. This two step procedure has been used successfully in certain applications (refs. 7, 9, 16) but has required two separate estimation algorithms. Taylor, on the other hand, incorporated a slight modification in a quasi-linearization response curve fitting algorithm which eliminated the necessity of using a separate procedure to obtain an initial estimate of the unknown parameters (ref. 17). This elimination simplified the total estimation problem and made the procedure more adaptable for the routine analysis of flight data.

Taylor showed satisfactory results for one application where measurements of all the output states were available. This thesis extends his procedure to the multivariable case where there may be fewer measurements than state variables in the system model. This technique uses an equations of motion procedure, which is similar to a linear observer, to obtain an initial estimate of the parameters, then switches to a quasi-linearization response curve fitting method. This particular equations of motion method can be applied to a general multi-input, multioutput, constant coefficient, linear system whereas, previously, equations of motion methods were generalized only to the single input, single-output system. In addition, the mathematical structure of this equations of motion method is nearly identical to the mathematical structure of the quasi-linearization implementation of the response curve fitting procedure. Because of this similarity, both procedures can be used in the same computational structure. This process will be referred to as the combined algorithm and is developed in Chapter IV. Some of this material has appeared in reference 21.

In Chapter VI, the combined algorithm is applied to the identification of the linearized longitudinal equations of motion of an airplane. Both simulated and flight data are used. Both single and multioutput examples are included. The effect of integration algorithms on the identification is illustrated. The effects of initial conditions and biases in the parameter identification are also illustrated. The parameters in a nonlinear set of differential equations representing the longitudinal response of a VTOL aircraft are estimated from simulated data. This latter problem was posed by personnel at Cornell Aeronautical Laboratory and was discussed at the 1970 JACC in the special session entitled, "Parameter Identification with Application to Aircraft Modeling" (refs. 22, 23).

1.3 IDENTIFIABILITY

Given a mathematical model it is usually not obvious whether or not the unknown parameters in the model are identifiable from input and response measurements.

There are two different problems in establishing the identifiability of the parameters. The first problem is to determine if the coefficients in the system transfer functions are identifiable after all cancelling poles and zeros have been eliminated. This problem is often referred to as the identifiability of the system's external description and depends on the type of test signal used in the identification. For example, if the input to a single-output, constant-coefficient, linear system is a single sinusoidal oscillation and if the initial conditions allow no transients, then the input and output can be realized by a first-order system regardless of the actual system dynamics. The importance of this problem in identifying aircraft parameters was recognized in 1947 - 1953 and a substantial amount of research was conducted in defining good input test signals. The results of these efforts were well summarized by Milliken in the following statement (ref. 2): "It would appear that an optimum input in a given case is that which best excites the frequency range of interest, and hence its harmonic content (the input signal) should be examined before the test to insure that it is suitable." Although this type of evaluation has been useful and is still the primary test used to define a good input signal, it is a qualitative procedure and does not define an optimum test signal. It would be interesting and perhaps useful to define a more quantitative procedure for designing input test signals, but this problem is not investigated here.

The second problem is to determine the identifiability of the coefficients or stability derivatives in the equations of motion for the system. This is referred to as the identifiability of an internal description of the system. Greenberg (ref. 7) pointed out that there are basic limitations in the determination of the stability derivatives in a particular set of differential equations as compared to the determination of the transfer function coefficients. In particular, he studied the fundamental mathematical limitations on the number of derivatives that can be isolated from flight records in the longitudinal case.

Although the identifiability of the system's external description implies the identifiability of the transfer function coefficients, these coefficients can be expressed in terms of a more fundamental set (with the trivial exception of the single-input, single-output system) called canonical parameters. These parameters can be used in a set of differential equations, called canonical equations, which relate the system's input to its response. The canonical equations can be obtained by linear transformations on the equations of motion for the vehicle. If the external description of a vehicle is identifiable, the parameters in the canonical equations are identifiable. It is therefore often convenient to put the equations into a canonical form. The canonical parameters are related to the stability derivatives and if the stability derivatives are identifiable, they can be computed as a function of the canonical parameters. Although there are many canonical forms, the parameters in many of them are not located in the matrices so that they can be identified directly by the combined algorithm (refs. 24, 25, 26).

In this report a canonical form for multioutput systems is presented which is analogous to a canonical form developed by Luenberger for multi-input systems (ref. 24). The final structure of the canonical form presented here is more defined than the one in reference 24, and the parameters can be uniquely identified from measurements of the system input and its response. In addition, the parameters are located so that they can be identified directly by the combined algorithm. This canonical form is useful in illustrating the generality of the combined algorithm and is presented in Chapter III, before the algorithm is developed.

1.4 COMPUTATIONAL METHODS

If response curve fitting methods are implemented by gradient algorithms, it is necessary to compute the system's sensitivity functions. These functions are the first-order variations of the system state due to unit perturbations in the unknown parameters. Each sensitivity function can generally be computed by the numerical solution of a set of differential equations of order equal to that of the system. Aström has shown that the computations required to obtain these sensitivities can be reduced for the time invariant, linear, single input, single output, discrete problem (ref. 20). Aström's results provided the motivation to investigate the possibility of reducing the computations for the time invariant, linear, multi-input, multioutput, continuous problem.

Wilkie and Perkins (ref. 27) also investigated this problem, but the method developed here requires the solution of p fewer nth order differential equations than their method (where p is the number of independent inputs to the system, and n is the system order).

It is shown in Chapter V that if the system is cyclic,[1] the sensitivity functions (for the system parameters and initial conditions) and the system response can be obtained by linear combinations of the solutions to (p + 2) nth order differential equations. Gopinath and Lange (ref. 28) have shown that if a system is not cyclic, it contains two or more identical and completely uncoupled subsystems imbedded in the original system. This is also manifest in the Jordan form for the state coefficient matrix of a noncyclic system. The technique presented can therefore be applied to each subsystem to obtain sensitivity functions for the noncyclic case. Because many of the sensitivity functions will be the same for the two independent subsystems, fewer than (p + 2) nth order equations may be required in the noncyclic case. Some suggestions are also made in Chapter V for simplifying the computation of the matrix of the integrated squares of the sensitivity functions. This matrix is used in the method of quasi-linearization and in the combined algorithm.

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[1]A system with state coefficient matrix F is cyclic if there is a vector l so that the n vectors $\left[F^{n-1}l \,\vdots\, \ldots \,\vdots\, l\right]$ are linearly independent.

II A SURVEY OF TECHNIQUES FOR ESTIMATING SYSTEM PARAMETERS FROM INPUT-OUTPUT DATA

2.1 EQUATIONS OF MOTION METHODS

2.1.1 Formulation

There are several different equations of motion methods but their main features are illustrated by the following two examples. A general discussion will then be presented.

Example 2.1 The Derivative Method (Refs. 7, 8)

Given measurements of the attitude rate, angle of attack, the derivatives of the attitude rate and angle of attack (these can be obtained indirectly from accelerometers), and the elevator deflection, consider the identification of the parameters in the short-period equations of motion for a conventional airplane.

The equations of motion are:

$$\left.\begin{aligned} &\text{Plunge equation:} \quad mu_o\dot{\alpha} \pm (L_\alpha + T)\alpha - mu_o q = -L_{\delta_e}\delta_e \\ &\text{Pitch equation:} \quad -M_{\dot{\alpha}}\dot{\alpha} - M_\alpha\alpha + I_y\dot{q} - M_q q = M_{\delta_e}\delta_e \end{aligned}\right\} \tag{2.1}$$

If $\dot{\alpha}$ is eliminated from the second equation, (2.1) can be rewritten:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dfrac{-(L_\alpha+T)}{mu_o} & 1 \\ \dfrac{M_\alpha}{I_y} - \dfrac{M_{\dot{\alpha}}(L_\alpha+T)}{I_y mu_o} & \dfrac{M_q+M_{\dot{\alpha}}}{I_y} \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} \dfrac{-L_{\delta_e}}{mu_o} \\ \dfrac{M_{\delta_e}}{I_y} - \dfrac{M_{\dot{\alpha}}L_{\delta_e}}{mu_o I_y} \end{bmatrix} \delta_e \tag{2.2}$$

Since measurements of all the system variables and their derivatives are available, they can be used in equation (2.2) at discrete times, t_i, to give a set of algebraic equations that are linear in the five unknown parameters;

$$\begin{bmatrix} \dot{\alpha}(t_i)-q(t_i) \\ \\ \dot{q}(t_i) \end{bmatrix} = \begin{bmatrix} \alpha(t_i) & 0 & 0 & \delta_e(t_i) & 0 \\ \\ 0 & \alpha(t_i) & q(t_i) & 0 & \delta_e(t_i) \end{bmatrix} \begin{bmatrix} \dfrac{-(L_\alpha+T)}{mu_o} \\ \dfrac{M_\alpha}{I_y} - \dfrac{M_{\dot{\alpha}}(L_\alpha+T)}{mu_o I_y} \\ \dfrac{M_q + M_{\dot{\alpha}}}{I_y} \\ \dfrac{-L_{\delta_e}}{mu_o} \\ \dfrac{M_{\delta_e}}{I_y} - \dfrac{M_{\dot{\alpha}} L_{\delta_e}}{mu_o I_y} \end{bmatrix} \tag{2.3}$$

If the first equation is used at two different times and the third at three different times, we will have five equations which, if independent, can be solved for the unknown parameters.

Typically, there will be more than five equations available if all the measurements are used. Because of modeling errors and uncertainty in the measurements, a solution to this enlarged set of equations will probably not exist. A method often used to define an estimate of the parameters is to choose them so that they minimize a weighted sum of the squared differences between the two sides of the equations. If equation (2.3) is written succinctly as

$$y(t_i) = A(t_i)\gamma + \varepsilon(t_i) \tag{2.4}$$

where γ is the vector of unknown parameters and $\varepsilon(t_i)$ is the error in these equations due to the uncertainty in the modeling or measurements, then an estimate of γ, $\hat{\gamma}$, is that γ which minimizes the function

$$J = \sum_{i=1}^{k} [y(t_i) - A(t_i)\gamma]^T W [y(t_i) - A(t_i)\gamma] \tag{2.5}$$

where W is a positive definite matrix used to express the relative confidence in the different measurements. To compute $\hat{\gamma}$ differentiate J with respect to the unknown parameters, set the resulting equations equal to zero and solve for γ. This gives the well-known solution

$$\hat{\gamma} = \left[\sum_{i=1}^{k} A^T(t_i)WA(t_i)\right]^{-1} \left[\sum_{i=1}^{k} A^T(t_i)Wy(t_i)\right] \tag{2.6}$$

In many applications measurements of some of the variables or derivatives of the variables are not available. If a variable but not its derivative is measured, it is tempting to differentiate the measured variable in order to use a procedure similar to that discussed in example 2.1. However, the differentiation of measured data introduces additional uncertainty so that this technique is usually inaccurate. The integral transform methods eliminate the difficulty.

Example 2.2 The Laplace Transform Method (Refs. 7, 8)

Consider the previous example with the exception that only the attitude rate and elevator deflection are measured.

The differential equation that relates the attitude rate to the elevator deflection is given by eliminating α from 2.1:

$$\ddot{q} - \left[\frac{M_q + M_{\dot{\alpha}}}{I_y} - \frac{(L_\alpha + T)}{mu_o}\right]\dot{q} - \left[\frac{(L_\alpha + T)}{mu_o}\frac{M_q}{I_y} + \frac{M_\alpha}{I_y}\right] q$$

$$= \left(\frac{M_{\delta_e}}{I_y} - \frac{M_{\dot{\alpha}} L_{\delta_e}}{mu_o I_y}\right)\dot{\delta}_e + \left[\frac{M_{\delta_e}}{I_y}\frac{(L_\alpha + T)}{mu_o} - \frac{M_\alpha L_{\delta_e}}{mu_o I_y}\right]\delta_e \tag{2.7}$$

and the Laplace transform is given by[2]

[2]Zero initial conditions have been assumed in this example. If the initial conditions are not zero, they could be included in equation (2.8) and treated as additional unknown parameters.

$$s^2q(s) - \left[\frac{M_q + M_{\dot{\alpha}}}{I_y} - \frac{(L_\alpha + T)}{mu_o}\right] sq(s) - \left[\frac{(L_\alpha + T)}{mu_o}\frac{M_q}{I_y} + \frac{M_\alpha}{I_y}\right] q(s)$$

$$= \left(\frac{M_{\delta_e}}{I_y} - \frac{M_{\dot{\alpha}}L_{\delta_e}}{mu_o I_y}\right) s\delta_e(s) + \left[\frac{M_{\delta_e}}{I_y}\frac{(L_\alpha + T)}{mu_o} - \frac{M_\alpha L_{\delta_e}}{mu_o I_y}\right]\delta_e(s) \qquad (2.8)$$

The Laplace transform of the measurements q and δ_e can be computed numerically for discrete values of s,

$$q(s_i) = \int_0^\infty e^{-s_i t} q(t)dt \approx \int_0^{t_f} e^{-s_i t} q(t)dt$$

$$\delta_e(s_i) = \int_0^\infty e^{-s_i t} \delta_e(t)dt \approx \int_0^{t_f} e^{-s_i t} \delta_e(t)dt$$

and used in equation (2.8) to obtain a set of algebraic equations that are linear in the unknown coefficients,

$$s_i^2 q(s_i) = \begin{bmatrix} s_i q(s_i) & q(s_i) & s_i\delta_e(s_i) & \delta_e(s_i)\end{bmatrix} \begin{bmatrix} \dfrac{M_q + M_{\dot{\alpha}}}{I_y} - \dfrac{(L_\alpha + T)}{mu_o} \\[2ex] \dfrac{(L_\alpha + T)M_q}{mu_o I_y} + \dfrac{M_\alpha}{I_y} \\[2ex] \dfrac{M_{\delta_e}}{I_y} - \dfrac{M_{\dot{\alpha}}L_{\delta_e}}{mu_o I_y} \\[2ex] \dfrac{M_{\delta_e}}{I_y}\dfrac{(L_\alpha + T)}{mu_o} - \dfrac{M_\alpha L_{\delta_e}}{mu_o I_y} \end{bmatrix} \qquad (2.9)$$

Since s is generally a complex number, each value of s results in two equations. If four independent equations can be obtained by using different values of s, they can be solved for the unknown coefficients.

As in example 2.1, more than four equations can be obtained by using additional values of s in (2.9). An estimate of the parameters can then be defined by a weighted least squares procedure identical to that discussed below example 2.1. The argument in equations (2.5) and (2.6) would be s_i instead of t_i.

The general formulation of the equations of motion method is now evident. A set of equations that describe the dynamic response of the system are hypothesized. These equations provide relationships among the system variables, their derivatives, and the system parameters. They are multiplied by a set of functions, called method functions, and are integrated over a time interval. In the derivative method, the method functions are delayed impulses, $\delta(t - t_i)$. In the Laplace transform method, the method functions are the exponential functions, $e^{-s_i t}$. Regardless of the type of method function, this procedure results in an arbitrarily large set of algebraic equations that can be solved for the unknown parameters. The particular set of method functions used determines the specific equations of motion method.

These ideas have been extended to nonlinear systems (ref. 8) and to time varying systems (refs. 11, 29). It is shown in appendix B that these ideas can also be used to design observers of reduced order.

2.1.2 Effects of Noise

A weighted least squares estimate for the parameters was introduced in the above discussion in order to estimate the parameters in the presence of uncertainty or noise. This technique works best when the errors, ε, in equations (2.4) are not dependent on the parameters, γ. (Errors in the determination of y would be of this type.) In the equations of motion method, A is also composed of measurements and is therefore subject to uncertainty. This uncertainty causes the error, ε, in equation (2.4) to depend on γ. This dependence causes bias in the parameter estimates (i.e., the expected value of the error in the parameter estimates is not zero) even though the uncertainty may be caused by system noise with zero mean (ref. 4). This idea is illustrated by the following example. The idea will then be generalized.

Example 2.3 Effect of Noise on an Equations of Motion Estimate

Consider a system described by

$$\dot{x} + ax = u$$

Let us assume that we have perfect measurements of u and $\dot{x}$ but that the measurement of x contains a small zero mean random bias which is not accounted for. Denote these measurements by the subscript m.

$$\dot{x}_m = \dot{x} \qquad E\{n_1\} = 0$$
$$x_m = x + n_1 \qquad E\{n_1^2\} = \sigma_n^2$$
$$u_m = u$$

If the derivative method is used, an estimate for the parameter a is obtained by solving the algebraic equation

$$\dot{x} + \hat{a}(x + n_1) = u$$

which implies

$$\hat{a} = \frac{u - \dot{x}}{x + n_1} = \frac{u - \dot{x}}{x\left(1 + \frac{n_1}{x}\right)} = a\left[\frac{1}{1 + \frac{n_1}{x}}\right]$$

If we assume that $n_1 << x$, the right side of this equation can be expanded in a power series

$$\hat{a} \approx a\left[1 - \frac{n_1}{x} + \frac{n_1^2}{x^2} + \ . \ . \ .\right]$$

and the expected value of $\hat{a}$ can be approximated by

$$E\{\hat{a}\} = a + E\left\{\frac{n_1^2}{x^2}\right\}$$

which implies that

$$E\{\hat{a} - a\} = E\left\{\frac{n_1^2}{x^2}\right\} \neq 0$$

and the estimate is said to be biased.

Let us define $A(\ )$ and $y(\ )$ in equation (2.4) for a discrete value of the argument by A_i and y^i, respectively. The components of the computed A_i and y^i can be broken into two parts

$$\left.\begin{aligned} A_i &= A_{t_i} + A_{\varepsilon_i} \\ y^i &= y_t^i + y_\varepsilon^i \end{aligned}\right\} \tag{2.10}$$

where A_{t_i} and y_t^i are defined as those portions of A_i and y^i for which the equality (2.4) holds with ε equal to zero,

$$y_t^i = A_{t_i}\gamma \tag{2.11}$$

If these definitions are used, then $\hat{\gamma}$ is the solution of

$$\left[\sum_{i=1}^{k}\left[A_{t_i} + A_{\varepsilon_i}\right]^T W \left[A_{t_i} + A_{\varepsilon_i}\right]\right]\hat{\gamma} = \sum_{i=1}^{k}\left[A_{t_i} + A_{\varepsilon_i}\right]^T W\left(y_t^i + y_\varepsilon^i\right) \tag{2.12}$$

and γ is the solution of

$$\left[\sum_{i=1}^{k} A_{t_i}^T W A_{t_i}\right]\gamma = \sum_{i=1}^{k} A_{t_i}^T W y_t^i \tag{2.13}$$

Set γ equal to $\hat{\gamma} + (\gamma - \hat{\gamma})$ in equation (2.13) and subtract equation (2.13) from (2.12) in order to obtain

$$\sum_{i=1}^{k} \left[A_{t_i}^T W A_{\varepsilon_i} + A_{\varepsilon_i}^T W A_{t_i} + A_{\varepsilon_i}^T W A_{\varepsilon_i}\right]\hat{\gamma} - \sum_{i=1}^{k} \left[A_{t_i}^T W A_{t_i}\right] [\gamma - \hat{\gamma}]$$

$$= \sum_{i=1}^{k} \left[A_{\varepsilon_i}^T W y_{t_i} + A_{\varepsilon_i}^T W y_{\varepsilon}^{i} + A_{t_i}^T W y_{\varepsilon}^{i}\right] \qquad (2.14)$$

The result implies that the error in the parameter estimates is given by

$$\delta\gamma_\varepsilon = [\gamma - \hat{\gamma}] = \left[\sum_{i=1}^{k} A_{t_i}^T W A_{t_i}\right]^{-1} \left[- \sum_{i=1}^{k} \left(A_{\varepsilon_i}^T W y_{t}^{i} + A_{\varepsilon_i}^T W y_{\varepsilon}^{i} + A_{t_i}^T W y_{\varepsilon}^{i}\right)\right.$$

$$\left. + \sum_{i=1}^{k} \left(A_{t_i}^T W A_{\varepsilon_i} + A_{\varepsilon_i}^T W A_{t_i} + A_{\varepsilon_i}^T W A_{\varepsilon_i}\right)\hat{\gamma}\right] \qquad (2.15)$$

The expected error is given by

$$E\{\delta\gamma_\varepsilon\} = \left[\sum_{i=1}^{k} A_{t_i}^T W A_{t_i}\right]^{-1} \left[- \sum_{i=1}^{k} E\left\{A_{\varepsilon_i}^T W y_{\varepsilon}^{i}\right\} + \sum_{i=1}^{k} E\left\{A_{\varepsilon_i}^T W A_{\varepsilon_i}\right\}\hat{\gamma}\right] \qquad (2.16)$$

where it has been assumed that y_ε^i and A_{ε_i} are zero mean and independent of A_{t_i} and y_t^i. It has also been assumed that A_{t_i} and y_t^i are deterministic quantities (although unknown). If there is any additive noise in the system, this expression is usually not equal to zero; therefore the parameter estimates are biased.

Comment: One well-known exception is the constant coefficient, linear, discrete problem with no numerator dynamics; with zero mean, independent, and gaussian process noise; and with no measurement noise. If the parameters are estimated by a procedure similar to the derivative method, the equivalent A_{ε_i} will be zero. This implies that the expected error in the parameter estimates is zero. These estimates are unbiased because the state at any instant of time is not dependent on the noise at that instant of time. This same idea can be extended to the analogous continuous system, if the integration algorithm is defined so that the state is not correlated with the noise at any instant of time.

2.2 RESPONSE CURVE FITTING METHODS

2.2.1 Formulation

Let us consider systems that are modeled by equations of the form

$$\left.\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u, \hat{\beta}, t) \qquad \hat{x}(0) = \hat{x}_o \\ \hat{y} &= h(\hat{x}, u, \hat{\beta}, t) \end{aligned}\right\} \tag{2.17}$$

where

$\hat{x}$ an $n \times 1$ state vector

u a $p \times 1$ input vector

$\hat{\beta}$ a vector of unknown parameters in f and h

$\hat{x}_o$ a vector of initial conditions, some of which may be unknown

$\hat{y}$ an $m \times 1$ model response vector

The response curve fitting methods are formulated by adjusting the parameters in $\hat{\beta}$ and the unknown initial conditions until the model response vector, $\hat{y}$, agrees, in some sense, with the measured response, y. The criterion often used to adjust the unknown parameters in the model is to minimize the function

$$J = \frac{1}{2} \sum_{i=1}^{k} [y(t_i) - \hat{y}(t_i)]^T W [y(t_i) - \hat{y}(t_i)] \tag{2.18}$$

in the discrete case, or the function

$$J = \frac{1}{2} \int_o^{t_f} [y(t) - \hat{y}(t)]^T W [y(t) - \hat{y}(t)] dt \tag{2.19}$$

in the continuous case. The positive definite weighting matrix W is used to express the relative confidence in the measurements.

Because the model response is generally a nonlinear function of the unknown parameters, equation (2.18) or (2.19) must be minimized by an iterative procedure. In this report, the method of quasi-linearization is used. An interesting relationship between the first-order gradient method, quasi-linearization, and the second-order Newton-Raphson method is illustrated in appendix A for this particular problem.

The basic idea behind the method of quasi-linearization is that the model response $\hat{y}$, which minimizes equation (2.18) or (2.19), can be approximated by a nominal response based on an initial estimate of the unknown parameters, plus a linearized correction about this nominal response (refs. 13, 15). This approximation is given by

$$\hat{y} \approx \hat{y}_A = y_N + \delta y \tag{2.20}$$

where

$$\left.\begin{aligned} \dot{x}_N &= f(x_N, u, \beta_N, t) \qquad x_N(0) = x_{N_o} \\ y_N &= h(x_N, u, \beta_N, t) \end{aligned}\right\} \tag{2.21}$$

and

$$\left.\begin{aligned} \delta\dot{x} &= \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_N \\ \beta=\beta_N}} \delta x + \left.\frac{\partial f}{\partial \beta}\right|_{\substack{x=x_N \\ \beta=\beta_N}} \delta\beta \qquad \delta x(0) = \delta x_o \\ \delta y &= \left.\frac{\partial h}{\partial x}\right|_{\substack{x=x_N \\ \beta=\beta_N}} \delta x + \left.\frac{\partial h}{\partial \beta}\right|_{\substack{x=x_N \\ \beta=\beta_N}} \delta\beta \end{aligned}\right\} \tag{2.22}$$

The subscript, N, refers to the initial estimate of the system parameters and the corresponding nominal response; $\hat{y}_A$ is the linearized approximation of $\hat{y}$ based on the initial estimate of the unknown parameters. Within this approximation δy is a linear function of the perturbations in the parameter $\hat{\beta}$ and unknown initial conditions. This is evident when δy is expressed in terms of the system transition matrix, $\Phi(t, \tau)$.

$$\delta y = \left[\left.\frac{\partial h}{\partial x}\right|_{\substack{x=x_N \\ \beta=\beta_N}} \int_0^t \Phi(t, \tau) \left.\frac{\partial f}{\partial \beta}\right|_{\substack{x=x_N \\ \beta=\beta_N}} d\tau + \left.\frac{\partial h}{\partial \beta}\right|_{\substack{x=x_N \\ \beta=\beta_N}} \right] \delta\beta + \left.\frac{\partial h}{\partial x}\right|_{\substack{x=x_N \\ \beta=\beta_N}} \Phi(t, 0)\delta x_o$$

If γ is a single vector containing both the unknown parameters in $\hat{\beta}$ and the unknown initial conditions, then δy can be expressed as

$$\delta y = A(t)\delta\gamma \tag{2.23}$$

The time histories in the matrix $A(t)$ are the numerical solutions of the differential equations

$$\dot{x}_{\gamma_i} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_N \\ \gamma=\gamma_N}} x_{\gamma_i} + \left.\frac{\partial f}{\partial \gamma_i}\right|_{\substack{x=x_N \\ \gamma=\gamma_N}} \qquad x_{\gamma_i}(0) = \frac{\partial \delta x_o}{\partial \gamma_i}$$

$$y_{\gamma_i} = \left.\frac{\partial h}{\partial x}\right|_{\substack{x=x_N \\ \gamma=\gamma_N}} x_{\gamma_i} + \left.\frac{\partial h}{\partial \gamma_i}\right|_{\substack{x=x_N \\ \gamma=\gamma_N}}$$

where γ_i is the ith parameter in the vector γ and $y_{\gamma_i}(t)$ is the ith column of $A(t)$.

If $\hat{y}_A$ is used in (2.18) or (2.19) in place of $\hat{y}$, the problem is reduced to the minimization of a quadratic form similar to that discussed in example 2.1. The estimate for $\delta\gamma$ is given by

$$\delta\hat{\gamma} = \left[\sum_{i=1}^{k} A^T(t_i)WA(t_i)\right]^{-1}\left[\sum_{i=1}^{k} A^T(t_i)W\big(y(t_i) - y_N(t_i)\big)\right] \tag{2.24}$$

or

$$\delta\hat{\gamma} = \left[\int_0^{t_f} A^TWA\,dt\right]^{-1}\left[\int_0^{t_f} A^TW(y - y_N)dt\right]$$

respectively. A new estimate of the parameters is obtained by correcting the initial estimates with the estimate of the error, $\delta\gamma$. In this way an iterative procedure is established for minimizing the function J. Kalaba (ref. 30) investigated various aspects of the convergence properties of this algorithm.

This procedure is applied to a nonlinear problem in Chapter VI.

Comment: Prior estimates of the unknown parameters can be incorporated in the identification by including this information in the cost function,

$$J = \frac{1}{2}(\hat{\gamma} - \gamma_p)^T\Lambda(\hat{\gamma} - \gamma_p) + \frac{1}{2}\int_0^{t_f}(y - \hat{y})^TW(y - \hat{y})dt$$

where Λ is a weighting which expresses the relative confidence in these prior estimates and γ_p is the prior estimate. The estimate for $\delta\gamma$ is given by

$$\delta\hat{\gamma} = \left[\Lambda + \int_0^{t_f} A^TWA\,dt\right]^{-1}\left[\Lambda(\gamma_p - \gamma_N) + \int_0^{t_f} A^TW(y - y_N)dt\right]$$

2.2.2 Effects of Noise (Maximum Likelihood Estimation)

The measured response can be considered as the summation of two components,

$$y = y_t + \varepsilon \tag{2.25}$$

where y_t is the system response caused by the known input $u(t)$ and initial conditions, and $\varepsilon(t)$ is the difference between the measured response and y_t. If the response curve fitting method has converged to a reasonably good estimate of the parameters, γ_N, the difference between the model response y_N and y_t can be approximated by the linearized equations,

$$y_t - y_N = A(t)\Big|_{\substack{x=x_N \\ \gamma=\gamma_N}} \delta\gamma \tag{2.26}$$

If equations (2.25) and (2.26) are used in (2.24), the estimate for $\delta\gamma$ is given by

$$\delta\hat{\gamma} = \left[\int_o^{t_f} A^TWA\ dt\right]^{-1} \left[\int_o^{t_f} A^TW(\varepsilon + A\delta\gamma)dt\right]$$

$$= \left[\int_o^{t_f} A^TWA\ dt\right]^{-1} \left[\int_o^{t_f} A^TW\varepsilon\ dt\right] + \delta\gamma \tag{2.27}$$

which implies that the error in the final parameter estimate is

$$\delta\gamma_\varepsilon = \delta\hat{\gamma} - \delta\gamma = \left[\int_o^{t_f} A^TWA\ dt\right]^{-1} \left[\int_o^{t_f} A^TW\varepsilon\ dt\right] \tag{2.28}$$

This linearized approximation is equivalent to assuming that the gradient of the model response, $A(t)$, is not affected by the errors in the parameter estimates and is therefore deterministic. The expected value of $\delta\gamma_\varepsilon$ is given by

$$E\{\delta\gamma_\varepsilon\} = \left[\int_o^{t_f} A^TWA\ dt\right]^{-1} \left[\int_o^{t_f} A^TWE\{\varepsilon\}dt\right] \tag{2.29}$$

which equals zero if $\varepsilon(t)$ is zero mean. This result does not depend on $\varepsilon(t)$ being white. Therefore, we can conclude that response curve fitting methods give unbiased estimates, to first order, whether there is process noise in the system or measurement noise. The above results do not, however, apply if there is noise in the measurements of the input u. This type of noise must be treated differently than process noise.

The variance of the errors in the parameter estimates is given by

$$E\{\delta\gamma_\varepsilon \delta\gamma_\varepsilon^T\} = \left[\int_0^{t_f} A^T WA\, dt\right]^{-1} \left[\int_0^{t_f}\int_0^{t_f} A^T(t) WE\{\varepsilon(t)\varepsilon^T(\tau)\} WA(\tau) dt\, d\tau\right] \left[\int_0^{t_f} A^T WA\, dt\right]^{-1} \tag{2.30}$$

An estimate of the variance can be computed since $\left[\int_0^{t_f} A^T WA\, dt\right]$ and $A(t)$ are computed during the identification and the $E\{\varepsilon(t)\varepsilon^T(\tau)\}$ can be estimated for ergodic processes by taking the autocorrelation of the residuals (the difference between the model response and the measurements). Under the special condition that the noise, $\varepsilon(t)$, is white and the weighting matrix is chosen so that

$$E\{\varepsilon(t)\varepsilon^T(\tau)\} = W^{-1}\delta(t - \tau) \tag{2.31}$$

equation (2.31) reduces to

$$E\{\delta\gamma_\varepsilon \delta\gamma_\varepsilon^T\} = \left[\int_0^{t_f} A^T WA\, dt\right]^{-1} \tag{2.32}$$

The results for the discrete problem are analogous and can be obtained by replacing the integrals in equations (2.27) to (2.32) by summations.

Maximum Likelihood Estimation: A maximum likelihood estimate for the parameters may be obtained if the probability density function for the measured response is known as a function of the unknown parameters.

If the probability density function is evaluated at the specific set of measurements, it becomes a function only of the unknown parameters and is called the likelihood function. The maximum likelihood estimate is the set of parameters that maximizes the likelihood function. The usual procedure for defining the likelihood function is to whiten the measured response by a causal and invertible transformation.

<u>Example 2.4 Maximum Likelihood Estimation in the Presence of Purely Random Gaussian Measurement Noise</u>

Consider a system described by the equations

$$\left.\begin{aligned} \dot{x} &= f(x,u,\beta,t) \qquad\qquad x(0) = x_o \\ y &= h(x,u,\beta,t) + \varepsilon(t) \end{aligned}\right\} \tag{2.33}$$

Let $y(t)$ be sampled at discrete times,

$$y(t_i) = h(x,u,\beta,t_i) + \varepsilon(t_i) \tag{2.34}$$

and assume that the joint probability density function of the sequence $\varepsilon(t_i)$, $i = 1, 2, \ldots$ is gaussian with correlation

$$E\{\varepsilon(t_i)\varepsilon^T(t_j)\} = R\delta_{ij} \tag{2.35}$$

If this system is modeled by equation (2.17) with $\hat{\beta} = \beta$ and $\hat{x}_o = x_o$, then the difference between y and $\hat{y}$ is equal to $\varepsilon(t)$. The difference between y and $\hat{y}$ will be denoted by $\nu(t)$,

$$y(t_i) - \hat{y}(t_i) = \nu(t_i) \tag{2.36}$$

where $\nu(t)$ is a function of $\hat{\gamma}$. If we evaluate the probability density function of $\varepsilon(t_i)$ using the sequence $\nu(t_i)$, the probability density function becomes a function of $\hat{\gamma}$ and is a likelihood function (L.F.) for the system,

$$\text{L.F.} = \left[e^{-(1/2)\sum_{i=1}^{k} \nu(t_i)^T R^{-1} \nu(t_i)} \right] \Big/ (2\pi)^{(km/2)} |R|^{(k/2)} \qquad (2.37)$$

Maximizing this function is equivalent to minimizing its logarithm or, in other words, the maximum likelihood estimate is obtained by the minimization of the function

$$J = \left(\frac{1}{2}\right) \left[k \log|R| + \sum_{i=1}^{k} \nu(t_i)^T R^{-1} \nu(t_i) \right] \qquad (2.38)$$

with respect to the unknown parameters in the constraint equations where

$$\nu(t_i) = y(t_i) - \hat{y}(t_i) \qquad (2.39)$$

and $\hat{y}(t_i)$ is given by equation (2.17). If R is known, the procedure is identical to the response curve fitting method. If R is unknown, it can be estimated iteratively by computing the mean square of the residuals.

Example 2.5 Maximum Likelihood Estimation in the Presence of Purely Random Gaussian Process Noise and Measurement Noise (Ref. 19)

Let us consider a system described on the equations

$$\left.\begin{aligned} \dot{x} &= Fx + Gu + v \qquad x(0) = x_o \\ y &= Hx + w \end{aligned}\right\} \qquad (2.40)$$

where v and w are zero mean, white noise, gaussian processes. Let $y(t)$ be sampled at discrete times,

$$y(t_i) = Hx(t_i) + w(t_i) \qquad (2.41)$$

and let the correlation of $w(t_i)$ be

$$E\{w(t_i)w^T(t_j)\} = R\delta_{ij} \qquad (2.42)$$

Let the correlation of $v(t)$ be

$$E\{v(t)v^T(\tau)\} = Q\delta(t - \tau) \tag{2.43}$$

The optimal filter for this system is given by (refs. 31 and 32):

$$\left.\begin{aligned}
&\dot{\hat{x}}(t/t_i) = F\hat{x}(t/t_i) + Gu(t) && \hat{x}(t_i/t_i) = \hat{x}(t_i/t_{i-1}) + K_i\varepsilon(t_i) \\
& && \hat{x}(0/-1) = \hat{x}_o \\
&\hat{y}(t_i/t_{i-1}) = H\hat{x}(t_i/t_{i-1}) && \\
&K_i = P(t_i/t_{i-1})H^T[HP(t_i/t_{i-1})H^T + R]^{-1} && \\
&\dot{P}(t/t_i) = FP(t/t_i) + P(t/t_i)F^T + GQG^T && P(t_i/t_i) = [I - K_iH]P(t_i/t_{i-1}) \\
& && P(0/-1) = P_o \\
&\varepsilon(t_i) = y(t_i) - \hat{y}(t_i/t_{i-1}) &&
\end{aligned}\right\} \tag{2.44}$$

It has been shown that the residuals, $\varepsilon(t_i)$, for the filter are gaussian and white (ref. 32). Therefore the probability density function for the residuals or innovations can be used to define a likelihood function.

Under the special assumption that the innovations are stationary, the equations for the optimal filter are simply

$$\left.\begin{aligned}
&\dot{\hat{x}}(t/t_i) = F\hat{x}(t/t_i) + Gu(t) && \hat{x}(t_i/t_i) = \hat{x}(t_i/t_{i-1}) + K\varepsilon(t_i) \\
& && \hat{x}(0/-1) = \hat{x}_o \\
&\hat{y}(t_i/t_{i-1}) = H\hat{x}(t_i/t_{i-1}) &&
\end{aligned}\right\} \tag{2.45}$$

and the parameters in K are constant. If we use these equations to model the unknown system, and define the difference between y and $\hat{y}$ as ν, $\nu(t_i) = y(t_i) - \hat{y}(t_i/t_{i-1})$, then the likelihood function for the system can be obtained by evaluating the probability density function of the innovations at the sequence of residuals, $\nu(t_i)$. The $\nu(t_i)$ are a function of the unknown system parameters as well as the filter gains, K, and the maximum likelihood estimate is obtained by minimizing the quantity

$$J = \left(\frac{1}{2}\right)\left[k \log|B| + \sum_{i=1}^{k} \nu(t_i)^T B^{-1} \nu(t_i)\right] \tag{2.46}$$

with respect to the unknown parameters in the constraint equations, (2.45). The matrix B in equation (2.46) is the covariance of the residuals.

The parameter estimates obtained by this procedure are consistent and asymptotically efficient. As the amount of data increases, the statistics of the errors in the parameter estimates approach

$$E\{\delta\gamma_\varepsilon\} = 0 \tag{2.47}$$

$$E\{\delta\gamma_\varepsilon \delta\gamma_\varepsilon^T\} = \left[\sum_{i=1}^{k} A^T(t_i) B^{-1} A(t_i)\right]^{-1} \tag{2.48}$$

where $A(t_i)$ is the gradient of the model response with respect to the unknown parameters. For a rigorous discussion on these properties, the reader should see references 20, 33, and 34.

The above results do not apply if there is noise in the measurements of the input u. This type of noise is not the same as process noise and its presence will cause a bias in the parameter estimates if the above procedure is used.

III STATIONARY LINEAR SYSTEMS

3.1 BACKGROUND

For many dynamic systems the relationship between input u, and output y is well described by a set of first-order constant-coefficient linear differential equations of the form[3]

$$\left.\begin{aligned} \dot{z} &= Az + Bu \qquad & z &= n \times 1,\ u = p \times 1 \\ y &= Cz \qquad & y &= m \times 1 \end{aligned}\right\} \tag{3.1}$$

A particular set of equations that relates the system input to its output with desired accuracy is called a realization for the system. A minimal realization is a realization of minimal order. Kalman has shown that a minimal realization is both controllable and observable (ref. 35). This property will be used extensively in the following discussion.

The minimal realization depends on the specific input to the system as well as on the structure of the system. For example, if the system (eq. (3.1)) is excited by a single sine wave, then the minimal realization for the steady-state response would be a first-order system. In another example, certain modes of a system may not be noticeably excited by a given input. The minimal realization would include only those modes that were excited and observed.

Even if we restrict our attention to minimal realizations, there are many choices of parameters in the matrices A, B, and C that give the

[3]When these equations are used, y(t) does not respond instantaneously to a step input in u(t). It is sometimes convenient to approximate a physical process by one that does respond instantaneously to a step change in the input. For example, the response of an accelerometer is often so fast that the sensor dynamics are negligible. The ideas presented here can be extended to include these situations.

same output for a given input. This is easily shown by introducing any nonsingular transformation of the state vector,

$$x = Tz \tag{3.2}$$

The input and output of the system can then be related by the equations

$$\left.\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \end{aligned}\right\} \tag{3.3}$$

where

$$\left.\begin{aligned} F &= TAT^{-1} \\ G &= TB \\ H &= CT^{-1} \end{aligned}\right\} \tag{3.4}$$

The system (3.3) is said to be equivalent to (3.1). Note that the matrices F, G, and H contain $n(n + m + p)$ parameters. Although the choice of F, G, and H that can be used in equations (3.3) to relate the system input to the system output is not unique, the transfer functions between $u(s)$ and $y(s)$ are unique (where $u(s)$ and $y(s)$ are the Laplace transforms of $u(t)$ and $y(t)$, respectively). If zero initial conditions are assumed, the transfer functions are given by

$$y(s) = C[Is - A]^{-1}Bu(s) = H[Is - F]^{-1}Gu(s) \tag{3.5}$$

There are mp individual transfer functions in equation (3.5) which would seem to imply that there might be as many as nmp numerator coefficients and n denominator coefficients. Although uniquely specified by the input-output measurements, all these coefficients are not independent. The computations for the transfer functions associated with different inputs are identical except for n parameters in the column of the B matrix associated with the different inputs. Consequently, the input and output can be related by a maximum of $n(m + p)$ independent parameters (i.e., $n(m + 1)$ for the first input and n additional parameters for each additional input). Since there are $n(m + p + n)$ parameters in F, G, and H, the above argument suggests that n^2 of these parameters might be specified and the remaining free parameters used to relate the system input to the output. The canonical form presented in this chapter contains a maximum of $n(m + p)$ parameters all of which are uniquely defined by the input-output behavior of the system. We will illustrate these ideas for a single-input single-output system prior to developing the canonical form for the general multivariable system.

<u>Example 3.1 Single-Input, Single-Output, Second-Order System</u>

Consider the single-input, single-output, second-order system given by

$$\left.\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix} u \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = 0 \\ y &= \begin{bmatrix} h_{11} & h_{21} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}\right\} \tag{3.6}$$

The Laplace transform of equation (3.6) is

$$\frac{y(s)}{u(s)} = \frac{d_1 s + d_o}{s^2 + c_1 s + c_o} \tag{3.7}$$

where

$$d_1 = g_{11}h_{11} + g_{21}h_{12}$$

$$d_0 = g_{11}\left[-h_{11}f_{22} + h_{12}f_{21}\right] + g_{21}\left[-h_{12}f_{11} + h_{11}f_{12}\right]$$

$$c_1 = \left[-f_{11} - f_{22}\right]$$

$$c_0 = \left[f_{22}f_{11} - f_{12}f_{21}\right]$$

Since $y(s)$ is completely specified by $u(s)$ and the four coefficients d_1, d_0, c_1, and c_0, it is clear that the eight parameters in F, G, and H are not uniquely defined. In fact, four of the parameters in F, G, and H can be determined in terms of the other four. By choosing four parameters we constrain the structure of F, G, and H so that the four remaining parameters are uniquely defined by the input and output relationships. One way of constraining the structure of F, G, and H is to set $h_{11} = 1$, $h_{12} = 0$, $f_{12} = 1$, and $f_{22} = 0$. The four remaining parameters in F, G, and H are then uniquely defined by the relationships

$$d_1 = g_{11}, \qquad d_0 = g_{21}, \qquad c_1 = -f_{11}, \qquad c_0 = -f_{21} \tag{3.8}$$

This particular choice of F, G, and H corresponds to a well-known canonical form for single-output systems. Other canonical forms can be used to represent this system but equation (3.8) is particularly well suited to the identification algorithm presented in this study.

3.2 A CANONICAL FORM FOR MULTIOUTPUT SYSTEMS

To the author's knowledge, none of the multivariable canonical forms currently available define a set of uniquely identifiable parameters and at the same time are suitable for use with the identification algorithm presented here. A canonical form which meets both of these criteria is presented in this section. It is analogous to a canonical form developed by Luenberger for multi-input systems.

In order to write our canonical form for the unknown system it is necessary to determine the first n linearly independent rows of the observability matrix for the system. If the system is described by a set of equations of the form (3.1), the observability matrix for the system is given by the matrix

$$O_b = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} \qquad (3.9)$$

If (3.1) is a minimal realization having order n, then there are n independent rows in O_b. Since the parameters in C and A are not known, however, it is not always clear how to determine the first n linearly independent rows in this matrix. If this information is not known, all possibilities should be considered. This procedure introduces additional uncertainty into the identification, and the combination of rows, which results in a model giving the "best fit" of the data, should be selected as the estimate of the system. In many applications, particularly in the identification of the parameters in the linearized equations of motion for an aircraft, the linear independence of the rows in the matrix O_b can be determined with a high degree of certainty on the basis of the dynamics of the problem without knowing the actual numerical values of the parameters. In the remainder of this report we will assume that this information, which will be referred to as the structure of the system, is known.

If the structure of the system is known, then the canonical form for the system is given by

$$\left.\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \end{aligned}\right\} \qquad (3.10)$$

where F and H are given in figure 1. There are no simplifications in the control coefficient matrix G and therefore this matrix has not been written out in detail. The numbers $p_1, p_2, \ldots, p_m$ in

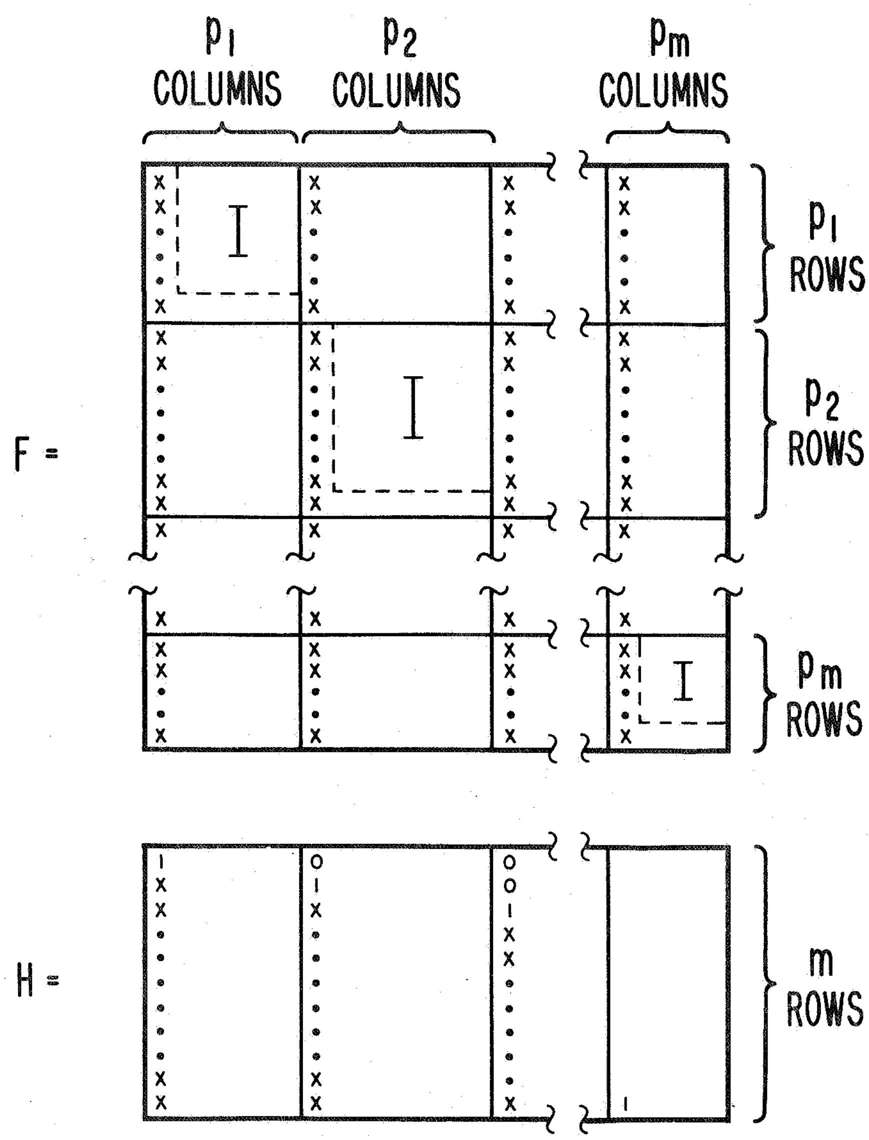

Figure 1.- General canonical structure.

figure 1 are equal to the number of rows in the first n linear independent rows of the observability matrix (3.9), that involve a multiplication by the first, second, . . ., and mth rows, respectively, of the matrix C. The symbol I in figure 1 is the identity matrix, the blank areas are all zeros, and the x's indicate nonzero elements.

If the unknown system is modeled by equation (3.10) where F and H are given in figure 1, the undefined parameters denoted by x are still not uniquely identifiable. It is shown in assertions 3.1 and 3.2 at the end of this section that some additional parameters in figure 1 can be set equal to zero by the relationships

if $p_i < p_j - k$ then

$$f_{s_j+k,s_i} = 0 \qquad k = 0, 1, \ldots, p_j - p_i - 1 \tag{3.11}$$

if $p_i \leqslant p_j$, $i \neq j$ then

$$h_{j,s_i} = 0 \tag{3.12}$$

where $f_{i,j}$ and $h_{i,j}$ are elements in F and H, respectively, and the subscript s_i is defined

$$s_i = \begin{cases} 1 & i = 1 \\ 1 + \sum_{j=1}^{i-1} p_j & i = 2, 3, \ldots \end{cases} \tag{3.13}$$

If (3.11) and (3.12) are used to set the corresponding parameters in figure 1, equal to zero, then a maximum of $n(m + p)$ parameters remain and can be uniquely identified from the measured data. These ideas are illustrated in the following example.

Example 3.2 A Fourth-Order System With Two Ouputs

Consider a system, (3.1), where A is a 4×4 matrix and C is a 2×4 matrix. The observability matrix is

$$
O_b = \begin{bmatrix} c_{(1)} \\ c_{(2)} \\ c_{(1)}A \\ c_{(2)}A \\ c_{(1)}A^2 \\ c_{(2)}A^2 \end{bmatrix}
$$

where $c_{(i)}$ is the ith row of C. Let the system be observable but assume that the fourth row of O_b, $c_{(2)}A$, is linearly dependent on $c_{(1)}$, $c_{(2)}$, and $c_{(1)}A$. The first four linearly independent rows of the observability matrix are then

$$
\left.\begin{matrix} c_{(1)} \\ c_{(2)} \\ c_{(1)}A \\ c_{(1)}A^2 \end{matrix}\right\} \tag{3.14}
$$

which implies that $p_1 = 3$ and $p_2 = 1$. Since $p_2 < p_1 - k$ for $k = 0$, 1, (3.11) implies that $f_{1,4} = f_{2,4} = 0$. Expression (3.12) gives no additional information about the parameters in H. The canonical form for the system is therefore given by the equations

$$
\left.\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \end{aligned}\right\} \tag{3.15}
$$

where

$$F = \begin{bmatrix} f_{11} & 1 & 0 & 0 \\ f_{21} & 0 & 1 & 0 \\ f_{31} & 0 & 0 & f_{34} \\ f_{41} & 0 & 0 & f_{44} \end{bmatrix} \quad G = \begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \\ g_{41} \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ h_{21} & 0 & 0 & 1 \end{bmatrix}$$

In many applications (all that we have considered), it is possible to order the measurements, y, so that the first n rows of O_b are linearly independent. The canonical form for this case is examined in detail because of its frequency of applicability. If r and q are defined as the remainder and quotient of n/m, respectively, then $p_i = q + 1$ for $i \leqslant r$, $p_i = q$ for $i > r$, and the canonical form for F and H is given in figure 2. As in figure 1, the F matrix has been partitioned into m^2 submatrices and H has been partitioned into m submatrices. Expression (3.11) implies that the parameter in the upper left corner of each submatrix in F having the dimension $q + 1 \times q$ is equal to zero and (3.12) implies that the H matrix is reduced to all 1's and 0's except for the last $m - r$ parameters in the first column of each submatrix having the dimension $m \times q + 1$.

If the state vector is an even multiple of the measurements and if the first n rows of the observability matrix are linearly independent, the parameters in the observation matrix H reduce to all zeros and ones.

Example 3.3 A Fourth-Order System With Two Outputs

Consider the system used in example 3.2, except that the first n rows of the observability matrix, O_b,

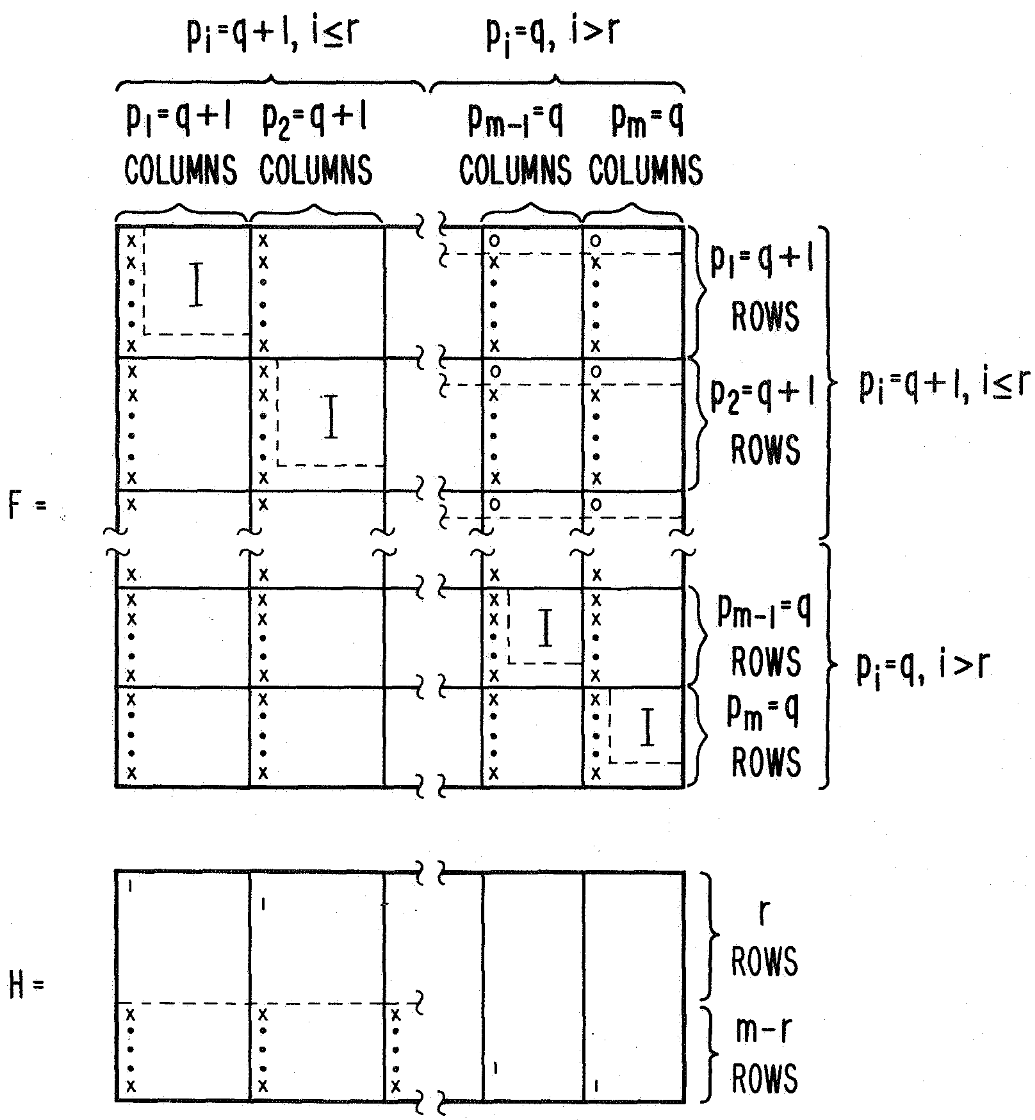

Figure 2.- Special case canonical structure.

$$\left.\begin{array}{l} c_{(1)} \\ c_{(2)} \\ c_{(1)}A \\ c_{(2)}A \end{array}\right\} \tag{3.16}$$

are linearly independent. Then $q = 2$, $r = 0$, and the input and output can be related by a realization having the form

$$F = \begin{bmatrix} f_{11} & 1 & f_{13} & 0 \\ f_{21} & 0 & f_{23} & 0 \\ f_{31} & 0 & f_{33} & 1 \\ f_{41} & 0 & f_{43} & 0 \end{bmatrix} \quad G = \begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \\ g_{41} \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.17}$$

The transformation that puts a system into its canonical form is constructed as follows: Arrange the first n linearly independent rows of the observability matrix (3.9) to form a nonsingular matrix P,

$$P = \begin{bmatrix} c_{(1)} \\ c_{(1)}A \\ \cdot \\ \cdot \\ \cdot \\ c_{(1)}A^{p_1-1} \\ c_{(2)} \\ \cdot \\ \cdot \\ \cdot \\ c_{(m)}A^{p_m-1} \end{bmatrix} \tag{3.18}$$

where p_i is the number of rows in this linearly independent set involving a multiplication by the ith row of C. Define $q^{(j)}$ as the jth column of $P^{-1} \triangleq Q$. The inverse of the canonical transformation matrix can then be constructed,

$$T^{-1} = \left[A^{(p_1-1)}q^{(l_1)}, \ldots, q^{(l_1)}, \ldots, A^{(p_m-1)}q^{(l_m)}, \ldots, q^{(l_m)}\right] \tag{3.19}$$

where l_i is defined by

$$l_i = \sum_{j=1}^{i} p_j$$

The remainder of this section illustrates that the assertions concerning the structure of the canonical form are correct.

Assertion 3.1 If a system is transformed according to expressions (3.2) to (3.4) where T^{-1} is constructed as in equation (3.19), then H will have the form given in figure 1 and if $p_i \leqslant p_j$ then $h_{j,s_i} = \delta_{ij}$.

Proof: The canonical form for the observation matrix is computed by means of the equation $H = CT^{-1}$ or

$$H = C\left[A^{(p_1-1)}q^{(l_1)}, \ldots, q^{(l_1)}, A^{(p_2-1)}q^{(l_2)}, \ldots, q^{(l_m)}\right] \tag{3.20}$$

If the jth row of the matrix C is denoted by $c_{(j)}$, the elements in H are computed by the matrix products

$$h_{j,l_i-k} = c_{(j)}A^k q^{(l_i)} \tag{3.21}$$

where $k \leqslant p_i - 1$. If $k < p_i - 1$ or ($k = p_i - 1$ and $j < i$) then the vector $c_{(j)}A^k$ is orthogonal to $q^{(l_i)}$ by the way the $q^{(l_i)}$ were chosen. This is illustrated by the following argument. The vector $c_{(j)}A^k$ is contained in the set of vectors denoted by the rows of the matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ c_{(1)}A^{p_i-1} \\ \vdots \\ c_{(j)}A^{p_i-1} \end{bmatrix} \tag{3.22}$$

Since the vector $c_{(i)}A^{p_i-1}$ is linearly independent of these rows and is the l_i row of the matrix P, the vector $c_{(j)}A^k$ can be expressed as a linear sum of the rows of P, excluding the l_i row,

$$c_{(j)}A^k = \sum_{s=1}^{n} \alpha_s p_{(s)} \qquad s \neq l_i \tag{3.23}$$

Taking the inner product of both sides of this equation with $q^{(l_i)}$ we obtain

$$c_{(j)}A^k q^{(l_i)} = \sum_{s=1}^{n} \alpha_s p_{(s)} q^{(l_i)} = \sum_{s=1}^{n} \alpha_s \delta_{s,l_i} = 0 \tag{3.24}$$

because $s \neq l_i$. If these elements are set equal to zero, H reduces to the form given in figure 1. If $p_i \leqslant p_j$ then $c_{(j)}A^{p_i-1}q^{(l_i)} = \delta_{ij}$ by the way the $q^{(l_i)}$ were chosen. This implies that $h_{j,l_i-p_i+1} = \delta_{i,j}$. But $l_i - p_i + 1 = s_i$ which implies that $h_{j,s_i} = \delta_{i,j}$.

Assertion 3.2.2 If a system is tranformed according to equations (3.2) to (3.4) where T^{-1} is constructed as shown in (3.19), then F will have the form given in figure 1 and if $p_i < p_j - k$ then $f_{s_j+k,s_i} = 0$ for $k = 0, 1, \ldots, p_j - p_i - 1$.

Proof: The canonical form for the state coefficient matrix is computed by the equation

$$F = TAT^{-1} \tag{3.25}$$

It is convenient to consider A as a linear transformation, σ, from a vector space U into itself with respect to some basis, $e_1, e_2, \ldots, e_n$. The elements in the ith column of A are the components of the transformed ith basis vector. If a new basis, $e_1', e_2', \ldots, e_n'$, is generated whose components are given in terms of the original basis by the columns of T^{-1}, the new basis vectors are related to each other by the transformation itself,

$$\sigma(e_i') = e_{i-1}' \qquad i \neq s_j,\ j = 1, 2, \ldots, m \tag{3.26}$$

Because the columns of the matrix contain the components of the transformed basis vector, the matrix F takes the form given in figure 1. The columns, excluding the s_i columns, contain all zeros except for a one on the superdiagonal.

Let us now consider the s_i columns of the state coefficient matrix. Again, using the fact that the columns of the matrix contain the components of the transformed basis vector, we can write

$$\sigma(e_{s_i}') = \sum_{j=1}^{n} f_{j,s_i} e_j' \tag{3.27}$$

In terms of the original basis, this implies that

$$A\left(A^{p_i-1} q^{(l_i)}\right) = A^{p_i} q^{(l_i)} = \sum_{j=1}^{m} \sum_{d=0}^{p_j-1} f_{s_j+d,s_i} A^{p_j-1-d} q^{(l_j)} \tag{3.28}$$

Associate with each row vector, $c_{(j)}A^d$, a number, $p_j - d$, where $j = 1, 2, \ldots, m$ and $d = 0, 1, 2, \ldots$. If there is a j so that $p_j - d > p_i$, take the inner products of both sides of equation (3.28) with the row vectors, $c_{(k)}A^b$, which maximize $p_k - b$ (i.e., $b = 0$ and k is such that $p_k = \max\{p_l:\ l = 1, \ldots, m\}$).

$$c_{(k)}A^{p_i}q^{(l_i)} = \sum_{j=1}^{m}\sum_{d=0}^{p_j-1} f_{s_j+d,s_i}\, c_{(k)}A^{p_j-1-d}q^{(l_j)} \qquad (3.29)$$

Note that because of the way the $q^{(l_j)}$ were chosen and because $p_k > p_i$,

$$\left.\begin{aligned} c_{(k)}A^{p_i}q^{(l_i)} &= 0 & & \\ c_{(k)}A^{p_j-1-d}q^{(l_j)} &= 0 & \quad p_j - 1 - d &< p_k - 1 \\ c_{(k)}A^{p_j-1-d}q^{(l_j)} &= \delta_{k,j} & \quad p_j - 1 - d &= p_k - 1 \end{aligned}\right\} \qquad (3.30)$$

In addition, note that $p_j - 1 - d$ cannot be greater than $p_k - 1$ because of the way the $c_{(k)}$ were chosen. Equation (3.29) therefore reduces to

$$0 = f_{s_k,s_i} \qquad (3.31)$$

This result can be stated as follows:

<u>Result 3.2.1</u>: If

$$\beta_{max} \triangleq \max_{\substack{j=1,2,\ldots,m \\ d=0,1,\ldots}}\{p_j - d\}$$

and if

$$p_k - b = \beta_{max}$$

then

$$f_{s_k+b,s_i} = 0$$

Assume $f_{s_j+d,s_i} = 0$ for all j and d such that $p_j - d > \beta > p_i$ for some β where β is a real number (if $\beta_{max} - 1 > p_i$ then

$\beta = \beta_{max} - 1$ is such a number). Take the inner products of equation (3.28) with all row vectors of the form $c_{(k)}A^b$ where $p_k - b = \beta$,

$$c_{(k)}A^{p_i+b}q^{(l_i)} = \sum_{j=1}^{m}\sum_{d=0}^{p_j-1} f_{s_j+d}A^{p_j-1-d+b}q^{(l_j)} \tag{3.32}$$

Note that because of the way the $q^{(l_j)}$ were chosen and because $p_k > p_i + b$,

$$\left.\begin{aligned} c_{(k)}A^{p_i+b}q^{(l_i)} &= 0 & & \\ c_{(k)}A^{p_j-1-d+b}q^{(l_j)} &= 0 & \quad p_j - 1 - d + b &< p_k - 1 \\ c_{(k)}A^{p_j-1-d+b}q^{(l_j)} &= \delta_{k,j} & \quad p_j - 1 - d + b &= p_k - 1 \end{aligned}\right\} \tag{3.33}$$

The only remaining terms in equation (3.32) are those for which $p_j - 1 - d + b > p_k - 1$. However, if $p_j - 1 - d + b > p_k - 1$ then $p_j - 1 - d > p_k - b - 1 = \beta - 1$ which implies that $p_j - d > \beta$ which implies by hypothesis that the coefficients of these terms equal zero. Equation (3.32) therefore reduces to

$$0 = f_{s_k+b,s_i} \tag{3.34}$$

and we establish the following result:

Result 3.2.2 If $f_{s_j+d,s_i} = 0$ for all j and d such that $p_j - d > \beta > p_i$ for some β, then $f_{s_j+d,s_i} = 0$ for all j and d such that $p_j - d = \beta > p_i$. Results 1 and 2 can be used to deduce by induction the original assertion that $f_{s_j+d,s_i} = 0$ for $d = 0, 1, \ldots, p_j - p_i - 1$.

3.3 IDENTIFIABILITY OF THE PARAMETERS IN THE CANONICAL FORM

If the minimal realization of a linear system is described by a set of equations of the form given in equations (3.10) through (3.13) and figure 1, then the undefined parameters in those equations are uniquely determined by measurements of the input and output. This assertion is proven in two parts: First, it is shown that the canonical realization for a system is unique (i.e., the canonical realizations of any two equivalent minimal realizations are identical). Second, it is shown that equations (3.10) through (3.13) and figure 1 are in the canonical form since the canonical transformation for these equations is the identity.

3.3.1 Uniqueness of the Canonical Realization

Consider two equivalent but different minimal realizations of a linear system

$$\left.\begin{aligned} \dot{z}^1 &= A_1 z^1 + B_1 u \\ y &= C_1 z^1 \end{aligned}\right\} \tag{3.35}$$

$$\left.\begin{aligned} \dot{z}^2 &= A_2 z^2 + B_2 u \\ y &= C_2 z^2 \end{aligned}\right\} \tag{3.36}$$

It is shown in reference 27 that under these conditions the states of the two systems are related by a nonsingular transformation

$$z^2 = Uz^1 \tag{3.37}$$

If the relations for equivalent systems presented in equations (3.2) through (3.4) are used, it is easy to see that the canonical transformations for these two realizations are related by

$$T_2^{-1} = UT_1^{-1} \tag{3.38}$$

where the subscripts 1 and 2 are used to distinguish between the canonical transformation of system 1 and system 2, respectively. The canonical realization of system 2 is related to the canonical realization of system 1 by

$$\left.\begin{aligned} F_2 &= T_2A_2T_2^{-1} = T_1U^{-1}UA_1U^{-1}UT^{-1} = F_1 \\ H_2 &= C_2T_2^{-1} = C_1U^{-1}UT^{-1} = H_1 \\ G_2 &= T_2B_2 = T_1U^{-1}UB_1 = G_1 \end{aligned}\right\} \tag{3.39}$$

and hence they are equal.

3.3.2 The Canonical Transformation for Equations (3.10) Through (3.13) is the Identity

It is shown in assertion 3.3.1 at the end of this section that the columns $p^{(l_1)}, p^{(l_2)}, \ldots, p^{(l_m)}$ of the P matrix contain all zeros except for a one on the main diagonal. This implies that the columns $q^{(l_1)}, q^{(l_2)}, \ldots, q^{(l_m)}$ of the Q matrix (the inverse of P) also contain all zeros except for a one on the main diagonal. It is then easy to see that the resulting canonical transformation T^{-1} is the identity. The difficult part of this derivation is to show that the columns, $p^{(l_i)}$, are of the asserted form. To facilitate the proof of this assertion, we will first prove results 3.3.1 through 3.3.5.

<u>Definition</u>: The element in the *ith* row and *jth* column of the matrix F^k will be denoted by $f^k_{i,j}$.

<u>Result 3.3.1</u>: If $1 \leqslant k \leqslant p_j - 1$ then $f^k_{i,l_j} = \delta_{i,l_j-k}$ and if $k = p_j$ then $f^{p_j}_{i,l_j} = f_{i,s_j}$.

<u>Proof</u>: If $p_j \geqslant 2$ then the parameters in the l_j column of F are given by

$$f_{i,l_j} = \delta_{i,l_j-1} \qquad \text{(see fig. 1)}$$

This implies that the parameters in the l_j column of F^2 are the parameters in the $(l_j - 1)$ column of F,

$$f^2_{i,l_j} = f_{i,l_j-2+1}$$

In general, if $1 \leqslant k \leqslant p_j$ then

$$f^k_{i,l_j} = f_{i,l_j-k+1}$$

which implies

$$f^k_{i,l_j} = \delta_{i,l_j-k} \qquad k \leqslant p_j - 1$$

$$f^{p_j}_{i,l_j} = f_{i,l_j-p_j+1} = f_{i,s_j}$$

Result 3.3.2: If $k = p_j \leqslant p_i - 1$ then $f^{p_j+e}_{s_i+d,l_j} = 0$ for $d = 0, 1, \ldots, p_i - p_j - 1 - e$ where $e = 0$.

Proof: If $k = p_j \leqslant p_i - 1$ then by result 3.3.1 $f^{p_j}_{i,l_j} = f_{i,s_j}$ which implies by assertion 3.2.2 that $f^{p_j}_{s_i+d,l_j} = 0$ for $d = 0, 1, \ldots, p_i - p_j - 1$. This can be restated as $f^{p_j+e}_{s_i+d,l_j} = 0$ for $d = 0, 1, \ldots, p_i - p_j - 1 - e$ where $e = 0$.

Result 3.3.3: If $f^{p_j+e}_{s_i+d,l_j} = 0$ for $d = 0, 1, \ldots, p_i - p_j - 1 - e$ and if $0 \leqslant q \leqslant p_i - p_j - 1 - (e+1)$, then $f^{p_j+e+1}_{s_i+q,l_j} = 0$ for $q = 0, 1, 2, \ldots, p_i - p_j - 1 - (e+1)$.

Proof:

$$f^{p_j+e+1}_{s_i+q,l_j} = \sum_{p=1}^{m} f_{s_i+q,p} f^{p_j+e}_{p,l_j} = \sum_{p=1}^{m} f_{s_i+q,s_p} f^{p_j+e}_{s_p,l_j} + f^{p_j+e}_{s_i+q+1,l_j}$$

(see fig. 1)

We will now show that each term on the right is zero. Let us first consider the first term. Either $p_i - q > p_p$ or $p_i - q \leqslant p_p$. If $p_i - q > p_p$ then $f_{s_i+q,s_p} = 0$ by assertion 3.2.2. If $p_i - q \leqslant p_p$ then $p_i - (p_i - p_j - 1 - (e+1)) \leqslant p_p \Rightarrow p_p - p_j - 1 - e - 1 \geqslant 0 \Rightarrow p_p - p_j - 1 - e > 0$

which implies $f^{p_j+e}_{s_p+d,l_j} = 0$ by hypothesis. The first term on the right of the equation is therefore equal to zero. The second term is equal to zero because

$$0 \leqslant q \leqslant p_i - p_j - 1 - (e+1) \Rightarrow 0 \leqslant q+1 \leqslant p_i - p_j - 1 - e \Rightarrow f^{p_j+e}_{s_p+q+1,l_j} = 0$$

by hypothesis.

Result 3.3.4: If $0 \leqslant d \leqslant (p_i - p_j - 1 - e)$ then

$$f^{p_j+e}_{s_i+d,l_j} = 0$$

Proof: It was shown in result 3.3.2 that this is true for $e = 0$. Then if $p_i - p_j - 1 - e \geqslant 0$ for $e = 1$, result 3.3.3 can be used to show that this result is true for $e = 1$. We can therefore proceed by introduction to establish the above result.

Result 3.3.5: If $1 \leqslant k \leqslant \max[p_i - 1, p_j - 1]$ then $f^k_{s_i,l_j} = 0$ except $f^{p_j-1}_{s_j,l_j}$ which equals 1.

Proof: Either $p_i \leqslant p_j$ or $p_i > p_j$. If $p_i \leqslant p_j$, then $1 \leqslant k \leqslant \max[p_i - 1, p_j - 1] \Rightarrow 1 \leqslant k \leqslant p_j - 1$ which implies by result 3.3.1 $f^k_{i,l_j} = \delta_{i,l_j-k}$ except if $k = p_j$ then $f^k_{i,l_j} = f_{i,s_j} \Rightarrow f^k_{s_i,l_j} = 0$ except $f^{p_j}_{s_j,l_j} = 1$ and the argument is completed. If $p_i > p_j$ then either $1 \leqslant k \leqslant p_j - 1$ or $p_j \leqslant k \leqslant p_i - 1$. If $1 \leqslant k \leqslant p_j - 1$ then the above reasoning completes the argument. If $p_j \leqslant k \leqslant p_i - 1$ then $0 \leqslant k - p_j \leqslant p_i - p_j - 1$ which implies $0 \leqslant p_i - p_j - 1 - (k - p_j)$. If $(k - p_j)$ is denoted by e then $f^{p_j+(k-p_j)}_{s_i+d,l_j} = f^k_{s_i+d,l_j} = 0$ by result 3.3.4 and the argument is completed.

Assertion 3.3.1 The parameters in the l_j column of the matrix P are given by

$$p_{i,l_j} = \delta_{i,l_j} \tag{3.40}$$

Proof: The parameters in the l_j column of P are computed by

$$p_{s_i+k,l_j} = \sum_{r=1}^{n} h_{i,r} f^k_{r,l_j} = \sum_{d=1}^{m} h_{i,s_d} f^k_{s_d,l_j} \tag{3.41}$$

where

$$k \leqslant p_i - 1 \tag{3.42}$$

p_d has to be less than p_i, equal to p_i, or greater than p_i. If $p_d < p_i$, then $h_{i,s_d} = 0$ by assertion 3.2.1. If $p_d = p_i$ then $h_{i,s_d} = \delta_{id}$ by assertion 3.2.1 and $p_d - 1 \geqslant k$ by (3.42) which implies by result 3.3.1 that $f^k_{s_d,l_j} = 0$ except $f^{p_j-1}_{s_j,l_j}$ which equals 1. If $p_d > p_i$ then $p_d - 1 > k$ by (3.42) which implies by result 3.3.1 that $f^k_{s_d,l_j} = 0$. If these results are used in (3.41), we obtain

$$p_{s_i+k,l_j} = f^k_{s_i,l_j} = 0 \quad \text{except for } i = j \text{ and } k = p_j - 1$$

in which case

$$p_{s_j+(p_j-1),l_j} = p_{l_j,l_j} = 1$$

This, therefore, implies that

$$p_{i,l_j} = \delta_{i,l_j}$$

and this completes the argument.

IV A NEW COMBINED ALGORITHM FOR ESTIMATING SYSTEM PARAMETERS FROM INPUT-OUTPUT DATA

4.1 STATEMENT OF THE PROBLEM

As stated in Chapter I, the problem is to minimize the function[4]

$$J = \int_0^{t_f} (y(t) - \hat{y}(t))^T W (y(t) - \hat{y}(t)) dt \tag{4.1}$$

with respect to the unknown parameters in F, G, H and x_o of the constraint equations

$$\left. \begin{array}{ll} \dot{\hat{x}} = F\hat{x} + Gu & \hat{x}(0) = x_o \\ \hat{y} = H\hat{x} & \end{array} \right\} \tag{4.2}$$

where $y(t)$ is the measured system response and $u(t)$ is the measured input. The main difficulty is that the model response $\hat{y}$ is a non-linear function of the unknown parameters in F and H. However, if the measurement errors (portions of measurements which are not correlated with u) are negligibly small, this problem can be formulated as a linear problem. The linear formulation can be used to obtain an initial estimate of the unknown parameters and this estimate can be used to initiate the iterative solution to the nonlinear problem. The linear formulation corresponds to an equations of motion method and the non-linear problem corresponds to a response curve fitting method.

4.2 THE EQUATIONS OF MOTION METHOD

For a perfect model and in the absense of noise, the output of (4.2) will equal the measurements exactly; therefore, the difference,

---

[4] In the case of discrete measurements, the problem is to minimize $J = \sum_{i=0}^{N} (y(t_i) - \hat{y}(t_i)) W (y(t_i) - \hat{y}(t_i))$ where $y(t_i)$ is the measurement of the system response at discrete times t_i.

$y(t) - \hat{y}(t)$, equals zero. Under these conditions this difference can be fed back to the model through arbitrary gains K and L without changing the model response $\hat{y}$. The equations for the model with this error feedback are

$$\left.\begin{aligned} \dot{\hat{x}} &= F\hat{x} + Gu + K[y - H\hat{x}] \qquad \hat{x}(0) = x_o \\ \hat{y} &= H\hat{x} + L[y - H\hat{x}] \end{aligned}\right\} \tag{4.3}$$

which when terms are combined can be rewritten

$$\left.\begin{aligned} \dot{\hat{x}} &= (F - KH)\hat{x} + Gu + Ky \qquad \hat{x}(0) = x_o \\ \hat{y} &= (I - L)H\hat{x} + Ly \end{aligned}\right\} \tag{4.4}$$

The latter set of equations is illustrated in block diagram form in figure 3.

The expressions for $\dot{\hat{x}}$ in equations (4.3) and (4.4) are identical to the state observer equation for deterministic systems as studied by Luenberger (refs. 36, 37). Because the choice of K and L is arbitrary, the parameters of $F - KH$ and $(I - L)H$ can be chosen independently of the unknown parameters in the system provided the structure of the system is known (i.e., the measurements can be arranged so that the values of p_i discussed in Chapter III are known). This can easily be demonstrated by writing the equations in the canonical form developed in Chapter III. If the choices for $F - KH$ and $(I - L)H$ are defined as F_N and H_N, respectively, and if these definitions are used in equations (4.4), $\hat{y}$ can be written

$$\left.\begin{aligned} \dot{\hat{x}} &= F_N\hat{x} + Gu + Ky \qquad \hat{x}(0) = x_o \\ \hat{y} &= H_N\hat{x} + Ly \end{aligned}\right\} \tag{4.5}$$

The advantage of using this formulation to model the unknown system is that since F_N and H_N can be chosen, the unknown parameters are contained in the matrices K, L, G, and x_o. These parameters are

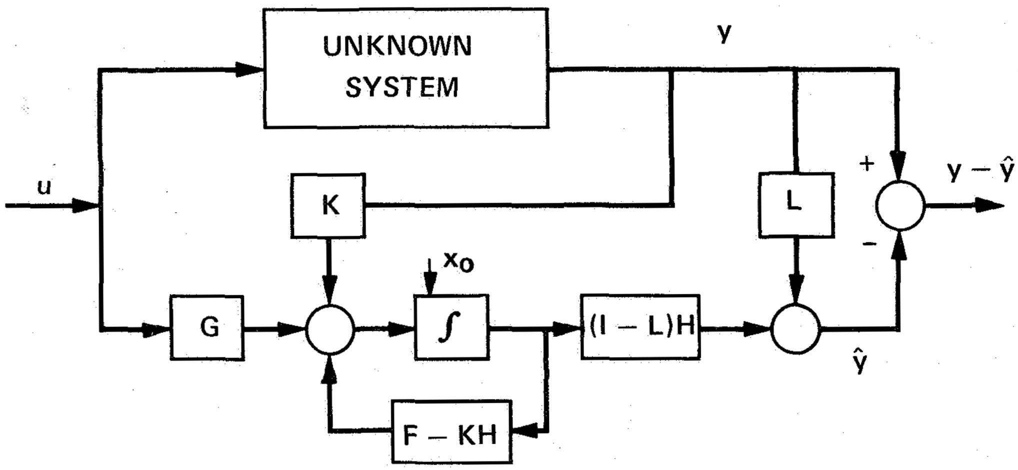

Figure 3.- Model with error feedback.

coefficients of known forcing functions and therefore affect the model response, $\hat{y}$, linearly. By formulating the problem in this manner we are constraining the allowable structure of the identified system to be related to our choice of F_N and H_N by

$$F_N = F - KH$$

$$H_N = (I - L)H$$

It is convenient to define $\delta G = G - G_N$ and $\delta x_o = x_o - x_{N_o}$ where G_N and x_{N_o} can be interpreted as initial estimates of G and x_o and can include any known parameters. Using these definitions in equations (4.5), we obtain

$$\dot{\hat{x}} = F_N\hat{x} + G_N u + \delta G u + Ky \qquad \hat{x}_o = x_{N_o} + \delta x_o$$

$$\hat{y} = H_N\hat{x} + Ly$$

By linear superposition, $\hat{y}$ can be expressed

$$\hat{y}(t) = y_N(t) + A(t)\delta\gamma \tag{4.6}$$

where $y_N(t)$ is the response of the equations

$$\left.\begin{aligned} \dot{x}_N &= F_N x_N + G_N u \qquad x_N(0) = x_{N_o} \\ y_N &= H_N x_N \end{aligned}\right\} \tag{4.7}$$

$\delta\gamma$ is a vector containing the unknown parameters in K, δG, L, and δx_o; and $A(t)$ is the gradient matrix of y_N with respect to these parameters. When (4.6) is substituted into (4.1), J becomes quadratic in the unknown parameters. The estimate of $\delta\gamma$ can be obtained as discussed in Chapter II and is given by

$$\delta\hat{\gamma} = \left[\int_o^{t_f} A^T(t)WA(t)dt\right]^{-1}\left[\int_o^{t_f} A^T(t)W\big(y(t) - y_N(t)\big)dt\right] \tag{4.8}$$

When the measurements are discrete the estimate of $\delta\gamma$ is given by

$$\delta\hat{\gamma} = \left[\sum_{i=1}^{N} A^T(t_i)WA(t_i)\right]^{-1} \left[\sum_{i=1}^{N} A^T(t_i)W\big(y(t_i) - y_N(t_i)\big)\right]$$

The individual components of $A(t)$ can be computed by the numerical solution of the differential equations

$$\left.\begin{aligned} \dot{x}_{\gamma_i} &= F_N x_{\gamma_i} + \frac{\partial K}{\partial \gamma_i} y + \frac{\partial \delta G}{\partial \gamma_i} u \qquad x_{\gamma_i}(0) = \frac{\partial \delta x_o}{\partial \gamma_i} \\ y_{\gamma_i} &= H_N x_{\gamma_i} + \frac{\partial L}{\partial \gamma_i} y \end{aligned}\right\} \tag{4.9}$$

where the partial derivatives of K, δG, L, and δx_o with respect to the parameter $\delta\gamma_i$ are zero except for a one in the location of the specific parameters $\delta\gamma_i$. More will be said about the computation of these equations in Chapter V.

The estimates for F, G, H, and x_o are determined from the estimates of K, δG, L, and x_o by the relationships

$$\left.\begin{aligned} \hat{H} &= (I - \hat{L})^{-1} H_N \\ \hat{F} &= F_N + \hat{K}\hat{H} \\ \hat{G} &= G_N + \delta\hat{G} \\ \hat{x}_o &= x_{N_o} + \delta\hat{x}_o \end{aligned}\right\} \tag{4.10}$$

In this way, the identification problem has been reduced to a sequence of operations involving the numerical solutions of (4.7), (4.8), (4.9), and (4.10). No iteration is required.

Example 4.1 Identification of a Fourth-Order System With Two Outputs

Consider the system used in example 3.2. If the matrices K and L,

$$K = \begin{bmatrix} k_{11} & 0 \\ k_{21} & 0 \\ k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}, \qquad L = \begin{bmatrix} 0 & 0 \\ l_{21} & 0 \end{bmatrix} \tag{4.11}$$

are used in (4.4), this system can be modeled by

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \end{bmatrix} = \begin{bmatrix} f_{11}-k_{11} & 1 & 0 & 0 \\ f_{21}-k_{21} & 0 & 1 & 0 \\ f_{31}-k_{31}-k_{32}h_{21} & 0 & 0 & f_{34}-k_{32} \\ f_{41}-k_{41}-k_{42}h_{21} & 0 & 0 & f_{44}-k_{42} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} + \begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \\ g_{41} \end{bmatrix} u + \begin{bmatrix} k_{11} & 0 \\ k_{21} & 0 \\ k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix} y$$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21}+h_{21} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ l_{21} & 0 \end{bmatrix} y \qquad \hat{x}(0) = x_o \tag{4.12}$$

Clearly, the parameters in $F - KH$ and $(I - L)H$ can be chosen independently of the numerical values of the parameters in F and H. If all the parameters in F, G, H, and x_o are to be identified, G_N and x_{N_o} can be chosen arbitrarily for use in equations (4.7) and the unknown parameters in the vector γ would be k_{11}, k_{21}, k_{31}, k_{41}, k_{32}, k_{42}, l_{21}, g_{11}, g_{21}, g_{31}, g_{41}, $x_1(0)$, $x_2(0)$, $x_3(0)$, and $x_4(0)$. An identification of F, G, H, and x_o can be obtained by using (4.7) through (4.10).

The above identification procedure has been referred to as an equations of motion method. This categorization is clear if the exponential function $e^{F_N(t-\tau)}$ is used as the method function in the integral transform of the assumed equations of motion (see section 2.1). The analogy between the integral transform approach and the concept of a linear observer is discussed in appendix B.

In the previous discussion the noise was assumed to be negligible in the unknown system. In the presence of noise, the output of (4.5) will not equal the measurements even for a perfect model. If the procedure described above is applied to a system with noise, the estimates of the parameters will be biased. The source of the bias is similar to that discussed in section 2.1.2 for the equations of motion method. In the case of discrete measurements, the bias is given by equation (2.16), which is rewritten here for convenience.

$$E\{\delta\gamma_\varepsilon\} = \left[\sum_{i=1}^{N} A_t^T(t_i)WA_t(t_i)\right]^{-1}\left[-\sum_{i=1}^{N} E\left\{A_\varepsilon^T(t_i)Wy_\varepsilon(t_i)\right\} + \sum_{i=1}^{N} E\left\{A_\varepsilon^T(t_i)WA_\varepsilon(t_i)\right\}\delta\hat{\gamma}\right] \tag{4.13}$$

The terms $A_t(t_i)$ and $\delta\hat{\gamma}$ have been used in place of A_{t_i} and $\hat{\gamma}$, respectively, for relevance to the present discussion. The term $A_t(t_i)$ is the gradient of $\hat{y}$ with respect to the parameters in $\delta\gamma$ if there were no noise in the system; $A_\varepsilon(t_i)$ is the difference between the gradient of $\hat{y}$ and $A_t(t_i)$. Finally, $y_\varepsilon(t_i)$ is that portion of the measurements which is not correlated with the input u or the initial conditions x_o. It can be seen from equation (4.13) that the size of the bias is equal to a constant plus a term proportional to the size of the estimate, $\delta\hat{\gamma}$. If the initial choices of F_N and H_N are such that the estimates of K and L are extremely large, then the bias error can often be reduced by choosing a new F_N and H_N equal to the estimates for F and H and repeating the process. If this procedure is repeated until the estimates, go to zero, the second term in equation (4.13) will vanish. However, the constant bias term usually cannot be eliminated by this process, as is illustrated in Chapter VI.

4.3 THE RESPONSE CURVE FITTING METHOD

The main reason for using a measurement error procedure is that unbiased noise in the system does not cause a bias in the parameter estimates (see Chapter II). One algorithm that can be used to minimize (4.1) subject to (4.2) is the method of *quasi-linearization*. The basic idea behind quasi-linearization has already been discussed in Chapter II. If the initial estimates of F, G, H, and x_o are defined as F_N, G_N, H_N, and x_{N_o}, respectively, then $\hat{x}$ and $\hat{y}$ can be approximated by

$$\hat{x} \approx x_N + \delta x$$
$$\hat{y} \approx y_N + \delta y$$

where

$$\dot{x}_N = F_N x_N + G_N u \qquad x_N(0) = x_{N_o}$$
$$y_N = H_N x_N$$

and where

$$\delta\dot{x} = F_N \delta x + \delta F x_N + \delta G u$$
$$\delta y = H_N \delta x + \delta H x_N$$

If these equations are added together we obtain

$$\left.\begin{aligned} \dot{\hat{x}} &= F_N \hat{x} + \delta F x_N + [G_N + \delta G]u \\ \hat{y} &= H_N \hat{x} + \delta H x_N \end{aligned}\right\} \tag{4.14}$$

If the system is modeled in the canonical form discussed in Chapter III, then the unknown parameters in δF and δH can be expressed in terms of matrices K and L by the relationships

$$\left.\begin{aligned} \delta F &= \hat{F} - F_N = K\hat{H} \approx KH_N \\ \delta H &= \hat{H} - H_N = L\hat{H} \approx LH_N \end{aligned}\right\} \tag{4.15}$$

where the approximations are based on δF and δH being small. If equation (4.15) is substituted into (4.14), we obtain.

$$\left.\begin{aligned} \dot{\hat{x}} &= F_N\hat{x} + Ky_N + [G_N + \delta G]u \qquad \hat{x}(0) = x_{N_o} + \delta x_o \\ \hat{y} &= H_N\hat{x} + Ly_N \end{aligned}\right\} \tag{4.16}$$

Equations (4.16) are identical to (4.6) except that y_N has replaced y. Parameter estimates can be obtained by the numerical solution of equations (4.7) through (4.10) with the exception that y_N is used in place of y in (4.9). New estimates of the unknown parameters are obtained by the solution of (4.10). If they differ significantly from the initial estimates, the procedure is repeated. In this way an iterative procedure is established for determining the unknown parameters, γ, that minimize (4.1).

4.4 THE COMBINED ALGORITHM

The idea for a combined algorithm is now evident. The structures of the equation error and measurement error problems are identical except for the computation of the components of $A(t)$. The only difference here is whether measured or estimated data are used in the sensitivity equations (4.9). If measured data are used, the procedure provides an estimate of F, G, H, and x_o in a single sequence of operations essentially independent of the initial choice of F_N, G_N, H_N, and x_{N_o}. In the absence of noise, this estimate is the same as the quasi-linearization estimate; but if there is noise in the system, this estimate will be biased. Choosing a new F_N, G_N, H_N, and x_{N_o} equal to the estimates of F, G, H, and x_o and repeating the equations of motion method usually reduces the bias error in the estimates. However, the bias cannot be eliminated completely by repeated application of this process. On achieving the best estimate by the equations of motion method, the combined algorithm replaces y by y_N in the sensitivity equations (4.9). This implements the response curve fitting method which generally converges to the unbiased estimate very rapidly.

This procedure is illustrated in figure 4. When the switch in the upper center of the diagram is in the (+) position, we are using the equations of motion method and when it is in the (-) position, we are using the response curve fitting method. For the initial set of iterations, the switch is in the (+) position. After that, it is in the (-) position. The rest of the computational structure remains unchanged. The components of $A(t)$ are computed by the numerical solution of the sensitivity equations (4.9) which are labeled in the figure. The outputs of the sensitivity equations are used to form the products $A(t)^T WA(t)$ and $f(t)^T W(y - y_N)$ which are integrated simultaneously in order to reduce storage requirements. The diagram is for continuous measurements. In the case of discrete measurements, the integrations over the interval (0, t) on the right hand side of the figure would be replaced by summations. The estimate $\delta\hat{\gamma}$ is obtained by the solution of (4.8) at the final time t_f, and the unknown parameters are computed using (4.10). The process is then repeated as indicated where the superscript (1) indicates the new estimate and the superscript (0) represents the previous estimate.

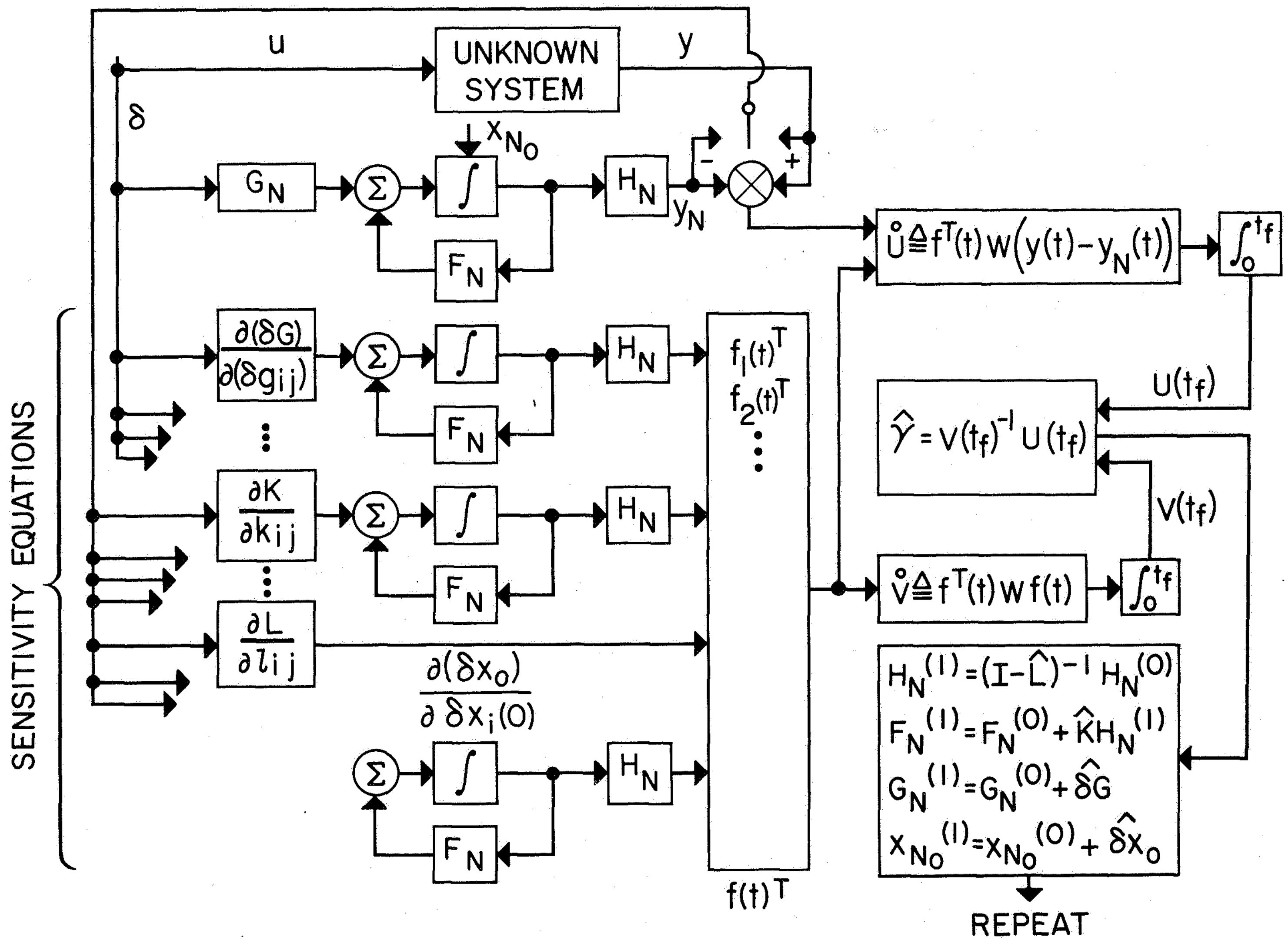

Figure 4.- Combined algorithm.

In order to apply the combined algorithm, it is necessary that the equations be written so that the unknown parameters can be affected independently by the parameters in the matrix products KH and LH.[5] It has been shown that this is always possible by going to the canonical form discussed in Chapter III. If the transformation results in fewer unknown parameters than were in the original equations, the transformed equations are preferable. However, if writing the equations in an appropriate form to apply the combined algorithm results in more unknown parameters[6] than in the original equations, it is clearly better to stay with the original equations. On the first few iterations those sensitivity equations that can be driven with the measured states are so driven, and the remaining sensitivity equations are driven with the estimated states.

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[5] See equations (4.3), (4.4), and (4.5).

[6] Constraints relating these additional parameters are available, but are generally difficult to take into account.

V A SIMPLIFICATION IN THE COMPUTATION OF THE SENSITIVITY FUNCTIONS AND INTEGRALS OF THE SQUARED SENSITIVITY FUNCTIONS

5.1 COMPUTATION OF THE SENSITIVITY FUNCTIONS

5.1.1 Statement of Problem

The problem can be stated as follows. Given a system described by the equations

$$\dot{x} = Fx + Gu \qquad x(0) = x_o \tag{5.1}$$

where x is an n-dimensional state vector and u is a p-dimensional input vector, and assuming that the system is cyclic,[7] find the first-order variations of the system state caused by unit perturbations of the parameters in F, G, and x_o. These sensitivity functions can be computed from the equations

$$\dot{x}_{g_{ij}}(t) = Fx_{g_{ij}}(t) + \frac{\partial G}{\partial g_{ij}} u_j(t) \qquad x_{g_{ij}}(0) = 0, \qquad \begin{array}{l} i = 1, \ldots, n \\ j = 1, \ldots, p \end{array} \tag{5.2}$$

$$\dot{x}_{x_i(0)}(t) = Fx_{x_i(0)}(t) \qquad x_{x_i(0)}(0) = \frac{\partial x_o}{\partial x_i(0)}, \qquad i = 1, \ldots, n \tag{5.3}$$

$$\dot{x}_{f_{ij}}(t) = Fx_{f_{ij}}(t) + \frac{\partial F}{\partial f_{ij}} x_j(t) \qquad x_{f_{ij}}(0) = 0, \qquad \begin{array}{l} i = 1, \ldots, n \\ j = 1, \ldots, n \end{array} \tag{5.4}$$

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[7] A system with state coefficient matrix F is cyclic if there is a vector l so that the n vectors $[F^{n-1}l \mid F^{n-2}l \mid \ldots \mid l]$ are linearly independent.

The notation $x_{g_{ij}}(t)$ denotes the sensitivity function for the parameter in the ith row and jth column of G. Similar definitions apply to $x_{f_{ij}}(t)$ and $x_{x_i(0)}(t)$. In this chapter we will establish the following important result.

Result: If the system (5.1) is cyclic, the system response and the sensitivity functions with respect to the system parameters and initial conditions can be obtained by linear combinations of the solutions to (p + 2) differential equations of order n.

5.1.2 Development

Since it is assumed that the system is cyclic, there is a nonsingular transformation T_c so that equations (5.1) can be written in companion form (ref. 38).[8]

$$z(t) = T_c x(t) \tag{5.5}$$

$$\dot{z}(t) = Az(t) + Bu(t) \qquad z(0) = z_o = T_c x_o \tag{5.6}$$

$$A = \left[\begin{array}{c|ccc} a_1 & & & \\ \cdot & & I & \\ \cdot & & & \\ \hline \cdot & & & \\ a_n & 0 & \cdots & 0 \end{array}\right] \tag{5.7}$$

The variations in $z(t)$ caused by unit perturbations in the parameters in A, B, and the initial conditions z_o can be computed by the numerical solution of

[8] The value of T_c is given by the inverse of the matrix $[F^{n-1}l \mid F^{n-2}l \mid \ldots \mid l]$ where l is any vector such that an inverse exists.

$$\dot{z}_{b_{ij}}(t) = Az_{b_{ij}}(t) + \frac{\partial b^{(j)}}{\partial b_{ij}} u_j(t) \qquad z_{b_{ij}}(0) = 0, \qquad \begin{array}{l} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, p \end{array} \tag{5.8}$$

$$\dot{z}_{z_i(0)}(t) = Az_{z_i(0)}(t) \qquad z_{z_i(0)}(0) = \frac{\partial z_o}{\partial z_i(0)}, \qquad i = 1, 2, \ldots, n \tag{5.9}$$

$$\dot{z}_{a_i}(t) = Az_{a_i}(t) + \frac{\partial a^{(1)}}{\partial a_i} z_i \qquad z_{a_i}(0) = 0, \qquad i = 1, 2, \ldots, n \tag{5.10}$$

where $b^{(j)}$ is the j*th* column of B and $a^{(1)}$ is the first column of A. The response $z(t)$ can be obtained by linear superposition from (5.8) and (5.9).

$$z(t) = \sum_{j=1}^{p} \sum_{i=1}^{n} b_{ij} z_{b_{ij}}(t) + \sum_{i=1}^{n} z_i(0) z_{z_i(0)}(t) \tag{5.11}$$

The system response, $x(t)$, and the sensitivities (5.2)-(5.4) can be obtained from (5.8)-(5.11) by the relationships

$$x(t) = T_c^{-1} z(t) \tag{5.12}$$

$$x_{g_{ij}}(t) = \cancelto{0}{\frac{\partial T_c^{-1}}{\partial g_{ij}}} \; z(t) + T_c^{-1}\left[z_{a_l}(t) \cancelto{0}{\frac{\partial a_l}{\partial g_{ij}}} + z_{b_{lk}}(t) \frac{\partial b_{lk}}{\partial g_{ij}} + z_{z_l(0)}(t) \cancelto{0}{\frac{\partial z_l(0)}{\partial g_{ij}}} \right] \tag{5.13}$$

$$x_{f_{ij}}(t) = \frac{\partial T_c^{-1}}{\partial f_{ij}} z(t)$$

$$+ T_c^{-1}\left[z_{a_l}(t) \frac{\partial a_l}{\partial f_{ij}} + z_{b_{lk}}(t) \frac{\partial b_{lk}}{\partial f_{ij}} + z_{z_l(0)}(t) \frac{\partial z_l(0)}{\partial f_{ij}} \right] \tag{5.14}$$

$$x_{x_i(0)}(t) = \cancelto{0}{\frac{\partial T_c^{-1}}{\partial x_i(0)}} z(t)$$

$$+ T_c^{-1}\left[z_{a_l}(t) \cancelto{0}{\frac{\partial a_l}{\partial x_i(0)}} + z_{b_{lk}}(t) \cancelto{0}{\frac{\partial b_{lk}}{\partial x_i(0)}} + z_{z_l(0)}(t) \frac{\partial z_l(0)}{\partial x_i(0)} \right] \tag{5.15}$$

where repeated subscripts imply summation. As a result of theorem 5.1 which is stated below, the $n(p + 2)$ sensitivity functions (5.8)-(5.11), hence the model response (5.11), can actually be computed by linear combinations of the solutions to the $(p + 2)$ nth-order single-input differential equations

$$\dot{\xi}^j(t) = A\xi^j(t) + l u_j(t) \qquad \xi^j(0) = 0, \quad j = 1, 2, \ldots, p \tag{5.16}$$

$$\dot{\xi}^{p+1}(t) = A\xi^{p+1}(t) \qquad \xi^{p+1}(0) = l \tag{5.17}$$

$$\dot{\xi}^{p+2}(t) = A\xi^{p+2}(t) + l z_1(t) \qquad \xi^{p+2}(0) = 0 \tag{5.18}$$

if the vector l is chosen so that these systems (5.16) to (5.18) are controllable. The result stated in section 5.1 of this chapter is thereby established.

<u>Theorem 5.1</u>: If the solution to the single input system

$$\dot{x}(t) = Fx(t) + gu(t) \qquad x(0) = 0 \tag{5.19}$$

is known, and if the system is controllable, then the solution to the system

$$\dot{z}(t) = Fz(t) + g'u(t) \qquad z(0) = 0 \tag{5.20}$$

for arbitrary g' can be obtained by a linear transformation, T, on the solution for $x(t)$ (i.e., $z(t) = Tx(t)$).

Proof: If $x(t)$ is the solution of (5.19) then $x^j(t) \triangleq Fx^j(t)$ is the solution to

$$\dot{x}^j(t) = Fx^j(t) + F^j gu(t) \qquad x^j(0) = 0 \tag{5.21}$$

where

$$j = 0, 1, 2, 3, \ldots, n-1$$

Since the system (5.19) is controllable, the control coefficient vectors in (5.21), $(g, Fg, \ldots, F^{n-1}g)$ are linearly independent and the control coefficient vector g' in (5.20) can be expressed as a linear combination of these vectors,

$$g' = \sum_{i=0}^{n-1} \alpha_i F^i g \tag{5.22}$$

Therefore, by linear superposition the response to equation (5.20) can be obtained from the solution to equation (5.19) by the relation

$$z(t) = \sum_{i=0}^{n-1} \alpha_i x^i(t) = \sum_{i=0}^{n-1} \alpha_i F^i x(t) \tag{5.23}$$

and this concludes the proof.

Comment: The α_i can be computed by the equation

$$\begin{bmatrix} \alpha_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_0 \end{bmatrix} = \left[F^{n-1}g \mid F^{n-2}g \mid \ldots \mid g\right]^{-1} g' \tag{5.24}$$

This is immediately evident from (5.22).

Corollary 1: Theorem 5.1 also applies if (5.19) and (5.20) are homogeneous differential equations with $x(0) = g$ and $z(0) = g'$, respectively.

5.1.3 Higher Order Derivatives

This same procedure can be used efficiently to generate higher-order sensitivity functions since the (n + 1) order sensitivity function is the first-order sensitivity of the *nth*-order sensitivity function. These higher-order sensitivity functions are used in certain numerical methods such as the Newton-Raphson procedure which require second or higher-order partial derivatives.

5.1.4 Special Case

Problem: Consider a single-output, multi-input system that is observable and controllable. The system can be modeled by equations of the form (5.6) and (5.7) where the measurement y is related to z by $y(t) = Hz(t)$ where $H = (1 \quad 0 , \ldots 0)$. Use the solutions to a minimal number of differential equations to compute the variations of the measurements $y(t)$ due to unit perturbations in the system parameters and initial conditions. These variations can be computed from the sensitivity functions (5.8) to (5.10) by the relationships

$$y_{b_{ij}}(t) = Hz_{b_{ij}}(t) \tag{5.25}$$

$$y_{z_i(0)}(t) = Hz_{z_i(0)}(t) \tag{5.26}$$

$$y_{a_i}(t) = Hz_{a_i}(t) \tag{5.27}$$

Solution: Choose the vector l in equations (5.16) to (5.18) to be

$$l^T = [0 \quad 0 \; . \; . \; . \; 1]$$

The corresponding controllability matrix $[A^{n-1}l \mid A^{n-2}l \mid . \; . \; . \mid l]$ is the identity matrix, and the systems are controllable. The solutions to equations (5.16) to (5.18) can therefore be used with the transformations defined in theorem 5.1 to obtain the sensitivity functions for the system. The variations of the system measurements, y, are related to the solutions, $\xi^k(t)$, k = 1, 2, . . ., p + 2, by the following transformations

$$\left.\begin{aligned} y_{b_{ij}}(t) &= HA^{n-i}\xi^j(t) && i = 1, 2, \ldots, n \\ & && j = 1, 2, \ldots, p \\ y_{z_i(0)}(t) &= HA^{n-i}\xi^{p+1}(t) && i = 1, 2, \ldots, n \\ y_{a_i}(t) &= HA^{n-i}\xi^{p+2}(t) && i = 1, 2, \ldots, n \end{aligned}\right\} \tag{5.28}$$

Equations (5.28) provide a solution to the stated problem; however, the computational savings due to the reduction in the number of required solutions to differential equations is somewhat offset by the algebraic transformations. These transformations can be eliminated from the computations by defining the vector

$$\rho^k(t) = \begin{bmatrix} HA^{n-1} \\ HA^{n-2} \\ . \\ . \\ . \\ H \end{bmatrix} \xi^k(t) \tag{5.29}$$

The variations in y are then given by the components of $\rho^k(t)$

$$\left.\begin{aligned} y_{b_{ij}}(t) &= \rho_i^j(t) && i = 1, 2, \ldots, n \\ & && j = 1, 2, \ldots, p \\ y_{z_i(0)}(t) &= \rho_i^{p+1}(t) && i = 1, 2, \ldots, n \\ y_{a_i}(t) &= \rho_i^{p+2}(t) && i = 1, 2, \ldots, n \end{aligned}\right\} \quad (5.30)$$

where $\rho_i^k(t)$ is the i*th* component of the vector $\rho^k(t)$. The vectors, $\rho^k(t)$, are the numerical solutions of the equations

$$\left.\begin{aligned} \dot{\rho}^j(t) &= A^T\rho^j(t) + l'u_j && \rho^j(0) = 0, \quad j = 1, 2, \ldots, p \\ \dot{\rho}^{p+1}(t) &= A^T\rho^{p+1}(t) && \rho^{p+1}(0) = l' \\ \dot{\rho}^{p+2}(t) &= A^T\rho^{p+2}(t) + l'z_1 && \rho^{p+2}(0) = 0 \end{aligned}\right\} \quad (5.31)$$

where

$$(l')^T = [1 \quad 0 \; . \; . \; . \; 0]$$

and where (by eqs. (5.11) and (5.30))

$$z_i(t) = y(t) = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij}\rho_i^j(t) + \sum_{i=1}^{n} z_i(0)\rho_i^{p+1}(t)$$

5.1.5 Application With the Combined Algorithm

In the combined algorithm, the system is modeled so that the measurements act as additional inputs to the model, and the sensitivities of the parameters in the state coefficient matrix, F_N, are not required. There are $p + m$ inputs to this model where p is the number of actual inputs and m is the number of measurements. A maximum of $(p + m + 1)$ n*th*-order differential equations are required to obtain all of the sensitivities used in the combined algorithm. If the system is cyclic then the solutions to any set of equations of the form

$$\left.\begin{array}{llll} \dot{\xi}^j = F_N\xi^j + l u_j & \xi^j(0) = 0 & j = 1, 2, \ldots, p \\ \dot{\xi}^{p+i} = F_N\xi^{p+i} + l y_i & \xi^{p+i}(0) = 0 & i = 1, 2, \ldots, m \\ \dot{\xi}^{p+m+1} = F_N\xi^{p+m+1} & \xi^{p+m+1}(0) = l & \end{array}\right\} \quad (5.32)$$

(where l is chosen so that the systems are controllable) can be used to obtain the required sensitivity functions.

5.2 COMPUTATION OF THE INTEGRALS OF THE SQUARED SENSITIVITY FUNCTIONS

5.2.1 Continuous Measurements

In addition to the sensitivity equations, the function

$$\sum_{i=1}^{n} A^T(t_i)WA(t_i) \quad \text{or} \quad \int_0^{t_f} A^T(t)WA(t)dt \qquad (5.33)$$

must be computed. If there are q unknown parameters, these matrices contain $q(q + 1)/2$ summations or integrations with each involving m summations. An alternative to the direct computation of these matrices is to take advantage of the relationships among the elements of these matrices provided by the sensitivity equations. First we will consider the continuous case and then extend the results to the case of discrete measurements by applying numerical integration approximations.

Because the components of the matrix $A(t)$ (the matrix of sensitivity functions) can be obtained by linear transformations of the solutions to $(p + m + 1)$ n*th*-order differential equations (which we will refer to by the vector $\theta^T = \left[\xi^{1^T} \mid \xi^{2^T} \mid \ldots \mid \xi^{p+m+1^T}\right]$, the elements of (5.33) can be obtained by linear combinations of the elements in the matrix

$$\int_0^{t_f} \theta\theta^T\, dt \qquad (5.34)$$

A differential equation for $\theta\theta^T$ can be obtained from equation (5.32)

$$\dot{\theta}\theta^T + \theta\dot{\theta}^T = \frac{d}{dt}(\theta\theta^T) = \begin{bmatrix} F_N & & 0 \\ & \ddots & \\ 0 & & F_N \end{bmatrix} \theta\theta^T + \theta\theta^T \begin{bmatrix} F_N^T & & 0 \\ & \ddots & \\ 0 & & F_N^T \end{bmatrix}$$

$$+ \begin{bmatrix} l & 0 & \cdots & 0 \\ 0 & l & \cdots & \cdot \\ & & \ddots & \vdots \\ & & & l \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \theta^T + \theta [u^T \quad y^T] \begin{bmatrix} l^T & 0 & & & 0 \\ 0 & l^T & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & l^T & 0 \end{bmatrix} \qquad (5.35)$$

This implies that

$$\theta\theta^T\Big|_{t=t_f} - \theta\theta^T\Big|_{t=0} = \begin{bmatrix} F_N & & 0 \\ & \ddots & \\ 0 & & F_N \end{bmatrix} \int_0^{t_f} \theta\theta^T \, dt + \int_0^{t_f} \theta\theta^T \, dt \begin{bmatrix} F_N & & 0 \\ & \ddots & \\ 0 & & F_N \end{bmatrix}$$

$$+ \int_0^{t_f} \begin{bmatrix} l & 0 & \cdots & 0 \\ 0 & l & & \vdots \\ \vdots & & \ddots & l \\ 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \theta^T + \theta [u^T \quad y^T] \begin{bmatrix} l^T & 0 & & 0 \\ 0 & l^T & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & l^T & 0 \end{bmatrix} dt \qquad (5.36)$$

which provides a set of $[(p + m + 1)n][(p + m + 1)n + 1]/2$ linear equations in the $[(p + m + 1)n][(p + m + 1)n + 1]/2$ unknowns of $\int_0^{t_f} \theta\theta^T \, dt$ and the $[(p + m)^2(2n - 1) + (p + m)(2n + 1)]/2$ unknowns of

$$\int_\theta^{t_f} \begin{bmatrix} l & 0 & \cdots & 0 \\ 0 & l & & \vdots \\ \vdots & & \ddots & l \\ 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \theta^T + \theta [u^T \quad y^T] \begin{bmatrix} l^T & 0 & & 0 \\ 0 & l^T & & 0 \\ \vdots & & \ddots & \\ 0 & & l^T & 0 \end{bmatrix} dt \qquad (5.37)$$

It is well known (ref. 38) that if F_N and $-F_N^T$ have no common eigenvalues, equation (5.36) can be solved uniquely for $\int_0^{t_f} \theta\theta^T \, dt$ in terms of

$$\theta\theta^T\Big|_{t=t_f} - \theta\theta^T\Big|_{t=0}$$

and (5.37). Clearly, if F_N is stable with no eigenvalues with zero real parts, it will have no eigenvalues in common with $-F_N^T$ and the above equation can be solved. In general, this procedure requires fewer integrations than would otherwise be required. There are algorithms available for solving the matrix equation (5.36) (see refs. 39 and 40), and it would appear that some advantage can be gained by using this idea.

Example 5.1 Single-Input, Single-Output, Second-Order System

Consider a stable (no eigenvalues with real parts greater than or equal to zero), single-input, single-output, second-order system modeled in its canonical form,

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \qquad x(0) = x_0 \\ y &= Cx \end{aligned}\right\} \tag{5.38}$$

where

$$A = \begin{bmatrix} a_1 & 1 \\ a_0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} \qquad C = [1 \quad 0]$$

Suppose that the two parameters in A, the two parameters in B, and the two initial conditions are to be identified. The sensitivity functions can be obtained by the numerical solution of equation (5.31) which were developed in section 5.1.4 entitled "Special Case." The matrix of sensitivity functions, A(t) (which is not to be confused with the state coefficient matrix A in this example), is given by

$$A(t) = \left[\rho^{1^T}, \ \rho^{2^T}, \ \rho^{3^T}\right]$$

and the matrix of integral squares of the sensitivity functions is equal to

$$\int_0^{t_f} A^T(t)WA(t)dt \tag{5.39}$$

Because this is a single-output system, W is a scalar and can be set equal to unity. Expression (5.39) is a 6 × 6 symmetric matrix and can be computed by performing 21 integrations.

If the procedure outlined in expressions (5.34) through (5.36) is followed, it can be shown that the matrix $\int_0^{t_f} A^T(t)A(t)dt$ must satisfy the equation

$$A^T(t)A(t)\Big|_{t=t_f} - A^T(t)A(t)\Big|_{t=0}$$

$$= \begin{bmatrix} A^T & 0 & 0 \\ 0 & A^T & 0 \\ 0 & 0 & A^T \end{bmatrix} \int_0^{t_f} A^T(t)A(t)dt + \int_0^{t_f} A^T(t)A(t)dt \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix}$$

$$+ \int_0^{t_f} \left\{ \begin{bmatrix} l' & 0 \\ 0 & l' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} A(t) + A(t)^T [u \quad y] \begin{bmatrix} l'^T & 0 & 0 \\ 0 & l'^T & 0 \end{bmatrix} \right\} dt \tag{5.40}$$

where

$$A^T(t)A(t)\Big|_{t=0} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and where

$$\int_0^{t_f} \left\{ \begin{bmatrix} l' & 0 \\ 0 & l' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} A(t) + A(t)^T [u \quad y] \begin{bmatrix} l'^T & 0 & 0 \\ 0 & l'^T & 0 \end{bmatrix} \right\} dt$$

$$= \int_0^{t_f} \begin{bmatrix} 2u\rho_1{}^1 & u\rho_2{}^1 & u\rho_1{}^2 + y\rho_1{}^1 & u\rho_2{}^2 & u\rho_1{}^3 & u\rho_2{}^3 \\ & 0 & y\rho_2{}^1 & 0 & 0 & 0 \\ & & 2y\rho_1{}^2 & y\rho_2{}^2 & y\rho_1{}^3 & y\rho_2{}^3 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \qquad (5.41)$$

Since the system is stable, we can compute the 11 integrals in equation (5.41) and then solve the algebraic equation (5.40) for the 21 integrals in $\int_0^{t_f} A^T(t)A(t)dt$. Another procedure is to use only the 10 equations provided by (5.40) for which the terms in (5.41) are equal to zero. If we compute the 11 integrals

$\int_0^{t_f} \rho_1{}^1\rho_1{}^1\, dt \qquad \int_0^{t_f} \rho_1{}^1\rho_1{}^2\, dt \qquad \int_0^{t_f} \rho_2{}^1\rho_2{}^1\, dt \qquad \int_0^{t_f} \rho_2{}^1\rho_1{}^2\, dt$

$\int_0^{t_f} \rho_2{}^1\rho_2{}^2\, dt \qquad \int_0^{t_f} \rho_2{}^1\rho_1{}^3\, dt \qquad \int_0^{t_f} \rho_2{}^1\rho_2{}^3\, dt \qquad \int_0^{t_f} \rho_1{}^2\rho_1{}^2\, dt$

$\int_0^{t_f} \rho_2{}^2\rho_2{}^2\, dt \qquad \int_0^{t_f} \rho_2{}^2\rho_1{}^3\, dt \qquad \int_0^{t_f} \rho_2{}^2\rho_2{}^3\, dt$

then the 10 pertinent equations in (5.40),

1. $$\rho_2{}^1\rho_2{}^1\big|_{t=t_f} = 2\int_0^{t_f} \rho_1{}^1\rho_2{}^1\, dt$$

2. $$\rho_2{}^2\rho_2{}^2\big|_{t=t_f} = 2\int_0^{t_f} \rho_1{}^2\rho_2{}^2\, dt$$

3. $$\rho_2{}^3\rho_2{}^3\big|_{t=t_f} = 2\int_o^{t_f} \rho_1{}^3\rho_2{}^3\, dt$$

4. $$\rho_2{}^1\rho_2{}^2\big|_{t=t_f} = \int_o^{t_f} (\rho_1{}^1\rho_2{}^2 + \rho_1{}^2\rho_2{}^1)dt$$

5. $$\rho_2{}^1\rho_2{}^3\big|_{t=t_f} = \int_o^{t_f} (\rho_1{}^1\rho_2{}^3 + \rho_1{}^3\rho_2{}^1)dt$$

6. $$\rho_2{}^2\rho_2{}^3\big|_{t=t_f} = \int_o^{t_f} (\rho_1{}^2\rho_2{}^3 + \rho_1{}^3\rho_2{}^2)dt$$

7. $$\rho_2{}^1\rho_1{}^3\big|_{t=t_f} = \int_o^{t_f} (\rho_1{}^1\rho_1{}^3 + a_1\rho_1{}^3\rho_2{}^1 + a_0\rho_2{}^3\rho_2{}^1)dt$$

8. $$\rho_2{}^2\rho_1{}^3\big|_{t=t_f} = \int_o^{t_f} (\rho_1{}^2\rho_1{}^3 + a_1\rho_1{}^3\rho_2{}^2 + a_0\rho_2{}^3\rho_2{}^2)dt$$

9. $$\rho_1{}^3\rho_1{}^3\big|_{t=t_f} - 1 = \int_o^{t_f} 2(a_1\rho_1{}^3\rho_1{}^3 + a_0\rho_2{}^3\rho_1{}^3)dt$$

10. $$\rho_1{}^3\rho_2{}^2\big|_{t=t_f} = \int_o^{t_f} (a_1\rho_1{}^3\rho_2{}^2 + a_0\rho_2{}^3\rho_2{}^3 + \rho_1{}^3\rho_1{}^3)dt$$

can be used to solve for the remaining 10 integrals as indicated

$$\int_o^{t_f} \rho_1{}^1\rho_2{}^1\, dt = \frac{1}{2}\,\rho_2{}^1\rho_2{}^1\big|_{t=t_f}$$

$$\int_o^{t_f} \rho_1{}^2\rho_2{}^2\, dt = \frac{1}{2}\,\rho_2{}^2\rho_2{}^2\big|_{t=t_f}$$

$$\int_o^{t_f} \rho_1{}^3\rho_2{}^3\, dt = \frac{1}{2}\,\rho_2{}^3\rho_2{}^3\big|_{t=t_f}$$

$$\int_o^{t_f} \rho_1{}^1\rho_2{}^2\, dt = \rho_2{}^1\rho_2{}^2\big|_{t=t_f} - \int_o^{t_f} \rho_2{}^1\rho_1{}^2\, dt$$

$$\int_o^{t_f} \rho_1{}^1\rho_2{}^3\, dt = \rho_2{}^1\rho_2{}^3\big|_{t=t_f} - \int_o^{t_f} \rho_2{}^1\rho_1{}^3\, dt$$

$$\int_0^{t_f} \rho_1{}^2\rho_2{}^3\, dt = \rho_2{}^2\rho_2{}^3\big|_{t=t_f} - \int_0^{t_f} \rho_2{}^2\rho_1{}^3\, dt$$

$$\int_0^{t_f} \rho_1{}^1\rho_1{}^3\, dt = \rho_2{}^1\rho_1{}^3\big|_{t=t_f} - \int_0^{t_f} (a_1\rho_2{}^1\rho_1{}^3 + a_0\rho_2{}^1\rho_2{}^3)dt$$

$$\int_0^{t_f} \rho_1{}^2\rho_1{}^3\, dt = \rho_2{}^2\rho_1{}^3\big|_{t=t_f} - \int_0^{t_f} (a_1\rho_2{}^2\rho_1{}^3 + a_0\rho_2{}^2\rho_2{}^3)dt$$

$$\int_0^{t_f} \rho_1{}^3\rho_1{}^3\, dt = \frac{1}{2a_1}\left(\rho_1{}^3\rho_1{}^3\big|_{t=t_f} - 1 - 2a_0 \int_0^{t_f} \rho_1{}^3\rho_2{}^3\, dt\right)$$

$$\int_0^{t_f} \rho_2{}^3\rho_2{}^3\, dt = \frac{1}{a_0}\left[\rho_1{}^3\rho_2{}^2\big|_{t=t_f} - \int_0^{t_f} (a_1\rho_2{}^2\rho_1{}^3 + \rho_1{}^3\rho_1{}^3)dt\right]$$

5.2.2 Extension to Discrete Measurements

Numerical integration approximations for the integrals in (5.36) can be used to compute the matrix $\sum_{i=1}^{N} A^T(t_i)WA(t_i)$ in the case of discrete measurements. Let $\dot{v}_{ij}(t)$ be an element in the matrix $A^T(t)WA(t)$ (see fig. 4) and let time be indexed from 1 to N where $t_1 = 0$ and $t_N = t_f$. The relationships between the integrations and the summations are given here for the rectangular, trapezoidal, and Simpson's rule integration routines.

1. Rectangular integration routine

$$\int_0^{t_f} \dot{v}_{lk}(t)dt \approx \Delta t\left[\sum_{i=1}^{N} \dot{v}_{lk}(t_i) - \dot{v}_{lk}(t_N)\right]$$

or

$$\sum_{i=1}^{N} \dot{v}_{lk}(t_i) \approx \frac{1}{\Delta t}\int_0^{t_f} \dot{v}_{lk}(t)dt + \dot{v}_{lk}(t_N)$$

2. Trapezoidal integration routine

$$\int_{o}^{t_f} \dot{v}_{lk}(t)dt \approx \Delta t \left[\sum_{i=1}^{N} \dot{v}_{lk}(t_i) - \frac{\dot{v}_{lk}(t_1) + \dot{v}_{lk}(t_N)}{2} \right]$$

or

$$\sum_{i=1}^{N} \dot{v}_{lk}(t_i) \approx \frac{1}{\Delta t} \int_{o}^{t_f} \dot{v}_{lk}(t)dt + \frac{\dot{v}_{lk}(t_1) + \dot{v}_{lk}(t_N)}{2}$$

3. Simpson's rule integration routine (modified)

The Simpson's rule routine requires that the total integration interval be divided into an even number of subintervals (the function is evaluated at an odd number of points). If the function is evaluated for an even number of points, the trapezoidal method can be used to integrate over one of the end subintervals and Simpson's rule used for the remainder of the integration. However, application of Simpson's rule does not provide a direct relationship between $\sum_{i=1}^{N} \dot{v}_{lk}(t_i)$ and $\int_{o}^{t_f} \dot{v}_{lk}(t)dt$.

Let us assume that the total integration interval is divided into an even number of intervals or that N is an odd number. Simpson's rule provides the relationship

$$\int_{o}^{t_f} \dot{v}_{lk}(t)dt$$

$$\approx \frac{1}{3} \Delta t \left[\dot{v}_{lk}(t_1) + \dot{v}_{lk}(t_N) + 4 \sum_{i=1}^{(N-1)/2} \dot{v}_{lk}(t_{2i}) + 2 \sum_{i=1}^{(N-3)/2} \dot{v}_{lk}(t_{2i+1}) \right] \quad \text{(i)}$$

This integration can also be approximated by using a trapezoidal integration over the first and last subintervals and using Simpson's rule for the points between. This procedure results in the relationship

$$\int_o^{t_f} \dot{v}_{lk}(t)dt \approx \left[\left(\dot{v}_{lk}(t_1) + \dot{v}_{lk}(t_2) + \dot{v}_{lk}(t_{N-1}) + \dot{v}_{lk}(t_N)\right)/2\right]\Delta t$$

$$+ \frac{1}{3}\Delta t\left[\dot{v}_{lk}(t_2) + \dot{v}_{lk}(t_{N-1}) + 4\sum_{i=1}^{(N-3)/2} \dot{v}_{lk}(t_{2i+1})\right.$$

$$\left. + 2\sum_{i=1}^{(N-3)/2} \dot{v}_{lk}(t_{2i})\right] \qquad \text{(ii)}$$

A direct relationship between the integration and summation can now be obtained by taking the average of approximations (i) and (ii),

$$\int_o^{t_f} \dot{v}_{lk}(t)dt \approx (i + ii)/2$$

$$= \Delta t\left[\sum_{i=1}^{N} \dot{v}_{lk}(t_i) - \left(\frac{7}{12}\right)\left(\dot{v}_{lk}(t_1) + \dot{v}_{lk}(t_N)\right)\right.$$

$$\left. + \left(\frac{1}{12}\right)\left(\dot{v}_{lk}(t_2) + \dot{v}_{lk}(t_{N-1})\right)\right] \qquad \text{(iii)}$$

or

$$\sum_{i=1}^{N} \dot{v}_{lk}(t_i) \approx \frac{1}{\Delta t}\int_o^{t_f} \dot{v}_{lk}(t)dt$$

$$+ \left(\frac{1}{12}\right)\left[7\left(\dot{v}_{lk}(t_1) + \dot{v}_{lk}(t_N)\right) - \left(\dot{v}_{lk}(t_2) + \dot{v}_{lk}(t_{N-1})\right)\right]$$

VI APPLICATIONS

6.1 LINEAR SYSTEMS

6.1.1 Simulated Data

The short-period dynamics of the C-8 airplane in the landing approach were simulated and the attitude rate response due to an elevator deflection was computed. The initial conditions were set equal to zero. Three different noise sequences, all having a variance of (0.005) rad^2 and a 0.2 second correlation time constant, were added to this attitude rate data tc give three different runs. These same three noise sequences were also subtracted from the attitude rate data to give three additional runs making a total of six runs.

For this particular example it is convenient to model the unknown system in its canonical form. The canonical form for the short-period equations of motion with only measurements of attitude rate is given by

$$\left.\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} f_{11} & 1 \\ f_{21} & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix} \delta_e \qquad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_0 = 0 \\ q &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned}\right\} \tag{6.1}$$

where z_1 is the attitude rate, z_2 is a linear combination of attitude rate and angle of attack, and δ_e is the elevator deflection. The set of parameters in K, L, δG, and δz_o which can be used to identify the unknown parameters in equations (6.1) are

$$K = \begin{bmatrix} k_{11} \\ k_{21} \end{bmatrix}, \qquad L = 0, \qquad \delta G = \begin{bmatrix} \delta g_{11} \\ \delta g_{21} \end{bmatrix}, \qquad \delta z_o = 0 \tag{6.2}$$

To illustrate the equations of motion method and response curve fitting method portions of the combined algorithm independently, the six runs were first analyzed by means of the equations of motion method portion of the algorithm. The estimated parameters of the canonical form were averaged over the six runs to reduce the error in these estimates due to variance and thereby illustrate the bias error. These averaged estimates are plotted in figure 5 against the number of iterations. The initial choice of the parameters, F_N, H_N, and G_N, denoted by the zero estimate, was purposely made considerably different from the actual values to emphasize the insensitivity of the convergence on this initial estimate. By the second iteration, the procedure has essentially reached a steady-state value for the unknown parameters, and subsequent iterations do not significantly change these estimates. The important point is that there is a very definite bias in these answers.

To illustrate that the bias observed in figure 5 can be eliminated by switching to the response curve fitting method, the final averaged estimates obtained by the equations of motion method in figure 5 were used to initiate the response curve fitting method for the same six runs. The average values of these estimates are plotted against the number of iterations in figure 6. As is shown, the bias is quickly removed. The final averaged parameter estimates are very close to the actual values.

6.1.2 Flight Data

The combined parameter estimation algorithm is also illustrated here by application to two sets of flight data. The first set of data included measurements of the attitude rate and elevator deflection for the C-8 airplane in the landing approach configuration over a period of 4 seconds. These data were used to identify the coefficients of the transfer function relating pitch rate to elevator deflection. The airplane was initially trimmed and therefore the initial conditions were

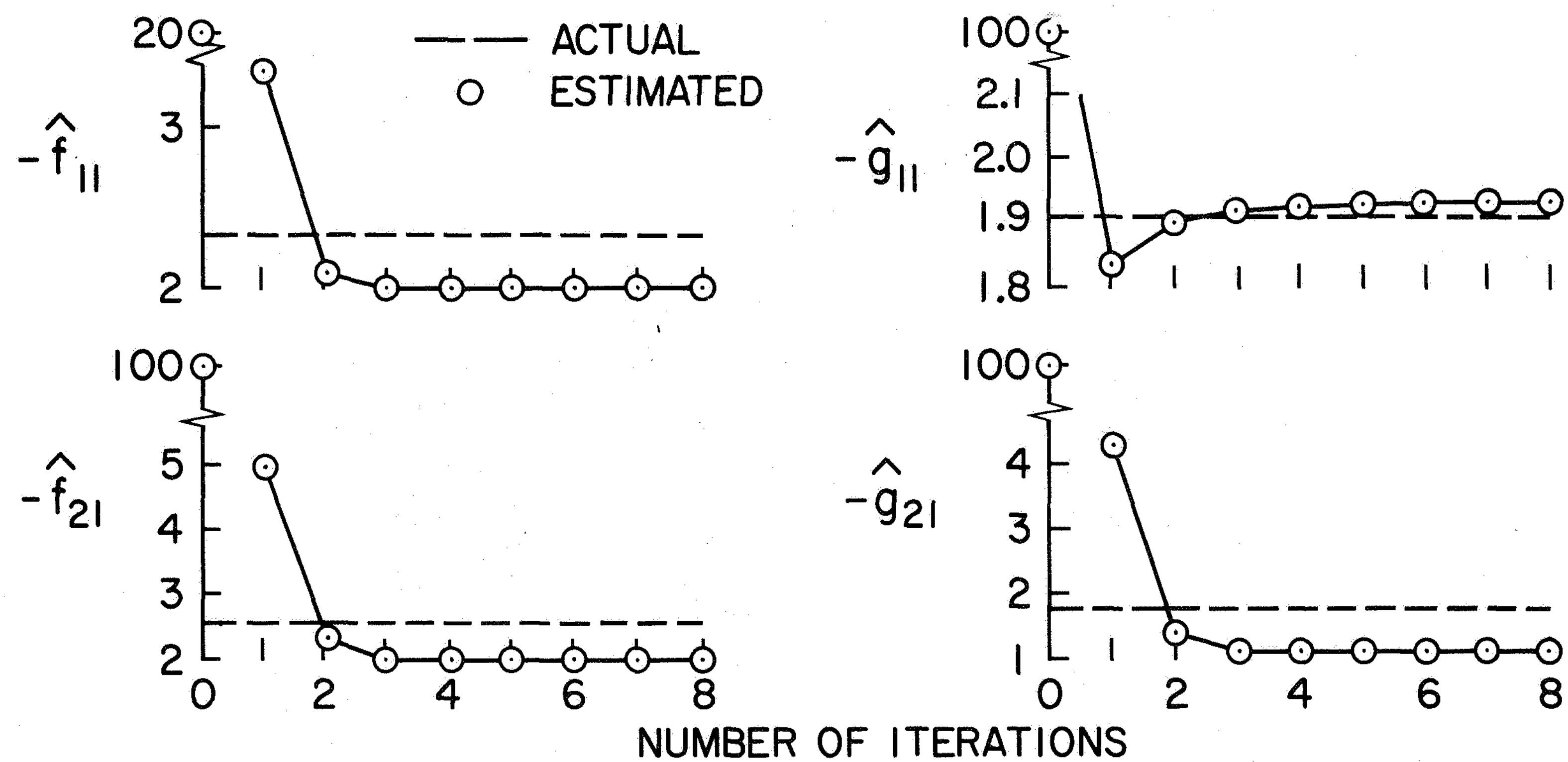


Figure 5.- Application of equations of motion method illustrating convergence and bias.

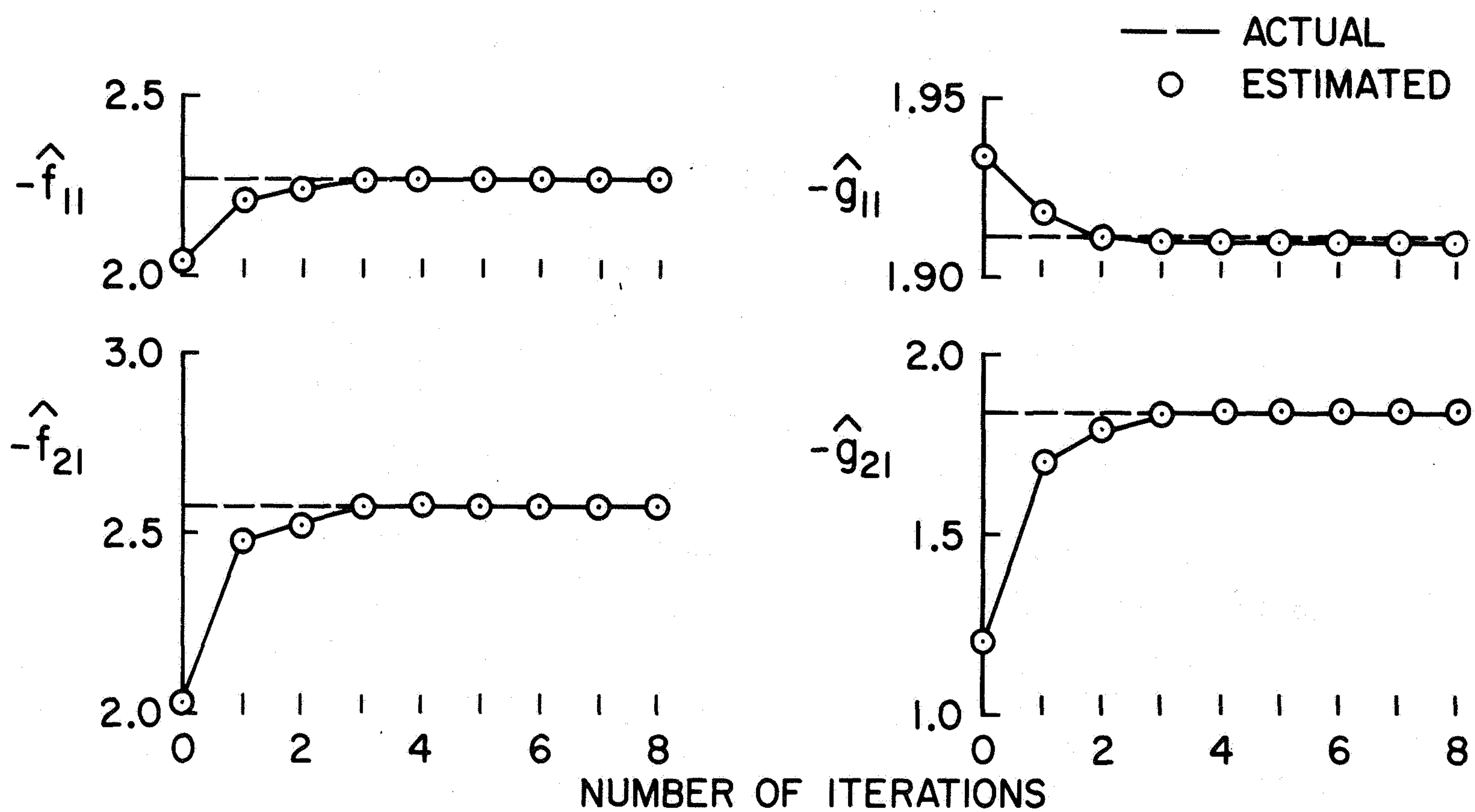


Figure 6.- Application of response curve fitting method illustrating elimination of bias.

assumed to be zero. The aircraft was excited by a doublet type elevator deflection. Because of the type of input and the short duration of data, the phugoid (long-period) mode was not noticeably excited. For this reason, the system was represented by the short-period dynamics and was modeled by the single-output canonical form (6.1). The estimated coefficients are plotted against the number of iterations in figure 7. The initial or zero estimate was purposely made considerably different from the expected system parameters to emphasize again the insensitivity of the method on this initial estimate. After four iterations, the parameters settled to a steady-state value. The equations of motion method portion of the combined algorithm was used during the first two iterations. The combined algorithm then switched to the response curve fitting method.

An indication of the accuracy of this identification is in figure 8. The time history of the elevator input is shown on the left side of the figure. This input was used together with the identified system dynamics

$$\ddot{q} + 2.276\,\dot{q} + 2.558\,q = -1.913\,\dot{\delta}_e - 1.82\,\delta_e \tag{6.3}$$

to compute an estimated attitude rate. The computed attitude rate is shown by the solid line on the right side of the figure and the measured attitude rate by the symbols. Clearly, the estimated transfer function provides a very good relationship between the input and output data.

The second set of data included measurements of the attitude rate, forward velocity, vertical acceleration, angle of attack, and elevator deflection for the C-8 airplane in the landing approach configuration over a period of 17 seconds. These data were used to identify the parameters in the linearized longitudinal equations of motion. In this case, the phugoid dynamics were definitely excited. The body axes of the

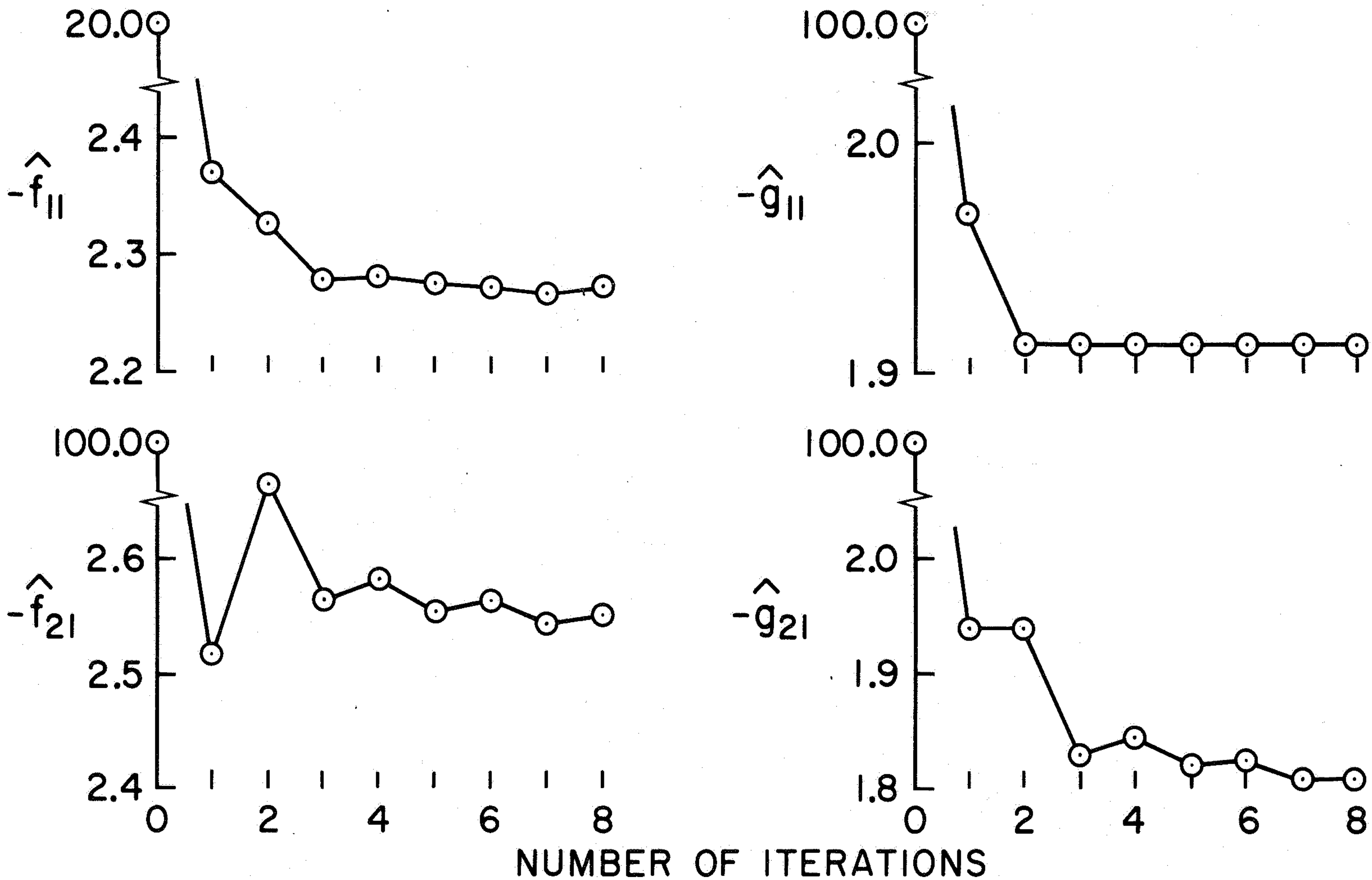


Figure 7.- Application of combined algorithm to flight test data short period dynamics.

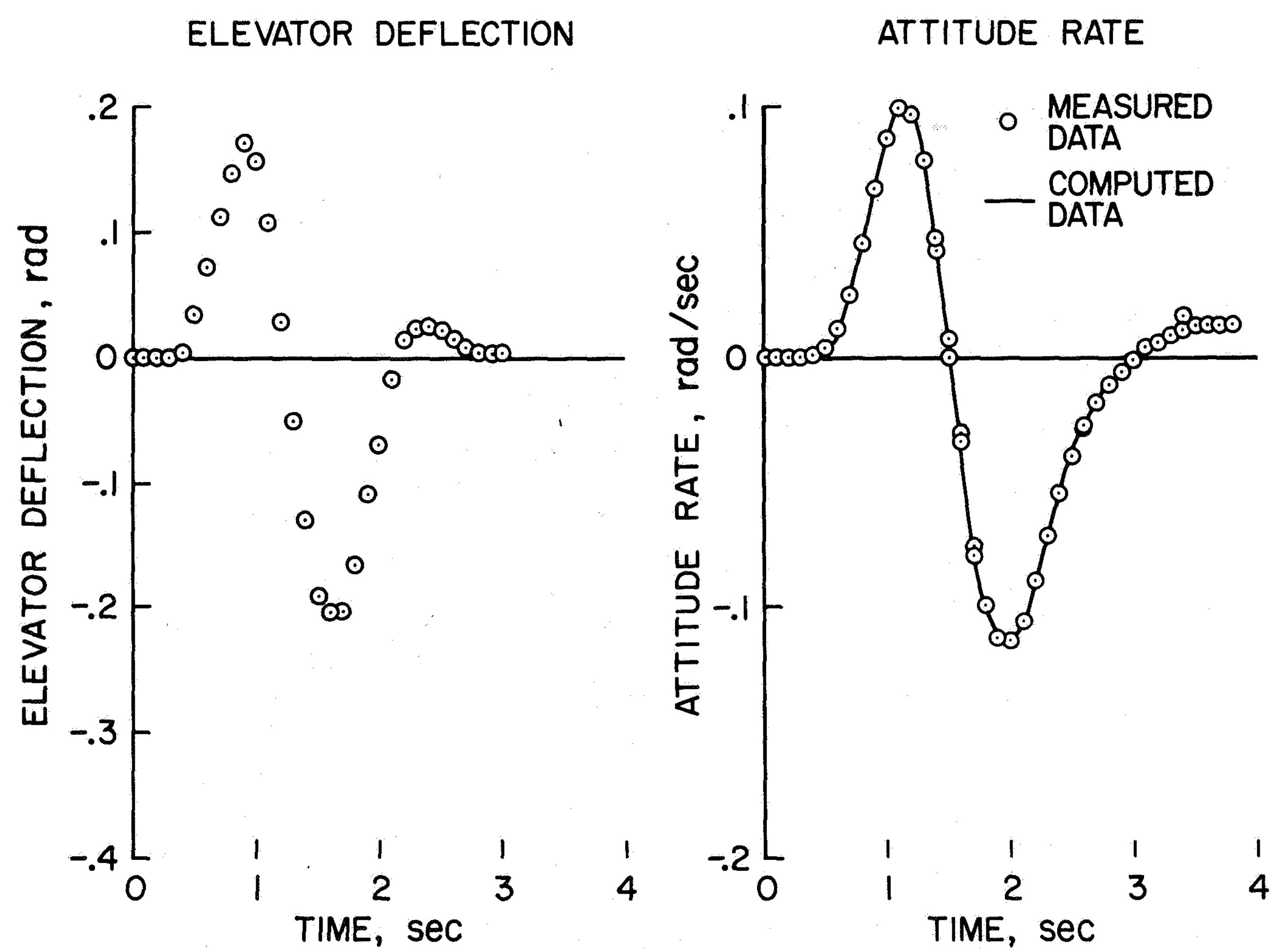


Figure 8.- Comparison of measured and estimated data.

airplane were nearly alined with the stability axes so that the vertical trim velocity, w_o, was set equal to zero. The vehicle and measurements were modeled by the equations

$$\begin{bmatrix} \dot{u} \\ \dot{\theta} \\ \dot{q} \\ \dot{a}_z \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g & -w_o & 0 & \frac{X_\alpha}{m} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{M_u}{Iy}+\frac{M_{\dot{\alpha}}Z_u}{Iymu_o} & 0 & \frac{M_{\dot{\alpha}}+M_q}{Iy} & 0 & \frac{M_{\dot{\alpha}}Z_\alpha}{Iymu_o}+\frac{M_\alpha}{Iy} \\ -20\,\frac{Z_u}{m} & 0 & 0 & -20 & -20\,\frac{Z_\alpha}{m} \\ \frac{Z_u}{mu_o} & 0 & 1 & 0 & \frac{Z_\alpha}{mu_o} \end{bmatrix} \begin{bmatrix} u \\ \theta \\ q \\ a_z \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{M_{\dot{\alpha}}Z_{\delta_e}}{Iymu_o}+\frac{M_{\delta_e}}{Iy} \\ -20\,\frac{Z_{\delta_e}}{m} \\ \frac{Z_{\delta_e}}{mu_o} \end{bmatrix} \delta_e \tag{6.4}$$

The states u, θ, q, and α are the perturbations in forward velocity, attitude, attitude rate, and angle of attack from level steady-state flight; a_z is a filtered measurement of the vertical acceleration. The filter time constant was 0.05 second, and this is indicated by the factor of 20.0 occurring in the equation for acceleration. The control variable, δ_e, is the elevator deflection. The trim velocity u_o and the gravitational constant g are assumed known. The vehicle was initially trimmed, so the initial conditions were assumed to be zero. The other parameters in the F and G matrices depend on the aerodynamic and mass characteristics of the vehicle and are considered unknown.

In this case it is not necessary to go to the canonical form. Since the unknown parameters in (6.4) are coefficients of the measured states, u, q, and α, a matrix F_N can be chosen identical to F except for the numeric values of the unknown parameters and still be related to F by the equation $F_N = F - KH$. The dependency between the parameters in the fourth and fifth rows of F and G in (6.4) (i.e., $f_{41} = -20u_of_{51}$,

$f_{45} = -20u_o f_{55}$, and $g_{41} = -20u_o g_{51}$) can be maintained by including this dependency in F_N and by defining K and δG as indicated below. There are only seven unknown parameters in K and two in δG. The set of parameters in K, L, δG, and δz_o to be used in the combined algorithm is given by

$$K = \begin{bmatrix} K_{11} & 0 & 0 & K_{14} \\ 0 & 0 & 0 & 0 \\ K_{31} & K_{32} & 0 & K_{34} \\ -20u_o K_{51} & 0 & 0 & -20u_o K_{54} \\ K_{51} & 0 & 0 & K_{54} \end{bmatrix}, \qquad L = 0 \tag{6.5}$$

$$\delta G = \begin{bmatrix} 0 \\ 0 \\ \delta g_{31} \\ -20u_o \delta g_{51} \\ \delta g_{51} \end{bmatrix}, \qquad \delta z_o = 0 \tag{6.6}$$

Since this is a multioutput situation, an appropriate weighting matrix, W, must be chosen for use in equation (4.1). For this example, the reciprocals of the weightings on u, q, a, and α were chosen to be $(1\ \text{ft/sec})^2$, $(1°/\text{sec})^2$, $(1\ \text{ft/sec}^2)^2$, and $(2°)^2$, respectively, and reflect the relative confidence in the measurements.

The results of this identification are shown in the 10 columns of figure 9. The parameter symbols are given in the first column. The initial estimates used to start the algorithm are given in the second column. The third and fourth columns give the estimates after the first two iterations using the equations of motion method. The remaining

PARAMETER AND INITIAL ESTIMATES		COMBINED ESTIMATION ALGORITHM							
SYMBOL	INITIAL ESTIMATES	EQUATION ERROR		OUTPUT ERROR					
$\frac{X_u}{m}$	-0.01	-0.017	-0.020	-0.020	-0.020	-0.020	-0.020	-0.020	-0.020
$\frac{M_u}{I_{yy}} + \frac{M_{\dot{\alpha}} Z_u}{I_{yy} m u_o}$	0.0	.002	.002	.003	.003	.003	.003	.003	.003
$\frac{Z_u}{m u_o}$	0.0	-.003	-.003	-.004	-.004	-.004	-.004	-.004	-.004
$\frac{M_{\dot{\alpha}} + M_q}{I_{yy}}$	-3.0	-1.373	-1.308	-1.602	-1.58	-1.584	-1.585	-1.587	-1.588
$\frac{X_\alpha}{m}$	0.0	27.837	33.144	33.887	33.685	33.705	33.720	33.731	33.736
$\frac{M_{\dot{\alpha}} Z_\alpha}{I_{yy} m u_o} + \frac{M_\alpha}{I_{yy}}$	0.0	-.404	-.578	-.541	-.567	-.564	-.564	-.563	-.562
$\frac{Z_\alpha}{m u_o}$	-1.0	-.83	-.808	-.739	-.736	-.737	-.737	-.737	-.737
$\frac{M_{\dot{\alpha}} Z_{\delta_e}}{I_{yy} m u_o} + \frac{M_{\delta_e}}{I_{yy}}$	-1.0	-1.466	-1.441	-1.651	-1.654	-1.656	-1.656	-1.657	-1.658
$\frac{Z_{\delta_e}}{m u_o}$	-1.0	-.007	.012	.005	.006	.005	.005	.005	.005
ITERATION No.	0	1	2	3	4	5	6	7	8

Figure 9.- Application of combined algorithm to multioutput data.

columns correspond to successive iterations using the response curve fitting method. Significant changes in the unknown parameters do not occur after the third or fourth iteration.

The identified parameters were used with the measured input to compute time histories of the velocity, attitude rate, vertical acceleration, and angle-of-attack perturbations. The computed and measured quantities are compared in figure 10. As in the first example, the estimated parameters provide a very good relationship between the input and output data.

6.1.3 Digital Modeling of Continuous Systems

There are many integration algorithms that can be used to solve differential equations on a digital computer. The identification algorithm discussed in this report has been implemented by the Adams-Moulton method, the Runge-Kutta method, and a discrete transition matrix method. All three methods were used to estimate the parameters in the short-period transfer function relating pitch rate to elevator deflection from flight data. Figure 11 shows a comparison of the results.

For the sample length (0.05 sec) and for the dynamics in this problem, the effect of the integration algorithm on the parameter estimates is negligible. However, when the Runge-Kutta and transition matrix methods are used, some care must be taken in interpreting the input.

If the Runge-Kutta method is used, then the solution of the differential equations at time $t_i + \Delta t$ depends on the solution of the equations at time t_i and on the input at times t_i, $t_i + \Delta t/2$, and $t_i + \Delta t = t_{i+1}$. The input $u(t)$ is measured only at times t_i, and t_{i+1}; it must therefore be approximated at time $t_i + \Delta t/2$. Since $u(t)$

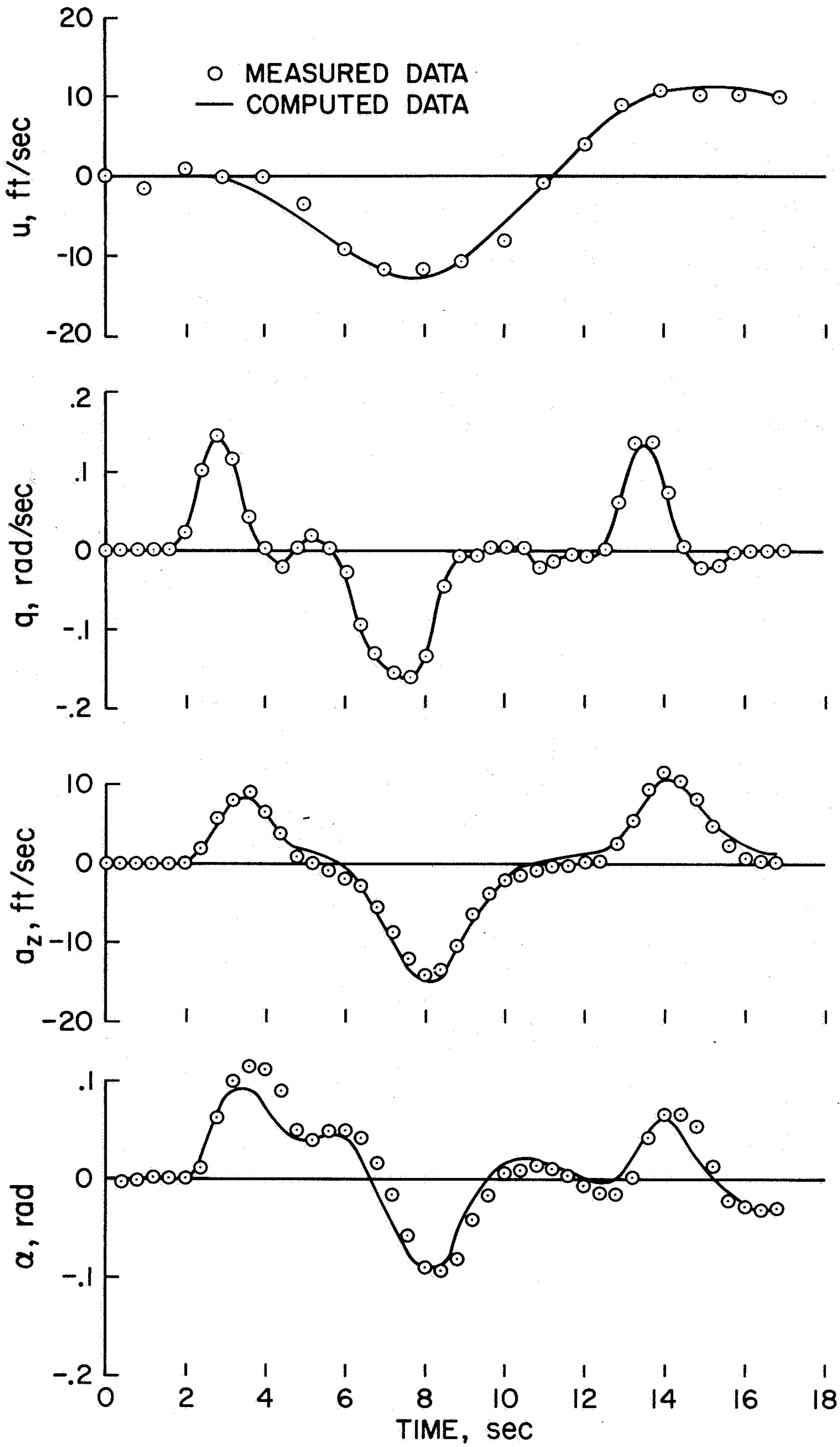


Figure 10.- Comparison of measured and estimated data.

$$\frac{q(s)}{\delta_e(s)} = \frac{b_1 S + b_0}{S^2 - a_1 S - a_0}$$

INTEGRATION ALGORITHM	PARAMETERS			
	b_1	b_0	a_1	a_0
ADAMS–MOULTON	−1.960	−.589	−1.976	−1.559
RUNGE–KUTTA	−1.966	−.593	−1.980	−1.560
TRANSITION MATRIX	−1.968	−.588	−1.982	−1.556

Figure 11.- Effect of integration algorithms on the parameter estimates.

is a continuous function and Δt is quite small, it may seem that $u(t_i + \Delta t/2)$ can be approximated by either $u(t_i)$ or $u(t_{i+1})$. It is clear that such an approximation will change the phase relationships between the input and the output. A better approximation for $u(t_i + \Delta t/2)$ is given by a linear interpolation of the measured data,

$$u(t_i + \Delta t/2) = \big(u(t_i) + u(t_{i+1})\big)/2 \tag{6.7}$$

The effect of these three interpretations of the input on the estimates for the parameters is illustrated in figure 12. The difference in the estimates is significant. The estimates presented in figure 11 were obtained by using (6.7).

In the transition matrix method, the differential equations are represented by the discrete equations

$$\left.\begin{aligned} x(t_{i+1}) &= \Phi x(t_i) + \Gamma\bar{u}(t_i) \\ y(t_i) &= Hx(t_i) \end{aligned}\right\} \tag{6.8}$$

where $\bar{u}(t_i)$ is a piecewise constant approximation of the input $u(t)$. The matrices Φ and Γ are related to the F and G matrices in the differential equations by

$$\Phi = e^{F\Delta t} \tag{6.9}$$

$$\Gamma\bar{u}(t_i) = \int_{t_i}^{t_{i+1}} e^{F(t-\tau)}Gu(\tau)d\tau = \int_{t_i}^{t_{i+1}} e^{F(t-\tau)}Gd\tau\bar{u}(t_i)$$

$$= F^{-1}[\Phi - I]G\bar{u}(t_i) \tag{6.10}$$

The parameters can be estimated from the estimates of Φ and Γ by the relationships

$$\left.\begin{aligned} F &= \frac{1}{\Delta t}\log[\Phi - I] \approx \frac{1}{\Delta t}\left\{[\Phi - I] - \frac{1}{2}[\Phi - I]^2 + \frac{1}{3}[\Phi - I]^3 + \ldots\right\} \\ G &= [\Phi - I]^{-1}F\Gamma \end{aligned}\right\} \tag{6.11}$$

$u\left(t_i + \frac{\Delta t}{2}\right)$ APPROXIMATED BY	PARAMETERS			
	b_1	b_0	a_1	a_0
$u(t_i)$	−2.050	−.390	−2.075	−1.355
$\frac{u(t_i) + u(t_i + \Delta t)}{2}$	−1.966	−.593	−1.98	−1.560
$u(t_i + \Delta t)$	−1.876	−.820	−1.907	−1.788

Figure 12.- Effect of input interpretation on parameter estimates - Runge-Kutta.

Again, since $u(t)$ is continuous and Δt is small, it may seem that $\bar{u}(t_i)$ can be approximated by either $u(t_i)$ or $u(t_{i+1})$. Both approximations will cause an error in the phase relationships between the input and output. A better approximation is to use an averaged value for $\bar{u}(t_i)$

$$\bar{u}(t_i) = [u(t_i) + u(t_{i+1})]/2 \tag{6.12}$$

The effect of these three interpretations of the input on the estimates for the parameters is illustrated in figure 13. As in the discussion of the Runge-Kutta method, the difference in the estimates is significant. The estimates in figure 11 were obtained by using equation (6.12).

6.1.4 Effects of Certain Model Errors

If one suspects that there are biases in the measurements and if there are uncertainties in the initial conditions, then these quantities should be estimated as well as the parameters in the differential equations. The effect of including these terms in the identification of the system parameters is illustrated in this section.

A maneuver similar to that discussed in the first part of section 6.1.2 was repeated eight times during a single flight of the C-8 airplane. The data from each maneuver were used to estimate the parameters in the transfer function relating pitch rate to elevator deflection. Although the plane approached steady-state trim conditions between maneuvers, the initial conditions and the trim elevator position were not quite zero. In the first identification of these parameters, the initial conditions and unknown biases were assumed to be zero. The results of this identification are shown in figure 14. The parameter symbols are given in the first column; the estimates for these parameters obtained from the individual maneuvers are given in the next eight columns. The last two columns contain the mean and mean squared error of these parameters. The mean was computed by averaging the estimated parameters, and

$\bar{u}(t_i)$ APPROXIMATED BY	PARAMETERS			
	b_1	b_0	a_1	a_0
$u(t_i)$	−2.091	−.290	−2.131	−1.255
$\frac{u(t_i) + u(t_i + \Delta t)}{2}$	−1.968	−.589	−1.982	−1.556
$u(t_i + \Delta t)$	−1.832	−.941	−1.881	−1.909

Figure 13.- Effect of input interpretation on parameter estimates - transition matrix method.

PARAMETER	MANEUVER								MEAN	M.S.E. PARAMETERS
	1	2	3	4	5	6	7	8		
b_1	-2.07	-1.87	DID NOT CONVERGE	-1.78	-1.69	-1.43	-1.61	-1.49	-1.75	.035
b_0	1.06	.12		-1.23	-.38	-4.97	-.79	-1.93	-.525	.915
a_1	-1.55	-1.80		-2.17	-1.91	-4.13	-2.09	-2.55	-2.012	.098
a_0	1.21	-.27		-1.67	-1.02	-5.73	-1.31	-2.88	-.99	1.58
M.S.E. SYSTEM RESPONSE, $(rad/sec)^2$	$.14 \times 10^{-4}$	$.1 \times 10^{-4}$		$.09 \times 10^{-4}$	$.33 \times 10^{-4}$	$.07 \times 10^{-4}$	$.12 \times 10^{-4}$	$.21 \times 10^{-4}$		

Figure 14.- Identification of system parameters - initial conditions and biases assumed to be zero.

the mean squared error was computed by averaging the square of difference between the estimated parameters and the computed mean. The parameters obtained from the sixth maneuver were not included in these computations because they were substantially different from those obtained during the other maneuvers. The mean square error of the system response is defined by

$$\text{M.S.E.} = \left(\frac{1}{N}\right) \sum_{i=1}^{N} (q(t_i) - \hat{q}(t_i))^2 \tag{6.13}$$

where q is the attitude rate and is indicated in the bottom row of the figure. Although the estimated parameters provide a good fit of the data as indicated by the M.S.E., there is a large variation in the estimated parameters. This is particularly true for the parameters identified in the first maneuver. The algorithm did not even converge for the data from the third maneuver.

In the second identification of the system parameters, the initial conditions were also treated as unknown parameters. The results are shown in figure 15. The initial conditions are indicated in the first column by the symbols $x_1(0)$ and $x_2(0)$. The estimated parameters in the first maneuver agree better with those obtained from the other maneuvers and the third maneuver converged without difficulty. Again, the estimates of the parameters for the sixth maneuver were not included in the computation of the mean and mean squared error for the parameter estimates. Including the initial conditions has reduced the computed variance in the parameter estimates by nearly a factor of 3.

PARAMETER	MANEUVER								MEAN	M.S.E. PARAMETERS
	1	2	3	4	5	6	7	8		
b_1	−1.77	−1.88	−1.73	−1.78	−1.73	−1.40	−1.56	−1.47	−1.70	.0168
b_0	−.38	.06	−.80	−1.00	−.59	−5.25	−.58	−1.91	−.743	.323
a_1	−1.96	−1.84	−2.08	−2.04	−2.04	−4.28	−1.98	−2.53	−2.07	.0408
a_0	−.98	−.31	−1.41	−1.46	−1.13	−6.12	−1.26	−2.89	−1.35	.523
$x_1(O)$	−.004	.000	−.001	−.000	.002	−.000	−.002	.003		
$x_2(O)$	.022	−.001	.035	.005	−.010	.012	.016	.009		
M.S.E. SYSTEM RESPONSE, $(rad/sec)^2$	$.65 \times 10^{-5}$	1.0×10^{-5}	$.41 \times 10^{-5}$	$.81 \times 10^{-5}$	3.3×10^{-5}	$.70 \times 10^{-5}$	1.0×10^{-5}	2.0×10^{-5}		

Figure 15.- Identification of system parameters and initial conditions - biases assumed to be zero.

In the third identification, the initial conditions and a bias error in the trim elevator position were treated as unknown parameters.[9] The results of this identification are illustrated in figure 16. The computed variances for the parameters b_o and a_o have been reduced from the results in figure 15 and this identification appears to be the best of the three considered.

6.2 IDENTIFICATION OF A NONLINEAR SYSTEM USING A COMBINED ALGORITHM

6.2.1 Problem Statement

This problem was posed by personnel of the Cornell Aeronautical Laboratory who supplied the equations of motion as well as the simulated data. The equations of motion describe the longitudinal response of VTOL type aircraft and are given in body axes by

$$\dot{x} = A\begin{bmatrix} x \\ \hdashline z \end{bmatrix}, \qquad y = \begin{bmatrix} x \\ \hdashline n_x \\ n_z \\ \dot{q} \end{bmatrix} + v(t) = \begin{bmatrix} I_{3\times 3} & \vdots & 0 \\ \hdashline & A' & \end{bmatrix}\begin{bmatrix} x \\ \hdashline z \end{bmatrix} + v(t)$$

$$\left.\begin{aligned}
&x^T = [\theta,\ u,\ w,\ q] \\
&z^T = [qu,\ u^2,\ uw,\ qw,\ \sin(\theta_o + \theta),\ \cos(\theta_o + \theta),\ u\delta_e,\ \delta_e,\ 1] \\
&A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_u & X_w & -w_o & 0 & X_{u^2} & X_{uw} & -1 & -g & 0 & X_{u\delta_e} & X_{\delta_e} & X_o \\ 0 & Z_u & Z_w & u_o & 1 & Z_{u^2} & Z_{uw} & 0 & 0 & g & Z_{u\delta_e} & Z_{\delta_e} & Z_o \\ 0 & M_u & M_w & M_q & M_{qu} & M_{u^2} & M_{uw} & 0 & 0 & 0 & M_{u\delta_e} & M_{\delta_e} & M_o \end{bmatrix} \\
&A' = A \ \text{ except that the circled terms are zero.}
\end{aligned}\right\} \tag{6.14}$$

---

[9]Because this is a single-input single-output problem, the effect of any biases in the measurements will be eliminated by treating the initial conditions and a bias in the input as unknown parameters.

PARAMETER	MANEUVER								MEAN	M.S.E. PARAMETERS
	1	2	3	4	5	6	7	8		
b_1	−1.78	−1.83	−1.72	−1.80	−1.69	−1.40	−1.50	−1.47	−1.68	.0178
b_0	−1.45	−1.47	−.53	−1.28	−1.41	−5.26	−2.04	−1.93	−1.44	.208
a_1	−2.46	−2.51	−1.95	−2.20	−2.41	−4.29	−2.72	−2.53	−2.40	.0539
a_0	−2.02	−1.94	−1.16	−1.65	−2.09	−6.13	−2.95	−2.94	−2.11	.364
$x_1(O)$	−.002	−.000	−.001	.001	.002	−.000	−.004	.002		
$x_2(O)$	.008	.011	.050	−.002	.003	.012	.035	.012		
BIAS (rad)	−.005	.005	.008	−.002	.005	.000	.005	.000		
M.S.E. SYSTEM RESPONSE, $(rad/sec)^2$	$.62 \times 10^{-5}$	$.92 \times 10^{-5}$	$.40 \times 10^{-5}$	$.81 \times 10^{-5}$	3.1×10^{-5}	$.69 \times 10^{-5}$	$.82 \times 10^{-5}$	2.0×10^{-5}		

Figure 16.- Identification of system parameters, initial conditions and biases.

θ, u, w, and q are deviations from the trim attitude, horizontal velocity, vertical velocity, and attitude rate, n_x and n_z are accelerations in ft/sec^2, δ_e is the elevator input, and the noise is gaussian with zero mean and covariance

$$E\{v(t_i)v(t_j)^T\} = R\delta_{ij} \tag{6.15}$$

The parameters w_o, u_o, and g in A are assumed known. The remaining parameters are to be estimated. It should be noted that the identifiability of these parameters is dependent on the input. For example, if δ_e is a step input there is no way of distinguishing between the coefficients of u and u_{δ_e}; these coefficients must therefore be combined.

6.2.2 Estimation Technique

Using the measurement error techniques, we will minimize the function

$$J = \left(\frac{1}{2}\right) \sum_{i=1}^{N} [y(t_i) - \hat{y}(t_i)]^T W[y(t_i) - \hat{y}(t_i) \tag{6.16}$$

with respect to the unknown parameters in the constraint equations

$$\left.\begin{aligned} \dot{\hat{x}} &= A \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} \\ \hat{y} &= \left[\begin{array}{c|c} I_{3\times3} & 0 \\ \hline \multicolumn{2}{c}{A'} \end{array}\right] \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} \triangleq C \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} \end{aligned}\right\} \tag{6.17}$$

where W is a diagonal matrix and the elements in W are updated by taking the inverse of the sample variances of the residuals; in other words, W is set equal to the inverse of the estimate for R. One method for solving this problem is the method of quasi-linearization discussed in Chapter 2. Although quasi-linearization provides fast convergence if initiated from a sufficiently accurate initial estimate, an initial estimate is not always available. The idea behind a combined algorithm is to modify the existing computational structure in order to implement an equation error method on the first iteration. In this example, an appropriate modification is particularly straightforward. The response, y_N, and the components of the matrix of sensitivity functions $A(t)$ are computed by

$$\left.\begin{aligned} \dot{x}_N &= A_N \begin{bmatrix} x_N \\ z_N \end{bmatrix} \\ y_N &= C_N \begin{bmatrix} x_N \\ z_N \end{bmatrix} \end{aligned}\right\} \tag{6.18}$$

and

$$\left.\begin{aligned} \dot{x}_{\gamma_i} &= A_N \begin{bmatrix} x_{\gamma_i} \\ \frac{\partial z}{\partial x} x_{\gamma_i} \end{bmatrix} + \frac{\partial A}{\partial \gamma_i} \begin{bmatrix} x_N \\ z_N \end{bmatrix} \triangleq F(t)\Big|_{x=x_N} x_{\gamma_i} + \frac{\partial A}{\partial \gamma_i} \begin{bmatrix} x_N \\ z_N \end{bmatrix} \\ y_{\gamma_i} &= C_N \begin{bmatrix} x_{\gamma_i} \\ \frac{\partial z}{\partial x} x_{\gamma_i} \end{bmatrix} + \frac{\partial C}{\partial \gamma_i} \begin{bmatrix} x_N \\ z_N \end{bmatrix} \triangleq H(t)\Big|_{x=x_N} x_{\gamma_i} + \frac{\partial C}{\partial \gamma_i} \begin{bmatrix} x_N \\ z_N \end{bmatrix} \end{aligned}\right\} \tag{6.19}$$

where y_{γ_i} is the *ith* column of $A(t)$. If on the first iteration, we set x_{γ_i} equal to zero for all i, and use the measured data to compute the vector $\begin{bmatrix} x_N \\ z_N \end{bmatrix}$, we have the derivative method which was discussed in example 2.1.

<u>6.2.3 Results</u>

The results of this identification are illustrated in figures 17(a) and 17(b). The parameter symbols are given in the first column. The

SYMBOL	SIMULATED VALUES	EQUATION ERROR	OUTPUT ERROR				ESTIMATED STANDARD DEVIATION
X_u	-.1697	-.1750	-.1703	-.1704	-.1703	-.1703	$.24 \times 10^{-2}$
X_w	.0148	.0190	.0151	.0151	.0151	.0151	$.94 \times 10^{-3}$
$X_u{}^2$	-.0003	-.0002	-.0003	-.0003	-.0003	-.0003	$.45 \times 10^{-4}$
X_{uw}	-.0016	-.0014	-.0015	-.0015	-.0015	-.0015	$.63 \times 10^{-4}$
$X_{u\delta_e}$	.0184	.0648	.0206	.0208	.0205	.0205	$.12 \times 10^{-1}$
X_{δ_e}	1.614	1.614	1.608	1.609	1.608	1.608	$.19 \times 10^{-1}$
X_o	1.312	1.290	1.314	1.314	1.316	1.316	$.20 \times 10^{-1}$
Z_u	-.9072	-.9638	-.9148	-.9146	-.9156	-.9159	$.58 \times 10^{-2}$
Z_w	-.6608	-.6338	-.6642	-.6644	-.6646	-.6643	$.30 \times 10^{-2}$
$Z_u{}^2$	-.0067	-.0055	-.0069	-.0069	-.0070	-.0070	$.12 \times 10^{-3}$
Z_{uw}	-.0025	-.00082	-.0026	-.0026	-.0028	-.0028	$.15 \times 10^{-3}$
$Z_{u\delta_e}$	.0167	.3381	.0216	.0214	.0353	.0372	$.25 \times 10^{-1}$
Z_{δ_e}	1.764	1.885	1.836	1.837	1.873	1.877	$.80 \times 10^{-4}$
Z_o	-32.14	-32.42	-32.21	-32.20	-32.22	-32.23	$.80 \times 10^{-1}$
M_u	-.0047	-.0054	-.0048	-.0048	-.0047	-.0047	$.24 \times 10^{-4}$

Figure 17(a).- Results of nonlinear identification - run 2A.

SYMBOL	SIMULATED VALUES	EQUATION ERROR	OUTPUT ERROR				ESTIMATED STANDARD DEVIATION
M_w	-.0089	-.0095	-.0091	-.0091	-.0089	-.0089	$.23 \times 10^{-4}$
M_q	-.6324	-.5901	-.625	-.6247	-.6312	-.6312	$.11 \times 10^{-2}$
M_{uq}	-.0012	-.0014	-.0016	-.0016	-.0011	-.0011	$.11 \times 10^{-3}$
M_{u^2}	$-.63 \times 10^{-5}$	$-.29 \times 10^{-4}$	$-.11 \times 10^{-4}$	$-.11 \times 10^{-4}$	$-.65 \times 10^{-5}$	$-.65 \times 10^{-5}$	$.80 \times 10^{-6}$
M_{uw}	$-.54 \times 10^{-4}$	$-.82 \times 10^{-4}$	$-.60 \times 10^{-4}$	$-.60 \times 10^{-4}$	$-.55 \times 10^{-4}$	$-.55 \times 10^{-4}$	$.76 \times 10^{-6}$
$M_{u\delta_e}$	.0012	.0025	.0011	.0011	.0012	.0012	$.44 \times 10^{-4}$
M_{δ_e}	.4792	.4760	.4789	.4789	.4792	.4792	$.90 \times 10^{-4}$
M_o	0.0	$-.88 \times 10^{-4}$	$.16 \times 10^{-3}$	$.14 \times 10^{-3}$	$-.20 \times 10^{-4}$	$-.21 \times 10^{-4}$	$.58 \times 10^{-4}$
$\frac{1}{R_{11}}$, $(1/\text{rad})^2$	$.36 \times 10^7$	$.10 \times 10^6$	$.10 \times 10^6$	$.25 \times 10^7$	$.31 \times 10^7$	$.33 \times 10^7$	
$\frac{1}{R_{22}}$, $(1/\text{ft/sec})^2$	4.0	10.0	10.0	4.2	4.2	4.178	
$\frac{1}{R_{33}}$, $(1/\text{ft/sec})^2$	177.8	100.0	100.0	197.1	199.8	197.6	
$\frac{1}{R_{44}}$, $(1/\text{rad/sec})^2$	$.33 \times 10^8$	$.1 \times 10^7$	$.1 \times 10^7$	$.22 \times 10^8$	$.29 \times 10^8$	$.32 \times 10^8$	
$\frac{1}{R_{55}}$, $(1/\text{ft/sec}^2)^2$	966.3	1000.0	1000.0	1062.	1071.	1068.	
$\frac{1}{R_{66}}$, $(1/\text{ft/sec}^2)^2$	38.65	100.0	100.0	37.6	37.6	37.35	
$\frac{1}{R_{77}}$, $(1/\text{rad/sec}^2)$	$.82 \times 10^9$	$.1 \times 10^7$	$.1 \times 10^7$	$.16 \times 10^9$	$.17 \times 10^9$	$.64 \times 10^9$	

Figure 17(b).- Results of nonlinear identification - run 2A.

second column contains the actual values of the parameters used to generate the data. The third column gives the estimates obtained by the equations of motion method. The next five columns contain estimates of the parameters obtained from successive iterations by the method of quasi-linearization. The last column contains estimates of the mean square error in the parameter estimates. These estimates were obtained by means of equation (2.32).

The numerical values of the parameters used in the W matrix do not affect the parameter estimates obtained in the equations of motion method for this problem. These parameters were set equal to the number $1/R_{ii}$, $i = 1, \ldots, 7$ listed in the first column of figure 17(b). These parameters were held fixed until the third iteration at which time they were estimated from the resulting residuals. These estimates were used to update the weighting matrix in the fourth iteration. Similarly, the estimates obtained for these parameters in the fourth iteration were used to update the weighting matrix in the fifth iteration.

VII CONCLUDING REMARKS

A method of parameter estimation has been presented that combines the best properties of the equations of motion and response curve fitting techniques. In the absence of noise, the procedure provides a weighted least-squares estimate for the unknown parameters in a single operation. If there is noise in the system, this estimate will be biased. The bias error can be removed by applying the procedure iteratively.

A canonical form is presented for multioutput systems. Modeling the system in this canonical form provides a set of identifiable parameters that can be estimated using the combined algorithm.

The combined algorithm has been applied successfully to the identification of the parameters in the longitudinal equations of aircraft motion using both simulated and flight data.

A method has been presented for computing the sensitivity functions for constant-coefficient linear systems, which requires fewer differential equation solutions than other methods. The method is based on linear transformations of solutions to a basic set of differential equations. For the single-output, multi-input system, these equations are particularly easy to implement. This technique for computing single-output sensitivity functions has been implemented and has substantially reduced computation time.

Some suggestions have been made for simplifying the computation of the integral square of the sensitivity functions. These integrals are used in the method of quasi-linearization and in the combined algorithm.

It has been shown that the generalized equations of motion theory discussed by Shinbrot can be used to derive the results presented by

Luenberger and Bryson for observers of lower order. The generalized equations of motion method also provides a useful method for designing such observers.

APPENDIX A

MINIMIZATION ALGORITHM

Many methods can be used to minimize the criterion

$$\left.\begin{aligned} J(\gamma) &= \frac{1}{2}\sum_{i=1}^{N}\left[y(t_i) - \hat{y}(t_i)\right]^T W_i\left[y(t_i) - \hat{y}(t_i)\right] \\ &\text{or} \\ J(\gamma) &= \frac{1}{2}\int_0^{t_f}\left[y(t) - \hat{y}(t)\right]^T W(t)\left[y(t) - \hat{y}(t)\right]dt \end{aligned}\right\} \tag{A1}$$

with respect to the unknown parameters, γ, in the constraint equations

$$\left.\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u, \gamma, t) \\ \hat{y} &= h(\hat{x}, u, \gamma, t) \end{aligned}\right\} \tag{A2}$$

Three of the more common methods are: (1) the first-order gradient method, (2) the method of quasi-linearization, and (3) the second-order gradient method. All of these methods can be related to the terms retained in a Taylor series expansion of J about an initial estimate of the unknown parameters denoted here by the subscript N,

$$J(\gamma) \approx J_N + \left.\frac{\partial J}{\partial \gamma}\right|_{x=x_N} \delta\gamma + \frac{1}{2}\,\delta\gamma^T \left.\frac{\partial^2 J}{\partial \gamma^2}\right|_{x=x_N} \delta\gamma + \text{higher order terms} \tag{A3}$$

By writing $\partial J/\partial\gamma|_{x=x_N}$ and $\partial^2 J/\partial\gamma^2|_{x=x_N}$ in terms of y_N we obtain

$$\left.\begin{aligned} \left.\frac{\partial J}{\partial \gamma}\right|_{x=x_N} &= -\int_0^{t_f} (y - y_N)^T W \frac{\partial y_N}{\partial \gamma}\,dt \\ \left.\frac{\partial^2 J}{\partial \gamma^2}\right|_{x=x_N} &= -\int_0^{t_f} \frac{\partial}{\partial \gamma}\left[(y - y_N)^T W \frac{\partial y_N}{\partial \gamma}\right]^T dt \\ &= \int_0^{t_f} \left(\frac{\partial y_N}{\partial \gamma}\right)^T W \frac{\partial y_N}{\partial \gamma}\,dt - \int_0^{t_f} \frac{\partial^2 y_N}{\partial \gamma^2} W(y - y_N)dt \end{aligned}\right\} \tag{A4}$$

Using (A4) in (A3) we obtain

$$\left.\begin{aligned} J(\gamma) - J_N = \Delta J = &-\int_0^{t_f} (y - y_N)^T W \frac{\partial y_N}{\partial \gamma} dt\ \delta\gamma && \text{first-order gradient} \\ &+ \frac{1}{2}\,\delta\gamma^T \int_0^{t_f} \left(\frac{\partial y_N}{\partial \gamma}\right)^T W \left(\frac{\partial y_N}{\partial \gamma}\right) dt\ \delta\gamma && \text{quasi-linearization} \\ &- \frac{1}{2}\,\delta\gamma^T \int_0^{t_f} \frac{\partial {y_N}^2}{\partial \gamma^2} W (y - y_N) dt\ \delta\gamma && \text{second-order gradient} \end{aligned}\right. \tag{A5}$$

The first-order gradient procedure retains only the first term in the expansion (A5). It provides information on which direction the parameters should be changed to reduce the cost J,

$$\delta\gamma^T = K \int_0^{t_f} (y - y_N)^T W \frac{\partial y_N}{\partial \gamma} dt \tag{A6}$$

An advantage of the first-order gradient method is that sufficiently small changes in the unknown parameters cause a reduction in the cost. However, the analyst has no way of determining the size of the parameter change. One method is to include a quadratic penalty function in the expression for the first-order gradient cost; in other words, choose $\delta\gamma$ to minimize

$$\Delta J = -\int_0^{t_f} (y - y_N)^T W \frac{\partial y_N}{\partial \gamma} dt\ \delta\gamma + \frac{1}{2}\,\delta\gamma^T B \delta\gamma \tag{A7}$$

where B is a positive definite weighting matrix. The choice of B is dependent on the analyst's experience with the specific problem.

The method of quasi-linearization contains one additional term in the Taylor series expansion. This term is quadratic and if the time histories of $\partial y_N/\partial\gamma$ are linearly independent (which is a condition for identifiability) it is positive definite. Quasi-linearization can therefore be considered a first-order gradient procedure with a special penalty function. It has the same advantages as the first-order gradient procedure in that the parameter changes will be in a direction that will reduce the function J, and only the first-order variations of the model response are required. It has the added advantage that near the minimum it begins to approach true second-order information since the last term in the second-order expansion, (A5), tends toward zero at the minimum (since $y \to y_N$).

The second-order procedure contains all the terms in the expansion (A5). It is the most efficient adjustment algorithm to use from points near the minimum. However, it has two disadvantages. First, it requires second-order variations of the model response, and second, if the initial estimate of the parameters is not near the minimum, the function may have a negative curvature so the parameters will change in the wrong direction.

The quasi-linearization procedure appears to provide a good parameter adjustment scheme for the parameter identification problem. In order to relate the formulation of the method as presented here to that discussed in section 2.2.1, it is only necessary to complete the square.

$$\Delta J = -\int_0^{t_f} (y - y_N)^T W \frac{\partial y_N}{\partial \gamma}\, dt\ \delta\gamma + \frac{1}{2}\, \delta\gamma^T \int_0^{t_f} \left(\frac{\partial y_N{}^T}{\partial \gamma}\right) W \frac{\partial y_N}{\partial \gamma}\, dt\ \delta\gamma$$

$$= \frac{1}{2} \int_0^{t_f} \left(y - y_N - \frac{\partial y_N}{\partial \gamma}\, \delta\gamma\right)^T W \left(y - y_N - \frac{\partial y_N}{\partial \gamma}\, \delta\gamma\right) dt$$

$$- \frac{1}{2} \int_0^{t_f} (y - y_N)^T W (y - y_N)\, dt \tag{A8}$$

Noting that $\partial y_N / \partial \gamma$ corresponds to $A(t)$ and that

$$\int_0^{t_f} (y - y_N)^T W (y - y_N)\, dt$$

is not affected by parameter variations $\delta\gamma$, the minimization of (A8) is equivalent to the minimization discussed in section (2.2.1).

APPENDIX B

THE DESIGN OF LINEAR OBSERVERS BY USING INTEGRAL TRANSFORMS

B.1 INTRODUCTION

The Kalman filter is a well-known technique for estimating the state of a system in the presence of noise (refs. 31, 32). This same structure can also be used to observe the state in the noise-free problem if there are unknown initial conditions. However, Luenberger and Bryson have developed elegant and explicit procedures for designing observers for the noise-free problem that are of lower order than the Kalman filter structure (refs. 36, 37, 41). They have shown that an estimate of the state can be reconstructed from the system measurements and the response of a $(n - m)th$ order filter where n is the order of the system and m is the number of independent measurements. Luenberger has also shown that a linear function of the state can be constructed from the system measurements and the response of an even lower order filter.

Although this report is primarily concerned with parameter estimation, there is a considerable similarity between this problem and the problem of state observation. In particular, Shinbrot's generalization of the equations of motion method through the use of integral transforms is very similar to the idea developed by Luenberger and Bryson for reconstructing the state by passing the measurements through a dynamic filter.

In this appendix, both the Kalman filter structure and the structure for observers of lower order are obtained through the use of integral transforms. Although the results are not basically new, an example will be used to show that this technique provides an alternative procedure for designing observers of lower order.

B.2 PRELIMINARY DISCUSSION

The problem can be stated as follows: Given a constant coefficient linear system described by the equations

$$\dot{x} = Fx + Gu \qquad x(o) = x_o \tag{B1}$$

$$y = Hx \tag{B2}$$

with unknown initial conditions, estimate the current state of the system from the measurements of the system input u and output y. The estimation of a linear function of the current state is a simple extension of this problem and will also be considered.

If there are m independent measurements (B2) provides m algebraic equations which are linear in the n unknown components of the state vector x. If $m < n$, these equations cannot be solved for x uniquely. The number of algebraic equations can be increased if both sides of (B2) are differentiated and (B1) is used to express the resulting equations in terms of x. This procedure results in the set of equations

$$\left.\begin{array}{llll} y^0 & \triangleq y & & = H\ x \\ y^1 & \triangleq \dot{y} - HGu & & = HF\ x \\ y^2 & \triangleq \dot{y}^1 - HFGu & & = HF^2x \\ \vdots & \vdots & \vdots & \vdots \\ y^j & \triangleq \dot{y}^{j-1} - HF^{j-1}Gu & & = HF^jx \end{array}\right\} \tag{B3}$$

where y^i (for each i) is defined as indicated and can be considered an additional measurement. If n independent equations are obtained by this procedure, we can, in principle, solve for the unknown state x and the system is said to be observable. The maximum superscript on y required to obtain n linearly independent equations is denoted by $\nu - 1$; ν is referred to as the observability index for the system. If the system is observable, ν must satisfy the inequality,

$$n/m - 1 \leqslant \nu - 1 \leqslant n - m \tag{B4}$$

If the first n linearly independent equations in (B3) are used to estimate the state, each additional measurement requires only a single differentiation of a previous measurement. The state can therefore be estimated by performing only $n - m$ differentiations.

The difficulty with this approach is that it is usually not possible to differentiate measured data even once, much less several times.

B.3 THE DESIGN OF SUPPLEMENTAL OBSERVERS OR OBSERVERS OF LOWER ORDER

An alternative to differentiating (B2) in order to obtain n linearly independent equations is to take integral transforms of (B2). This was the idea suggested by Shinbrot in the identification problem. The Laplace transform method, illustrated in example (2.2), illustrates this type of procedure. In this section we will use the convolution function, $e^{s_i(t-\tau)}$, as the method function. In order to simplify the equations we will introduce a notation used by Lessing (ref. 29).

Notation B.1

$$T_i z(t) \triangleq \int_0^t e^{s_i(t-\tau)} z(\tau)d\tau \tag{B5}$$

where $z(t)$ is either a vector or a scalar function of time, $e^{s_i(t-\tau)}$ is a scalar, and s_i is a complex or real number.

Two identities which will prove helpful are:

Identity B.1

$$T_i\dot{x}(t) = x(t) + s_iT_ix(t) - x_oe^{s_it} \tag{B6}$$

(Integration by parts)

Identity B.2

$$T_i x(t) = F_i^{-1} x(t) - F_i^{-1} G T_i u(t) - F_i^{-1} x_o e^{s_i t} \tag{B7}$$

where

$$F_i \triangleq [F - s_i I]$$

and s_i is not an eigenvalue of F.

Proof: Take the integral transform of both sides of (B1). Because $e^{s_i t}$ is a scalar, this transform can be written

$$T_i \dot{x} = F T_i x + G T_i u \tag{B8}$$

If identity B1 is used, equation (B8) can be written

$$x(t) = [F - s_i I] T_i x(t) + G T_i u + x_o e^{s_i t} \tag{B9}$$

Since s_i is not an eigenvalue of F, this equation can be solved for $T_i x(t)$ as given in equation (B7) and this concludes the argument.

Let us now augment the set of algebraic equations (B2) as suggested in the beginning of this section. The integral transform of (B2)

$$T_1 y(t) = H T_1 x(t) \tag{B10}$$

can be expressed in terms of x by means of identity (B2). This procedure can then be repeated a number of times in order to obtain the following sequence of equations:

$$\left.\begin{array}{lll}
y^0 \triangleq y & = H & x + 0 \\
y^1 \triangleq T_1 (y^0 + H F_1^{-1} G u) & = H F_1^{-1} & x + \varepsilon_1 \\
y^2 \triangleq T_2 (y^1 + H F_1^{-1} F_2^{-1} G u) & = H F_1^{-1} F_2^{-1} & x + \varepsilon_2 - T_2 \varepsilon_1 \\
\vdots \qquad \vdots & \vdots & \vdots \\
y^{\nu-1} \triangleq T_{\nu-1}\left(y^{\nu-2} + H F_1^{-1} F_2^{-1} \ldots F_{\nu-1}^{-1} G u\right) & = H F_1^{-1} F_2^{-1} \ldots F_{\nu-1}^{-1} & x + \varepsilon_{\nu-1} - T_{\nu-1} \varepsilon_{\nu-2} \\
 & & - \ldots T_{\nu-1} T_{\nu-2} \ldots T_2 \varepsilon_1
\end{array}\right\} \tag{B11}$$

where

$$\varepsilon_j = HF_1^{-1} \ . \ . \ . \ F_j^{-1} x_o e^{s_j t}$$

and y^j (for each j) is defined as indicated and can be considered an additional measurement. If the system is observable, this sequence will have n independent equations (see Lemma B1 and theorem B1 at the end of this paragraph), which can be solved for x as a function of the measurements y^i and the functions ε_i. The y^i can be computed but the ε_i are not known because the initial conditions are not known. Because the terms ε_i decay with a time constant dependent on s_i and because the s_i can be chosen almost arbitrarily, x can be approximated by solving (B11) with the ε_i set equal to zero. This approximation of x is referred to as an estimate of x and is denoted by $\hat{x}$. The error in the estimate would be proportional to the ε_i which are, in turn, proportional to the initial conditions.

<u>Comment</u>: If some of the initial conditions are known or approximately known, they can be included in the measurements y^i in order to reduce the error in the estimate for x. Let x_{N_o} be a best initial guess at the initial conditions. The state can then be estimated by solving n independent equations in the sequence

$$\left.\begin{array}{llll}
y^0 \triangleq y & & = & Hx \\
y^1 \triangleq T_1(y^0 + HF_1^{-1}Gu) & +y_{N_o}^1 e^{s_1 t} & = & HF_1^{-1}x + \varepsilon_1 \\
y^2 \triangleq T_2(y^1 + HF_1^{-1}F_2^{-1}Gu) & +y_{N_o}^2 e^{s_2 t} & = & HF_1^{-1}F_2^{-1}x + \varepsilon_2 - T_2\varepsilon_1 \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
y^{\nu-1} \triangleq T_{\nu-1}\left(y^{\nu-2} + HF_1^{-1}F_2^{-1}\ldots F_{\nu-1}^{-1}Gu\right) & +y_{N_o}^{\nu-1} e^{s_{\nu-1} t} & = & HF_1^{-1}F_2^{-1}\ldots F_{\nu-1}^{-1}x + \varepsilon_{\nu-1} \\
 & & & -T_{\nu-1}\varepsilon_{\nu-2} - \ldots - T_{\nu-1}T_{\nu-2}\ldots T_2\varepsilon_1
\end{array}\right\}$$

(B12)

where

$$\varepsilon_j = HF_1^{-1}F_2^{-1} \ . \ . \ . \ F_j^{-1}(x_o - x_{N_o})e^{s_j t}$$

$$y_{N_o}^j = HF_1^{-1}F_2^{-1} \ . \ . \ . \ F_j^{-1}x_{N_o}$$

with the ε_i set equal to zero.

We will now show that if the system has observability index ν, then there are n linearly independent equations in the sequence (B11).

Lemma B.1 If A, B, and C are three matrices such that the matrix product ABC is defined, then the rank of this matrix product is related to the ranks of A, B, and C by the inequality

$$r_A + r_B + r_C - q - p \leqslant r_{ABC} \leqslant \min(r_A, r_B, r_C)$$

where r_A, r_B, r_C, and r_{ABC} denote the rank of A, B, C, and ABC, respectively, and where q and p are the number of columns in A and B, respectively.

Proof: Sylvester's inequality for the rank of the product of two matrices, AB, states that

$$r_A + r_B - q \leqslant r_{AB} \leqslant \min(r_A, r_B)$$

This implies

$$r_A + r_B + r_C - q - p \leqslant r_{AB} + r_C - p \leqslant r_{ABC} \leqslant \min(r_{AB}, r_C) \leqslant \min(r_A, r_B, r_C)$$

and this concludes the argument.

Theorem B.1: If the observability index for a system (B.1) is ν, then the matrix

$$O_I = \begin{bmatrix} H \\ HF_1^{-1} \\ \cdot \\ \cdot \\ \cdot \\ HF_1^{-1} \cdots F_{\nu-1}^{-1} \end{bmatrix}$$

has rank n.

<u>Proof</u>:

$$\begin{bmatrix} H \\ HF_1^{-1} \\ \cdot \\ \cdot \\ \cdot \\ HF_1^{-1} \cdots F_{\nu-1}^{-1} \end{bmatrix} = \begin{bmatrix} HF_1F_2 \cdots F_{\nu-1} \\ \cdot \\ \cdot \\ \cdot \\ HF_1 \\ H \end{bmatrix} \left[F_1F_2 \cdots F_{\nu-1}\right]^{-1} \tag{B13}$$

because the F_i commute. In addition

$$\begin{bmatrix} HF_1F_2 \cdots F_{\nu-1} \\ \cdot \\ \cdot \\ \cdot \\ HF_1 \\ H \end{bmatrix} = \begin{bmatrix} I & a_1^1 I & a_2^1 I & \cdots & a_{\nu-1}^1 I \\ 0 & I & a_1^2 I & & a_{\nu-1}^2 I \\ \cdot & & I & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & & & \\ 0 & 0 & 0 & & I \end{bmatrix} \begin{bmatrix} HF^{\nu-1} \\ HF^{\nu-2} \\ \cdot \\ \cdot \\ \cdot \\ HF \\ H \end{bmatrix} \tag{B14}$$

where I is an $m \times m$ identity matrix and where the a_i^j are the coefficients of the polynomials.

$$(\lambda + s_j)(\lambda + s_{j+1}) \cdots (\lambda + s_{\nu-1}) = \lambda^{\nu-j} + a_1^j \lambda^{\nu-j-1} + a_2^j \lambda^{\nu-j-2} + \cdots + a_{\nu-j}^j$$

The first matrix on the right of (B14) has rank nm, the second matrix on the right of (B14) has rank n, and because each F_i has rank n, the second matrix on the right of (B13) has rank n. By Lemma B.1

$$nm + n + n - nm - n \leqslant r_{O_I} \leqslant \min(nm, n, n)$$

which implies r_{O_I} (the rank of O_I) equals n and this concludes the argument.

If the first n linearly independent equations in (B12) are used to solve for $\hat{x}$, each equation in addition to the first m equations can be realized by passing a linear combination of previously generated "measurements" through a first-order filter with initial condition defined by the appropriate component of y_{N_0}. A total of $n - m$ additional equations is required. We can therefore estimate the state by using a filter having order $n - m$ with $\nu - 1$ distinct and almost arbitrarily chosen eigenvalues. The filter can also be designed with $n - m$ distinct eigenvalues by using different transformations on each of the measurements. One method involves a tedious selection procedure; another method involves putting the system into the canonical form discussed in Chapter III and applying the selection procedure defined above for each individual single-output subsystem. This latter procedure was used in reference 37. We therefore obtain the following important result.

Theorem B.2: Given a n*th*-order system (B1) which is observable through m independent measurements (B2), an estimate of the system state can be constructed from the measurements of the input and output and the response of a $(n - m)$*th*-order filter. The error in the estimate will decay with a time constant equal to the negative of the real part of the inverse of the eigenvalues in the filter. The eigenvalues of the filter can be chosen arbitrarily, provided they do not equal any of the system's eigenvalues.

Comment: The eigenvalues of the observer can equal the eigenvalues of the system, but in this case the transformation acts only on the system input and not on the measurement. If s_i is an eigenvalue of the system, an equation relating $x(t)$ and $T_i u$ can be obtained by taking the inner product of equation (B9) with the eigenvector associated with F_i.

In many cases a complete estimate of the state is not required. For example, if we are designing a single input feedback control law, we may require only a single linear combination of the states. It is therefore only necessary to estimate this linear function. It has been shown in this section that an estimate of the state is given by a linear transformation on the augmented measurements (y^i, $i = 0, 1, \ldots, \nu - 1$). This can be denoted by the matrix equation.

$$\hat{x} = D \begin{bmatrix} y^0 \\ y^1 \\ \cdot \\ \cdot \\ \cdot \\ y^{\nu-1} \end{bmatrix}$$

where D is a $n \times (\nu \times m)$ matrix. A linear combination of the states can therefore be estimated by a linear combination of the augmented measurements. This linear combination can be written

$$u(t) = \sum_{i=1}^{m} \left[\alpha_i^0 y_i^0 + \sum_{j=1}^{\nu-1} \alpha_i^j y_i^j \right]$$

where y_i^j is the i*th* component of y^j. The individual terms in the above summation can be defined

$$z_i = \sum_{j=1}^{\nu-1} \alpha_i^j y_i^j$$

and are the solutions to the single output systems

$$\begin{bmatrix} \dot{y}_i^1 \\ \dot{y}_i^2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{y}_i^{\nu-1} \end{bmatrix} = \begin{bmatrix} S_1 & 0 & 0 & \cdots & 0 \\ 1 & S_2 & 0 & & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & & 1 & s_{\nu-1} \end{bmatrix} \begin{bmatrix} y_i^1 \\ y_i^2 \\ \\ \\ \\ y_i^{\nu-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} y_i + \begin{bmatrix} h_{(i)}F_1^{-1} \\ h_{(i)}F_1^{-1}F_2^{-1} \\ \cdot \\ \cdot \\ \cdot \\ h_{(i)}F_1^{-1}F_2^{-1} \cdots F_{\nu-1}^{-1} \end{bmatrix} Gu$$

$$z_i = \begin{bmatrix} \alpha_i^1 \alpha_i^2 & \cdots & \alpha_i^{\nu-1} \end{bmatrix} \begin{bmatrix} y_i^1 \\ \cdot \\ \cdot \\ \cdot \\ y_i^{\nu-1} \end{bmatrix}$$

Because these systems have identical dynamics for all i, the summation of the z_i can be realized by a single system or order $\nu - 1$. This provides a second important result, which was also first proved by Luenberger.

Theorem B.3: If the system is observable, then an estimate of an arbitrary linear function of the state can be constructed from the measurements of the input and output and the response of a $(\nu - 1)$*th* order filter.

Example B.1 Longitudinal Equations of Motion for an Aircraft

Consider the linearized longitudinal equations of motion for an aircraft including both the short and long period modes. Assume that the attitude rate, forward velocity, and the elevator input are the only measured variables. The equations describing the system and its measurements can be written

$$\begin{bmatrix} \dot{u} \\ \dot{\theta} \\ \dot{q} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 1 & 0 \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & 1 & a_{44} \end{bmatrix} \begin{bmatrix} u \\ \theta \\ q \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_3 \\ b_4 \end{bmatrix} \delta_e \qquad \text{(B14)}$$

$$y_1 = u$$

$$y_2 = q$$

i) Construct a state estimation of order $n - m = 2$ having time constants of 0.5 second.

ii) Construct a system of order $\nu - 1 = 1$ that can be used to estimate an arbitrary linear combination of the states.

Solution:

i) If $e^{-2(t-\tau)}$ is used as the method function and definition 1 and identity 1 are applied, then the integral transform of (B15) is given by

$$\begin{bmatrix} u(t) \\ \theta(t) \\ q(t) \\ \alpha(t) \end{bmatrix} = \begin{bmatrix} a_{11}+2 & a_{12} & a_{13} & a_{14} \\ 0 & 2 & 1 & 0 \\ a_{31} & 0 & a_{33}+2 & a_{34} \\ a_{41} & 0 & 1 & a_{44}+2 \end{bmatrix} \begin{bmatrix} T_1 u \\ T_1 \theta \\ T_1 q \\ T_1 \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_3 \\ b_4 \end{bmatrix} T_1 \delta_e + \begin{bmatrix} u(0) \\ \theta(0) \\ q(0) \\ \alpha(0) \end{bmatrix} e^{-2t} \qquad \text{(B16)}$$

If the initial conditions are known, then, because $u(t)$ and $q(t)$ are measured and since we can generate $T_1 u(t)$ and $T_1 q(t)$, equation (B16) provides four equations in the four unknowns $\theta(t)$, $\alpha(t)$, $T_1\theta(t)$, and $T_1\alpha(t)$. These equations can be solved for $T_1\alpha(t)$, $T_1\theta(t)$, $\theta(t)$, and $\alpha(t)$ if the third, first, second and fourth equations, respectively, are used as indicated below.

$$T_1\alpha(t) = q(t)/a_{34} - (a_{31}/a_{34})T_1 u - (a_{33}+2)/a_{34} T_1 q - (b_3/a_{34})T_1\delta_e - q(0)/a_{34} e^{-2t} \qquad \text{(B17)}$$

$$T_1\theta(t) = u(t)/a_{12} - (a_{11}+2)/a_{12}T_1u - (a_{13}/a_{12})T_1q - (a_{14}/a_{12})T_1\alpha - u(0)/a_{12}e^{-2t}$$

$$= u(t)/a_{12} - (a_{14}/a_{12}a_{34})q(t) - [(a_{11}+2)/a_{12} - a_{14}a_{31}/a_{12}a_{34}]T_1u$$

$$-[a_{13}/a_{12} - a_{14}(a_{33}+2)/a_{12}a_{34}]T_1q + a_{14}b_3/a_{12}a_{34}T_1\delta_e$$

$$-[u(0)/a_{12} - (a_{14}/a_{12}a_{34})q(0)]e^{-2t} \quad \text{(B18)}$$

$$\theta(t) = 2T_1\theta(t) + T_1q(t) + \theta(0)e^{-2t} = 2(u(t)/a_{12} - (a_{14}/a_{12}a_{34})q(t))$$

$$-2[(a_{11}+2)/a_{12} - a_{14}a_{31}/a_{12}a_{34}]T_1u$$

$$-[2(a_{13}/a_{12} - a_{14}(a_{33}+2)/a_{12}a_{34}) + 1]T_1q + 2a_{14}b_3/a_{12}a_{34}T_1\delta_e$$

$$-[2(u(0)/a_{12} - (a_{14}/a_{12}a_{34})q(0)) - \theta(0)]e^{-2t} \quad \text{(B19)}$$

$$\alpha(t) = a_{41}T_1u + T_1q + (a_{44}+2)T_1\alpha + b_4T_1\delta_e + \alpha(0)e^{-2t}$$

$$= [(a_{44}+2)/a_{34}]q(t) + [a_{41} - (a_{44}+2)a_{31}/a_{34}]T_1u$$

$$+[1 - (a_{44}+2)(a_{33}+2)/a_{34}]T_1q + [b_4 - (a_{44}+2)b_3/a_{34}]T_1\delta_e$$

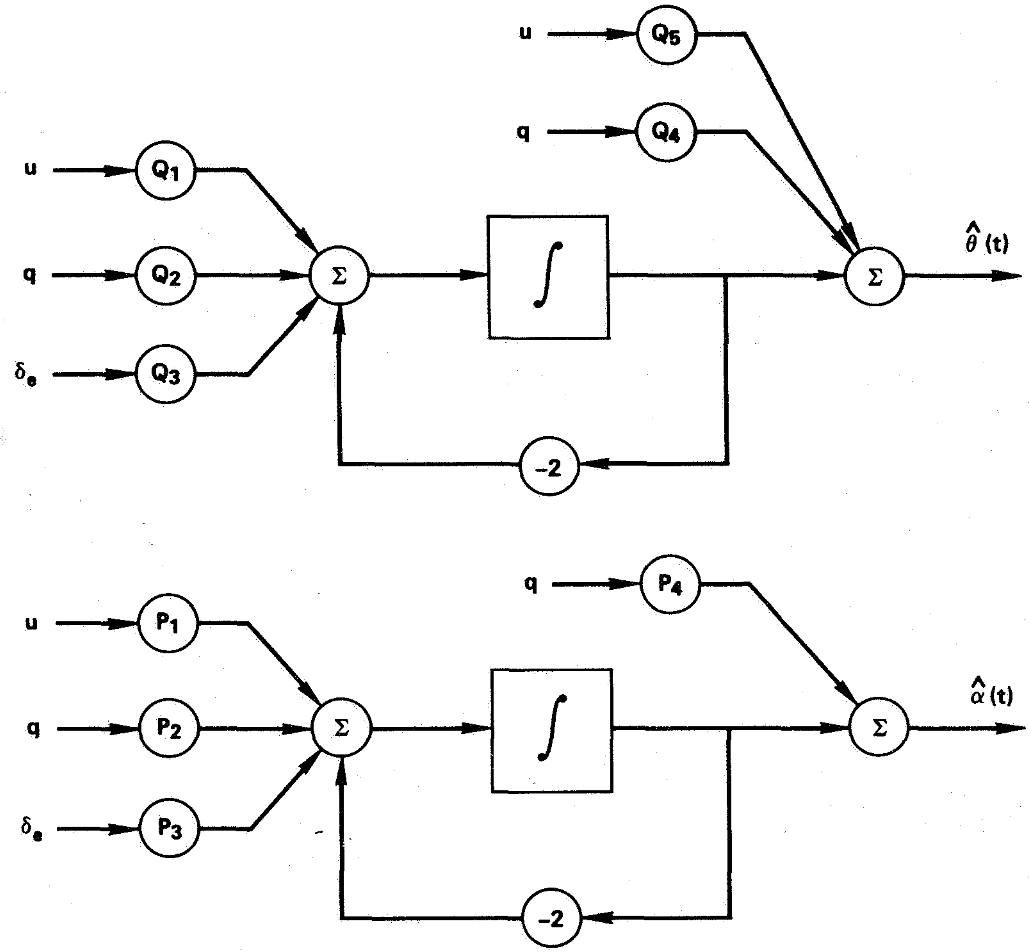
$$+[\alpha(0) - ((a_{44}+2)/a_{34})q(0)]e^{-2t} \quad \text{(B20)}$$

Since all of the terms on the right hand side of these equations are known except for the initial conditions, estimates for $\theta(t)$ and $\alpha(t)$ can be obtained by neglecting the initial conditions. The errors in the estimates would be

$$\alpha(t) - \hat{\alpha}(t) = [\alpha(0) - ((a_{44} + 2)/a_{34})q(0)]e^{-2t}$$

$$\theta(t) - \hat{\theta}(t) = [\theta(0) - 2(u(0)/a_{12} - (a_{14}/a_{12}a_{34})q(0))]e^{-2t}$$

The state observer therefore consists of two identical and uncoupled first-order systems. Their structures are indicated in figure 18.



$$Q_1 = -2\left[\frac{(a_{11} + 2)}{a_{12}} - \frac{a_{14}\,a_{31}}{a_{12}\,a_{34}}\right] \qquad P_1 = a_{41} - (a_{44} + 2)\,\frac{a_{31}}{a_{34}}$$

$$Q_2 = -\left[2\left(\frac{a_{13}}{a_{12}} - \frac{a_{14}}{a_{12}}\,\frac{(a_{33} + 2)}{a_{34}}\right) + 1\right] \qquad P_2 = 1 - (a_{44} + 2)\,\frac{(a_{33} + 2)}{a_{34}}$$

$$Q_3 = 2\,\frac{a_{14}}{a_{12}}\,\frac{b_3}{a_{34}} \qquad P_3 = b_4 - (a_{44} + 2)\,\frac{b_3}{a_{34}}$$

$$Q_4 = -2\,\frac{a_{14}}{a_{12}\,a_{34}} \qquad P_4 = \frac{(a_{44} + 2)}{a_{34}}$$

$$Q_5 = \frac{2}{a_{12}}$$

Figure 18.- Estimation of attitude and angle of attack from measurements of attitude rate and forward velocity.

ii) Because the state observer was constructed so that both unmeasured states were estimated by identical first-order systems, a linear combination of the estimated states can be obtained by using a single first-order system. The estimation of an arbitrary linear combination of the system states, $c_1u(t) + c_2\theta(t) + c_3q(t) + c_4\alpha(t)$, can therefore be estimated as shown in figure 19.

B.4 LINEAR OBSERVER OF ORDER n

Consider the integral transform of (B1) with the $n \times n$ matrix $e^{F_N(t-\tau)}$ as the method function,

$$\int_0^t e^{F_N(t-\tau)}\dot{x}(\tau)d\tau = \int_0^t e^{F_N(t-\tau)}[Fx(\tau) + Gu(\tau)]d\tau \tag{B21}$$

If the left hand side of (B21) is integrated by parts and the terms are combined, this equation can be written

$$\int_0^t e^{F_N(t-\tau)}\{[F - F_N]x(\tau) + Gu(\tau)\}d\tau + e^{F_Nt}x_o \tag{B22}$$

If F_N is chosen so that $F - F_N = KH$, then (B2) can be used in (B22) to obtain a relationship between $x(t)$, $y(t)$, $u(t)$, and x_o,

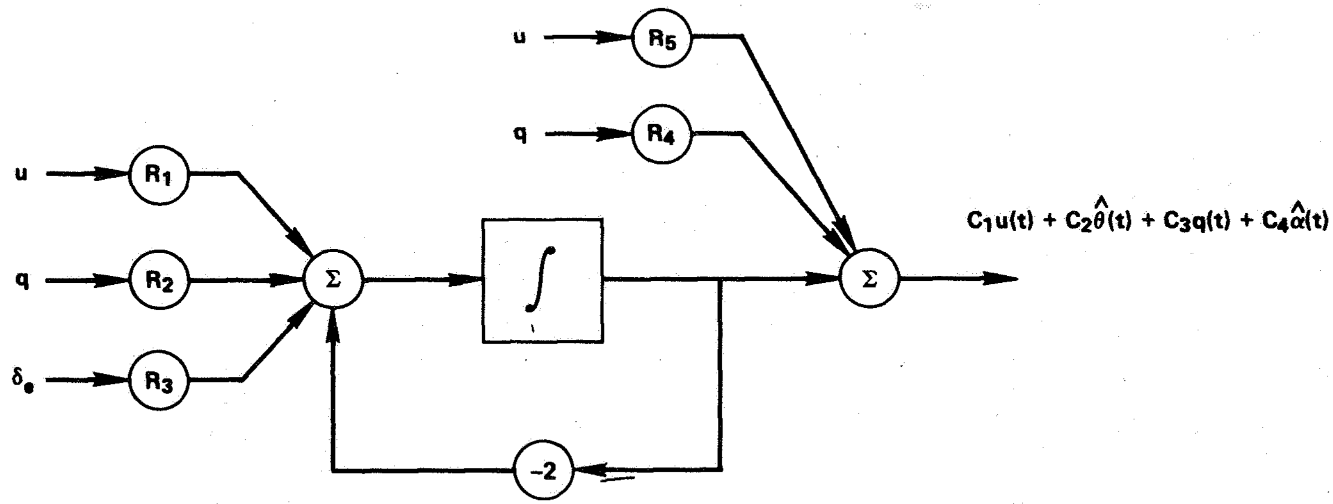
$$x(t) = \int_0^t e^{F_N(t-\tau)}\{Ky(\tau) + Gu(\tau)\}d\tau + e^{F_Nt}x_o \tag{B23}$$

so that an estimate of $x(t)$ is given by

$$\hat{x}(t) = \int_0^t e^{F_N(t-\tau)}\{Ky(\tau) + Gu(\tau)\}d\tau \tag{B24}$$

and the error in the estimate is given by

$$x(t) - \hat{x}(t) = e^{F_Nt}x_o$$



$R_1 = C_2Q_1 + C_4P_1$
$R_2 = C_2Q_2 + C_4P_2$
$R_3 = C_2Q_3 + C_4P_3$
$R_4 = C_2Q_4 + C_3 + C_4P_4$
$R_5 = C_1 + C_2Q_5$

NOTE: Q_i AND P_i ARE DEFINED AS IN FIGURE 18

Figure 19.- Estimation of a linear functional of the system states from measurements of attitude rate and forward velocity.

Since it is possible to arbitrarily place the eigenvalues of $F - KH$ by an appropriate choice of K, the error in the estimate can be made to go to zero arbitrarily fast. Equation (B24) is the solution to the differential equation

$$\dot{\hat{x}} = F\hat{x} + Gu + K[y - H\hat{x}] \qquad \hat{x}(0) = x_o \tag{B25}$$

which has the same structure as the state estimators studied by Kalman and Luenberger.

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