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A FIRST-ORDER THEORY FOR ROTATING GRAVITY GRADIOMETERS

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SUMMARY

The equations of motion for a rotating gradiometer are developed and specialized to a forced harmonic oscillator model that may be solved analytically. Numerical results are given comparing the analytical approximation to numerical integration of the exact equations.

INTRODUCTION

A rotating gradiometer may be visualized as two perpendicular dumbells coupled at their centers by a mechanical torsional spring pivot. Gradiometers have been studied experimentally and theoretically for application to a variety of problems involving the measurement of gravitational fields [1-3].

Since the gravitational torque on each dumbell of the rotating gradiometer is opposite (and equal if they have equal moments of inertia), there will be a deflection of the spring pivot coupling. This deflection is the quantity that is observed and related to the torque which may then be related to the external gravitational field and mass.

The general relationship for the torque on any rigid body because of an external gravitational field is presented in the Appendix. This report begins with that basic relationship and develops the specific equations for a rotating gradiometer. Conditions at resonance are also discussed since the gradiometer may be rotated at a speed commensurate with the fundamental frequency of the spring to give a resonant condition.

GENERAL EQUATIONS

As shown in the Appendix, the gravitational torque on any rigid body is given by

$$\vec{N} = -G \int_m V, \Lambda I \, dm \quad , \tag{1}$$

where G is the universal gravitational constant, m is the external mass, I is the moment of inertia matrix of the rigid body, and V, is the gravity-gradient matrix of the gravitational field generated by m. It is also shown in the Appendix that equation (1) is invariant under orthonormal coordinate transformations.

If m is considered a spherical homogeneous mass and $\vec{N}$, V, and I are referred to the principal axes of the small body, equation (1) becomes

$$\vec{N}' = \frac{3mG}{R^5} \vec{R}' \times I' \vec{R}' \quad , \tag{2}$$

where the primes indicate that the quantities are referred to the principal axes of the small body and I' is the diagonalized moment of inertia matrix of the small body; i.e.,

$$I' = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \quad .$$

Figure 1 gives the geometrical relationship of the prime system (fixed along the principal axes of dumbell no. 1), the unprimed system (fixed in the laboratory), and the $\vec{R}$ vector.

The torque on dumbell no. 1 referred to the x, y, z system may be written as

$$\vec{N} = \frac{3mG}{R^3} \vec{R} \times T I' T^T \vec{R} \quad , \tag{3}$$

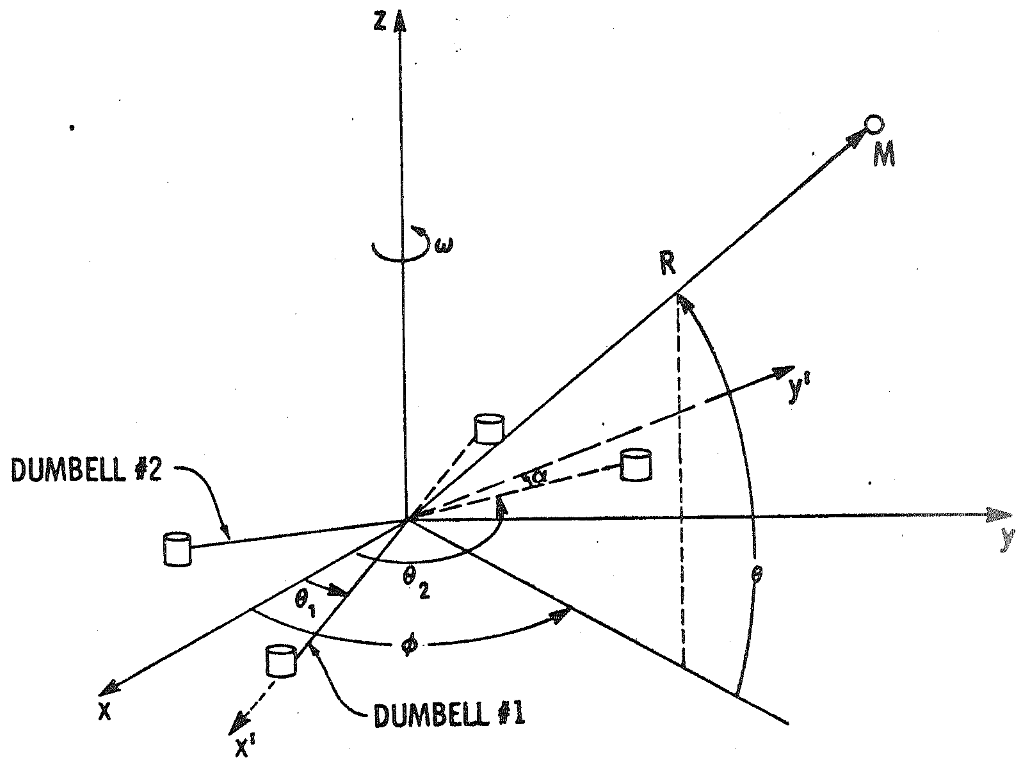

Figure 1. Geometry of principal coordinates and laboratory coordinates.

where T is the transformation matrix from the primed (principal) axes of dumbell no. 1; i.e.,

$$T = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\vec{R} = \begin{bmatrix} R\cos\theta & \cos\phi \\ R\cos\theta & \sin\phi \\ R\sin\theta' & \end{bmatrix} \equiv \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} .$$

After denoting the principal moments of inertia of dumbell no. 1 as A_1, B_1, and C_1, it is assumed that $A_1 \cong 0$ and $B_1 = C_1$. Thus, I_1' becomes

$$I_1' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & C_1 \end{bmatrix} .$$

Using these results in equation (3) gives

$$\vec{N}_1 = \begin{bmatrix} N_x^{(1)} \\ N_y^{(1)} \\ N_z^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{3mGC_1}{R^5} (R_y R_z \sin^2\theta_1 + R_x R_z \sin\theta_1 \cos\theta_1) \\ \frac{3mGC_1}{R^5} (-R_x R_z \cos^2\theta_1 + R_y R_z \sin\theta_1 \cos\theta_1) \\ \frac{3mGC_1}{R^5} \left[(R_y^2 - R_x^2) \frac{1}{2} \sin 2\theta_1 + R_x R_y \cos 2\theta_1\right] \end{bmatrix} . \quad (4)$$

Similarly, the gravity-gradient torque on dumbell no. 2 is

$$\vec{N}_2 = \begin{bmatrix} N_x^{(2)} \\ N_y^{(2)} \\ N_z^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{3mGC_2}{R^5} (R_y R_z \sin^2\theta_2 + R_x R_z \sin\theta_2 \cos\theta_2) \\ \frac{3mGC_2}{R^5} (-R_x R_z \cos^2\theta_2 + R_y R_z \sin\theta_2 \cos\theta_2) \\ \frac{3mGC_2}{R^5} \left[(R_y^2 - R_x^2) \frac{1}{2} \sin 2\theta_2 + R_x R_y \cos 2\theta_2\right] \end{bmatrix} . \quad (5)$$

Now assuming that the deflection of one of the dumbells with respect to the other is restricted to the x, y plane that coincides with the x', y' plane, then only the N_z components of the torque contribute to this deflection. This allows equating the third component of the torque to the rate of change of angular momentum along the z axis. Before doing this, however, the restoring and damping torques will be formulated.

The restoring torque on dumbell no. 2 is

$$N_2^{(R)} = -k\alpha \quad , \tag{6}$$

and the damping torque on dumbell no. 2 is

$$N_2^{(D)} = -h\dot{\alpha} \quad , \tag{7}$$

where k and h are assumed constant. Likewise, the restoring torque on dumbell no. 1 is

$$N_1^{(R)} = k\alpha \quad , \tag{8}$$

and the damping torque on dumbell no. 1 is

$$N_1^{(D)} = h\dot{\alpha} \quad . \tag{9}$$

The sum of all the torques on dumbell no. 1 may be equated to $C_1\ddot{\theta}_1$, and a similar operation may be performed for dumbell no. 2. Thus,

$$C_1\ddot{\theta}_1 = h\dot{\alpha} + k\alpha + \frac{3mGC_1}{R^5}\left[(R_y^2 - R_x^2)\frac{1}{2}\sin 2\theta_1 + R_xR_y\cos 2\theta_1\right] \tag{10}$$

and

$$C_2\ddot{\theta}_2 = -h\dot{\alpha} - k\dot{\alpha} + \frac{3mGC_2}{R^5}\left[(R_y^2 - R_x^2)\frac{1}{2}\sin 2\theta_2 + R_xR_y\cos 2\theta_2\right] \quad . \tag{11}$$

It is observed from Figure 1 that

$$\theta_2 - \theta_1 = \frac{\pi}{2} + \alpha \quad , \tag{12}$$

then

$$\ddot{\theta}_2 - \ddot{\theta}_1 = \ddot{\alpha} \quad .$$

Using these results in equations (10) and (11), a differential equation for α and θ_1 may be written as

$$\ddot{\alpha} = -\left(\frac{h}{C_2} + \frac{h}{C_1}\right)\dot{\alpha} - \left(\frac{k}{C_1} + \frac{k}{C_2}\right)\alpha + \frac{3mG}{2R^5}(R_y^2 - R_x^2)(\sin 2\theta_2 - \sin 2\theta_1)$$

$$+ \frac{3mG}{R^5} R_x R_y (\cos 2\theta_2 - \cos 2\theta_1) \tag{13}$$

and

$$\ddot{\theta}_1 = \frac{h}{C_1}\dot{\alpha} + \frac{k}{C_1}\alpha + \frac{3mG}{R^5}\left[(R_y^2 - R_x^2)\frac{1}{2}\sin 2\theta_1 + R_x R_y \cos 2\theta_1\right] . \tag{14}$$

Replacing θ_2 with $\theta_1 + \alpha + \frac{\pi}{2}$ from equation (12) gives the following differential equations for $\ddot{\alpha}$ and $\ddot{\theta}_1$ as functions of α and θ_1:

$$\ddot{\alpha} = -H\dot{\alpha} - K\alpha + \frac{3mG}{2R^5}(R_y^2 - R_x^2)(-\sin 2\theta_1 \cos 2\alpha - \cos 2\theta_1 \sin 2\alpha$$

$$- \sin 2\theta_1) + \frac{3mG}{R_5} R_x R_y (-\cos 2\theta_1 \cos 2\alpha$$

$$+ \sin 2\theta_1 \sin 2\alpha$$

$$- \cos 2\theta_1) \tag{15}$$

and

$$\ddot{\theta}_1 = \frac{h}{C_1}\dot{\alpha} + \frac{k}{C_1}\alpha + \frac{3mG}{R^5}\left[(R_y^2 - R_x^2)\frac{1}{2}\sin 2\theta_1 + R_x R_y \cos 2\theta_1\right] , \tag{16}$$

where

$$H = \frac{h}{C_1} + \frac{h}{C_2} \quad \text{and} \quad K = \frac{k}{C_1} + \frac{k}{C_2} .$$

These are the exact differential equations for a spherical homogeneous perturbing mass generating a torque on the coupled dumbells where the restoring torque is proportional to the deflection (α) of the spring coupling and the damping torque is proportional to the angular velocity ($\dot{\alpha}$) of the spring coupling.

FORCED HARMONIC OSCILLATOR EQUATIONS

Equations (15) and (16) are not forced harmonic equations for the deflection, and certain assumptions are necessary before they can be reduced to forced harmonic equations. First, the small angle approximation must be made for α so that $\cos 2\alpha \cong 1$ and $\sin 2\alpha \cong 2\alpha \cong 0$. The second assumption that is necessary is to assume that R_x, R_y, and R_z are constant. A third assumption is that $\theta_1 = \omega t$ where ω = constant angular rate.

If the above assumptions are made, then

$$\ddot{\alpha} + H\dot{\alpha} + K\alpha = -\frac{3mG}{R^5}(R_y^2 - R_x^2)\sin 2\omega t - \frac{6mG}{R^5}R_xR_y\cos 2\omega t \tag{17}$$

and

$$\theta_1 = \omega t \quad . \tag{18}$$

Since $\vec{R} = (R_x, R_y, R_z)$ is assumed constant, it may be assumed that $\vec{R}$ lies along the x axis without any loss of generality since this is equivalent to assuming that the x axis is along $\vec{R}$ initially; then,

$$\ddot{\alpha} + H\dot{\alpha} + K\alpha = \frac{3mG}{R^3}\sin 2\omega t \quad , \tag{19}$$

which is the forced harmonic oscillator equation.

SOLUTION OF THE FORCED HARMONIC OSCILLATOR EQUATION

The solution to equations of the form of equation (19) are well known. An excellent discussion of the solutions may be found in Reference 5.

The solution to equation (19) for $\frac{H}{2} < K$ may be written as

$$\alpha = e^{-bt}(A_1 \cos qt + A_2 \sin qt) + \frac{a}{p} \sin(2\omega t - \beta) \quad , \tag{20}$$

where

$$b = \frac{H}{2} \ , \quad q = \sqrt{n^2 - b^2} \quad , \quad n^2 = K \quad ,$$

$$a = \frac{3mG}{R^3} \ , \quad p = \sqrt{[n^2 - (2\omega)^2]^2 + (4b\omega)^2} \quad ,$$

$$\tan\beta = \frac{4b\omega}{n^2 - (2\omega)^2} \ ,$$

and A_1 and A_2 are initial integration constants.

One quantity that is usually of interest is the time constant τ, which is sometimes referred to as the relaxation time or the $\frac{1}{e}$ folding time. This is the time required for the first term (transient term) to reach $\frac{1}{e}$ of its original value. This occurs when $b\tau = 1$; i.e., $\tau = \frac{1}{b} = \frac{2}{H}$ which is seen to depend on the damping of the oscillator. When $t \to \infty$, the transient term goes to zero. This is the transient time that is seen to take several time constants to get down to one-hundredth or one-thousandth of the initial value. After the transient time, the oscillator is considered to be in steady state with amplitude

$$\frac{a}{p} = \frac{\frac{3mG}{R^3}}{\left\{[n^2-(2\omega)^2]^2 + (4b\omega)^2\right\}^{1/2}} \quad . \tag{21}$$

or for $n >> \frac{1}{\tau} = b$,

$$Q = \frac{n}{2b} = \frac{n\tau}{2} \quad ; \qquad (26)$$

and from equation (24) ,

$$f = \frac{1}{2n\tau} \quad , \qquad (27)$$

which gives

$$2f = \frac{1}{2Q} \quad . \qquad (28)$$

This indicates that a high f and high Q are incompatible.

COMPARISONS AND RESULTS

The results of the forced harmonic solution and numerical integration of equations (15) and (16) are shown in Table 1. The initial conditions assumed were as follows:

$$K = (4\pi)^2 \text{ radians/sec}^2 \quad ,$$

$$H = 0.404 \text{ radian/sec} \quad ,$$

and

$$\omega = 2\pi \text{ radians/sec} \quad .$$

As can be seen, the steady-state results agree through the third significant figure. From these results, it may be concluded that the forced harmonic oscillator approximation is sufficient for design purposes. It should be noted that the forced harmonic oscillator theory assumes a linear spring constant and damping coefficient.

Another point of interest is the driving frequency ω required to give maximum steady-state amplitude. This occurs when p is a minimum; thus, differentiating p with respect to 2ω and equating the result to zero gives the conditions

$$(2\omega)^2 = n^2 - 2b^2$$

or

$$p = 2b\sqrt{n^2 - b^2} \tag{22}$$

for maximum amplitude. These results may be used for determining the best operating conditions of a gradiometer.

Savet [3] has suggested a figure of merit f for various sensors defined as the sensitivity σ divided by the time constant squared τ^2. The sensitivity is defined as the steady-state displacement amplitude divided by the amplitude of the forcing term. This figure of merit in the present discussion turns out to be

$$f \equiv \frac{\sigma}{\tau^2} = \left(\frac{a}{p} \div a\right) \div \tau^2 = \frac{1}{p} \div \tau^2 = \frac{1}{p\tau^2} \quad . \tag{23}$$

Using equation (22) in the above,

$$f = \frac{1}{2b\tau^2\sqrt{n^2-b^2}} = \frac{1}{2\tau\sqrt{n^2 - \frac{1}{\tau^2}}} \quad . \tag{24}$$

If the quality amplification factor Q is defined as the energy stored in the spring divided by the energy dissipated per cycle at resonance and steady state, then

$$Q = \frac{2\pi}{1-e^{-4\pi/\tau\sqrt{n^2-\frac{1}{\tau^2}}}} = \frac{2\pi}{1-e^{-8\pi f}} \tag{25}$$

TABLE 1. POINTS IN STEADY STATE FOR 1 c/s

Numerical Integration		Forced Harmonic Oscillation Solution	
Time (sec)	Delta (radians $\times 10^{-9}$)	Time (sec)	Delta (radians $\times 10^{-9}$)
30.0125	-0.59040669	30.0000	-0.59015672
30.2625	0.59056961	30.2500	0.59034407
32.0125	-0.59143408	32.0000	-0.59142200
32.2625	0.59152283	32.2500	0.59154729
33.0125	-0.59174430	33.0000	-0.59188750
33.2625	0.59180415	33.2500	0.59198996
42.025	-0.59316773	42.0000	-0.59363485
42.275	0.59323604	42.2500	0.5936516
Steady State			
29.5125	-0.59004683	30.0000	-0.59015672

Figure 2 shows the waveform for 20 sec, which indicates that the transient term lasts for about 4τ since τ is 5 sec for this case.

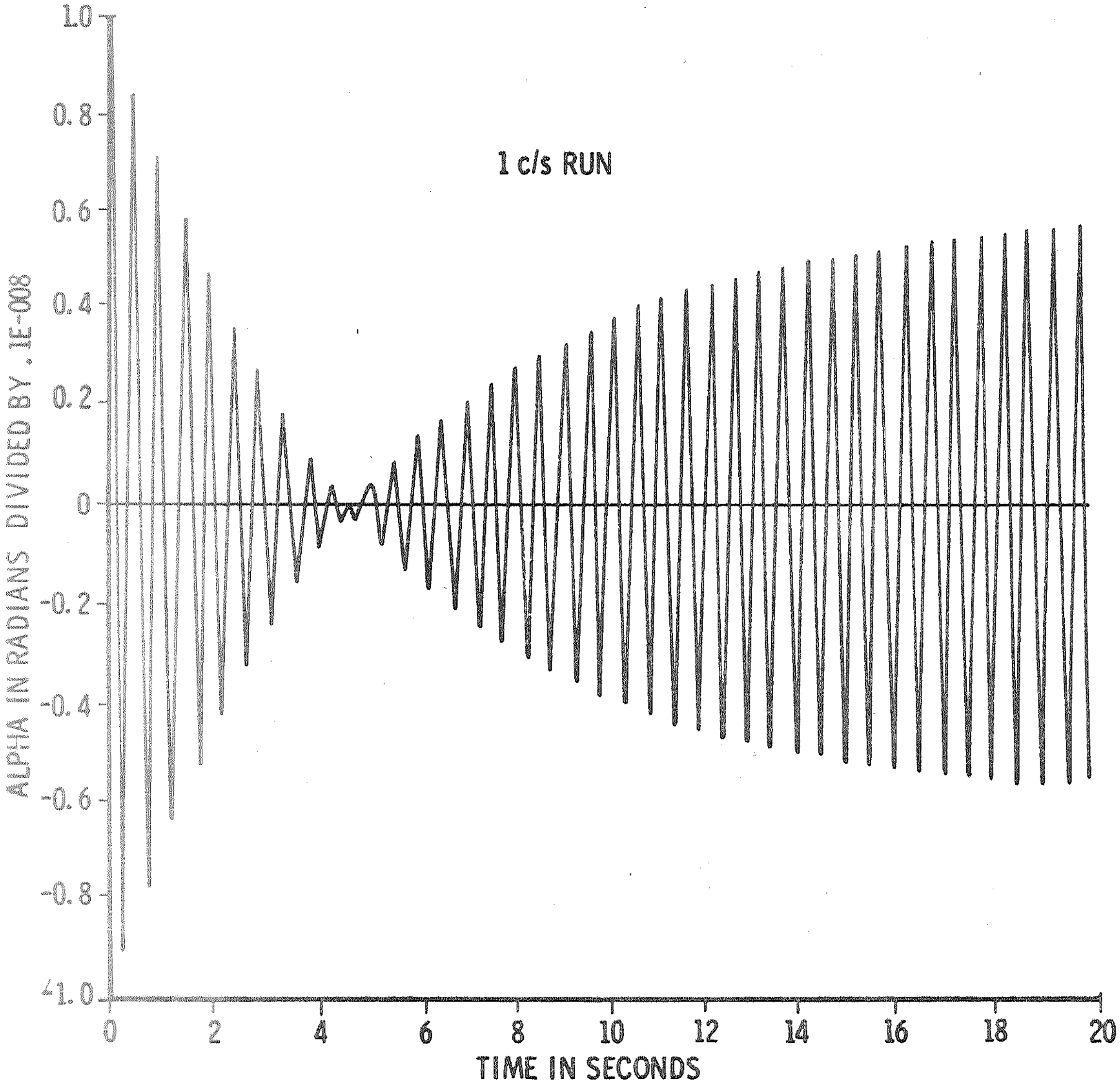

Figure 2. Waveform for 20 sec for 1 c/s run.

APPENDIX

THE GRAVITY-GRADIENT TORQUE AS A CONTRACTED ALTERNATING PRODUCT

THE GRAVITATIONAL TORQUE OF ONE RIGID BODY ON ANOTHER

To establish explicit invariant relationships between the moment of inertia of a small body, the gravity gradient because of the presence of larger bodies, and the torque on the small body, it is necessary to define the moment of inertia and gravity gradient in such a way that their transformation properties are known.

The Moment of Inertia

From Figure 1-A, the moment of inertias usually are defined as

$$I_{ij} = \begin{cases} -\int_m \rho_i \rho_j \, dm & \text{for } i \neq j \\ \\ \int_m (\rho^2 - \rho_i^2)\, dm & \text{for } i = j \end{cases} , \qquad \text{(A-1)}$$

where $\vec{\rho} \equiv (\rho_1, \rho_2, \rho_3)$, $i \equiv \{1, 2, 3\}$, and $j \equiv \{1, 2, 3\}$. To check the transformation properties of I_{ij}, an orthonormal Cartesian coordinate transformation M is defined such that $\vec{\rho} = M\vec{\rho}\,'$; i.e.,

$$\rho_i = M_{ip'} \rho_{p'}$$

and

$$\rho_j = M_{jq'} \rho_{q'} \quad .$$

Then, equation (A-1) becomes for $i \neq j$

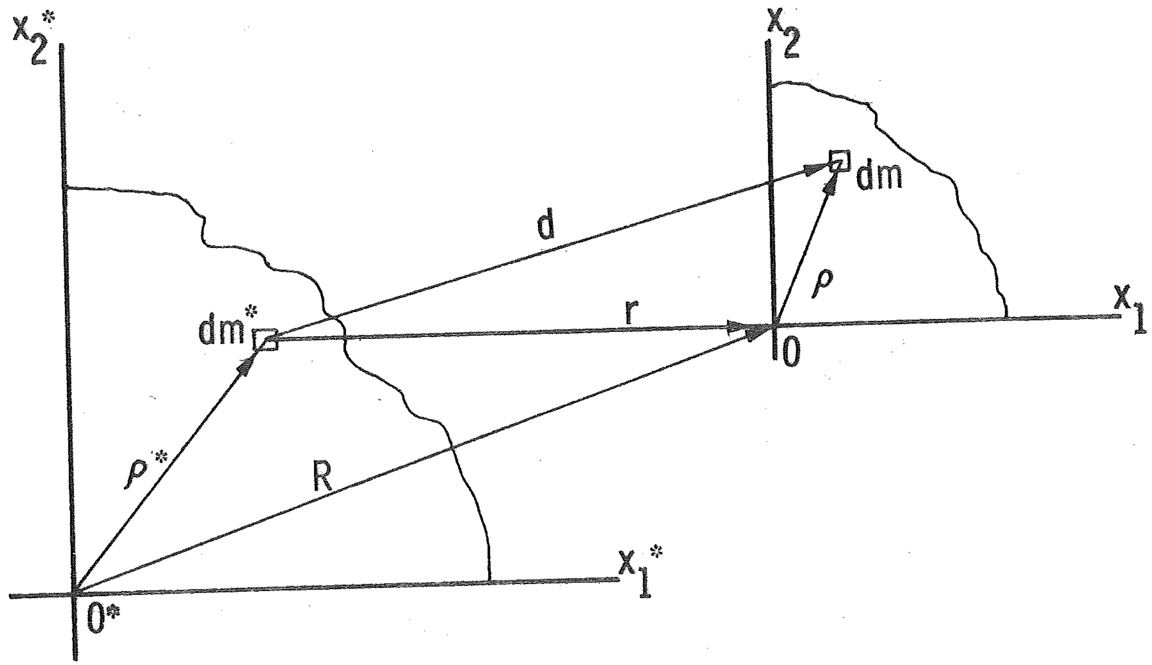

FIGURE 1-A. Geometry of Inertial Coordinates Fixed at the Center of Mass of Large and Small Rigid Bodies

$$I_{ij} = -\int_m M_{ip'}\rho_{p'} M_{jq'}\rho_{q'}\, dm = -\int_m M_{ip'}\rho_{p'}\rho_{q'} M_{iq'}\, dm$$

$$= -M_{ip'}\left(\int_m \rho_{p'}\rho_{q'}\, dm\right) M_{jq} = M_{ip'} I_{p'q'} M_{jq'}$$

$$= (MI')_{iq'} M_{jq'} = (MI'M^T)_{ij} \quad . \qquad (A-2)$$

Thus, $I = MI'M^T$ for $i \neq j$. For $i = j$, the integrals in equation (A-1) are all of the form $\int_m \rho_i\rho_i\, dm$, which will also transform according to equation (A-2) .

The Gravity Gradient

The gravity gradient is usually defined as the second partial derivatives of a potential function, say $V(x_1, x_2, x_3)$. Then,

$$V,_{ij} \equiv \frac{\partial^2 V}{\partial x_j\, \partial x_i} \quad . \qquad (A-3)$$

To check the transformation properties of $V,_{ij}$ an orthonormal Cartesian coordinate transformation M is defined such that $\vec{x} = M\vec{x}'$. Thus,

$$x_i = M_{ip'} x_{p'} \quad , \quad x_{p'} = M_{p'i}\, x_i \quad ,$$

$$x_j = M_{jq'} x_{q'} \quad , \text{ and } x_{q'} = M_{q'j}\, x_j \quad .$$

Now,

$$V,_{ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial V}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial V}{\partial x_{p'}}\,\frac{\partial x_{p'}}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\,\frac{\partial V}{\partial x_{p'}}\,M_{p'i}$$

$$= \frac{\partial}{\partial x_{q'}}\,\frac{\partial V}{\partial x_{p'}}\,\frac{\partial x_{q'}}{\partial x_j}\,M_{p'i} = \frac{\partial^2 V}{\partial x_{q'}\partial x_{p'}}\,M_{q'j}\,M_{p'i}$$

$$= V,_{p'q'}M_{q'j}\,M_{p'i} = M_{p'i}\,V'_{,p'q'}\,M_{q'j}$$

$$= M_{p'i}\,(V,M^T)_{p'j} = \left[(M^T)^T\;V'_, M^T\right]_{ij} = (MV'_,M^T)_{ij} \quad .$$

Thus, $V, = MV'_,M^T \quad .$ (A-4)

The Torque as the CAP Product of V, and I

From Figure 1, the element of force $d(\vec{dF})$ on the element of mass dm because of the element of mass dm* is

$$d(\vec{dF}) = -Gdmdm^* \frac{\vec{d}}{d^3} \equiv -Gdmdm^* \vec{f}(\vec{d}) \quad . \qquad \text{(A-5)}$$

However, since $\vec{d} = \vec{r} + \vec{\rho}$ and it is assumed that $\rho << r$, then the i^{th} component of $\vec{f}(\vec{d})$ is [4]

$$f_i(\vec{d}) = f_i(\vec{r} + \vec{\rho}) \doteq f_i(\vec{r}) + \frac{\partial f_i}{\partial x_j}\rho_j \equiv f_i(\vec{r}) + V,_{ij}\rho_j \quad . \qquad \text{(A-6)}$$

Where

$$\vec{r} \equiv (x_1, x_2, x_3), \qquad \vec{\rho} \equiv (\rho_1, \rho_2, \rho_3) \quad ,$$

$$f_i \equiv \frac{\partial V}{\partial x_i} \quad , \text{ and } \quad V \equiv \frac{1}{d} \quad .$$

Now, substituting equation (A-6) into equation (A-5),

$$d(d\vec{F}) = -Gdmdm^* \left[\vec{f}(\vec{r}) + V,\vec{\rho}\right] \quad .$$

The torque about 0 because of the force of dm^* on dm is

$$d(d\vec{N}) = \vec{\rho} \times d(d\vec{F}) = -Gdmdm^* \, \vec{\rho} \times \left[f(\vec{r}) + V,\vec{\rho}\right]$$

$$= -Gdmdm^* \vec{\rho} \times \vec{f}(\vec{r}) - Gdmdm^* \, \vec{\rho} \times V, \vec{\rho} \quad .$$

Integrating first over m gives

$$d\vec{N} = -Gdm^* \int_m \vec{\rho} \times \vec{f}(\vec{r})\, dm - Gdm^* \int_m \vec{\rho} \times V, \vec{\rho}\, dm \quad . \qquad \text{(A-7)}$$

The first integral on the right is zero because of the choice of 0 as the center of mass for m. The cross product in equation (A-7) is

$$\vec{\rho} \times V, \vec{\rho} = \begin{bmatrix} \rho_2 V,_{3j}\rho_j & - \rho_3 V,_{2j}\rho_j \\ \rho_3 V,_{1j}\rho_j & - \rho_1 V,_{3j}\rho_j \\ \rho_1 V,_{2j}\rho_j & \rho_2 V,_{1j}\rho_j \end{bmatrix} \quad .$$

Expanding and integrating over M gives

$$d\vec{N} = -Gdm^* \begin{bmatrix} V,_{2j} I_{3j} & - V,_{3j} I_{2j} \\ V,_{3j} I_{1j} & - V,_{1j} I_{3j} \\ V,_{1j} I_{2j} & - V,_{2j} I_{1j} \end{bmatrix}$$

$$= - Gdm* \begin{bmatrix} (V,I)_{23} - (V,I)_{32} \\ (V,I)_{31} - (V,I)_{13} \\ (V,I)_{12} - (V,I)_{21} \end{bmatrix} . \tag{A-8}$$

As a notation convenience, it may be observed that the result in equation (A-8) may be obtained by placing the column or row vectors of V, and I in a determinant and using the conventional cross product rule except that the dot product is used when multiplying the column vectors. Using the Contracted Alternating Product (CAP) symbol $\textcircled{\Lambda}$ to indicate this operation between V, and I, one has

$$V,\textcircled{\Lambda} I = \begin{bmatrix} e_1 & e_2 & e_3 \\ \vec{V}_1 & \vec{V}_2 & \vec{V}_3 \\ \vec{I}_1 & \vec{I}_2 & \vec{I}_3 \end{bmatrix} = \begin{bmatrix} \vec{V}_2 \cdot \vec{I}_3 - \vec{V}_3 \cdot \vec{I}_2 \\ \vec{V}_3 \cdot \vec{I}_1 - \vec{V}_1 \cdot \vec{I}_3 \\ \vec{V}_1 \cdot \vec{I}_2 - \vec{V}_2 \cdot \vec{I}_1 \end{bmatrix} , \tag{A-9}$$

which is the same as equation (A-8) since $\vec{V}_i \equiv (V,_{i1}, V,_{i2}, V,_{i3})$, $\vec{I}_i \equiv (I_{1i}, I_{2i}, I_{3i})$ and both V, and I are symmetric. Thus, the torque given in equation (A-8) may be written as the CAP product of V, and I as

$$d\vec{N} = - Gdm* \ V,\textcircled{\Lambda} I$$

or

$$\vec{N} = - G \int_{m*} V,\textcircled{\Lambda} I dm* \quad . \tag{A-10}$$

This is considered to be desirable since it has the effect of separating or isolating those effects that depend on the field which is in V, and those that depend on the physical characteristics of the body which is in I. It also gives their relationship to the torque.

The Torque Referred to a Coordinate System in Which I is Diagonal

It may be shown that applying an orthonormal transformation to the CAP product of two matrices (or tensors) is the same as the CAP product of the transformed matrices (or tensors). It has already been seen that V, and I transform according to equations (A-2) and A-4); then, if M is the transformation from $\overrightarrow{N'}$ to N,

$$d\vec{N} = Md\vec{N}' = -Gdm^* MV_,' \Lambda I = -Gdm^* \; MV_,' M^T \Lambda MI'M^T$$

$$= -Gdm^* \; V_, \Lambda I$$

or

$$\vec{N} = -G \int_{m^*} V_, \Lambda I dm^* \quad . \qquad \text{(A-11)}$$

From the preceding discussion it is clear that to refer the torque $\vec{N}$ to a particular coordinate system, both V, and I must also be referred to that system. For example, I may be diagonalized by referring I to the principal axes of M. In this system, the CAP product becomes

$$V_,' \Lambda I' = \begin{bmatrix} V_{,23}'(I'_{22} - I'_{33}) \\ V_{,23}'(I'_{33} - I'_{11}) \\ V_{,12}'(I_{11} - I'_{22}) \end{bmatrix} \quad , \qquad \text{(A-12)}$$

where the primes indicate that the quantities are referred to the principal axes of m. From equation (A-12), the torque in the primed system depends on the three cross gradients; however, if the torque is transformed by a general rotation of coordinate axes, each component of the torque depends on all five of the independent gradient elements as indicated by equation (A-9).

The Torque of a Spherically Symmetric Homogeneous Body on a Small Distant Body

If m^* is spherically symmetric, then $\vec{\rho}^* = (0, 0, 0)$ in Figure 1-A so that $\vec{r} = \vec{R} - \vec{\rho}^* = \vec{R}$; then,

$$\vec{f}(\vec{r}) = f(\vec{R} - \vec{\rho}^*) = \vec{f}(\vec{R}) = \frac{\vec{R}}{R^3}$$

$$V'_{,23} = \frac{\partial f_2}{\partial R_3} = -\frac{3R_3R_2}{R^5}$$

$$V'_{,13} = \frac{\partial f_1}{\partial R_3} = -\frac{3R_3R_1}{R^5}$$

$$V'_{,12} = \frac{\partial f_1}{\partial R_2} = -\frac{3R_1R_2}{R^5} \quad .$$

Substituting the above into equation (A-10) and integrating gives

$$\vec{N} = \frac{3MG}{R^5} \vec{R}' \times I'\vec{R}' \quad , \qquad \text{(A-13)}$$

where

$$\vec{R} \equiv (R_1, R_2, R_3) \quad .$$

This is the same result as used in Reference 6.

SUMMARY

An explicit relationship has been developed between the gravity gradient, moment of inertia, and torque of one large rigid body on another small body. From this theory, conclusions can be drawn concerning observable quantities and derived information from the observables. Also various applications, measurement concepts, and analysis techniques can be developed.

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APPROVAL

A FIRST-ORDER THEORY FOR ROTATING GRAVITY GRADIOMETERS

By Robert L. Holland

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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