N71-34518

Bellcomm

955 L'Enfant Plaza North, S.W.
Washington, D. C. 20024

B71 08018

date: August 16, 1971

to: Distribution

from: W. G. Heffron, S. L. Levie, Jr.

subject: Unnormalized Associated Legendre Functions: A Compendium of Graphs, Expansions, Properties, and Integrals -- Case 310

MEMORANDUM FOR FILE

Attached is a useful compendium of graphs, expansions, properties and integrals of the unnormalized associated Legendre functions, denoted $P_\ell^m(x)$, for $|x| \leq 1$. The material which is new (i.e., not available from the compendiums included in the references) is

1. Expressions for $P_\ell^m(x)$ up to P_{10}^{10}.
2. Graphs of $P_\ell^m(x)$ up to P_{10}^{10}.
3. Fourier decompositions of $P_\ell^m(\sin\theta)$ and $P_\ell^m(\cos\theta)$ up to P_{10}^{10}.
4. Least upper bound information.
5. Derivatives of solid spherical harmonics.
6. The integrals $I_a(\ell, i, n, j)$ and $J_p(s, n, m)$.

The term "unnormalized" refers to a particular convention in the definition of $P_\ell^m(x)$. The convention adopted is the one used by Battin (1964), Emde and Jahnke (1945), Kaula (1966), and Korn and Korn (1961), as well as others. It differs by a normalization factor from other conventions in use.

Additional copies of this compendium are available from the authors. Corrections and suggestions are solicited.

Mrs. Sheryl Watson provided the graphs in Figures 1 through 12.

W. G. Heffron

W. G. Heffron

Sterling Levie Jr.

S. L. Levie, Jr.

2014-$\frac{WGH}{SLL}$-slr

Attachment

Bellcomm, Inc.
955 L'Enfant Plaza North, S. W.
Washington, D. C. 20024

W. G. Heffron
and
S. L. Levie, Jr.

Unnormalized Associated Legendre Functions: A Compendium of Graphs, Expansions, Properties, and Integrals

Unnormalized Associated Legendre Functions: A Compendium of Graphs, Expansions, Properties, and Integrals

TABLE OF CONTENTS

TABLE OF CONTENTS

(Continued)

UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS

Definitions, Functional Forms, and Graphs

The associated Legendre functions satisfy the differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]u = 0.$$

It will be assumed that ℓ and m are natural numbers, $0 \leq m \leq \ell$, and $|x| \leq 1$. With these restrictions, the unnormalized associated Legendre functions are completely characterized by

$$u \equiv P_\ell^m(x) = \frac{(1-x^2)^{m/2}}{2^\ell \ell!}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell \qquad \text{RODRIGUES' FORMULA}$$

$$= (1-x^2)^{m/2}\sum_{k=0}^{\left[\frac{\ell-m}{2}\right]} T_{\ell mk}\, x^{\ell-m-2k},$$

in which

$[b] \equiv$ greatest integer in b

$$T_{\ell mk} = \frac{(-)^k\,(2\ell-2k)!}{2^\ell k!(\ell-k)!(\ell-m-2k)!}.$$

When $m=0$ these solutions are called Legendre's polynomials, denoted $P_\ell(x)$.

The definitions above specify the "unnormalized" associated Legendre functions, which are used throughout this compendium. The formulas herein may be converted to other fairly common normalizations with the following relations:

$$P_\ell^m = (-)^m P_{\ell m}^{I} = \sqrt{\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}}\; P_{\ell m}^{II}$$

$$= \sqrt{\frac{(\ell+m)!}{(2-\delta_{m0})(2\ell+1)(\ell-m)!}}\; P_{\ell m}^{III} = \sqrt{\frac{(\ell+m)!}{(2-\delta_{m0})(\ell-m)!}}\; P_{\ell m}^{IV}$$

where

$P_{\ell m}^{I}$ is common in theoretical physics

$P_{\ell m}^{II}$ gives value unity for $\frac{1}{2}\int_{-1}^{1}\left[P_{\ell m}^{II}(x)\right]^2 dx$
(latitude normalization)

$P_{\ell m}^{III}$ gives value unity for
$\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\cos^2 m\phi\left[P_{\ell m}^{III}(x)\right]^2 dx\,d\phi$ (sphere normalization)

$P_{\ell m}^{IV}$ gives value $1/(2\ell+1)$ for
$\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\cos^2 m\phi\left[P_{\ell m}^{IV}(x)\right]^2 dx\,d\phi$ (another sphere normalization).

The unnormalized associated Legendre functions up to $P_{10}^{10}(x)$ are listed in Table 1 as functions of x. They are plotted in Figures 1 through 12.

The functions $P_\ell^m(\sin\theta)$ may be Fourier-decomposed with the following formula:

$$P_\ell^m(\sin\theta) = \sum_{\substack{q=c \\ \text{by 2's}}}^{\ell} \left\{\begin{matrix}\cos q\theta \\ \sin q\theta\end{matrix}\right\}\begin{matrix}(\ell-m)\,\text{even} \\ (\ell-m)\,\text{odd}\end{matrix} \sum_{\substack{p=\max(m-\ell,q-m) \\ \text{by 2's}}}^{\min(\ell-m,q+m)} \sum_{\substack{k=|p| \\ \text{by 2's}}}^{\ell-m}$$

$$\left\{\frac{(-)^{[(3k+\ell-m-p+1)/2]}}{2^{k+\ell+m}} \frac{(k+\ell+m)!}{k!\,\ell!} (2-\delta_{0q}) \times \binom{\ell}{\frac{k+\ell+m}{2}} \binom{k}{\frac{k-p}{2}} \binom{m}{\frac{m+p-q}{2}}\right\},$$

where $c = \begin{cases} 0 \text{ if } \ell = \text{even} \\ 1 \text{ if } \ell = \text{odd}\end{cases}$

and $[b] \equiv$ greatest integer in b.

These decompositions are given explicitly in Table 2 up to $P_{10}^{10}(\sin\theta)$. Table 2 also lists the Fourier decompositions of $P_\ell^m(\cos\theta)$ up to $P_{10}^{10}(\cos\theta)$, which are related to the decompositions of $P_\ell^m(\sin\theta)$ through the replacement of θ by $\frac{\pi}{2}-\theta$.

TABLE 1

EXPRESSIONS FOR UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS UP TO $P_{10}^{10}(x)$

EACH OF THESE FUNCTIONS CAN BE WRITTEN IN THE FORM:

$$P_\ell^{\,m}(x) = (1-x^2)^p \sum_{i=0}^{\ell-m} a_i x^i$$

THE TABLE LISTS ℓ, m, p, AND THE a_i FOR EACH OF THE FUNCTIONS UP TO $P_{10}^{10}(x)$.

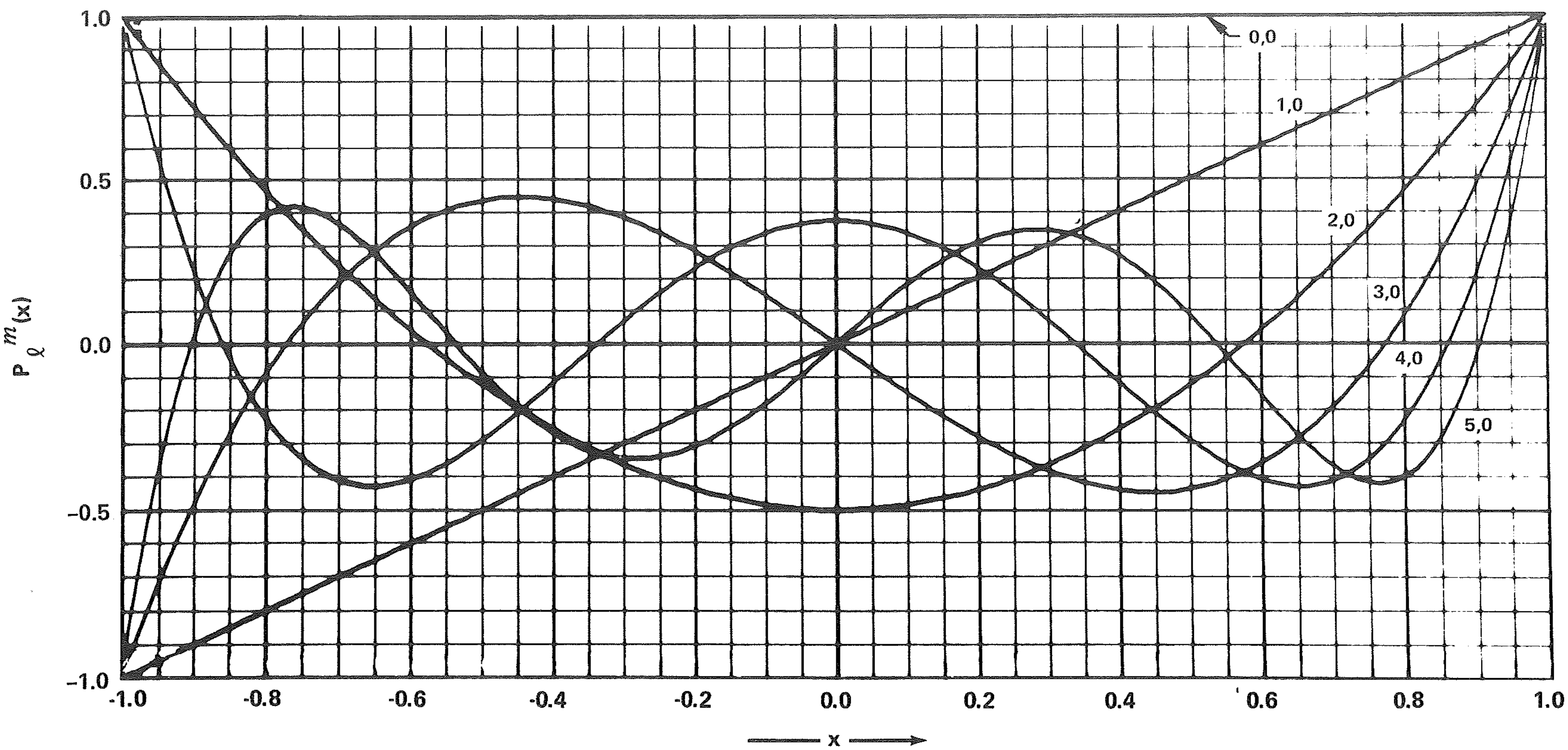
ℓ	m	p	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
0	0	0	1										
1	0	0	0	1									
2	0	0	−1/2	0	3/2								
3	0	0	0	−3/2	0	5/2							
4	0	0	3/8	0	−15/4	0	35/8						
5	0	0	0	15/8	0	−35/4	0	63/8					
6	0	0	−5/16	0	105/16	0	−315/16	0	231/16				
7	0	0	0	−35/16	0	315/16	0	−693/16	0	429/16			
8	0	0	35/128	0	−315/32	0	3465/64	0	−3003/32	0	6435/128		
9	0	0	0	315/128	0	−1155/32	0	9009/64	0	−6435/32	0	12155/128	
10	0	0	−63/256	0	3465/256	0	−15015/128	0	45045/128	0	−109395/256	0	46189/256
1	1	1/2	1										
2	1	1/2	0	3									
3	1	1/2	−3/2	0	15/2								
4	1	1/2	0	−15/2	0	35/2							
5	1	1/2	15/8	0	−105/4	0	315/8						
6	1	1/2	0	105/8	0	−315/4	0	693/8					
7	1	1/2	−35/16	0	945/16	0	−3465/16	0	3003/16				
8	1	1/2	0	−315/16	0	3465/16	0	−9009/16	0	6435/16			
9	1	1/2	35/128	0	−3465/32	0	45045/64	0	−45045/32	0	109395/128		
10	1	1/2	0	3465/128	0	−15015/32	0	135135/64	0	−109395/32	0	230945/128	

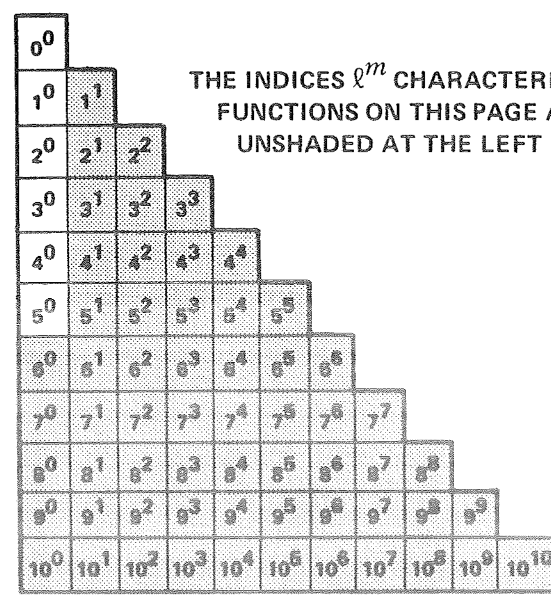
TABLE 1 (CONTINUED)

ℓ	m	p	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
2	2	1	3										
3	2	1	0	15									
4	2	1	−15/2	0	105/2								
5	2	1	0	−105/2	0	315/2							
6	2	1	105/8	0	−945/4	0	3465/8						
7	2	1	0	945/8	0	−3465/4	0	9009/8					
8	2	1	−315/16	0	10395/16	0	−45045/16	0	45045/16				
9	2	1	0	−3465/16	0	45045/16	0	−135135/16	0	109395/16			
10	2	1	3465/128	0	−45045/32	0	675675/64	0	−765765/32	0	2078505/128		
3	3	3/2	15										
4	3	3/2	0	105									
5	3	3/2	−105/2	0	945/2								
6	3	3/2	0	−945/2	0	3465/2							
7	3	3/2	945/8	0	−10395/4	0	45045/8						
8	3	3/2	0	10395/8	0	−45045/4	0	135135/8					
9	3	3/2	−3465/16	0	135135/16	0	−675675/16	0	765765/16				
10	3	3/2	0	−45045/16	0	675675/16	0	−2297295/16	0	2078505/16			
4	4	2	105										
5	4	2	0	945									
6	4	2	−945/2	0	10395/2								
7	4	2	0	−10395/2	0	45045/2							
8	4	2	10395/8	0	−135135/4	0	675675/8						
9	4	2	0	135135/8	0	−675675/4	0	2297295/8					
10	4	2	−45045/16	0	2027025/16	0	−11486475/16	0	14549535/16				
5	5	5/2	945										
6	5	5/2	0	10395									
7	5	5/2	−10395/2	0	135135/2								
8	5	5/2	0	−135135/2	0	675675/2							

TABLE 1 (CONCLUDED)

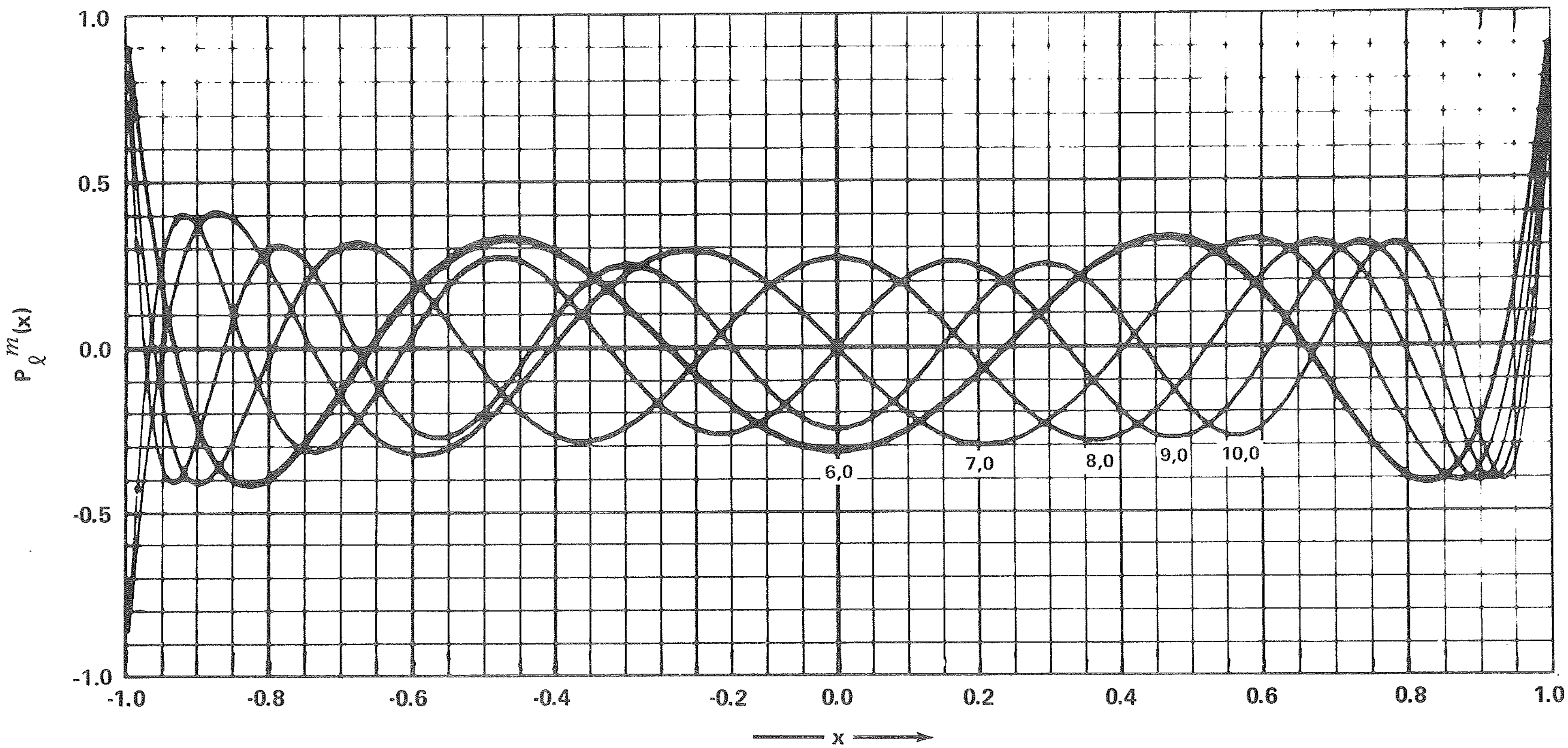
ℓ	m	p	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
9	5	5/2	135135/8	0	−2027025/4	0	11486475/8						
10	5	5/2	0	2027025/8	0	−11486475/4	0	43648605/8					
6	6	3	10395										
7	6	3	0	135135									
8	6	3	−135135/2	0	2027025/2								
9	6	3	0	−2027025/2	0	11486475/2							
10	6	3	2027025/8	0	−34459425/4	0	218243025/8						
7	7	7/2	135135										
8	7	7/2	0	2027025									
9	7	7/2	−2027025/2	0	34459425/2								
10	7	7/2	0	−34459425/2	0	218243025/2							
8	8	4	2027025										
9	8	4	0	34459425									
10	8	4	−34459425/2	0	654729075/2								
9	9	9/2	34459425										
10	9	9/2	0	654729075									
10	10	5	654729075										

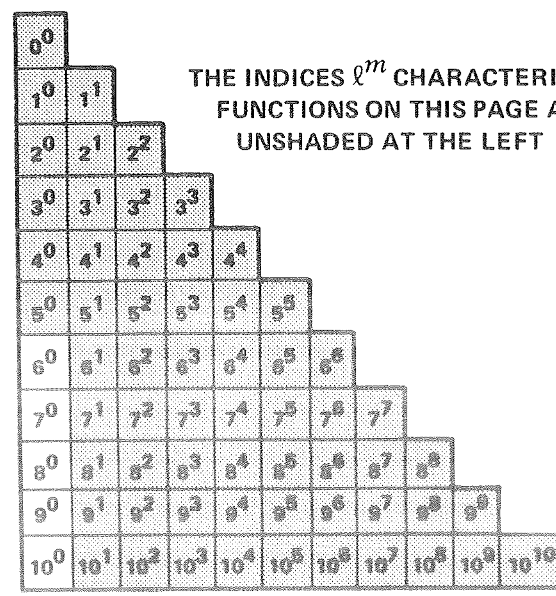




THE INDICES ℓ^m CHARACTERIZING THE FUNCTIONS ON THIS PAGE ARE UNSHADED AT THE LEFT

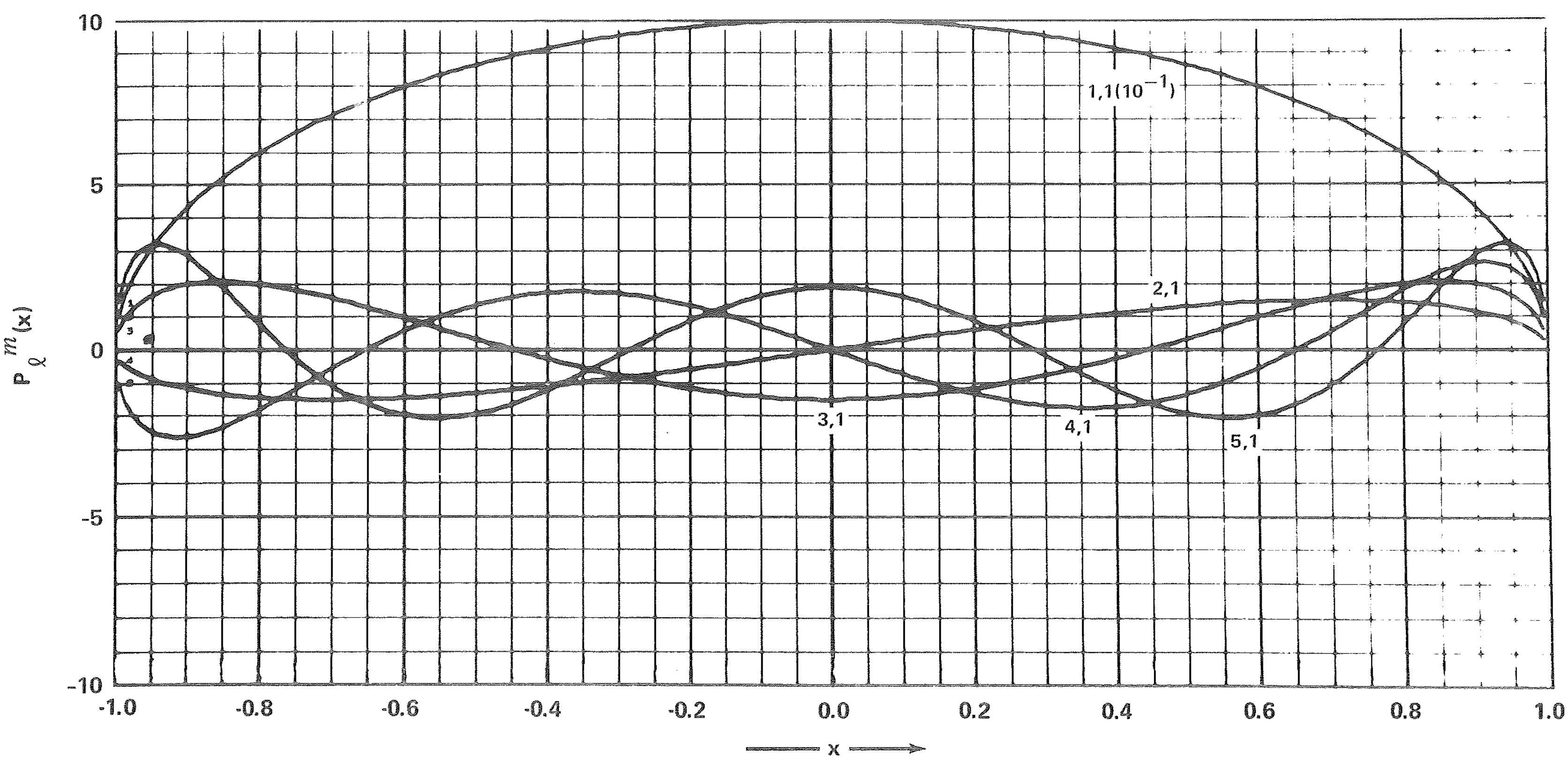
FIGURE 1 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS

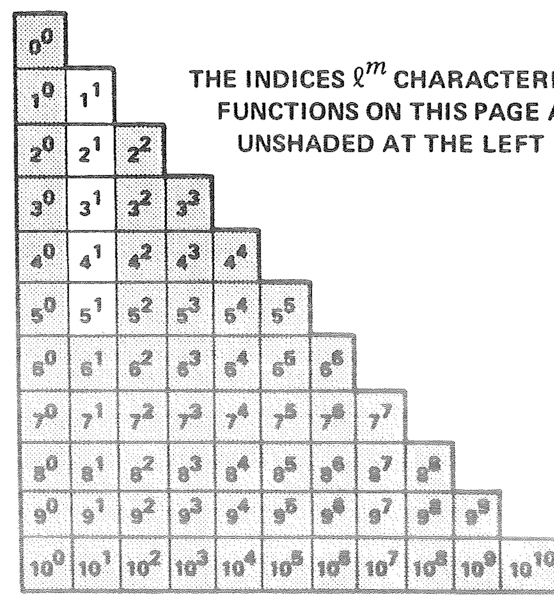




THE INDICES ℓ^m CHARACTERIZING THE FUNCTIONS ON THIS PAGE ARE UNSHADED AT THE LEFT

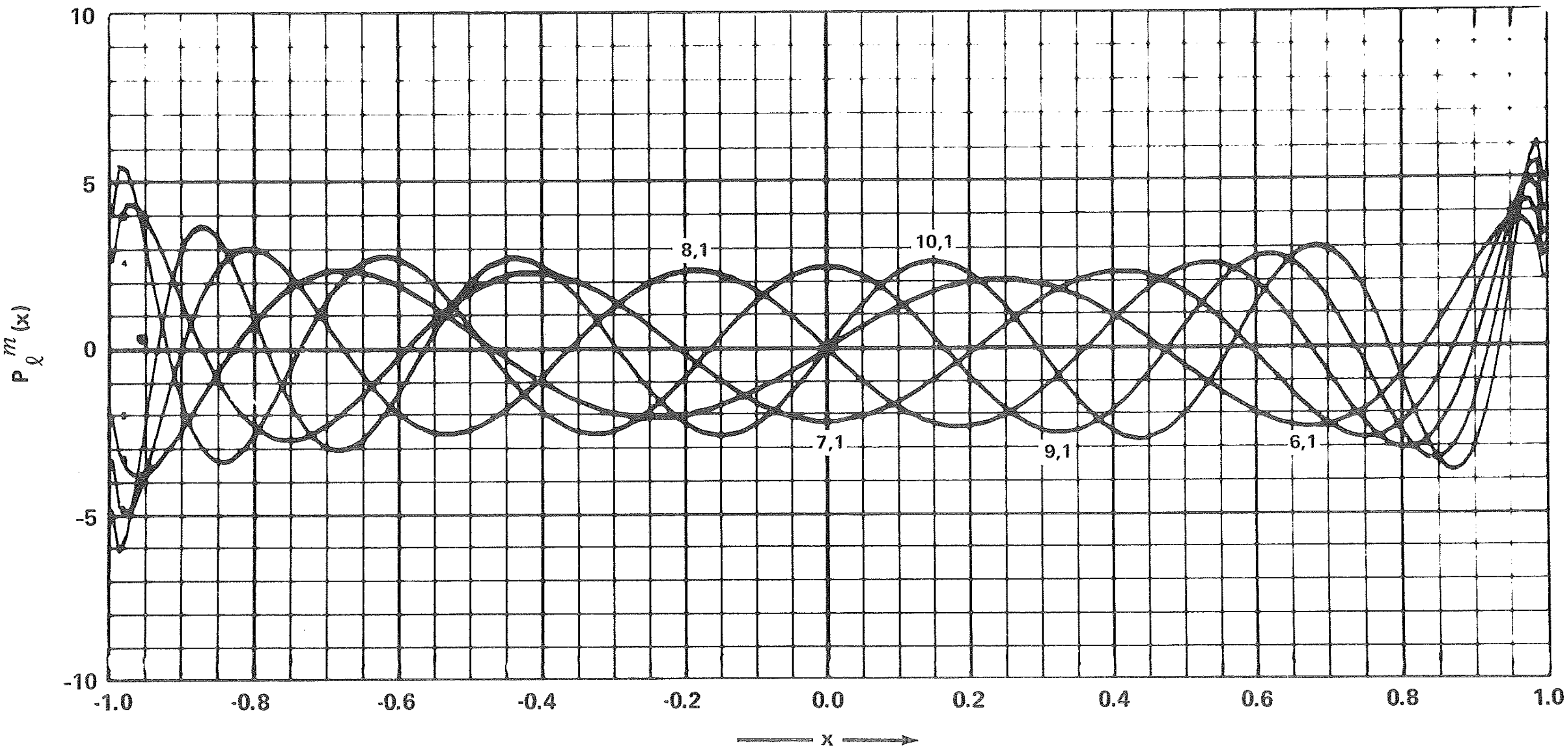
FIGURE 2 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS





THE INDICES ℓ^m CHARACTERIZING THE FUNCTIONS ON THIS PAGE ARE UNSHADED AT THE LEFT

FIGURE 3 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE



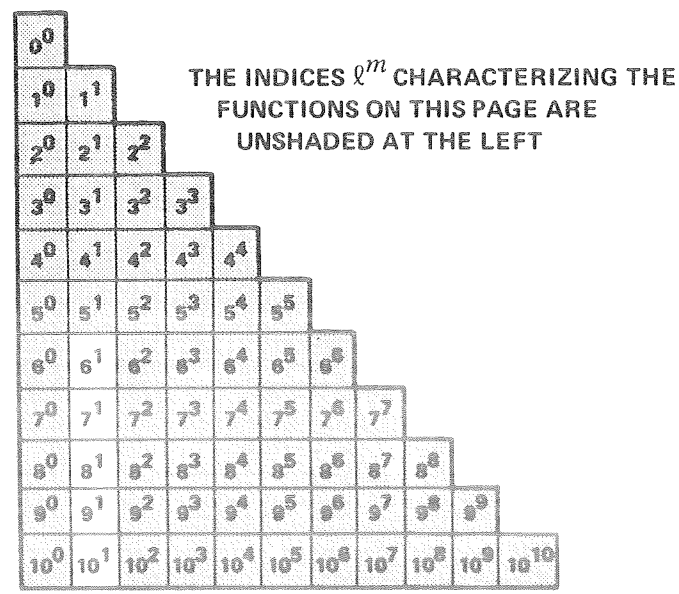
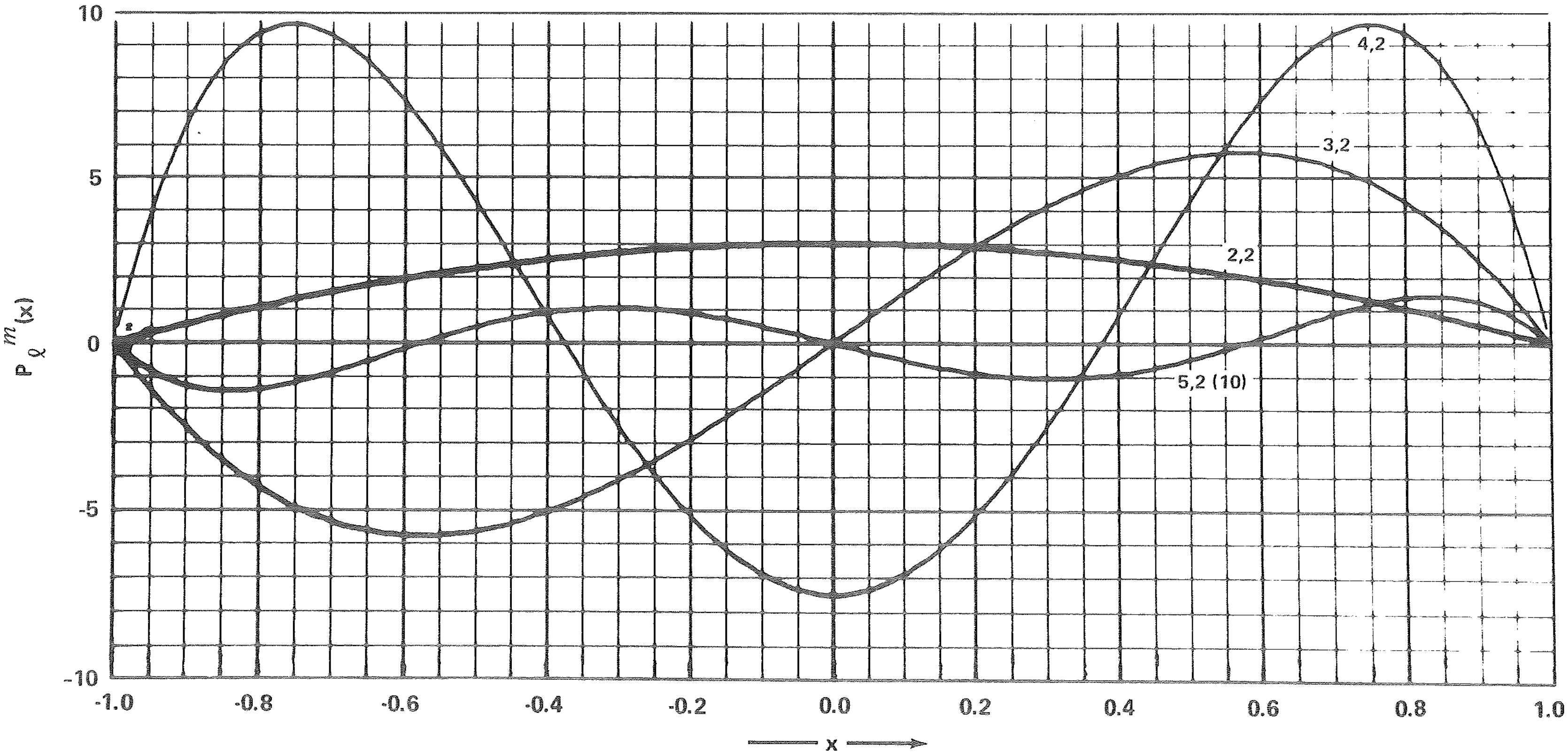


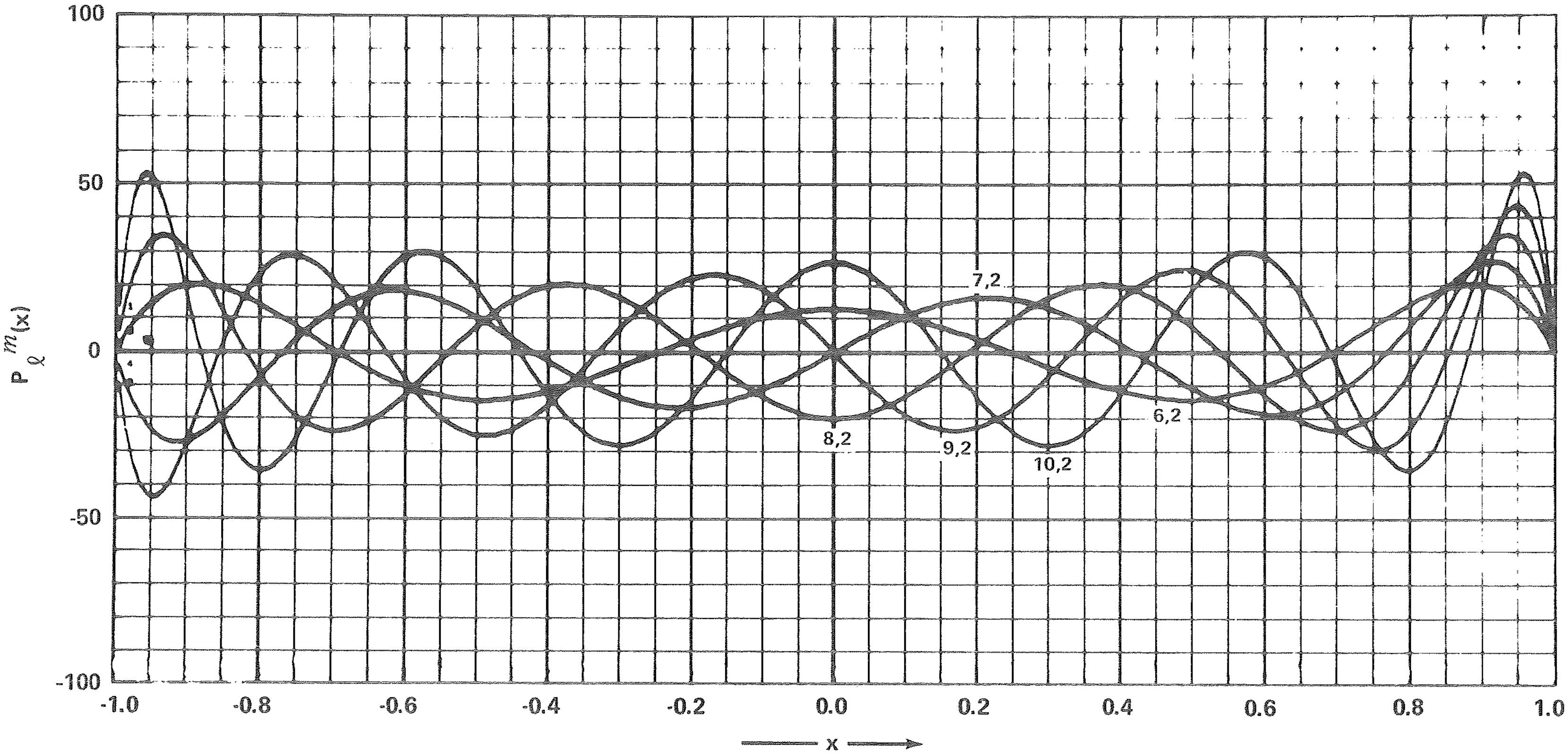
FIGURE 4 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS

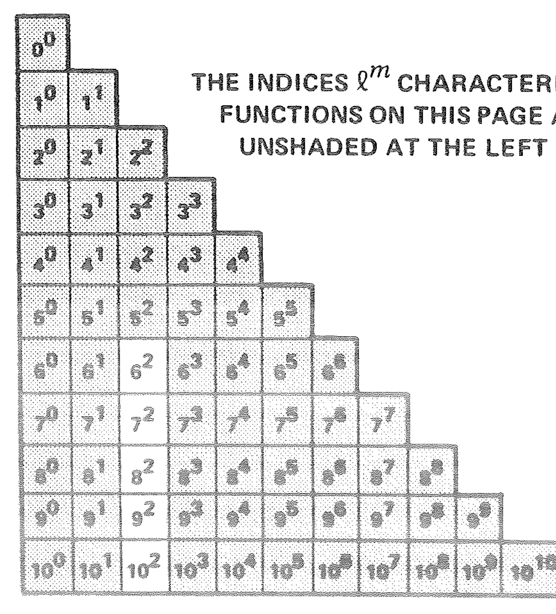


0^0										
1^0	1^1									
2^0	2^1	2^2								
3^0	3^1	3^2	3^3							
4^0	4^1	4^2	4^3	4^4						
5^0	5^1	5^2	5^3	5^4	5^5					
6^0	6^1	6^2	6^3	6^4	6^5	6^6				
7^0	7^1	7^2	7^3	7^4	7^5	7^6	7^7			
8^0	8^1	8^2	8^3	8^4	8^5	8^6	8^7	8^8		
9^0	9^1	9^2	9^3	9^4	9^5	9^6	9^7	9^8	9^9	
10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}

THE INDICES ℓ^m CHARACTERIZING THE FUNCTIONS ON THIS PAGE ARE UNSHADED AT THE LEFT

FIGURE 5 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE





THE INDICES ℓ^m CHARACTERIZING THE FUNCTIONS ON THIS PAGE ARE UNSHADED AT THE LEFT

FIGURE 6 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS

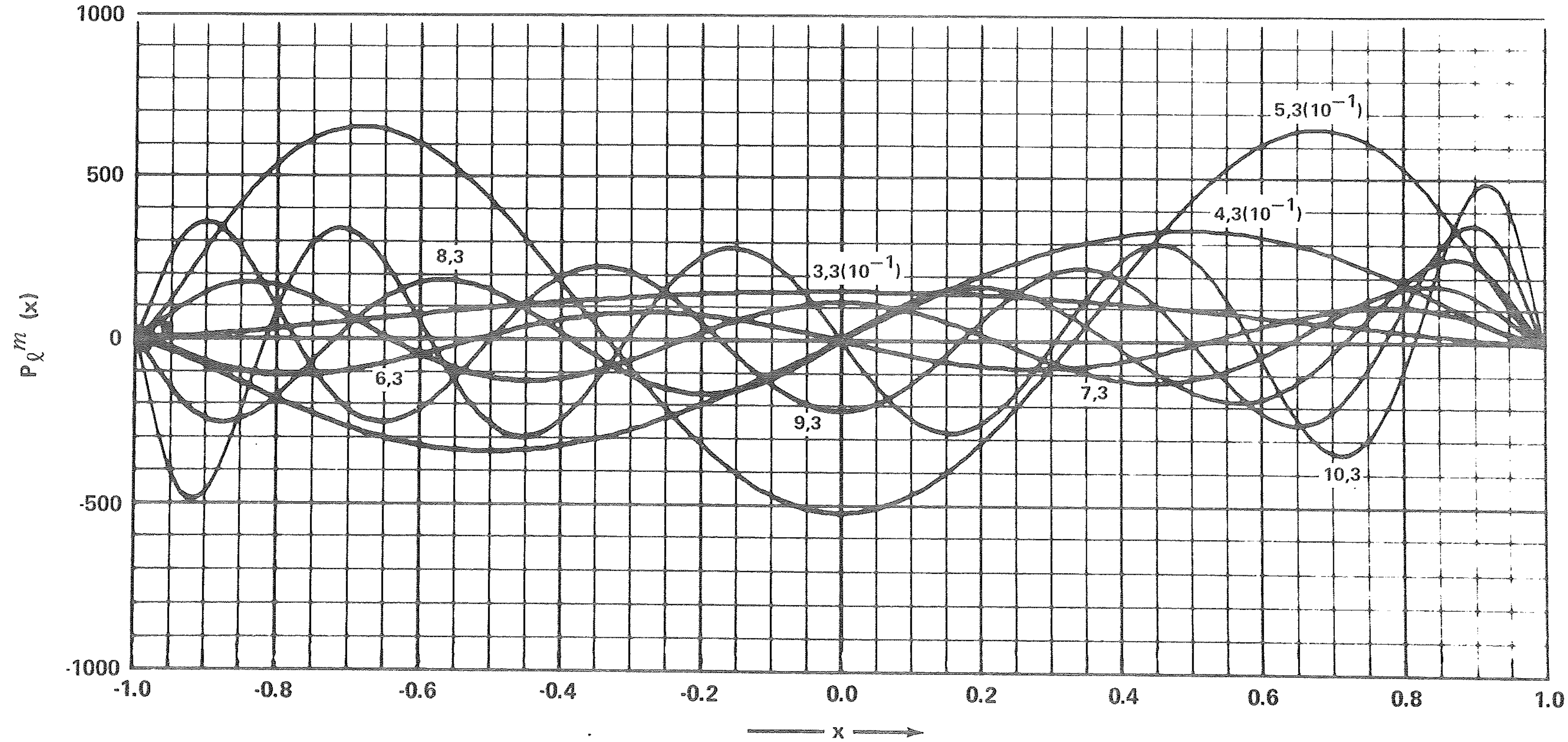
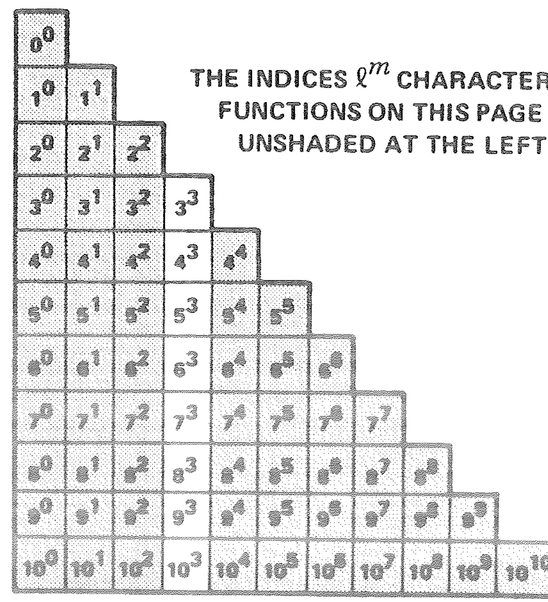
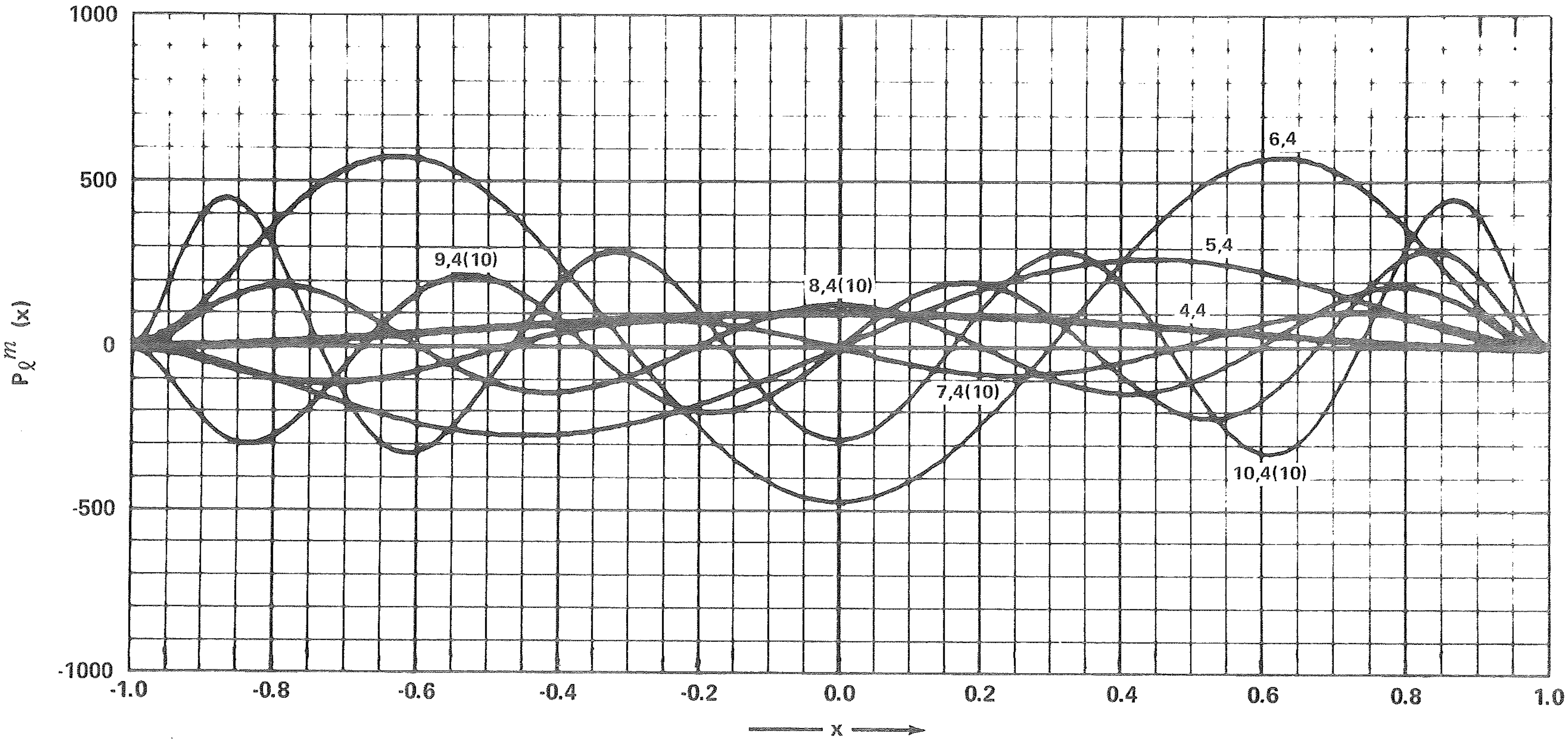
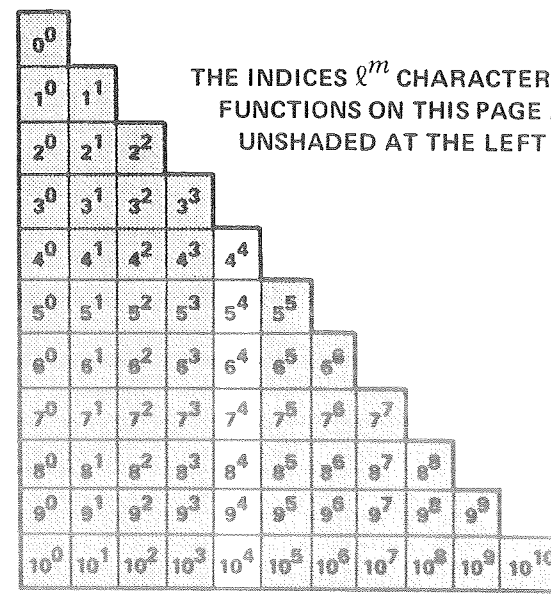


FIGURE 7 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE



THE INDICES ℓ^m CHARACTERIZING THE FUNCTIONS ON THIS PAGE ARE UNSHADED AT THE LEFT





THE INDICES ℓ^m CHARACTERIZING THE FUNCTIONS ON THIS PAGE ARE UNSHADED AT THE LEFT

FIGURE 8 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE

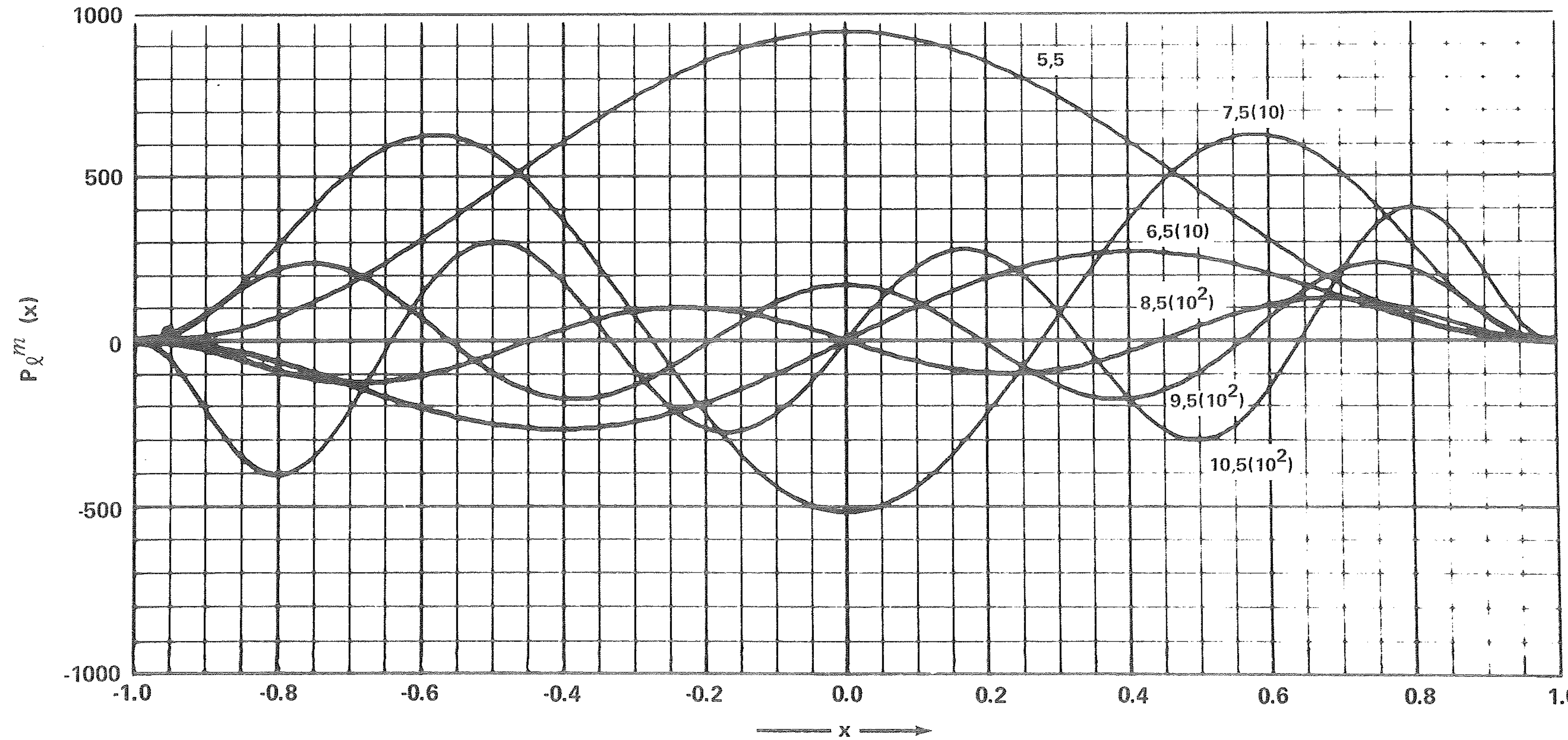
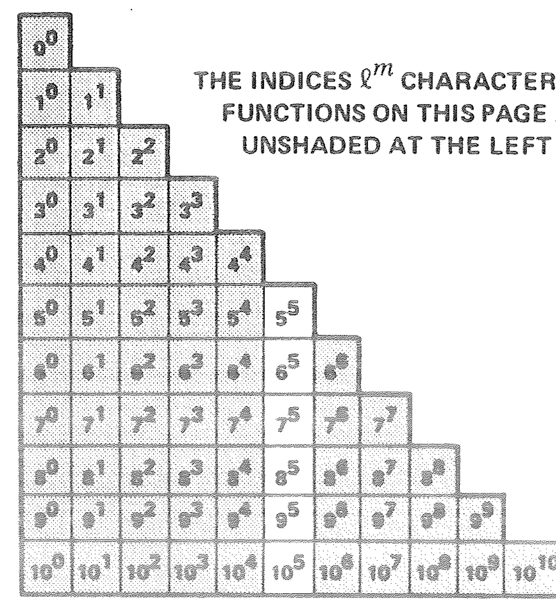
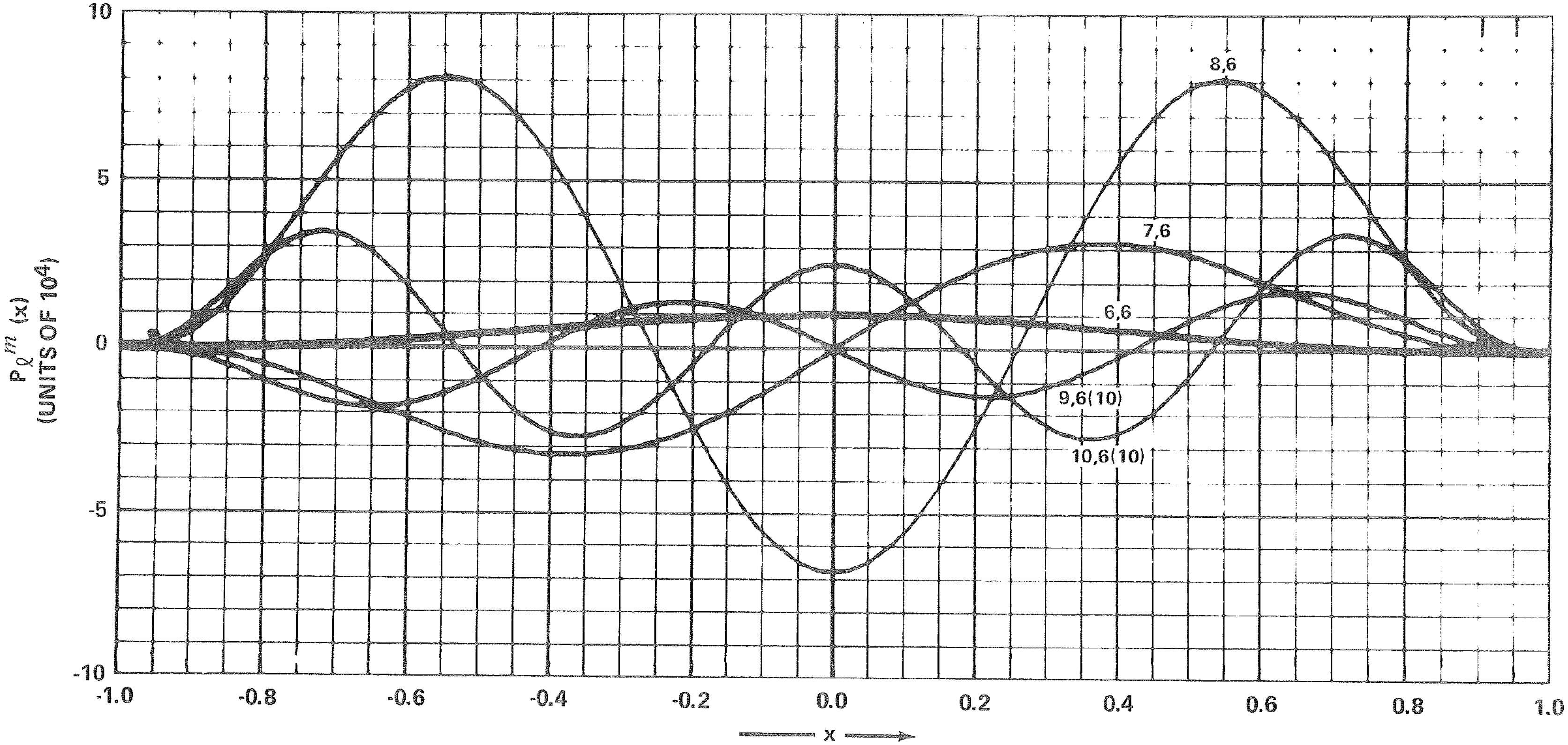
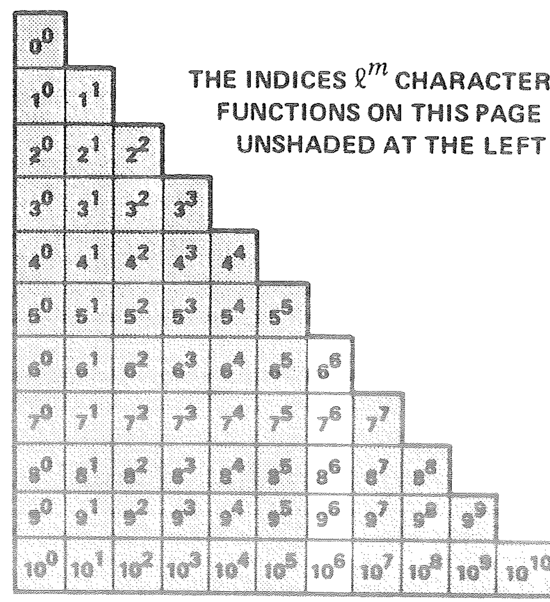


FIGURE 9 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE



THE INDICES ℓ^m CHARACTERIZING THE FUNCTIONS ON THIS PAGE ARE UNSHADED AT THE LEFT





THE INDICES ℓ^m CHARACTERIZING THE FUNCTIONS ON THIS PAGE ARE UNSHADED AT THE LEFT

FIGURE 10 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE

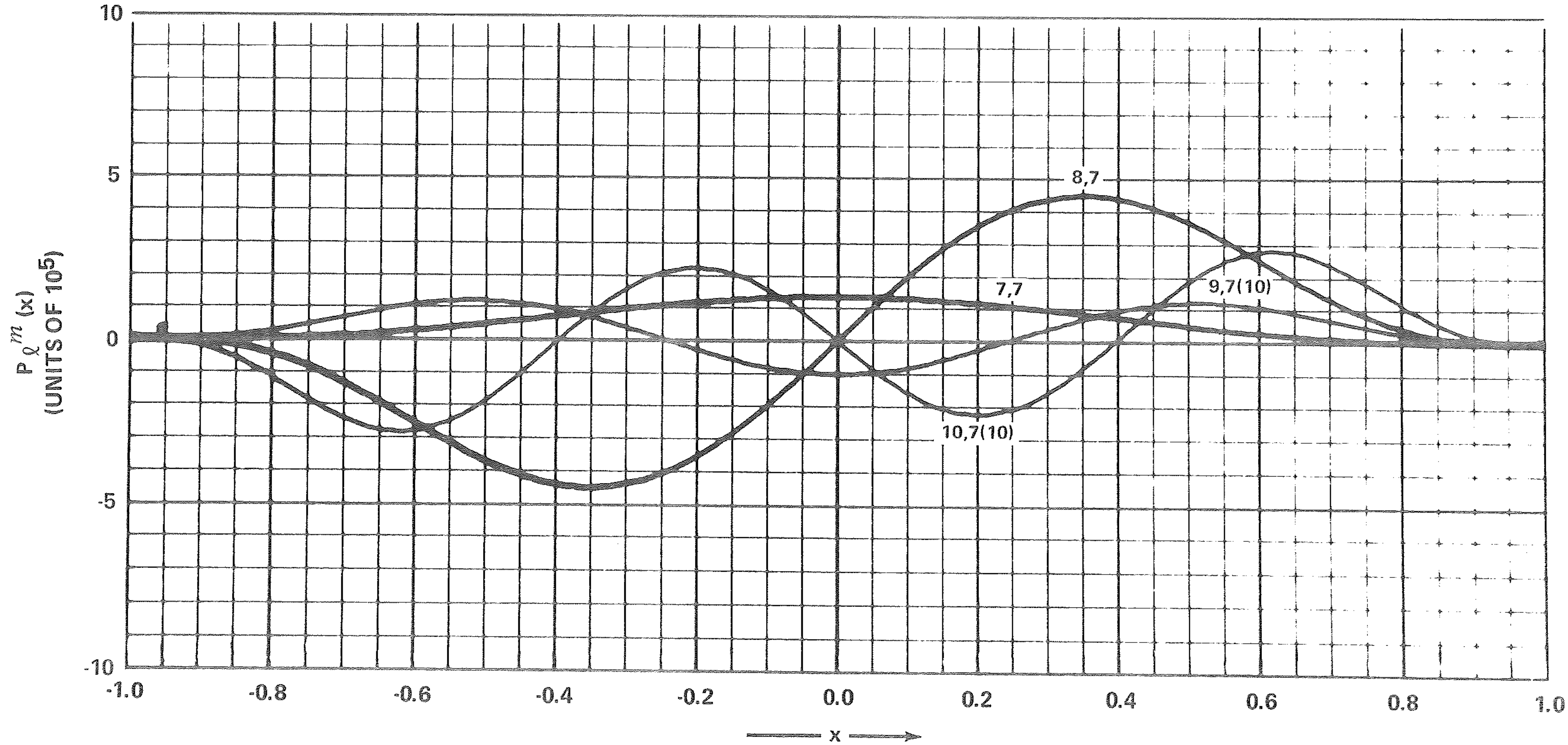
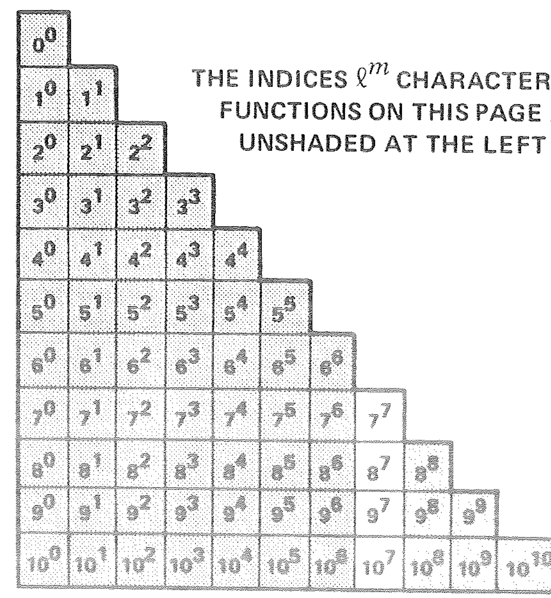
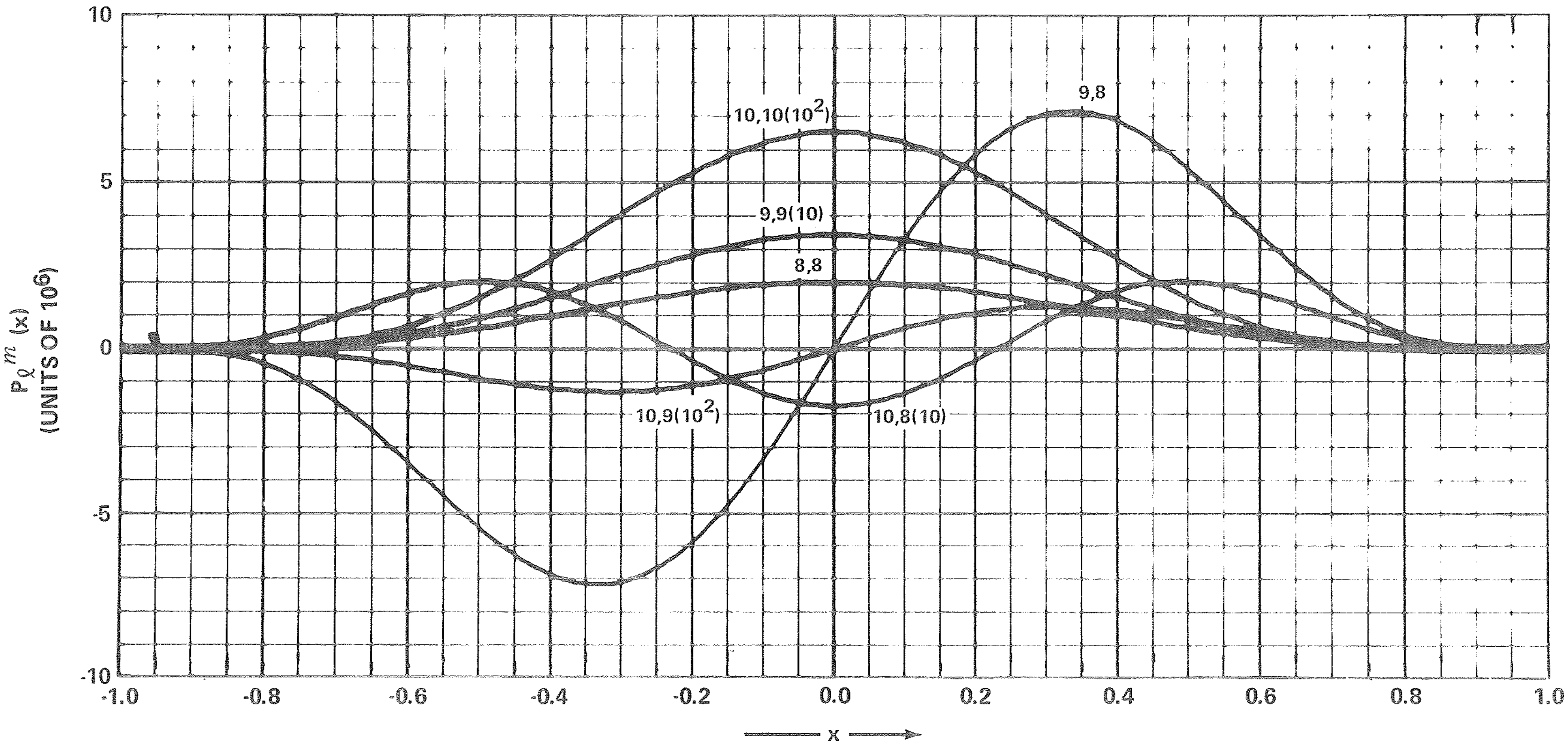
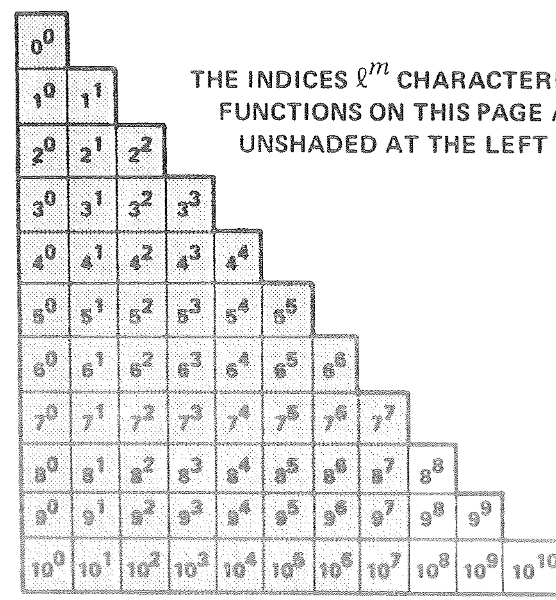


FIGURE 11 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE



THE INDICES ℓ^m CHARACTERIZING THE FUNCTIONS ON THIS PAGE ARE UNSHADED AT THE LEFT





THE INDICES ℓ^m CHARACTERIZING THE FUNCTIONS ON THIS PAGE ARE UNSHADED AT THE LEFT

FIGURE 12 - UNNORMALIZED ASSOCIATED LEGENDRE FUNCTIONS. ACTUAL VALUES MAY BE RECOVERED BY MULTIPLYING THE GRAPHICAL VALUES BY THE FACTORS IN PARENTHESES, WHERE APPLICABLE

TABLE 2

FOURIER DECOMPOSITIONS OF $P_\ell^m(\mathrm{SIN}\theta)$ AND $P_\ell^m(\mathrm{COS}\theta)$ UP TO $\ell = m = 10$

BOTH $P_\ell^m(\mathrm{SIN}\theta)$ AND $P_\ell^m(\mathrm{COS}\theta)$ DECOMPOSE INTO SUMS OF COSINES AND/OR SINES OF EITHER EVEN OR ODD MULTIPLES OF θ, THE HIGHEST MULTIPLE BEING ALWAYS $\ell\theta$. FOR EACH ℓ AND m, THE TABLE GIVES THE FORM FOR THE EXPANSIONS OF $P_\ell^m(\mathrm{SIN}\theta)$ AND $P_\ell^m(\mathrm{COS}\theta)$ AND SUPPLIES THE COEFFICIENTS FOR EACH TERM.

		1	SINθ	COS2θ	SIN3θ	COS4θ	SIN5θ	COS6θ	SIN7θ	COS8θ	SIN9θ	COS10θ	P_ℓ^m (SINθ)
ℓ	m	1	COSθ	–COS2θ	–COS3θ	COS4θ	COS5θ	–COS6θ	–COS7θ	COS8θ	COS9θ	–COS10θ	P_ℓ^m (COSθ)
0	0	1											
1	0	0	1										
2	0	1/4	0	-3/4									
3	0	0	3/8	0	-5/8								
4	0	9/64	0	-5/16	0	35/64							
5	0	0	15/64	0	-35/128	0	63/128						
6	0	25/256	0	-105/512	0	63/256	0	-231/512					
7	0	0	175/1024	0	-189/1024	0	231/1024	0	-429/1024				
8	0	1225/16384	0	-315/2048	0	693/4096	0	-429/2048	0	6435/16384			
9	0	0	2205/16384	0	-1155/8192	0	1287/8192	0	-6435/32768	0	12155/32768		
10	0	3969/65536	0	-8085/65536	0	2145/16384	0	-19305/131072	0	12155/65636	0	-46189/131072	

TABLE 2 (CONTINUED)

ℓ	m	$\cos\theta$	$\sin 2\theta$	$\cos 3\theta$	$\sin 4\theta$	$\cos 5\theta$	$\sin 6\theta$	$\cos 7\theta$	$\sin 8\theta$	$\cos 9\theta$	$\sin 10\theta$	$P_\ell^m(\sin\theta)$
		$\sin\theta$	$\sin 2\theta$	$-\sin 3\theta$	$-\sin 4\theta$	$\sin 5\theta$	$\sin 6\theta$	$-\sin 7\theta$	$-\sin 8\theta$	$\sin 9\theta$	$\sin 10\theta$	$P_\ell^m(\cos\theta)$
1	1	1										
2	1	0	3/2									
3	1	3/8	0	-15/8								
4	1	0	5/8	0	-35/16							
5	1	15/64	0	-105/128	0	315/128						
6	1	0	105/256	0	-63/64	0	693/256					
7	1	175/1024	0	-567/1024	0	1155/1024	0	-3003/1024				
8	1	0	315/1024	0	-693/1024	0	1287/1024	0	-6435/2048			
9	1	2205/16384	0	-3465/8192	0	6435/8192	0	-45045/32768	0	109395/32768		
10	1	0	8085/32768	0	-2145/4096	0	57915/65536	0	-12155/8192	0	230945/65536	

ℓ	m	1	$\sin\theta$	$\cos 2\theta$	$\sin 3\theta$	$\cos 4\theta$	$\sin 5\theta$	$\cos 6\theta$	$\sin 7\theta$	$\cos 8\theta$	$\sin 9\theta$	$\cos 10\theta$	$P_\ell^m(\sin\theta)$
		1	$\cos\theta$	$-\cos 2\theta$	$-\cos 3\theta$	$\cos 4\theta$	$\cos 5\theta$	$-\cos 6\theta$	$-\cos 7\theta$	$\cos 8\theta$	$\cos 9\theta$	$-\cos 10\theta$	$P_\ell^m(\cos\theta)$
2	2	3/2	0	3/2									
3	2	0	15/4	0	15/4								
4	2	45/16	0	-15/4	0	-105/16							
5	2	0	105/16	0	-105/32	0	-315/32						
6	2	525/128	0	-1785/256	0	315/128	0	3465/256					
7	2	0	4725/512	0	-3591/512	0	693/512	0	9009/512				
8	2	11025/2048	0	-315/32	0	3465/512	0	0	0	-45045/2048			
9	2	0	24255/2048	0	-10395/1024	0	6435/1024	0	6435/4096	0	-109395/4096		
10	2	218295/32768	0	-412335/32768	0	83655/8192	0	-366795/65536	0	-109395/32768	0	2078505/65536	

TABLE 2 (CONTINUED)

		COSθ	SIN2θ	COS3θ	SIN4θ	COS5θ	SIN6θ	COS7θ	SIN8θ	COS9θ	SIN10θ	P_ℓ^m (SINθ)
ℓ	m	SINθ	SIN2θ	–SIN3θ	–SIN4θ	SIN5θ	SIN6θ	–SIN7θ	–SIN8θ	SIN9θ	SIN10θ	P_ℓ^m (COSθ)
3	3	45/4	0	15/4								
4	3	0	105/4	0	105/8							
5	3	315/16	0	-1365/32	0	-945/32						
6	3	0	2835/64	0	-945/16	0	-3465/64					
7	3	14175/512	0	-36855/512	0	38115/512	0	45045/512				
8	3	0	31185/512	0	-51975/512	0	45045/512	0	135135/1024			
9	3	72765/2048	0	-100485/1024	0	135135/1024	0	-405405/4096	0	-765765/4096		
10	3	0	315315/4096	0	-70785/512	0	1332045/8192	0	-109395/1024	0	-2078505/8192	

		1	SINθ	COS2θ	SIN3θ	COS4θ	SIN5θ	COS6θ	SIN7θ	COS8θ	SIN9θ	COS10θ	P_ℓ^m (SINθ)
ℓ	m	1	COSθ	–COS2θ	–COS3θ	COS4θ	COS5θ	–COS6θ	–COS7θ	COS8θ	COS9θ	–COS10θ	P_ℓ^m (COSθ)
4	4	315/8	0	105/2	0	105/8							
5	4	0	945/8	0	2835/16	0	945/16						
6	4	4725/32	0	-4725/64	0	-12285/32	0	-10395/64					
7	4	0	51975/128	0	10395/128	0	-86625/128	0	-45045/128				
8	4	363825/1024	0	-51975/128	0	-93555/256	0	135135/128	0	675675/1024			
9	4	0	945945/1024	0	-135135/512	0	-405405/512	0	3108105/2048	0	2297295/2048		
10	4	2837835/4096	0	-4099095/4096	0	-45045/1024	0	11216205/8192	0	-8423415/4096	0	-14549535/8192	

TABLE 2 (CONTINUED)

		COSθ	SIN2θ	COS3θ	SIN4θ	COS5θ	SIN6θ	COS7θ	SIN8θ	COS9θ	SIN10θ	P_ℓ^m (SINθ)
ℓ	m	SINθ	SIN2θ	–SIN3θ	–SIN4θ	SIN5θ	SIN6θ	–SIN7θ	–SIN8θ	SIN9θ	SIN10θ	P_ℓ^m (COSθ)
5	5	4725/8	0	4725/16	0	945/16						
6	5	0	51975/32	0	10395/8	0	10395/32					
7	5	259875/128	0	-343035/128	0	-446985/128	0	-135135/128				
8	5	0	675675/128	0	-405405/128	0	-945945/128	0	-675675/256			
9	5	4729725/1024	0	-4729725/512	0	1216215/512	0	27702675/2048	0	11486475/2048		
10	5	0	23648625/2048	0	-3378375/256	0	-2027025/4096	0	11486475/512	0	43648605/4096	

		1	SINθ	COS2θ	SIN3θ	COS4θ	SIN5θ	COS6θ	SIN7θ	COS8θ	SIN9θ	COS10θ	P_ℓ^m (SINθ)
ℓ	m	1	COSθ	–COS2θ	–COS3θ	COS4θ	COS5θ	–COS6θ	–COS7θ	COS8θ	COS9θ	–COS10θ	P_ℓ^m (COSθ)
6	6	51975/16	0	155925/32	0	31185/16	0	10395/32					
7	6	0	675675/64	0	1216215/64	0	675675/64	0	135135/64				
8	6	4729725/256	0	0	0	-2837835/64	0	-135135/4	0	-2027025/256			
9	6	0	14189175/256	0	4729725/128	0	-10135125/128	0	-42567525/512	0	-11486475/512		
10	6	127702575/2048	0	-89864775/2048	0	-56081025/512	0	480404925/4096	0	356080725/2048	0	218243025/4096	

TABLE 2 (CONTINUED)

		$\cos\theta$	$\sin 2\theta$	$\cos 3\theta$	$\sin 4\theta$	$\cos 5\theta$	$\sin 6\theta$	$\cos 7\theta$	$\sin 8\theta$	$\cos 9\theta$	$\sin 10\theta$	$P_\ell^m(\sin\theta)$
ℓ	m	$\sin\theta$	$\sin 2\theta$	$-\sin 3\theta$	$-\sin 4\theta$	$\sin 5\theta$	$\sin 6\theta$	$-\sin 7\theta$	$-\sin 8\theta$	$\sin 9\theta$	$\sin 10\theta$	$P_\ell^m(\cos\theta)$
7	7	4729725/64	0	2837835/64	0	945945/64	0	135135/64				
8	7	0	14189175/64	0	14189175/64	0	6081075/64	0	2027025/128			
9	7	99324225/256	0	-42567525/128	0	-83108025/128	0	-180405225/512	0	-34459425/512		
10	7	0	562837275/512	0	-11486475/64	0	-1481755275/1024	0	-126351225/128	0	-218243025/1024	

		1	$\sin\theta$	$\cos 2\theta$	$\sin 3\theta$	$\cos 4\theta$	$\sin 5\theta$	$\cos 6\theta$	$\sin 7\theta$	$\cos 8\theta$	$\sin 9\theta$	$\cos 10\theta$	$P_\ell^m(\sin\theta)$
ℓ	m	1	$\cos\theta$	$-\cos 2\theta$	$-\cos 3\theta$	$\cos 4\theta$	$\cos 5\theta$	$-\cos 6\theta$	$-\cos 7\theta$	$\cos 8\theta$	$\cos 9\theta$	$-\cos 10\theta$	$P_\ell^m(\cos\theta)$
8	8	7094578/128	0	14189175/16	0	14189175/32	0	2027025/16	0	2027025/128			
9	8	0	241215975/128	0	241215975/64	0	172297125/64	0	241215975/256	0	34459425/256		
10	8	2170943775/512	0	723647925/512	0	-1137161025/128	0	-9614179575/1024	0	-2033106075/512	0	-654729075/1024	

TABLE 2 (CONCLUDED)

		$\cos\theta$	$\sin 2\theta$	$\cos 3\theta$	$\sin 4\theta$	$\cos 5\theta$	$\sin 6\theta$	$\cos 7\theta$	$\sin 8\theta$	$\cos 9\theta$	$\sin 10\theta$	$P_\ell^m(\sin\theta)$
ℓ	m	$\sin\theta$	$\sin 2\theta$	$-\sin 3\theta$	$-\sin 4\theta$	$\sin 5\theta$	$\sin 6\theta$	$-\sin 7\theta$	$-\sin 8\theta$	$\sin 9\theta$	$\sin 10\theta$	$P_\ell^m(\cos\theta)$
9	9	2170943775/128	0	723647925/64	0	310134825/64	0	310134825/256		34459425/256		
10	9	0	-3435635385/16	0	1964187225/32	0	17677685025/512	0	654729075/64	0	654729075/512	

		1	$\sin\theta$	$\cos 2\theta$	$\sin 3\theta$	$\cos 4\theta$	$\sin 5\theta$	$\cos 6\theta$	$\sin 7\theta$	$\cos 8\theta$	$\sin 9\theta$	$\cos 10\theta$	$P_\ell^m(\sin\theta)$
ℓ	m	1	$\cos\theta$	$-\cos 2\theta$	$-\cos 3\theta$	$\cos 4\theta$	$\cos 5\theta$	$-\cos 6\theta$	$-\cos 7\theta$	$\cos 8\theta$	$\cos 9\theta$	$-\cos 10\theta$	$P_\ell^m(\cos\theta)$
10	10	-13735772505/128	0	27076139/256	0	9820936125/64	0	29462808375/512	0	3273645375/256	0	654729075/512	

Zeros, Symmetry, and Special Values

The function $P_\ell^m(x)$ has $(\ell-m)$ zero crossings on the interval (-1, 1). Its properties include the following:

$$P_\ell^m(-x) = (-)^{\ell-m} P_\ell^m(x) \qquad \text{SYMMETRY}$$

$$P_\ell^m(1) = \delta_{m0}$$

$$P_\ell^m(0) = \frac{(-)^{\frac{\ell-m}{2}} \quad (\ell+m)!}{2^\ell \left(\frac{\ell-m}{2}\right)! \quad \left(\frac{\ell+m}{2}\right)!} \times \begin{cases} 0 & [\ell+m = \text{odd}] \\ 1 & [\ell+m = \text{even}] \end{cases}$$

$$P_\ell^\ell(x) = \frac{(2\ell)!}{2^\ell \ell!} (1-x^2)^{\ell/2}$$

Asymptotic Values

Let $\qquad \ell >> \frac{1}{\varepsilon} >> \frac{6}{\pi} \quad$ and $\quad 0 < \varepsilon \leq \theta \leq \pi - \varepsilon,$

and define $\qquad \phi = \left(\ell+\frac{1}{2}\right)\theta + \frac{\pi}{4}.$

Then

$$P_\ell^m(\cos\theta) \approx (-n)^m \sqrt{\frac{2}{\ell\pi \sin\theta}} \sin\left(\phi+\frac{m\pi}{2}\right)$$

and

$$P_\ell (\cos\theta) \approx \sqrt{\frac{2}{\ell\pi \sin\theta}} \left[\left(1-\frac{1}{4\ell}\right)\sin\phi - \frac{1}{8\ell} \text{ctn}\,\theta \cos\phi\right].$$

Upper Bounds

Table 3 gives the least upper bounds (LUB's) for the unnormalized associated Legendre functions up to $P_{10}^{10}(x)$. The values were taken from Figures 1 through 12. Empirically, up to $P_{10}^{10}(x)$ it is found that

$$\mathrm{LUB}\,|P_\ell^m(x)| > (1+\delta_{m0})\sqrt{\frac{2}{2\ell+1}\,\frac{(\ell+m)!}{(\ell-m)!}} \quad .$$

Analytical relations are

$$\mathrm{LUB}\,|P_\ell^\ell(x)| = \frac{(2\ell)!}{2^\ell \ell!}$$

$$\mathrm{LUB}\,|P_\ell^m(x)| \leq \mathrm{LUB}\,|P_\ell^\ell(x)| \qquad \text{CONJECTURED}$$

$$|P_\ell^m(x)| < \frac{2}{\sqrt{\pi\ell}}\ \frac{(\ell+m)!}{\ell!}\ \frac{1}{(1-x^2)^{\ell+1/2}} \qquad [\ell\neq 0].$$

$\ell \downarrow$ \ $m \rightarrow$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	1.5	3								
3	1	2.2	5.8	15							
4	1	2.7	9.6	34	110						
5	1	3.3	15	66	270	950					
6	1	3.8	20	110	580	2800	1.1E4				
7	1	4.3	27	180	1100	6300	3.2E4	1.4E5			
8	1	4.9	35	260	1900	13,000	8.1E4	4.6E5	2.1E6		
9	1	5.7	44	360	3000	24,000	1.8E5	1.2E6	7.2E6	3.5E7	
10	1	6.3	53	490	4500	41,000	3.5E5	2.9E6	2.1E7	1.3E8	6.6E8

TABLE 3. Least Upper Bounds for $P_\ell^m(x)$.
Values are extracted from Figures 1 through 12.
NOTE: The notation aEb means a x 10^b.

Recurrence Relations

The argument x is understood in the following recurrence relations, which have been taken from [Magnus and Oberhettinger, 1949] and [Emde and Jahnke, 1945]. Restrictions on m may arise due to the convention on page 1.

$$P_\ell^{-m} = \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m$$

$$P_{-\ell-1}^m = P_\ell^m$$

$$x(2\ell+1) P_\ell^m = (\ell-m+1) P_{\ell+1}^m + (\ell+m) P_{\ell-1}^m$$

$$P_\ell^{m+2} - 2(m+1) \frac{x}{\sqrt{1-x^2}} P_\ell^{m+1} + (\ell-m)(\ell+m+1) P_\ell^m = 0$$

$$(\ell-m)(\ell-m+1) P_{\ell+1}^m = (\ell+m)(\ell+m+1) P_{\ell-1}^m - (2\ell+1) \sqrt{1-x^2} P_\ell^{m+1}$$

$$P_{\ell-1}^m - P_{\ell+1}^m + (2\ell+1) \sqrt{1-x^2} P_\ell^{m-1} = 0$$

$$P_{\ell-1}^m - x P_\ell^m + (\ell-m+1) \sqrt{1-x^2} P_\ell^{m-1} = 0$$

$$x P_\ell^m - P_{\ell+1}^m + (\ell+m) \sqrt{1-x^2} P_\ell^{m-1} = 0$$

$$x (\ell-m) P_\ell^m - (\ell+m) P_{\ell-1}^m + \sqrt{1-x^2} P_\ell^{m+1} = 0$$

$$(\ell-m+1)\ P_{\ell+1}^{\ m} - x\ (\ell+m+1)\ P_{\ell}^{\ m} + \sqrt{1-x^2}\ P_{\ell}^{\ m+1} = 0$$

$$(\ell+1)\ P_{\ell+1}^{\ m} + \ell P_{\ell-1}^{\ m} = (2\ell+1)\left[x\ P_{\ell}^{\ m} + m\sqrt{1-x^2}\ P_{\ell}^{\ m-1}\right] - \ell P_{\ell-1}^{\ m}$$

$$(1-x^2)\ \frac{d\ P_{\ell}^{\ m}}{dx} = x\ (\ell+1)\ P_{\ell}^{\ m} - (\ell-m+1)\ P_{\ell+1}^{\ m}$$

$$= -x\ell\ P_{\ell}^{\ m} + (\ell+m)\ P_{\ell-1}^{\ m}$$

$$= \sqrt{1-x^2}\ P_{\ell}^{\ m+1} - xm\ P_{\ell}^{\ m}$$

$$= -(\ell-m+1)(\ell+m)\ \sqrt{1-x^2}\ P_{\ell}^{\ m-1} + xm\ P_{\ell}^{\ m}$$

$$\frac{xdP_{\ell}}{dx} - \frac{dP_{\ell-1}}{dx} = \ell\ P_{\ell}$$

$$\frac{-xdP_{\ell}}{dx} + \frac{dP_{\ell+1}}{dx} = (\ell+1)\ P_{\ell}$$

$$\frac{dP_{\ell+1}}{dx} - \frac{dP_{\ell-1}}{dx} = (2\ell+1)\ P_{\ell}$$

<u>Recurrence Relations for Solid Spherical Harmonics</u>

The infinite series

$$u(h,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \left(a_{\ell m}\, U_\ell^{\,m} + b_\ell^{\,m}\, V_\ell^{\,m} \right),$$

with

$$\begin{pmatrix} U_\ell^{\,m} \\ V_\ell^{\,m} \end{pmatrix} = \frac{P_\ell^{\,m}(\cos\theta)}{r^{\ell+1}} \begin{pmatrix} \cos\, m\phi \\ \sin\, m\phi \end{pmatrix},$$

is a solution of Laplace's equation in spherical polar coordinates. The functions $U_\ell^{\,m}$ and $V_\ell^{\,m}$ are called solid spherical harmonics. These have the following useful recurrence properties [DeWitt, 1962]:

$$U_{\ell+1}^{\,m} = \frac{1}{r(\ell-m+1)} \left[(2\ell+1)\ \cos\theta\ U_\ell^{\,m} - \frac{\ell+m}{r}\, U_{\ell-1}^{\,m} \right]$$

$$V_{\ell+1}^{\,m} = \frac{1}{r(\ell-m+1)} \left[(2\ell+1)\ \cos\theta\ U_\ell^{\,m} - \frac{\ell+m}{r}\, U_{\ell-1}^{\,m} \right]$$

$$U_{\ell+1}^{\,\ell+1} = \frac{2\ell+1}{r^2} \left[U_\ell^{\,\ell}\ C(\phi) - V_\ell^{\,\ell}\ S(\phi) \right]$$

$$V_{\ell+1}^{\,\ell+1} = \frac{2\ell+1}{r^2} \left[V_\ell^{\,\ell}\ C(\phi) + V_\ell^{\,\ell}\ S(\phi) \right],$$

where

$$\begin{pmatrix} C(\phi) \\ S(\phi) \end{pmatrix} = \sin\,\theta \begin{pmatrix} \cos\ \phi \\ \sin\ \phi \end{pmatrix}.$$

Defining $\bar{\nabla}$ as the gradient operator, then

$$2\bar{\nabla}\, U_\ell^m = \begin{bmatrix} C_\ell^m\, U_{\ell+1}^{m-1} - U_{\ell+1}^{m+1} \\ -C_\ell^m\, V_{\ell+1}^{m-1} - V_{\ell+1}^{m+1} \\ -2(\ell-m+1)\, U_{\ell+1}^{m} \end{bmatrix}$$

$$2\bar{\nabla}\, V_\ell^m = \begin{bmatrix} C_\ell^m\, V_{\ell+1}^{m-1} - V_{\ell+1}^{m-1} \\ C_\ell^m\, U_{\ell+1}^{m-1} + U_{\ell+1}^{m+1} \\ -2(\ell-m+1)\, V_{\ell+1}^{m} \end{bmatrix},$$

where $$C_\ell^m = (\ell-m+1)(\ell-m+2).$$

Recurrence relations on $u(r,\theta,\phi)$ may be found in [James, 1969].

<u>Important Expansions</u>

$$\frac{1}{\sqrt{1 - 2hx + h^2}} = \left\{ \begin{array}{lll} \sum\limits_{\ell=0}^{\infty} & h^\ell P_\ell(x) & [h<1] \\ \sum\limits_{\ell=0}^{\infty} & \frac{1}{h^{\ell+1}} P_\ell(x) & [h>1] \end{array} \right\} \text{GENERATING FUNCTION}$$

Let α be the angle between two points with spherical polar coordinates (θ, ϕ) and (θ', ϕ').

$$P_\ell(\cos\alpha) = \sum_{m=0}^{\ell} (2-\delta_{m0}) \frac{(\ell-m)!}{(\ell+m)!}$$

$$\times P_\ell^{m}(\cos\theta)\, P_\ell^{m}(\cos\theta') \cos m(\phi-\theta')$$

ADDITION THEOREM

Short Table of Integrals

$$\int_{-1}^{1} P_\ell^{m}(x)\, P_{\ell'}^{m}(x)\, dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'}$$

ORTHOGONALITY INTEGRAL MO

$$\int_{-1}^{1} \frac{P_\ell^{m}(x)\, P_\ell^{m'}(x)}{1-x^2}\, dx = \frac{1}{m} \frac{(\ell+m)!}{(\ell-m)!} \delta_{mm'} \qquad [0<m\leq\ell]$$

MO

$$I_a(\ell, i, n, j) \equiv \int_{-a}^{a} P_\ell^{i}(x)\, P_n^{j}(x)\, dx$$

Let $\quad 2p \equiv 2q - 2(t+\nu) + (\ell+n) - (i+j)$

and $\quad T_{\alpha\beta\gamma} \equiv \dfrac{(-)^{\gamma}(2\alpha-2\gamma)!}{2^{\alpha}\gamma!(\alpha-\gamma)!(\alpha-\beta-2\gamma)!}$

and $\quad [b] \equiv$ greatest integer in b

$$I_a(\ell, i, n, j) = 0 \qquad [\text{if } (\ell+n) - (i+j) = \text{odd}]$$

SL

$$I_a(\ell, i, n, j) = 2 \sum_{t=0}^{\left[\frac{\ell-i}{2}\right]} \sum_{\nu=0}^{\left[\frac{n-j}{2}\right]} \sum_{q=0}^{\left[\frac{i+j}{2}\right]}$$

SL

$$(-)^q \, T_{\ell i t} \, T_{n j \nu} \begin{pmatrix} \frac{i+j}{2} \\ q \end{pmatrix} \frac{a^{1+2p}}{1+2p}$$

[if $(i+j)$ = even and $(\ell+n)$ = even]

$$I_a(\ell, i, n, j) = 2 \sum_{t=0}^{\left[\frac{\ell-i}{2}\right]} \sum_{\nu=0}^{\left[\frac{n-j}{2}\right]} \sum_{q=0}^{\frac{i+j-1}{2}} (-)^q \, T_{\ell i t} \, T_{n j \nu} \begin{pmatrix} \frac{j+\ell-1}{2} \\ q \end{pmatrix}$$

SL

$$\times \int_0^{\arcsin(a)} \sin^{2p}(\theta) \cos^2(\theta) \, d\theta$$ [if $(i+j)$ = odd and $(\ell+n)$ = odd]

$$I_1(\ell, i, n, j) = \pi \sum_{t=0}^{\left[\frac{\ell-i}{2}\right]} \sum_{\nu=0}^{\left[\frac{n-j}{2}\right]} \sum_{q=0}^{\frac{i+j-1}{2}} (-)^q \, T_{\ell i t} \, T_{n j \nu} \begin{pmatrix} \frac{j+\ell-1}{2} \\ q \end{pmatrix}$$

SL

$$\times \frac{1}{(2p+1) \; 2^{2p+2}} \begin{pmatrix} 2p+2 \\ p+1 \end{pmatrix}$$

[if $(i+j)$ = odd and $(\ell+n)$ = odd]

$$J_p(s,\ n,\ m) \equiv \int_0^{\pi} (\sin\theta)^{2s-3}\ (\cos\theta)^p\ P_n^{\ m}(\cos\theta)\ P_{n-p}^{\ m}(\cos\theta)\ d\theta$$

$$J_0(0,\ n,\ m) = \frac{(n+m)!}{2m(m^2-1)(n-m)!}\ (n^2+n+m^2-1) \qquad [n-p \geq m \geq 2] \qquad \text{HK}$$

$$J_0(1,\ n,\ m) = \frac{(n+m)!}{m(n-m)!} \qquad [n-p \geq m \geq 1] \qquad \text{HK}$$

$$J_0(2,\ n,\ m) = \frac{2(n+m)!}{(2n+1)(n-m)!} \qquad [n-p \geq m \geq 0] \qquad \text{HK}$$

$$J_1(0,\ n,\ m) = \frac{n(n+m)!}{2m(m^2-1)(n-m-1)!} \qquad [n-p \geq m \geq 2] \qquad \text{HK}$$

$$J_1(1,\ n,\ m) = \frac{(n+m-1)!}{n(n-m-1)!} \qquad [n-p \geq m \geq 1] \qquad \text{HK}$$

$$J_1(2,\ n,\ m) = \frac{2(n+m)!}{(4n^2-1)(n-m-1)!} \qquad [n-p \geq m \geq 0] \qquad \text{HK}$$

$$\int_0^{\pi} \left[\frac{dP_n^{\ m}(\cos\theta)}{d\theta}\right]^2 \sin\theta\ d\theta = \left(\frac{2n(n+1)}{2n+1} - m\right) \frac{(n+m)!}{(n-m)!} \qquad \text{HK}$$

$$\int_0^{\pi} \left[\frac{dP_n^{\,m}(\cos\theta)}{d\theta} \right]^2 \frac{d\theta}{\sin\theta} =$$

HK

$$\begin{cases} n(n+1) & [m=0] \\ \dfrac{m(n^2+n+1-m^2)}{2m^2-1} \quad \dfrac{(n+m)!}{(n-m)!} & [m\neq 0] \end{cases}$$

$$\int_0^{\pi} \left[\frac{d^2P_n^{\,m}(\cos\theta)}{d^2\theta} \right]^2 \sin\theta \; d\theta =$$

HK

$$\begin{cases} \dfrac{n(n+1)(2n^2-1)}{2n+1} & [m=0] \\ 2n^2(1-m) + \dfrac{1}{2}\, m(n^2-3n+m^2+1) & \\ & [m\neq 0] \\ + \dfrac{2n^4}{2n+1}\,\dfrac{(n+m)!}{(n-m)!} & \end{cases}$$

Other integrals involving the associated Legendre functions may be found in [Bateman, 1953], in [Gradshteyn and Ryzhik, 1965], and in [Magnus and Oberhettinger, 1949].

The references abbreviated beside the formulas listed above are:

MO: [Magnus and Oberhettinger, 1949]
SL: [Levie, 1971]
HK: [Higgens and Kopal, 1968]

REFERENCES

1. Bateman, H., Higher Transcendental Functions, Vol. 1, ed. A. Erdélyi, 302 pp. McGraw-Hill, New York, 1953.
2. Battin, R. H., Astronautical Guidance, 400 pp., McGraw-Hill, New York, 1964.
3. DeWitt, R. N., "Derivations of Expressions Describing the Gravitational Field of the Earth," U.S. Naval Weapons Laboratory, Technical Memorandum K-35/62, November, 1962.
4. Emde, F., and Jahnke, E., Tables of Functions with Formulae and Curves, 382 pp., Dover, New York, 1945.
5. Gradshteyn, I. S., and Ryzhik, I. M., Tables of Integrals, Series, and Products, ed. A. Jeffrey, 1086 pp., Academic, New York, 1965.
6. Higgens, T. P., and Kopal, Z., "Volume Integrals of Products of Spherical Harmonics and Their Application to Viscous Dissipation Phenomena in Fluids," Boeing Scientific Research Laboratories document D1-82-0733, July, 1968.
7. James, R. W., "Transformation of Spherical Harmonics Under Change of Reference Frame," Geophys. J. R. Astr. Soc., 17, 305-316, 1969.

8. Kaula, W. M., Theory of Satellite Geodesy, 120 pp., Blaisdell, Waltham, Massachusetts, 1966.

9. Korn, G. A., and Korn, T. M., Mathematical Handbook for Scientists and Engineers, 943 pp., McGraw-Hill, New York, 1961.

10. Levie, S. L. Jr., "Transformation of a Potential Function Under Coordinate Rotations," Bellcomm, Inc., Technical Report TR-70-310-1, Washington, D.C., 1970a (available as N70-32736 from National Technical Information Service, Springfield, Virginia).

11. Levie, S. L. Jr., "Transformation of a Potential Function Under Coordinate Translations," Bellcomm, Inc., Technical Memorandum TM-2014-7, Washington, D.C., 1970b (available as N70-41194 from National Technical Information Service, Springfield, Virginia).

12. Levie, S. L. Jr., "Fitting a Mass Distribution to a Potential," Bellcomm, Inc., Memorandum for File B71-02002, Washington, D.C., 1971 (available as N71-17428 from National Technical Information Service, Springfield, Virginia).

13. Magnus, W., and Oberhettinger, F., Formulas and Theorems for the Functions of Mathematical Physics, trans. J. Werner, 172 pp., Chelsea, New York, 1949.

Subject: Unnormalized Assoicated Legendre Functions: A Compendium of Graphs, Expansions, Properties and Integrals

From: W. G. Heffron, S. L. Levie, Jr.

DISTRIBUTION LIST

NASA Headquarters

V. N. Huff/MTE
A. S. Lyman/MR
L. R. Scherer/MAL
W. E. Stoney/MAE

Manned Spacecraft Center

J. Funk/FM8
J. C. McPherson/FM4
R. K. Osburn/FM4
E. R. Schiesser/FM4
W. R. Wollenhaupt/FM4

Marshall Space Flight Center

R. F. Arenstorf/S&E-Comp-T
H. G. Krause/S&E-Aero-T
H. J. Sperling/S&E-Aero-T

Goddard Space Flight Center

J. Barsky/832
L. H. Carpenter/554
B. Kaufman/551.2
J. P. Murphy/552
P. Musen/641

Jet Propulsion Laboratory

J. H. Born/97
R. A. Broucke/156-220
P. Gottlieb/156-220
J. H. Lieske/156-220
A. Liu/156-217
J. Lorell/156-217
W. G. Melbourne/156-203
W. L. Sjogren/156-251

Langley Research Center

W. H. Michael, Jr./1152A
R. H. Tolson/1192C

National Oceanic and Atmospheric Administration

F. Morrison

Smithsonian Astrophysical Observatory

E. M. Gaposchkin
C. A. Lundquist
B. G. Marsden

The Ohio State University

R. H. Rapp

Aeronautical Chart and Information Center

R. L. Ealum
J. Hopkins

Massachusetts Institute of Technology

R. H. Battin
S. J. Madden, Jr.

Computer Sciences Corporation

R. M. L. Baker, Jr.
D. H. Novak

DISTRIBUTION LIST CONT.

University of California in Los Angelos

W. M. Kaula

University of Texas at Austin

V. Szebehely
B. D. Tapley

U. S. Naval Observatory

A. D. Fiala
T. C. Van Flandern

Analytical Mechanics, Assocs.

M. Payne

Bellcomm, Inc.

G. M. Anderson
R. A. Bass
A. P. Boysen, Jr.
M. V. Bullock
J. O. Cappelari, Jr.
D. A. Corey
D. A. DeGraaf
F. El-Baz
W. W. Ennis
A. J. Ferrari
D. R. Hagner
H. A. Helm
N. W. Hinners
T. B. Hoekstra
W. W. Hough
M. Liwshitz
H. S. London
J. L. Marshall, Jr.
K. E. Martersteck
J. Z. Menard
G. T. Orrok
P. E. Reynolds
R. D. Sharma
W. R. Sill
R. V. Sperry

Bellcomm Inc. Cont.

J. W. Timko
R. L. Wagner
M. P. Wilson
D. B. Wood
T. L. Yang
M. T. Yates
Department 1024 File
Central Files
Library

Memorandum only

G. C. Bill
J. P. Downs
D. P. Ling