N71-34530

THEORETICAL CHEMISTRY INSTITUTE

THE UNIVERSITY OF WISCONSIN

ON THE NONITERATIVE SOLUTION OF INTEGRAL EQUATIONS

FOR SCATTERING OF ELECTROMAGNETIC WAVES

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WIS-TCI-446 23 July 1971

MADISON, WISCONSIN

ON THE NONITERATIVE SOLUTION OF INTEGRAL EQUATIONS FOR SCATTERING OF ELECTROMAGNETIC WAVES *

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ABSTRACT

The scattering of electromagnetic waves is considered using the integral equation form of Maxwell's equations for the electric field. These equations are of a form similar but not identical to those arising in quantum mechanical scattering theory. The homogeneous integral solution procedure of Sams and Kouri is adapted to these field equations to derive Volterra integral equations of the second kind for "modified field functions". A quadrature solution procedure is examined for the solution of the Volterra equations and its merits discussed.

* This research was completed during a visit to the Theoretical Chemistry Institute, University of Wisconsin, Madison, Wisconsin, July 1971. Support of this research by National Science Foundation Grant GP-18872 and National Aeronautics and Space Administration Grant NGL 50-002-001 is gratefully acknowledged.

I. INTRODUCTION

The purpose of this note is to point out a computational procedure for solving Maxwell's equations for scattering of electromagnetic waves. We employ Maxwell's equations expressed in integral form. The present section does not contain anything new but is presented to establish notation and the approach to be pursued. Our formulation of the equations is essentially that presented by Newton in his reference work on wave and particle scattering.[1-5]

Maxwell's equations in differentiated form may be written as

$$\vec{\nabla} \times \vec{\nabla} \times \vec{\mathcal{E}} - k^2 \vec{\mathcal{E}} = k^2 (n'^2 - 1) \vec{\mathcal{E}} + \vec{\nabla} \times [(1 - \mu^{-1}) \vec{\nabla} \times \vec{\mathcal{E}}] \tag{1}$$

where the index of refraction n' is given by

$$n'^2 = \epsilon + 4\pi i \sigma/\omega. \tag{2}$$

Here μ represents the magnetic permeability, ϵ is the dielectric constant, σ is the conductivity and ω the circular frequency of the radiation. The refractive index n' and permeability μ in general are tensor quantities and need not be uniform. Thus, the scattering medium need not be homogeneous or isotropic. For the case where μ is not a tensor (isotropic medium), Eq. (1) may be written as

$$\vec{\nabla} \times \vec{\nabla} \times \vec{\mathcal{E}} - k^2 \vec{\mathcal{E}} = k^2 (n^2 - 1) \vec{\mathcal{E}} + \mu^{-1} (\vec{\nabla}\mu) \times (\vec{\nabla} \times \vec{\mathcal{E}}) \tag{3}$$

and

$$n^2 = \mu n'^2. \tag{4}$$

To convert Eqs. (1) or (3) into integral equations, we employ the tensor Green's function $\overleftrightarrow{\Gamma}(k;\vec{r},\vec{r}\,')$ satisfying

$$\vec{\nabla}\times(\vec{\nabla}\times\overleftrightarrow{\Gamma}) - k^2\overleftrightarrow{\Gamma} = \overleftrightarrow{1}\,\delta(\vec{r}-\vec{r}\,') \tag{5}$$

which may be shown to be[1-3]

$$\overleftrightarrow{\Gamma}(k;\vec{r},\vec{r}\,') = (\overleftrightarrow{1} + k^{-2}\vec{\nabla}\vec{\nabla})\,\frac{\exp(ik|\vec{r}-\vec{r}\,'|)}{4\pi|\vec{r}-\vec{r}\,'|}\,. \tag{6}$$

(This of course is the causal Green's function so that purely outgoing scattered waves will result). The resulting integral equation for the electric field is

$$\vec{\mathcal{E}}(\vec{k}\nu,\vec{r}) = \vec{\mathcal{E}}_0(\vec{k}\nu,\vec{r}) + \int d\vec{r}\,'\,\overleftrightarrow{\Gamma}(k;\vec{r},\vec{r}\,')\cdot\left\{k^2(n'^2-1)\cdot\vec{\mathcal{E}}(\vec{k}\nu,\vec{r}\,') + \vec{\nabla}'\times[(1-\mu^{-1})\vec{\nabla}'\times\vec{\mathcal{E}}(\vec{k}\nu,\vec{r}\,')]\right\} \tag{7}$$

where $\vec{\mathcal{E}}_0(\vec{k}\nu,\vec{r})$ is the field of the incident electromagnetic radiation and the label ν denotes the initial polarization (relative to $\vec{k}$, the propagation vector, as the Z axis). Thus, one type initial radiation could be

$$\vec{\mathcal{E}}_0(\vec{k}\nu,\vec{r}) = \exp(i\vec{k}\cdot\vec{r})\,\vec{\chi}'_\nu \tag{8}$$

which is a plane wave with circular polarization vector $\vec{\chi}'_\nu\ (\nu = \pm 1)$. For the present discussion, it is convenient to expand the electric fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{E}}_0$ and the Green's function in the basis of vector spherical harmonics[1,3] so that

$$\overleftrightarrow{\Gamma}(k;\vec{r},\vec{r}') = (krr')^{-1}\sum_{\substack{JM\\ \lambda\lambda'}} \vec{Y}^{(\lambda)}_{JM}(\hat{r})\,\Gamma^{J}_{\lambda\lambda'}(k;r_<,r_>)\,\vec{Y}^{(\lambda')*}_{JM}(\hat{r}')(-1)^J \tag{9}$$

and

$$\vec{\mathcal{E}}(\vec{k}\nu,\vec{r}) = \frac{4\pi}{kr}\sum_{\substack{JM\lambda\\ J'M'\lambda'}} \vec{Y}^{(\lambda)}_{JM}(\hat{r})\,\mathcal{E}^{\lambda\lambda'}_{JM,J'M'}(r)\,i^{J}\,\vec{Y}^{(\lambda')}_{J'M'}(\hat{k})\cdot\vec{\chi}'_\nu \tag{10}$$

where

$$\Gamma^{J}_{ee}(k;r_<,r_>) = u'_J(kr_<)\,w^{(+)\prime}_J(kr_>), \tag{11}$$

$$\Gamma^{J}_{mm}(k;r_<,r_>) = -u_J(kr_<)\,w^{(+)}_J(kr_>), \tag{12}$$

$$\Gamma^{J}_{eo}(k;r_<,r_>) = \Gamma^{J}_{oe}(k;r_>,r_<) = \frac{[J(J+1)]^{1/2}}{k^2 r'}\frac{\partial}{\partial r}\left[u_J(kr_<)\,w^{(+)}_J(kr_>)\right] \tag{13}$$

and

$$\Gamma^{J}_{00}(k; r_<, r_>) = -\frac{(J+1)}{k^2 r r'} u_J(kr_<)\,\omega^{(+)}_J(kr_>) \tag{14}$$

and all others are zero.

In the above expressions, $u_J(kr)$ is the Ricatti-spherical Bessel function[3] of order J, $\omega^{(+)}_J(kr)$ is the Ricatti-spherical Hankel function[3] of the first kind of order J and by u'_J (or $\omega^{(+)\prime}_J$) we mean the derivative of u_J (or $\omega^{(+)}_J$) with respect to r.

These results are then employed to express the partial wave components of the integral equation for the electric field (for an isotropic medium) as

$$\mathcal{E}^{\lambda\lambda'}_{JM,J'M'}(r) = u_J(kr)\,\delta_{JJ'}\delta_{MM'}\delta_{\lambda\lambda'}(\delta_{\lambda 0}-1)$$
$$+k^2 \sum_{\substack{J''M'' \\ \lambda'''\lambda''}} \int_0^\infty dr'\, \Gamma^{J}_{\lambda\lambda'''}(r_<, r_>)\, \eta^{\lambda'''\lambda''}_{JM,J''M''}(r')\, \mathcal{E}^{\lambda''\lambda'}_{J''M'',J'M'}(r') \tag{15}$$

where

$$\eta^{\lambda'''\lambda''}_{JM,J''M''}(r) = i^{J+J''} \int d\hat{r}\, \vec{Y}^{(\lambda''')*}_{JM}(\hat{r}) \cdot \overleftrightarrow{1}(n^2-1) \cdot \vec{Y}^{(\lambda'')}_{J''M''}(\hat{r}). \tag{16}$$

For spherically symmetric scatterers, Eq. (16) reduces to

$$\eta^{\lambda'''\lambda''}_{JM,J''M''}(r) = (-1)^J \delta_{JJ''}\,\delta_{MM''}\,\delta_{\lambda'''\lambda''}\left[n^2(r)-1\right], \tag{17}$$

and the integral equation for $\mathcal{E}^{\lambda\lambda'}_{JM}(r)$ reduces to a set of three coupled equations given by

$$\mathcal{E}^{\lambda\lambda'}_{JM}(r) = u_J(kr)\,\delta_{\lambda\lambda'}\,(\delta_{\lambda 0}-1)$$
$$+ (-1)^J k^2 \sum_{\lambda''} \int_0^\infty dr'\, \Gamma^J_{\lambda\lambda''}(r_<, r_>)\left[n^2(r')-1\right]\mathcal{E}^{\lambda''\lambda'}_{JM}(r') \tag{18}$$

for $\lambda, \lambda', \lambda''$ equal to e, m and o. For purposes of simplicity, we shall couch our discussion of a solution procedure in terms of Eq. (18). However, the noniterative solution method presented herein can be applied to the general case of a nonisotropic nonhomogeneous medium without any essential change. Finally, we comment that these integral equations are very similar to those encountered in quantum mechanical scattering where now the role of the potential is taken by the quantity $\left[n^2(r) - 1\right]$. We keep in mind that in the absence of any dispersive medium, $n = 1$ so that the "potential" in Eq. (18) tends to zero as one moves from the scattering region into free space.

II. THE NONITERATIVE SOLUTION PROCEDURE

We now discuss the adaptation of the Sams-Kouri homogeneous integral solution formalism[6-7] to the present problem of scattering of electromagnetic waves. It is convenient to express Eq. (18) in a matrix form as

$$\underset{\approx}{\mathcal{E}}_{JM}(r) = \underset{\approx}{U}_J(r) + (-1)^J k^2 \int_0^\infty dr' \, \underset{\approx}{\Gamma}^J(r_<, r_>) \cdot \underset{\approx}{m}(r') \cdot \underset{\approx}{\mathcal{E}}_{JM}(r') \tag{19}$$

where

$$[\underset{\approx}{\mathcal{E}}_{JM}(r)]_{\lambda\lambda'} = \mathcal{E}^{\lambda\lambda'}_{JM}(r), \tag{20}$$

$$[\underset{\approx}{\Gamma}^J(r_<, r_>)]_{\lambda\lambda'} = \Gamma^J_{\lambda\lambda'}(r_<, r_>), \tag{21}$$

$$[\underset{\approx}{U}_J(r)]_{\lambda\lambda'} = U_J(kr)\,\delta_{\lambda\lambda'}[\delta_{\lambda 0} - 1], \tag{22}$$

and

$$[\underset{\approx}{m}(r)]_{\lambda\lambda'} = \delta_{\lambda\lambda'}[n^2(r) - 1]. \tag{23}$$

It is next convenient to express the matrix $\underset{\approx}{\Gamma}^J$ in the form

$$\underset{\approx}{\Gamma}^J(r_<, r_>) = \underset{\approx}{\mathcal{F}}{}_1^J(r_<) \cdot \underset{\approx}{\mathcal{F}}{}_2^J(r_>) + \frac{1}{r'}\frac{d}{dr}\, \underset{\approx}{\mathcal{F}}{}_3^J(r_<) \cdot \underset{\approx}{\mathcal{F}}{}_4^J(r_>) \tag{24}$$

where $\underset{\approx}{\mathcal{F}}{}_1^J$, $\underset{\approx}{\mathcal{F}}{}_2^J$ and $\underset{\approx}{\mathcal{F}}{}_3^J$ are diagonal matrices given by

$$\mathcal{F}_{100}^J(r) = i(J+1)^{1/2}\, u_J(kr)/kr, \tag{25}$$

$$\mathcal{F}_{1ee}^J(r) = \frac{d}{dr} u_J(kr), \tag{26}$$

$$\mathcal{F}_{1mm}^J(r) = i\, u_J(kr), \tag{27}$$

$$\mathcal{F}_{200}^J(r) = i(J+1)^{1/2}\, w_J^{(+)}(kr)/kr, \tag{28}$$

$$\mathcal{F}_{2ee}^J(r) = \frac{d}{dr} w_J^{(+)}(kr), \tag{29}$$

$$\mathcal{F}_{2mm}^J(r) = i\, w_J^{(+)}(kr), \tag{30}$$

$$\mathcal{F}^{J}_{3oo}(r) = i\,[J(J+1)]^{1/4}\, u_J(kr)/k, \tag{31}$$

$$\mathcal{F}^{J}_{3ee}(r) = i\,[J(J+1)]^{1/4}\, u_J(kr)/k, \tag{32}$$

$$\mathcal{F}^{J}_{3mm}(r) = 0, \tag{33}$$

while $\underset{\approx}{\mathcal{F}}^{J}_{4}$ has only two nonzero elements given by

$$\mathcal{F}^{J}_{4oe}(r) = \mathcal{F}^{J}_{4eo} = i\,[J(J+1)]^{1/4}\, w^{(+)}_J(kr)/k. \tag{34}$$

We may now employ Eq. (24) and explicitly eliminate the $r_<$ and $r_>$ variables to obtain for Eq. (19) the result

$$\begin{aligned}
\underset{\approx}{\mathcal{E}}_{JM}(r) = \underset{\approx}{u}_J(r) &+ (-1)^J k^2\, \underset{\approx}{\mathcal{F}}^{J}_{2}(r)\cdot\int_0^r \underset{\approx}{\mathcal{F}}^{J}_{1}(r')\cdot \underset{\approx}{m}(r')\cdot \underset{\approx}{\mathcal{E}}_{JM}(r')\,dr' \\
&+ (-1)^J k^2\, \underset{\approx}{\mathcal{F}}^{J}_{1}(r)\cdot\int_r^\infty \underset{\approx}{\mathcal{F}}^{J}_{2}(r')\cdot \underset{\approx}{m}(r')\cdot \underset{\approx}{\mathcal{E}}_{JM}(r')\,dr' \\
&+ (-1)^J k^2\, \underset{\approx}{\mathcal{F}}^{J\prime}_{4}(r)\cdot\int_0^r dr'\, \underset{\approx}{\mathcal{F}}^{J}_{3}(r')\cdot \underset{\approx}{m}(r')\cdot \underset{\approx}{\mathcal{E}}_{JM}(r')/r' \\
&+ (-1)^J k^2\, \underset{\approx}{\mathcal{F}}^{J\prime}_{3}(r)\cdot\int_r^\infty dr'\, \underset{\approx}{\mathcal{F}}^{J}_{4}(r')\cdot \underset{\approx}{m}(r')\cdot \underset{\approx}{\mathcal{E}}_{JM}(r')/r',
\end{aligned} \tag{35}$$

where by $\underset{\approx}{\mathcal{F}}{}_3^{J\prime}(r)$ and $\underset{\approx}{\mathcal{F}}{}_4^{J\prime}(r)$ we mean

$$\underset{\approx}{\mathcal{F}}{}_3^{J\prime}(r) = \frac{d}{dr}\,\underset{\approx}{\mathcal{F}}{}_3^{J}(r), \tag{36}$$

$$\underset{\approx}{\mathcal{F}}{}_4^{J\prime}(r) = \frac{d}{dr}\,\underset{\approx}{\mathcal{F}}{}_4^{J}(r). \tag{37}$$

(Here it is important to note that the matrices $\underset{\approx}{\mathcal{F}}{}_1^{J}$ and $\underset{\approx}{\mathcal{F}}{}_2^{J}$ commute and the matrices $\underset{\approx}{\mathcal{F}}{}_3^{J}$ and $\underset{\approx}{\mathcal{F}}{}_4^{J}$ commute). We now follow Sams and Kouri[6] and simply add and subtract the integrals

$$(-1)^J k^2\,\underset{\approx}{\mathcal{F}}{}_1^{J}(r)\cdot\int_0^r \underset{\approx}{\mathcal{F}}{}_2^{J}(r')\cdot\underset{\approx}{m}(r')\cdot\underset{\approx}{\mathcal{E}}{}_{JM}(r')\,dr' \tag{38}$$

and

$$(-1)^J k^2\,\underset{\approx}{\mathcal{F}}{}_3^{J\prime}(r)\cdot\int_0^r \underset{\approx}{\mathcal{F}}{}_4^{J}(r')\cdot\underset{\approx}{m}(r')\cdot\underset{\approx}{\mathcal{E}}{}_{JM}(r')\,dr' \tag{39}$$

in Eq. (35) to obtain

$$\underset{\approx}{\mathcal{E}}_{JM}(r) = \underset{\approx}{U}_J(r) + (-1)^J k^2 \int_0^r dr' \left[\underset{\approx}{\mathcal{F}}_2^{J}(r) \cdot \underset{\approx}{\mathcal{F}}_1^{J}(r') - \underset{\approx}{\mathcal{F}}_1^{J}(r) \cdot \underset{\approx}{\mathcal{F}}_2^{J}(r') \right] \cdot \underset{\approx}{M}(r') \cdot \underset{\approx}{\mathcal{E}}_{JM}(r')$$

$$+ (-1)^J k^2 \int_0^r dr' \left[\underset{\approx}{\mathcal{F}}_4^{J'}(r) \cdot \underset{\approx}{\mathcal{F}}_3^{J}(r') - \underset{\approx}{\mathcal{F}}_3^{J'}(r) \cdot \underset{\approx}{\mathcal{F}}_4^{J}(r') \right] \cdot \underset{\approx}{M}(r') \cdot \underset{\approx}{\mathcal{E}}_{JM}(r')$$

$$+ \underset{\approx}{\mathcal{F}}_1^{J}(r) \cdot \underset{\approx}{C}^J + \underset{\approx}{\mathcal{F}}_3^{J'}(r) \cdot \underset{\approx}{D}^J .$$

(40)

Here the constant matrices $\underset{\approx}{C}^J$ and $\underset{\approx}{D}^J$ are defined by

$$\underset{\approx}{C}^J = (-1)^J k^2 \int_0^\infty dr\, \underset{\approx}{\mathcal{F}}_2^{J}(r) \cdot \underset{\approx}{M}(r) \cdot \underset{\approx}{\mathcal{E}}_{JM}(r)$$

(41)

and

$$\underset{\approx}{D}^J = (-1)^J k^2 \int_0^\infty dr\, \underset{\approx}{\mathcal{F}}_4^{J}(r) \cdot \underset{\approx}{M}(r) \cdot \underset{\approx}{\mathcal{E}}_{JM}(r) .$$

(42)

In order to solve Eq. (40), we now write the electric field $\underset{\approx}{\mathcal{E}}_{JM}(r)$ as

$$\underset{\approx}{\mathcal{E}}_{JM}(r) = \underset{\approx}{\mathcal{E}}_{JM}(0|r) + \underset{\approx}{\mathcal{E}}_{JM}(1|r) \cdot \underset{\approx}{C}^J + \underset{\approx}{\mathcal{E}}_{JM}(2|r) \cdot \underset{\approx}{D}^J$$

(43)

where clearly

$$\underset{\approx}{\mathcal{E}}_{JM}(p|r) = \underset{\approx}{I}_J(p|r)$$

$$+(-1)^J k^2 \int_0^r dr' \left[\underset{\approx}{\mathcal{F}}_2^J(r) \cdot \underset{\approx}{\mathcal{F}}_1^J(r') - \underset{\approx}{\mathcal{F}}_1^J(r) \cdot \underset{\approx}{\mathcal{F}}_2^J(r') \right] \cdot \underset{\approx}{m}(r') \cdot \underset{\approx}{\mathcal{E}}_{JM}(p|r')$$

$$+(-1)^J k^2 \int_0^r dr' \left[\underset{\approx}{\mathcal{F}}_4^{J\prime}(r) \cdot \underset{\approx}{\mathcal{F}}_3^J(r') - \underset{\approx}{\mathcal{F}}_3^{J\prime}(r) \cdot \underset{\approx}{\mathcal{F}}_4^J(r') \right] \cdot \underset{\approx}{m}(r') \cdot \underset{\approx}{\mathcal{E}}_{JM}(p|r') \quad ; \quad p = 0, 1, 2 \tag{44}$$

and

$$\underset{\approx}{I}_J(0|r) = \underset{\approx}{U}_J(r), \tag{45}$$

$$\underset{\approx}{I}_J(1|r) = \underset{\approx}{\mathcal{F}}_1^J(r), \tag{46}$$

$$\underset{\approx}{I}_J(2|r) = \underset{\approx}{\mathcal{F}}_3^{J\prime}(r). \tag{47}$$

We now note that Eqs. (41) - (42) constitute a set of algebraic equations for the constant matrices $\underset{\approx}{C}^J$ and $\underset{\approx}{D}^J$ given by

$$\underset{\approx}{C}^J = (-1)^J k^2 \int_0^\infty dr\, \underset{\approx}{\mathcal{F}}_2^J(r)\cdot \underset{\approx}{M}(r)\cdot \Big[\underset{\approx}{\mathcal{E}}_{JM}(0|r) + \underset{\approx}{\mathcal{E}}_{JM}(1|r)\cdot \underset{\approx}{C}^J + \underset{\approx}{\mathcal{E}}_{JM}(2|r)\cdot \underset{\approx}{D}^J\Big] \tag{48}$$

and

$$\underset{\approx}{D}^J = (-1)^J k^2 \int_0^\infty dk\, \underset{\approx}{\mathcal{F}}_4^J(r)\cdot \underset{\approx}{M}(r)\cdot \Big[\underset{\approx}{\mathcal{E}}_{JM}(0|r) + \underset{\approx}{\mathcal{E}}_{JM}(1|r)\cdot \underset{\approx}{C}^J + \underset{\approx}{\mathcal{E}}_{JM}(2|r)\cdot \underset{\approx}{D}^J\Big]. \tag{49}$$

These clearly may be solved as soon as the $\underset{\approx}{\mathcal{E}}_{JM}(p|r)$, $p = 0, 1, 2$ are known by solving Eqs. (44), $p = 0, 1, 2$. The Eqs. (44) are of course recognized as Volterra integral equations of the second kind[3] and there are a variety of procedures which can be employed in their solution. We here wish to point out one particular approach which is noniterative in nature and which has been applied to quite similar equations occuring in quantum mechanical scattering with considerable success.[6-7] We now insert a Newton-Cotes quadrature (e.g., the trapezoidal rule) for the integration to obtain

$$\underset{\approx}{\mathcal{E}}_{JM}(p|r_m) = \underset{\approx}{I}_J(p|r_m) + (-1)^J k^2 \sum_{t=1}^{m} W_t \begin{bmatrix} \underset{\approx}{\mathcal{F}}_2^J(r_m)\cdot \underset{\approx}{\mathcal{F}}_1^J(r_t) - \underset{\approx}{\mathcal{F}}_1^J(r_m)\cdot \underset{\approx}{\mathcal{F}}_2^J(r_t) \\ + \underset{\approx}{\mathcal{F}}_4^{J\prime}(r_m)\cdot \underset{\approx}{\mathcal{F}}_3^J(r_t) - \underset{\approx}{\mathcal{F}}_3^{J\prime}(r_m)\cdot \underset{\approx}{\mathcal{F}}_4^J(r_t) \end{bmatrix} \cdot \underset{\approx}{M}(r_t)\cdot \underset{\approx}{\mathcal{E}}_{JM}(p|r_t). \tag{50}$$

It is of interest to examine the quantity in brackets above for $t = m$. It is readily seen to equal $\underset{\approx}{V}_m$ where

$$\underset{\approx}{V}_m = \left\{ \frac{d}{dr} \left[\underset{\approx}{\mathcal{F}}_4^J(r) \cdot \underset{\approx}{\mathcal{F}}_3^J(r) \right] \right\}_{r=r_m} \tag{51}$$

since the first term vanishes. (Here we see an important difference between the electromagnetic scattering problem and the quantum mechanical problem. Unless there are velocity dependent potentials occuring in the quantum mechanical problem, only the term $\underset{\approx}{\mathcal{F}}_2^J(r_m) \cdot \underset{\approx}{\mathcal{F}}_1^J(r_t) - \underset{\approx}{\mathcal{F}}_1^J(r_m) \cdot \underset{\approx}{\mathcal{F}}_2^J(r_t)$ occurs and this is readily seen to vanish at $r_t = r_m$. The result is that for quantum mechanical problems, the analogue of Eq. (50) may be solved without necessitating any matrix inversions). Thus, in order to solve Eq. (50) for the "modified field functions" $\underset{\approx}{\mathcal{E}}_{JM}(p/r)$ at the point r_m, it is necessary to compute the inverse of the matrix $\underset{\approx}{1} - \underset{\approx}{V}_m \cdot \underset{\approx}{M}(r_m)(-1)^J k^2 W_m$. However, it is stressed that, for the present problem, this is simply a 3x3 matrix and in general, the dimensionality of the matrix to be inverted is the same as the dimensionality of the matrix $\underset{\approx}{\mathcal{E}}(r)$. This is to be contrasted with what one encounters in a direct quadrature solution of Eq. (19) where one obtains

$$\underset{\approx}{\mathcal{E}}_{JM}(r_m) = \underset{\approx}{U}_J(r_m) + (-1)^J k^2 \sum_{t=1}^{N} \widetilde{W}_t \underset{\approx}{\Gamma}^J(r_m, r_t) \cdot \underset{\approx}{M}(r_t) \cdot \underset{\approx}{\mathcal{E}}_{JM}(r_t). \tag{52}$$

Even if one uses a quadrature scheme which takes account of the cusp occuring in $\underline{\underline{\Gamma}}^{J}$, one must invert a matrix whose order is 3NxN where N is the number of quadrature points employed.

It should also be noted that once the functions $\underline{\underline{\mathcal{E}}}_{JM}(\rho|r)$ are obtained out into the free space region, the integrals required to solve Eqs. (48) - (49) for $\underline{\underline{C}}^{J}$ and $\underline{\underline{D}}^{J}$ are known. Thus, the complete specification of the field $\underline{\underline{\mathcal{E}}}_{JM}(r)$ in space is then possible.

Finally, it is noted that one might also desire to attempt the solution of Eqs. (44) by iteration. Then in contrast to an iterative solution of Eq. (19), the convergence of which is governed by the same conditions as the Born-Neumann procedure in quantum mechanics,[1] the conditions for convergence of iteration of Eqs. (44) are those of an integral equation with a triangular kernal (i.e., $K(r, r') = 0$ for $r' > r$). These conditions for the quantum mechanical scattering problem are discussed by Newton.[8] Thus one may hope for convergence under more general conditions in the case of Eqs. (44).

ACKNOWLEDGEMENT

The author gratefully acknowledges the kind hospitality of the Theoretical Chemistry Institute of the University of Wisconsin during July, 1971 enabling the completion of this work.

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