DESIGN OF COMPUTATIONAL ALGORITHMS FOR OPTIMAL CONTROL BY HILBERT SPACE METHODS

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16. Abstract

A new classification of computational methods and three new computational algorithms for optimal control are formulated. In this report we treat the optimal control problem of minimizing a cost function J on a bounded open subset of Hilbert space subject to dynamical system and terminal equality constraints. Basic concepts from the theory of linear vector spaces are used to obtain a minimizing sequence for J in the control space U, the state space X, and the dual space X* (the space of bounded linear functionals on X).

A computational solution may be obtained by the following alternate viewpoints: (1) Approximation in U — choosing a sequence of approximate control functions in U, (2) Approximation in X* — choosing a sequence of approximate adjoint functions that correspond to linear functions in X*, (3) Approximation in X — choosing a sequence of approximate state functions in X. The preceding classification provides unification of first- and second-variation methods previously published and geometric insight for the design of new techniques.

Three new computational algorithms are developed, each belonging to one of the above categories. The first is based on Approximation in U. Concepts of asymptotic stability and Hamilton-Jacobi theory are used to obtain a sequence of functionals convergent to the optimal cost. The method is applied to the state regulator problem, and an algorithm is obtained that requires solving linear differential equations instead of nonlinear Riccati equations. The method possesses both monotone and quadratic convergence. The second is based on Approximation in X* and is an extension of the quasi-Newton methods to boundary condition iteration. This algorithm inherits the advantages of the quasi-Newton method employed including rapid and stable descent to a minimum. The inverse Jacobian is obtained by iteration, and only first-order partial derivatives of the cost functional are required. The third is based on Approximation in X. An inverse mapping of X into U is defined and used to develop a new second-variation algorith. In contrast to previous second-variation methods, we avoid the use of adjoint functions and the Hamiltonian formulation. Two methods for solving the associated boundary value problem are developed in detail, and necessary and sufficient conditions for a solution to the latter are given. A number of computational examples describe the application of each new algorithm, and comparisons are made with the exact solutions.

Finally, we obtain the solution to a class of state regulator problems. These problems involve first-order time-varying systems and are designed for testing computational algorithms.

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DESIGN OF COMPUTATIONAL ALGORITHMS FOR OPTIMAL CONTROL BY HILBERT SPACE METHODS

SUMMARY

A new classification of computational methods for optimal control problems is given that includes the first variation methods of steepest descent, quasilinearization, boundary condition iteration, and the second variation method. The classification is based on viewing the solutions as a sequence of approximate control functions in the control space U, a sequence of approximate adjoint functions corresponding to linear functionals in the dual space X^*, or a sequence of approximate state functions in the state space, X.

Three new computational algorithms for optimal control problems are developed, each belonging to one of the above categories. The first is based on Approximation in X^* and is an extension of the quasi-Newton methods to boundary condition iteration. In contrast to former second-order methods for boundary condition iteration involving the Newton-Raphson method, the inverse Jacobian matrix of second partial derivatives is obtained by iteration. Only first partial derivatives of the cost functional are required, and a detailed technique for obtaining these derivatives is given.

The second is based on Approximation in U. Concepts of asymptotic stability and Hamilton-Jacobi theory are used to obtain a sequence of functionals convergent to the optimal cost and which specify a minimizing sequence of control functions for the cost functional. A detailed algorithm for the state regulator problem is given. The control sequence is obtained by solving linear equations instead of nonlinear Riccati equations and possesses both monotonic and quadratic convergence. Numerical studies indicate that the algorithm is preferable to existing techniques, including Runge-Kutta integration of the Riccati equations and the Automatic Synthesis Program (ASP).

The third is based on Approximation in X and exploits the concept of an inverse mapping of X into U. A second variation algorithm is formulated, and a detailed solution for the resulting boundary value problem is given. As an important special case the state regulator problem is treated, and an approximation to the optimal control is obtained. Comparison with the existing techniques involving the second variation indicates reduced computational requirements for problems in which the Euclidean dimension of U is less than that of X.

A number of computational examples demonstrate the application of the new algorithms, and comparisons are made with the exact solutions.

Finally, we obtain the solution to a class of state regulator problems. These problems involve first-order time-varying systems and are designed for testing computational algorithms.

CHAPTER 1

INTRODUCTION

Statement of Problem and Research Objectives

The development of the Maximum Principle by Pontryagin and his colleagues in the 1950's is acknowledged as one of the most significant results in modern control theory. The method is a culmination of research into the Calculus of Variations by Bliss and the "Chicago school" more than 20 years earlier, and provides necessary (and often sufficient) conditions for an optimum. During the past 10 years, considerable effort has been devoted to exploiting the Maximum Principle for the design of computational algorithms for the numerical solution of control problems.

A more fundamental approach is through the application of "functional analysis" and the concept of minimizing a linear functional on a normed linear vector space. Pontryagin, in fact, used these tools in the proof of his celebrated result. The abstraction of the vector space approach yields greater insight into the problem structure, and the geometric character of the results provides a unified framework for further extensions.

Basically the optimal control problem consists of the following items: a cost functional J to be minimized; dynamical system constraints described by differential, difference, or integral equations relating the state x and contol u; and constraints on the domain and range of these functions. A control u is said to be optimal if J is minimized and the constraints are satisfied. Throughout this report we assume that x and u belong to bounded open sets X and U, respectively, which are subsets of infinite dimensional Hilbert space. The system dynamics will be modeled by nonlinear ordinary differential equations. Terminal equality constraints on the state and unspecified terminal time will be considered.

A computational solution to the optimal control problem stated above may be obtained by the following alternate viewpoints: (1) Approximation in U — choosing a sequence of approximate control functions in U by expressing the state in X as a bounded linear mapping of U into X and employing adjoint functions (representing elements in X*) to determine the mapping, (2) Approximation in X* — choosing a sequence of approximate adjoint functions which correspond to linear functionals in X*, (3) Approximation in X — choosing a sequence of approximate state functions in X, provided that a mapping of X into U can be found.

The first objective of this research is to classify previous computational methods for optimal control problems according to the above scheme. The second objective is to formulate the optimal control problem as Approximation in X by careful development of an "inverse" mapping of X into U. The last objective of the study is to develop new algorithms in each of the three categories and demonstrate their application by computational examples.

Organization of the Report

In Chapter 1, the problem and research objectives are stated, and the organization of the report is presented. Basic terminology and definitions are introduced.

In Chapter 2, the optimal control problem is defined, and assumptions are given. Using basic concepts from the theory of linear vector spaces (linear operators, Frechet differentials, adjoints, etc.) necessary conditions for a minimum of J on $X \times U \times T$ are derived. As an example, the state regulator problem is reviewed, and known results are obtained and discussed.

In Chapter 3, the computational solution of the optimal control problem is treated as Approximation in X^* and U. These two categories are treated together because of their natural dependence on the necessary conditions for an optimal control. Well-known methods of descent for finite dimensional minimization are reviewed as a prototype for methods in infinite dimensions which follow, including the methods of steepest descent, boundary condition iteration, quasilinearization, and second variation. A computational algorithm for Approximation in X^* is formulated based on boundary condition iteration by a quasi-Newton search. Next we describe a method for Approximation in U based on concepts of asymptotic stability and Hamilton-Jacobi theory. As an important special case of the latter method, a computational algorithm for the state regulator problem is developed.

In Chapter 4, the inverse mapping from X into U is defined, and several important properties of the mapping are proved. A second variation method is developed based on the inverse mapping. The basic second variation algorithm is formulated and, as a special case, the state regulator problem is treated.

In Chapters 3 and 4, the application of each new algorithm is demonstrated by computational examples, and comparisons are made with the exact solutions.

In Chapter 5, conclusions and recommendations for future research related to the report topic are given.

In Appendix A, methods are given for solving the linear differential equation associated with the algorithm for monotone Approximation in U .

In Appendix B, methods are given for solving the Accessory Problem associated with the second variation algorithm for Approximation in X.

In Appendix C, we present a class of state regulator problems with nontrivial solutions that may be used to check computational algorithms.

Terminology

For the purpose of introducing relevant terminology and notation, the following definitions from the theory of linear vector spaces [1] are provided:

Euclidean space $\mathbb{R}^n$ is the space of real valued n-tuples $(x_1,\ldots,x_n)$, $(y_1,\ldots,y_n)$ with norm $\| x \|_1 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$ induced by the inner product $\left[x,y\right]_1 = \sum_{i=1}^{n} x_i y_i$. Hilbert space $L_n^2(T)$ is the (complete) space of $\mathbb{R}^n$ - valued functions x, y which are Lebesgue square integrable on the closed interval $T = [t_0, t_1]$ of $\mathbb{R}$, the extended real line, with norm $\| x \| = \left(\int_{t_0}^{t_1} \sum_{i=1}^{n} x_i^2(t)\, dt \right)^{1/2}$ induced by the inner product $\left[x,y\right] = \int_{t_0}^{t_1} \sum_{i=1}^{n} x_i(t) y_1(t)\, dt$. Note that $\mathbb{R}^n$ is a special case of Hilbert space.

The interval $[t_0, t_1]$ is denoted by T. The terms mapping and operator will be used interchangeably. A functional is a mapping of Hilbert space into $\mathbb{R}$. A linear mapping K of Hilbert space X into Y is bounded if there exists a constant α such that $\| Kx \| \leq \alpha \| x \|$ for all $x \in X$. The dual space X^* of Hilbert space X consists of all bounded linear functionals on X. The adjoint K^* of a bounded linear mapping $K : X \to Y$ where X and Y are Hilbert space is a mapping $K^* : Y \to X$ defined by $\left[y, Kx\right] = \left[K^* y, x\right]$ for $x \in X$, $y \in Y$. A bounded linear operator K is self adjoint if $K = K^*$. A self adjoint matrix operator K on $\mathbb{R}^n$ is positive definite (positive semidefinite) if $\left[x, Kx\right]_1 > 0 \ (\geq 0)$ for all $x \neq \theta \left(\left[x, Kx\right]_1 = 0 \text{ for } x = \theta \right)$ and will be denoted by $K > 0 \ (\geq 0)$. A self adjoint matrix operator on $\mathbb{R}^n$ is called symmetric. The transpose of a

matrix operator M on $\mathbb{R}^n$ is denoted by M* in accordance with the definition of adjoint. The null operator is denoted Θ or θ and identity by I where the space is obvious by the context.

Differentiation of a function x on T with respect to the real variable t will be denoted by $\dot{x}$. Partial derivatives will be denoted by subscripts as in the following example:

$$L_x = \left(\frac{\partial L}{\partial x_1}, \ldots, \frac{\partial L}{\partial x_n} \right), L_{xx} = \begin{pmatrix} \frac{\partial^2 L}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_n \partial x_n} \end{pmatrix}$$

where $L : \mathbb{R}^n \to \mathbb{R}$. The inverse of matrix M is denoted by M^{-1}. If X and Y are subsets of Hilbert space, the cartesian product $X \times Y$ is the collection of ordered pairs (x, y) with $x \in X$, $y \in Y$. Addition and scalar multiplication are defined by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $\alpha(x, y) = (\alpha x, \alpha y)$. $C^{(k)}(T)$ is the class of continuously k-differentiable functions on the interval T. Given X and Y Hilbert spaces, the first Gateaux differential (the first variation) $\delta F(x;h)$ of $F : X \to Y$ at x with increment h is defined by

$$\delta F(x;h) = \lim_{\alpha \to 0} \frac{1}{\alpha} (F(x + \alpha h) - F(x)) = \frac{d}{d\alpha} F(x + \alpha h) \Big|_{\alpha = 0}$$

if the limit exists in the sense of convergence in the norm. If for fixed $x \in X$ and each $h \in X$ there exists $\delta F(x;h)$ which is linear and bounded and

$$\lim_{\|h\| \to 0} \frac{1}{\|h\|} \left(\|F(x + h) - F(x) - \delta F(x, h)\| \right) = 0 \quad ,$$

then F is said to be first Frechet differentiable at x, and the unique mapping $dF(x;h) = \delta F(x;h)$ is called the first Frechet differential of F at x with increment h. Higher-order differentials are defined in a similar manner. We shall say that F has a relative weak (strong) minimum at $\hat{x} \in X$ if there exists an $\alpha > 0$ such that $F(x) \geq F(\hat{x})$ whenever $\|x - \hat{x}\| < \alpha$ and $x \in C^1(T)$ $(x \in C^0(T))$. For brevity we shall often say that F has a minimum over

all X if the above condition holds for all $x \in X$. A sequence of functions $x^{(k)}$, $k = 0, 1, \ldots$ in Hilbert space X is called a minimizing sequence for the functional $V : X \to \mathbb{R}$ if

$$\lim_{k \to \infty} V\left(x^{(k)}\right) = \inf_{x \in X} V(x)$$

CHAPTER 2

ANALYSIS OF THE OPTIMAL CONTROL PROBLEM IN HILBERT SPACE

In this chapter, the optimal control problem is defined. Using elementary concepts from the theory of linear vector spaces, necessary conditions for a minimum of J on $X \times U \times T$ are derived. As an example, the state regulator problem is reviewed, and known results are obtained and analyzed.

Definition of the Optimal Control Problem

Consider the problem of determining a control function u in the control space $U = L_r^2(T)$, $T = [t_0, t_1]$ such that a cost functional $J : X \times U \times T \rightarrow \mathbb{R}$ defined by

$$J(u, x) = \int_{t_0}^{t_1} L(x, u, t)\, dt \tag{1}$$

is minimized for all x in the state space $X = L_n^2(T)$, subject to nonlinear dynamical system constraints

$$\dot{x} = f(x, u, t) \quad , \quad x(t_0) = c \quad , \tag{2}$$

where $f : X \times U \times T \rightarrow X$ and subject to nonlinear terminal equality constraints

$$\psi(x, t_1) = \theta \quad , \tag{3}$$

where $\psi : X \times T \rightarrow \mathbb{R}^p$. Restrictions on the problem are considered in the following section.

Basic Assumptions

The basic assumptions are:

1. The pair of admissible functions (u, x) and their increments (δu, δx) that satisfy the linearized dynamical system

$$\delta\dot{x} = f_x(x, u, t)\,\delta x + f_u(x, u, t)\,\delta u \quad , \quad \delta x(t_0) = \theta \tag{4}$$

are continuously differentiable on T .

2. Mappings L, f, and ψ are continuously second differentiable on their domains.

3. The functional J possesses first and second Frechet differentials and is a convex function of its arguments over all $X \times U \times T$.

4. The linearized dynamical system [eq. (4)] is uniformly completely controllable on T [2]. This condition is equivalent to requiring that the controllability matrix

$$\int_{t_0}^{t_1} \Phi(t_0, s)\, f_u(x, u, s)\, f_u^*(x, u, s)\, \Phi^*(t_0, s)\, ds \tag{5}$$

is positive definite where $\Phi(t, s)$ is the transition matrix of equation (4) and satisfies the matrix differential equation

$$\frac{d}{dt}\Phi(t, t_0) = f_x(x, u, t)\,\Phi(t, t_0) \quad , \quad \Phi(t_0, t_0) = I \quad . \tag{6}$$

5. The terminal equality constraints are linearly independent. Thus

$$\text{rank} \left\{ \psi_x(x, t_1) \right\} = p \quad . \tag{7}$$

Condition 3 is a sufficient condition for a minimum of J and may be too restrictive for applications. Furthermore, conditions 1 and 2 may also be too strong and should be examined in specific cases.

If conditions 1 through 5 are satisfied, the following development provides both necessary and sufficient conditions for a weak minimum of J on $X \times U \times T$.

Necessary Conditions

To facilitate the presentation of the concepts to follow, let us assume that the terminal time t_1 is specified in advance. In general, it is only

available implicitly through the terminal equality constraint [eq. (3)]. Necessary conditions for the more general case have been given in References 3 and 4.

Let the Lagrangian functional $F : X \times U \times X^* \times \mathbb{R}^p \times T \to \mathbb{R}$ be defined by

$$F(x, u, \lambda, \nu, t) = J(x, u) + \left[\lambda, f(x, u, t) - \dot{x}\right] + \left[\nu, \psi(x, t_1)\right]_1 , \tag{8}$$

where $x \in X = L_n^2(T)$ and $u \in U = L_r^2(T)$. Elements, λ, ν are the unique "Lagrange multiplier" or adjoint functions, which represent elements in the dual spaces X^* and $(\mathbb{R}^p)^*$, respectively.[1]

Suppose we define the bounded mapping, $K : X \to X$, for all $x \in X$ by

$$(Kz)(t) = \int_{t_0}^{t} z(s)\, ds \quad . \tag{9}$$

Then the state function may be expressed in terms of its derivative as

$$x(t) = c + (K\dot{x})(t) \tag{10}$$

where c is the initial condition for the dynamical system [eq. (2)]. Equation (10) implies that F as defined by equation (8) may be viewed as a function of $\dot{x}$.

If equations (2) and (3) are satisfied, a necessary condition for a weak minimum of F, and consequently a minimum of J over the product space $X \times U \times T$, is that the first Frechet differential of F at $\dot{x}$ and u vanish.

---

1. Because of the (isomorphic) equivalence of λ, ν with elements of their dual spaces [1], we shall write $\lambda \in X^*$, $\nu \in (\mathbb{R}^p)^*$ and we shall identify X^* and $(\mathbb{R}^p)^*$ with $L_n^2(T)$ and $\mathbb{R}^p$, respectively.

Existence of the Frechet differential implies the existence of the Gateaux differential [5], and the latter provides a convenient means for computing dF . Hence, for all $\delta\dot{x} \in X$,

$$
\begin{aligned}
dF(\dot{x};\delta\dot{x}) &= \frac{d}{d\alpha}\, F\left(c + K\,(\dot{x} + \alpha\delta\dot{x})\,,\, u\,,\, \lambda\,,\, \nu\,,\, t\right) \Big|_{\alpha = 0} \\
&= \frac{d}{d\alpha} \left(\int_{t_0}^{t_1} \left(L_0 + \left[L_x(x\,,\,u\,,\,t)\,,\,\delta x \right]_1 + 0(\|\,\delta x\,\|^2) \right) dt \right. \\
&+ \left[\lambda\,,\, f_0 + f_x(x\,,\,u\,,\,t)\delta x \right] + 0(\|\,\delta x\,\|^2) - \left[\lambda\,,\, \dot{x} + \alpha\delta\dot{x} \right] \\
&+ \left. \left[\nu\,,\, \psi_0 + \psi_x(x, t_1)\,\delta x \right]_1 + 0(\|\,\delta x\,(t_1)\,\|_1^2) \right) \Big|_{\alpha = 0} \\
&= \frac{d}{d\alpha} \left(\int_{t_0}^{t_1} \left(\left[L_x(x\,,\,u\,,\,t)\,,\,\delta x \right]_1 \right. \right. \qquad (11) \\
&\qquad + \left[\lambda\,,\, f_x(x\,,\,u\,,\,t)\,\delta x \right]_1 \\
&\qquad - \left[\lambda\,,\, \dot{x} + \alpha\delta\dot{x} \right]_1 + 0(\|\,\delta x\,\|_1^2) \Big) \, dt \\
&\quad + \left. \left[\nu\,,\, \psi_x(x\,,\,t_1)\,\delta x(t_1) \right]_1 + 0(\|\,\delta x\,(t_1)\,\|_1^2) \right) \Big|_{\alpha = 0} \\
&= \frac{d}{d\alpha} \Big(\left[L_x(x\,,\,u\,,\,t)\,,\,\delta x \right] \\
&\quad + \left[\lambda\,,\, f_x(x\,,\,u\,,\,t)\,\delta x \right] \\
&\quad - \left[\lambda\,,\, \dot{x} \right] - \left[\lambda\,,\, \alpha\delta\dot{x} \right] + 0(\|\,\delta x\,\|^2) \\
&\quad + \left[\nu\,,\, \psi_x(x\,,\,t_1)\,\delta x(t_1) \right]_1 + 0(\|\,\delta x(t_1)\,\|_1^2) \Big) \Big|_{\alpha = 0}
\end{aligned}
$$

where L_0 , f_0 , ψ_0 are constants, and $0(\|\,\delta x\,\|^2)$, $0(\|\,\delta x(t_1)\,\|_1^2)$ are remainder terms for first-order Taylor series expansions of L , f , ψ at x . By substituting $K\delta\dot{x}$ for δx and differentiating with respect to α , we obtain for all $\delta\dot{x}$ X ,

$$
\begin{aligned}
dF(\dot{x};\delta\dot{x}) &= \left[L_x(x,u,t)\ ,\ K\delta\dot{x} \right] + \left[\lambda\ ,\ f_x(x,u,t)\ K\delta\dot{x} \right] \\
&\quad - \left[\lambda\ ,\ \delta\dot{x} \right] + \left[\nu\ ,\ \psi_x(x,t_1)\ (K\delta\dot{x})(t_1) \right]_1 \\
&= \left[K^*\ \ L_x(x,u,t) + f_x^*(x,u,t)\lambda\ \ - \lambda\ ,\ \delta\dot{x} \right] \\
&\quad + \left[\psi_x^*(x,t_1)\ \nu\ ,\ (K\delta\dot{x})(t_1) \right]_1 \\
&= \left[K^* \left(L_x(x,u,t) + f_x^*(x,u,t)\lambda \right) - \lambda + \psi_x^*(x,t_1)\nu\ ,\ \delta\dot{x} \right] \\
&= \theta \qquad .
\end{aligned}
\tag{12}
$$

In computing $dF(\dot{x},\ \delta\dot{x})$ we have performed the following steps: (1) expanded L, f, ψ in a first-order Taylor series about x; (2) dropped the constant terms L_0, f_0, ψ_0, which are independent of α; (3) used the Hilbert space inner product notation $[\,,\,]$ for $\int_{t_0}^{t_1} [\,,\,]\ dt$; (4) used equation (10) to write δx as a function of its derivative $\delta\dot{x}$; (5) performed the indicated differentiation with respect to α ; (6) used the identity

$$
\left[\psi_x^*(x,t_1)\,\nu\ ,\ (K\delta\dot{x})(t_1) \right]_1 = \left[\psi_x^*(x,t_1)\,\nu\ ,\ \delta\dot{x} \right] \qquad . \tag{13}
$$

Similarly, for all $\delta u \in U$,

$$
\begin{aligned}
dF(u;\delta u) &= \frac{d}{d\alpha} F(x,\ u+\alpha\delta u,\ \lambda,\ \nu,\ t)\Big|_{\alpha=0} \\
&= \left[L_u(x,u,t)\ ,\ \delta u \right] + \left[\lambda\ ,\ f_u(x,u,t)\ \delta u \right] \\
&= \theta \qquad .
\end{aligned}
\tag{14}
$$

A well-known theorem in the Calculus of Variations is the Euler-Lagrange Lemma, which states if β is a continuous function on T and if $\left[\beta,\ \delta z \right] = 0$ for every δz in $C^1(T)$ where $\delta z(t_0) = \delta z(t_1) = \theta$, then $\beta \equiv \theta$ on T [6, 7]. This theorem is of fundamental importance and will be used frequently throughout this report.

Therefore, by the Euler-Lagrange Lemma, equations (12) and (14) imply that an optimal control function $\hat{u}$ satisfies

$$L_u(x, \hat{u}, t) + f_u^*(x, \hat{u}, t)\lambda = \theta \quad , \tag{15}$$

and the adjoint function λ satisfies

$$\lambda = K^* \left(L_x(x, \hat{u}, t) + f_x^*(x, \hat{u}, t)\lambda \right) + \psi_x^*(x, t_1)\nu \quad . \tag{16}$$

The adjoint operator K* exists since K is bounded on X. The mapping K has been used by Mitter [8] to obtain necessary conditions and to develop an iterative procedure for determining the optimal control function. This section is based on Gruver [9] and is an extension of the results contained therein. A less formal treatment is given by Blum [10].

State Regulator Problem

A well-known problem in optimal control concerns minimizing on $X \times U \times T$ a quadratic cost functional

$$J(u, x) = \tfrac{1}{2}\left([x, Qx] + [u, Ru] \right) \tag{17}$$

subject to linear dynamical system constraints

$$\dot{x} = Ax + Bu \quad , \quad x(t_0) = c \quad , \tag{18}$$

where Q, R, A, B are real valued $n \times n$, $r \times r$, $n \times n$, $n \times r$ matrix operators, respectively, on T. It is assumed that $Q \geqq 0$ and $R > 0$ and that the pair (A, B) is uniformly completely controllable on T. Hence, the conditions of the section, Basic Assumptions, are satisfied, and equations (15) and (16) are both necessary and sufficient for a unique minimum of J.

From equation (15) the optimal control $\hat{u}$ is

$$\hat{u} = -R^{-1} B^* \lambda \quad . \tag{19}$$

Equations (16), (18), and (19) imply that x and λ satisfy the integral equations

$$x(t) = c + \int_{t_0}^{t} (Ax - BR^{-1}B^*\lambda)\, ds \tag{20}$$

$$\lambda(t) = \int_{t}^{t_1} (Qx + A^*\lambda)\, ds \quad , \tag{21}$$

where elements of A, B, Q, x, λ are functions of the real variable s. In the last equation the adjoint of the integral operator K in equation (9) has been computed [1, p. 154]. In 1960 Kalman showed that equations (20) and (21) may be decoupled by employing the "Riccati transformation"

$$\lambda = Px \quad , \tag{22}$$

where P is an n × n real symmetric positive definit matrix operator on T. By substituting equations (20) and (21) into equation (22), differentiating equations (20) and (20) with respect to t, and eliminating λ using equation (22), the optimal control function may be expressed as

$$\hat{u} = -R^{-1}B^* Px \quad ,$$

where P satisfies the matrix Riccati equation

$$-\dot{P} = PA + A^* P - PBR^{-1}B^* P + Q \quad , \quad P(t) = \Theta \quad . \tag{23}$$

Furthermore, the minimum cost functional has the explicit representation

$$J(\hat{u}, x) = \tfrac{1}{2}\left[\dot{x}, Px\right]_1 \quad ,$$

and $\lim_{t_1 \to +\infty} J(\hat{u}, x)$ is a Lyapunov functional for the optimally controlled system [2]. Suppose that A, B, Q, R are constant matrices on T and $t_1 \to +\infty$. Because of the time invariance, we may consider the problem on the infinte interval $[0, \infty]$ by translation of the orgin. Then, Kalman has shown that $\overline{P} = \lim_{t \to -\infty} P(t)$ exists, is unique, and satisfies the matrix equation

$$\overline{P}A + A^* \overline{P} - \overline{P}BR^{-1}B^* \overline{P} + Q = \Theta \quad . \tag{24}$$

Furthermore, there exists a solution to equation (24) such that the controlled system is asymptotically stable.

As an alternative to solving equation (23), the adjoint function λ can be obtained in terms of a 2n × 2n transition matrix [11]. One difficulty is that the procedure requires inversion of submatrices of the transition matrix, which may become ill conditioned because of roundoff error in computation.

Kalman has also obtained an algorithm for computing discrete time optimal control functions by solving Riccati-type difference equations [12, 13]. In the section, The State Regulator Problem Revisited, I, Chapter 3, we describe another technique that was developed by the author and that leads to a monotone and quadratic convergent approximation of the minimum cost functional and the optimal control for equations (17) and (18) by solving linear matrix differential equations instead of nonlinear Riccati equations. The method is applicable to both time-invariant and time-varying systems.

CHAPTER 3

APPROXIMATION IN X^* AND U

In this chapter, we consider the computational solution of optimal control problems first, as Approximation in X^* and second, as Approximation in U. These two categories are treated together because of their natural relationship to the necessary conditions [eqs. (15) and (16)]. Methods of descent for finite dimensional minimization are reviewed as prototypes for infinite dimensional spaces. As realizations of the latter, we examine the well-known "direct" methods of steepest descent and second variations and also the "indirect" methods of boundary condition iteration and quasilinearization.

A second-order computational algorithm for Approximation in X^* by conjugate direction search is formulated. An accurate means for computing the gradient is given. Next, we describe a method for Approximation in U based on concepts of Hamilton-Jacobi theory and asympotic stability. As an important special case, a computational algorithm for the state regulator problem is developed that requires solution of linear matrix equations instead of nonlinear Riccati equations. Methods for solving these equations are discussed.

Descent Algorithms for Euclidean Space

In this section we offer a preview of the functional minimization problem based on well-known methods for unconstrained minimization in finite dimensional Euclidean space. Suppose there is given a functional $J : \mathbb{R}^n \to \mathbb{R}$, and the problem is to find a minimum of J on $\mathbb{R}^n$. It is assumed that the gradient $g = J_x$ may be obtained analytically as a function. Throughout this section the gradient, the $\mathbb{R}^n$ valued function J_x defined by $dF(x: h) = \left[J_x, h \right]_1$, will be denoted by g.

A basic consideration in the design of minimization algorithms is finding a systematic method for updating an element $x^{(k)}$ in $\mathbb{R}^n$ such that $\left\{ x^{(k)} \right\}$ converges to a point where J is minimized. Essentially, the update consists of a step direction in $\mathbb{R}^n$ and a step length. In general, the step length must be determined by a one-dimensional search procedure such as

quadratic or cubic interpolation, Fibonacci search, or golden section minimization [14, 15]. Choice of the step direction is more complicated, and one classification scheme is as follows: (1) first-order methods; (2) second-order methods; (3) conjugate direction methods. In (1) the step direction in $\mathbb{R}^n$ for the update is restricted to the negative gradient of J (the direction of "steepest descent" in $\mathbb{R}^n$) and, hence, only first-order partial derivatives of J are needed. In (2) the step direction employs second-order partial derivatives of J. In (3) the step directions are chosen g_x - conjugate as defined below and require only first-order partial derivatives of J. Moreover, convergence of a minimizing sequence is quadratic in the sense of minimizing a quadratic functional on $\mathbb{R}^n$ in at most n iterations.

FIRST-ORDER METHODS

A model for first-order methods is steepest descent. In this technique the step direction is restricted to the negative gradient of J. The procedure may be implemented either in a continuous or a discrete manner. The latter is well-suited for digital computer calculation, and the steps in the algorithm are as follows:

1. Choose an initial element, $x^{(0)}$; let $k = 0$.

2. Compute the step direction,

$$s^{(k)} = -g(x^{(k)}) \quad . \tag{25}$$

3. Compute (by a one-dimensional search) a step length $\alpha^{(k)}$ which minimizes $J\left(x^{(k)} + \alpha^{(k)} s^{(k)}\right)$.

4. Update $x^{(k)}$ according to

$$x^{(k+1)} = x^{(k)} + \alpha^{(k)} s^{(k)} \quad . \tag{26}$$

5. Let $k \rightarrow k + 1$ and repeat steps 2 through 4 until $||g(x^{(k)})||_1$ is less than a predetermined positive number.

The method of steepest descent is extremely stable in a large neighborhood of a minimum. However, computational results indicate slow convergence,

except for the case in which the "level curves" (loci of constant J in $\mathbb{R}^n$) are hyperspheres. Since successive steps are orthogonal, the procedure often leads to inefficient "zig-zagging". Mathematically, it has been shown that convergence of $\left\{ x^{(k)} \right\}$ is at least as fast as a geometric series with ratio $(M-m)/M$ where $mI < g_x < MI$, $0 < m < M$ [16].

SECOND-ORDER METHODS

A model for second-order methods is the classical Newton-Raphson method. Basically, it is an iterative technique for solving for the "roots" of an equation $T(x) = \theta$ where T is a nonlinear mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$. Suppose $J : \mathbb{R}^n \rightarrow \mathbb{R}$, then the gradient equation,

$$g(\hat{x}) = \theta \quad , \tag{27}$$

is a necessary condition for a minimum of J at $\hat{x}$, and the Newton-Raphson method is a convenient means for obtaining the minimizing element $\hat{x}$. The algorithm is identical to that given for steepest descent except that steps 3 and 4 are replaced by the following iteration:

$$x^{(k+1)} = x^{(k)} - g_x^{-1}(x^{(k)})\, g(x^{(k)}) \quad , \tag{28}$$

where g_x is the Jacobian matrix of g.[2] Conditions for existence of the inverse Jacobian, and convergence of the minimizing sequence $\left\{ x^{(k)} \right\}$ are given in Reference 17. These conditions are rather lengthy, however, and in practice are usually not checked. The Newton-Raphson method is very efficient close to the minimum since g_x is a measure of curvature in $\mathbb{R}^n$. In fact, if J is a quadratic functional on $\mathbb{R}^n$, then under certain conditions the algorithm converges in one iteration. A disadvantage, however, is that the inverse Jacobian may fail to exist during the descent, whereas if the problem

2. The $n \times n$ Jacobian matrix g_x of the $\mathbb{R}^n$-valued function $g = J_x$ is also called the Hessian of the quadratic functional J.

possesses a well-defined gradient, it could probably be solved by steepest descent. An additional disadvantage is that the iteration equation (28) may be unstable for certain choices of $x^{(0)}$ and fail to converge. This instability may sometimes be corrected by adjustment of the step length.

CONJUGATE DIRECTION METHODS

A compromise between the methods of steepest descent and Newton-Raphson is the following class of minimization techniques, which possesses quadratic convergence — in the sense of minimizing a quadratic functional on $\mathbb{R}^n$ in at most n steps.

The class has the advantage that only first-order partial derivatives of J are needed. The earliest conjugate direction method, called the method of conjugate gradients, was developed by Hestenes and Stieffel [18].

The name, "conjugate direction methods," arises because successive directions $s^{(k)}$, $k = 0, 1, 2, \ldots$ in $\mathbb{R}^n$ are chosen to be conjugate (orthogonal with respect to the Jacobian matrix, g_x) as defined by

$$\left[s^{(j)}, g_x\left(x^{(k)}\right) s^{(k)} \right]_1 = 0 \quad , \quad j \neq k \tag{29}$$

As in the Newton-Raphson technique, an essential simplification occurs if J is a quadratic functional on $\mathbb{R}^n$. In fact, if J is of the form

$$J(x) = c + \left[b, x \right]_1 + \frac{1}{2} \left[x, Ax \right]_1 \tag{30}$$

where $b \in \mathbb{R}^n$ and A is a positive definite matrix, then the main theorem for the method specifies a minimizing sequence $\left\{ x^{(k)} \right\}$ for J with the property that $\left\{ x^{(k)} \right\} \rightarrow A^{-1} b$ [1, p. 294]. Convergence of the method is far superior to steepest descent. Details concerning the construction of the minimizing sequence, the step direction, and step length are available in the references and will not be needed in the sequel.

Another important class of conjugate direction techniques is the quasi-Newton methods. Suppose J is a quadratic functional as in equation (30).

Then this class of methods generates a sequence of positive definite $n \times n$ matrices $\left\{ H^{(k)} \right\}$, which converge to the inverse Jacobian A^{-1}. Many different schemes for updating $H^{(k)}$ have been proposed, although most computational results and the greatest success has been reported for that given by Davidon and later refined by Fletcher and Powell [19]. The authors of the latter reference have also proved the stability and convergence of the algorithm for nonlinear J. Details concerning the construction of the minimizing sequence are given in the references [19, 15].

The method of conjugate gradients requires saving the gradient between iterations, and computer storage requirements are about the same as for the method of steepest descent. Quasi-Newton methods, however, provide faster convergence for nonlinear problems in return for storage between iterations of both the gradient and the $n \times n$ matrix $H^{(k)}$.

Descent Algorithms for Hilbert Space

In the following section, computational methods for optimal control are classified according to (1) methods based on the first variation of the cost functional; (2) methods based on the second variation of the cost functional. Within these categories, particularly in (1), the first, second order, and conjugate direction methods of the section, Descent Algorithm for Euclidean Space, may be used to compute the control function update.

METHODS BASED ON THE FIRST VARIATION

We now consider minimizing a functional $J : X \times U \times T \to \mathbb{R}$ where a minimizing sequence $\left\{ u^{(k)} \right\}$ for J lies in a bounded open subset U of Hilbert space. Suppose we are given the optimal control problem of the section, Definition of the Optimal Control Problem, Chapter 2, of minimizing a cost functional

$$J(u, x) = \int_{t_0}^{t_1} L(x, u, t)\, dt \qquad (31)$$

subject to nonlinear dynamical system constraints

$$\dot{x} = f(x, u, t) \quad , \quad x(t_0) = c \qquad (32)$$

and terminal equality constraints

$$\psi(x, t_1) = \theta \quad . \tag{33}$$

In general the terminal time t_1 may be free and, consequently, determined implicitly through equation (33).

Necessary conditions for a solution to a special case of this problem, (t_1 specified) were derived in the section, Necessary Conditions, Chapter 2. Suppose we define the Hamiltonian functional $H : X \times X^* \times U \times T \rightarrow \mathbb{R}$ by

$$H(x, \lambda, u, t) = L(x, u, t) + \left[\lambda, f(x, u, t)\right]_1 \tag{34}$$

where $\lambda \in X^*$ (see footnote 1 in Chapter 2). Necessary conditions for a weak minimum of J over the product space $X \times U \times T$ are derived in References 3 and 4, and the results are as follows:

$$\dot{x} = H_\lambda(x, \lambda, \hat{u}, t) \quad , \quad x(t_0) = c \tag{35}$$

$$\dot{\lambda} = -H_x(x, \lambda, \hat{u}, t) \tag{36}$$

$$\theta = H_u(x, \lambda, \hat{u}, t) \tag{37}$$

$$\lambda(t_1) = \psi_x^*(x, t_1)\, \nu \tag{38}$$

$$0 = H(x, \lambda, \hat{u}, t_1) + \left[\psi_t(x, t_1) \, , \, \nu\right]_1 \tag{39}$$

$$\theta = \psi(x, t_1) \tag{40}$$

where ν is a constant adjoint function in $(\mathbb{R}^p)^*$. Notice that if t_1 is specified, equations (39) and (40) are not needed, and equations (36) through (38) are equivalent to equations (15) and (16). Equations (35) through (40) constitute a two-point boundary problem since $\lambda(t_0)$ is unknown. Thus a

control function $\hat{u} \in U$ is optimal provided that an adjoint function λ can be found such that the following conditions are satisfied:

1. The canonical equations (35) and (36).

2. The min-H requirement, equation (37).

3. The terminal boundary conditions, equations (38) through (40).

Example 1

As a concrete example of the necessary conditions just developed, consider the minimization of a quadratic cost functional

$$J(u, x) = \frac{1}{2} \int_0^{t_1} (x^2 + u^2)\, dt$$

subject to a linear first-order dynamical constraint

$$\dot{x} = -x + u , \qquad x(0) = 1$$

and terminal equality constraint

$$x(t_1) - 5 = 0 ,$$

where t_1 is free. For this simple problem, the Hamiltonian functional is

$$H(x, \lambda, u, t) = \frac{1}{2}(x^2 + u^2) + \lambda(-x + u) ,$$

and equations (35) through (40) are as follows:

1. The canonical equations

$$\dot{x} = -x + u , \qquad x(0) = 1 \tag{35'}$$

$$\dot{\lambda} = -x + \lambda . \tag{36'}$$

2. The min-H requirement

$$u + \lambda = 0 \quad . \tag{37'}$$

3. The terminal boundary conditions

$$\lambda(t_1) = \nu \tag{38'}$$

$$0 = \left(\frac{1}{2}(x^2 + u^2) + \lambda(-x + u) + \dot{x}\nu \right) \Big|_{t_1} \tag{39'}$$

$$0 = x(t_1) - 5 \quad . \tag{40'}$$

In the past, an iterative solution to equations (35) through (40) has involved satisfying any two of the preceding conditions while adjusting the third. Thus we obtain the well-known methods of quasilinearization (adjust the adjoint functions in condition 1) [20], steepest descent (adjust the control function in condition 2) [21,22], and boundary condition iteration (adjust the adjoint functions in condition 3) [23,24]. Quasilinearization and boundary condition iteration are called "indirect" methods since a minimizing sequence $\left\{ u^{(k)} \right\}$ for J is obtained indirectly by iteration of the canonical equations (35) and (36), usually by means of a generalized Newton-Raphson search [17]. In this work we feel that Approximation in X^* is a more natural classification since the basic iteration occurs in X^*. On the other hand, steepest descent is referred to as a "direct" method since the Hamiltonian functional and, consequently, the cost functional are minimized directly in the control space U. Thus steepest descent may be classified as Approximation in U.

Usually the adjustment in X^* or U is enforced by a form of gradient descent. In some cases the canonical equations (35) and (36) can be expressed as a contraction mapping on the Hilbert space $X \times X^* \times U \times T$, and one could employ the method of successive approximations attributed to Picard and formalized by Banach to obtain a fixed point. Some results using this concept have been obtained by deJong [25].

Let us consider the boundary condition iteration method in more detail for the special case t_1 specified and unconstrained terminal state. The general

case is given in References 26 and 24. Suppose equation (37) may be solved explicitly for u as a function of x and λ. Then, by substituting $u(x, \lambda)$ into equations (35) and (36), the canonical equations may also be expressed in terms of x and λ as follows:

$$\dot{x} = H_{\lambda}(x, \lambda, u(x, \lambda), t) \quad , \quad x(t_0) = c \tag{41}$$

$$\dot{\lambda} = -H_{x}(x, \lambda, u(x, \lambda), t) \quad , \quad \lambda(t_0) = d \tag{42}$$

where d is the initial condition for the adjoint function and is unknown. The basic boundary condition iteration algorithm for Approximation in X^* proceeds as follows:

1. Choose an initial value for $d^{(0)}$; let $k = 0$.

2. Integrate equations (41) and (42) forward from $x(t_0)$, $\lambda(t_0)$.

3. Compute the function $u^{(k)}$ from equation (37) and store $x^{(k)}(t_1)$, $\lambda^{(k)}(t_1)$.

4. Update $d^{(k)}$ in the direction of steepest descent of the terminal boundary conditions.

5. Let $k \rightarrow k + 1$, and repeat steps 2 through 4 until equations (38) through (40) are satisfied to within a predetermined bound.

In step 4, we usually minimize a functional E which involves the terminal boundary conditions. A procedure for choosing this functional will be explained in Example 2 and, for a more general case, in the forthcoming section, A Conjugate Direct Method for Approximation in X^* .

Example 2

Use of the boundary condition iteration algorithm will now be demonstrated by a special case of Example 1 given earlier in this section. Suppose that the terminal time is specified at $t_1 = 1$, and the terminal state is unconstrained. Then, equations (39') and (40') are not needed. From the min-H requirement, equation (37'), the control function may be expressed as $u = -\lambda$ and used to eliminate u in equations (35') and (36'). The result is

$$\dot{x} = -x - \lambda \quad , \quad x(0) = 1 \tag{41'}$$

$$\dot{\lambda} = -x + \lambda \quad , \qquad \lambda(0) = d \quad , \tag{42'}$$

where d is the unknown initial condition for the adjoint function. A convenient choice for a functional of terminal error is $E(x,\lambda,t) = \frac{1}{2}\lambda^2(t)$ since if $E(x,\lambda,t_1) = 0$, the terminal boundary condition is satisfied. To implement a descent algorithm, the gradient $E_d = E_x x_d + E_\lambda \lambda_d$ is needed. Let us differentiate equations (41') and (42') with respect to d. The result is the "sensitivity equations,"

$$\dot{x}_d = -x_d - \lambda_d \quad , \qquad x_d(0) = 0$$

$$\dot{\lambda}_d = -x_d + \lambda_d \quad , \qquad \lambda_d(0) = 1 \quad ,$$

a set of coupled first-order differential equations that may be solved for λ_d. A rigorous derivation of these equations has been given by Levine [23]. Suppose, for example, that $d = 0$. Then by solving equations (41') and (42') for the functions x, λ and solving the "sensitivity equations" for x_d, λ_d, the gradient is

$$E_d = \lambda \lambda_d$$

$$= -\frac{1}{8}\left(e^{-\sqrt{2}\,t} - e^{\sqrt{2}\,t}\right)\left((1-\sqrt{2})\,e^{-\sqrt{2}\,t} - (1+\sqrt{2})\,e^{\sqrt{2}\,t}\right) \quad .$$

Now that the preliminary equations have been set up, the first iteration of the boundary condition algorithm proceeds as follows:

1. Let $d = 0$.

2. Integrate equations (41') and (42') forward from

$$x^{(0)} = \frac{1}{2\sqrt{2}}\left((1-\sqrt{2})\,e^{\sqrt{2}\,t} - (1+\sqrt{2})\,e^{-\sqrt{2}\,t}\right)$$

$$\lambda^{(0)} = \frac{1}{2\sqrt{2}}\left(e^{\sqrt{2}\,t} - e^{-\sqrt{2}\,t}\right) \quad .$$

3. Compute u from equation (37'):

$$u^{(0)} = -\frac{1}{2\sqrt{2}}\left(e^{\sqrt{2}\,t} - e^{-\sqrt{2}\,t}\right) \quad ,$$

and store

$$x^{(0)}(1) = \frac{1}{2\sqrt{2}}\left(-0.586\, e^{\sqrt{2}} - 2.414\, e^{-\sqrt{2}}\right)$$

$$\lambda^{(0)}(1) = \frac{1}{2\sqrt{2}}\left(e^{\sqrt{2}} - e^{-\sqrt{2}}\right) \quad .$$

4. Update $d^{(0)}$ according to

$$d^{(1)} = d^{(0)} + \alpha^{(0)} E_d\left(x^{(0)}, \lambda^{(0)}, 1\right)$$

$$= \frac{\alpha^{(0)}}{8}\left(e^{-\sqrt{2}} - e^{\sqrt{2}}\right)\left(0.586\, e^{-\sqrt{2}} + 2.144\, e^{\sqrt{2}}\right)$$

where $\alpha^{(0)}$ is a constant chosen to minimize $E(x, \lambda, 1)$.

Step 4 involves a one-dimensional search for $\alpha^{(0)}$, which requires repeated evaluation of $E(x, \lambda, 1)$ using the functions $x^{(0)}$ and $\lambda^{(0)}$ from equations (41') and (42') with initial conditions $x^{(0)}(0) = 1$, $\lambda^{(0)}(0) = d^{(0)}$. Steps 2 through 4 are repeated until $E(x^{(k)}, \lambda^{(k)}, t_1) = \frac{1}{2}\left(\lambda^{(k)}(t_1)\right)^2 < \alpha_1$ where α_1 is a small positive number. Notice that integrating the canonical equations has given rise to terms of the form $e^{\sqrt{2}\,t}$, functions which increase

with t. This illustrates an inherent weakness of the boundary condition iteration method, because the technique is applicable only to those optimal control problems for which the adjoint equation (36) is stable when integrated forward on T, or at least the instabilities do not predominate. Fortunately, this includes a large class of trajectory optimization problems for which atmospheric effects are neglibible. The algorithm is particularly attractive since only storage of the terminal values of $x(t_1)$ and $\lambda(t_1)$ is required between iterations. In contrast, steepest descent methods require storing at least the entire adjoint function λ on T.

In the past, choice of the update in step 4 was based on minimizing a functional of terminal error (such as a quadratic) using steepest descent [23, 27], conjugate gradient search [28], or the Newton-Raphson method [26, 24] to satisfy the terminal boundary conditions. In the section, A Conjugate Direction Method for Approximation in X^*, we will describe the application of quasi-Newton methods described in the section, Conjugate Direction Methods, to the boundary condition iteration technique.

Finally, for comparison with the latter algorithm the basic steepest descent algorithm for Approximation in U is formulated as follows:

1. Choose a nominal control function $u^{(0)}$; let $k = 0$.

2. Integrate equation (35) forward from $x(t_0)$. Then compute $\lambda(t_1)$, and store the function $x^{(k)}$.

3. Integrate equation (36) backward from $\lambda(t_1)$, and store the function $\lambda^{(k)}$.

4. Update the control function in the direction of steepest descent of H according to

$$u^{(k+1)} = u^{(k)} - \alpha^{(k)} H_u\left(x^{(k)}, \lambda^{(k)}, u^{(k)}, t\right)$$

where $\alpha^{(k)}$ is chosen to minimize $||H\left(x^{(k)}, \lambda^{(k)}, u^{(k+1)}, t\right)||^2$.

5. Let $k \rightarrow k + 1$ and repeat steps 2 through 4 until $||H_u\left(x^{(k)}, \lambda^{(k)}, u^{(k)}, t\right)||^2$ is less than a predetermined positive number.

Example 3

Use of the steepest descent algorithm is now demonstrated by the problem introduced in Example 2. The first iteration proceeds as follows:

1. Let $u^{(0)} = 0$.

2. Integrate equation (35') forward from $x(0) = 1$:

$$x^{(k)} = e^{-t} \quad .$$

3. Integrate equation (36') backward[3] from $\lambda(1) = 0$.

$$\lambda^{(0)} = \frac{1}{2}\left(e^{t-1} - e^{-(t+1)}\right) \quad .$$

4. Update the control function according to

$$u^{(1)} = u^{(0)} - \alpha^{(0)} H_u = \frac{\alpha^{(0)}}{2}\left(e^{t-1} - e^{-(t+1)}\right) \quad ,$$

where $\alpha^{(0)}$ is chosen to minimize

$$\| H\left(x^{(0)}, \lambda^{(0)}, u^{(0)}, t\right)\|^2 = \int_0^1 \left(\frac{1}{2}\left(\left(x^{(0)}\right)^2 + \left(u^{(1)}\right)^2\right) + \lambda^{(0)}\left(-x^{(0)} + u^{(1)}\right)\right)^2 dt$$

using a one-dimensional search procedure [14, 15].

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3. To integrate backward, we make the change of variable $t = 1 - s$ in equation (36'). Therefore, $d\lambda/dt = -d\lambda/ds$, and equation (36') becomes $d\lambda/ds + \lambda = e^{s-1}$.

Steps 2 through 4 of the algorithm are repeated until $||H_u||^2 = \int_0^1 \left(u^{(k)} + \lambda^{(k)}\right)^2 dt < \alpha_2$. It is significant that the adjoint equation is integrated in its "stable" direction. Consequently, the steepest descent method may be applied to a wider class of problems than the boundary condition iteration method in return for storage of at least the entire adjoint function λ over the interval T. In terms of numerical integration using a digital computer, this requires discretizing the interval T and storing numbers at many points.

In step 4, the conjugate direction methods with their inherent advantages discussed in the section, Conjugate Direction Methods, can be used. Conjugate direction algorithms for computational solution of optimal control problems by Approximation in U have been formulated by Lasdon, Mitter, and Waren [29]; Sinnott and Luenberger [30], and Lynch [31]. In the section, A Conjugate Direction Method for Approximation in X^*, we present a new technique for Approximation in U based on concepts from Hamilton-Jacobi theory and asymptotic stability. As an important special case, the state regulator problem is treated in detail.

METHODS BASED ON THE SECOND VARIATION

The following is a sketch of a second variation method that leads to an algorithm for Approximation in U. Because of the mathematical complexity of the results, we shall restrict this discussion to the special case of t_1 specified and unconstrained terminal state. Details of the derivation and more general cases are given in the references [32, 33].

Essentially, the method consists of minimizing a second-order approximation to the Lagrangian equation (8) over the control space U and using a Riccati transformation to decouple the resulting boundary value problem. In fact, the entire development is similar to that given in the sections, Necessary Conditions, and State Regulator Problem in Chapter 2. Let us assume that the first-order necessary conditions, equations (35), (36), and (38), are satisfied at the kth iteration by a control function $u = u^{(k)}$ and corresponding state function $x = x^{(k)}$ with increments $\delta u = \delta u^{(k)}$ and $\delta x = \delta x^{(k)}$, respectively. Suppose the cost functional in equation (1) is written in the terms of the Hamiltonian functional, equation (34),

$$J(u, x) = \int_{t_0}^{t_1} \left(H - \left[\lambda, f\right]_1\right) dt$$

where the arguments of H and f have been omitted for brevity. Let $J(u,x)$ be approximated by a constant term plus the sum of its first and second Frechet differentials (the first and second variations) at x and u. The procedure for obtaining the differentials is similar to that given in the section, Necessary Conditions, and the result is $J(u,x) \cong J_0 + E(\delta u, \delta x)$ where

$$E(\delta u, \delta x) = \left[H_u, \delta u\right] + \frac{1}{2}\left[\delta x, H_{xx}\, \delta x\right] + \frac{1}{2}\left[\delta x, H_{xu}\, \delta u\right]$$

$$+ \frac{1}{2}\left[\delta u, H_{ux}\, \delta x\right] + \frac{1}{2}\left[\delta u, H_{uu}\, \delta u\right] \quad ,$$

and J_0 is a constant.

The increments δx and δu satisfy the linearized dynamical system

$$\dot{\delta x} = f_x\, \delta x + f_u\, \delta u \quad , \qquad \delta x(t_0) = \theta \quad , \tag{43}$$

and superscripts denoting iteration are omitted. For convenience, we are using the Hilbert space inner product notation as defined in the Terminology section, Chapter 1.

The design of a second variation algorithm is based on finding a control increment δu which minimizes $E(\delta u, \delta x)$ subject to equation (43). This "accessory minimization problem" is solved by the usual technique — equating the first Frechet differential E at δu in an arbitrary direction ξ to zero, integrating by parts, and applying the Euler-Lagrange Lemma (see the section, Necessary Conditions, Chapter 2). The result is as follows:[4]

$$\dot{\delta x} = H_{\lambda x}\, \delta x + H_{\lambda u}\, \delta u \quad , \qquad \delta x(t_0) = \theta \tag{44}$$

$$\dot{\delta \lambda} = -H_{xx}\, \delta x - H_{xu}\, \delta u \quad , \qquad \delta \lambda(t_1) = \theta \tag{45}$$

4. Recall that $H_\lambda = f$.

$$\theta = H_u + H_{uu}\,\delta u + H_{ux}\,\delta x + H_{u\lambda}\,\delta\lambda \quad , \tag{46}$$

where $\delta\lambda$ is the adjoint function increment.

Moreover, sufficient conditions for a weak minimum are the strong Legendre and the Jacobi (conjugate point) conditions:

$$\text{(a)}\quad H_{uu} > 0 \tag{47a}$$

$$\text{(b)}\quad H_{xx} - H_{xu}\,H_{uu}^{-1}\,H_{ux} \geqq 0 \quad . \tag{47b}$$

(An alternate approach is taken in the section, A Second Variation Method for Approximation in X , Chapter 4, where the minimization is performed in the state space X after applying an "inverse mapping.")

If H_{uu} is positive definite on T , its inverse exists, and equation (46) may be solved for the control increment as

$$\delta u = -H_{uu}^{-1}\left(H_u + H_{ux}\,\delta x + H_{u\lambda}\,\delta\lambda\right) \tag{48}$$

By eliminating δu in equations (44) and (45), we obtain the following linear boundary value problem in the product space $X \times X^* \times T$:

$$\delta\dot{x} = A_1\,\delta x + A_2\,\delta\lambda + v \quad , \qquad \delta x(t_0) = \theta \tag{49}$$

$$\delta\dot{\lambda} = -A_3\,\delta x - A_1^*\,\delta\lambda + w \quad , \qquad \delta\lambda(t_1) = \theta \tag{50}$$

where

$$A_1 = f_x - f_u\,H_{uu}^{-1}\,H_{ux} \quad , \qquad A_2 = f_u\,H_{uu}^{-1}\,f_u$$

$$A_3 = H_{xx} - H_{xu}\,H_{uu}^{-1}\,H_{ux} \quad , \qquad v = f_u\,H_{uu}\,H_u$$

$$w = H_{xu}\,H_{uu}^{-1}\,H_u \quad .$$

In order to decouple equations (49) and (50), we introduce the "nonhomogeneous" Riccati transformation

$$\delta\lambda = P\delta x + q \quad . \tag{51}$$

Then the solution to the boundary value problem may be expressed in terms of a matrix Riccati equation

$$-\dot{P} = PA_1 + A_1^* P - PA_2 P + A_3 \quad , \quad P(t_1) = \Theta \quad , \tag{52}$$

and the linear equation

$$-\dot{q} = (PA_2 + A_1^*)q + Pv + w \quad , \quad q(t_1) = \theta \quad . \tag{53}$$

The sufficiency conditions [eq. (47)] insure that H_{uu}^{-1} exists and that equation (52) possesses a solution on T.

The basic second variation algorithm proceeds as follows:

1. Choose a nominal control function $u^{(0)}$; let $k = 0$.

2. Integrate equation (35) forward from $x(t_0)$ and store the function $x^{(k)}$.

3. Integrate equation (36) backward from $\lambda(t_1)$ and store the function $\lambda^{(k)}$.

4. Compute the partial derivative matrices; check equation (47a), compute H_{uu}^{-1}, and check equation (47b).

5. Integrate equations (52) and (53) backward from $P(t_1)$, $q(t_1)$, and store the functions $P^{(k)}$, $q^{(k)}$.

6. Compute the gain matrices

$$Y = -H_{uu}^{-1}(H_{ux} + H_{u\lambda} P)$$

$$z = -H_{uu}^{-1}(H_u + H_{u\lambda} q)$$

7. Integrate $\delta\dot{x} = (f_x + f_u Y)\,\delta x + z$ from $\delta x(t_0) = \theta$ and store $\delta x^{(k)}$.

8. Compute the control update $u^{(k+1)} = u^{(k)} + Y\delta x^{(k)} + z$.

9. Let $k \rightarrow k + 1$, and repeate steps 2 through 8 until $||H_u^{(k)}||^2$ is less than a predetermined positive number.

Example 4

Use of the second variation algorithm is now demonstrated by the computational example introduced in Example 2. The first iteration proceeds as follows:

1. Let $u^{(0)} = 0$.

2. Integrate equation (35') forward from $x(0) = 1$:

$$x^{(0)} = e^{-t} \quad .$$

3. Integrate equation (36') backward from $\lambda(1) = 0$:

$$\lambda^{(0)} = \frac{1}{2}\left(e^{t-1} - e^{-(t+1)}\right) \quad .$$

4.

$$H_u^{(0)} = te^{-t} \qquad H_{\lambda u}^{(0)} = H_{u\lambda}^{(0)} = 1$$

$$H_{xx}^{(0)} = 1 \qquad H_{\lambda x}^{(0)} = -1$$

$$H_{xu}^{(0)} = H_{ux}^{(0)} = 0 \qquad f_x = -1$$

$$H_{uu}^{(0)} = 1 \qquad f_u = 1 \quad .$$

5. Integrate equation (51) backward:

$$\dot{P} - 2P - P^2 + 1 = 0 \quad , \quad P(1) = 0 \quad ,$$

for which a solution by separation of variables is

$$P = \frac{(\sqrt{2} - 1) + (1 - \sqrt{2})\, e^{2\sqrt{2}(t-1)}}{1 - (1 - \sqrt{2})\, e^{2\sqrt{2}(t-1)}}$$

(Note: The function q is zero because v and w are zero.)

6. Compute the gain matrices $Y = -P$ and $z = -te^{-t}$.

Equation (43) may be integrated to obtain δx and, hence, the control increment δu as specified in steps 7 and 8.

7. Compute $\delta u^{(0)}$, and update $u^{(0)}$:

$$u^{(1)} = -\left(\frac{e^{t-1} - e^{-(t-1)}}{2} + e^{-t} P(t)\right) \quad .$$

Steps 2 through 7 are repeated until

$$|| H_u^{(k)} ||^2 = \int_0^1 \left(u^{(k)} + \lambda^{(k)}\right)^2 dt < \alpha_3 \quad .$$

The second variation algorithm requires integration of $\frac{1}{2} n(n+9)$ equations at each control function update and consequently suffers from the "curse of dimensionality." In other words, high-order problems may require an excessive amount of computation. In the last example, $n = 1$, and five equations must be solved. Another disadvantage is the complexity of the computer program since the algorithm requires a significantly greater number of instructions than first-variation methods. In compensation for the latter difficulties, the second variation algorithm results in rapid convergence to a minimum. However,

the occurrence of conjugate points [violation of the Jacobi condition, equation (47b)] would invalidate the procedure, whereas first-variation methods could possibly be used to obtain a solution. Usually the full increment is not used, and adjustment of step length should be provided to insure stable descent. For example, let $u^{(k+1)} = u^{(k)} + \alpha^{(k)} \delta u^{(k)}$ where $\alpha^{(k)} \leqq 1$ [32].

A Conjugate Direction Method for Approximation in X^*

A detailed formulation of the boundary condition iteration algorithm using conjugate direction search in X^* is developed for the optimal control problem of the section, Definition of the Optimal Control Problem, in Chapter 2. In contrast to former methods for boundary condition iteration involving the Newton-Raphson method [26, 24], the inverse Jacobian matrix is obtained by iteration, and only first-order partial derivatives of the cost functional are required. This formulation has been reported by Gruver [9].

The most difficult aspect of setting up a boundary condition iteration algorithm is choosing a function of the terminal boundary conditions to be minimized. In the section, Descent Algorithms for Hilbert Space, we employed a positive definite functional involving the terminal adjoint function. A generalization of the latter to the case of t_1 free and terminal equality constraints is to choose the Euclidean norm of an $\mathbb{R}^{n+p+1}$-valued function of terminal boundary conditions, equations (38) through (40). Suppose we define the mapping $E : X \times X^* \times \mathbb{R}^P \times T \to \mathbb{R}$ by

$$E(x, \lambda, \nu, t) = \frac{1}{2} \| \omega \|_1^2 \tag{54}$$

where ω is the $\mathbb{R}^{n+p+1}$-valued function,

$$\omega = \left(H(x, \lambda, u(x, \lambda), t) + \left[\psi_t(x, t), \nu \right]_1 , \quad \lambda(t) - \psi_x^*(x, t)\nu , \quad \psi(x, t) \right) ,$$

and $\| \cdot \|_1$ is the usual Euclidean norm on $\mathbb{R}^{n+p+1}$. As before in the development of the section, Descent Algorithms for Hilbert Space, d is the unknown

initial condition for the adjoint function in λ in X^*, ν is the unknown constant adjoint function with values in $(\mathbb{R}^p)^*$, and t_1 is the unknown terminal time. It is assumed that there exists a solution to equation (35) through (40). Then, if equations (35) through (37) are satisfied, a necessary condition for a weak minimum of J over the product space $X \times U \times T$ is $E(x, \lambda, \nu, t_1) = 0$. We shall consider the simpler problem of choosing a minimizing sequence $\left\{ d^{(k)}, \nu^{(k)}, t_1^{(k)} \right\}$ such that

$$\lim_{k \to \infty} E\left(x^{(k)}, \lambda^{(k)}, \nu^{(k)}, t_1^{(k)} \right) = 0 \tag{55}$$

and perform the minimization by conjugate direction search using the quasi-Newton method of Fletcher and Powell [19].

Next, the gradient is computed in a manner similar to that described in Example 2 in the section, Descent Algorithms for Hilbert Space. The gradient of E is the $\mathbb{R}^{n+p+1}$-valued function

$$g(d, \nu, t) := (E_d, E_\nu, E_t)\Big|_{t_1} \quad . \tag{56}$$

Calculation of E_ν and E_t may be performed explicitly. The ith element of the $\mathbb{R}^n$-valued function E_d is

$$(E_d)_i = \sum_{j=1}^{n+p+1} \sum_{k=1}^{n} \omega_j \left(\frac{\partial \omega_j}{\partial x_k} \frac{\partial x_k}{\partial d_i} + \frac{\partial \omega_j}{\partial \lambda_k} \frac{\partial \lambda_k}{\partial d_i} \right) \quad , \tag{57}$$

where $x = (x_1, \ldots, x_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, and $\omega = (\omega_1, \ldots, \omega_{n+p+1})$. Thus, evaluation of E_d requires the $n \times n$ matrix functions x_d and λ_d. By differentiating equations (41) and (42) with respect to the initial condition d, we obtain the matrix "sensitivity equations,"

$$\dot{x}_d = H_{\lambda x}(x, \lambda, u(x, \lambda), t)\, x_d + H_{\lambda\lambda}(x, \lambda, u(x, \lambda), t)\, \lambda_d \quad ,$$

$$x_d(t_0) = \Theta \tag{58}$$

$$\dot{\lambda}_d = -H_{xx}(x, \lambda, u(x, \lambda), t)\, x_d - H_{x\lambda}(x, \lambda, u(x, \lambda), t)\, \lambda_d \quad ,$$

$$\lambda_d(t_0) = I \quad . \tag{59}$$

Equations (58) and (59) provide an accurate means for computing E_d and, hence, the gradient of E in terms of (d, ν, t_1). Knowledge of the gradient enables us to implement the conjugate direction methods described in the section, Descent Algorithms for Euclidean Space.

The complete conjugate direction algorithm for Approximation in X^* is as follows: Assume that equation (37) has been used to eliminate the dependence of equations (35) and (36) on the control function u as in equations (41) and (42). Then

1. Choose an initial $\pi^{(0,\,0)} = \left(d^{(0)}, \nu^{(0)}; t_1^{(0)} \right)$; let $i = j = 0$.

2. Integrate equations (41) and (42) forward from $x(t_0)$, $\lambda(t_0) = d^{(i)}$ until $t = t_1^{(i)}$. Compute $u(x, \lambda)$ and store $x^{(i)}\left(t_1^{(i)}\right)$, $\lambda^{(i)}\left(t_1^{(i)}\right)$.

3. Integrate equations (58) and (59) forward from $x_d(t_0)$, $\lambda_d(t_0)$ until $t = t_1^{(i)}$, and store the functions $x_d^{(i)}$, $\lambda_d^{(i)}$.

4. Compute and store $g\left(\pi^{(i,\,j)}\right) = \left(E_d, E_\nu, E_t\right)\Big|_{t_1}$ where $\pi^{(i,\,j)} = \left(d^{(i)}, \nu^{(i)}, t_1^{(i)}\right)$.

5. Compute $\pi^{(i,\,j)}$ by the quasi-Newton method with Fletcher-Powell update.

   a. Choose $H^{(0)} = I$.

   b. Compute $s^{(j)} = -H^{(j)}\, g\left(\pi^{(i,\,j)}\right)$.

   c. Determine $\alpha^{(j)}$ such that $E\left(\pi^{(i,\,j)} + \alpha^{(j)}\, s^{(j)}\right)$ is minimized.

   d. $\pi^{(i,\,j+1)} = \pi^{(i,\,j)} + \alpha^{(j)}\, s^{(j)}$.

   e. Compute $g\left(\pi^{(i,\,j+1)}\right) = \left(E_d, E_\nu, t_1\right)\Big|_{t_1}$.

f. $$H^{(j+1)} = H^{(j)} + \alpha^{(j)} \frac{\| s^{(j)} \|^2}{\left[\Delta g^{(j)} , s^{(j)}\right]} - \frac{\| \Delta g^{(j)} H^{(j)} \|^2}{\left[\Delta g^{(j)} , H^{(j)} \Delta g^{(j)}\right]}$$

where

$$\Delta g^{(j)} = g\left(\pi^{(i , j+1)}\right) - g\left(\pi^{(i , j)}\right) .$$

g. Let $j \rightarrow j+1$ and repeat steps b through f until $\| g\left(\pi^{(i , j)}\right) \|^2$ is less than a predetermined positive number.

6. Let $i \rightarrow i+1$ and repeat steps 2 through 5 until $E\left(\pi^{(i , \infty)}\right)$ is less than a predetermined positive number.

Determination of the parameter $\alpha^{(j)}$ in step 5c may be accomplished by a one-dimensional search procedure [14, 15, 29]. In step 5f the quasi-Newton update as given by Fletcher and Powell [19] is used to update $H^{(j)}$, an $n \times n$ matrix of functions on T which converges to the inverse Jacobian matrix of $g\left(\pi^{(i , j)}\right)$ as $j \rightarrow \infty$. Notice that the indicated norms and inner products in step 5f are in the Hilbert space $L^2_{n+p+1}(T)$.

The conjugate direction algorithm just presented may be applied to a wide class of optimal control problems and inherits the advantages of both the boundary condition method (moderate amount of storage) and the quasi-Newton method (rapid and stable descent). The algorithm requires fewer instructions than second-variation methods. Since the algorithm is based upon satisfying only first-order necessary conditions for a minimum of J, it does not depend upon the sufficiency conditions [eqs. (47a) and (47b)] second variation methods. Two very important advantages of the algorithm are the stability and rapid convergence resulting from the Fletcher-Powell update, step 5f, of the quasi-Newton method. The Newton-Raphson method locates the minimum of a quadratic functional $J : \mathbb{R}^n \rightarrow \mathbb{R}$ in only one iteration and could be used to compute $\pi^{(i , j)}$ in step 5. However, the iteration may be unstable for certain initial elements. Steepest descent methods, although very stable, may never converge if the eigenvalues of the Jacobian matrix are far apart. Quasi-Newton methods require at most n iterations to minimize a quadratic functional. The Fletcher-Powell quasi-Newton method results in rapid and stable descent even for highly nonlinear terminal boundary conditions and is acknowledged as the most powerful minimization technique currently available.

The most difficult aspect of using the algorithm is choosing the one-dimensional search procedure in step 5c. Quasi-Newton methods usually require accurate determination of the one-dimensional minimum. However, since each evaluation of E in step 5c requires forward integration of equations (41) and (42), direct search or higher techniques such as cubic interpolation may require too many evaluations of the functional E and, consequently, an excessive amount of computer time. It is suggested that a quadratic interpolation scheme [15] or golden section minimization [14] be employed, although the final choice of the best one-dimensional search must be resolved by numerical testing of the algorithm with nonlinear problems.

EXAMPLE 5

Use of the conjugate direction algorithm for Approximation in X^* will be demonstrated by the problem given in Example 2. The first iteration proceeds as follows:

1 through 4 are identical to Example 2 with $\pi^{(0\,,\,0)} = d^{(0)} = 0$.

5. Update $\pi^{(i\,,\,j)}$ by the Fletcher-Powell method:

a. $H^{(0)} = 1$.

b. $s^{(0)} = -g\left(\pi^{(0\,,\,0)}\right)$

$$= -\frac{1}{8}\left(e^{-\sqrt{2}\,t} - e^{\sqrt{2}\,t}\right)\left(0.586\, e^{-\sqrt{2}\,t} + 2.414\, e^{\sqrt{2}\,t}\right).$$

c. Determine $\alpha^{(0)}$ such that $E\left(d^{(0)} + \alpha^{(0)}\, s^{(0)}\right)$ is minimized (see Example 2, step 4, in the section, Descent Algorithms for Hilbert Space).

d. $\pi^{(0\,,\,1)} = \pi^{(0\,,\,0)} + \alpha^{(0)}\, s^{(0)} = \alpha^{(0)}\, s^{(0)}$.

e. Compute $g\left(\pi^{(0\,,\,1)}\right)$ from equations (58) and (59) as in step 4.

f. $$H^{(1)} = -\frac{\alpha^{(0)} \int_0^1 \left(s^{(0)}(t)\right)^2 dt}{\int_0^1 \left(g\left(\pi^{(0\,,\,1)}\right) - g\left(\pi^{(0\,,\,0)}\right)\right)(t)\, s^{(0)}(t)\, dt}.$$

Steps 5b through 5f of the algorithm are repeated until $\| \left(g\ \pi^{(i\ ,\ j)} \right) \|^2 = \int_0^1 g\left(\pi^{(i\ ,\ j)}\right)^2\ dt < \alpha_4$. After convergence of the sequence $\left\{ \pi^{(i\ ,\ j)} \right.$, $j = 1, 2, \ldots$, steps 2 through 5 are repeated until $E\left(\pi^{(i\ ,\ \infty)}\right) < \alpha_5$. Notice that the first iteration of the quasi-Newton method in step 5, for $H^{(0)} = I$, results in minimizing in the direction of steepest descent. However, successive step directions $s^{(j)}$ are modified in step 5b by the matrix $H^{(j)}$ and eventually result in a Newton-Raphson iteration.

A Method for Monotone Approximation in U

In this section, a sequence of functional approximations to the optimal cost functional is obtained by using concepts from the Hamilton-Jacobi theory to generate a minimizing sequence for J in the control space U. This technique was first reported by Gruver [13] and Puri and Gruver [34].[5] Similar results for a more general case were given by Leake and Liu [35]. An alternate proof for the case of the state regulator problem has been given by Kleinman and Athans [36] and Wonham [37].

Consider the optimal control problem as stated in the section, Definition of the Optimal Control Problem, in Chapter 2. Let us assume that the terminal time is specified, and the terminal state is unconstrained. The following method is based on obtaining a (k+1)st control function approximation to the optimal control by minimizing a certain Hamiltonian functional containing the kth control function approximation. Let the kth approximation to the minimum cost functional $V^{(k)} : X \times U \times T \rightarrow \mathbb{R}$ be defined by

$$V^{(k)} = \int_t^{t_1} L\,(x\,,\,u^{(k)}\,,\,s)\,ds \tag{60}$$

and the Hamiltonian functional $H : X \times U \times X^* \times T \rightarrow \mathbb{R}$ by

$$H\,(x\,,\,u\,,\,V_x^{(k)}\,,\,t) = L(x\,,\,u\,,\,t) + \left[V_x^{(k)}\,,\,f(x\,,\,u\,,\,t) \right]_1 \tag{61}$$

5. This section and the next one contain an expansion and correction to References 13 and 34.

By defining the Hamiltonian as in equation (61), we are treating $V_x^{(k)}$ as an adjoint function in X^*. This choice provides a convenient method for determining the adjoint function from the cost functional. For example, suppose that L is a quadratic functional on $X \times U \times T$. Then $V^{(k)}$ has the representation $V^{(k)} = \frac{1}{2}\left[x, P^{(k)} x\right]_1$ for some $n \times n$ matrix $P^{(k)}$ whose elements are functions on T and, therefore, $V_x^{(k)} = P^{(k)} x$. Consequently, $P^{(k)}$ specifies a Riccati transformation as in equation (22) and, it can be shown, also satisfies a matrix Riccati equation. Further results on the correspondence of $V_x^{(k)}$ and the adjoint functions in X^* are given by Kalman [11] although, for our purpose in the following section, the latter discussion provides adequate motivation for use in the method that follows.

Given the initial approximation $u^{(1)}$, suppose that we have obtained the kth approximation $u^{(k)}$ and the corresponding cost functional $V^{(k)}$. By the min-H condition a better approximation $u^{(k+1)}$ to the optimal control $\hat{u}$ is obtained by minimizing $H\left(x, u, V_x^{(k)}, t\right)$ over all $u \in U$. In fact, the Maximum Principle [38] insures that $u^{(k+1)}$ is a better approximation even if the control function is restricted to a closed proper subset of U. In the latter case, however, there are no simple methods for performing the minimization since the gradient of H may not exist.

We shall now show that the latter procedure for obtaining $u^{(k+1)}$ from $u^{(k)}$ results in a sequence of cost functional approximations, which is monotone decreasing to the minimum cost functional. By definition of $u^{(k+1)}$ we have

$$H\left(x, u^{(k+1)}, V_x^{(k)}, t\right) = \inf_{u \in U} H\left(x, u, V_x^{(k)}, t\right) ,$$

which implies

$$H\left(x, u^{(k+1)}, V_x^{(k)}, t\right) \leqq H\left(x, u^{(k)}, V_x^{(k)}, t\right) . \qquad (62)$$

Furthermore, it may be shown [11] that $V^{(k)}$ satisfies the "Hamilton-Jacobi" partial differential equation,

$$H(x, u^{(k)}, V_x^{(k)}, t) + V_t^{(k)} = 0 \quad . \tag{63}$$

Consider the control function $u^{(k+1)}$ and the induced state function from equation (2). Then by using equations (60) through (63) we obtain

$$\begin{aligned} -\frac{d}{dt} V^{(k+1)} &= L(x, u^{(k+1)}, t) \\ &\leq L(x, u^{(k+1)}, t) - \left(H(x, u^{(k+1)}, V_x^{(k)}, t) + V_t^{(k)} \right) \\ &= - V_t^{(k)} + \left[V_x^{(k)}, f(x, u^{(k+1)}, t) \right]_1 \\ &= -\frac{d}{dt} V^{(k)} \quad . \end{aligned} \tag{64}$$

The second line in equation (64) follows since equations (62) and (63) imply that

$$H(x, u^{(k+1)}, V_x^{(k)}, t) + V_t^{(k)} \leq 0$$

We are using the fact that the third line of equation (64) is the total derivative of $V^{(k)}$ with respect to t around the state function x as determined by $u^{(k+1)}$. By integrating the first and last terms in equation (64) on $(t, t_1]$ and using the fact that $V^{(k+1)}(t_1) = V^{(k)}(t_1) = 0$, we obtain the inequality

$$V^{(k+1)} \leq V^{(k)} \quad , \qquad k = 1, 2, \ldots \tag{65}$$

which proves that $\left\{ V^{(k)} \right\}$ is monotone decreasing.

Since $V^{(\infty)}$ also satisfies equations (62) and (63) and, therefore, for $k = 1, 2, \ldots$

$$H\left(x, u, V_x^{(\infty)}, t\right) + V_t^{(\infty)} \leqq H\left(x, u, V_x^{(k)}, t\right) + V_t^{(k)} \quad , \tag{66}$$

the limit $V^{(\infty)}$ of the sequence $\left\{ V^{(k)} \right\}$ can be established by the same reasoning leading to equation (65). It is more difficult to show, in general, that $V^{(\infty)} = \hat{V}$, the minimum cost functional. Leak and Liu [35] proved it by assuming continuity of the operator that transforms the functional $V^{(k)}$ into $V^{(k+1)}$. In the following section, we consider the special case of linear dynamical constraints and quadratic cost functional (the state regulator problem described in the section, State Regulator Problem, Chapter 2). It is shown that $V^{(k)}$ may be expressed in terms of a positive definite matrix $P^{(k)}$, which satisfies a sequence of linear differential equations that converges as $k \to \infty$ to the matrix Riccati equation (23). Since it has been shown that equation (23) possesses a unique positive definite solution [2], existence of the optimal control for the original problem implies that $\left\{ V^{(k)} \right\}$ converges to the minimum cost functional. In the more general case, however, explicit conditions which insure that $\lim_{k \to \infty} V^{(k)} = \hat{V}$ are not yet available.

At this point we shall assume that $L(x, u, t)$ is positive semidefinite[6]. In addition, suppose $t_1 \to + \infty$. Then we may guarantee the stability of equation (2) resulting from the approximation sequence $\left\{ u^{(k)} \right\}$ as follows. Assume that there exists an initial control function $u^{(1)} \in U$ such that equation (2) is uniformly asymptotically stable [39, p. 395]. Then it is shown in Reference 2 that $V^{(1)}$ is a Lyapunov functional for equation (2) with $u = u^{(1)}$. Suppose that we have obtained the kth approximation $u^{(k)}$, and the corresponding cost functional $V^{(k)}$ is a Lyapunov functional for equation (2). Then by applying the previous method of choosing $u^{(k+1)}$ by minimizing $H(x, u, V_x^{(k)}, t)$, we can show that $u^{(k+1)}$ is also a "stable"

6. A functional $L : X \times U \times T \to \mathbb{R}$ is positive semidefinite if $L(x, u, t) \geqq 0$ for all $x \in X$, $u \in U$, $t \in T$ with equality if, and only if, $x = \theta$, $u = \theta$.

control. By assumption $L(x, u^{(k+1)}, t)$ is positive semidefinite, which implies that $V^{(k+1)}$ is positive definite. Moreover, $\frac{d}{dt}V^{(k+1)} = -L(x, u^{(k+1)}, t)$, which is negative semidefinite. Thus $V^{(k+1)}$ is a Lyapunov functional for equation (2) with $u = u^{(k+1)}$, which implies that equation (2) is uniformly asymptotically stable.

The preceding development may be summarized by the following:
Theorem: Given the cost functional [eq. (1)] and dynamical system constraints [eq. (2)], let us define the mappings $V^{(k)} : X \times U \times T \to \mathbb{R}$ by $V^{(k)} = \int_t^{t_1} L\left(x, u^{(k)}, s\right) ds$ and $H : X \times U \times X^* \times T \to \mathbb{R}$ by $H(x, u, V_x^{(k)}, t) = L(x, u, t) + \left[V_x^{(k)}, f(x, u, t)\right]$. Then given an initial $u^{(1)}$, a sequence of control function approximations $\left\{u^{(k)}\right\}_1$ may be generated by minimizing $H(x, u, V_x^{(k-1)}, t)$, $k = 2, 3, \ldots$ over all $u \in U$, and this sequence possesses the following property of monotone approximation:

$$\hat{V} \leqq V^{(\infty)} \leqq V^{(k+1)} \leqq V^{(k)}$$

for $k = 1, 2, \ldots$ where $V^{(\infty)} = \lim_{k \to \infty} V^{(k)}$ exists, and $\hat{V}$ is the minimum cost functional corresponding to the optimal control function. Moreover, if $L(x, u, t)$ is positive definite, $t_1 \to \infty$, and if there exists an initial $u^{(1)}$ such that equation (2) is uniformly asymptotically stable, the dynamical system, equation (2) is uniformly asymptotically stable for $u = u^{(k)}$, $k = 2, 3, \ldots$

The State Regulator Problem Revisited, I

As an application of the method for monotone Approximation in U, let us consider a special case of linear dynamical system constraints

$$\dot{x} = Ax + Bu \quad , \quad x(t_0) = c \tag{67}$$

and the quadratic cost functional

$$J(u, x) = \frac{1}{2}\left(\left[x, Qx\right] + \left[u, Ru\right]\right) \tag{68}$$

where A, B, Q, R are matrices whose elements are functions on T and satisfy conditions given in the section, State Regulator Problem, Chapter 2. Suppose that $u^{(1)}$ is an initial control function. Then by the previous theorem, a minimizing sequence for J can be obtained by selecting $u = u^{(k+1)}$ in U such that the Hamilton functional

$$H(x, u, V_x^{(k)}, t) = \frac{1}{2}\left(\left[x, Qx\right]_1 + \left[u, Ru\right]_1\right) + \left[V_x^{(k)}, Ax + Bu\right]_1 \tag{69}$$

is minimized. Since U is an open set, a weak minimum of H can be found by equating its gradient with respect to u with zero. Hence,

$$\begin{aligned} \theta &= H_u(x, u, V_x^{(k)}, t) \\ &= B^* V_x^{(k)} + Ru^{(k+1)} \end{aligned} \tag{70}$$

and since $R > 0$ we may solve equation (70) for $u^{(k+1)} = -R^{-1}B^* V_x^{(k)}$. To evaluate $V_x^{(k)}$ in terms of known quantities, we shall use an approach similar to that described in Reference 2, which originally motivated the definition of $V^{(k)}$. Assume that $V^{(k)}$ can be represented as $V^{(k)} = \frac{1}{2}\left[x, P^{(k)}x\right]_1$ where $P^{(k)}$ is a positive definite, symmetric, $n \times n$ matrix whose elements are functions on T. Then the kth approximation to the optimal control function is

$$u^{(k+1)} = -R^{-1} B^* P^{(k)} x \qquad , \qquad k = 1, 2, 3, \ldots \quad . \tag{71}$$

Using equation (68) and changing the lower limit of integration to $t \epsilon [t_0, t_1]$, $V^{(k)}$ may be written as

$$\frac{1}{2}\left[x, P^{(k)}x\right]_1 = \frac{1}{2}\int_t^{t_1}\left(\left[x, Qx\right]_1 + \left[-R^{-1} B^* P^{(k-1)} x, -B^* P^{(k-1)} x\right]_1\right) ds \quad .$$

Differentiating the last equation with respect to t and substituting $A - BR^{-1} B^* P^{(k-1)} x$ for $\dot{x}$, we obtain the following linear matrix differential equation:

$$-\dot{P}^{(k)} = P^{(k)} A^{(k)} + \left(A^{(k)}\right)^* P^{(k)} + Q^{(k)} \quad , \quad P^{(k)}(t_1) = \Theta \tag{72}$$

where

$$A^{(k)} = A - BR^{-1} B^* P^{(k-1)} \tag{73}$$

$$Q^{(k)} = Q + P^{(k-1)} BR^{-1} B^* P^{(k-1)} \tag{74}$$

Equations (71) through (74) define a recursive solution for the optimal control functions which may be summarized in the following algorithm:

1. Choose an initial control function $u^{(1)}$ and determine $P^{(0)}$; let $k = 1$.

2. Compute $A^{(k)}$, $Q^{(k)}$ from equations (73) and (74).

3. Integrate $-\dot{P}^{(k)} = P^{(k)} A^{(k)} + (A^{(k)})^* P^{(k)} + Q^{(k)}$ backward from $P^{(k)}(t_1)$, and store the function $P^{(k)}$.

4. Integrate $\dot{x} = A^{(k)} x$ forward from $x(t_0)$, and store the function $x^{(k)}$.

5. Compute the next control function according to $u^{(k+1)} = -R^{-1} B^* P^{(k)} x^{(k)}$.

6. Let $k \rightarrow k+1$ and repeat steps 2 through 4 until $|| B^* V_x^{(k)} + Ru^{(k+1)} ||^2$ is less than a predetermined positive number.

In general, step 3 involves integrating a time-varying linear differential equation.

If a closed-loop control function is desired, step 4 may be replaced by computation of the feedback gain matrix $R^{-1}B^* P^{(k)}$. For the systems of order $n \geq 3$, finding the required initial matrix $P^{(0)}$ is nontrivial. The pole-shifting technique used in References 40 and 41 is an effective method for constant coefficient systems. However, a general method for selecting $P^{(0)}$ is not available at the present time.

The greatest advantage of the algorithm just presented is that equation (72) specifies a linear matrix differential equation that must be solved to compute the new control function $u^{(k+1)}$. Methods for solving equation (72) are given in Appendix A. (In contrast, the standard approach given in the section, State Regulator Problem, Chapter 2, involves solving a nonlinear matrix Riccati equation for which a solution may not even exist.) The algorithm requires a modest amount of computer instructions and computation time. It was recently shown by Kleinman and Athans [36] that $\left\{ V^{(k)} \right\}$ also possesses quadratic convergence, whereas most other algorithms for computing the optimal control, such as ASP or Runge-Kutta integration of equation (72), display only linear convergence to the minimum cost functional. Although further testing of the algorithm with high-order linear time-varying systems having finite time interval $(t_1 < \infty)$ is needed, numerical studies by this author and application of the algorithm by others [36, 41, 40] have indicated that the method for monotone Approximation in U described in this section is the most efficient technique for the case of constant coefficient systems and infinite time interval.

EXAMPLE 6

Given the quadratic cost functional

$$J(u, x) = \frac{1}{2} \int_0^{t_1} (qx^2 + ru^2)\, dt \tag{75}$$

and the linear first-order system

$$\dot{x} = -ax + u \quad , \qquad x(0) = c \quad , \tag{76}$$

where $q \geq 0$, $r > 0$, and $t_1 \leq \infty$. The problem is to determine a minimizing sequence for J . Computation of an approximation to the closed-loop optimal

control function by the previous algorithms for monotone Approximation in U proceeds for the first iteration as follows:

1. Choose $u^{(1)} = 0$, which implies $P^{(0)} = 0$.

2. Compute $A^{(1)} = -a$, $Q^{(1)} = q$.

3. Integrate equation (72):

$$-\dot{P}^{(1)} = -2aP^{(1)} + q \quad , \qquad P^{(1)}(t_1) = 0 \quad ,$$

which has the solution,

$$P^{(1)} = \frac{q}{2a}\left(1 - \exp(-2a\,(t_1 - t))\right) \quad .$$

4. Update the control function:

$$u^{(2)} = -\frac{1}{r}\dot{P}^{(1)}x^{(1)}$$

$$= -\frac{q}{2ar}\left(1 - \exp(-2a\,(t_1 - t))\right)x^{(1)} \quad .$$

The second iteration requires integrating the time-varying linear equation

$$\dot{P}^{(2)} = -2A^{(2)}P^{(2)} - Q^{(2)} \quad , \qquad P^{(2)}(t_1) = 0 \quad ,$$

where

$$A^{(2)} = a + \frac{q}{2ar}\left(1 - \exp(-2a\,(t_1 - t))\right)$$

$$Q^{(2)} = q + \frac{q^2}{4a^2r}\left(1 - 2\exp(-2a\,(t_1 - t)) + \exp(-4a\,(t_1 - t))\right) \quad .$$

The solution to the last equation is

$$P^{(2)} = \sum_{n=0}^{\infty} P_n^{(2)} \exp(-2an\ (t_1 - t)) \quad ,$$

where

$$P_0^{(2)} = \frac{q(4a^2r + q)}{4a(q + 2a^2r)}$$

$$P_1^{(2)} = \frac{q\left(2aP_0^{(2)} - q\right)}{2a(4a^2r + q)}$$

$$P_2^{(2)} = \frac{q\left(4aP_1^{(2)} + q\right)}{4a(6a^2r + q)}$$

and

$$\sum_{n=0}^{\infty} P_n^{(2)} = 0 \quad .$$

Therefore, the resulting approximation to the optimal control function is

$$u^{(3)} = -\frac{1}{r} \sum_{n=0}^{\infty} P_n^{(2)} \exp(-2an(t_1 - t))\ x^{(2)} \quad .$$

Suppose that $t_1 \to +\infty$. Then if $a > 0$, a minimizing sequence of asymptotically stable functions for J is as follows:

$$u^{(1)} = 0 \tag{77}$$

$$u^{(2)} = -\frac{1}{2a} x^{(1)} \tag{78}$$

$$u^{(3)} = -\frac{4a^2 + 1}{4a(2a^2 + 1)} x^{(2)} \quad . \tag{79}$$

EXAMPLE 7

Given the cost functional

$$J(u, x) = \frac{1}{2} \int_0^\infty (x_1^2 + u^2)\, dt \quad , \tag{80}$$

and the linear system

$$\dot{x}_1 = x_2 \quad , \qquad x_1(0) = c_1 \tag{81}$$

$$\dot{x}_2 = -x_1 - a x_2 + u \quad , \qquad x_2(0) = c_2 \quad , \tag{82}$$

where $a > 0$, we apply the algorithm of this section to find a minimizing sequence for J . Let us define the state function $x = (x_1, x_2)$ and the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -a \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Q = \mathrm{diag}(1, 0) \qquad R = 1 \quad .$$

A minimizing sequence $u^{(k)}$ is obtained from

$$u^{(k+1)} = -R^{-1} B^* \overline{P}^{(k)} x^{(k)} \quad . \tag{83}$$

The constant matrix $\overline{P}^{(k)}$ is the solution of the linear matrix equation

$$\overline{P}^{(k)} A^{(k)} + \left(A^{(k)}\right)^* \overline{P}^{(k)} + Q^{(k)} = \Theta \quad , \tag{84}$$

where

$$A^{(k)} = A - BR^{-1}B^{*}\overline{P}^{(k-1)}$$

$$Q^{(k)} = Q + \overline{P}^{(k-1)}BR^{-1}B^{*}\overline{P}^{(k-1)} \quad .$$

Since $a > 0$ we may choose $u^{(1)} = 0$; hence $\overline{P}^{(0)} = \Theta$, and from equations (83) and (84) the result of the first three iterations is

$$u^{(1)} = 0 \tag{85}$$

$$u^{(2)} = -\frac{1}{2}x_1 - \frac{1}{2a}x_2 \tag{86}$$

$$u^{(3)} = -\frac{5}{12}x_1 - \frac{10a^2 + 3}{12a(2a^2 + 1)}x_2 \quad . \tag{87}$$

EXAMPLE 8

Given the cost functional

$$J(u, x) = \frac{1}{2}\int_0^{\infty} (q_{11}x_1^2 + q_{22}x_2^2 + ru^2)\,dt \quad , \tag{88}$$

where $q_{11} \geq 0$, $q_{22} \geq 0$, $r > 0$, and the linear third-order system

$$\dot{x}_1 = x_2 \tag{89}$$

$$\dot{x}_2 = x_3 \tag{90}$$

$$\dot{x}_3 = -0.1x_1 - 1.2x_2 - 2.1x_3 + u \tag{91}$$

determine a minimizing sequence for J. Applying the previous algorithm, we define the state function $x(t) \in \mathbb{R}^3$, control function $u(t) \in \mathbb{R}$, and the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.1 & -1.2 & -2.1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Q = \operatorname{diag}(q_{11}, q_{22}, 0) \qquad R = r \quad .$$

The problem was programmed for the digital computer, and representative results are shown below. In each case the closed-loop approximation to the optimal control is $u = -\sum_{n=1}^{3} k_n x$.

q_{11}	q_{22}	r	k_1	k_2	k_3	min cost
1	0	0.5	1.3177	1.8090	0.7334	2.0730
1	0	1	0.9050	1.3215	0.5558	2.4141
1	0	2	0.6141	0.9461	0.4104	2.8252
1	1	0.5	1.3177	2.1787	0.8610	2.3351
1	1	2	0.6141	1.0756	0.4615	3.0103

In each case, four iterations were required to obtain convergence of the algorithm. As a check on the computer results, the exact solution was obtained for $q_{11} = 1$, $q_{22} = 0$, $r = 1$ using a spectral factorization method devised by Puri and described by Puri and Gruver [34]. For this case the exact solution to three-decimal accuracy is:

$$k_1 = 0.905 \qquad k_2 = 1.32 \qquad k_3 = 0.556 \quad ,$$

which compares favorably with the approximation shown above.

Further examples of the latter approximation method to the state regulator problem have been given recently by McLane [41] and Yu , Vongsuriya, and Wedman [40]. In the latter reference, the authors treat the optimization of a power system involving an eighth-order constant coefficient system with two control inputs. Analog simulation was used to check the computer solution, and the results appear favorable.

CHAPTER 4

APPROXIMATION IN X

In this chapter, we formulate a new class of computational methods for the solution of the optimal control problem stated in the section, Definition of the Optimal Control Problem, Chapter 2. Given the cost functional [eq. (1)] subject to dynamical system constraints [eq. (2)], we seek a minimizing sequence for J by Approximation in X. Central to the theory is a representation for the mapping from X into U. A new second variation algorithm is developed for the nonlinear control problem. In contrast to previous second variation methods, we avoid the use of adjoint functions and the Hamiltonian formulation. At each control function update, the control increment is obtained in terms of the state increment in a least-squares sense. An accessory boundary value problem is obtained, the solution of which specifies the control function update. As an important special case, the state regulator problem is treated, and several computational examples illustrate the use of the technique.

The Inverse Mapping

Let us assume that the dynamical system [eq. (2)] may be linearized about an initial control function $u = u^{(k)}$ in U and corresponding state function $x = x^{(k)}$ in X. Hence,

$$\delta\dot{x}(t) = f_x(t)\,\delta x(t) + f_u(t)\,\delta u(t) \quad , \quad \delta x(t_0) = \theta \quad , \tag{92}$$

where $\delta x = \delta x^{(k)}$ in X and $\delta u = \delta u^{(k)}$ in U are increments at x and u, respectively. For brevity, the arguments x, u of f and superscripts denoting iteration are dropped.

If the matrix $f_u(t)$ were invertible and nonzero for all $t \in T$, we could solve for $\delta u(t)$ simply by premultiplying equation (92) by $f_u^{-1}(t)$. In general, however, $f_u(t)$ is not invertible and the pseudo inverse [42],

$$f_u^{+}(t) = \left(f_u^{*}(t)\, f_u(t)\right)^{-1} f_u^{*}(t) \quad ,$$

yields the least-squares solution for $\delta u(t)$ [in terms of $\delta x(t)$] to the "overdetermined" system [eq. (92)], which has more equations than unknowns. If $f_u(t)$ is invertible, the psuedo inverse reduces to the usual inverse.

In general, $f_u(t)$ may be zero for some $t \in T$. If $f_u^*(t)\, f_u(t) \geq 0$, it is convenient to define the ϵ-pseudo inverse of $f_u(t)$ as

$$g_\epsilon(t) = \left(f_u^*(t)\, f_u(t) + \epsilon I \right)^{-1} f_u^*(t) \quad , \tag{93}$$

where ϵ is a small positive (real) number. The parameter ϵ insures that the indicated inverse in equation (93) exists. If $f_u^*(t)\, f_u(t) > 0$ for all $t \in T$, we shall choose $\epsilon = 0$.

Based on the above remarks, equation (92) may be solved for the control increment in terms of the state increment as

$$\delta u(t) = g_\epsilon(t)\ (\mathcal{L}\, \delta x)(t) \quad , \tag{94}$$

where

$$(\mathcal{L}\, \delta x)(t) = \delta \dot{x}(t) - f_x(t)\, \delta x(t) \quad . \tag{95}$$

At this point of the development, we must restrict the function δx to be at least continuous and possess a derivative in X. This condition is satisfied by assumption 1 of the section, Basic Assumptions, Chapter 2. Let us define the residual

$$(\rho(\delta u))(t) = (\mathcal{L}\, \delta x)(t) - f_u(t)\, \delta u(t) \quad .$$

The functional $||\rho(\delta u)||^2$ is a measure of the error between the acutal and desired solution to equation (92). For example, if f_u is invertible, then $||\rho(\delta u)||^2 = 0$. It is possible for $||\rho(\delta u)||^2 = 0$ even if f_u is singular. An important example of the latter is a linear time-varying system having a single control function input. Use of the usual pseudo inverse ($\epsilon = 0$) in determining the control increment results in minimizing the functional $||\rho(\delta u)||^2$. The following theorem specifies the function that is minimized by the use of the ϵ-pseudo inverse:

Theorem: Given equation (92), suppose that the function $\mathcal{L}\delta x$ and the matrix operator f_u are specified in advance, where $\mathcal{L}\delta x \in X$, $\delta u \in U$, and f_u induces a bounded linear mapping from U to X. Let ϵ be chosen such that $\det(f_u^* f_u + \epsilon I) \neq 0$ for all $t \in T$ and let $||\rho_\epsilon(\delta u)||^2$ be defined by

$$|| \rho_\epsilon (\delta u) ||^2 = || f_u \delta u - \mathcal{L}\delta x ||^2 + \epsilon || \delta u ||^2 \quad . \tag{96}$$

Then,

$$|| \rho_\epsilon (\hat{\delta u}) ||^2 \leq || \rho_\epsilon (\delta u) ||^2 \quad \text{for all} \quad \delta u \in U \quad ,$$

where

$$\hat{\delta u} = (f_u^* f_u + \epsilon I)^{-1} f_u^* \mathcal{L}\delta x \quad .$$

Proof: $||\rho_\epsilon(\delta u)||^2 = || f_u \delta u - \mathcal{L}\delta x ||^2 + \epsilon || \delta u ||^2$

$$= \left[f_u \delta u - \mathcal{L}\delta x, f_u \delta u - \mathcal{L}\delta x \right] + \epsilon \left[\delta u, \delta u \right]$$

$$= \left[f_u^* f_u \delta u, \delta u \right] + \left[\mathcal{L}\delta x, \mathcal{L}\delta x \right]$$

$$- \left[f_u \delta u, \mathcal{L}\delta x \right] - \left[\mathcal{L}\delta x, f_u \delta u \right] + \epsilon \left[\delta u, \delta u \right]$$

$$= \left[f_u^* f_u \delta u, \delta u \right] - 2 \left[\delta u, (f_u^* f_u + \epsilon I) \delta v \right]$$

$$+ || \mathcal{L}\delta x ||^2 + \epsilon \left[\delta u, \delta u \right]$$

where

$$\delta v = (f_u^* f_u + \epsilon I)^{-1} f_u^* \mathcal{L}\delta x \quad .$$

Finally, by collecting terms, $||\rho_\epsilon(\delta u)||^2$ may be written as

$$||\rho_\epsilon(\delta u)||^2 = \left[(f_u^* f_u + \epsilon I)(\delta u - \delta v), (\delta u - \delta v)\right] - \left[(f_u^* f_u + \epsilon I)\delta v, \delta v\right] + ||\mathcal{L}\delta x||^2 . \quad (98)$$

By assumption $||\mathcal{L}\delta x||^2$, a fixed number, is finite. Since $f_u^* f_u + \epsilon I > 0$, $||\rho_\epsilon(\delta u)||^2$ is minimized by $\delta u = \delta v$. QED

In summary, we have obtained in inverse mapping in the sense that the control increment may be expressed in terms of the state increment by a linear mapping of X into U as defined by equation (94). Under certain conditions as specified in the last theorem, use of the ϵ-pseudo inverse results in minimizing the square of the norm of the residual plus an additional term involving the control increment "energy." If $f_u(t) \neq \theta$ for all $t \in T$, we shall choose $\epsilon = 0$.

A Second Variation Method fo Approximation in X

In the section, Descent Algorithms for Hilbert Space, Chapter 3, we described the well-known "direct" second variation algorithm based on minimizing, in the control space U, the sum of the first and second variations of the cost functional. In this section, we exploit the inverse mapping, defined in the preceding section, for the purpose of developing a new second variation algorithm that is based on performing the minimization in the state space X.

Suppose the terminal time is specified, and the terminal state is unconstrained. Then let the cost functional

$$J(u, x) = \int_{t_0}^{t_1} L(x, u, t)\, dt \quad (99)$$

be approximated by a constant term J_0 plus the sum of the first and second Frechet differentials of J at $u = u^{(k)}$ and $x = x^{(k)}$, which satisfy equation (1) with increments $= \delta u^{(k)}$ and $\delta x = \delta x^{(k)}$, respectively,

which satisfy equation (92). This sum is computed by taking the following Gateaux differentials:

$$E(\delta u, \delta x) = \frac{\partial}{\partial \alpha_1}\left(\int_{t_0}^{t_1} L(x + \alpha_1 \delta x,\ u + \alpha_1 \delta u,\ t)\ dt\right)\Bigg|_{\alpha_1 = 0}$$
$$+ \frac{1}{2}\frac{\partial^2}{\partial x_1 \partial x_2}\left(\int_{t_0}^{t_1} L(x + \alpha_1 \delta x + \alpha_2 \delta^2 x,\ u + \alpha_1 \delta u + \alpha_2 \delta^2 u,\ t)\ dt\right)\Bigg|_{\substack{\alpha_1 = 0 \\ \alpha_2 = 0}}$$

where $\delta^2 u$ and $\delta^2 x$ are second-order increments at u and x, respectively. For brevity, arguments of L and superscripts denoting iteration are dropped. By expanding L in a second-order Taylor series and performing the indicated differentiation, E may be rewritten using the Hilbert space inner product notation as follows:

$$E(\delta u, \delta x) = \left[L_x, \delta x\right] + \left[L_u, \delta u\right] + \frac{1}{2}\left[\delta x, L_{xx}\, \delta x\right]$$
$$+ \frac{1}{2}\left[\delta x, L_{xu}\, \delta u\right] + \frac{1}{2}\left[\delta u, L_{ux}\, \delta x\right] \qquad (100)$$
$$+ \frac{1}{2}\left[\delta u, L_{uu}\, \delta u\right] .$$

The inverse mapping defined by equation (94) is now used to eliminate explicit dependence in equation (100) with respect to δu. Thus we obtain

$$
\begin{aligned}
E(\cdot, \delta x) = &\left[L_x, \delta x \right] + \left[L_u, g_\epsilon (\delta \dot{x} - f_x \delta x \right] \\
&+ \frac{1}{2} \left[\delta x, L_{xx} \delta x \right] + \frac{1}{2} \left[\delta x, L_{xu} g_\epsilon (\delta \dot{x} - f_x \delta x) \right] \\
&+ \frac{1}{2} \left[g_\epsilon (\delta \dot{x} - f_x \delta x), L_{ux} \delta x \right] \\
&+ \frac{1}{2} \left[g_\epsilon (\delta \dot{x} - f_x \delta x), L_{uu} g_\epsilon (\delta \dot{x} - f_x \delta x) \right] \\
= &\left[L_x, \delta x \right] + \left[L_u, g_\epsilon \, \delta \dot{x} \right] - \left[L_u, g_\epsilon \, f_x \delta x \right] \qquad (101) \\
&+ \frac{1}{2} \left[\delta x, L_{xx} \delta x \right] + \frac{1}{2} \left[\delta x, L_{xu} g_\epsilon \, \delta \dot{x} \right] \\
&- \frac{1}{2} \left[\delta x, L_{xu} g_\epsilon \, f_x \delta x \right] + \frac{1}{2} \left[g_\epsilon \, \delta \dot{x}, L_{ux} \delta x \right] \\
&- \frac{1}{2} \left[g_\epsilon \, f_x \delta x, L_{ux} \delta x \right] + \frac{1}{2} \left[g_\epsilon \, \delta \dot{x}, L_{uu} g_\epsilon \, \delta \dot{x} \right] \\
&- \frac{1}{2} \left[g_\epsilon \, \delta \dot{x}, L_{uu} g_\epsilon \, f_x \delta x \right] - \frac{1}{2} \left[g_\epsilon \, f_x \delta x, L_{uu} g_\epsilon \, \delta \dot{x} \right] \\
&+ \frac{1}{2} \left[g_\epsilon \, f_x \delta x, L_{uu} g_\epsilon \, f_x \delta x \right] \quad .
\end{aligned}
$$

The development of the second variation method for Approximation in X is based on finding a state increment δx in X which minimizes E. Since E in equation (101) is an explicit function δx, this minimization may be performed without the use of adjoint functions and the Hamiltonian formulation used in the section, Descent Algorithms for Hilbert Space, Chapter 3. A necessary condition for a weak minimum of E over the state space X is that its first Frechet differential at δx with increment ξ vanish. Since it is assumed that the Frechet differential of E exists, it may be computed by using the Gateaux differential. The latter is linear in ξ and can be written in the form $dE(\delta x; \xi) = \left[E_{\delta x}, \xi \right]$, where $E_{\delta x}$ is called the gradient of E with respect to δx. Rather than compute the Gateaux differential as in the

section, Necessary Conditions, Chapter 2, we shall take the following alternative approach and (1) simply compute $E_{\delta x}$;[7] (2) use the definition of adjoint from the section, Terminology, Chapter 1, which permits moving an operator from one side of an inner product to the other side; (3) integrate terms involving $\delta\dot{x}$ by parts, which results in "boundary terms" involving $\delta x(t_1)$ and $\delta\dot{x}(t_1)$ only, since $\delta x(t_0) = \theta$. For example

$$\left[L_u, g_\epsilon \delta\dot{x}\right] = \left[g_\epsilon^* L_u, \delta\dot{x}\right]$$

$$= \left[(g_\epsilon^* L_u)\big|_{t_1}, \delta x(t_1)\right]_1 - \left[\mathscr{D}_t(g_\epsilon^* L_u), \delta x\right],$$

where $\mathscr{D}_t = I\frac{d}{dt}$ is defined as the matrix differentiation operator on $C^1(T)$. Then if we define

$$h = g_\epsilon L_{uu} g_\epsilon \quad ,$$

the differential of E is computed as follows:

$$\begin{aligned} dE(\delta x; \xi) = \Big[& (L_x - \mathscr{D}_t(g_\epsilon^* L_u) - f_x^* g_\epsilon^* L_u \\ & + L_{xx}\delta x + L_{xu} g_\epsilon \delta\dot{x} - \mathscr{D}_t((L_{xu} g_\epsilon)^* \delta x) \\ & - (L_{xu} g_\epsilon f_x + (L_{xu} g_\epsilon f_x)^*)\delta x \\ & - \mathscr{D}_t(h\,\delta\dot{x}) - \frac{1}{2} f_x^* h\,\delta\dot{x} \\ & + \frac{1}{2}\mathscr{D}_t(h f_x \delta x) - \frac{1}{2} f_x^* h\,\delta\dot{x} \\ & + \frac{1}{2}\mathscr{D}_t(h f_x \delta x) + f_x^* h f_x \delta x, \xi \Big] \\ & + \Big[(g_\epsilon^* L_u)\big|_{t_1} + (g_\epsilon^* L_{xu} - L_{xu} g_\epsilon)\big|_{t_1} \delta x(t_1) \\ & + (h f_x)\big|_{t_1} \delta\dot{x}(t_1), \xi(t_1)\Big]_1 \quad . \end{aligned}$$

7. Note: The gradients of $\left[x, Ay\right]_1$ and $\left[y, Ax\right]$ with respect to x are Ay and A^*y, respectively.

By collecting terms involving δx and its derivatives, $dE(\delta x;\xi)$ may be written as

$$dE(\delta x;\xi) = \Big[L_x - \mathscr{D}_t (g_\epsilon^* L_u) - f_x^* g_\epsilon L_u$$
$$+ (L_{xx} - \mathscr{D}_t (L_{xu} g_\epsilon)^* - (L_{xu} g_\epsilon f_x + (L_{xu} g_\epsilon f_x)^*)$$
$$- \mathscr{D}_t (h f_x) + f_x^* h f_x) \delta x + ((L_{xu} g_\epsilon - (L_{xu} g_\epsilon)^*)$$
$$- \mathscr{D}_t h - f_x^* h + h f_x) \delta \dot{x}$$
$$- h \delta \ddot{x}, \xi \Big] + \text{boundary terms} \quad . \tag{102}$$

Suppose we arbitrarily set the boundary terms to zero. Then by the Euler-Lagrange Lemma in the section, Necessary Conditions, $dE(\delta x;\ \xi) = \theta$ implies that the state increment $\delta x = \delta x^{(k)}$ satisfies the following linear second-order differential equation:

$$h \delta \ddot{x} = F \delta \dot{x} + G \delta x + k \quad , \tag{103}$$

where[8]

$$h = g_\epsilon^* L_{uu} g_\epsilon \tag{104}$$

$$F = L_{xu} g_\epsilon - g_\epsilon^* L_{xu}^* + h f_x - f_x^* h - \mathscr{D}_t h \tag{105}$$

$$G = L_{xx} - L_{xu} g_\epsilon f_x - f_x^* g_\epsilon^* L_{xu}^* + f_x^* h f_x - \mathscr{D}_t (g_\epsilon^* L_{xu}^* + h f_x) \tag{106}$$

$$k = L_x - f_x^* g_\epsilon^* L_u - \mathscr{D}_t (g_\epsilon^* L_u) \tag{107}$$

8. Recall that $g_\epsilon = (f_u^* f_u + \epsilon I)^{-1} f_u^*$ and $\mathscr{D}_t = I \frac{d}{dt}$.

and subject to the following set of "self-adjoint" boundary conditions:

$$\delta x(t_0) = \theta \tag{108}$$

$$h(t_1)\ \delta\dot{x}(t_1) = N(t_1)\ \delta x(t_1) + r(t_1) \quad , \tag{109}$$

where

$$N = h f_x - L_{xu}\, g_\epsilon + g_\epsilon^* L_{xu} \tag{110}$$

$$r = g_\epsilon^* L_u \tag{111}$$

To compute the control increment $\delta u = \delta u^{(k)}$ from equation (94), we must solve equations (103) through (111). Because of the similarity of the latter two-point boundary value problem to equations (49) and (50) of the section, Descent Algorithms for Hilbert Space, Chapter 3 (in which we obtained second-variation necessary conditions for Approximation in U), we shall refer to equations (103) through (111) as the "Accessory Problem." Computational solution of the Accessory Problem is discussed in Appendix B. The solution for the special case of the state regulator problem is given in the section, The State Regulator Problem Revisited, II. Sufficient conditions for the existence of a solution to the Accessory Problem, and hence existence of a weak minimum of E , follow from the Calculus of Variations and involve the classical strong Legendre and the Jacobi conditions [6, 7].

EXAMPLE 9

Consider the cost functional

$$J(u, x) = \frac{1}{2} \int_0^1 \left(x + xu + \frac{1}{2} u^2 \right) dt \tag{112}$$

and the first-order linear dynamical system constraint

$$\dot{x} = u \quad , \quad x(0) = 1 \quad . \tag{113}$$

For this simple case

$$L_x = u + 1 \qquad L_{xx} = 0 \qquad f_x = 0$$

$$L_u = x + u \qquad L_{xu} = L_{xu} = 1 \qquad f_u = 1 \quad .$$

$$L_{uu} = 1$$

Since $(f_u^* f_u)(t) > 0$, $t \in [0, 1]$, we may choose $\epsilon = 0$. Then $g = 1$ and

$$h = 1 \qquad G = 0 \qquad N = 0$$

$$F = 1 \qquad k = u + 1 - \dot{x} - \dot{u} \qquad r = x + u \quad .$$

Therefore, equation (103) becomes

$$\delta\ddot{x} = u + 1 - \dot{x} - \dot{u} \tag{103'}$$

subject to equations (108) and (109), the "self adjoint" boundary conditions

$$\delta x(0) = 0 \tag{108'}$$

$$\delta\dot{x}(1) = u(1) + x(1) \quad . \tag{109'}$$

A general algorithm for the computational solution of problems of this kind is given in the next section.

The Basic Computational Algorithm

The steps in computing optimal control functions by the methods described in the sections, The Inverse Mapping, and a Second Variation Method for Approximation in X, may be summarized in the following algorithm:

1. Choose an initial control function $u^{(0)}$; let $k = 0$.

2. Integrate $\dot{x} = f(x, u, t)$ forward from $x(t_0)$ and store the functions $x^{(k)}$, $\dot{x}^{(k)}$.

3. Compute the following partial-derivative matrices evaluated at $x^{(k)}$ and $u^{(k)}$: f_x, f_u, L_u, L_x, L_{ux}, L_{xu}, L_{uu}, L_{xx}.

4. Solve the Accessory Problem, equations (103) through (111), for $\delta x^{(k)}$, $\delta \dot{x}^{(k)}$ (see Appendix B).

5. Compute the control function increment $\delta u^{(k)} = g_{\epsilon}(\delta \dot{x}^{(k)} - f_x \delta x^{(k)})$.

6. Update the control function according to $u^{(k+1)} = u^{(k)} + \alpha^{(k)} \delta u^{(k)}$, where $\alpha^{(k)} \leq 1$ is a constant.

7. Let $k \rightarrow k+1$ and repeat steps 2 through 6 until $|J(u^{(k)}, x^{(k)}) - J(u^{(k+1)}, x^{(k+1)})|$ is less than a predetermined positive number.

As was mentioned for the second variation algorithm described in the section, Descent Algorithms for Hilbert Space, Chapter 3, the full increment $\delta u^{(k)}$ should not be used, and the constant $\alpha^{(k)}$ is included in step 6 to provide stable descent.

EXAMPLE 10

As an example of the second variation algorithm of this section consider the problem introduced in Example 9 of minimizing the cost functional

$$J(u, x) = \int_0^1 \left(x + xu + \frac{1}{2} u^2\right) dt$$

subject to a first-order linear system

$$\dot{x} = u \quad , \quad x(0) = 1 \quad .$$

It was shown in the previous example that the Accessory Problem is

$$\delta\ddot{x} = u + 1 - \dot{u} - \dot{x} \qquad (103')$$

subject to "self adjoint" boundary conditions

$$\delta x(0) = 0 \qquad (108')$$

$$\delta\dot{x}(1) = u(1) + x(1) \quad . \qquad (109')$$

The steps in computing the optimal control are as follows:

1. Let $u^{(0)} = 0$

2. Integrating equation (113) yields $x^{(0)} = 0$.

3-4. Solve equation (103') by letting $\delta x = \frac{1}{2}t^2 + c_1 t + c_2$ and evaluating constants $c_1 = -\frac{5}{4}$, $c_2 = 0$ by using equations (108') and (109'):

(4.17'-4.18'):

$$\delta x = \frac{1}{2}t\left(t - \frac{5}{2}\right) \quad .$$

5. Compute the control increment from equation (94):

$$\delta u = t - \frac{5}{4}$$

6. Update the control function:

$$u^{(1)} = u^{(0)} + \delta u^{(0)} = t - \frac{5}{4} \quad .$$

In this example, J is a true quadratic, and the system is linear. Therefore, the differential is also linear in x and u , and the optimal control $u = u^{(1)}$ has been obtained in a single iteration, as may be verified by comparison with the solution given in Reference 43.

EXAMPLE 11

Next, we consider the minimization of the quadratic cost functional

$$J(u,x) = \frac{1}{2} \int_1^2 \left(x + \frac{1}{2} u^2 \right) dt$$

subject to a first-order nonlinear system

$$\dot{x} = -x^2 + u \quad , \qquad x(1) = 1 \quad .$$

The Accessory Problem for this example is

$$\begin{aligned} \delta\ddot{x} &= (4x^2 - 2\dot{x})\,\delta x + 1 + 2xu - \dot{u} \\ &= (6x^2 - 2u)\,\delta x + 1 + 2xu - \dot{u} \end{aligned}$$

subject to boundary conditions

$$\begin{aligned} \delta x(1) &= 0 \\ \delta\dot{x}(2) &= -2x(2)\,\delta x(2) + u(2) \end{aligned}$$

The steps in computing the first approximation to an optimal control are as follows:

1. Let $u^{(0)} = 0$.

2. Integrating $\dot{x} = -x^2$ yields $x^{(0)} = \frac{1}{t}$.

3-4. Solve the Accessory Problem,

$$\delta\dot{x} = \left(\frac{6}{t^2}\right) \delta x + 1$$

$$\delta x(1) = 0$$

$$\delta\dot{x}(2) = -\,\delta x(2)$$

The general solution to the latter equations may be obtained by integration and is follows:

$$\delta x = \frac{1}{10} t^3 + \frac{3}{20} t^{-2} - \frac{1}{4} t^2$$

5. Compute the control increment from equation (94):

$$\delta u = \frac{1}{10} t^2 - \frac{3}{5} t^{-3}$$

6. Update the control function:

$$u^{(1)} = u^{(0)} + \delta u^{(0)} = \frac{1}{10} t^2 - \frac{3}{5} t^{-3} \quad .$$

In this example involving a nonlinear system, we obtained a second-order linear, time-varying equation for the Accessory Problem which could be solved analytically in closed form. In general, the accessory problem must be solved numerically on a digital computer using the methods given in Appendix B.

The State Regulator Problem Revisited, II

Consider the state regulator problem of minimizing the quadratic cost functional

$$J(u, x) = \frac{1}{2} \left(\left[x, Qx \right] + \left[u, Ru \right] \right) \tag{114}$$

subject to linear dynamical system constraints

$$\dot{x} = Ax + Bu \quad , \quad x(t_0) = c \quad , \tag{115}$$

where Q, R, A, B are constant matrices on T , $Q \geq 0$, $R > 0$, and the pair (A, B) is completely controllable. A computational solution to this

problem is now formulated by Approximation in X , using the methods described in the sections, The Inverse Mapping, and A Second Variation Method for Approximation in X .

Since the cost functional is a true quadratic and the system is linear in x and u , the differential is also linear in x and u . Therefore, as shown below, equations (103) through (11) may be solved exactly in a single iteration. Consequently, we shall replace δu and δx by u and x , respectively, and derive the Accessory Problem for this special case. By means of the inverse mapping defined by equation (94), the cost functional may be expressed explicitly in terms of the state function. Since $B^* B > 0$ for all $t \in T$, the parameter ϵ may be chosen zero. Then the cost functional may be written as

$$J(\cdot, x) = \frac{1}{2}\left(\left[x, Qx\right] + \left[B^{+}\mathcal{L}x, RB^{+}\mathcal{L}x\right]\right) \tag{116}$$

where

$$B^{+} = (B^*B)^{-1} B^* \tag{117}$$

$$\mathcal{L}x = \dot{x} - Ax \quad . \tag{118}$$

A necessary condition for a weak minimum of J over the state space X is that its first Frechet differential at x with increment ξ vanish. By performing the same steps as used in equations (103) through (111), we obtain the following linear second-order differential equation

$$h\ddot{x} = (hA - A^* h)\,\dot{x} + (A^* h A + Q)\,x \quad , \tag{119}$$

where

$$h = (B^{+})^* R\, B^{+} \tag{120}$$

and subject to the "self adjoint" boundary conditions

$$x(t_0) = c \tag{121}$$

$$h\,\dot{x}(t_1) = hA\,x(t_1) \quad . \tag{122}$$

After solving equations (119) through (122) for x and $\dot{x}$, the optimal control function $\hat{u}$ may be recovered, using the inverse mapping, as

$$\hat{u} = B^{+}(\dot{x} - A x) \quad . \tag{123}$$

For the solution of the Accessory Problem, equations (119) through (122), it is convenient to consider the two cases: (1) $h > 0$; and (2) $h \geq 0$. When applicable, the solution for Case 1 is easier to implement. In the following, we shall obtain closed-loop control functions by using the Riccati transformation. In Appendix B, both open and closed-loop control functions are obtained for the more general problem of the section, Definition of the Optimal Control Problem, Chapter 2.

CASE 1: $h > 0$

Since $R > 0$, the condition $h > 0$ implies that the Euclidean dimension of the control space U and the state space X is equal; that is, dim U = dim X . Consider the Riccati transformation

$$\dot{x} = P_1 x \quad , \tag{124}$$

where P_1 is an $n \times n$ matrix whose elements are functions on T . By substituting in equation (119) for $\dot{x}$ and $\ddot{x}$ using equation (124), we find that P_1 satisfies the following matrix Riccati equation:

$$-h \dot{P}_1 = h P_1^2 + (A^{*}h - hA) P_1 - (A^{*}hA + Q) \quad , \quad P_1(t_1) = A \quad . \tag{125}$$

From equations (123), (124) and (117), the optimal control for this case is

$$\hat{u} = (B^{*}B)^{-1} B^{*}(P_1 - A) x \quad , \tag{126}$$

where P_1 is obtained by solving equation (125).

CASE 2. $h \geq 0$.

The condition that $h \geq 0$ implies that dim $U \leq$ dim X . Let equations (119) through (122) be rewritten as the following pair of first-order vector differential equations:

$$\dot{x}^a = A^{11} x^a + A^{12} x^b \tag{127}$$

$$\dot{x}^b = A^{21} x^a + A^{22} x^b \quad , \tag{128}$$

where

$$x^a = x = (x_1, \ldots, x_n) \quad , \quad x^b = \dot{x} = (x_{n+1}, \ldots, x_{2n})$$

$$x_{i+1} = \dot{x}_i \ , \ i = 1, 2, \ldots, 2n - 1$$

with boundary conditions

$$x^a(t_0) = c \tag{129}$$

$$C\, x^a(t_1) = x^b(t_1) \quad . \tag{130}$$

Matrices A^{ij}, C are $n \times n$ with elements that are constant on T. Consider the Riccati transformation

$$x^b = P_2 x^a \quad , \tag{131}$$

where P_2 is $n \times n$ and has elements that are functions on T. By using equation (131) in equations (127) through (130) we obtain the following matrix Riccati equation:

$$-\dot{P}_2 = P_2 A^{11} - A^{22} P_2 + P_2 A^{12} P_2 - A^{21} \quad , \quad P_2(t_1) = C \quad . \tag{132}$$

Having obtained P_2 by solving equation (132), the closed-loop optimal control for this case is obtained from equations (123), (131), and (117) as

$$\hat{u} = (B^*B)^{-1} B^*(A^{11} + A^{12} P_2 - A)\, x \quad . \tag{133}$$

EXAMPLE 12

Consider the minimization of

$$J(u, x) = \int_0^1 \left(x^2 + \frac{1}{2} u^2\right) dt \qquad (134)$$

subject to the first-order linear system

$$\dot{x} = -\frac{1}{2} x + u \quad , \quad x(0) = 1 \quad . \qquad (135)$$

Using the methods described for the state regulator problem of this section, the Accessory Problem is

$$\ddot{x} = \frac{9}{4} x \qquad (136)$$

$$x(0) = 1 \qquad (137)$$

$$\dot{x}(1) = -\frac{1}{2} x(1) \quad . \qquad (138)$$

Since $h = 1$ is positive, Case 1 of this section applies. Solution of equations (136) through (138) by the Riccati transformation $x = P(t)x$ results in the Riccati equation

$$-\dot{P} = P^2 - \frac{9}{4} \quad , \quad P(1) = -\frac{1}{2} \quad . \qquad (139)$$

The latter equation may be integrated backward by replacing the independent variable t by $-s$ and letting $\frac{dP}{dt} = -\frac{dP}{ds}$. Then by separating variables and integrating, we obtain

$$P = \frac{3}{2} \coth\left(-\frac{3}{2} t + \xi_1\right) \quad ,$$

where ξ_1 is adjusted such that $P(1) = -\frac{1}{2}$. Hence, from equation (126), the optimal control is

$$\hat{u} = \left(P - \frac{1}{2}\right)x = \left(-\frac{1}{2} + \frac{3}{2}\coth\left(-\frac{3}{2}t + \xi_1\right)\right)x \quad . \tag{140}$$

To check the validity of the solution just obtained, we employ the standard method for solving such problems as described in the section, State Regular Problem, Chapter 2. The Riccati equation (47) is

$$\dot{P} = P + P^2 - 2 \quad , \quad P(1) = 0 \quad ,$$

which has a solution

$$P = -\frac{1}{2} + \frac{3}{2}\coth\left(-\frac{3}{2}t + \xi_2\right) \quad ,$$

where ξ_2 is adjusted such that $P(1) = 0$. Therefore, the optimal control is

$$\hat{u} = -Px = \left(-\frac{1}{2} + \frac{3}{2}\coth\left(-\frac{3}{2}t + \xi_2\right)\right)x \quad ,$$

which is identical to equation (140).

EXAMPLE 13

Given the quadratic cost functional

$$J(u, x) = \frac{1}{2}\int_0^1 (x_1^2 + u^2)\,dt \tag{141}$$

and the second-order linear system

$$\dot{x}_1 = x_2 \quad , \quad x_1(0) = 1 \tag{142}$$

$$\dot{x}_2 = -x_1 - x_2 + u \quad , \quad x_2(0) = 0 \quad , \tag{143}$$

we seek the optimal control by Approximation in X .

A necessary condition for a minimum of J in X is given by the system of equations (119), (121), and (122), which for this problem is

$$0 = \dot{x}_2 + 2x_1 + x_2 \tag{144}$$

$$\ddot{x}_2 = -\dot{x}_1 + x_1 + x_2 \quad , \tag{145}$$

where

$$x_2(0) = 0 \tag{146}$$

$$x_1(0) = 1 \tag{147}$$

$$\dot{x}_2(1) + x_1(1) + x_2(1) = 0 \tag{148}$$

For the preceding boundary value problem the matrix

$$h = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

from equation (120) is positive semidefinite. Thus, Case II of this section applies and equations (144) through (148) are written as

$$\begin{pmatrix} \dot{x}^a \\ \dot{x}^b \end{pmatrix} = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} x^a \\ x^b \end{pmatrix} \tag{149}$$

where $x^a = (x_1, x_2)$, $x^b = (\dot{x}_1 , \dot{x}_2)$. and

$$A^{11} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad A^{12} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad A^{21} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} , \quad A^{22} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

subject to the boundary conditions

$$x^a(0) = (1, 0) \tag{150}$$

$$x^b(1) = C x^a(1) \quad , \tag{151}$$

where

$$C = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad .$$

Equations (149) through (151) may be solved on the finite interval T by using the Riccati transformation (the sweep method) described in Appendix B. The optimal control is given by equation (133), where elements of the 2×2 matrix P_2 satisfy equation (B.23), which results in the following equations:

$$-\dot{p}_{11} = -p_{21} + p_{11}\, p_{12} \quad , \quad p_{11}(1) = -1$$

$$-\dot{p}_{12} = p_{11} - p_{22} + p_{12}^2 \quad , \quad p_{12}(1) = -1$$

$$-\dot{p}_{21} = p_{11} + p_{11}\, p_{22} + 1 \quad , \quad p_{21}(1) = 1$$

$$-\dot{p}_{22} = p_{21} + p_{12} + p_{12}\, p_{22} \quad , \quad p_{22}(1) = 0$$

Equations (144) and (145) correspond to a single fourth-order equation in x_1 for which a solution can also be obtained by the classical Laplace transform method. The procedure is straightforward; however, evaluating the residues at the poles s_i, $i=1, 2, 3, 4$ is very tedious since the initial values of $\ddot{x}_1$ and $\dddot{x}_1$ are not specified. A computer algorithm for evaluating the inverse transform would be helpful. Unfortunately, the solution does not exist on the infinite interval $[0, \infty]$ since the system possesses poles in the right-half complex plan

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

Few classification schemes are perfect since there is usually overlap, and membership in a single category is often fuzzy. However, the scheme given in this report appears to be logical and easy to apply. Most of the well-known computational methods for optimal control belong to one of the categories, including steepest descent, quasilinearization, boundary condition iteration, and the "direct" second variation method. The classification provides geometric insight for the design of new algorithms, as evidenced by the second variation method for Approximation in X.

The conjugate direction algorithm for Approximation in X^* requires a moderate amount of instructions and computation. The method inherits all the advantages of the particular quasi-Newton method which is employed, including rapid and stable descent to a minimum. In contrast, methods involving the Newton-Raphson method may suffer from ill conditioning of the Jacobian matrix and unstable descent, in addition to the difficulty in estimating second partial derivatives of the cost functional by divided differences.

The monotone convergence algorithm for Approximation in U requires a modest amount of instructions and computation, even for high-order systems. Numerical results indicate that the algorithm is preferable to existing techniques, including Runge-Kutta integration of the Riccati equations or the "Automatic Synthesis Program" (ASP). In the latter two methods, convergence is linear, whereas in the new method convergence is both monotonic and quadratic.

The second variation algorithm for Approximation in X requires a large amount of instructions and computation. However, the transition matrix approach offers the advantage that only functions evaluated at the terminal time are saved between control updates. For problems in which the Euclidean dimension of the control space U is less than that of the state space X, storage requirements are less than former second variation methods. Solution of the Accessory Problem by the sweep method, on the other hand, requires nearly the same amount of computation as former techniques.

Future research in the area of this report should concern extending the algorithms for Approximation in X and Approximation in U to handle terminal equality constraints on the state with unspecified terminal time. For

the second variation method, it is expected that a slight redefinition of the Hilbert space inner product as in Reference 10 would be needed for the latter purpose. Further consequences of the inverse mapping should be investigated for high-order time-varying systems by application of the algorithm to physical problems.

Further research should involve the application of the algorithms to constrained nonlinear problems. At present there are a number of methods for handling constraints that restrict elements of the spaces U and X to closed subsets of Hilbert space [44]. Among these are (1) methods of feasible directions such as the "projected gradient scheme" ; (2) penalty function methods; (3) methods of set approximation such as the Ritz method; (4) duality methods such as the "moment methods" based on the Krein L-problem and also methods based on the Kuhn Tucker conditions; (5) methods of optimal evolution including the well-known Dynamic Programming technique. Methods (3) and (4) appear to be most promising for future development.

APPENDIX A

SOLUTION OF THE LINEAR EQUATION $-\dot{P}^{(k)} = P^{(k)} A^{(k)} + \left(A^{(k)}\right)^{*} P^{(k)} + Q^{(k)}$

In this section we consider the computational solution of the linear differential equation that was obtained by application of the method for monotone Approximation in U .

Suppose A, B, Q, R , are constant matrices, and $t_1 \rightarrow + \infty$. Consider a translation of the time origin such that $t_0 = 0$. Since $A^{(k)}$ is a stability matrix (that is, the system equation (67) with the control function defined by equation (71) is asymptotically stable), $\lim_{t \rightarrow - \infty} \dot{P}^{(k)}(t) = \Theta$, and we may obtain the optimal control approximation from equation (71) by solving the linear matrix equation

$$\overline{P}^{(k)} A^{(k)} + \left(A^{(k)}\right)^{*} \overline{P}^{(k)} + Q^{(k)} = \Theta \quad , \tag{A.1}$$

where $A^{(k)}$ and $Q^{(k)}$ are given by equations (73) and (74), and

$$\overline{P}^{(k)} = \lim_{t \rightarrow - \infty} P^{(k)}(t) \quad .$$

Then equation (A.1) may be written as $\frac{1}{2}n(n+1)$ linear algebraic equations whose solution is

$$p^{(k)} = -\left(D^{(k)}\right)^{-1} q^{(k)} \tag{A.2}$$

where

$$p^{(k)} = \left(p_{11}^{(k)}, \ldots, p_{1n}^{(k)}; p_{22}^{(k)}, \ldots, p_{2n}^{(k)}; \ldots, p_{nn}^{(k)}\right)$$

$$q^{(k)} = \left(q_{11}^{(k)}, \ldots, q_{1n}^{(k)}; q_{22}^{(k)}, \ldots, q_{2n}^{(k)}; \ldots, q_{nn}^{(k)}\right)$$

and $D^{(k)}$ is an $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$ constant matrix whose elements are linear combinations of elements of $A^{(k)}$. For example, if $n = 3$, $D^{(k)}$ is as follows:

$$D^{(k)} = \begin{pmatrix} 2a_{11}^{(k)} & 2a_{21}^{(k)} & 2a_{31}^{(k)} & 0 & 0 & 0 \\ a_{12}^{(k)} & a_{11}^{(k)} + a_{22}^{(k)} & a_{21}^{(k)} & a_{21}^{(k)} & a_{31}^{(k)} & 0 \\ a_{13}^{(k)} & a_{23}^{(k)} & a_{11}^{(k)} + a_{33}^{(k)} & 0 & a_{23}^{(k)} & a_{31}^{(k)} \\ 0 & 2a_{12}^{(k)} & 0 & 2a_{22}^{(k)} & 2a_{32}^{(k)} & 0 \\ 0 & a_{13}^{(k)} & a_{12}^{(k)} & a_{23}^{(k)} & a_{22}^{(k)} + a_{33}^{(k)} & a_{32}^{(k)} \\ 0 & 0 & 2a_{13}^{(k)} & 0 & 2a_{23}^{(k)} & 2a_{33}^{(k)} \end{pmatrix}$$

For the finite interval case, $t_1 - t_0 < \infty$ and the elements of $A^{(k)}$ and $Q^{(k)}$ are functions on T. Let us consider the following expansions:

$$P^{(k)}(t) = \sum_{n=1}^{\infty} P_n^{(k)} (t_1 - t)^n \tag{A.3}$$

$$A^{(k)}(t) = \sum_{n=0}^{\infty} A_n^{(k)} (t_1 - t)^n \tag{A.4}$$

$$Q^{(k)}(t) = \sum_{n=0}^{\infty} Q_n^{(k)} (t_1 - t)^n \tag{A.5}$$

where $P_n^{(k)}$, $A_n^{(k)}$, and $Q_n^{(k)}$ are constant $n \times n$ matrices. Substituting the latter expressions for $P^{(k)}$, $A^{(k)}$, and $Q^{(k)}$ into equation (72) and equating coefficients of $(t_1 - t)^n$, $n = 0, 1, 2, \ldots$, we obtain the following set of equations:

$$P_0^{(k)} = \Theta$$

$$P_{n+1}^{(k)} = \frac{1}{n+1}\left(Q_n^{(k)} + \sum_{\nu=1}^{n} P_\nu^{(k)} A_{n-\nu}^{(k)} + \left(A_{n-\nu}^{(k)}\right)^* P^{(k)} \right) \quad . \quad (A.6)$$

Equation (A.6) may be solved recursively for each value of k and provides one means for solving the linear matrix differential equation (72). Motivated by this result, a similar approach was used to generate simple optimal test problems and is described in Appendix C.

In the general case of time-varying A, B, Q, or R, equation (72) must be integrated numerically using a digital computer. Note that equation (72) is integrated backward from the final condition $P^{(k)}(t_1) = \Theta$.

APPENDIX B

SOLUTION OF THE ACCESSORY PROBLEM

For the computational solution of the boundary value problem, equation (103) through (111), it is convenient to consider the following two cases: (1) $h > 0$; (2) $h \geq 0$ and $L_{uu} > 0$. When applicable, the procedure described for Case 1 is easier to implement and requires less computation.

Case 1. $h > 0$.

TRANSITION MATRIX APPROACH

Let us rewrite equation (103) as the following $\mathbb{R}^{2n}$-valued system of equations:

$$\delta\dot{z} = S\delta z + w \quad , \tag{B.1}$$

where

$$\delta z = \begin{pmatrix} \delta x \\ \delta \dot{x} \end{pmatrix} \quad , \quad S = \begin{pmatrix} \Theta & I \\ h^{-1}G & h^{-1}F \end{pmatrix}, \quad w = \begin{pmatrix} \theta \\ h^{-1}k \end{pmatrix}$$

and subject to the boundary conditions, equations (108) and (109),

$$\delta x(t_0) = \theta \tag{B.2}$$

$$\delta \dot{x}(t_1) = h^{-1}(t_1)\, N(t_1)\, \delta x(t_1) + h^{-1}(t_1)\, r(t_1) \quad . \tag{B.3}$$

The solution to equation (B.1) can be written in terms of the initial conditions $\delta x(t_0)$ and $\delta \dot{x}(t_0)$ as

$$\delta x(t) = \Phi^{11}(t, t_0)\, \delta x(t_0) + \Phi^{12}(t, t_0)\, \delta \dot{x}(t_0) + \delta \tilde{x}(t) \quad , \tag{B.4}$$

$$\delta \dot{x}(t) = \Phi^{21}(t, t_0)\, \delta x(t_0) + \Phi^{22}(t, t_0)\, \delta \dot{x}(t_0) + \delta \tilde{\dot{x}}(t) \quad , \tag{B.5}$$

where $\Phi^{ij}(t, t_0)$ are $n \times n$ submatrices of the transition matrix for the linear system [eq. (B.1)], $\delta\dot{x}(t_0)$ is unknown, and $\delta\tilde{x}$, $\delta\tilde{\dot{x}}$, represent the unknown "forced solutions" of equation (B.1). Evaluating equation (B.5) at $t = t_1$ and using equations (B.2) and (B.3), we obtain

$$\begin{aligned}\Phi^{22}(t_1 , t_0)\, \delta\dot{x}(t_0) &= \delta\dot{x}(t_1) - \delta\tilde{\dot{x}}(t_1) \\ &= h^{-1}(t_1)\, N(t_1)\, \delta x(t_1) + h^{-1}(t_1)\, r(t_1) - \delta\tilde{\dot{x}}(t_1) \\ &= h^{-1}(t_1)\, N(t_1)\, \Phi^{12}(t_1 , t_0)\, \delta\dot{x}(t_0) \\ &\quad + h^{-1}(t_1)\, N(t_1)\, \delta\tilde{x}(t_1) - \delta\tilde{\dot{x}}(t_1) + h^{-1}(t_1)\, r(t_1) \quad .\end{aligned} \tag{B.6}$$

Equation (B.6) may be solved for the "missing" initial condition,

$$\delta\dot{x}(t_0) = \Big(h(t_1)\, \Phi^{22}(t_1 , t_0) - N(t_1)\, \Phi^{12}(t_1 , t_0)\Big)^{-1} \times \Big(N(t_1)\, \delta\tilde{x}(t_1) - \delta\tilde{\dot{x}}(t_1) + r(t_1)\Big) \quad . \tag{B.7}$$

If we can determine $\delta\tilde{x}(t_1)$, $\delta\tilde{\dot{x}}(t_1)$ and $\Phi^{ij}(t_1 , t_0)$, $i = 1, 2$, the missing initial condition $\delta\dot{x}(t_0)$ is specified by equation (B.7). A procedure for computing these unknown functions is as follows. Let $\delta x(t_0) = \delta\dot{x}(t_0) = \theta$ in equation (B.1). Then $\delta\tilde{x}(t) = \delta x(t)$ and $\delta\tilde{\dot{x}}(t) = \delta\dot{x}(t)$. By integrating equation (B.1) with these initial conditions until $t = t_1$, we obtain the "forced solution" $\delta\tilde{\dot{x}}(t_1)$. To compute the submatrices $\Phi^{ij}(t_1 , t_0)$, $i = 1, 2$, let $k = \theta$ in equation (B.1) so that the "forced solutions in equations (B.4) and (B.5) are zero. Then by integrating equation (B.1) with the initial conditions $\delta x(t_0) = \theta$ and $\dot{x}(t_0) = \{\Delta_{1j} , \Delta_{2j} , \ldots , \Delta_{nj}\}$, where Δ_{ij} is defined by

$$\Delta_{ij} = \begin{cases} 1 , & i = j \\ 0 , & i \neq j \end{cases} ,$$

the jth column of $\Phi(t_1 , t_0)$ is obtained. By repeating this procedure for $j = 1, 2, \ldots , n$, we obtain the last n columns of $\Phi(t_1 , t_0)$ as required.

A total of $2n(m + 2)$ differential equations must be integrated where $m = \text{rank } h$. Existence of the inverse in equation (B.7) is insured by the

absence of conjugate points in the interval T [45, 46] and, together with the strong Legendre condition $h > 0$, constitutes sufficient conditions for a solution to the Accessory Problem.

THE SWEEP METHOD

Alternatively, boundary value problems of this type may be solved by the sweep method [6]. Consider the inhomogeneous Riccati transformation

$$\delta\dot{x} = P_1\delta x + q_1 \quad , \tag{B.8}$$

where P_1 is an $n \times n$ matrix, and q_1 is an $\mathbb{R}^n$-valued function. It is sufficient to assume that elements of P_1 and q_1 are real-valued continuous functions on T. Then

$$\begin{aligned}\delta\ddot{x} &= \dot{P}_1\delta x + P_1\delta\dot{x} + \dot{q}_1 \\ &= (\dot{P}_1 + P_1^2)\,\delta x + (P_1q_1 + \dot{q}_1) \\ &= h^{-1}(F\delta\dot{x} + G\delta x + k) \quad .\end{aligned}$$

In the last step we have used equation (103). Thus, the following equations for the "backward sweep" are obtained:

$$-h\dot{P}_1 = hP_1^2 - FP_1 - G \quad , \quad h(t_1)\,P_1(t_1) = N(t_1) \tag{B.9}$$

$$-h\dot{q}_1 = (hP_1 - F)\,q_1 - k \quad , \quad h(t_1)\,q_1(t_1) = r(t_1) \quad . \tag{B.10}$$

During the "forward sweep" we must integrate

$$\dot{\delta x} = P_1\delta x + q_1 \quad , \quad \delta x(t_0) = \theta \tag{B.11}$$

to obtain δx on T. From equations (94) and (B.8) the control increment may be expressed in terms of the state increment as follows:

$$\delta u = Y\delta x + z_1 \quad , \tag{B.12}$$

where the "feedback" and "feedforward" gains are, respectively,

$$\Upsilon_1 = g_\epsilon (P_1 - f_x) \tag{B.13}$$

$$z_1 = g_\epsilon q_1 \tag{B.14}$$

Existence of a solution to equation (117) is insured by the conjugate point condition, [6,45]. In contrast to Riccati equations encountered in the sections, State Regulator Problems, Chapter 2, and Methods Based on the Second Variation, Chapter 3, P_1 in equation (B.9) is not necessarily symmetric, and hence $n(n+2)$ equations must be integrated to obtain δu. In comparison, the Riccati transformation for the second variation method described in the section, Descent Algorithms for Hilbert Space, Chapter 3, results in $\frac{1}{2}n(n+9)$ equations

Case 2. $h \geq 0$ and $L_{uu} > 0$.

In this case equations (103) through (11) may be written as the IR^{2n}-valued system of linear equations

$$\begin{pmatrix} \delta\dot{x}^a \\ \delta\dot{x}^b \end{pmatrix} = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} \delta x^a \\ \delta x^b \end{pmatrix} + \begin{pmatrix} d^a \\ d^b \end{pmatrix} \tag{B.15}$$

where $\delta x^a = (\delta x_1, \ldots, \delta x_n)$, $\delta x^b = (\delta x_{n+1}, \ldots, \delta x_{2n})$ satisfy $\delta x_{i+1} = \delta\dot{x}_i$, $i = 1, 2, \ldots 2n-1$ with boundary conditions

$$\delta x^a(t_0) = \theta \tag{B.16}$$

$$C(t_1)\, \delta x^a(t_1) = D(t_1)\, \delta x^b(t_1) - e(t_1) \tag{B.17}$$

Notice that by definition $\delta x^a = \delta x$ and $\delta x^b = \delta\dot{x}$. Matrices A^{ij}, C, D and d^a, d^b are $n \times n$ and $n \times 1$, respectively, with elements that are functions on T. It is assumed that $D(t_1)$ is invertible.

TRANSITION MATRIX APPROACH

In analogy to Case 1 let

$$\begin{pmatrix} \delta x^a(t_1) \\ \\ \delta x^b(t_1) \end{pmatrix} = \begin{pmatrix} \Phi^{11}(t_1, t_0) & \Phi^{12}(t_1, t_0) \\ \\ \Phi^{21}(t_1, t_0) & \Phi^{22}(t_1, t_0) \end{pmatrix} \begin{pmatrix} \delta x^a(t_0) \\ \\ \delta x^b(t_0) \end{pmatrix}$$

$$+ \int_{t_0}^{t_1} \begin{pmatrix} \Phi^{11}(t_1, s) & \Phi^{12}(t_1, s) \\ \\ \Phi^{21}(t_1, s) & \Phi^{22}(t_1, t_0) \end{pmatrix} \begin{pmatrix} d^a(s) \\ \\ d^b(s) \end{pmatrix} ds \qquad \text{(B. 18)}$$

where $\Phi^{ij}(t_1, t_0)$ are $n \times n$ submatrices of the transition matrix for the linear system equation (B. 15). Combining equations (B. 18), (B. 16), and (B. 17), we obtain

$$C(t_1)\left(\Phi^{12}(t_1, t_0)\ \delta x^b(t_0) + \int_{t_0}^{t_1} \left(\Phi^{11}(t_1, s)\ d^a(s) + \Phi^{12}(t_1, s)\ d^b(s) \right) ds \right)$$

$$= D(t_1)\left(\Phi^{22}(t_1, t_0)\ \delta x^b(t_0) + \int_{t_0}^{t_1} \left(\Phi^{21}(t_1, s)\ d^a(s) + \Phi^{22}(t_1, s)\ d^b(s) \right) ds \right) - e(t). \qquad \text{(B. 19)}$$

Equation (B. 19) may be solved for the missing initial condition as

$$x^b(t_0) = \left(D(t_1)\ \Phi^{22}(t_1, t_0) - C(t_1)\ \Phi^{12}(t_1, t_0) \right)^{-1} \times$$

$$\left(\int_{t_0}^{t_1} \left(\left(C(t_1)\ \Phi^{11}(t_1, s) - D(t_1)\ \Phi^{21}(t_1, s) \right) d^a(s) \right.\right.$$

$$\left.\left. + \left(C(t_1)\ \Phi^{12}(t_1, s) - D(t_1)\ \Phi^{22}(t_1, s) \right) d^b(s) \right) ds + e(t_1) \right) . \qquad \text{(B. 20)}$$

The linear system equation (B. 15), together with initial conditions in equations (B. 16) and (B. 17), constitutes an initial value problem, which, if the inverse exists, may be integrated forward to obtain δx^a and δx^b on T.

Existence of the inverse in equation (B. 20) is insured by the absence of conjugate points in the interval T [45, 46] and, together with the strong Legendre condition $L_{uu} > 0$, represents sufficient conditions for a solution to the Accessory Problem.

SWEEP METHOD

Consider the following inhomogeneous Riccati transformation:

$$\delta x^b = P_2 \delta x^a + q_2 \quad , \tag{B. 21}$$

where P_2 is an $n \times n$ matrix on T, and q_2 is an $\mathbb{R}^n$-valued function on T. Then

$$\begin{aligned} \delta \dot{x}^b &= \dot{P}_2 \delta x^a + P_2 \delta \dot{x}^a + \dot{q}_2 \\ &= \dot{P}_2 \delta x^a + P_2 \left[A_{11} \delta x^a + A_{12} \left[P_2 \delta x^a + q_2 \right] + d^a \right] + \dot{q}_2 \\ &= A^{21} \delta x^a + A^{22} \left[P_2 \delta x^a + q_2 \right] + d^a \quad , \end{aligned} \tag{B. 22}$$

where we have used equations (B. 21) and (B. 15). Thus the following equations for the "backward sweep" are obtained:

$$\dot{P}_2 = - P_2 A^{11} + A^{22} P_2 - P_2 A^{12} P_2 + A^{21} \quad , \quad D(t_1)\ P_2(t_1) = C(t_1) \tag{B. 23}$$

$$\dot{q}_2 = (A^{22} - P_2 A^{12}) q_2 - P_2 d^a + d^b \quad , \quad D(t_1)\ q_2(t_1) = e(t_1) \quad . \tag{B. 24}$$

During the "forward sweep" we must solve

$$\delta \dot{x}^a = (A^{11} + A^{12} P_2)\ \delta x^a + A^{12} q_2 + d^a \quad , \quad \delta x^a(t_0) = \theta \quad . \tag{B. 25}$$

Having obtained the $\mathbb{R}^n$-valued function δx^a, the control increment is

$$\delta u = Y_2 \delta x + z_2 \quad , \tag{B.26}$$

where the "feedback" and "feedforward" gains are, respectively,

$$Y_2 = g_\epsilon (A^{11} + A^{12} P_2 - f_x) \tag{B.27}$$

$$z_2 = g_\epsilon A^{12} q_2 \quad . \tag{B.28}$$

Existence of a solution to equation (B.23) is insured by the conjugate point condition [45, 6]. Since A^{12} and A^{21} are not necessarily symmetric, $n(n+2)$ equations must be integrated to obtain $\delta x = \delta x^a$.

APPENDIX C

A CLASS OF STATE REGULATOR PROBLEMS FOR TESTING COMPUTATIONAL ALGORITHMS

This section describes a class of simple first-order time-varying dynamical systems and the optimal solution with respect to quadratic cost as specified by the Riccati solution of the section, State Regulator Problem, Chapter 2. These problems have been designed to provide a class of simple problems with nontrivial closed-form solutions for the testing of computational algorithms and are based on Reference 47.

The system dynamics are assumed to be modeled by a linear first-order differential equation,

$$\dot{x} = Ax + u \quad , \tag{C.1}$$

where A is a continuous function on $[0 , t_1]$ that will be determined; $x \in L_1^n(T)$ and $u \in L_1^n(T)$ are the state and control functions, respectively. The cost functional is

$$J(u , x) = \tfrac{1}{2} \int_0^{t_1} (Qx^2 + u^2)\, dt \quad , \tag{C.2}$$

where Q is a continuous function $[0 , t_1]$ which will also be determined. From the section, State Regulator Problem, the optimal feedback solution to equations (A.1) and (A.2) is

$$\hat{u} = - Px \quad , \tag{C.3}$$

where P satisfies the Riccati differential equation

$$- \dot{P} = 2PA - P^2 + Q \tag{C.4}$$

with boundary condition $P(t_1) = 0$. Equation (C.4) must be solved in reverse time starting with the boundary condition and integrating backwards.

Example 1

Let a_n , q_n , p_n be constants and suppose that

$$A(t) = a_0 + a_1 (t_1 - t) \tag{C.5}$$

$$Q(t) = \sum_{n=0}^{2} q_n (t_1 - t)^n \tag{C.6}$$

$$P(t) = p_1 (t_1 - t) \tag{C.7}$$

By substituting equations (C.5) and (C.7) into equation (C.4) and collecting coefficients of $(t_1 - t)^n$, $n = 0, 1, \ldots$ we obtain the following constraints on our choice of system parameters:

$$p_1 = q_0 \tag{C.8}$$

$$q_1 + 2a_0 p_1 = 0 \tag{C.9}$$

$$q_2 + 2a_1 p_1 - p_1^2 = 0 \tag{C.10}$$

Hence, given the system equations (C.1) and (C.5), equations (C.2) and (C.6) specify the cost functional for linear feedback. For example, suppose that

$$A(t) = \left(-1 + \tfrac{1}{4}(t_1 - t) \right) \tag{C.11}$$

Then $a_0 = -1$, $a_1 = \frac{1}{4}$. If $p_1 = \frac{1}{2}$, $q_0 = \frac{1}{2}$, then from equations (C.9) and (C.10) we find that the controlled system is specified by

$$\dot{x} = \left(-1 - \tfrac{1}{4}(t_1 - t) \right) x \tag{C.12}$$

$$J(\hat{u}, x) = \tfrac{1}{2} \int_0^{t_1} \left(\tfrac{1}{2} + (t_1 - t) + \tfrac{1}{4}(t_1 - t)^2 \right) x^2 dt = \tfrac{1}{4} x^2(0) \tag{C.13}$$

$$\hat{u} = -\tfrac{1}{2}(t_1 - t) x \tag{C.14}$$

Example 2

Let a_n , q_n , p_n be constants and suppose that

$$A(t) = a_0 + a_1 e^{\alpha(t_1 - t)} \tag{C.15}$$

$$Q(t) = q_0 + q_1 e^{\alpha(t_1 - t)} + q_2 e^{2\alpha(t_1 - t)} \tag{C.16}$$

$$P(t) = p_0 + p_1 e^{\alpha(t_1 - t)} \quad . \tag{C.17}$$

Substituting equations (C.15) through (C.17) into equation (C.4) and collecting coefficients of $e^{n\alpha(t_1 - t)}$, $n = 0, 1, \ldots$ we obtain the following constraints

$$q_2 + 2p_1 a_1 - p_1^2 = 0 \tag{C.18}$$

$$\alpha p_1 = q_1 + 2a_0 p_1 + 2a_1 p_0 - 2p_0 p_1 \tag{C.19}$$

$$q_0 + 2a_0 p_0 - p_0^2 = 0 \quad . \tag{C.20}$$

In addition, since $P(t_1) = 0$,

$$p_0 + p_1 = 0 \quad . \tag{C.21}$$

Equations (C.18) through (C.21) specify the requirements on our choice of cost functional equation (C.16) for the system equation (C.15) and control in terms of equation (C.17). For $\alpha < 0$ and $t_1 \to +\infty$ the optimal control will approach that for the constant coefficient system

$$\dot{x} = a_0 x + u \tag{C.22}$$

$$J(u, x) = \int_0^\infty (q_0 x^2 + u^2)\, dt \quad . \tag{C.23}$$

Example 3

Let a_n, q_n, p_n be constants and suppose that

$$A = a_0 + a_1 e^{\alpha t} \tag{C. 24}$$

$$Q = q_0 + q_1 e^{\alpha t} + q_2 e^{2\alpha t} \tag{C. 25}$$

$$P = p_0 + p_1 e^{\alpha t} \quad . \tag{C. 26}$$

Substituting in equation (C. 4) and collecting coefficients as above, we obtain

$$2a_0 p_0 + q_0 - p_0^2 = 0 \tag{C. 27}$$

$$-\alpha p_1 = 2a_1 p_0 + 2a_0 p_1 + q_1 - 2a_0 p_1 \tag{C. 28}$$

$$2a_1 p_1 + q_2 - p_1^2 = 0 \tag{C. 29}$$

$$p_0 + p_1 e^{\alpha t_1} = 0 \quad . \tag{C. 30}$$

Consider the case of infinite time interval $[0\ ,\ \infty]$ and $\alpha = -1$. Moreover, suppose that

$$a_0 = \tfrac{1}{4} \qquad q_0 = 0 \qquad q_1 = \tfrac{1}{2} \qquad p_0 = 0$$

$$a_1 = \tfrac{1}{2} \qquad q_2 = 0 \qquad p_1 = 1 \quad .$$

Then the optimal solution is characterized by

$$\dot{x} = \tfrac{1}{2}\left(\tfrac{1}{2} + e^{-t}\right)x + \hat{u} \tag{C. 31}$$

$$Q(t) = \tfrac{1}{2} e^{-t} \tag{C. 32}$$

$$P(t) = e^{-t} \tag{C. 33}$$

$$\hat{u} = -e^{-t} x \tag{C. 34}$$

It is interesting that the dynamics of the closed-loop system are

$$\dot{x} = \tfrac{1}{2}\left(\tfrac{1}{2} - e^{-t}\right)x \quad , \tag{C.35}$$

an optimal system which becomes unstable within one second! Another example for this case is generated by the following choice:

$$a_0 = \tfrac{1}{4} \qquad q_0 = 0 \qquad q_1 = \frac{3}{2} \qquad p_0 = 0$$

$$a_1 = \tfrac{1}{2} \qquad q_2 = 0 \qquad p_1 = 1 \quad ,$$

for which the optimal system is characterized by

$$\dot{x} = \tfrac{1}{2}\left(-\tfrac{1}{2} + e^{-t}\right)x + \hat{u} \tag{C.36}$$

$$Q(t) = \frac{3}{2}e^{-t} \tag{C.37}$$

$$P(t) = e^{-t} \tag{C.38}$$

$$\hat{u} = -e^{-t}x \quad . \tag{C.39}$$

Example 4

Let

$$A(t) = a_0 + a_1 \sin \omega t + a_2 \cos \omega t \tag{C.40}$$

$$Q(t) = q_0 + q_1 \sin \omega t + q_2 \cos \omega t + q_3 \sin 2\omega t + q_4 \cos 2\omega t \tag{C.41}$$

$$P(t) = p_0 + p_1 \sin \omega t + p_2 \cos \omega t \quad . \tag{C.42}$$

The resulting constraints are obtained as follows:

$$2a_0p_0 + q_0 - p_0^2a_1p_1 + a_2p_2 - \tfrac{1}{2}(p_1^2 - p_2^2) = 0 \tag{C.43}$$

$$\omega p_2 + 2(a_0 p_1 + a_1 p_0 - p_0 p_1) + q_1 = 0 \tag{C.44}$$

$$-\omega p_1 + 2(a_0 p_2 + a_2 p_0 - p_0 p_2) + q_2 = 0 \tag{C.45}$$

$$a_1 p_2 + a_2 p_1 + q_3 + p_1 p_2 = 0 \tag{C.46}$$

$$-a_1 p_1 + a_2 p_2 + q_4 + \tfrac{1}{2}(p_1^2 - p_2^2) = 0 \tag{C.47}$$

$$p_0 + p_1 \sin \omega t_1 + p_2 \cos \omega t_1 = 0 \tag{C.48}$$

Equation (C.43) through (C.48) may be used to generate optimal test problems, as in the previous example.

Example 5

Let a_n, q_n, p_n be constants and suppose that

$$A(t) = \sum_{n=0}^{\infty} a_n e^{n\alpha t} \tag{C.49}$$

$$Q(t) = \sum_{n=0}^{\infty} q_n e^{n\alpha t} \tag{C.50}$$

$$P(t) = \sum_{n=0}^{\infty} p_n e^{n\alpha t} \tag{C.51}$$

The constraints on a_n and q_n are obtained by substituting equations (C.49) through (C.51) into equation (C.4). Thus

$$p_0 = 0$$

$$-\alpha p_{n+1} = \frac{1}{n+1}\left(q_n + \sum_{\nu=1}^{n}\left(2p_\nu a_{n-\nu} - p_\nu p_{n-\nu}\right)\right), \quad n = 0, 1, 2, \ldots \tag{C.52}$$

where the terminal constraint $p_0 = 0$ has been combined with equation (C. 51) to yield

$$P(t) = \sum_{n=1}^{\infty} \left(p_n \left(e^{n\alpha t} - e^{n\alpha t_1} \right) \right) \tag{C. 53}$$

Example 6

Let a_n, q_n, p_n be constants and suppose that

$$A(t) = \sum_{n=0}^{\infty} a_n t^n \tag{C. 54}$$

$$Q(t) = \sum_{n=0}^{\infty} q_n t^n \tag{C. 55}$$

$$P(t) = \sum_{n=1}^{\infty} p_n (t_1 - t)^n \quad . \tag{C. 56}$$

In a similar manner we obtain constraints on a_n and q_n in the form of the recursion relations:

$$p_0 = 0 \quad ,$$

$$-\alpha p_{n+1} = \frac{1}{n+1} \left(q_n + \sum_{\nu=1}^{n} \left(2p_\nu a_{n-\nu} - p_\nu p_{n-\nu} \right) \right) , \quad n = 0, 1, 2, \ldots \tag{C. 57}$$

which may be used to generate optimal solutions to the problem.

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APPROVAL

DESIGN OF COMPUTATIONAL ALGORITHMS FOR OPTIMAL CONTROL BY HILBERT SPACE METHODS

By William A. Gruver III

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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