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SYNTHESIS OF DYNAMIC SYSTEMS USING FORMA (FORTRAN MATRIX ANALYSIS)

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SYNTHESIS OF DYNAMIC SYSTEMS USING FORMA - FORTRAN MATRIX ANALYSIS

Volume I - Methodology

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FOREWORD

This report presents the results of a study on the synthesis of dynamic systems using FORMA - FORTRAN Matrix Analysis. The study, performed from May 18, 1970 to July 18, 1971 was conducted by the Dynamics and Loads Section, Martin Marietta Corporation, Denver Division under Contract NAS8-25922. The program was administered by the National Aeronautics and Space Administration, George C. Marshall Space Flight Center, Huntsville, Alabama under the direction of Dr. John R. Admire, Analytical Mechanics Division, Astronautics Laboratory.

This report is published in four volumes:

Volume I - Methodology

Volume II - Programming Manual

Volume III - Subroutine Explanations

Volume IV - Subroutine Listings

CONTENTS

ABSTRACT

This study presents techniques for the solution of structural dynamic systems on an electronic digital computer using FORMA (FORTRAN Matrix Analysis).

FORMA is a library of subroutines coded in FORTRAN IV for the efficient solution of small and medium size structural dynamics problems of up to 150 degrees of freedom. These subroutines are in the form of building blocks that can be put together to solve a large variety of structural dynamics problems. The obvious advantage of the building block approach is that programming and checkout time are limited to that required for putting the blocks together in the proper order.

The FORMA method has advantageous features such as:

(1) Subroutines in the library have been used extensively for many years and as a result are well checked out and debugged;

(2) Method will work on any computer with a FORTRAN IV compiler;

(3) Incorporation of new subroutines is no problem;

(4) Basic FORTRAN statements may be used to give extreme flexibility in writing a program.

ACKNOWLEDGEMENTS

The author expresses his appreciation to those individuals whose assistance was necessary for the successful completion of the study. Dr. John R. Admire was instrumental in the definition of the program scope and contributed many valuable comments. Messrs. Carl Bodley, Wilcomb Benfield, Richard Hruda, and Herbert Wilkening, all of the Dynamics and Loads Section, Denver division of Martin Marietta Corporation, have contributed ideas, as well as subroutines, in the formulation of the FORMA library.

The author also expresses his appreciation to those persons who developed FORTRAN, particularly the subroutine concept of that programming tool.

I. INTRODUCTION

The formulation and solution of most structural dynamics problems involves the use of matrix analysis and an electronic digital computer. Matrix analysis is used because it allows complicated arithmetical operations to be formulated systematically and provides a compact form of bookkeeping. The electronic digital computer is used in the solution of the problem because of its low cost per calculation.

After the analyst has formulated a problem in matrix notation, he is faced with the practical consideration of obtaining numerical answers using numerical input to the equation. The analyst must therefore translate (i.e., program) the equations into a form recognizable by the computer. Two computer programming approaches are available to the analyst. One is to program the computer to solve a specific type problem using a basic programming language such as ALGOL or FORTRAN. This approach can yield a very efficient computer program but the development of such a program is very time consuming. Thus, such an approach is practical only if the program will be used extensively. The second approach involves a library of matrix analysis operations in subroutine form that allows the analyst to set up his own program using a "building block" concept. This second approach allows the acquisition of quick results from problems of quite different types. The latter approach is considered in this report.

The validity of the second approach becomes evident from a study of structural dynamic analysis methods. This study reveals that for most types of problems, the mathematical operations required for solutions are limited in number. Thus, these mathematical operations can be programmed in the form of computer subroutines resulting in a library of "building blocks" that can be put together to solve a large variety of structural dynamics problems. The obvious advantage of the building block approach is that the only programming and checkout time required is putting the necessary blocks together in the proper order.

The building block approach described in this report uses FORTRAN call statements with subroutines from a library of subroutines entitled FORMA (FORTRAN Matrix Analysis). Development of subroutines in the FORMA library was started in 1964 by engineers in the Dynamics and Loads Section of Martin Marietta Corporation, Denver Division, to solve a wide variety of structural dynamics analyses of aerospace vehicles such as the Titan booster and Skylab orbiting laboratory. These subroutines were programmed specifically for the solution of small and medium size structural dynamics problems of up to 150 degrees of freedom. Currently, 86 subroutines are in the FORMA library. Explanations and listings of these subroutines are given in Volumes III and IV, respectively. The FORMA library includes subroutines for beam mass matrix calculations, beam stiffness matrix calculations, eigenvalue-vector solutions, time response solutions as well as the basic matrix algebra subroutines. A list of available subroutines is given as Appendix A, Volume II. The subroutines in this library have been used extensively and as a result are well checked out and debugged. The FORMA method has advantageous features such as:

1) Method will work on any computer with a FORTRAN IV compiler. It has been used on the IBM 7044, IBM 7094, IBM 360, GE 625/635, CDC 6400/6500, and UNIVAC 1108 with only minor modifications;
2) Computer times are reasonable;
3) Incorporation of new subroutines is no problem;
4) Basic FORTRAN statements may be used to give extreme flexibility in writing a program;
5) An analyst can program relatively complex problems with very little programming experience;
6) The method of programming is closely related to the manner of the mathematical formulation of the physical problem.

Before discussing the solution of structural dynamics problems using FORMA, a few techniques for the synthesis of dynamic systems will be presented in this volume. Some of these techniques are in common use throughout the aerospace industry while others are unique to the Dynamics and Loads Section. All techniques have been proved through repeated use in project studies at Martin Marietta Corporation, Denver Division.

The basic technique used in this report for the synthesis of dynamic systems is the finite element method. In this approach the mass, stiffness, and external forces of a continuous structural system with an infinite number of degrees of freedom are represented by a discrete mathematical model having a finite number of degrees of freedom. This approximation of a continuous structural system by a discrete coordinate model is accomplished using kinetic energy, strain energy, and virtual work concepts. The assumption made in this type of approach is that any displacement in the continuous system can be represented as an interpolation function operating on the displacement of the discrete coordinates. The accuracy of this approximation depends upon the choice of the interpolation function and the discrete coordinates selected. This displacement assumption is essentially a modification of the Rayleigh-Ritz method that defines the deflection as a summation of assumed displacement functions, each of which is weighted by a generalized coordinate representing the contribution of the assumed function. Each displacement function is assumed over the entire structure and considerable judgment is necessary in choosing these functions. The modification used here is to use a simple displacement function over only a portion of the structure and then use the discrete point (also referred to as a panel point) displacements as the generalized coordinates. No claim is made for any originality of this technique; it is in common use throughout the industry. Historically, the Lagrangian method in a general finite degree of freedom approach utilizing matrices was in use as early as 1936 (Ref 1, page 5).

In conclusion, this report emphasizes assistance to a structural dynamicist to obtain a computer solution to a problem using FORMA subroutines. It does not teach a person to become a structural dynamicist.

II. TECHNIQUE

A. EQUATIONS OF MOTION

The equations of motion to be used in this report are derived om Lagrange's equations. Lagrange's equations of motion give convenient method of setting up the differential equations of uilibrium for a system whose configuration can be expressed in rms of generalized coordinates. Lagrange's equations of motion panded to include dissipated energy are given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{h}_i}\right) - \frac{\partial T}{\partial h_i} + \frac{\partial R}{\partial \dot{h}_i} + \frac{\partial U}{\partial h_i} = F_i \quad (i=1, \ldots, N) \tag{A-1}$$

here T = kinetic energy of the system,

R = dissipated energy of the system,

U = potential energy of the system,

F = generalized force,

h = generalized coordinate, and

N = number of degree of freedom.

'o important definitions of terms introduced above will be made ıw.

Generalized Coordinates -- Any independent set of functions ıich will prescribe the position of every particle in the sys-:m is called a set of generalized coordinates.

Degrees of Freedom - In a static or dynamic system, the num-:r of degrees of freedom is the number of independent functions :oordinates) required to completely specify the configuration the system.

The kinetic energy is of quadratic form and can be expressed ı matrix notation as

$$T = \frac{1}{2} \{\dot{h}\}^T [A] \{\dot{h}\} \tag{A-2}$$

ıere [A] is an N×N matrix of mass coefficients, a_{ij}. The poten-.al strain energy is also of quadratic form and can be expressed

$$U = \frac{1}{2} \{h\}^T [S] \{h\} \tag{A-3}$$

where [S] is an N×N matrix of stiffness coefficients, s_{ij}. The energy dissipation function can be approximated by (Ref 1)

$$R = \frac{1}{2} \{\dot{h}\}^T [B] \{\dot{h}\} \tag{A-4}$$

where [B] is an N×N matrix of damping coefficients. The generalized force, F_i, associated with the generalized coordinate, h_i, is determined from the virtual work, δW, of the external forces, that is, from

$$\delta W = \{\delta h\}^T \{F\}. \tag{A-5}$$

Substituting Equations (A-2), (A-3), (A-4) into Equation (A-1), and using Equation (A-5), we obtain the equations of motion as

$$[A] \{\ddot{h}(t)\} + [B] \{\dot{h}(t)\} + [S] \{h(t)\} = \{F(t)\}. \tag{A-6}$$

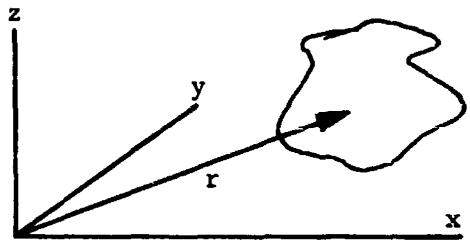
The technique of obtaining these matrices and the solution of this equation for $\{\ddot{h}(t)\}$, $\{\dot{h}(t)\}$, and $\{h(t)\}$ will be discussed in other sections of this report.

B. MASS MATRICES

The mass matrix of a structure will be determined using a kinetic energy approach. For small deformations, the kinetic energy of the body in the sketch below is given by

$$T = \frac{1}{2} \iiint \rho(x, y, z) \left(\frac{\partial r(x,y,z,t)}{\partial t}\right)^2 dx \; dy \; dz \tag{B-1}$$

where ρ is the material density.



Because it is difficult to work with the integral form for nonuniform structures, the deflection (r(x,y,z,t)) of the structure is represented by the deflection of a discrete number of points (panel points) on the structure. The displacement between the points is obtained by assuming any arbitrary function. As mentioned in the introduction, the approach advocated in this report is to assume a simple function between adjacent panel points. This approach is a modification of the Rayleigh-Ritz method in which the deflection is defined as a summation of assumed displacement functions, each of which is weighted by a generalized coordinate representing the contribution of the assumed function. Each displacement function is assumed over the entire structure and considerable skill is required to choose these functions. The modification in this report is to use a simple displacement function over only a portion of the structure and then use the panel point displacements as the generalized coordinates.

Various assumed displacement functions will yield uniquely different mass matrices. For example, for a structure idealized as a beam, a uniform displacement function between pairs of the panel points will yield a diagonal mass matrix; a linear displacement function will yield a tridiagonal mass matrix.

The technique mentioned here can be applied to any type of structural element; for example, plates, shells, or beams. A detailed explanation of the application of this method to a beam is given in the explanations of Subroutine MASS1 (assumed linear deflection) and Subroutine MASS2 (assumed cubic deflection) in Volume III of this report.

Other techniques for calculation of the mass matrix are also given in References 2, 3 and 4.

C. STIFFNESS MATRICES

The stiffness matrix of a structure can be determined using a strain potential energy technique. For small deflections, the potential energy of a structure can be given by

$$U = \frac{1}{2}\iiint \Big(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\Big)\, dx\, dy\, dz, \tag{C-1}$$

where σ is the normal stress,

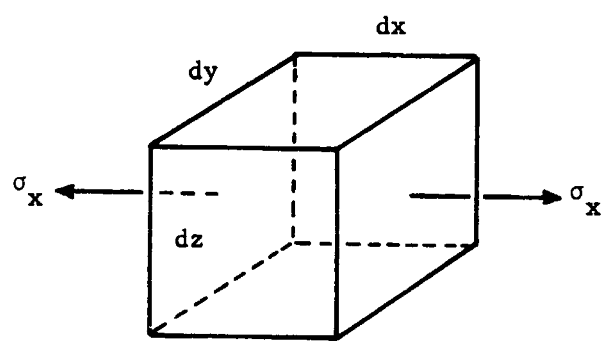
τ is the shear stress,

ε is the normal strain, and

γ is the shear strain.

Two basic approaches can be used to obtain the stiffness matrix. Either the deflection can be defined in terms of the panel point deflections, as was done in calculating the mass matrix, or the stress distribution can be assumed.

As an example in the use of Equation (C-1), consider a beam with longitudinal axis x and an infinitesimal element with normal stress σ_x as shown in the sketch.



Due to the force (σ_x dy dz) the element elongates a distance $\varepsilon_x$dx where ε_x is the strain. Assuming the element is initially unstrained, the work done is the average force on the element during the deformation. In a perfectly elastic body, no energy is dissipated; thus the work done is stored as recoverable internal strain energy and is given as

$$U = \iiint \left(\frac{1}{2} \sigma_x \, dy \, dz \right) \left(\varepsilon_x \, dx \right) \tag{C-2}$$

Using Hooke's law, that is,

$$\sigma = \varepsilon E$$

where E is the modulus of elasticity of the material, we obtain

$$U = \frac{1}{2} \iiint E \, \varepsilon_x^2 \, dx \, dy \, dz \tag{C-3}$$

If q_y (x,t) is the lateral displacement of the neutral axis of the beam, the strain is given by

$$\varepsilon_x = y \frac{\partial^2 q_y}{\partial x^2} \quad . \tag{C-4}$$

Substituting Equation (C-4) into (C-3) gives the strain energy as

$$U = \frac{1}{2} \int_0^L \iint E \, y^2 \left(\frac{\partial^2 \, q_y}{\partial x^2} \right)^2 dy \, dz \, dx$$

$$= \frac{1}{2} \int_0^L E \, I \, (x) \left(\frac{\partial^2 \, q_y}{\partial x^2} \right)^2 dx \tag{C-5}$$

where the cross-sectional moment of inertia, I, has been defined as

$$I = \iint y^2 \, dy \, dz. \tag{C-6}$$

In the above approach, the displacement function $q_y(x,t)$ would be assumed as some function of the displacements of panel points on the structure.

Rather than assume the displacement function to obtain the stiffness matrix, an alternative procedure is available whereby the stress function is assumed. In this approach, the bending stress is given by

$$\sigma_x = \frac{M\,y}{I} \tag{C-7}$$

where y is the distance from the neutral axis to the element,

I is the cross-sectional moment of inertia about the neutral axis, and

M is the bending moment.

Substituting Equation (C-7) into (C-2) along with Hooke's equation as $\varepsilon = \sigma/E$, gives the strain energy as

$$U = \frac{1}{2}\iiint \frac{M^2 y^2}{E\,I^2}\,dy\,dz\,dx \tag{C-8}$$

$$= \frac{1}{2}\int_0^L \frac{M^2}{EI(x)}\,dx \tag{C-9}$$

where the cross-sectional moment of inertia is given by Equation (C-6).

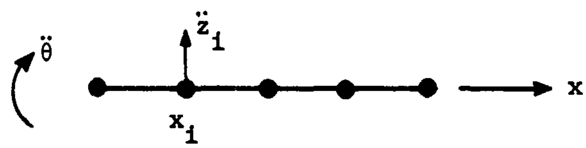
This latter approach is explained in detail in the explanations of Subroutine STIF1 (a longitudinal rod or torsional bar) and Subroutine STIF2 (a beam).

Much attention has been given to the subject of finite element representation of structural stiffness properties. For example, see References 12 thru 16.

D. INERTIAL REACTION TRANSFORMATION

The inertial reaction transformation relates inertia plus applied forces to applied forces for an ungrounded structure.

A physical interpretation of this transformation matrix may be obtained from the following example. Consider a beam with a single degree of freedom (translation) at each panel point.



The inertia forces are defined by

$$\{F_I\} = -[A]\{\ddot{z}\} \tag{D-1}$$

where [A] is the mass matrix of the beam and $\{\ddot{z}\}$ is the panel point accelerations. The panel point accelerations are given by

$$\{\ddot{z}\} = \left[\{1\}\ \{x_R - x\}\right] \begin{bmatrix} \ddot{z}_R \\ \ddot{\theta} \end{bmatrix} \tag{D-2}$$

where x_R denotes a reference station that may have any value and may or may not coincide with a panel point station. Let us define

$$[RBM] = \left[\{1\}\ \{x_R - x\}\right] , \tag{D-3}$$

that is, the rigid body modes for the beam. The rigid body translation mode is given by {1} and the rigid body rotation mode is given by $\{x_R - x\}$. Substituting Equations (D-2) and (D-3) into (D-1) gives

$$\{F_I\} = -[A][RBM]\begin{bmatrix} \ddot{z}_R \\ \ddot{\theta} \end{bmatrix} \tag{D-4}$$

For loads equilibrium, the sum of the forces and the moments about a reference station must be zero. That is

$$\begin{bmatrix} \{1\}^T \\ \{x_R - x\}^T \end{bmatrix} \left[\{F_A\} + \{F_I\} \right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(D-5)}$$

where $\{F_A\}$ is the applied external forces on the beam. From Equation (D-5), noting that $\begin{bmatrix} \{1\}^T \\ \{x_R - x\}^T \end{bmatrix} = [RBM]^T$, we have

$$-[RBM]^T \{F_I\} = [RBM]^T \{F_A\} \qquad \text{(D-6)}$$

Substituting Equation (D-4) into (D-6) gives

$$[RBM]^T [A] [RBM] \begin{bmatrix} \ddot{z}_R \\ \ddot{\theta} \end{bmatrix} = [RBM]^T \{F_A\}$$

from which

$$\begin{bmatrix} \ddot{z}_R \\ \ddot{\theta} \end{bmatrix} = \left[[RBM]^T [A] [RBM] \right]^{-1} [RBM]^T \{F_A\}. \qquad \text{(D-7)}$$

Substituting Equation (D-7) into (D-4) gives the inertia loads in terms of the applied loads as

$$\{F_I\} = -[A][MESS] \{F_A\} \qquad \text{(D-8)}$$

where we have defined

$$[MESS] = [RBM] \left[[RBM]^T [A] [RBM] \right]^{-1} [RBM]^T. \qquad \text{(D-9)}$$

Now the applied plus inertia loads can be expressed as

$$\begin{aligned} \{F_A\} + \{F_I\} &= \{F_A\} - [A][MESS] \{F_A\} \\ &= \left[[I] - [A][MESS] \right] \{F_A\} \qquad \text{(D-10)} \end{aligned}$$

Define [1-AMESS] = $\left[[I] - [A][MESS] \right]$. [1-AMESS] is the matrix relating the applied plus inertia loads to the applied loads. It can be shown that [1-AMESS] is independent of the reference station x_R.

It can be shown that any number of rigid body modes may be used even though this demonstration used two rigid body modes.

Subroutine UMAM1 is programmed to calculate the transformation matrix [1-AMESS].

Several important features of [1-AMESS] exist. The first deals with the triple matrix product

$$[1\text{-AMESS}]^T \left[E_R\right] [1\text{-AMESS}]$$

where $\left[E_R\right]$ is the structural flexibility matrix of a structure restrained in a statically determinant fashion. The result of this operation is a "free-free" structural influence coefficient matrix where the grounding restraints of the structure have been removed. It is somewhat helpful to think that the grounding restraints of the structure have been replaced by inertial restraints.

A second important feature is that the product $\left[E_F\right]\left[K_F\right]$ = [1-AMESS] where $\left[E_F\right]$ is the free-free structural flexibility matrix and $\left[K_F\right]$ is the free-free structural stiffness matrix.

An interesting property of [1-AMESS] is that it is an *idempotent* matrix, that is, a matrix whose product with itself is equal to itself. This can be easily shown as follows:

$$\Big[[I] - [A][MESS]\Big]\Big[[I] - [A][MESS]\Big]$$

$$= [I] - [A][MESS] - [A][MESS] + [A][MESS][A][MESS]$$

Now

$$[A][MESS][A][MESS]$$

$$= [A][RBM]\Big[[RBM]^T[A][RBM]\Big]^{-1}\ [RBM]^T[A][RBM]$$

$$\Big[[RBM]^T[A][RBM]\Big]^{-1}\ [RBM]^T$$

$$= [A][RBM]\Big[[RBM]^T[A][RBM]\Big]^{-1}\ [RBM]^T$$

$$= [A][MESS]$$

$$\therefore \Big[[I] - [A][MESS]\Big]\Big[[I] - [A][MESS]\Big] = [I] - [A][MESS].$$

This technique was formulated by Mr. David Lang at Chance Vought Aircraft (now a division of LTV) in an informal memorandum in 1959. Later investigation showed a similar technique (although obscure) in Reference 3.

To demonstrate that the triple matrix product $[1\text{-AMESS}]^T \left[E_R\right]$ $[1\text{-AMESS}]$ results in a free-free structural flexibility matrix, consider the following discussion:

If the supports of a structure can provide equilibration of the applied forces, then the structural flexibility matrix, [E], is simply obtained by matrix inversion of the structural stiffness matrix, [S]. If, however, there are no supports, or the supports are not adequate to prevent rigid body motion (i.e., a hinged-free beam), then $[E] = [S]^{-1}$ does not exist since there are one or more nontrivial solutions that satisfy

$$[S]\left[h_R\right] = [0] \qquad \text{(D-11)}$$

The columns of $\left[h_R\right]$ satisfying Equation (D-11) are rigid body displacements.

Let us consider a quasi-static problem wherein time invariant forces act on an unsupported structure. If structural damping is present, then we may conceive of a steady-state solution in which the structure moves with constant acceleration (rigid-body) as a deformed body. Now, the general motion of an unrestrained structure might be thought of as a linear combination of the rigid body modes plus a linear combination of the "elastic" modes, or

$$\{h\} = \left\{h_R\right\} + \left\{h_E\right\}, \text{ and}$$

$$\left\{h_R\right\} = \left[\phi_R\right]\left\{q_R\right\},$$

$$\left\{h_E\right\} = \left[\phi_E\right]\left\{q_E\right\}.$$

The equations of motion for this situation would be

$$[A]\left\{\ddot{h}_R\right\} + [S]\left\{h_E\right\} = \left\{F_A\right\} \qquad \text{(D-12)}$$

Because we have stated that $\ddot{q}_E = \dot{q}_E = 0$, $[B]\{\dot{h}_R\} = [S]\{h_R\} = \{0\}$. If we premultiply Equation (D-12) through by $\left[\phi_R\right]^T$, we obtain

$$\{\ddot{q}_R\} = \left(\left[\phi_R\right]^T[A]\left[\phi_R\right]\right)^{-1}\left[\phi_R\right]^T\{F_A\}$$

and we may write

$$[S]\{h_E\} = \left([\ I\ ] - [A]\left[\phi_R\right]\left(\left[\phi_R\right]^T[A]\left[\phi_R\right]\right)^{-1}\left[\phi_R\right]^T\right)\{F_A\}. \tag{D-13}$$

Equation (D-13) is the relationship between the elastic restoring forces $[S]\ \{h_E\}$ and the applied forces $\{F_A\}$. For conciseness, let us define again

$$[1\text{-AMESS}] \equiv [\ I\ ] - [A]\left[\phi_R\right]\left(\left[\phi_R\right]^T[A]\left[\phi_R\right]\right)^{-1}\left[\phi_R\right]^T.$$

Let us turn our attention back to Equation (D-13). It reads

$$[S]\{h_E\} = [1\text{-AMESS}]\{F_A\}, \tag{D-14}$$

$$\text{or } ([S] + [\Delta S])\{h_E\} = [1\text{-AMESS}]\{F_A\} + [\Delta S]\{h_E\}. \tag{D-15}$$

If $[\Delta S]$ is a diagonal matrix with only nonzero values corresponding to "ground springs" sufficient to provide support to the structure, then by adding $[\Delta S]$ to $[S]$, the rank of the sum has been made equal to N, the number of equations, and $[E] = ([S] + [\Delta S])^{-1}$ exists. The matrix $[E]$ corresponds to the flexibility of the original structure with sufficient support springs included to prevent rigid-body motion.

It is clear that the minimum number of support springs required for this equals the defficiency of rank of $[S]$ for $[S]^{-1}$ to exist, and also is the number of rigid body modes involved in the unrestrained structure.

If we now premultiply Equation (D-15) by $[1\text{-AMESS}]^T[E]$, we obtain

$$[1\text{-AMESS}]^T\{h_E\} = [1\text{-AMESS}]^T[E][1\text{-AMESS}]\{F_A\}$$

$$+ [1\text{-AMESS}]^T[E][\Delta S]\{h_E\}. \tag{D-16}$$

The left-hand side of Equation (D-16) is

$$[1\text{-AMESS}]^T\{h_E\} = \left([\ I\ ] - [A]\left[\phi_R\right]\left(\left[\phi_R\right]^T[A]\left[\phi_R\right]\right)^{-1}\left[\phi_R\right]^T\right)^T\left[\phi_E\right]\{q_E\}$$

$$= \left([\ I\ ] - \left[\phi_R\right]\left(\left[\phi_R\right]^T[A]\left[\phi_R\right]\right)^{-1}\left[\phi_R\right]^T[A]\right)\left[\phi_E\right]\{q_E\}.$$

Using $\left[\phi_R\right]^T[A]\left[\phi_E\right] = [0]$ due to orthogonality,

$$[1\text{-AMESS}]^T\{h_E\} = \left[\phi_E\right]\{q_E\}$$

$$= \{h_E\}. \tag{D-17}$$

The second term of Equation (D-16) contains the factor

$$[E][\Delta S]\{h_E\}.$$

The $[\Delta S]$ matrix contains only nonzero elements at diagonal locations corresponding to support spring locations. The product $[\Delta S]\{h_E\}$ represents a vector of forces all zero except for the locations corresponding to the support springs. Now the product $[E][\Delta S]\{h_E\}$ corresponds to a displacement vector wherein *only* the support springs have elastic deformation, because the forces $[\Delta S]\{h_E\}$ act only on the support springs. Hence, define the displacement vector as some linear combination of the rigid-body modes.

$$[E][\Delta S]\{h_E\} \equiv \left[\phi_R\right]\{\eta_R\} \tag{D-18}$$

It is clear that if *more* than enough support springs were added than necessary to provide support, then the above argument would not hold. Hence, the structural flexibility matrix [E] must reflect a *statically determinant* set of supports.

From these considerations, the second term of Equation (D-16) becomes

$$[1\text{-AMESS}]^T \left[\phi_R\right] \{\eta_R\}$$

$$= \left([\ I\ ] - [A] \left[\phi_R\right] \left(\left[\phi_R\right]^T [A] \left[\phi_R\right] \right)^{-1} \left[\phi_R\right]^T \right)^T \left[\phi_R\right] \{\eta_R\}$$

$$= \left([\ I\ ] - \left[\phi_R\right] \left(\left[\phi_R\right]^T [A] \left[\phi_R\right] \right)^{-1} \left[\phi_R\right]^T [A] \right) \left[\phi_R\right] \{\eta_R\}$$

$$= \left(\left[\phi_R\right] - \left[\phi_R\right] \right) \{\eta_R\}$$

$$= \{0\}. \tag{D-19}$$

Substituting these results, that is Equations (D-17) and (D-19) into (D-16) yields

$$\{h_E\} = [1\text{-AMESS}]^T [E] [1\text{-AMESS}] \{F_A\}. \tag{D-20}$$

$$= \left[E_F\right] \{F_A\} \tag{D-21}$$

where $\left[E_F\right] \equiv [1\text{-AMESS}]^T [E][1\text{-AMESS}]$ fits the definition of a structural flexibility matrix that is unrestrained or more correctly, ungrounded. The ground springs in [E] have been replaced by inertial forces to restrain the structure.

An interesting aspect of Equation (D-20) is that the force vector [1-AMESS] $\{F_A\}$ acts on a grounded structure having influence coefficients [E] associated with it. The deflection vector [E] [1-AMESS] $\{F_A\}$ is translated and rotated by the transformation $[1\text{-AMESS}]^T$ such that the resulting elastic deflections $\{h_E\}$ are measured with respect to the principal axes of inertia of the *deformed structure.*

We might ask what are the loads in the support springs due to application of the load vector [1-AMESS] $\{F_A\}$. We know that we can solve for support reactions from equations of static equilibrium *if* the reactions are statically determinant. These equations of static equilibrium are of the form

$$\{R\} = [B]\left[\phi_R\right]^T\{P\}, \qquad \text{(D-22)}$$

where {R} is a vector of reactions, and {P} is *any* set of external forces. We have replaced {P} by [1-AMESS] $\{F_A\}$, therefore

$$\{R\} = [B]\left[\phi_R\right]^T\left([\ I\ ] - [A]\left[\phi_R\right]\left(\left[\phi_R\right]^T[A]\left[\phi_R\right]\right)^{-1}\left[\phi_R\right]^T\right)\{F_A\}$$

$$= [B]\left(\left[\phi_R\right]^T - \left[\phi_R\right]^T\right)\{F_A\}$$

$$= \{0\}.$$

This is all to say that the reactions in the support springs vanish because [1-AMESS] $\{F_A\}$ are the applied plus inertial forces which are a set of self-equilibrated forces.

E. VIBRATION ANALYSES

The vibration analysis of a structure consists of calculating the natural frequencies and mode shapes of the conservative (undamped) and homogeneous (unforced or free), second order differential equations obtained from equation (A-6) as

$$[A]\{\ddot{h}(t)\} + [S]\{h(t)\} = \{0\} \tag{E-1}$$

The purpose of this calculation is to obtain

1) A set of orthogonal vectors (mode shapes) so that the coupled equations can be reduced to a set of uncoupled equations;
2) Information as to critical frequencies and optimum locations of sensors for control system design.

As before, $[A]$ is the mass matrix and $[S]$ is the stiffness matrix. $\{h(t)\}$ is a column of discrete coordinate displacements about a position of stable equilibrium.

The time dependence in Equation (E-1) can be removed by the transformation $\{h(t)\} = \alpha\{\phi\}e^{j\omega t}$ because the solutions of Equation (E-1) are assumed to be harmonic in time. Thus, for a nontrivial solution, Equation (E-1) becomes

$$(-\omega^2[A] + [S])\{\phi\} = \{0\}. \tag{E-2}$$

Equation (E-2) is recognized as a general eigenvalue problem of order N with eigenvector $\{\phi\}$ (mode shape) and eigenvalue ω^2 (circular frequency squared). There are N values of ω^2 and $\{\phi\}$ so that the expansion of Equation (E-2) to include all solutions can be expressed as

$$[A]\left[\{\phi\}_1 \,\vdots\, \{\phi\}_2 \,\vdots\, \cdots \,\vdots\, \{\phi\}_N\right]\begin{bmatrix}\omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_N^2\end{bmatrix} = [S]\left[\{\phi\}_1 \,\vdots\, \{\phi\}_2 \,\vdots\, \cdots \,\vdots\, \{\phi\}_N\right] \tag{E-3}$$

$$\text{or } [A][\Phi]\lceil \omega^2 \rfloor = [S][\Phi]. \tag{E-3a}$$

If the flexibility matrix [E] of the structure exists, the following equation is used for the vibration analysis.

$$[E][A][\Phi] = [\Phi]\left[\frac{1}{\omega^2}\right] \tag{E-4}$$

where [E] is obtained as follows. Assume the structure is grounded so that the stiffness matrix is nonsingular and its inverse exists. The inverse of the stiffness matrix is defined to be the flexibility matrix, that is,

$$[E] = [S]^{-1}. \tag{E-5}$$

Using this relationship in Equation (E-3a) and postmultiplying by $\left[\omega^2\right]^{-1}$ gives Equation (E-4).

If the structure is not grounded, the stiffness matrix, [S], is singular, and Equation (E-5) is not applicable. The flexibility matrix, [E], can still be obtained by other techniques (either directly as mentioned in Section C or by operation on the stiffness matrix as mentioned in Section D) so that Equation (E-4) is still valid.

Vibration analyses can be calculated using Subroutines MODE1, MODE1A (for a system with mass and stiffness matrices) and Subroutine MODE1B (for a system with mass and flexibility matrices).

Many different methods are available for calculating the natural frequencies and mode shapes of a structure. For example, see Reference 23.

F. MODAL COUPLING

The synthesis of complex dynamic systems through the use of modal coupling provides a powerful tool for the analyst. Modal coupling is an analytical technique by which the modal characteristics of a complete structural system are formulated by coupling the modal properties of a number of component subsystems.

The primary application of modal coupling is to calculate the modal characteristics of a large structural system in a more economical manner than by calculating the modal characteristics using the mass and stiffness matrices of the complete structure. Also, in the analysis of large structural systems, the number of degrees of freedom may exceed the computer capacity. In both of these cases, it is feasible to separate the total system into subsystems, calculate the modal properties of the subsystems, and then combine these results to obtain the modal properties of the complete structure.

An energy approach for problem formulation is basic to the derivation of equations used for modal coupling. The energy approach offers unique advantages in the formulation of the differential equations of motion for complex structural systems. This approach allows any system to be divided into a number of subsystems that may then be considered independently. The energy of each subsystem can be expressed in a convenient set of coordinates and, since energy is a scalar quantity, the total system energy can be expressed as the sum of the energies of the several subsystems. The differential equations of motion follow from the application of Lagrange's formulation. The usual coordinates for this type of solution are the normal modes of the various subsystems and the resulting equations of motion are therefore also expressed in terms of subsystem normal modes. This procedure forms the basis for the modal coupling analysis.

The subject of modal coupling has received much attention lately and good documentation has been given to it in the literature. For example, see References 5 thru 11.

As an example of some elementary techniques of modal coupling, consider the problem of a structural system composed of substructures denoted as body (a) and body (b) as shown in the sketch. The modal characteristics, both frequency and mode shapes, are wanted for the complete structure.

The kinetic energy of the complete structure is given in terms of the substructures as

$$T = \frac{1}{2}\left(\{\dot{h}\}_a^T\,[A]_a\,\{\dot{h}\}_a + \{\dot{h}\}_b^T\,[A]_b\,\{\dot{h}\}_b\right)$$

$$= \frac{1}{2}\begin{bmatrix}\{\dot{h}\}_a^T & \{\dot{h}\}_b^T\end{bmatrix}\begin{bmatrix}[A]_a & 0 \\ 0 & [A]_b\end{bmatrix}\begin{bmatrix}\{\dot{h}\}_a \\ \{\dot{h}\}_b\end{bmatrix} . \qquad \text{(F-1)}$$

The strain energy is likewise given as

$$U = \frac{1}{2}\left(\{h\}_a^T\,[S]_a\,\{h\}_a + \{h\}_b^T\,[S]_b\,\{h\}_b\right)$$

$$= \frac{1}{2}\begin{bmatrix}\{h\}_a^T & \{h\}_b^T\end{bmatrix}\begin{bmatrix}[S]_a & 0 \\ 0 & [S]_b\end{bmatrix}\begin{bmatrix}\{h\}_a \\ \{h\}_b\end{bmatrix} , \qquad \text{(F-2)}$$

where

$[A]_a$ = mass matrix of substructure a,

$[S]_a$ = stiffness matrix of substructure a, and

$\{h\}_a$ = *discrete absolute* displacements.

Subscript b has similar meaning. The displacement of substructure b is given by

$$\{h\}_b = [T]_{ba}\,\{h\}_a + \{h_R\}_b , \qquad \text{(F-3)}$$

where $[T]_{ba}$ is a transformation giving the displacement of body b in terms of body a, and $\{h_R\}_b$ is the displacement of body b *relative* to the interface between body b and body a. Therefore,

$$\begin{bmatrix} \{h\}_a \\ \{h\}_b \end{bmatrix} = \begin{bmatrix} [I] & [0] \\ [T]_{ba} & [I] \end{bmatrix} \begin{bmatrix} \{h\}_a \\ \{h_R\}_b \end{bmatrix} . \tag{F-4}$$

When uncoupled mode shapes are obtained for bodies a and b,

$$\{h\}_a = [\phi]_a \{q\}_a \tag{F-5a}$$

$$\{h_R\}_b = [\phi]_b \{q\}_b \tag{F-5b}$$

where

$\{q\}_a$ is a *normal absolute* displacement; and

$\{q\}_b$ is a *normal relative* displacement.

$[\phi]_a$ are normal modes of body a. May be obtained using $[A]_a$ only, or $[A]_a + [T]^T_{ba}[A]_b[T]_{ba}$ or any assumed shape, or even random numbers. It has been observed (Ref 5) however, that better coupled modes result from using $[A]_a + [T]^T_{ba}[A]_b[T]_{ba}$ as the mass matrix for calculating $[\phi]_a$. Substitution of Equations (F-5a) and (F-5b) into (F-4) gives

$$\begin{bmatrix} \{h\}_a \\ \{h\}_b \end{bmatrix} = \begin{bmatrix} [\phi]_a & [0] \\ [T]_{ba}[\phi]_a & [\phi]_b \end{bmatrix} \begin{bmatrix} \{q\}_a \\ \{q\}_b \end{bmatrix} \tag{F-6}$$

Substitution of Equation (F-6) into (F-1) gives

$$T = \frac{1}{2} \begin{bmatrix} \{\dot{q}\}^T_a & \{\dot{q}\}^T_b \end{bmatrix} \begin{bmatrix} [\phi]^T_a\left([A]_a + [T]^T_{ba} [A]_b [T]_{ba}\right)[\phi]_a & [\phi]^T_a [T]^T_{ba} [A]_b [\phi]_b \\ [\phi]^T_b [A]_b [T]_{ba} [\phi]_a & [\phi]^T_b [A]_b [\phi]_b \end{bmatrix} \begin{bmatrix} \{\dot{q}\}_a \\ \{\dot{q}\}_b \end{bmatrix} \tag{F-7}$$

$$= \frac{1}{2} \{\dot{q}\}^T [A] \{\dot{q}\} \tag{F-8}$$

Substitution of Equation (F-6) into (F-2) gives the same form for the kernal matrix as previously obtained in Equation (F-7). However, because $[S]_b\ [T]_{ba} = [0]$, the strain energy equation reduces to

$$U = \frac{1}{2}\begin{bmatrix}\{q\}_a^T & \{q\}_b^T\end{bmatrix}\begin{bmatrix}[\phi]_a^T\ [S]_a\ [\phi]_a & [0] \\ [0] & [\phi]_a^T\ [S]_b\ [\phi]_b\end{bmatrix}\begin{bmatrix}\{q\}_a \\ \{q\}_b\end{bmatrix} \tag{F-9}$$

$$= \frac{1}{2}\ \{q\}^T\ [S]\ \{q\} \tag{F-10}$$

Another way to obtain Equation (F-9) is to consider the strain energy as

$$U = \frac{1}{2}\left(\{h\}_a^T\ [S]_a\ \{h\}_a + \left\{h_R\right\}_b^T\ [S]_b\ \left\{h_R\right\}_b\right)$$

$$= \frac{1}{2}\begin{bmatrix}\{h\}_a^T & \left\{h_R\right\}_b^T\end{bmatrix}\begin{bmatrix}[S]_a & [0] \\ [0] & [S]_b\end{bmatrix}\begin{bmatrix}\{h\}_a^T \\ \left\{h_R\right\}_b^T\end{bmatrix}$$

Substitution of Equations (F-5a) and (F-5b) into the above equation gives Equation (F-9).

It is obvious that $[S]_b\ [T]_{ba} = 0$ when the interface tie between the substructures is statically determinant. It is shown in Reference 5 that this is also true even for a statically indeterminant tie.

Using Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0$$

with Equations (F-8) and (F-10) gives

$$[A]\ \{\ddot{q}\} + [S]\ \{q\} = \{0\}. \tag{F-11}$$

Assuming $\{q\} = [\Phi]\{\zeta\}$, the resulting eigen problem is

$$[A][\Phi][\Omega^2] = [S][\Phi] \tag{F-12}$$

where $[\Omega]$ is the desired *coupled* frequency. Because $\{q\} = [\Phi]$ $\{\zeta\}$, then in Equation (F-6)

$$\begin{bmatrix} \{h\}_a \\ \{h\}_b \end{bmatrix} = \begin{bmatrix} [\phi]_a & [0] \\ [T]_{ba}[\phi]_a & [\phi]_b \end{bmatrix} [\Phi]\{\zeta\}, \tag{F-13}$$

where

$$\begin{bmatrix} [\phi]_a & [0] \\ [T]_{ba}[\phi]_a & [\phi]_b \end{bmatrix} [\Phi]$$

are the desired coupled mode shapes.

G. TIME RESPONSE ANALYSIS

The time response analysis of a structure consists of the solution of Equation (G-1) (repeated here from Section A) for $\{\ddot{h}(t)\}$, $\{\dot{h}(t)\}$, and $\{h(t)\}$ using numerical integration.

$$[A]\{\ddot{h}(t)\} + [B]\{\dot{h}(t)\} + [S]\{h(t)\} = \{F(t)\} \tag{G-1}$$

where [A] is the mass matrix of the structure,
[B] is the damping matrix,
[S] is the stiffness matrix,
{F(t)} is the forces acting on the structure, and
{h(t)} is the displacements at selected points on the structure.

Two numerical integration techniques are included in this report: the Runge-Kutta technique, [see explanations (Volume III) and listings (Volume IV) of Subroutines TR1, TR1A, TR1B, and TR1C], and the Newmark-Beta technique given by Subroutines TR2 and TR2A.

The solution of Equation (G-1) is not easy however. First, the damping matrix, [B], for the discrete coordinate representation is not well defined (Ref 17). Second, high frequency content implicit in the discrete coordinate model can result in divergence of the numerical integration procedure. Third, a less expensive solution can be obtained by using a modal representation of the structure because a lesser number of degrees of freedom will be used.

From a vibration analysis of the undamped and unforced (or free) equations of motion, we obtain

$$\{h(t)\} = [\phi]\,\{q(t)\} \tag{G-2}$$

as mentioned in Section E.

{h(t)} is the discrete coordinate displacements,
[ϕ] is the mode shapes, and
{q(t)} is the modal coordinate displacements.

Substitution of Equation (G-2) into (G-1) and premultiplying $[\phi]^T$ gives

$$[\phi]^T[A][\phi]\{\ddot{q}(t)\} + [\phi]^T[B][\phi]\{\dot{q}(t)\} + [\phi]^T[S][\phi]\{q(t)\} = [\phi]^T\{F(t)\}. \quad \text{(G-3)}$$

Now, $[\phi]^T[A][\phi] = \left\lceil G_M \right\rfloor$ (generalized mass)

$$= \lceil I \rfloor$$

because of the normalization used in this report. Therefore, $[\phi]^T[S][\phi] = \lceil \omega^2 \rfloor$ from inspection of Equation (E-3a). It is assumed that

$$[\phi]^T[B][\phi] = \left\lceil 2\zeta\omega G_M \right\rfloor$$

$$= \lceil 2\zeta\omega \rfloor$$

because $G_M = 1$.

Now, $\zeta = \frac{c}{c_{CR}}$, the ratio of viscous damping, c, to critical damping, c_{CR}. It should be noted that structural damping, g, is related to the damping ratio by

$$g = 2\zeta$$

as given in Reference 18. From these expressions, Equation (G-3) can be written as

$$\lceil I \rfloor\{\ddot{q}(t)\} + \lceil 2\zeta\omega \rfloor\{\dot{q}(t)\} + \lceil \omega^2 \rfloor\{q(t)\} = [\phi]^T\{F(t)\}. \quad \text{(G-4)}$$

This set of equations is uncoupled differential equations and can be conveniently solved for $\{\ddot{q}(t)\}$, $\{\dot{q}(t)\}$, and $\{q(t)\}$ using Subroutine TR3. The solution is obtained in closed form and avoids any difficulties associated with numerical integration.

The nature of the structural problem being solved may not permit a solution using Equation (G-4). For example, inclusion of a control system or aerodynamic forces dependent on structural elastic displacement and velocity will introduce coupling terms in the coefficient matrices of $\{\ddot{q}(t)\}$, etc. When this occurs, the solution of the coupled differential equations cannot be obtained in closed form but must again be obtained by numerical integration techniques, as in the Subroutines TR1, TR1A, TR1B, TR1C and Subroutines TR2, TR2A.

As an example of the control system and aerodynamic coupling introduced in the equations of motion with modal coordinates, that is Equation (G-4), consider the following gust analysis.

The following assumptions and ground rules were used:

1) The gust had a maximum amplitude of 30 ft/sec and a shape of one minus cos ωt;

2) The minimum base wavelength of the gust was 200 ft and its maximum base wavelength was 1500 ft;

3) The gust was assumed to envelop the entire vehicle at once;

4) Quasi-steady state aerodynamic theory was assumed to be valid.

The minimum base wavelength considered was that which would tune the gust frequency to the first analytical bending mode. The minimum gust period considered was 0.35 sec, and the maximum considered was 2.4 sec; however, the maximum period for a base wavelength of 1500 ft is 1.4 sec at the headwind $q\alpha_t$ flight time. Since the gust-response loads were found to peak beyond the 1.4-sec period, the higher loads (corresponding to a period of 1.8 sec) were used to determine the total loads on the vehicle.

Beginning with the generalized equation of motion, and scaling the modal amplitudes to produce a normalized mass of one, we have Equation (G-4) again.

$$\lceil I \rfloor \{\ddot{q}(t)\} + \lceil 2\zeta\omega \rfloor \{\dot{q}(t)\} + \lceil \omega^2 \rfloor \{q(t)\} = [\phi]^T \{F(t)\}.$$

The vector $\{F\}$ is composed of two basic parts:

$\{F_A\}$ = Aerodynamic vector;

$\{F_E\}$ = Engine vector.

Excluding the steady-state aerodynamic forces, the force vector takes the following general form:

$$\left\{F_A\right\} = \left[\int_0^L \frac{\partial N(x,\alpha,N)}{\partial \alpha}\, dx\right] \{\Delta\alpha(x)\} \tag{G-5}$$

where

$$\left[\int_0^L \frac{\partial N(x,\alpha,N)}{\partial \alpha}\right] dx = \left[\frac{N}{\alpha}\, ;\right] \tag{G-6}$$

and, for any panel point Sta j:

$$\Delta\alpha_j(x) = \frac{V_{g(t)}}{V_{rw}} - \frac{\dot{h}_j}{V_{rw}} + \theta_j; \tag{G-7}$$

where:

Q = Dynamic pressure;

$\frac{N}{Q\alpha_j}$ = Slope, with respect to α, of the lift coefficient at Sta j;

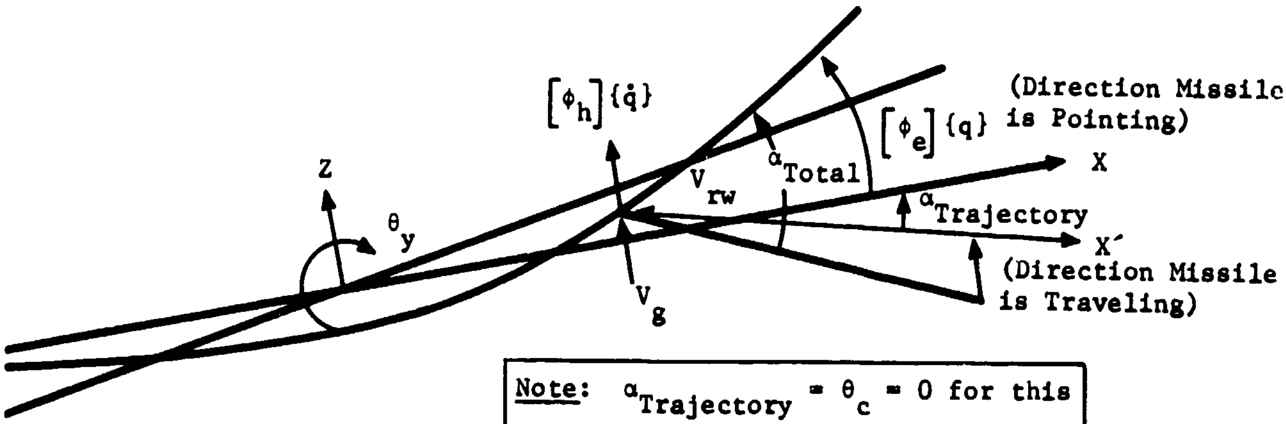
α_j = Angle-of-attack at Sta j;

$\dot{h}_j$ = Lateral velocity at Sta j;

θ_j = Lateral rotation at Sta j.

Now let $\dot{h}_j = \phi_{hj}\,\dot{q}_j$ and $\theta_j = \phi_{\theta j}\,q_j$ and substitute these expressions into Eq (G-7), over all panel-point stations. Then,

$$\{\Delta\alpha(x)\} = \overbrace{\frac{1}{V_{rw}}\left(\underbrace{\{1\}\, V_g\,(t) - \left[\phi_h\right]\{\dot{q}\}}_{\text{Term B}}\right)}^{\text{Term A}} + \underbrace{\left[\phi_\theta\right]\{q\}}_{\text{Term C}}, \tag{G-8}$$



<u>Note</u>: $\alpha_{Trajectory} = \theta_c = 0$ for this analysis.

where:

Term A = Tangent of the angle between the wind and the trajectory = $\frac{\text{Lateral}}{\text{Longitudinal}}$;

Term B = Difference between the velocity of the gust and the lateral velocity of the missile panel points;

Term C = Sine of the angle that the deformed missile makes with the trajectory;

V_{rw} = Relative wind velocity due to flight along the trajectory;

V_g = Gust velocity;

$[\phi_h]$ = Modal deflection in the lateral plane;

$[\phi_\theta]$ = Modal slopes;

$\{\dot{q}\}$ = Modal* velocities due to the gust response;

$\{q\}$ = Modal* displacements due to the gust.

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*The term "modal" refers both to rigid-body and to elastic-body modes.

By substituting Eq (G-8) into Eq (G-5) and premultiplying by $[\phi]^T$, we obtain an equation for the normalized aerodynamic force:

$$[\phi]^T\{F_A\} = [\phi]^T \left[\int_0^L \frac{\partial N(x,\alpha,N)}{\partial\alpha}\, dx\right]\left[\frac{1}{V_{rw}}\{1\}\left(V_g - \left[\phi_h\right]\{\dot{q}\}\right)\right.$$

$$\left. + \left[\phi_e\right]\{q\}\right]. \qquad \text{(G-9)}$$

Abbreviating Eq (G-9)

$$[\phi]^T\{F_A\} = \left\{N_g\right\} V_g(t) - [\dot{N}]\{\dot{q}\} + [N]\{q\}; \qquad \text{(G-10)}$$

where:

$$\left\{N_g\right\} = [\phi]^T \frac{Q}{V_{rw}}\left[\frac{N}{Q\alpha}\right];$$

$$\left\{F_A\right\} = \frac{V_g}{V_{rw}}\, Q\left[\frac{N}{Q\alpha}\right].$$

$$\left[\frac{N}{Q\alpha}\right] = \frac{1}{Q}\left[\int_0^L \frac{\partial N(x,\alpha,N)}{\partial\alpha}\, dx\right],$$

that is, the representation of the distributed air forces on a beam is calculated by Subroutine ALOD1.

By substituting Eq (G-10) into Eq (G-4), we obtain:

$$\left[1\right]\{\ddot{q}\} + \left(\left[2\zeta\omega\right] + [\dot{N}]\right)\{\dot{q}\} + \left(\left[\omega^2\right] - [N]\right)\{q\}$$

$$= \left\{N_g\right\} V_g(t) - [\phi]^T\left\{F_E\right\},$$

where:

$$[\dot{N}] = \frac{Q}{V_{rw}}\left[\phi_h\right]^T\left[\frac{N}{Q\alpha}\right]\left[\phi_h\right];$$

$$[N] = Q\left[\phi_h\right]^T\left[\frac{N}{Q\alpha}\right]\left[\phi_\theta\right];$$

$$[\phi]^T\left\{F_E\right\} = [\phi]^T\left\{P_{os}\right\} T\,\delta_E(t) = \left\{\phi_{hg}\right\} T\,\delta_E(t);$$

and

ϕ_{hg} = Modal deflection of the gimbal in the lateral plane;

T = Thrust;

δ_E = Engine rotation command from the autopilot.

Thus, $\delta_E(t)$ should be included on the left side of the equation as a variable itself, while the time-dependent term $V_g(t)$ is assigned by specification and included on the right side:

$$\left[\begin{array}{c:c} 1 & 0 \\ \hdashline 0 & 0 \end{array}\right] \begin{Bmatrix} \ddot{q} \\ \ddot{\delta}_E \end{Bmatrix} + \left[\begin{array}{c:c} \left[2\zeta\omega\right] + [\dot{N}] & 0 \\ \hdashline 0 & 0 \end{array}\right] \begin{Bmatrix} \dot{q} \\ \dot{\delta}_E \end{Bmatrix}$$

$$+ \left[\begin{array}{c:c} \left[\omega^2\right] - [N] & \{T\phi_{hg}\} \\ \hdashline 0 & 0 \end{array}\right] \begin{Bmatrix} q \\ \delta_E \end{Bmatrix} = \begin{Bmatrix} N_g \\ \hdashline 0 \end{Bmatrix} V_g(t). \qquad \text{(G-11)}$$

Since more unknowns exist than equations, autopilot equations must be included to define the relationship between the structural response and the engine response.

Each block in the flight controls diagram contains the Laplace representation of the following transfer function:

$$E_{OUT} = \frac{(as + b)}{f(s)} E_{IN},$$

where f(s) is of quadratic or lower order.

This expression can be rewritten in matrix form as:

$$\left[\begin{array}{c:c} F(s) & (-as - b) \end{array}\right] \begin{Bmatrix} E_{OUT} \\ \hline E_{IN} \end{Bmatrix} = \{0\}.$$

For the several transfer functions in question, a series of equations can be stacked in matrix form as follows:

$$[G]\ \{E\} = \{0\}$$

where

$$G(I,J) = \alpha_{ij}\ s^2 + \beta_{ij}\ s + \gamma_{ij}.$$

Then,

$$G\{E\} = \alpha\{E\} + \beta\{E\} + \gamma\{E\} = 0.$$

Since the variable $\{E\}$ includes engine and autopilot variables $\{\delta\}$ and airframe modal coordinates $\{q\}$, we have:

$$\begin{bmatrix}\alpha_q & \alpha_\delta\end{bmatrix}\begin{Bmatrix}\ddot{q}\\ \ddot{\delta}\end{Bmatrix} + \begin{bmatrix}\beta_q & \beta_\delta\end{bmatrix}\begin{Bmatrix}\dot{q}\\ \dot{\delta}\end{Bmatrix} + \begin{bmatrix}\gamma_q & \gamma_\delta\end{bmatrix}\begin{Bmatrix}q\\ \delta\end{Bmatrix} = \{0\}. \qquad \text{(G-12)}$$

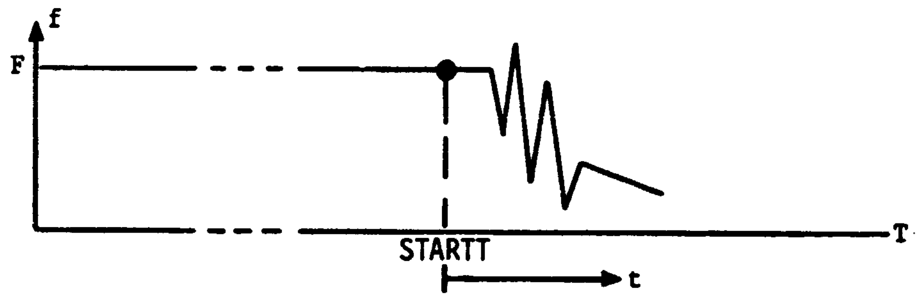
Then equation (G-11) may be expanded to include autopilot equation (G-12):

$$\left[\begin{array}{c|c}1 & 0\\ \hline \alpha_q & \alpha_\delta\end{array}\right]\begin{Bmatrix}\ddot{q}\\ \ddot{\delta}\end{Bmatrix} + \left[\begin{array}{c|c}\left[2\zeta\omega\right] + [\dot{N}] & 0\\ \hline \beta_q & \beta_\delta\end{array}\right]\begin{Bmatrix}\dot{q}\\ \dot{\delta}\end{Bmatrix}$$

$$+ \left[\begin{array}{c|c}\left[\omega^2\right] - [N] & \left\{T\phi_{hg}\right\}\\ \hline \gamma_q & \gamma_\delta\end{array}\right]\begin{Bmatrix}q\\ \delta\end{Bmatrix} = \begin{Bmatrix}N_g\\ 0\end{Bmatrix} V_g(t) \qquad \text{(G-13)}$$

Another common time response analysis is the thrust termination problem.

Assume the time history of the force is given below.



The second order differential equation to be solved is $m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$.

The force (F) is assumed to have been acting for such a long time prior to t = STARTT that all transients have damped so that:

$$x\ (t = \text{STARTT}) = \frac{F}{k} \tag{G-14}$$

$$\dot{x}\ (t = \text{STARTT}) = 0 \tag{G-15}$$

$$\ddot{x}\ (t = \text{STARTT}) = 0 \tag{G-16}$$

If the problem is started at t = STARTT, we will get step force response due to the force (F). To counteract this incorrect step force response, supply initial conditions for x and $\dot{x}$ given by equations (G-14) and (G-15) above. If k = 0, and c = 0 (rigid body mode), use x (t = STARTT) = anything.

Proof that the response due to the above initial conditions balances the response due to a step force is given as follows. From the explanation of Subroutine TR3 in Volume III, the response due to step force F is:

$$x_S(t) = \frac{F}{m\omega^2}\left[1 - \frac{e^{-at}}{b}\ (a \sin bt + b \cos bt)\right] \tag{G-17}$$

$$\dot{x}_S(t) = \frac{F}{mb}\ e^{-at} \sin bt \tag{G-18}$$

$$\ddot{x}_S(t) = \frac{F}{mb}\ e^{-at}\ (- a \sin bt + b \cos bt) \tag{G-19}$$

where

$$a = \frac{c}{2m}$$

$$b = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

The response due to initial displacement $x(t = \text{STARTT}) = \frac{F}{k}$ is:

$$x_D(t) = \frac{x(t = \text{STARTT})}{b}\ e^{-at}\ (a \sin bt + b \cos bt)$$

$$= \frac{F}{kb}\ e^{-at}\ (a \sin bt + b \cos bt) \tag{G-20}$$

$$\dot{x}_D(t) = -\frac{F}{kb}\,\omega^2\, e^{-at} \sin bt \tag{G-21}$$

$$\ddot{x}_D(t) = \frac{F}{kb}\,\omega^2\, e^{-at} (a \sin bt - b \cos bt) \tag{G-22}$$

Using $k = \omega^2 m$

Equations (G-17) + (G-20) gives $x(t) = \frac{F}{k}$ (G-23)

Equations (G-18) + (G-21) gives $\dot{x}(t) = 0$ (G-24)

Equations (G-19) + (G-22) gives $\ddot{x}(t) = 0$ (G-25)

These equations, (G-23), (G-24), and (G-25), agree with equations (G-14), (G-15), and (G-16), respectively.

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APPENDIX A

MATRIX ALGEBRA

Because the synthesis of dynamic systems using FORMA implies matrix analysis, a few fundamental concepts of matrices are given here. For more complete information, References 19 thru 21 are recommended.

1. A matrix is defined to be a rectangular array of numbers arranged in m rows and n columns as shown:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Each number a_{ij} is called an element of the matrix where the subscripts i, j denote the row, column, respectively, of the element. The order or size of the matrix is m x n.

1a. A matrix composed of a single row is referred to as a row or row vector. For example,

$$\{B\}^T = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \end{bmatrix}.$$

The nomenclature used here is the bracket { } for a column, thus the transpose form $\{\ \}^T$ for a row.

1b. A matrix composed of a single column is referred to as a column or column vector. For example,

$$\{C\} = \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \\ \cdot \\ \cdot \\ \cdot \\ C_{m1} \end{bmatrix}.$$

1c. A matrix composed of a single row and column obeys the rules of ordinary numbers and is called a scalar.

2. A matrix that has the same number of rows as columns is called a square matrix. Among the class of square matrices are:

2a. If the element a_{ij} of a matrix equals the element a_{ji} for all i, j, the matrix is called symmetric. This operation is accomplished by Subroutines SYMLH or SYMUH.

2b. If all elements other than those on the main diagonal are zero, the matrix is called a diagonal matrix and is denoted by [ ].

2c. A diagonal matrix with all elements on the main diagonal equal to unity is called a unity or identity matrix and is denoted by [I]. A unity matrix is generated by Subroutine UNITY.

3. If all elements of a matrix are zero, it is called a null or zero matrix. A zero matrix is generated by Subroutine ZERO.

4. The transposed matrix of a matrix [A] is the matrix whose rows correspond to the columns of [A] and is denoted as $[A]^T$. For example, if

$$[A] = \begin{bmatrix} 2. & 1. & -1. \\ 6. & -4. & 0. \end{bmatrix}$$

then

$$[A]^T = \begin{bmatrix} 2. & 6. \\ 1. & -4. \\ -1. & 0. \end{bmatrix}.$$

This operation is accomplished by Subroutine TRANS.

5. Two matrices are defined to be equal if they are of the same size and their corresponding elements are equal.

6. A matrix whose elements are obtained by multiplying all elements of the matrix [A] by a scalar α is called the product of the scalar α and the matrix [A].

$$\alpha[A] = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}.$$

This calculation is accomplished by Subroutine ALPHAA.

7. If [A] and [B] are matrices of the same size, their sum is defined as the matrix [Z] whose element

$$z_{ij} = a_{ij} + b_{ij}.$$

For example,

$$\begin{aligned} [Z] &= [A] + [B] \\ &= \begin{bmatrix} 2. & 6. \\ -1. & 3. \end{bmatrix} + \begin{bmatrix} 5. & -2. \\ 3. & 4. \end{bmatrix} \\ &= \begin{bmatrix} 7. & 4. \\ 2. & 7. \end{bmatrix}. \end{aligned}$$

Subroutine AABB calculates the sum of matrices [A] and [B], each multiplied by a scalar α and β, respectively. That is,

$$[Z] = \alpha\ [A] + \beta\ [B].$$

8. The multiplication of matrices differs inherently from ordinary or scalar algebraic multiplication. First, matrix multiplication is not in general commutative, that is, in general [A] [B] $\neq$ [B] [A]. Second, multiplication of two matrices is valid only if the number of columns in the first matrix is equal to the number of rows in the second matrix. Such matrices are called conformable matrices. The matrix of the resulting product will have the same number of rows as the first matrix and the same number of columns as the second matrix. The actual process of multiplication is defined as follows: The element z_{ij} in the product matrix [Z] = [A] [B] is equal to the sum of the products of the elements of the i^{th} row of [A] times the elements of the j^{th} column of [B], beginning at the left-hand end and top, respectively; thus

$$z_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

where n is the number of columns of [A] and the number of rows of [B]. To illustrate matrix multiplication, consider the following example:

$$[Z] = [A] \times [B]$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}.$$

Matrix multiplication is calculated by Subroutines MULT, MULTA, and MULTB.

9. Matrix [Z] is called the inverse (or reciprocal) of the matrix [A] if

$$[Z]\ [A] = [I].$$

There are two major conditions which must be satisfied for a matrix [A] to have a reciprocal matrix. First, the matrix must be square, and second, the matrix must be nonsingular. A nonsingular matrix is defined as a matrix whose determinant does not vanish. Matrix inversion is calculated by Subroutines INV1, INV2, INV3, INV4, and INV5.