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STABILITY OF A PLANE VORTEX SHEET BETWEEN GASES EXCHANGING HEAT BY RADIATION

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16. Abstract

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SUMMARY

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The effect of base flow thermal radiation across the vortex sheet is also analyzed, by considering a uniformly heat generative, symmetric inner gas with non-uniform base temperature and a faster moving, semi-infinite outer gas. It is shown that the base flow temperature variation has a stabilizing influence. An approximate application of this analysis is made to a nuclear rocket with the conclusion that the disturbed vortex sheet is less unstable because of the radiative transfer.

INTRODUCTION

Background

This investigation was prompted by concern for the containment of the uranium gas core in a coaxial jet nuclear rocket. In particular, we wish to know the influence of radiative heat transfer upon the integrity of the interface between the inner heat generative gas and the outer, faster moving, coolant gas. A schematic of the rocket chamber is sketched in Figure 1 (see, for example, Putre [30]).

For tractability of our problem, certain simplifying assumptions will be required. At the outset we shall assume that the interface separating the two dissimilar gases degenerates to a vortex sheet of infinitesimal thickness, across which there will be no molecular diffusion. Further, we shall choose to analyze only two-dimensional parallel flow in an x-y plane. The results and conclusions from this analysis should carry over to the axisymmetric flow in view of the qualitative agreement existing between results for the two geometries as exhibited by Gill [7] and Lessen, Fox and Zien [17] in their studies of jet and wake instabilities in isentropic gases. Other simplifying assumptions will be mentioned as needed in the development of succeeding sections.

Literature Survey

Hydrodynamic stability. - Several authors have studied the vortex sheet stability problem, nearly always in the presence of complete local thermodynamic equilibrium flow. Helmholtz, Rayleigh and Kelvin (see Lamb [14]) investigated the plane vortex sheet separating incompressible inviscid flows of semi-infinite extent and found the sheet to be unstable to small disturbances. Landau [15], Hatanaka [10], Pai [25] and Miles [23] extended this to compressible flows with the conclusion that the sheet would be neutrally stable when $|U_2-U_1| > (a_1^{2/3}+a_2^{2/3})^{3/2}$ where (U,a) refer to the speed and isentropic speed of sound in the base flow of the two gases. The specific heat ratios of the two gases were assumed equal. It should be noted that Miles demonstrated that Landau, Hatanaka and Pai generated spurious eigenvalues in addition to the correct ones in their approach to the problem. Miles, in using an initial value problem approach where disturbances were restricted to those initiated at the vortex sheet and propagated outward, was able to show the appropriate number and stability character of the eigenvalues generated from the principal branch of his characteristic equation. He demonstrated that there exist two principal modes for subsonic disturbances, which appear when $|U_2-U_1| < (a_1+a_2)$ and three principal modes for supersonic disturbances, which appear when $|U_2-U_1| > (a_1+a_2)$. Only one mode is unstable for $|U_2-U_1| < (a_1^{2/3}+a_2^{2/3})^{3/2}$ while all three modes are neutrally stable for $|U_2-U_1| > (a_1^{2/3}+a_2^{2/3})^{3/2}$.

All seven of the above investigators considered the vortex sheet to be separating semi-infinite gases. Other authors have restricted the dimension of one of the gases. For instance, Gill and Lessen et al considered "top-hat" type velocity profiles for compressible jet and wake type flows in both two-dimensional planar and axisymmetric cylindrical geometries. Gill's approach was purely analytical, thus requiring him to restrict his attention to certain asymptotic limits to make the problem tractable. Specifically, he considered disturbances along the vortex sheet with a small length scale (short waves) compared to the dimension of the inner gas. Obviously, this identifies with the above case of a vortex sheet separating semi-infinite gases except when $|U_2-U_1| > (a_1^{2/3}+a_2^{2/3})^{3/2}$. In the latter region Gill claims that there is an "enhanced instability" by virtue of the ability of waves to reflect back and forth at one of the resonant angles (see Fejer and Miles [6])with a growth rate of the order log (αM), where $M = |\frac{U_2-U_1}{a}|$, $a = a_1 = a_2$, α is the dimensionless wave number, and αM is taken as large. He notes that, for these short waves, the wave speed is close to M/2 which complies with Miles' explicit result provided for the vortex sheet in an infinite domain. We must be careful when considering Gill's results since the short wave limit would be highly affected in the presence of viscosity, an ingredient neglected in his investigation.

Lessen et al restricted their attention to jets (or wakes) moving supersonically relative to the surrounding gas and looked for unstable solutions. Their numerical results consisted of plots of the real (wave speed) and imaginary (amplification factor) parts of the eigenvalues versus a dimensionless wave number (disturbance wave number times inner gas half-width) for various supersonic inner jet Mach numbers. Their paper provides evidence that the vortex sheet is unstable to small disturbances at all supersonic speeds in the presence of a plane of symmetry for both symmetrical and anti-symmetrical disturbances.

For incompressible flow Betchov and Criminale [2] showed where placement of a wall near a shear layer has a stabilizing influence but does not afford stability. They note the fact that for short waves (compared to shear layer to wall distance) the eigenfunctions diminish to a small amplitude into the free stream on either side of the shear layer while for long waves there results a buildup of pressure fluctuations toward the wall.

In each of the above cases the investigators took the base flow properties of each gas to be uniform with discontinuities occurring only at the interface separating them. This assumption results in a constant coefficient differential acoustic disturbance equation, allowing simple exponential solutions. Early investigators pursued parallel flow stability problems wherein the base flow properties varied across streamlines in some continuous manner. Although the resulting governing equations are linear on the basis of the small disturbance approach, they are of variable coefficient form and only tractable in some relatively simple cases. Tollmein [34] considered the incompressible boundary layer while Lees and Lin [16] did the same for the compressible boundary layer. Pai [26] extended the method of Lees and Lin to a jet flow of a single gas. His supersonic disturbance stability criterion in this case lends support to his vortex sheet stability criterion. Each of the above three studies included viscosity in a large Reynolds number expansion but did not account for any other non-equilibrium mechanism.

In recent years interest has been shown in the stability of a disturbed vortex sheet separating two fluids subject to some form of molecular non-equilibrium. For instance, Wang and Maslen [39] investigated the stability of a vortex sheet separating two perfectly conducting semi-infinite compressible fluids in the presence of uniform, parallel magnetic fields. They show that, when the ratio between Alfvén speed and sound speed is unity in both fluids, the sheet is completely stable. Wang [38] also investigated the stability of the vortex sheet separating two semi-infinite compressible chemically relaxing gases and found that a measure of non-equilibrium in the perturbation problem affords instability at all speeds. However, should the relaxation times be particularly fast (equilibrium) or slow (frozen) the sheet is neutrally stable for $|U_2-U_1| > (\bar{a}_1^{-2/3}+\bar{a}_2^{-2/3})^{3/2}$ where $\bar{a}$ is either the equilibrium or frozen speed of sound. We should note that, in each of these two investigations, the non-equilibrium was assumed to occur in the small disturbance only while the base flows were in complete equilibrium.

<u>Thermal radiation</u>. - In the last decade, with the advent of higher speed projectiles and re-entry vehicles operating under high temperatures, a new interest has been generated in the non-equilibrium phenomenon of thermal radiation in gaseous flows. Although Couette and Rayleigh type flows have been solved, the mainstream of interest has seemingly been in wave structure problems. In particular, investigators such as Zel'dovich [42], Raizer [32], Heaslet and Baldwin [11] and Pearson [28] have considered the steady flow deterministic problem of the hot compressed gas downstream of a shock to be radiating heat back upstream tending to smear the discontinuity. The latter two papers were the culmination of numerical work wherein the non-linear effects of large heat transfer across strong shocks could be accounted for, that is, there was no necessity to assume uniform conditions upstream and downstream. The latter conditions were assumed, however, for relatively weak shocks and consequent small radiation in the analytical linearized treatment shown in Vincenti and Kruger [37].

Another brand of deterministic wave-type problem which has received considerable attention is that of the propagation of a small disturbance into a uniform hot gas. For instance, Baldwin [1], Lick [18] and Moore [24] each attacked the initial value problem of a piston suddenly set in motion with an infinitesimal speed, forcing a small disturbance to be propagated outward into a semi-infinite gas. Although each author assumed uniform properties in the undisturbed gas, affording exponential solutions, they still found it necessary to make some rather stringent assumptions or to attack the problem numerically. They did, however, exhibit the fact that the acoustic disturbance decays and disperses as it propagates into the semi-infinite medium. The dispersion is due to the fact that radiation can travel at the speed of light, thus spreading the disturbance out behind and ahead of the wave center. The decay of the disturbance exhibits the relaxation character of the radiative non-equilibrium phenomenon. These authors also discussed the appearance in their problem of the transparent and opaque limits, near and far from the piston respectively, pointing out that the acoustic wave propagates at the isentropic speed of sound in these limits and at the isothermal speed of sound in the transition region between these limits. For a fairly opaque gas, then, the transparent region close to the piston may be considered as a boundary layer, a point which will have importance to a part of the present study.

Vincenti and Baldwin [36] made observations of similar nature to those of the above authors in their study of the response of the semi-infinite hot gas to small sinusoidal oscillations in both position and temperature of the piston. In this case, however, the frequency of the propagating disturbance is established by that of the oscillating piston. However, depending upon the impedance of the radiating gas, the disturbance undergoes decay and phase shift as it propagates deeper into the semi-infinite domain. An important accomplishment of these authors is their exhibition of two types of waves making up the disturbance, a modified classical wave and a radiation-induced wave, the latter being a peculiarity of the higher order of the governing equations. The intensity of each of these waves depends upon the mechanical and thermal boundary conditions in the problem, although the radiation-induced wave will vanish in the limits of a completely opaque, transparent or cold gas. In such limits the classical wave propagates at the isentropic speed of sound and becomes purely a function of mechanical boundary conditions. In the very hot limit, both waves may exist to an intensity which depends upon the nature of the boundary conditions, the modified classical wave propagating at the isothermal speed of sound and the radiation-induced wave propagating at speeds ranging from zero to infinity depending upon the opacity of the gas and the induced frequency of the disturbance.

Long and Vincenti [21] investigated, by numerical methods, the pressure response in a finite, uniformly hot gas situated between fixed walls with one wall having a sinusoidally oscillating temperature of small amplitude. Their results exhibit the resonance character of their finite chamber in affording a peaked response to the standing modified classical wave. However, because of the ability of radiation to travel at all speeds, the off-resonance response to the modified classical wave is not zero but rather smoothly varying about some lower amplitude while the response due solely to the radiation-induced wave is quite uniform for the hot gas.

In nearly all of the above mentioned deterministic problems involving thermal radiation, and particularly so in the non-numerical analyses, it was found necessary to apply restrictions such as

(a) grey gas,

(b) differential approximation to the kernel of the radiation integral in a one-dimensional problem,

(c) opaque or transparent limits on absorptivity,

(d) small perturbations to a uniform base "flow".

Thus, it should not be surprising that some such simplifying assumptions will be necessary when combining the difficult problems of thermal radiation and stability. Furthermore, as is the practice of most authors, we shall assume that, while radiation is being considered, other forms of relaxation, such as vibration and chemistry, will be ignored.

BASIC EQUATIONS

Equations of Motion

We shall here derive and set forth the equations governing a small disturbance potential existing in a two-dimensional radiating, heat generative gas. For simplicity, we shall assume applicability of the equation of state of an ideal gas. We shall allow for variation in base flow temperature and density in the transverse direction (y-dir.) only. The resulting equations will be specialized when adapted to particular eigenvalue problems appearing in subsequent chapters.

The basic governing gasdynamic equations include continuity, momentum, energy and state

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} \tag{2}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} \tag{3}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} - \frac{\partial q_i}{\partial x_i} + Q \tag{4}$$

$$p = \rho RT \tag{5}$$

where $\frac{\partial q_i}{\partial x_i}$ represents the divergence of radiant heat flux and Q represents volumetric heat generation rate in the gas. We shall consider the applicability of linear superposition of small disturbances upon a suitable base flow such that $p = \bar{p} + p'$, etc. We shall also make the important assumption of a steady, uniform, parallel base flow in a field of uniform pressure $\bar{p}$ and heat generation $\bar{Q}$. With the parallel flow assumption we are at liberty to choose any particular x-direction speed U and adjust all perturbation equations by applying the simple Galilean transformation $(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})$ in place of $\frac{\partial}{\partial t}$ for a stationary fluid. Thus, for convenience, we shall temporarily take $\bar{u} = 0$ in the base flow and superimpose later a non-zero value as needed. Allowing for the transverse variation of $\bar{T}$ and $\bar{\rho}$, linearization yields the small perturbation counterparts of eqns. (1)-(5) as

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x}(\bar{\rho}u') + \frac{\partial}{\partial y}(\bar{\rho}v') = 0 \tag{6}$$

$$\bar{\rho}\,\frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \tag{7}$$

$$\bar{\rho}\,\frac{\partial v'}{\partial t} + \frac{\partial p'}{\partial y} = 0 \tag{8}$$

$$\bar{\rho}c_p\left(\frac{\partial T'}{\partial t} + v'\,\frac{d\bar{T}}{dy}\right) - \frac{\partial p'}{\partial t} = -\frac{\partial q_i'}{\partial x_i} + Q' \tag{9}$$

$$\frac{p'}{\bar{p}} = \frac{\rho'}{\bar{\rho}} + \frac{T'}{\bar{T}} \tag{10}$$

We may define a disturbance potential ϕ according to

$$u' = \frac{\partial \phi}{\partial x} \qquad p' = -\bar{\rho}\,\frac{\partial \phi}{\partial t} \tag{11}$$

in order to satisfy the first momentum equation. Substituting these into the second momentum equation yields

$$v' = \frac{1}{\bar{\rho}}\,\frac{\partial}{\partial y}(\bar{\rho}\phi) \tag{12}$$

If we differentiate state eqn. (10) with respect to t and substitute from continuity eqn. (6) in terms of ϕ while maintaining that $\bar{\rho}$ is variable in y only, we get

$$\frac{1}{\bar{T}} \frac{\partial T'}{\partial t} = -\frac{\bar{\rho}}{\bar{p}} \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{\bar{\rho}} \frac{\partial^2}{\partial y^2} (\bar{\rho}\phi) \tag{13}$$

Substituting this and state eqn. (10) into energy eqn. (9) in terms of ϕ gives us

$$\frac{\partial q_i'}{\partial x_i} - Q' = \frac{\bar{\rho}}{\gamma-1} \frac{\partial^2 \phi}{\partial t^2} - \frac{\gamma\bar{p}}{\gamma-1} \frac{\partial^2 \phi}{\partial x^2} - \frac{\gamma R}{\gamma-1} \frac{\partial}{\partial y} [\bar{T} \frac{\partial}{\partial y} (\bar{\rho}\phi)] \tag{14}$$

We shall assume that there are some fixed characteristic temperature T_o and density ρ_o such that $\bar{T} = T_o f(y)$ and, by virtue of the ideal gas law, $\bar{\rho} = \rho_o/f(y)$ where $f(y)$ is a base flow function to be specified later. We shall also define a reference value of the isentropic speed of sound as

$$a = \sqrt{\gamma \frac{\bar{p}}{\rho_o}} \tag{15}$$

where γ is the ratio of specific heats. Substituting our definitions into eqns. (13) and (14) now gives us

$$\frac{1}{T_o} \frac{\partial T'}{\partial t} = -\frac{\gamma}{a^2} \frac{\partial^2 \phi}{\partial t^2} + f \frac{\partial^2 \phi}{\partial x^2} + f^2 \frac{\partial^2}{\partial y^2} (\frac{\phi}{f}) \tag{16}$$

$$\frac{\gamma-1}{\gamma\bar{p}} (\frac{\partial q_i'}{\partial x_i} - Q') = \frac{1}{a^2 f} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial}{\partial y} [f \frac{\partial}{\partial y}(\frac{\phi}{f})] \tag{17}$$

Heat Generation

We need rate equations for q' and Q'. For the latter we need to make considerations with respect to the neutron physics of the problem. The local heat generation rate in a reactor core is given by

$$Q = N\sigma_f E\Gamma\rho \tag{18}$$

where N is the ratio of Avagadro's number to the atomic number of the gas, σ_f is the fission cross-section, E is the energy released per fission that goes into heat generation, Γ is the local neutron flux and ρ is the gas density. As shown by Glasstone and Edlund [9], energy released per fission for the great majority of fission modes is approximately distributed as follows:

Kinetic energy of fission fragments	162 Mev
Beta decay energy	5
Gamma decay energy	5
Neutrino energy	11
Energy of fast neutrons	6
Instantaneous gamma-ray energy	6
Total fission energy	195 Mev

N and σ_f are constants for a given core gas. E is some fraction of the total fission energy. Ordinarily, neutrinos are lost from the system while the energy associated with the fission fragments and instantaneous gamma-rays appears almost immediately as heat, and the energy associated with beta and gamma decay appears as heat over a much longer period of time. The disposition of the neutron energy depends upon whether the reactor is fast or thermal. Since a fast gas core reactor would be difficult to control, we should think more in terms of a thermal reactor where more than 90% of the neutrons must be thermalized before being absorbed for further fissions. This is accomplished by letting the fast neutrons diffuse outside the core into a moderator (which may also be a reflector) of light atoms. There, elastic scattering slows the neutrons to about 0.25ev whereupon they diffuse back into the core. This slowing down process takes considerable time, perhaps in excess of one millisecond, according to Podney and Smith [29]. Thus, in a thermal reactor, a power excursion (excursion in Γ) is relatively slow.

The concern for characteristic times expressed above is important when considering our acoustic perturbation problem which has a characteristic time dependent upon the speed of sound in the gas. This time is short and therefore perturbations in heat generation can be taken as quite independent of the long life neutron flux excursions. Thus, we may view heat generation perturbation Q' as varying linearly with local density perturbation ρ' with E taken to be approximately 168 Mev, the sum of fission fragment kinetic energy and instantaneous gamma-ray energy.

Therefore

$$Q' = N\sigma_f E\Gamma\rho' \tag{19}$$

If we differentiate eqn. (19) with respect to t and substitute eqn. (6) in terms of ϕ we have

$$\frac{\partial Q'}{\partial t} = -N\sigma_f E\Gamma\rho_o\left[\frac{\partial^2}{\partial x^2}\left(\frac{\phi}{f}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\phi}{f}\right)\right] \tag{20}$$

Thermal Radiation

We now need a rate equation for thermal radiation. Assuming local thermodynamic equilibrium in the gas, this may be expressed in terms of intensity I_ν as

$$l_i \frac{\partial I_\nu}{\partial x_i} = -\alpha_\nu (I_\nu - B_\nu) \tag{21}$$

where l_i is the direction cosine of the ith coordinate, α_ν is a volumetric absorption coefficient, B_ν is the Planck function and the subscript ν indicates frequency dependence. The frequency dependence may be dropped by making the assumption of a grey gas, which implies that the absorption coefficient is frequency independent. Rather than attempt to solve the multi-dimensional radiation transfer equation in explicit form,* it may be satisfactory only to satisfy certain moments of the equation. This was done formally by Cheng [3] where he substituted for I an infinite series of spherical harmonics. Subsequent integration yielded an infinite set of equations equivalent to the original transfer equation. Truncation to a first approximation and certain simplification led to the set of equations

$$\frac{\partial q_i}{\partial x_i} = -\alpha_o (I_o - 4\sigma T^4) \tag{22}$$

$$\frac{\partial I_o}{\partial x_i} = -3\alpha_o q_i \tag{23}$$

where α_o is the grey gas absorption coefficient, I_o is the zeroth moment of intensity, the space-integrated intensity defined as

$$I_o = \int_o^{4\pi} I(\Omega)\, d\Omega \tag{24}$$

and q_i is the directional dependent first moment of intensity, the radiation heat flux vector defined as

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* It is to be understood that a formal solution of eqn. (17) can be obtained for the one-dimensional case in terms of integrals. A purely differential equation can be derived from this by approximating the exponential-integral kernel E_2 by a purely exponential function. A recent paper by Gillis, Cogley and Vincenti [8] propose a non-grey gas substitute kernel whereby they claim that existing grey-gas solutions need only be reinterpreted in terms of the non-grey gas case.

$$q_i = \int_0^{4\pi} I(\Omega)\, l_i d\Omega \; * \qquad (25)$$

Vincenti and Kruger [37] have shown that eqns. (22) and (23) can be derived by assuming that radiation pressure and energy density are related as if the radiation were isotropic; this constitutes the so-called Milne-Eddington approximation.

Elimination of I_o from eqns. (22) and (23) will lead to a differential equation for the heat flux vector expressed as

$$\frac{\partial}{\partial x_i}\left(\frac{\partial q_j}{\partial x_j}\right) - \frac{4\sigma}{\lambda}\frac{\partial T^4}{\partial x_i} - \frac{3}{\lambda^2} q_i = 0 \qquad (26)$$

where we have replaced the grey gas absorption coefficient by the reciprocal of the photon mean free path λ, defined as the distance in which a beam of radiant flux diminishes to 1/e of its original value.

Eqn. (26) has some interesting features worth discussing at this point. Note first that, if the Planck function $\sigma T^4/\pi$ is uniform, an appropriate solution is $q_i = 0$. Next, consider a medium of high absorptivity (fairly opaque). This means that photons emitted at one point are absorbed at another point close by. In this case we may drop the first term of eqn. (26) and note that the transfer of heat now depends upon the gradient of T^4, requiring a rapid and continuous variation in temperature of the medium to accomplish substantial heat transfer. On the other hand, a medium of small absorptivity (fairly transparent) implies that photons are free to travel large distances before being reabsorbed. This restriction allows us to drop the third term of eqn. (26), indicating that the medium temperature need not be rapidly and continuously varying to admit passage of substantial heat transfer. It is also useful to note that in each of the above restricted ranges, the heat transfer depends upon the photon mean free path in such a way as to be less than a maximum for a given gradient of the Planck function. A maximum rate of heat transfer (maximum non-equilibrium) would therefore occur for a given Planck function gradient when λ is of an intermediate value and all three terms of eqn. (26) are in balance. The above features of the radiation heat flux will carry over to its small disturbance counterpart and to the perturbation potential equation.

Now, if we linearize each of eqns. (22), (23), and (26) for the sake of our small disturbance theory we shall have

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* If the integration is performed over each half-space we may identify one-sided heat fluxes q_i^+ and q_i^- where these are the components perpendicular to the plane separating the respective half-spaces. The net heat flux is then related to these by $q_i = q_i^+ - q_i^-$. The integrated intensity defined by eqn. (24) is then related according to $I_o = 2(q_i^+ + q_i^-)$.

$$\frac{\partial q_i'}{\partial x_i} = -\frac{1}{\lambda}(I_o' - 16\sigma T_o^3 f^3 T') \qquad (27)$$

$$\frac{\partial I_o'}{\partial x_i} = -\frac{3}{\lambda} q_i' \qquad (28)$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial q_j'}{\partial x_j}\right) - \frac{16\sigma T_o^3}{\lambda}\frac{\partial}{\partial x_i}(f^3 T') - \frac{3}{\lambda^2} q_i' = 0 \qquad (29)$$

Acoustic Perturbation Equation

We may now combine eqns. (16), (17), (20) and (29) into a single fifth order differential equation governing our disturbance potential. First, take the divergence of eqn. (29) and differentiate it with respect to t. Then, take the Laplacian of eqn. (16), differentiate eqn. (17) with respect to t and substitute the results along with eqn. (20) into the differentiated form of eqn. (29) to arrive at

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{3}{\lambda^2}\right)\left\{\frac{\partial}{\partial t}\left[\frac{1}{a^2}\frac{\partial^2}{\partial t^2} - f\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}\left(f\frac{\partial}{\partial y}\right)\right]\frac{\phi}{f} - \frac{aG}{\lambda}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\phi}{f}\right\}$$

$$+ \frac{aK}{\lambda}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left\{f^4\left[\frac{\gamma}{a^2}\frac{\partial^2}{\partial t^2} - f\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\frac{\phi}{f}\right\} = 0 \qquad (30)$$

where we have defined radiation and heat generation parameters according to

$$K = \frac{16(\gamma-1)\sigma T_o^4}{\gamma \bar{p} a} \qquad\qquad G = \frac{\gamma-1}{a^3}\lambda N \sigma_f E \Gamma$$

It will also be useful to get expressions for the integrated intensity I_o' and the net heat flux q_i' in terms of the disturbance potential. We may accomplish the first of these by differentiating eqn. (17) with respect to t and substituting it along with eqns. (16) and (20) into eqn. (27), after differentiating the latter with respect to t. The result is

$$\frac{\partial I'_o}{\partial t} = -\frac{\gamma\bar{p}\lambda}{\gamma-1}\{\frac{\partial}{\partial t}[\frac{1}{a^2}\frac{\partial^2}{\partial t^2} - f\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}(f\frac{\partial}{\partial y})]\frac{\phi}{f} - \frac{aG}{\lambda}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\frac{\phi}{f}$$

$$+ \frac{aK}{\lambda}f^4[\frac{\gamma}{a^2}\frac{\partial^2}{\partial t^2} - f(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})]\frac{\phi}{f}\} \quad (31)$$

The heat flux expression may now be derived by simply taking the gradient of eqn. (31) and substituting it into eqn. (28) to get

$$\frac{\partial q'_i}{\partial t} = \frac{\gamma\bar{p}\lambda^2}{3(\gamma-1)}\frac{\partial}{\partial x_i}\{\frac{\partial}{\partial t}[\frac{1}{a^2}\frac{\partial^2}{\partial t^2} - f\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}(f\frac{\partial}{\partial y})]\frac{\phi}{f} - \frac{aG}{\lambda}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\frac{\phi}{f}$$

$$+ \frac{aK}{\lambda}f^4[\frac{\gamma}{a^2}\frac{\partial^2}{\partial t^2} - f(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})]\frac{\phi}{f}\} \quad (32)$$

ONE-DIMENSIONAL PROBLEM

Nature of Problem

There are several features of our stability problem which can affect the results, such as geometry (including location and type of boundary conditions), presence of mechanical or thermal non-equilibrium in the base flow and/or non-equilibrium effects in the perturbed flow. Two of these effects which are new to the question of stability are the thermal radiation and the heat generation. Since it is not our objective to seek solely an answer to that specific problem which prompted this study, but rather to gain some fundamental understanding on a more general level, it will prove instructive to attack simple problems first in order to more or less isolate one or two of the above discussed effects at a time; thus, our interest in a simple one-dimensional problem. In this way we shall be able to show that the dissipative effect of thermal radiation upon propagating disturbances discussed in papers reviewed in the INTRODUCTION will carry over in terms of an eigenvalue problem.

Governing Equations

Our eigenvalue problem will be characterized by a length y_o which will be identified later. Non-dimensionalizing with respect to this length and assuming a uniform base "flow" the one-dimensional counterpart of eqns. (30), (11), (12), (31) and (32) become

$$\left(\frac{\partial^2}{\partial \eta^2} - 3\frac{y_o^2}{\lambda^2}\right)\left[\frac{\partial}{\partial t}\left(\frac{y_o^2}{a^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial \eta^2}\right) - \frac{aG}{\lambda}\frac{\partial^2 \phi}{\partial \eta^2}\right]$$

$$\frac{aK}{\lambda}\frac{\partial^2}{\partial \eta^2}\left(\frac{\gamma y_o^2}{a^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial \eta^2}\right) = 0 \qquad (33)$$

$$p' = -\rho_o \frac{\partial \phi}{\partial t} \qquad (34)$$

$$v' = \frac{1}{y_o}\frac{\partial \phi}{\partial \eta} \qquad (35)$$

$$\frac{\partial I_o'}{\partial t} = -\frac{\gamma \bar{p} \lambda}{(\gamma-1)y_o^2}\left[\frac{\partial}{\partial t}\left(\frac{y_o^2}{a^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial \eta^2}\right) - \frac{aG}{\lambda}\frac{\partial^2 \phi}{\partial \eta^2}\right.$$

$$\left. + \frac{aK}{\lambda}\left(\frac{\gamma y_o^2}{a^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial \eta^2}\right)\right] \qquad (36)$$

$$\frac{\partial q'}{\partial t} = \frac{\gamma \bar{p} \lambda^2}{3(\gamma-1)y_o^3}\frac{\partial}{\partial \eta}\left[\frac{\partial}{\partial t}\left(\frac{y_o^2}{a^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial \eta^2}\right) - \frac{aG}{\lambda}\frac{\partial^2 \phi}{\partial \eta^2}\right.$$

$$\left. + \frac{aK}{\lambda}\left(\frac{\gamma y_o^2}{a^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial \eta^2}\right)\right] \qquad (37)$$

where we have defined $\eta = y/y_o$ and set $f = 1$.

Signaling Equation Concept

Before picking a particular geometry and associated boundary conditions for a one-dimensional eigenvalue problem, it might be interesting to investigate the governing differential equation for "stability" via a signaling equation idea proposed by Whitham [40]. Such a signaling equation would take the form

$$\left(\frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial x}\right)----\left(\frac{\partial}{\partial t} + c_n \frac{\partial}{\partial x}\right)\phi$$

$$+ P\left(\frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial x}\right)----\left(\frac{\partial}{\partial t} + a_m \frac{\partial}{\partial x}\right)\phi = 0 \qquad (38)$$

where, if $m = n - 1$ and $P \geq 0$, then $c_1 > a_1 > c_2 > \cdots > a_{n-1} > c_n$ will indicate "stability", that is damped propagating waves.

We note that our governing eqn. (33) contains three independent parameters, namely y_o/λ, K and G. If we can find limits where only one parameter appears at a time, we may apply Whitham's idea. In the limit of a completely cold (K=0) and non-heat generative (G=0) gas a sufficient solution to eqn. (33) is provided by solving the isentropic acoustic equation.* In this limit, then, eqn. (33) may be written

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{a^2}{y_o^2}\frac{\partial^2 \phi}{\partial \eta^2} = \left(\frac{\partial}{\partial t} + \frac{a}{y_o}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial t} - \frac{a}{y_o}\frac{\partial}{\partial \eta}\right)\phi = 0 \tag{39}$$

Obviously, in this simple case, $a > -a$ indicating that we have right and left-running standing (undamped) waves with signaling speeds of $\pm$ a.

In the non-heat generative, infinitely hot ($K \to \infty$) gas limit (Stokes' flow) eqn. (33) reduces to an isothermal wave equation**

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{a^2}{\gamma y_o^2}\frac{\partial^2 \phi}{\partial \eta^2} = \left(\frac{\partial}{\partial t} + \frac{a}{\sqrt{\gamma} y_o}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial t} - \frac{a}{\sqrt{\gamma} y_o}\frac{\partial}{\partial \eta}\right)\phi = 0 \tag{40}$$

which is "stable", giving right and left-running standing waves with signaling speeds of $\pm\, a/\sqrt{\gamma}$.

Cogley [4] has already applied Whitham's idea to eqn. (32) for G=0 for short time (fairly transparent gas) and long time (fairly opaque gas) when considering waves propagating into a semi-infinite gas. We shall do the same for waves of both families. In Whitham's form, the fairly transparent restriction allows reduction of eqn. (33) to

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \frac{a}{y_o}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial t} + 0\,\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial t} - \frac{a}{y_o}\frac{\partial}{\partial \eta}\right)\phi \\ + \frac{\gamma a K}{\lambda}\left(\frac{\partial}{\partial t} + \frac{a}{\sqrt{\gamma} y_o}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial t} - \frac{a}{\sqrt{\gamma} y_o}\frac{\partial}{\partial \eta}\right)\phi = 0\end{aligned} \tag{41}$$

---

* It can be shown that the isentropic acoustic equation is derivable directly by omitting thermal radiation and heat generation from the original perturbation eqns. (6)-(10).

** As Vincenti and Baldwin have indicated, the modified classical wave and the radiation-induced wave are possible in the hot limit. Which one will exist and abide by eqn. (40) depends upon the boundary conditions.

where

$$a > a/\sqrt{\gamma} > 0 > -a/\sqrt{\gamma} > -a$$

while the fairly opaque restriction yields

$$\left(\frac{1}{\infty}\frac{\partial}{\partial t} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial t} + \frac{a}{\sqrt{\gamma}y_o}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial t} - \frac{a}{\sqrt{\gamma}y_o}\frac{\partial}{\partial \eta}\right)\left(\frac{1}{\infty}\frac{\partial}{\partial t} - \frac{\partial}{\partial \eta}\right)\phi$$
$$+ \frac{3y_o^2}{a\lambda K}\left(\frac{\partial}{\partial t} + \frac{a}{y_o}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial t} + 0\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial t} - \frac{a}{y_o}\frac{\partial}{\partial \eta}\right)\phi = 0 \qquad (42)$$

where

$$\infty > a > a/\sqrt{\gamma} > 0 > -a/\sqrt{\gamma} > -a > -\infty$$

Since $\gamma aK/\lambda$ and $3y_o^2/a\lambda K$ are positive we may conclude that radiative non-equilibrium has a damping effect upon both right and left-running propagating waves, implying a favorable influence upon stability.

If we now set $K = 0$ while $G \neq 0$, eqn. (33) may be reduced to

$$\left(\frac{\partial}{\partial t} + \frac{a}{y_o}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial t} + 0\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial t} - \frac{a}{y_o}\frac{\partial}{\partial \eta}\right)\phi$$
$$+ \frac{aG}{\lambda}\left(\frac{1}{\infty}\frac{\partial}{\partial t} + \frac{\partial}{\partial \eta}\right)\left(\frac{1}{\infty}\frac{\partial}{\partial t} - \frac{\partial}{\partial \eta}\right)\phi = 0 \qquad (43)$$

where

$$a \not> \infty > 0 > -\infty \not> -a$$

Thus, the density dependent heat generation should have a growth effect upon propagating waves, implying an adverse influence upon stability.

Single, Heat Generative, Confined Gas

Eigenvalue problem. - For utmost simplicity, we shall establish the eigen behavior of a single, heat generative, radiating gas with uniform properties* by subjecting it to small disturbances which can be treated by the theory of normal modes. We may establish a characteristic modal frequency for this one-dimensional problem by fixing the dimension of the gas, leaving it confined between walls at $\eta = 0$ and $\eta = 1$. Consistent with the theory of normal modes we may assume a solution in time of the form

$$\phi(\eta,t) = \Phi(\eta)e^{-i\omega t} \tag{44}$$

where $\omega = \omega_r + i\omega_i$ is the complex frequency of the disturbance, the real part being wave speed divided by a length and the imaginary part being an amplification factor. For $\omega_i > 0$ the disturbance is exponentially growing (unstable) while for $\omega_i < 0$ it is exponentially decaying (stable), in which case we may call its magnitude a damping factor.

Substituting eqn. (44) into eqns. (33)-(37) gives us

$$\left(\frac{d^2}{d\eta^2} - 3\frac{y_o^2}{\lambda^2}\right)\left[\left(1 + \frac{iy_oG}{\lambda\bar{\omega}}\right)\frac{d^2\Phi}{d\eta^2} + \bar{\omega}^2\Phi\right] + \frac{iy_oK}{\lambda\bar{\omega}}\frac{d^2}{d\eta^2}\left[\frac{d^2\Phi}{d\eta^2} + \gamma\bar{\omega}^2\Phi\right] = 0 \tag{45}$$

$$p'(\eta) = i\,\frac{\gamma\bar{p}}{ay_o}\,\bar{\omega}\Phi \tag{46}$$

$$v'(\eta) = \frac{1}{y_o}\,\frac{d\Phi}{d\eta} \tag{47}$$

$$I_o'(\eta) = \frac{\gamma\,\bar{p}\,\lambda}{(\gamma-1)y_o^2}\left[\left(1 + \frac{iy_oG}{\lambda\bar{\omega}}\right)\frac{d^2\Phi}{d\eta^2} + \bar{\omega}^2\Phi + \frac{iy_oK}{\lambda\bar{\omega}}\left(\frac{d^2\Phi}{d\eta^2} + \gamma\bar{\omega}^2\Phi\right)\right] \tag{48}$$

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* We are assuming that the uniform base heat generation is removed in some uniform, fictitious manner. Obviously, in the gas rocket problem, the generated energy is radiated to the coolant gas which convects it downstream.

$$q'(\eta) = -\frac{\gamma\bar{p}\lambda^2}{3(\gamma-1)y_o^3}\frac{d}{d\eta}\left[\left(1 + \frac{iy_oG}{\gamma\bar{\omega}}\right)\frac{d^2\Phi}{d\eta^2} + \bar{\omega}^2\Phi + \frac{iy_oK}{\lambda\bar{\omega}}\left(\frac{d^2\Phi}{d\eta^2} + \gamma\bar{\omega}^2\Phi\right)\right] \tag{49}$$

where we have defined the dimensionless frequency

$$\bar{\omega} = \frac{\omega y_o}{a} \tag{50}$$

and let $p'(\eta)$, $v'(\eta)$, $I_o'(\eta)$ and $q'(\eta)$ represent the η-direction mode shapes of these disturbance quantities.

We note that eqn. (45) is of fourth order in space, requiring two boundary conditions (one mechanical and one thermal) to be specified at each wall. If we take the walls to be immovable and perfectly reflecting (or adiabatic) the boundary conditions become $v'(0) = v'(1) = q'(0) = q'(1) = 0$. The first two of these, in view of eqn. (47), require

$$\Phi'(0) = \Phi'(1) = 0 \tag{51}$$

where the primes on Φ will hereafter indicate differentiation with respect to η. Substituting eqn. (51) into eqn. (49) will allow our thermal boundary conditions to be expressed as

$$\Phi'''(0) = \Phi'''(1) = 0 \tag{52}$$

Since eqns. (45), (51) and (52) are homogeneous we have the makings of an eigenvalue problem.

A formal solution of eqn. (45) is of the form $e^{ic\eta}$. Defining

$$m = 1 + \frac{iy_oK}{\lambda\bar{\omega}}\left(1 + \frac{G}{K}\right) \tag{53a}$$

$$\mu = -\left[3\frac{y_o^2}{\lambda^2}\left(1 + \frac{iy_oG}{\lambda\bar{\omega}}\right) - \left(1 + \frac{i\gamma y_oK}{\lambda\bar{\omega}}\right)\bar{\omega}^2\right]\Big/m \tag{53b}$$

$$\nu = -\frac{3\frac{y_o^2}{\lambda^2}\bar{\omega}^2}{m} \tag{53c}$$

and substituting into eqn. (45) will yield the characteristic equation

$$c^4 - \mu c^2 + \nu = 0 \tag{54}$$

Since this is bi-quadratic, we have only two distinct values of c; call them

$$\left.\begin{matrix}a\\b\end{matrix}\right\} = \left[\frac{\mu \pm (\mu^2 - 4\nu)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}} \tag{55}$$

The general solution of eqn. (45) is then

$$\Phi(\eta) = Ae^{ia\eta} + Be^{-ia\eta} + Ce^{ib\eta} + De^{-ib\eta} \tag{56}$$

Substitution of eqn. (56) into eqns. (51) and (52) and successive elimination of the coefficients will give us the eigenvalue equation

$$\sin a \sin b = 0 \tag{57}$$

which has non-trivial zeroes when

$$a \text{ or } b = n\pi \qquad\qquad \text{where } n = 1,2,3,\cdots\cdots \tag{58}$$

It is interesting to note that our solution contains both waveforms, the modified classical wave associated with a and the radiation-induced wave associated with b.

In this very simple case it is only necessary to substitute eqns. (58) and (53) into eqn. (55) and invert to find the complex frequency. This procedure yields a cubic in $\bar{\omega}$.

$$\left(\frac{\bar{\omega}}{n\pi}\right)^3 + i\,\frac{\gamma K}{\sqrt{3}\left(\frac{n\pi\lambda}{\sqrt{3}y_o} + \frac{\sqrt{3}y_o}{n\pi\lambda}\right)}\left(\frac{\bar{\omega}}{n\pi}\right)^2 - \frac{\bar{\omega}}{n\pi} - i\left[\frac{y_o G}{n\pi\lambda} + \frac{K}{\sqrt{3}\left(\frac{n\pi\lambda}{\sqrt{3}y_o} + \frac{\sqrt{3}y_o}{n\pi\lambda}\right)}\right] = 0 \tag{59}$$

If we define

$$\delta = \frac{\bar{\omega}}{n\pi} \qquad \alpha = \frac{y_o G}{n\pi\lambda} \qquad \beta = \frac{(\gamma-1)K}{\sqrt{3}\left(\frac{n\pi\lambda}{\sqrt{3}y_o} + \frac{\sqrt{3}y_o}{n\pi\lambda}\right)} \tag{60}$$

substitution into eqn. (59) will yield

$$\delta^3 + i\,\frac{\gamma\beta}{\gamma-1}\,\delta^2 - \delta - i\beta\left(\frac{\alpha}{\beta} + \frac{1}{\gamma-1}\right) = 0 \tag{61}$$

Discussion. - One of the three roots to eqn. (61) yields zero frequency and a damped disturbance for all γ, α and β. Because the other two roots yield identical amplification factors δ_i with frequencies δ_r of equal magnitude but of opposite sign, it will be considered sufficient to display results for only the positive frequency root.

Neutral stability is ascertained from eqn. (61) by setting $\delta_i = 0$ and satisfying real and imaginary parts. For the positive frequency root this occurs when $\delta_r = 1$ and $\alpha/\beta = 1$. This neutral curve is plotted in Figure 2 in terms of a relation between the parameters y_o/λ, K and G. When $\alpha/\beta \neq 1$, the positive frequency disturbance either grows or decays in time. This behavior is exhibited in Figure 3 for $\gamma = 1.4$.

Most of the interesting features of Figure 3 are associated with ranges of β. Large β corresponds to large K, a very hot gas. Small β, on the other hand, may correspond to small K (cool gas) or, alternatively, to large or small $\sqrt{3}y_o/n\pi\lambda$ (low frequency, opaque gas or high frequency, transparent gas).

It should be clear that, when $\beta = 0$ (very cold gas or zero or infinite absorptivity or frequency), the disturbance will travel as a standing wave at the isentropic speed of sound. On the other hand, when $\beta \to \infty$ (infinite temperature), the disturbance will again travel as a standing wave but at the isothermal speed of sound. The greatest degree of non-equilibrium behavior (largest amplification or damping) will occur at intermediate β.

Certainly, we picked a convenient example which resulted in a very simple eigenvalue equation. Other boundary conditions on the walls could have been chosen, such as symmetry in v' or perhaps zero perturbation in wall temperature. The variations are numerous and the resulting eigenvalue equations would often have to be solved numerically for ω without the simplicity of eqns. (58) and (61).

We shall discuss in the next chapter how, in certain limits of the radiation parameters, the full non-equilibrium governing differential equation reduces to lower order with a consequent reduction in the available number of boundary conditions. For certain small values of these parameters there will be possible, then, a small perturbation of the equilibrium acoustic theory to include a small measure of non-equilibrium. Since this perturbation is regular, it should also be possible to accomplish it by formal expansion of the solution. In the present case, we may formally expand eqn. (61) for a cool gas with small heat generation. To second degree the solution for the positive frequency root is

$$\delta_r \approx 1 + \frac{3}{8}\alpha^2 + \frac{3-\gamma}{4(\gamma-1)}\alpha\beta - \frac{\gamma+3}{8(\gamma-1)}\beta^2 \qquad \delta_i \approx \frac{\alpha-\beta}{2} \tag{62}$$

which shows clearly that $\alpha/\beta = 1$ marks neutral stability. We have already noted that $\beta << 1$ means either $\lambda K/y_o << 1$ or $y_o K/\lambda << 1$ while $\alpha << 1$ requires $y_o G/\lambda << 1$ for moderate $n\pi$.

Two Immiscible Gases

Analysis. - Since we are truly interested in a problem which contains two adjacent gases of a different nature, it might be instructive to extend our one-dimensional problem to include two immiscible stagnant gases with different radiation absorption capability. Although we are at present pursuing study of a disturbance mode which is of a very different nature than that which characterizes a vortex sheet problem, nevertheless, the new version of the one-dimensional problem will afford us some further understanding of the radiation phenomenon and some knowledge of how to apply appropriate boundary conditions across the interface separating the two gases.

For the sake of clarity, substitute the definitions of eqns. (53) into eqns. (45)-(49) and rewrite the latter here.

$$\Phi^{iv} + \mu\Phi'' + \nu\Phi = 0 \tag{63}$$

$$p'(\eta) = i\,\frac{\gamma\bar{p}}{a y_o}\,\bar{\omega}\Phi \tag{64}$$

$$v'(\eta) = \frac{1}{y_o}\,\Phi' \tag{65}$$

$$I_o'(\eta) = \frac{\gamma\bar{p}\lambda}{(\gamma-1)y_o^2}\,[m\Phi'' + (m+\gamma-1)\,\bar{\omega}^2\Phi] \tag{66}$$

$$q'(\eta) = -\frac{\gamma\bar{p}\lambda^2}{3(\gamma-1)y_o^3}\,[m\Phi''' + (m+\gamma-1)\,\bar{\omega}^2\Phi'] \tag{67}$$

The properties in each of these, of course, will assume uniform values peculiar to the gas in which it is being applied.

Eqn. (63) has a general solution of the form

$$\Phi = A \sin a\eta + B \cos a\eta + C \sin b\eta + D \cos b\eta \tag{68}$$

analogous to eqn. (56) where a and b are given by eqn. (55). If we restrict the "inner" gas (gas 1) to $0 \leq \eta \leq \eta_1$ and the "outer" gas (gas 2) to $\eta_1 \leq \eta \leq 1$ and require the same immovable and reflecting walls at $\eta = 0$ and $\eta = 1$, the boundary conditions of eqns. (51) and (52) still apply. Application of these will reduce eqn. (68) to

$$\Phi_1 = B_1 \cos a_1\eta + D_1 \cos b_1\eta \tag{69}$$

for the inner gas and to

$$\Phi_2 = B_2 \cos a_2(1-\eta) + D_2 \cos b_2(1-\eta) \tag{70}$$

for the outer gas.

We have left four unknown constants. Thus, we must apply four boundary conditions (two mechanical and two thermal) at the interface between the gases. The mechanical conditions would amount to matching pressure and velocity. From eqns. (64) and (65) these may be expressed in terms of the potential function as

$$\Phi_1(\eta_1) = \Phi_2(\eta_1) \tag{71}$$

$$\Phi_1'(\eta_1) = \Phi_2'(\eta_1) \tag{72}$$

where we shall, for the sake of simplicity, limit our analysis to the case where the gases have equal specific heat ratios and sound speeds. This allows ϖ to be common to both gases.

The thermal conditions require matching integrated intensity and heat flux across the transparent interface between the gases. The first of these amounts to a conservation of photons while the second conserves energy. From eqns. (66) and (67) these conditions may be expressed as

$$m_1\Phi_1''(\eta_1) + (m_1 + \gamma-1)\,\bar{\omega}^2\Phi_1(\eta_1)$$
$$= \frac{\lambda_2}{\lambda_1}\,[m_2\Phi_2''(\eta_1) + (m_2 + \gamma-1)\,\bar{\omega}^2\Phi_2(\eta_1)] \tag{73}$$

$$m_1\Phi_1'''(\eta_1) + (m_1 + \gamma-1)\,\bar{\omega}^2\Phi_1'(\eta_1)$$
$$= \left(\frac{\lambda_2}{\lambda_1}\right)^2 [m_2\Phi_2'''(\eta_1) + (m_2 + \gamma-1)\,\bar{\omega}^2\Phi_2'(\eta_1)] \tag{74}$$

Substitution of eqns. (69) and (70) into eqns. (71)-(74) will result in four equations homogeneous in B_1, D_1, B_2 and D_2. Setting the coefficient determinant of these equal to zero constitutes the eigenvalue equation. In terms of present nomenclature this determinant may be written in the form

$$|a_{ij}| = 0 \qquad i,j = 1, 2, 3, 4 \tag{75}$$

where we are setting

$a_{11} = \cos a_1\eta_1$ $\qquad$ $a_{12} = \cos b_1\eta_1$

$a_{13} = -\cos a_2(1-\eta_1)$ $\qquad$ $a_{14} = -\cos b_2(1-\eta_1)$

$a_{21} = a_1 \sin a_1\eta_1$ $\qquad$ $a_{22} = b_1 \sin b_1\eta_1$

$a_{23} = a_2 \sin a_2(1-\eta_1)$ $\qquad$ $a_{24} = b_2 \sin b_2(1-\eta_1)$

$a_{31} = [m_1a_1^2 - (m_1 + \gamma-1)\bar{\omega}^2]\cos a_1\eta_1$ $\qquad$ $a_{32} = [m_1b_1^2 - (m_1 + \gamma-1)\bar{\omega}^2]\cos b_1\eta_1$

$$a_{33} = -\frac{\lambda_2}{\lambda_1}[m_2a_2^2 - (m_2 + \gamma-1)\bar{\omega}^2]\cos a_2(1-\eta_1)$$

$$a_{34} = -\frac{\lambda_2}{\lambda_1}[m_2b_2^2 - (m_2 + \gamma-1)\bar{\omega}^2]\cos b_2(1-\eta_1)$$

$$a_{41} = a_1[m_1a_1^2 - (m_1 + \gamma-1)\bar{\omega}^2]\sin a_1\eta_1$$

$$a_{42} = b_1\ [m_1b_1^2 - (m_1 + \gamma-1)\bar{\omega}^2]\sin b_1\eta_1$$

$$a_{43} = (\frac{\lambda_2}{\lambda_1})^2\ a_2[m_2a_2^2 - (m_2 + \gamma-1)\ \bar{\omega}^2]\sin a_2(1-\eta_1)$$

$$a_{44} = (\frac{\lambda_2}{\lambda_1})^2\ b_2[m_2b_2^2 - (m_2+\gamma -1)\ \bar{\omega}^2]\sin b_2(1-\eta_1)$$

The roots of such an eigenvalue equation can be found by Muller's numerical relaxation method (see, for instance, Wilkinson [41]).

Limitation upon radiation parameter. - It is extremely important to have compatible perturbed and base flows. For this reason we must consider any restrictions imposed upon the base "flow" by the assumption of uniformity of properties in each gas. Complete uniformity of temperature in a gas requires that either it has zero absorptivity, in which case heat can be transferred across it without interference from the gas, or it is in complete equilibrium and not subject to heat transfer. The former case is not very interesting at present because it yields only isentropic behavior in the perturbation problem. The latter would require no temperature slip at the interface in the base flow; otherwise there would have to be some temperature gradient in the gases on either side of the interface to be in accord with our analysis of eqn. (26). Thus, adhering strictly to the uniform temperature requirement forces us to consider only $K_1 = K_2$ for the time being. A problem involving base flow heat transfer will be considered in a later chapter.

Results. - In pursuing a solution to the eigenvalue problem described in eqn. (75) we shall choose to neglect heat generation since, as will subsequently be shown, its parameter takes on rather small values in any practical problem and thus has small effect upon the question of stability. Having assumed that the gases have equal specific heat ratios and sound speeds and established that $K_1 = K_2$, eqns. (53), (55) and (75) yield the fact that we have left only five free parameters, namely γ, y_o/λ_1, λ_2/λ_1, η_1 and K. If we wish to relate the present problem to the single-gas problem* analyzed earlier in this chapter we may formulate a combination of these parameters to yield one of the form of β in eqn. (60). Thus, we may now consider our free parameters to be γ, β, λ_2/λ_1, η_1 and K.

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* By setting $m_1/m_2 = a_1/a_2 = b_1/b_2 = 1$ it can be shown that eqn. (75) reduces from the two-gas problem to the one-gas problem of eqn. (57).

Figure 4 shows a comparison of frequency and damping of the fundamental acoustic mode for different values of λ_2/λ_1, the $\lambda_2/\lambda_1 = 1$ case being taken as the $\alpha/\beta = 0$ result from Figure 3. This figure shows clearly that a maximum of damping (maximum non-equilibrium) occurs for a given value of K when there exists an intermediate value of y_o/λ such that there remains a balance between the transparent and opaque operators upon the wave operator on the left of eqn. (33). Considering some deviation from this intermediate value of y_o/λ in one of our two gases, either toward the transparent limit or toward the opaque limit, will reduce the non-equilibrium effect. These observations agree with those made with regard to eqn. (26) in the last chapter.

PLANE VORTEX SHEET IN INFINITE DOMAIN

Problem Description

We shall begin consideration of the question of stability of a vortex sheet separating two thermally radiating gases. For the present we shall restrict attention to the problem of uniform base flow properties and zero internal heat generation. We saw in the last chapter where, if the gases are absorbent and we assume equal specific heat ratios and sound speeds, we must adhere to the condition of $K_1 = K_2$. No such restriction need be applied to the x-direction speeds of the two gases, however, since we are assuming that our gases are completely inviscid. In fact, it is exactly the existence of this velocity slip which produces a new disturbance mode in the perturbation problem with a characteristic frequency which orders itself to this speed difference. In the limit of diminishing velocity slip this mode vanishes (zero frequency) in a manner described in Lamb [14].

To avoid the complication of additional lengths we shall, for the present, consider the plane vortex sheet to be separating semi-infinite gases. In the isentropic limit, then, our problem will reduce to that considered by Pai and Miles. A schematic sketch of the problem we have just described is shown in Figure 5.

Governing Equations

By taking the Galilean transformation $\partial/\partial t = \partial/\partial t + U\,\partial/\partial x$ we may superimpose a uniform speed U upon either gas in our problem. Doing this and taking $f = 1$ and $G = 0$ in eqns. (30), (11), (12), (16), (17), (31) and (32) yields the equations governing the present problem.

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{3}{\lambda^2}\right)\left[\frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right]$$
$$+ \frac{aK}{\lambda}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[\frac{\gamma}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right] = 0 \qquad (76)$$

$$p' = -\frac{\gamma\bar{p}}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi \tag{77}$$

$$u' = \frac{\partial\phi}{\partial x} \tag{78}$$

$$v' = \frac{\partial\phi}{\partial y} \tag{79}$$

$$\frac{1}{T_o}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)T' = -\frac{\gamma}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \tag{80}$$

$$\frac{\gamma-1}{\gamma\bar{p}}\frac{\partial q_i'}{\partial x_i} = \frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} \tag{81}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)I_o' = -\frac{\gamma\bar{p}\lambda}{\gamma-1}\left\{\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right]\right.$$
$$\left. + \frac{aK}{\lambda}\left[\frac{\gamma}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right]\right\} \tag{82}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)q_i' = \frac{\gamma\bar{p}\lambda^2}{3(\gamma-1)}\frac{\partial}{\partial x_i}\left\{\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right]\right.$$
$$\left. + \frac{aK}{\lambda}\left[\frac{\gamma}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^3}\right]\right\} \tag{83}$$

Note that the group of terms in the square bracket to the left of eqn. (76) constitutes the Prandtl-Glauert operator. The group of terms in the square bracket to the right constitutes a Prandtl-Glauert operator with an isothermal speed of sound, hereafter referred to as an isothermal Prandtl-Glauert operator.

Apparent Singular Behavior

In certain limits of our radiation parameters the governing eqn. (76) appears to contain singular behavior. Similar to the observations of Vincenti and Baldwin, we see that, when $\lambda \to 0$, $\lambda \to \infty$ or $K \to 0$, the terms to the right of eqn. (76) go to zero and a correct solution can be found from the isentropic Prandtl-Glauert equation, which is a reduction in order from the original differential equation. Likewise, when $K \to \infty$, the terms to the right in eqn.(76) dominate and a correct solution is found from the isothermal Prandtl-Glauert equation, again a reduction in order from the original differential equation. As Van Dyke [35] points out, such behavior in a small parameter limit signals singular behavior unless the boundary conditions are consistent with the reduced equations. This consistency is exactly the case here and it is most easily seen by returning to the original equations governing the thermal radiation, eqns. (22), (23) and (26).

In the limit $\lambda \to 0$, eqn. (26) infers that $q_i = 0$ while eqn. (22) says $I_o = 4\sigma T^4$. Thus, in a completely opaque gas, the intensity can be a non-zero value depending upon the local temperature, but there can be no heat transfer because photons emitted at a location are immediately reabsorbed an infinitesimal distance away. Now, since continuity of temperature is consistent with the isentropic equilibrium solution and intensity is proportional to it, we conclude that thermal boundary conditions will vanish in a manner consistent with the reduction of the governing differential equation. It is only necessary, then, to consider the Prandtl-Glauert equation subject to appropriate mechanical boundary conditions in pressure and/or velocity.

In the limit $\lambda \to \infty$ eqn. (22) yields $\partial q_i/\partial x_i = 0$ or q_i = const. while eqn. (23) yields $\partial I_o/\partial x_i = 0$ or I_o = const. This says that radiant heat energy of a given intensity can traverse a completely transparent gas without alteration of its value, i.e., without interaction with the gas itself. Therefore, any thermal boundary conditions independently imposed at the boundaries of such a gas will have no effect upon the the gas. Once again the Prandtl-Glauert equation applies, but in this case, there can be temperature slip between the gas and its boundaries since there is, in effect, no thermal contact.

In the limit $K \to 0$, or zero temperature, eqn. (26) can only admit the homogeneous solution $q_i = 0$. From eqn. (22) this says $I_o = 0$. Therefore, in a completely cold gas the thermal boundary conditions must vanish in a manner consistent with the reduction to an isentropic governing Prandtl-Glauert equation.

In the remaining limit of $K \to \infty$, an infinitely hot gas, eqn. (26) implies an infinite rate of heat transfer and therefore an isothermal state in the gas, regardless of the thermal boundary conditions. Thus, mechanical disturbances in this gas are governed by an isothermal form of the Prandtl-Glauert equation and are influenced only by mechanical boundary conditions.

The observations discussed above with respect to eqns. (22), (23) and (26) can be shown to hold with respect to their small disturbance counterparts, eqns. (76), (82) and (83). The overall conclusion from this discussion is that no singularity really exists and, as Vincenti and Baldwin have pointed out, solutions to the perturbation potential eqn. (76) will pass over smoothly into those of equilibrium acoustic theory at these parameter limits. Of course, this will mean the loss of the radiation-induced wave, leaving only the classical wave-form.

Cool Gas Limits

Reduction of equations. - We mentioned in the last chapter the possibility of perturbing the equilibrium acoustic theory to include a small measure of non-equilibrium. We may perform this perturbation directly upon the full governing non-equilibrium differential equation for certain small values of the radiation parameters. The result will be a differential equation of lower order subject only to mechanical boundary conditions. We shall demonstrate this for a gas which is either opaque, cool or transparent, cool.

In the completely transparent limit, there is no absorption in the gas and eqn. (76) reduces to the isentropic Prandtl-Glauert equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = \frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \phi \tag{84}$$

If we use this as an approximation in the right-hand group of terms in eqn. (76) we may recover some measure of non-equilibrium behavior in the transparent, cool limit with the second order governing differential equation

$$\left[\frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\phi + \frac{(\gamma-1)K}{\lambda a}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi = 0 \tag{85}$$

In the completely opaque limit, there is infinite absorptivity and eqn. (84) again applies. Using the latter as an approximation in the right-hand group of terms in eqn. (76) will provide a small measure of non-equilibrium behavior in the opaque, cool limit governed by

$$\left[\frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\phi - \frac{(\gamma-1)\lambda K}{3a^3}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^3\phi = 0 \tag{86}$$

It is sufficient for the question of stability to assume a solution periodic in the x-direction. Thus, consider the vortex sheet to be perturbed from its nominal position according to

$$h(x,t) = h_o e^{i\alpha(x-ct)} \tag{87}$$

where $\alpha h_o << 1$ in accord with our requirement of small disturbances to a uniform stream, α is the real wave number and $c = c_r + ic_i$ is a complex wave speed. Note that $c_i > 0$ indicates exponential growth in time.

Compatible with eqn. (87) we would assume a solution for our disturbance potential in x, t of the form

$$\phi(x,y,t) = \Phi(y)e^{i\alpha(x-ct)} \tag{88}$$

and a solution for Φ as

$$\Phi(y) = Ae^{-\alpha\sigma y} + Be^{\alpha\sigma y} \tag{89}$$

where, for a transparent, cool gas

$$\sigma = [1 - (\frac{c-U}{a})^2 - \frac{i(\gamma-1)K}{\alpha\lambda} (\frac{c-U}{a})]^{\frac{1}{2}} \tag{90}$$

and for the opaque, cool gas

$$\sigma = \{1 - (\frac{c-U}{a})^2 [1 + \frac{i(\gamma-1)\alpha\lambda K}{3} (\frac{c-U}{a})]\}^{\frac{1}{2}} \tag{91}$$

where we restrict the branch such that $\sigma_r \geq 0$.

<u>Eigenvalue problem</u>. - Eqn. (89) demands application of two boundary conditions in each gas. Placing the vortex sheet nominally at $y = 0$, we may require vanishing disturbance as $y \to \infty$.* This leaves us with

$$\Phi_1 = B_1 e^{\alpha\sigma_1 y} \tag{92}$$

for the gas in $y < 0$ and

$$\Phi_2 = A_2 e^{-\alpha\sigma_2 y} \tag{93}$$

for the gas in $y > 0$.

---

* In the limits of $K/\alpha\lambda = 0$ for a transparent gas or $\alpha\lambda K = 0$ for an opaque gas, eqn. (84) goversn and we have the possibility of undamped outgoing waves in the case of supersonic disturbances ($|c-U| > a$). Lin [20] remarks that, unless we impose some restriction at infinity in this case, we have no discrete characteristic value problem. In the presence of thermal radiation, however, the decay of the disturbance as it propagates to infinity is a natural consequence, even for supersonic disturbances, and the imposition of a vanishing condition at infinity is not unduly restrictive.

The remaining conditions are satisfied by matching the pressure and normal velocity component across the vortex sheet at all times. The second of these requires

$$\frac{\partial h}{\partial t} = v_1'(0) - U_1 \frac{\partial h}{\partial x} = v_2'(0) - U_2 \frac{\partial h}{\partial x} \tag{94}$$

Using eqns. (77) and (79) and substituting eqns. (87) and (88) into eqn. (94) will allow the two matching conditions to be expressed as

$$\frac{\gamma_1}{a_1^2}(c-U_1)\,\Phi_1(0) = \frac{\gamma_2}{a_2^2}(c-U_2)\,\Phi_2(0) \tag{95}$$

$$\frac{1}{c-U_1}\,\Phi_1'(0) = \frac{1}{c-U_2}\,\Phi_2'(0) \tag{96}$$

Finally, if we substitute eqns. (92) and (93) into eqns. (95) and (96) and eliminate coefficients, we will be left with the eigenvalue equation

$$\frac{-\sigma_1}{\gamma_1\left(\dfrac{c-U_1}{a_1}\right)^2} = \frac{\sigma_2}{\gamma_2\left(\dfrac{c-U_2}{a_2}\right)^2} \tag{97}$$

We remark that this equation is similar to the one treated by Wang for chemical non-equilibrium in the perturbed flow and that it is reducible to that treated by Pai and Miles for isentropic flow.

Stability. - We first question whether or not eqn. (97) contains a neutral stability curve. We should first check to see if there is any region where eqn. (97) cannot govern when $\sigma_r \geq 0$. Let $c = c_r$ and consider

$$\sigma = \sqrt{a_r + ia_i} = (a_r^2 + a_i^2)^{1/4}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \tag{98}$$

where $\theta = \tan^{-1}(a_i/a_r)$. Define $-\pi < \theta < \pi$ for the desired branch for single-valuedness. Our present problem is to determine when $\sigma = i\sigma_i$ only, and this would occur only when $a_r < 0$ and $a_i = 0$. This happens only when $|(c_r-U)/a| > 1$ and $k = 0$ where, for the sake of argument, we have defined $k = \alpha\lambda K$ for the opaque, cool gas or $k = K/\alpha\lambda$ for the transparent, cool gas. Thus, so long as $k > 0$ in both σ_1 and σ_2 we are including all possible neutral disturbances.

Now, we separate real and imaginary parts of eqn. (97) for $c_i = 0$.

$$\frac{-\sigma_{1r}}{\gamma_1\left(\frac{c_r-U_1}{a_1}\right)^2} = \frac{\sigma_{2r}}{\gamma_2\left(\frac{c_r-U_2}{a_2}\right)^2} \qquad \frac{-\sigma_{1i}}{\gamma_1\left(\frac{c_r-U_1}{a_1}\right)^2} = \frac{\sigma_{2i}}{\gamma_2\left(\frac{c_r-U_2}{a_2}\right)} \tag{99}$$

Since $\sigma_r > 0$ for all cases of $k > 0$, it is immediately apparent that the first of eqns. (99) cannot be satisfied when $k > 0$ in either σ_1 or σ_2. Therefore, we conclude that there is no neutral stability.

Next, we must determine whether the vortex sheet is completely stable or unstable. We can do so by mapping to a Cauchy-Nyquist diagram. Define

$$G(c) = \frac{\gamma_2\left(\frac{c-U_2}{a_2}\right)^2 \sigma_1}{\gamma_1\left(\frac{c-U_1}{a_1}\right)^2 \sigma_2} \tag{100}$$

so that $F(c) = 1 + G(c) = 0$. If, in plotting the whole upper half c-plane onto the G-plane, we encircle $G = -1$, then the sheet is unstable. Note the double zero at $c = U_2$ and the double pole at $c = U_1$. We must find the location of branch points of $G(c)$ which will be, of course, the branch points of the σ's.

For the transparent, cool gas we have branch points at

$$\frac{c}{a} = \frac{U}{a} \pm \sqrt{1 - \left[\frac{(\gamma-1)K}{2\alpha\lambda}\right]^2} - i\,\frac{(\gamma-1)K}{2\alpha\lambda} \tag{101}$$

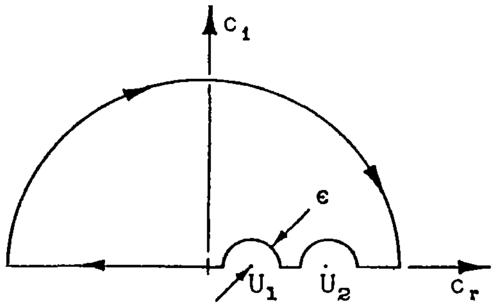
both of which are in the lower half c-plane.

For the opaque, cool gas we have branch points at

$$\frac{c}{a} = \frac{U}{a} \pm \sqrt{1 - \left[\frac{(\gamma-1)\alpha\lambda K}{6}\right]^2} - i\,\frac{(\gamma-1)\alpha\lambda K}{6} \tag{102}$$

both of which are in the lower half c-plane.

Thus, mapping the upper half c-plane onto the G-plane we have only one real concern, which involves the double pole at $c = U_1$. We need not actually map the G-plane but only notice that if we set $(c-U_1)/a_1 = \varepsilon e^{i\theta}$ and move counter-clockwise about $c = U_1$ from $\theta = 0$ to $\theta = \pi$, the argument of $G(c)$ goes



correspondingly from 0 to 2π, thus encircling the G = -1 point once. Therefore, we conclude that the vortex sheet is unstable for k_1 or $k_2 > 0$ regardless of whether the gases are transparent or opaque, cool. This will be taken to mean that there is at least one mode which is unstable; there can be other modes which remain stable.

Static considerations. - It would be interesting to ascertain why a vortex sheet between two semi-infinite isentropic gases can have a region of neutral stability according to Miles' criterion but complete instability occurs if either gas contains a measure of non-equilibrium behavior. We may do this in a crude way by treating the vortex sheet by static considerations as did Ackeret (see Liepmann and Puckett [19]). What we are about to do is treat the vortex sheet as though it were a flexible wall with a prescribed motion

$$h = h_o \mathrm{Re}\,[e^{i\alpha(x-c_r t)}] \tag{103}$$

where h_o is a small constant and c_r is a real wave speed only. Thus, we are now treating a determinate problem rather than an indeterminate eigenvalue problem. Consider the gases to be moving in opposite directions and each at a speed relative to the wall equal to half their velocity difference.

Consider first the gas in $y > 0$. If we differentiate the disturbance potential eqn. (85) or (86) once with respect to y we can just as well express either of them as operators on the disturbance velocity component v'. The corresponding solution to eqns. (88) and (93) for $y > 0$ is

$$v' = A\,\mathrm{Re}\,[e^{-\alpha\sigma y + i\alpha(x-c_r t)}] \tag{104}$$

where σ is given by either eqn. (90) or (91). The prescribed motion of the wall will now fix the value of A, that is

$$v'(0) = \frac{dh}{dt} = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} = h_o\alpha(U-c_r)\ \mathrm{Re}\ [e^{i\alpha(x-c_rt + \frac{\pi}{2\alpha})}] \tag{105}$$

Therefore, eqn. (104) becomes

$$v' = h_o\alpha(U-c_r)\ \mathrm{Re}\ [e^{-\alpha\sigma y + i\alpha(x-c_rt + \frac{\pi}{2\alpha})}] \tag{106}$$

We may now integrate eqn. (106) with respect to y and differentiate it with respect to x and t in accord with eqns. (77) and (79) to get the perturbed pressure field

$$p' = -\frac{\gamma\bar{p}h_o\alpha(U-c_r)^2}{a^2\sqrt{\sigma_r^2 + \sigma_i^2}}\ \mathrm{Re}\ [e^{-\alpha\sigma y + i\alpha(x-c_rt - \frac{1}{\alpha}\tan^{-1}\frac{\sigma_i}{\sigma_r})}] \tag{107}$$

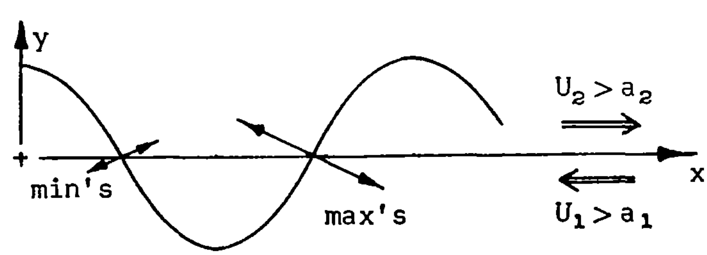
On the wall on the $y = 0^+$ side eqns. (103), (106) and (107) become

$$h(x,t) = h_o\cos\alpha(x-c_rt) \tag{108}$$

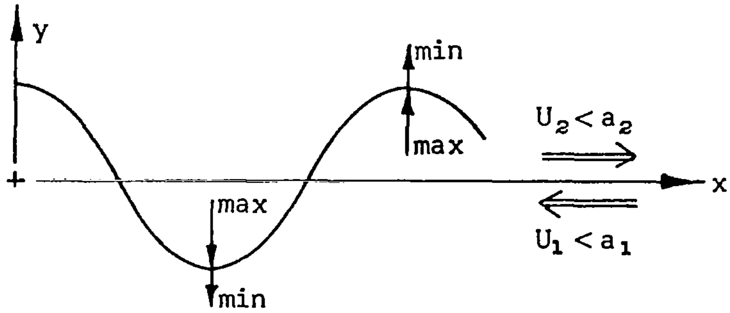
$$v'(x,t) = h_o\alpha(c_r-U)\sin\alpha(x-c_rt). \tag{109}$$

$$p'(x,t) = -\frac{\gamma\bar{p}h_o\alpha(c_r-U)^2}{a^2\sqrt{\sigma_r^2 + \sigma_i^2}}\cos\alpha(x-c_rt - \frac{1}{\alpha}\tan^{-1}\frac{\sigma_i}{\sigma_r}) \tag{110}$$

Now, if we look at the $y = 0^-$ side of the wall, the phase changes by π, thus making a sign change in p'. In the case of isentropic gases $\sigma = \sqrt{1 - (\frac{c_r-U}{a})^2}$ so that if $|\frac{c_r-U}{a}| < 1$ then $\sigma = \sigma_r$ only and if $|\frac{c_r-U}{a}| > 1$ then $\sigma = i\sigma_i$ only. Consequently, for supersonic wave speeds (relative to the stream) there is a phase shift of $\pi/2$ in the pressure. If $|\frac{c_r-U}{a}| > 1$ for both streams, but in opposing directions with respect to the wall, we may have cancellation of forces on the wall. At time t this appears as shown below.

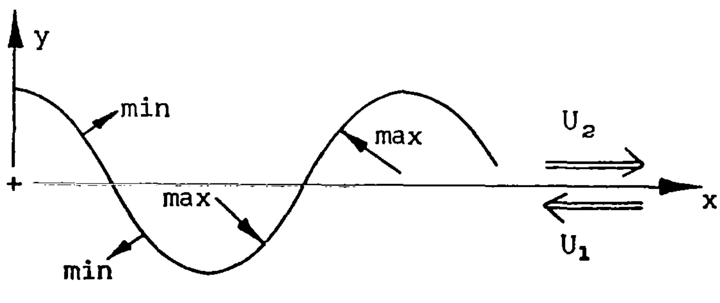


Otherwise, if the streams are subsonic with respect to the wall, there is reinforcement rather than cancellation of pressure forces. This is also sketched below.



We should perhaps keep in mind that, in accordance with Miles' theory, it is a necessary condition for neutral stability that $|U_2 - U_1| > (a_1 + a_2)$, but not sufficient. In other words, a single mode only obtains neutral stability, having the above pressure shift, when $|U_2 - U_1| > (a_1 + a_2)$, thus causing the eigenvalue equation to be satisfied. However, when $|U_2 - U_1| > (a_1^{2/3} + a_2^{2/3})^{3/2}$ all modes are neutrally stable and would then correspond to the pressure balance shown.

Now, when $k \neq 0$ we always have $\sigma = \sigma_r + i\sigma_i$, that is $\sigma_r, \sigma_i \neq 0$. In this case, the $\tan^{-1}(\sigma_i/\sigma_r)$ term causes the phase to vary continuously within 0 to $\pi/2$. With $k > 0$, the phase shift must be less than $\pi/2$, even for supersonic wave speeds. This condition is exhibited below.



Stabilizing or de-stabilizing effects. - From the preceding argument we may infer that a measure of non-equilibrium is de-stabilizing to supersonic disturbances but stabilizing to subsonic disturbances (although both remain unstable). Unfortunately, however, we can show that the stabilizing effect is small while the de-stabilizing effect is large. We shall argue in the following way:

For subsonic flow relative to the wall in the above static analysis, $\tan^{-1}(\sigma_i/\sigma_r) = 0$ when $k = 0$. Upon increasing k from the zero value we cause a corresponding increase in σ_i/σ_r but the value of $\tan^{-1}(\sigma_i/\sigma_r)$ increases from zero very slowly. This means that the pressure shift is quite small for an increase in k (of either gas) and consequently, the subsonic stabilizing effect is small.

For supersonic flow relative to the wall in the above static analysis, $\tan^{-1}(\sigma_i/\sigma_r) = \pi/2$ since $(\sigma_i/\sigma_r) \to \infty$ when $k = 0$. A small increase in k brings about a great change in σ_i/σ_r and a correspondingly substantial decrease in $\tan^{-1}(\sigma_i/\sigma_r)$. Therefore, the pressure shift (from the isentropic shift of $\pi/2$) is large and consequently, the supersonic de-stabilizing effect is large.

The above argument, on the basis of static considerations, is crude but qualitatively correct. In a subsequent section of this chapter we shall see that these observations for small k prove out.

Physical interpretation. - Perhaps the most pertinent question at this point is "What is the physical explanation for the fact that the presence of a measure of thermal radiation non-equilibrium in the perturbation of either gas has the stabilizing or de-stabilizing effects discussed above?" We can offer an explanation by way of comparison to the case of isentropic gases. For the latter, when $|U_2-U_1| > (a_1 + a_2)$, a disturbance propagating away from the vortex sheet cannot radiate acoustic energy back across the sheet into the other gas and consequently, the gas into which the disturbance is propagating acts in a spring-like manner to the disturbance. On the other hand, when $|U_2-U_1| < (a_1 + a_2)$, there can be a feed-back of acoustic energy and a consequent loss of some of the spring-like behavior.

In the presence of thermal radiation, some energy (of a disturbance) can be transmitted at the speed of light. This means that, when a disturbance is propagated into one of the gases, this gas has the ability to "relax" the disturbance. When $|U_2-U_1| > (a_1 + a_2)$, this diminishes the spring-like resistance of the gas whereas, when $|U_2-U_1| < (a_1 + a_2)$, this "relaxation" reduces the amount of acoustic energy feed-back across the vortex sheet to a small degree.

Infinitely Hot Gas

In the last section we discussed the fact that there is a region of neutral stability ($c_i = 0$) for supersonic disturbances when $|U_2-U_1| > (a_1^{2/3} + a_2^{2/3})^{3/2}$ in the limits of completely opaque ($\lambda = 0$), completely transparent ($\lambda = \infty$) or completely cold ($K = 0$) gases. Another region of neutral stability can be shown to exist in the limit of an infinitely hot ($K = \infty$) gas. We have already discussed Vincenti and Baldwin's observation that either the modified classical wave or the radiation-induced wave can exist in this limit depending upon the boundary conditions. Since, in the isothermal limit, the radiation-induced wave produces a field which is uniform in space (due to its infinite propagation speed) and our boundary conditions are homogeneous at $|y| \to \infty$, we cannot expect any disturbance propagating with this wave to be maintained. On the other hand, a disturbance propagating with the modified classical wave at the isothermal speed of sound can produce variations in the field properties of pressure and velocity, contributing to disturbed motion of the vortex sheet. Thus, we need only study the stability of disturbances governed by the isothermal form of the Prandtl-Glauert equation*

$$\frac{\gamma}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \phi - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0 \qquad (111)$$

and the mechanical boundary conditions of eqns. (95) and (96).

The problem we have now described is completely analogous to that of Pai and Miles but for the replacement of the isentropic sound speed with the isothermal sound speed. Taking $\gamma_1 = \gamma_2 = \gamma$, subsonic disturbances now appear when $|U_2-U_1| < (a_1 + a_2)/\sqrt{\gamma}$ with one of the two principal modes being unstable. Supersonic disturbances appear in three principal modes with one unstable when $(a_1 + a_2)/\sqrt{\gamma} < |U_2-U_1| < (a_1^{2/3} + a_2^{2/3})^{3/2}/\sqrt{\gamma}$ and all three neutrally stable when $|U_2-U_1| > (a_1^{2/3} + a_2^{2/3})^{3/2}/\sqrt{\gamma}$.

In the next section of this chapter we shall show numerically that the solution to the isentropic (cold gas) limit goes smoothly over into the solution in the isothermal (hot gas) limit.** In view of the fact that we have instability for the plane vortex sheet in the isentropic and isothermal limits for $|U_2-U_1| < (a_1^{2/3} + a_2^{2/3})^{3/2}$ and $|U_2-U_1| < (a_1^{2/3} + a_2^{2/3})^{3/2}/\sqrt{\gamma}$ respectively, and that a measure of non-equilibrium is de-stabilizing for supersonic disturbances in the cool gas cases, we may project that there will be no complete stability in any intermediate region of non-equilibrium.

Non-Equilibrium Region

<u>Problem formulation</u>. - We shall now extend our study of the stability of the plane vortex sheet in an infinite domain through the region of radiative non-equilibrium. To do this we must derive our eigenvalue problem from the full differential equation and boundary conditions. The equations which govern our problem are eqns. (76)-(83). If we apply the small disturbance solution in (x,t) given by eqn. (88) these may be expressed as

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* This can be derived directly from the original perturbation equations starting with assumption of zero perturbation temperature.

** It appears that the subsonic and supersonic phase relations described for Ackeret's wavy wall in the isentropic limit are destroyed with the introduction of radiative non-equilibrium but restored upon reaching the isothermal limit.

$$m\,\phi^{iv} - \left[n + m + \frac{3}{(\alpha\lambda)^2}\right]\phi'' + \left\{n + \frac{3}{(\alpha\lambda)^2}\left[1 - \left(\frac{c-U}{a}\right)^2\right]\right\}\phi = 0 \tag{112}$$

$$p'(n) = \frac{i\alpha\gamma\bar{p}}{a^2}(c - U)\Phi \tag{113}$$

$$u'(n) = i\alpha\Phi \tag{114}$$

$$v'(n) = \alpha\Phi' \tag{115}$$

$$\frac{T'(n)}{T_o} = \frac{i\alpha}{c-U}\left\{\phi'' - \left[1 - \gamma\left(\frac{c-U}{a}\right)^2\right]\phi\right\} \tag{116}$$

$$\frac{\partial q'}{\partial n}(n) = -\frac{\alpha\gamma\bar{p}}{\gamma-1}\left\{\phi'' - \left[1 - \left(\frac{c-U}{a}\right)^2\right]\phi\right\} \tag{117}$$

$$I_o'(n) = \frac{\alpha^2\gamma\bar{p}\lambda}{\gamma-1}(m\Phi'' - n\Phi) \tag{118}$$

$$q'(n) = -\frac{\alpha^3\gamma\bar{p}\lambda^2}{3(\gamma-1)}(m\Phi''' - n\Phi') \tag{119}$$

where we have defined $\eta = \alpha y$,

$$m = 1 + \frac{iaK}{\alpha\lambda(c-U)} \tag{120}$$

$$n = m - [\gamma(m-1) + 1] \left(\frac{c-U}{a}\right)^2 \tag{121}$$

and dropped x-direction heat flux and its derivative in accord with our small disturbance requirement. Heat flux boundary conditions can then be applied to leading order at the nominal position (y=0 in the present case) of the interface.

Defining

$$\mu = \left[n + m + \frac{3}{(\alpha\lambda)^2}\right] \Big/ m \tag{122}$$

$$\nu = \left\{n + \frac{3}{(\alpha\lambda)^2}\left[1 - \left(\frac{c-U}{a}\right)^2\right]\right\} \Big/ m \tag{123}$$

$$\left.\begin{matrix} a \\ b \end{matrix}\right\} = \left[\frac{\mu \pm (\mu^2 - 4\nu)^{1/2}}{2}\right]^{1/2} \tag{124}$$

where, for uniqueness, $\mathrm{Re}(a,b) \geq 0$, we may write a general solution to eqn. (112) as

$$\Phi = Ae^{a\eta} + Be^{-a\eta} + Ce^{b\eta} + De^{-b\eta} \tag{125}$$

Eigenvalue problem. - Eqn. (125) requires four boundary conditions in each gas. Requiring finiteness as $|y| \to \infty$ leaves the solution

$$\Phi_1 = A_1 e^{a_1\eta} + C_1 e^{b_1\eta} \tag{126}$$

for the gas in $y < 0$ and

$$\Phi_2 = B_2 e^{-a_2\eta} + D_2 e^{-b_2\eta} \tag{127}$$

for the gas in $y > 0$.

The remaining conditions consist in matching pressure, normal velocity component, integrated intensity and heat flux across the vortex sheet. The first two of these are given by eqns. (95) and (96). Using eqns. (118) and (119) the latter two may be expressed as

$$\frac{\gamma_1\lambda_1}{\gamma_1-1}\left[m_1\phi_1''(0) - n_1\phi_1(0)\right] = \frac{\gamma_2\lambda_2}{\gamma_2-1}\left[m_2\phi_2''(0) - n_2\phi_2(0)\right] \tag{128}$$

$$\frac{\gamma_1\lambda_1^2}{\gamma_1-1}\left[m_1\phi_1'''(0) - n_1\phi_1'(0)\right] = \frac{\gamma_2\lambda_2^2}{\gamma_2-1}\left[m_2\phi_2'''(0) - n_2\phi_2'(0)\right] \tag{129}$$

Substituting eqns. (126) and (127) into eqns. (95), (96), (128) and (129) will yield a set of four homogeneous equations in A_1, C_1, B_2 and D_2. Setting the coefficient determinant of these equal to zero constitutes the eigenvalue equation. This will take the form

$$|a_{ij}| = 0 \qquad i, j = 1, 2, 3, 4 \tag{130}$$

If we simplify by setting $\gamma_1/\gamma_2 = a_1/a_2 = 1$ and $U_1 = 0$ (only the relative speed of the gases is important) the individual components of eqns. (130) become

$$a_{11} = 1 \qquad a_{12} = 1 \qquad a_{13} = -\frac{\bar{c}-\bar{U}}{\bar{c}} \qquad a_{14} = -\frac{\bar{c}-\bar{U}}{\bar{c}}$$

$$a_{21} = a_1 \qquad a_{22} = b_1 \qquad a_{23} = \frac{\bar{c}a_2}{\bar{c}-\bar{U}} \qquad a_{24} = \frac{\bar{c}b_2}{\bar{c}-\bar{U}}$$

$$a_{31} = m_1a_1^2 - n_1 \qquad a_{32} = m_1b_1^2 - n_1$$

$$a_{33} = -\frac{\lambda_2}{\lambda_1}(m_2a_2^2 - n_2) \qquad a_{34} = -\frac{\lambda_2}{\lambda_1}(m_2b_2^2 - n_2)$$

$$a_{41} = a_1(m_1a_1^2 - n_1) \qquad a_{42} = b_1(m_1b_1^2 - n_1)$$

$$a_{43} = \left(\frac{\lambda_2}{\lambda_1}\right)^2 a_2(m_2a_2^2 - n_2) \qquad a_{44} = \left(\frac{\lambda_2}{\lambda_1}\right)^2 b_2(m_2b_2^2 - n_2)$$

where we have defined $\bar{c} = c/a$ and $\bar{U} = U_2/a$.

<u>Results</u>. - In producing numerical results it is sufficient to choose certain typical values for our parameters. We have proven in the last chapter that, for a given value of K, maximum non-equilibrium occurs when the transparent and opaque operators in eqn. (76) are balanced. Therefore, we may for

the present purposes, consider it sufficient to select $\alpha\lambda = \mathcal{O}(1)$ for each gas and pursue the complete range of K.

Since we are concerned with the question of instability we shall choose to produce results only for the unstable mode emanating from the principal branch described by Miles. The question remains as to what slip speeds to choose. Figure 6 shows the stability behavior of the most unstable mode in the isentropic and isothermal gas limits. Since the behavior changes between subsonic and supersonic disturbances and again when supersonic neutral stability is achieved, we should choose at least one value of $\bar{U}$ in each of the ranges $\bar{U} < 2/\sqrt{\gamma}$, $2 < \bar{U} < 2\sqrt{2/\gamma}$, $2\sqrt{2/\gamma} < \bar{U} < 2\sqrt{2}$ and $\bar{U} > 2\sqrt{2}$.

Figures 7 through 11 display the stability behavior of our selected mode through a large range of K for chosen values of γ, $\alpha\lambda_1$, $\alpha\lambda_2$ and $\bar{U}$. It is readily seen that there is no region of complete stability in the presence of radiative non-equilibrium. The wave speed c_r remains equal to U/2 throughout the non-equilibrium range while the amplification factor undergoes a smooth transition between the isentropic and isothermal limits. Subsonic disturbances are monotonically stabilized to some extent depending upon $\bar{U}$ while supersonic disturbances are de-stabilized and then stabilized as K increases. Disturbances in the range $\bar{U} > 2\sqrt{2}$ lose their neutral stability as K increases from zero but become neutrally stable again as $K \to \infty$. Note especially that, for the case where $2\sqrt{2/\gamma} < \bar{U} < 2\sqrt{2}$, the mode is unstable at the $K = 0$ end but becomes neutrally stable as $K \to \infty$.

Discussion. - Earlier in this chapter we explained the physical consequence of introducing a measure of radiative non-equilibrium to initially isentropic flows. The relaxation effect upon the disturbance as it propagated into either gas explained the de-stabilization to supersonic disturbances. Now, however, as the gases become hot enough, thermal radiation is quite profuse, tending to smooth out all temperature variations and reduce the intensity of the mechanical disturbance throughout the field. Thus, a new (isothermal) relationship is established between pressure and velocity at each point in the field and the vortex sheet becomes less unstable for a given velocity slip across it.

With regard to the stability of the vortex sheet between semi-infinite isentropic gases Pai remarked that, in general, the characteristic equation has infinite roots. However, as Miles has shown, the principal branch of the solution to the characteristic equation has only two roots for subsonic disturbances and three roots for supersonic disturbances. It has been our objective to identify in our problem only these principal branch roots and, in particular, to follow the behavior of the most unstable one. There remains a question, however, with regard to the possibility of modes generated in conjunction with the radiation-induced wave in a region of radiative non-equilibrium. A purposeful attempt was therefore made to alter the parameters of our problem in solving the characteristic equation to see if new unstable modes, not identifiable with the classical acoustic wave, could be found. This effort did not bear fruit and perhaps it is not surprising. Vincenti and Baldwin have indicated that disturbances of a mechanical nature (for instance, pure harmonic wall motion) give rise predominantly to the modified classical wave while the radiation-induced wave is present to a lesser extent and has high speed and large damping over most of the parameter ranges. Thus, it is highly likely that any modes that might be associated with the radiation-induced wave are very stable and not of concern.

VORTEX SHEET NEAR SYMMETRY PLANE

Equilibrium Limits

In this chapter we shall extend our problem to include the stability of a plane vortex sheet near a virtual boundary, which may be a plane of symmetry for a two-dimensional jet or wake. We shall continue our assumption of an isothermal base flow, thus allowing radiative non-equilibrium to occur only in the perturbations. We shall speak of the "inner" gas as that which is confined and finite in dimension and the "outer" gas as that which is bounded only at infinity. An appropriate sketch is shown in Figure 12.

Betchov and Criminale has indicated a stabilizing influence upon the vortex sheet in incompressible flow when introducing a wall into one of the isentropic streams near the sheet. The nearness of the wall is measured in terms of αy_1 where α is the wave number of the disturbance and y_1 is the distance between the vortex sheet and the wall. The smaller αy_1, the more the stabilizing influence.

In contrast, as Gill and Lessen et al make clear, introducing a virtual boundary to supersonic disturbances in isentropic gases will have just the opposite effect. The disturbances are now able to travel without decay to the boundary and reflect back toward the gas interface. For supersonic disturbances which are initially unstable the introduction of the boundary is less and less stabilizing at higher and higher slip velocities until finally, when $|U_2-U_1| > (a_1^{2/3} + a_2^{2/3})^{3/2}$, there is a de-stabilization. The latter occurs by virtue of the fact that waves can reflect back and forth within the inner gas at resonance angles releasing large amounts of energy to the sheet. The over-all consequence of a virtual boundary existing near the vortex sheet in isentropic flow will be to afford instability at all slip speeds.

The consequences of placing a virtual boundary within "sight" of the vortex sheet as outlined above will carry over completely to the isothermal equilibrium state when the gases are very hot. In pursuing the argument, we need only to replace the isentropic sound speed with the isothermal sound speed.

An interesting feature of the presence of a virtual boundary is the fact that the wave speed of the principal unstable mode is reduced, no longer allowing us to identify subsonic and supersonic disturbances ($|c-U| < a$ and $|c-U| > a$ respectively) strictly with particular ranges of slip speed. Now, a disturbance may be subsonic relative to the inner stream but supersonic relative to the outer stream, for example.

Eigenvalue Problem

The equations governing either the inner or outer gas are listed as eqns. (112)-(121). The solution expressed in eqn. (125), or more particularly eqn. (127), still is appropriate for the outer gas. The inner gas, however, has a virtual boundary a distance αy_1 away from the interface. Thus, it is more convenient to redefine μ from eqn. (122) as

$$\mu = -[n + m + \frac{3}{(\alpha\lambda)^2}]/m \tag{131}$$

leaving eqns. (123) and (124) as they are, and giving the general solution for the inner gas as

$$\Phi_1 = A_1 \sin a_1\eta + B_1 \cos a_1\eta + C_1 \sin b_1\eta + D_1 \cos b_1\eta \tag{132}$$

It will be convenient to transfer the origin of our coordinate system to the plane of symmetry and consider only the problem bounded in $0 \leq \eta < \infty$. Thus, the inner gas is confined in $0 \leq \eta \leq \alpha y_1$ while the outer gas exists in $\alpha y_1 \leq \eta < \infty$. The choice of boundary conditions to be specified at the plane of symmetry is arbitrary. Since the results for symmetrical and anti-symmetrical disturbances in isentropic gases are qualitatively the same, as exhibited by Lessen et al [17], we shall limit our choice to symmetry in pressure and integrated intensity. Thus, our boundary conditions at $\eta = 0$ become $v_1'(0) = q_1'(0) = 0$. Analogous to our one-dimensional problem these are expressed as $\Phi_1'(0) = \Phi_1'''(0) = 0$ in terms of the potential function. Applying these to eqn. (132) leaves

$$\Phi_1 = B_1 \cos a_1\eta + D_1 \cos b_1\eta \tag{133}$$

As before, we match pressure, normal velocity component, integrated intensity and heat flux across the interface. From eqns. (87), (113), (115), (118) and (119) these are

$$\frac{\gamma_1}{a_1^2}(c-U_1)\Phi_1(\alpha y_1) = \frac{\gamma_2}{a_2^2}(c-U_2)\,\Phi_2\,(\alpha y_1) \tag{134}$$

$$\frac{1}{c-U_1}\Phi_1'(\alpha y_1) = \frac{1}{c-U_2}\Phi_2'(\alpha y_1) \tag{135}$$

$$\frac{\gamma_1\lambda_1}{\gamma_1-1}\left[m_1\Phi_1''(\alpha y_1) - n_1\Phi_1(\alpha y_1)\right] = \frac{\gamma_2\lambda_2}{\gamma_2-1}\left[m_2\Phi_2''(\alpha y_1) - n_2\Phi_2(\alpha y_1)\right] \tag{136}$$

$$\frac{\gamma_1\lambda_1^2}{\gamma_1-1}\left[m_1\Phi_1'''(\alpha y_1) - n_1\Phi_1'(\alpha y_1)\right] = \frac{\gamma_2\lambda_2^2}{\gamma_2-1}\left[m_2\Phi_2'''(\alpha y_1) - n_2\Phi_2'(\alpha y_1)\right] \tag{137}$$

If we once again assume, for simplicity, that $\gamma_1/\gamma_2 = a_1/a_2 = 1$, $U_1 = 0$, $\bar{c} = c/a$ and $\bar{U} = U_2/a$ and substitute eqns. (133) and (127) into eqns. (134)-(137), we shall have the eigenvalue equation

$$|a_{ij}| = 0 \qquad i, j = 1, 2, 3, 4 \tag{138}$$

where

$$a_{11} = \cos a_1\alpha y_1 \qquad a_{12} = \cos b_1\alpha y_1 \qquad a_{13} = -\frac{\bar{c}-\bar{U}}{\bar{c}} \qquad a_{14} = -\frac{\bar{c}-\bar{U}}{\bar{c}}$$

$$a_{21} = a_1 \sin a_1\alpha y_1 \quad a_{22} = b_1 \sin b_1\alpha y_1 \quad a_{23} = -\frac{\bar{c}a_2}{\bar{c}-\bar{U}} \qquad a_{24} = -\frac{\bar{c}b_2}{\bar{c}-\bar{U}}$$

$$a_{31} = (m_1a_1^2 + n_1)\cos a_1\alpha y_1 \qquad a_{32} = (m_1b_1^2 + n_1)\cos b_1\alpha y_1$$

$$a_{33} = \frac{\lambda_2}{\lambda_1}(m_2a_2^2 - n_2) \qquad a_{34} = \frac{\lambda_2}{\lambda_1}(m_2b_2^2 - n_2)$$

$$a_{41} = a_1(m_1a_1^2 + n_1)\sin a_1\alpha y_1 \qquad a_{42} = b_1(m_1b_1^2 + n_1)\sin b_1\alpha y_1$$

$$a_{43} = \left(\frac{\lambda_2}{\lambda_1}\right)^2 a_2(m_2a_2^2 - n_2) \qquad a_{44} = \left(\frac{\lambda_2}{\lambda_1}\right)^2 b_2(m_2b_2^2 - n_2)$$

Results

Eigenvalue behavior. - We have now posed a problem which depends upon the proximity of a plane of symmetry to the vortex sheet. Since our objective is primarily to discern the effect of thermal radiation upon the vortex sheet stability, we shall not attempt to exhaust solutions for a complete range of the new length αy_1. This has been done by Betchov and Criminale for subsonic flow and by Lessen et al for supersonic flows. We shall consider it sufficient to choose a single fixed value of αy_1 such that the proximity of the plane of symmetry does cause a discernible effect upon our unstable principal acoustic mode, and then proceed to study the effect of the thermal radiation upon this mode at different slip speeds. We shall see that the radiation effect is superimposed in such a way as to allow generalization for other reasonable choices of αy_1.

Figures 7 through 11 present the results for the present problem. In the $K << 1$ limit approaching the isentropic case we find that the results agree with those of Betchov and Criminale and Lessen et al. The subsonic and supersonic flows for $U < 2\sqrt{2}$ have been stabilized by the proximity of the plane of symmetry while the supersonic flows for $U > 2\sqrt{2}$ appear to have been de-stabilized. Taking account of the lower speed of sound in the isothermal limit, these results carry over as previously described. It is interesting to note that the disturbance wave speed is reduced in the isothermal limit by virtue of the presence of the plane of symmetry. This is apparently due to the fact that the acoustic wave traveling in the inner gas does so at the lower sound speed and thus takes longer to reflect back to the interface where it reinforces the wavy nature of the disturbance. This drop in wave speed with higher temperature is much less pronounced for the subsonic cases since the disturbance decays while propagating within the inner gas.

Through the transition region of radiative non-equilibrium between the isentropic and isothermal limits the behavior of the unstable mode is smoothly varying, effecting the transition from the higher isentropic values to the lower isothermal values, primarily through the region of maximum non-equilibrium. Changing the order of the opacity in either gas will delay this transition since, as we have previously observed, a gas with either a high or a low opacity will act more in an isentropic manner at moderate temperatures. For instance, letting either gas be fairly opaque will allow us to drop the transparent operator in the left of eqn. (76), leaving us with the parameter product λK before the isothermal term. It is then apparent that higher temperatures are required to achieve an isothermal state in the gas. Such an example, allowing the inner gas to be more opaque, is exhibited in Figure 10. On the other hand, if we allow one of the gases to be fairly transparent, we may drop the opaque operator on the left of eqn. (76), leaving us with the parameter product K/λ before the isothermal term. Again we would require higher temperatures to achieve an isothermal state in the gas.

Profile functions. - We may make a few remarks of interest with regard to the profile functions of perturbation pressure, velocity and temperature. From eqns. (113), (115) and (116) we may define profile functions for the inner and outer gases as

$$\hat{p}_1 = \frac{ap_1'(\eta)}{\alpha\gamma\bar{p}} = i\,\bar{c}\,\Phi_1 \qquad \hat{v}_1 = \frac{v_1'(\eta)}{\alpha} = \Phi_1'$$

$$\hat{T}_1 = \frac{aT_1'(\eta)}{\alpha\, T_o} = \frac{i}{\bar{c}}\,[\Phi_1'' - (1 - \gamma\,\bar{c}^2)\Phi_1] \tag{139}$$

$$\hat{p}_2 = \frac{ap_2'(\eta)}{\alpha\,\gamma\bar{p}} = i\,(\bar{c}-\bar{U})\Phi_2 \qquad \hat{v}_2 = \frac{v_2'(\eta)}{\alpha} = \Phi_2'$$

$$\hat{T}_2 = \frac{aT_2'(\eta)}{\alpha T_o} = \frac{i}{\bar{c}-\bar{U}}\,\{\Phi_2'' - [1 - \gamma(\bar{c}-\bar{U})^2]\Phi_2\} \tag{140}$$

Figures 13 through 16 show selected plots of these functions in terms of their magnitude and phase. The profile of pressure has been normalized at the interface. One of the interesting features concerning the distribution of the profile functions is that they are not purely exponentially decaying into the semi-infinite outer gas as they would be for isentropic flow. Because the thermal radiation acts only as a second order modification of the classical (isentropic) acoustic behavior this is not readily apparent from the figures, but it is a fact that there is less than an exponential decay near to the interface. Mathematically, this is due to the contribution of the radiation-induced wave-form to the solution in eqn. (127). Physically, this is caused by the fact that energy of the vortex sheet disturbance can be transmitted directly by thermal radiation into the outer gas to a depth on the order of the photon mean free path length.

The effect of the plane of symmetry is readily apparent. It can be seen that there is a build up of pressure and temperature fluctuations in the inner gas, a natural result of the reflected waves.

All of the figures exhibit slip in $\hat{v}$ and $\hat{T}$ at the interface. The former is simply due to the x-direction slip speed across the vortex sheet. The latter is characteristic of radiation problems if we neglect molecular conduction. The amount of this temperature slip can vary within the limits of isentropic and isothermal behavior, depending upon the relative opacity of the two gases and the temperature level reflected in the value of K.

Comparing plots for the different temperature levels ($K = 0.5, 50.0$) shows a greatly reduced temperature fluctuation relative to the pressure fluctuation for the higher temperature. This exhibits the approach toward isothermal acoustic behavior as the terms to the right of eqn. (76) begin to dominate.

Comparing plots for the different slip speeds ($\bar{U} = 0.5, 2.5$) shows that the velocity fluctuation relative to the pressure fluctuation is less for the supersonic case. This reflects the effect of compressibility.

VORTEX SHEET IN PRESENCE OF BASE FLOW HEAT TRANSFER

Base Flow

Problem description. - We shall now extend our two-dimensional jet or wake problem to one which includes heat transfer in the base flow. We shall consider the source for the thermal non-equilibrium as volumetric heat generation within the inner gas. Then, if we fix our coordinates with the inner gas and claim that the outer gas has a much greater speed relative to it, we can envision one-dimensional heat transfer across the inner gas in the y-direction to the outer gas which then absorbs the heat and convects it downstream. Thus, we may continue our assumption of negligible x-direction variation in base flow properties.

It remains our only problem in the base flow to determine appropriate temperature distributions across the inner and outer gases. This requires solution of the energy eqn. (4) in conjunction with the radiation transfer eqn. (26). As applied to the present problem these are

$$\bar{\rho} c_p U \frac{\partial \bar{T}}{\partial x} = - \frac{\partial \bar{q}}{\partial y} + \bar{Q} \tag{141}$$

$$\frac{\partial^2 \bar{q}}{\partial y^2} - \frac{4\sigma}{\lambda} \frac{\partial \bar{T}^4}{\partial y} - \frac{3}{\lambda^2} \bar{q} = 0 \tag{142}$$

where U = 0 for the inner gas in the present coordinate system and $\bar{Q} = 0$ for the outer gas.

Inner gas. - If we neglect the convection term in the inner gas the energy eqn. (141) becomes

$$\frac{d\bar{q}}{dy} = \bar{Q} \tag{143}$$

If we assume uniform heat generation we note that the radiation transfer eqn. (142) reduces to

$$\frac{d\bar{T}^4}{dy} = - \frac{3}{4\sigma\lambda} \bar{q} \tag{144}$$

Integration of eqn. (143) now yields

$$\bar{q} = \bar{Q}y + C_1 \tag{145}$$

But, for symmetry, let $\bar{q}(0) = 0$ so that $C_1 = 0$. Substitution of eqn. (145) into eqn. (144) and integration yields

$$\bar{T}^4 = -\frac{3\bar{Q}}{8\sigma\lambda} y^2 + C_2 \tag{146}$$

where we have taken λ to be a constant across the inner gas. If we set as a reference temperature $\bar{T}(0) = T_o$ and substitute $f = \bar{T}/T_o$ and $\eta = \alpha y$, eqn. (146) may be written

$$f^4 = 1 - \Delta\eta^2 \tag{147}$$

where we have defined

$$\Delta = \frac{3\bar{Q}}{8\sigma\alpha^2\lambda T_o^4} \tag{148}$$

If we choose to let $\bar{Q}$ be uniform at

$$\bar{Q} = N\sigma_f E\Gamma\rho_o$$

from eqn. (18) and utilize the definitions of the heat generation and radiation parameters, G and K respectively, as defined below eqn. (30) we may get for eqn. (148)

$$\Delta = \frac{6G}{(\alpha\lambda)^2 K} \tag{149}$$

thus fixing the relationship between the heat generation and radiation parameters in the base flow.

Outer gas. - Since there is convection in the outer gas there is need to concern ourselves with the x-direction development of the temperature profile in the base flow in order to select a typical y-direction temperature distribution. If the outer gas is initially uniform at temperature T_∞ we expect the greatest rate of change of its temperature to occur at its common surface with the inner gas, a sort of radiation boundary layer growth. The magnitude of temperature slip which may exist at this interface is a function of the opacities of the two gases, the greater the opacities, the less the slip. If we assume the inner gas to be quite opaque, its surface may be considered as an opaque radiating wall at temperature T_W. In this case, the magnitude of temperature slip is determined by the opacity of the outer gas, the greater its opacity, the more rapid the boundary layer growth and the more rapid the development of a non-uniform temperature profile in the outer gas. Note that there is uniform heat flux crossing the interface at all x-locations since there is uniform internal heat generation. Thus, there is an essential temperature difference $(T_w - T_\infty)$ maintained between the inner gas outer surface and the free stream of the outer gas, particularly if the x-direction development of outer gas temperature is slow.

The problem pertaining to the outer gas that we have formulated above has the form of a Rayleigh problem, one which Sparrow and Cess [33] have treated. They found solutions for the temperature distribution in x and y for short time by means of a series expansion in x and for long time by means of the radiation slip method. A linearization was made in the solution by claiming that $(\bar{T} - T_\infty) << (T_w - T_\infty)$, where $\bar{T}$ is the base temperature at any point during its development in the outer flow. Thus, the long time solution would have to abide by this restriction. They have also shown a solution for temperature slip for all time (all x) by using an exponential approximation to the exponential integrals in the integral form of the radiation transfer equation. This exponential approximation is effectively what we have done in arriving at eqn. (26) if the latter is restricted to one dimension. There is a small difference, however, since Sparrow and Cess have assumed $E_2(t) = e^{-2t}$ where we have effectively taken $E_2(t) = e^{-\sqrt{3}t}$. Their results will, nevertheless, be taken as applicable to our present problem. Defining

$$\xi = \frac{2\sigma T_\infty^3 x}{\bar{\rho} c_p \lambda U} \tag{150}$$

their solution for the slip temperature $\bar{T}(\xi, \alpha y_1)$ is

$$\frac{\bar{T}(\xi, \alpha y_1) - T_w}{T_\infty - T_w} = I_o(4\xi)\, e^{-4\xi} \tag{151}$$

where I_o is the modified Bessel function of the first kind. The short time solution yields

$$\frac{\bar{T}(\xi, \alpha y_1) - T_w}{T_\infty - T_w} = 1 - 4\xi + 12\xi^2 + ---- \tag{152}$$

In applying the above analysis to our stability problem a desirable result would be that only short time need be considered so that we may use a uniform outer gas base temperature. This would require small ξ according to the above solution. Putting ξ in terms of our usual nomenclature it may be rewritten in the form

$$\xi = \frac{(\alpha x)K}{8(\alpha\lambda)\bar{U}} \tag{153}$$

Thus, we see that for $\xi << 1$ we are only allowed to consider low temperature, high speed, small absorptivity or short length for the outer gas.

One factor that has been left out of the above analysis is a measure of the original temperature slip. If this is not very large we need not be concerned with the above criterion for short time. The original slip can be measured either in terms of the total heat generation of the inner gas or the associated temperature drop through the inner gas. The heat transferred from the interface to the outer gas at T_∞ in the opaque wall assumption is

$$\bar{q}_w = \sigma(T_w^4 - T_\infty^4) \tag{154}$$

where this is the accumulation over the source within the inner gas, that is $\bar{q}_w = \bar{Q}y_1$. Therefore, we may express the initial slip as

$$1 - \left(\frac{T_\infty}{T_w}\right)^4 = \frac{\bar{Q}y_1}{\sigma T_w^4} \tag{155}$$

From eqn. (146) we can get

$$T_o^4 - T_w^4 = \frac{3\bar{Q}y_1^2}{8\sigma\lambda_1} \tag{156}$$

Elimination of $\bar{Q}$ from eqns. (155) and (156) yields

$$1 - \left(\frac{T_\infty}{T_w}\right)^4 = \frac{8\lambda_1}{3y_1}\left[\left(\frac{T_o}{T_w}\right)^4 - 1\right] \tag{157}$$

Thus, as long as

$$\left(\frac{T_o}{T_w}\right)^4 << \frac{3y_1}{8\lambda_1} \tag{158}$$

the initial temperature slip in the outer gas is small and we need not be concerned with the short time restriction of our Rayleigh problem.

Inner gas opacity. - We have taken the inner gas to be quite opaque for the benefit of the Rayleigh analysis. It was not necessary to do so in order to effect a solution for temperature distribution within the inner gas in terms of $\bar{Q}$. However, it is true that a fissioning gas which has the capability of producing a substantial volumetric heat generation rate does have a high fission cross-section. Such a gas also normally has a high radiation absorptivity. Thus, it was not unreasonable for us to think in terms of a fairly opaque inner gas.

If we carry this assumption one step further we may combine eqns. (155) and (156) to yield

$$T_o^4 = T_\infty^4 + \frac{3\bar{Q}y_1^2}{8\sigma\lambda_1}\left(1 + \frac{8\lambda_1}{3y_1}\right) \tag{159}$$

Using eqn. (148) we may get

$$\Delta = \frac{1 - \left(\frac{T_\infty}{T_o}\right)^4}{(\alpha y_1)^2\left(1 + \frac{8\lambda_1}{3y_1}\right)} = \frac{1 - K_2/K_1}{(\alpha y_1)^2\left(1 + \frac{8\lambda_1}{3y_1}\right)} \tag{160}$$

assuming equal specific heat ratios and sound speeds of the inner gas at T_o and the outer gas at T_∞. In the case of $y_1/\lambda_1 >> 1$ the slip may be ignored in Δ, leaving us with

$$\Delta = \frac{1 - K_2/K_1}{(\alpha y_1)^2} \tag{161}$$

Perturbation Problem

Governing equations. - Our perturbation problem must remain compatible to the base flow that we have described above. Thus, we must account for the transverse variation of the base flow density and temperature. Upon applying the Galilean transformation $\partial/\partial t = \partial/\partial t + U\,\partial/\partial x$, eqns. (30), (11), (12), (16), (17), (20), (31) and (32) will govern the present problem. Assuming a solution in x,t for the modified potential function of the form

$$\frac{\phi}{f}(x,y,t) = \Phi(y)e^{i\alpha(x-ct)} \tag{162}$$

and letting $\eta = \alpha y$, these governing equations become

$$\left[\frac{d^2}{d\eta^2} - 1 - \frac{3}{(\alpha\lambda)^2}\right]\left[\frac{d}{d\eta}\left(f\frac{d\Phi}{d\eta}\right) - f\Phi + \left(\frac{c-U}{a}\right)^2\Phi + \frac{i a G}{\alpha\lambda(c-U)}\left(\frac{d^2\Phi}{d\eta^2} - \Phi\right)\right]$$
$$+ \frac{i a K}{\alpha\lambda(c-U)}\left(\frac{d^2}{d\eta^2} - 1\right)\left[f^5\frac{d^2\Phi}{d\eta^2} - f^5\Phi + \gamma f^4\left(\frac{c-U}{a}\right)^2\Phi\right] = 0 \tag{163}$$

$$p'(\eta) = \frac{i\alpha\gamma\bar{p}}{a^2}(c-U)\Phi \tag{164}$$

$$u'(\eta) = i\alpha\Phi \tag{165}$$

$$v'(\eta) = \alpha f \frac{d\Phi}{d\eta} \tag{166}$$

$$\frac{T'(\eta)}{T_o} = \frac{i\alpha f}{c-U}\left[f \frac{d^2\Phi}{d\eta^2} - f\Phi + \gamma\left(\frac{c-U}{a}\right)^2 \Phi\right] \tag{167}$$

$$\frac{\partial q'}{\partial \eta}(\eta) = -\frac{\alpha\gamma\bar{p}}{\gamma-1}\left[\frac{d}{d\eta}\left(f\frac{d\Phi}{d\eta}\right) - f\Phi + \left(\frac{c-U}{a}\right)^2\Phi + \frac{iaG}{\alpha\lambda(c-U)}\left(\frac{d^2\Phi}{d\eta^2} - \Phi\right)\right] \tag{168}$$

$$\begin{aligned} I_o'(\eta) = \frac{\alpha\gamma\bar{p}(\alpha\lambda)}{\gamma-1}\Big\{&\frac{d}{d\eta}\left(f\frac{d\Phi}{d\eta}\right) - f\Phi + \left(\frac{c-U}{a}\right)^2\Phi + \frac{iaG}{\alpha\lambda(c-U)}\left(\frac{d^2\Phi}{d\eta^2} - \Phi\right) \\ &+ \frac{iaK}{\alpha\lambda(c-U)}\left[f^5\frac{d^2\Phi}{d\eta^2} - f^5\Phi + \gamma f^4\left(\frac{c-U}{a}\right)^2\Phi\right]\Big\} \end{aligned} \tag{169}$$

$$\begin{aligned} q'(\eta) = -\frac{\alpha\gamma\bar{p}(\alpha\lambda)^2}{3(\gamma-1)}\frac{d}{d\eta}\Big\{&\frac{d}{d\eta}\left(f\frac{d\Phi}{d\eta}\right) - f\Phi + \left(\frac{c-U}{a}\right)^2\Phi + \frac{iaG}{\alpha\lambda(c-U)}\left(\frac{d^2\Phi}{d\eta^2} - \Phi\right) \\ &+ \frac{iaK}{\alpha\lambda(c-U)}\left[f^5\frac{d^2\Phi}{d\eta^2} - f^5\Phi + \gamma f^4\left(\frac{c-U}{a}\right)^2\Phi\right]\Big\} \end{aligned} \tag{170}$$

What makes the present problem different from that of the last chapter is the influence of base flow heat transfer upon the perturbations and the presence of a heat generation perturbation. Since the former influences the perturbation problem only through the base flow properties, in particular the dimensionless temperature f, and the parameter of the latter is related to f through eqn. (149), the whole effect is measurable solely in terms of f. Thus, we are concerned with evaluating the effect upon the vortex sheet stability of non-uniformity in f. It is apparent from eqn. (156) that this non-uniformity can only be substantial if the gas is either quite opaque or has high fission cross-section. Since, as we have pointed out, these two properties are entirely compatible, we shall subsequently make the assumption of a fairly opaque inner gas and use this restriction to advantage in the perturbation analysis.

Eqn. (156) makes clear the fact that internal heat generation leads to non-uniformity in f which will carry a measurable influence upon the perturbation problem. We may show, however, that the direct appearance of a heat generation term in the perturbation equations has little influence in the opaque gas. Eqn. (149) shows G to be proportional to $\Delta(\alpha\lambda)^2K$. Estimating Δ from eqn. (161) and noting that $(1 - K_2/K_1) < 1$ we find that $G < K_1/(y_1/\lambda_1)^2$. Therefore, it appears that, for moderate values of K_1, the heat generation terms may be dropped from our perturbation equations.* This discovery means that the base flow temperature profile in the inner gas may be produced by other than a fissioning gas in experiments used to verify the present stability theory.

With the above discussed restrictions we may now write the somewhat simplified equations governing the inner gas as

$$[(f\Phi_1')' - f\Phi_1 + \bar{c}^2\Phi_1] - \frac{i\alpha\lambda_1 K_1}{3\bar{c}}\left(\frac{d^2}{d\eta^2} - 1\right)[f^5\Phi_1'' - f^5\Phi_1 + \gamma f^4\bar{c}^2\Phi_1] = 0 \tag{171}$$

$$p_1'(\eta) = \frac{i\alpha\gamma\bar{p}}{a}\,\bar{c}\Phi_1 \tag{172}$$

$$u_1'(\eta) = i\alpha\Phi_1 \tag{173}$$

$$v_1'(\eta) = \alpha f\Phi_1' \tag{174}$$

$$\frac{T_1'(\eta)}{T_{o1}} = \frac{i\alpha f}{a\,\bar{c}}[f\Phi_1'' - f\Phi_1 + \gamma\bar{c}^2\Phi_1] \tag{175}$$

$$\frac{dq_1'}{d\eta}(\eta) = -\frac{\alpha\gamma\bar{p}}{\gamma-1}[(f\Phi_1')' - f\Phi_1 + \bar{c}^2\Phi_1] \tag{176}$$

$$I_{o1}'(\eta) = \frac{i\alpha\gamma\,\bar{p}K_1}{(\gamma-1)\bar{c}}[f^5\Phi_1'' - f^5\Phi_1 - \gamma f^4\bar{c}^2\Phi_1] \tag{177}$$

$$q_1'(\eta) = -\frac{i\alpha\gamma\,\bar{p}(\alpha\lambda_1)K_1}{3(\gamma-1)\bar{c}}\frac{d}{d\eta}[f^5\Phi_1'' - f^5\Phi_1 - \gamma f^4\bar{c}^2\Phi_1] \tag{178}$$

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* The influence of this term upon the eigenvalue was spot checked at several points and found to be nearly indiscernible.

where we are taking $U_1 = 0$ and defining $\bar{c}$, the eigenvalue, as in the last chapter.

The equations which govern the outer gas are those of eqns. (112)-(121). Requiring the finiteness condition as $\eta \to \infty$ and utilizing the definitions of eqns. (122)-(124) will afford the solution of eqn. (127) with two unknown coefficients remaining.

Eigenvalue problem. - Requiring symmetry in the inner gas such that $v_1'(0) = q_1'(0) = 0$ and matching pressure, normal velocity component, integrated intensity and heat flux at the interface at $\eta = \alpha y_1$ provides us with a problem similar in nature to that of the last chapter. The only difference involves the solution of eqn. (171) for the inner gas. This has been accomplished by a numerical procedure which is described in the Appendix.

Projected results. - In the formulation of the present problem we have added a new parameter Δ associated with the base flow heat transfer. With the assumption of a fairly opaque inner gas it might at first appear that the product $\alpha\lambda_1 K_1$ in eqn. (171) constitutes only one free parameter but, in the absence of any such restriction upon the outer gas, the parameters $\alpha\lambda_1$ and K_1 appear separately in the boundary conditions at the interface. Thus, if the outer gas is radiating with an opacity of unit order, we have the free parameters γ, αy_1, $\bar{U}$, $\alpha\lambda_1$, $\alpha\lambda_2$, K_1 and K_2 after letting $\gamma_1/\gamma_2 = a_1/a_2 = 1$.

Certain observations may be made prior to actual solution of the present perturbation problem. We have learned that maximum radiative non-equilibrium occurs in the perturbations for a given temperature level (reflected in the value of K) when the opaque and transparent operators in the radiative transfer equation are of the same order. This notion still applies with regard to the outer gas for which we have made no opacity restriction. For the inner gas, however, we have required it to be fairly opaque. Thus, comparable non-equilibrium must occur at higher K.

The influence of variable base flow temperature can be seen in eqn. (171) in both the isentropic and the isothermal groups of terms (with regard to perturbations only). In a region where $f = 1$ we have the usual isentropic and isothermal sound speeds but in regions where $f < 1$ the sound speeds are effectively reduced, leading to a lesser acoustic impedance in the gas. In the last chapter we saw where the lesser isothermal sound speed contributed to some stabilization. If this carries over to the present case, we might expect that the influence of lesser sound speeds (by comparison to our reference value of $a = \sqrt{\gamma R T_0}$) in certain regions of the inner gas will be one of stabilization.

We might point out also that the product $\alpha\lambda_1 K_1 f^5$ appears as a coefficient to the highest derivative in eqn. (171). When this is small it would appear to make our problem singular. But, as we have previously discussed, the solution goes over smoothly to that indicative of equilibrium acoustics in this limit while the thermal boundary conditions vanish in a regular manner. Pursuance of this limit by a numerical procedure, however, can be troublesome because of multiplication and division by small numbers in matching boundary conditions at the interface. In regions where f varies rapidly further numerical difficulty can be encountered because of the demand for a small integration step size.

Results

Eigenvalue behavior. - Figures 17 through 20 display the effect of a variable base flow temperature in the inner gas upon the eigenvalue for selected values of opacity, temperature level and slip speed. It can be concluded that the base flow heat transfer which accounts for the transverse temperature distribution considered in eqn. (147) offers some reduction to the instability of the disturbed vortex sheet.

Several observations may be made with regard to the eigenvalue behavior as a function of the several parameters which appear in the present problem. All four figures plot the complex wave speed $\bar{c}$ as a function of K_2/K_1 for $\alpha\lambda_1 = 10^{-2}$. Figures 17 and 19 also consider $\alpha\lambda_1 = 10^{-3}$ and 10^{-4}. In all cases, it is necessary to insure the validity of our numerical results in light of the opaque inner gas assumption. This assumption, if it breaks down, will do so adjacent to the interface between the gases, where there exists a transparent boundary layer. The limiting criterion used is based upon a calculation of the terms neglected in the analysis and a comparison of them to the terms retained. Quite arbitrarily, the solid curves of Figures 17 through 20 have been made broken when the neglected terms reach approximately 20 per cent of the retained terms for a location immediately adjacent to the interface. Combinations of $\alpha\lambda_1$ and K_1 were limited to those which would not yield too low a number for their product so as to avoid the numerical difficulties previously discussed. It is to be noted that greater instability occurs for lower $\alpha\lambda_1$. This is as it should be since a lower $\alpha\lambda_1$ indicates a more opaque inner gas which approaches classical equilibrium behavior.

A higher temperature level (reflected in K_1) offers less instability because of the transition toward isothermal equilibrium behavior as we discussed in the last chapter. This is true for all K_1 when $K_2/K_1 = 1$ but it is not without limit when $K_2/K_1 < 1$. Figure 19 shows that, when $\alpha\lambda_1 = 10^{-2}$, $\alpha\lambda_2 = 1$ and $K_2/K_1 = 10^{-1}$ for instance, there is again a de-stabilization as K_1 exceeds about 10^2.

Because the effect of the "nearness" of the plane of symmetry is superimposed, as previously discussed, it was considered sufficient to limit αy_1 to one value while investigating the influence of the remaining parameters. It can be seen from the figures that the effect of varying K_2/K_1 is qualitatively similar for the four different slip speeds chosen.

Profile functions. - Figures 21 and 22 offer a typical comparison of the profile functions for cases with and without base flow heat transfer. The equations governing the functions for the inner gas are those of eqns. (A4), (A6) and (A7) of the Appendix. The equations governing the functions for the outer gas come from eqns. (140) where the profile for temperature must be altered by a factor of $(K_2/K_1)^{\frac{1}{4}}$ in order to reference it to the symmetry plane base temperature of T_{o1}. The latter profile is then governed by

$$\hat{T}_2 = \frac{aT_2'(\eta)}{\alpha T_{o1}} = \frac{i(K_2/K_1)^{\frac{1}{4}}}{\bar{c}-\bar{U}}\{\Phi_2'' - [1 - \gamma(\bar{c}-\bar{U})^2]\Phi_2\} \tag{179}$$

In general, the observations made in the last chapter regarding the profile functions carry over to the present case. The same arguments apply to slip at the interface and the smaller temperature fluctuations at the higher base temperature levels. As before, variation of the parameters associated with thermal radiation carry a primary influence upon the temperature fluctuations and a secondary influence upon the pressure and velocity fluctuations.

The major difference exists in the influence of the base flow temperature variation (reflected in K_2/K_1). A lower K_2/K_1 leads to a greater temperature fluctuation relative to pressure fluctuation because of the function f in the terms to the right in the governing eqn. (171). A lower K_2/K_1 also leads to steeper perturbation profiles adjacent to the interface in the inner gas, that for temperature being more pronounced than that for velocity.

APPLICATION TO GASEOUS NUCLEAR ROCKET

Rocket Parameters

In this chapter we shall make an approximate application of the above analysis to the gaseous nuclear rocket problem which prompted this investigation. At present, there appears to be no firmly established design, either in a mechanical or a thermal sense, but we shall derive approximations to the appropriate parameters based upon recent available reports.

Kascak [13] suggests a fuel core of approximately R = 1.6 ft. radius containing uranium gas with opacities equivalent to $R/\lambda_1 = 10^3$ or higher. McLafferty [22] and Kascak suggest power levels which yield approximately $\bar{Q} = 10^8$ Btu/hr-ft^3. Putre [30] and McLafferty suggest that the inlet temperature of the coolant gas is approximately $T_\infty = 5000°R$ after regeneratively cooling the nozzle. Based upon this information we can calculate an approximate centerline temperature T_o. The cylindrical form equivalent to eqn. (159) is

$$T_o^4 = T_\infty^4 + \frac{3\bar{Q}R^2}{16\sigma\lambda_1}\left(1 + \frac{8\lambda_1}{3R}\right) \tag{180}$$

Substitution of the above data yields approximately $T_o = 64{,}000°R$. This calls for approximately $K_2/K_1 = 3.5 \times 10^{-5}$.

Parks, Lane, Stewart and Peyton [27] give specific heat values for gaseous uranium such that $\gamma = 1.4$ approximately. Putre considers fuel to propellant density ratios ranging from 1.0 to 4.7. An intermediate value would yield a sound speed in the fuel lower than in the propellant. As a compromise and to satisfy our assumption of $a_1/a_2 = 1$ in our perturbation analysis, we shall take a = 5000 fps. Assuming an operating pressure of 100 atm. will yield a radiation parameter for the propellant at inlet of approximately $K_2 = 10^{-3}$.

Kascak has suggested opacities equivalent to $R/\lambda_2 = 5$ or higher for the propellant. Putre has suggested a slip speed on the order of 100 fps. We shall arbitrarily take for our purpose 125 fps and have as a consequence $\bar{U} = 0.025$.

Rayleigh Analysis

The heat transfer per unit area to the surface of a cylindrical gas core generating heat uniformly is $q_w = \bar{Q}R/2$. Substituting this into eqn. (154) will yield an estimate of the initial temperature slip from the opaque surface of the inner gas to the coolant gas as

$$\frac{T_\infty}{T_w} = \left(1 + \frac{\bar{Q}R}{2\sigma T_\infty^4}\right)^{-1/4} = .34$$

Using $y_1/\lambda_2 = 5$ and $\alpha y_1 = 0.7$ we find from eqn. (153)

$$\xi = \frac{.001}{8(.14)(.025)}(\alpha x) = .0357(\alpha x)$$

which indicates that we can consider the occurrence of a few wavelengths in the downstream direction before appreciable radiation boundary layer development in the outer gas.

A few remarks about this result are in order. Conceptual designs of the rocket indicate relatively small chamber length to diameter ratios. Kascak has suggested a length of 6.0 ft. which yields a length to core diameter ratio of 3.75. This means that, if $\alpha y_1 = 0.7$, only 2.62 wavelengths exist in the chamber. It is the whole objective of the rocket to gain substantial enthalpy in the propellant as it traverses the chamber length. Then it would be desirable to consider higher coolant gas opacity or slower speed, both of which would yield a more substantial radiation boundary layer development over a shorter chamber length for a given αy_1. On the other hand, shorter wavelengths may be of concern, such as those more indicative of turbulence. In that case, the chamber would contain many more wavelengths and it would thus take many more wavelengths for development of the radiation boundary layer. Then, it is permissable to consider a uniform outer gas temperature for our perturbation analysis.

We have already observed in prior work that subsonic disturbances to the vortex sheet, existing for subsonic slip speeds, will tend not to be influenced by a real or virtual boundary existing more than a few wavelengths away. We shall see evidence of this fact upon reviewing the results of the perturbation problem analysis of the next section.

Perturbation Problem Results

Figure 23 displays the behavior of the complex wave speed $\bar{c}$ as a function of wave number for fixed values of γ, $\bar{\bar{U}}$, y_1/λ_1, y_1/λ_2, K_2 and K_2/K_1 as listed on the figure, considering a confined inner gas and a semi-infinite outer gas. The value of K_2/K_1 selected for the perturbation analysis was 10^{-3} which corresponds to an approximate centerline temperature of T_o = 28,000°R. This was done in an attempt to avoid the necessity of using an excessively small step size in the numerical integration in the inner gas near the interface. The consequences of this substitution will be discussed later in this section.

For smaller wave numbers, the virtual boundary of the symmetry plane in the inner gas obviously has some effect upon the eigenvalue. As we would expect, there is a build up of pressure and temperature disturbances in the inner gas with particularly steep profiles adjacent to the interface because of the influence of the function f in the governing equations. In the outer gas there is decay of the disturbances to a vanishing magnitude as they propagate outward from the vortex sheet. A question remains, however, as to the direction of propagation to determine the number of wavelengths in the downstream direction before substantial decay takes place. This point is important to the question of radiation boundary layer growth. From eqns. (88) and (127) the solution in the semi-infinite outer gas takes the form

$$\phi = (Ae^{-a\alpha y} + Be^{-b\alpha y})e^{i\alpha(x - ct)}$$

For the parameters selected and a choice of $\alpha y_1 = 0.7$, the numerics yield

$$a = 10.3 + 2.5i \qquad\qquad b = 1.0 + 0.00014i$$

where a is associated with the modified classical wave and b is associated with the radiation-induced wave. Then, the angle of propagation of these is

$$(dy/dx)_a = 1/a_i = 0.4 \qquad\qquad (dy/dx)_b = 1/b_i = 7000$$

while the associated damping in the y-direction is

$$|\phi|_a \sim e^{-10.3\alpha y} \qquad\qquad |\phi|_b \sim e^{-\alpha y}$$

Thus, the damping is quite substantial in the x-direction.

Realistically, we realize that the outer gas in the rocket is not unconfined. In fact, it is expected that the dimension of the outer gas would be of the same order as that of the inner gas. Then, the question remains as to the influence of an outer boundary. Figure 23 displays the behavior of our eigenvalue upon placing a wall, where v_2' and q_2' vanish, at a distance from the interface equal to the half-width of the core gas. This alters the form of the solution in the outer gas to one similar to that of the inner gas but with uniform base temperature. For the longer wavelengths this has a slightly de-stabilizing influence rather than stabilizing as with one boundary. This is due to the fact that disturbances generated at the vortex sheet and propagated outward in both directions tend to cancel each other after being reflected back to the interface, thus destroying the stabilizing influence of a single boundary. This is more clearly seen from the plots on this figure of the eigenvalue behavior of purely isentropic gases with like properties.

Comparison of the isentropic cases to the radiating cases shows that the latter are less unstable, exhibiting the fact that radiative transfer between the gases does have a stabilizing influence.

Figure 23 shows that, as we move to shorter wavelengths, the eigenvalue approaches a fixed asymptotic value, independent of the wave number. This means, of course, that the boundaries no longer influence the vortex sheet instability, the eigenfunctions decaying to very small magnitudes before encountering the boundaries.

A check was made of the transparent terms neglected in our opaque inner gas perturbation analysis. For our particular choice of K_2/K_1 it was found that the neglected terms were on the same order of magnitude as the retained terms immediately adjacent to the interface. A value of $K_2/K_1 \approx 3.5 \times 10^{-5}$ would have exceeded applicability of our assumption and we would be forced to consider the effect of the transparent boundary layer adjacent to the interface. Then, of course, it would not be permissible to consider the outer gas as receiving radiation purely from an opaque inner gas wall.

As we have mentioned before, it was not necessary to assume that we were dealing with an opaque gas in order to drop the first term of the radiative transfer eqn. (26) in the base flow. The loss of this term was purely a consequence of the fact that the source term $\bar{Q}$ was taken as uniform across the gas.* The result is that the Planck function $\sigma\bar{T}^4/\pi$ is parabolically distributed. It would likewise be true that the first term of eqn. (26) could be dropped in the opaque assumption, regardless of the distribution of the source function, so that the radiative transfer becomes a diffusion process. Given a high magnitude of source function, however, the temperature distribution becomes particularly steep and the short photon path diffusion process no longer applies. Kascak based his heat transfer analysis for the base flow in the inner gas upon the diffusion approximation, arguing that the gas is quite opaque. His analysis would appear to be in error, however, because he considers non-uniform source distributions and power levels which give rise to such a steeply varying Planck function distribution that he exceeds the

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* The uniform source problem was considered by Heaslet and Warming [12].

diffusion approximation. To be correct, then, he should consider the effect of keeping the first term of eqn. (26).

Our choice of a higher K_2/K_1 corresponds to a lower $\bar{Q}$ so that the Planck function distribution is not so steep as to exceed the diffusion approximation, the latter being of importance in our perturbation analysis even though $\bar{Q}$ may be uniform. A lower K_2/K_1 would offer a more steeply varying function f which would lead to an increased stabilization effect. However, because the transparent boundary layer adjacent to the interface would then have to be included in our perturbation analysis, this increased stabilization effect would not be expected to be very appreciable.

Related Experiments

Two experiments which appear to be somewhat related to the present work are reported by Ragsdale and Lanzo [31]. One of these was carried out to determine the effect of inlet velocity profiles and turbulence levels upon the mixing of coaxial streams of air, at normal temperatures, in a rocket-like chamber such as that sketched in Figure 1. The inner stream, colored with iodine, is injected at low speed parallel to the high speed outer stream. In one case, visual observation showed that a large scale mixing effect took place with a downstream recirculation of the propellant into the core region. A foamy, porous material, was then introduced across the inlet to break up the large scale turbulence and provide a more uniform, laminar-like flow at inlet. In this case the recirculation no longer existed and the mixing seemed to be confined to a relatively thin layer between the streams.

The second experiment involved the coaxial flow of argon (slow moving inner gas) and air (faster moving outer gas) where the argon was inductively heated in a plasma state by coupling to a high frequency alternating current field, the latter created by passing a current through a copper coil embedded in the outer wall. No care was taken to provide a laminar-like inlet so that large scale turbulence was probably introduced. Argon concentration profiles were measured across the cylindrical cavity before and after the argon was heated. Extremely high concentrations of air were in evidence in the central portion for the cold flow but mixing was retarded when the argon was heated. This implies that the heated inner gas (with a presumed thermal radiation to the outer gas) suppresses the turbulence and reduces the mixing.

If the laminar-like inlet of the first experiment had been provided for the second experiment, turbulence would have to be confined to that generated at the gas interface and a better analogy to the present work could be drawn. It is expected, however, that the vortex sheet will be unstable and turbulence will be generated but, in view of the experimental results, it will be diminished to an extent in the presence of thermal radiation. This implies that the amplification of the disturbance is reduced, an effect which the present linear perturbation theory predicts.

CONCLUSION

It is known that thermal radiation is a thermodynamic non-equilibrium process that acts to damp acoustic disturbances as they propagate in a gas. It is shown in the present work that standing, undamped one-dimensional waves can propagate back and forth between fixed boundaries when the gas is either completely cold or infinitely hot, in the former case traveling at the isentropic speed of sound and in the latter case traveling at the isothermal speed of sound. Maximum damping is incurred when the gas is in a state of maximum non-equilibrium, which occurs at an intermediate temperature depending upon the opacity of the gas. Density dependent heat generation introduced into the perturbations of the gas has an amplification effect upon the standing waves.

Previous investigators have shown the plane vortex sheet separating semi-infinite isentropic gases to be unstable to small disturbances except when the slip speed across the sheet exceeds a certain supersonic value. For equal specific heat ratios and sound speeds in the gases the sheet becomes neutrally stable for slip speeds in excess of $2\sqrt{2}$ times the isentropic speed of sound. The present study shows that this criterion also applies to the isothermal equilibrium state if the isentropic sound speed is replaced by the isothermal sound speed. In addition, it is shown that the amplification factor is somewhat less at the isothermal limit for a given unstable slip speed, the effect being smaller for smaller speeds. Uniformly increasing the temperature level from the isentropic limit to the isothermal limit in the gases gives rise to a monotonic decrease in amplification factor for all subsonic disturbances while, for supersonic disturbances, the amplification factor increases and then decreases. Thus, thermal radiation in the perturbations has a de-stabilizing influence upon otherwise neutrally stable disturbances.

Prior work has shown that the introduction of a virtual boundary near the vortex sheet in isentropic flow stabilizes it for slip speeds less than $2\sqrt{2}$ times the sound speed but de-stabilizes it for slip speeds greater than $2\sqrt{2}$ times the sound speed. The present work demonstrates that these observations carry over to the isothermal limit. In the transitional non-equilibrium region the presence of the boundary appears to give rise to more of a monotonic decrease in the amplification factor with lesser variation between the isentropic and isothermal limits.

Introducing uniform heat generation into a confined inner gas gives rise to a parabolic distribution of temperature to the fourth power in the gas, implying a transfer of heat from the inner gas, across the vortex sheet to the outer gas in the base flow. The present investigation indicates that this variation in base temperature is somewhat stabilizing (lesser amplification factor) to small disturbances of the vortex sheet at all slip speeds. Therefore, it appears that radiative non-equilibrium in the base flow has a stabilizing influence. Introducing heat generation in the perturbations in the inner gas in proportion to perturbed density, consistent with the base flow heat generation which gives rise to the base temperature variation, appears to have no discernible effect upon the stability question.

An approximate application of the present analysis was made to a coaxial flow gaseous nuclear rocket. The result exhibited that the vortex sheet between the slow moving core gas and the faster moving propellant gas is unstable to small disturbances but with a lesser amplification factor than that which befits isentropic gases.

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APPENDIX

The solution of eqn. (171) for a radiating gas with non-uniform base temperature can be accomplished by numerical integration. This integration becomes part of an overall numerical iteration procedure for the proper value of the complex wave speed.

If we define

$$h_1(\eta) = f^5 \qquad\qquad h_2(\eta) = h_1'$$
$$h_3(\eta) = h_1 - \gamma \bar{c}^2 f^4 \qquad\qquad h_4(\eta) = h_3' \tag{A1}$$

$$g_o(\eta) = h_1 \qquad\qquad g_1(\eta) = 2h_2$$
$$g_2(\eta) = h_2' - h_1 - h_3 + \frac{3i\bar{c}}{\alpha\lambda_1 K_1} f$$
$$g_3(\eta) = 2h_4 - \frac{3i\bar{c}}{\alpha\lambda_1 K_1} f' \tag{A2}$$
$$g_4(\eta) = h_4' - h_3 + \frac{3i\bar{c}}{\alpha\lambda_1 K_1} (f - \bar{c}^2)$$

and let p, q and r be the first, second and third derivatives, respectively, of the function Φ_1, we may replace eqn. (171) with four first order ordinary differential equations in the form

$$\Phi_1' = p \qquad\qquad p' = q \qquad\qquad q' = r$$
$$r' = -\frac{1}{g_o}(g_1 r + g_2 q - g_3 p - g_4 \Phi_1) \tag{A3}$$

where ( )' denotes differentiation with respect to η. The η-direction mode functions from eqns. (172)-(178) may be written as

$$p_1'(\eta) = \frac{i\alpha\gamma\bar{p}}{a} \bar{c}\, \Phi_1 \tag{A4}$$

$$u_1'(\eta) = i\alpha\Phi_1 \tag{A5}$$

$$v_1'(\eta) = \alpha f p \tag{A6}$$

$$\frac{T_1'(\eta)}{T_{o1}} = \frac{i\alpha f}{a\bar{c}} [fq - (f - \gamma\bar{c}^2)\,\Phi_1] \tag{A7}$$

$$\frac{dq_1'}{d\eta}(\eta) = -\frac{\alpha\gamma\bar{p}}{\gamma-1} [fq + f'p - (f - \bar{c}^2)\,\Phi_1] \tag{A8}$$

$$I_{o1}'(\eta) = \frac{i\alpha\gamma\bar{p}K_1}{(\gamma-1)\bar{c}} (h_1 q - h_3\Phi_1) \tag{A9}$$

$$q_1'(\eta) = -\frac{i\alpha\gamma\bar{p}(\alpha\lambda_1)K_1}{3(\gamma-1)\bar{c}} (h_1 r + h_2 q - h_3 p - h_4\Phi_1) \tag{A10}$$

Eqns. (A3) may be forward integrated by a Runge-Kutta method (see, for instance, Conte [5]) if we have starting values for $\Phi_1(0)$, $p(0)$, $q(0)$ and $r(0)$. The boundary conditions of vanishing normal velocity component and heat flux due to symmetry at $\eta = 0$ lead to the conditions $p(0) = r(0) = 0$ from eqns. (A1), (A6) and (A10) where $(f^4)'(0) = (f^5)'(0) = 0$ from eqn. (147). We must yet provide starting values for $\Phi_1(0)$ and $q(0)$. We may accomplish this by performing our integration twice, arbitrarily setting $\Phi_1(0) = 0$ and $q(0) \neq 0$ in one problem, then reversing these assignments in the second problem, and finally providing the solution as a linear combination of the two. This is permissable, of course, only by virtue of the fact that our perturbation equations are linear.

Thus, the starting values for problem <u>a</u> may be taken as

$$\Phi_a(0) = 1 \qquad p_a(0) = q_a(0) = r_a(0) = 0 \tag{A11}$$

with the Runge-Kutta integration being performed upon eqns. (A3). The result will be numerical values of $\Phi_a(\alpha y_1)$, $p_a(\alpha y_1)$, $q_a(\alpha y_1)$ and $r_a(\alpha y_1)$. The starting values of problem <u>b</u> may be assigned as

$$q_b(0) = 1 \qquad \Phi_b(0) = p_b(0) = r_b(0) = 0 \tag{A12}$$

allowing numerical results for $\Phi_b(\alpha y_1)$, $p_b(\alpha y_1)$, $q_b(\alpha y_1)$ and $r_b(\alpha y_1)$.

In an eigenvalue problem we have the freedom to arbitrarily set one of the solution integration constants, considering that the eigenvalue $\bar{c}$ is an unknown. Thus, our linear combination may take the form

$$\left.\begin{aligned}\Phi_1 &= \Phi_a + C\Phi_b \\ p &= p_a + Cp_b \\ q &= q_a + Cq_b \\ r &= r_a + Cr_b\end{aligned}\right\} \tag{A13}$$

In the case of a radiating outer gas the solution to the perturbation potential function is given in eqn. (127) where finiteness at $\eta \to \infty$ has been applied, leaving two unknown coefficients. The boundary conditions applied at the interface between the gases are taken as the usual matching of pressure, normal velocity component, integrated intensity and heat flux. Utilizing eqns. (87), (94), (113), (115), (118), (119), (A4), (A6), (A9) and (A10), these may be written for the present case as

$$\Phi_1(\alpha y_1) = \frac{\bar{c} - \bar{U}}{\bar{c}} \Phi_2(\alpha y_1) \tag{A14}$$

$$f(\alpha y_1)\ p(\alpha y_1) = \frac{\bar{c}}{\bar{c} - \bar{U}} \ \Phi_2'(\alpha y_1) \tag{A15}$$

$$h_1(\alpha y_1)\ q(\alpha y_1) - h_3(\alpha y_1)\ \Phi_1(\alpha y_1) = - \frac{i(\alpha\lambda_2)\bar{c}}{K_1} [m\Phi_2''(\alpha y_1) - n\Phi_2(\alpha y_1)] \tag{A16}$$

$$h_1(\alpha y_1)r(\alpha y_1) + h_2(\alpha y_1)q(\alpha y_1) - h_3(\alpha y_1)p(\alpha y_1) - h_4(\alpha y_1)\ \Phi_1(\alpha y_1) = - \frac{i(\alpha\lambda_2)^2\bar{c}}{(\alpha\lambda_1)K_1} [m\Phi_2'''(\alpha y_1) - n\Phi_2'(\alpha y_1)] \tag{A17}$$

Substitution from eqns. (127) and (A13) into eqns. (A14)-(A17) will result in four equations in the four unknowns B_2, D_2, C and the eigenvalue $\bar{c}$.

The interation procedure starts by making two guesses of the eigenvalue $\bar{c}$ for a given set of parameters γ, αy_1, $\bar{U}$, $\alpha\lambda_1$, $\alpha\lambda_2$, K_1 and K_2. For each guess the Runge-Kutta numerical integration through the inner gas is performed, utilizing eqns. (A1)-(A3), (A11) and (A12), to arrive at Φ_a and Φ_b and their derivatives at $\eta = \alpha y_1$. Then, the two complex exponential solutions of eqn. (127) and their derivatives at $\eta = \alpha y_1$ are calculated while using eqns. (120)-(124). Upon substituting into the three eqns. (A14)-(A16) we may eliminate

B_2, D_2 and C. Eqn. (A17) may then be used for linear extrapolation to a new estimate of $\bar{c}$. The process is repeated until eqn. (A17) is satisfied within an established convergence criterion, whereupon the last estimate of the eigenvalue is considered the correct value.

Once the correct value of the complex wave speed $\bar{c}$ is found by the above procedure, the Runge-Kutta integration process may be repeated with the starting values

$$\Phi_1(0) = 1 \qquad p(0) = 0 \qquad q(0) = C \qquad r(0) = 0 \tag{A18}$$

in order to provide the profile functions for the inner gas. The profile functions for the outer gas can be calculated by substitution into eqn. (127) for Φ_2 and its derivatives.

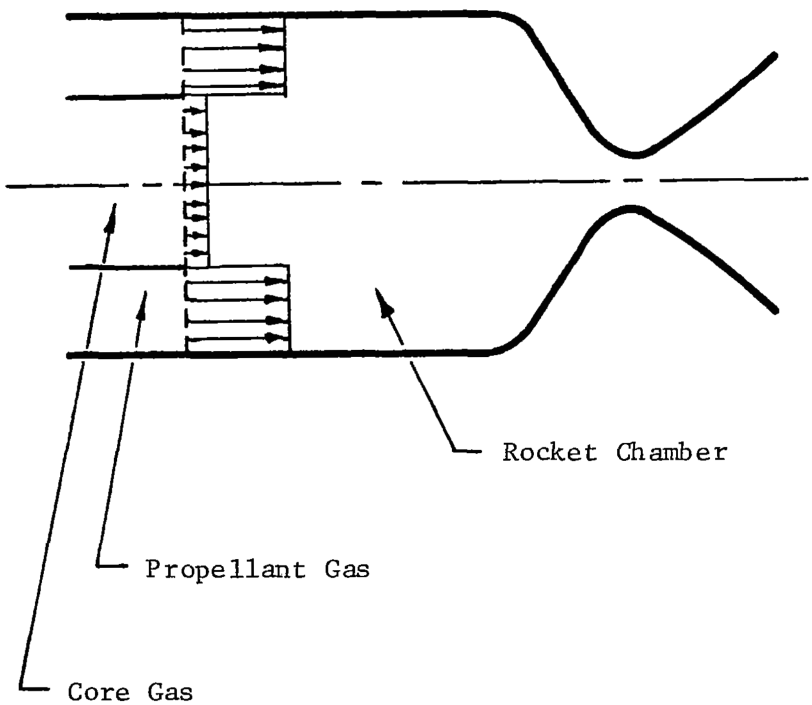

Figure 1. Schematic of coaxial flow rocket chamber.

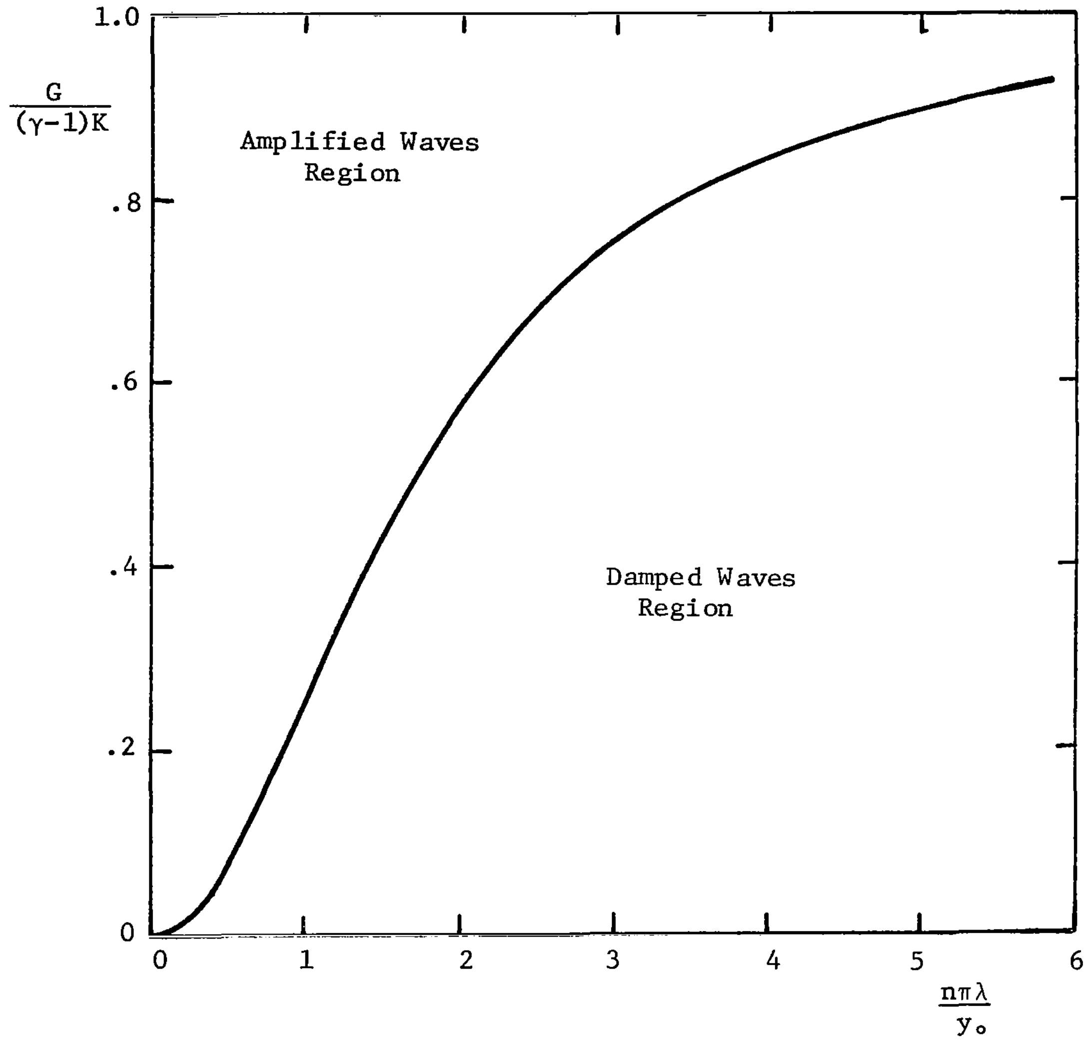

Figure 2. Neutral stability curve for 1-D radiating and heat generative gas with uniform properties confined between rigid adiabatic walls.

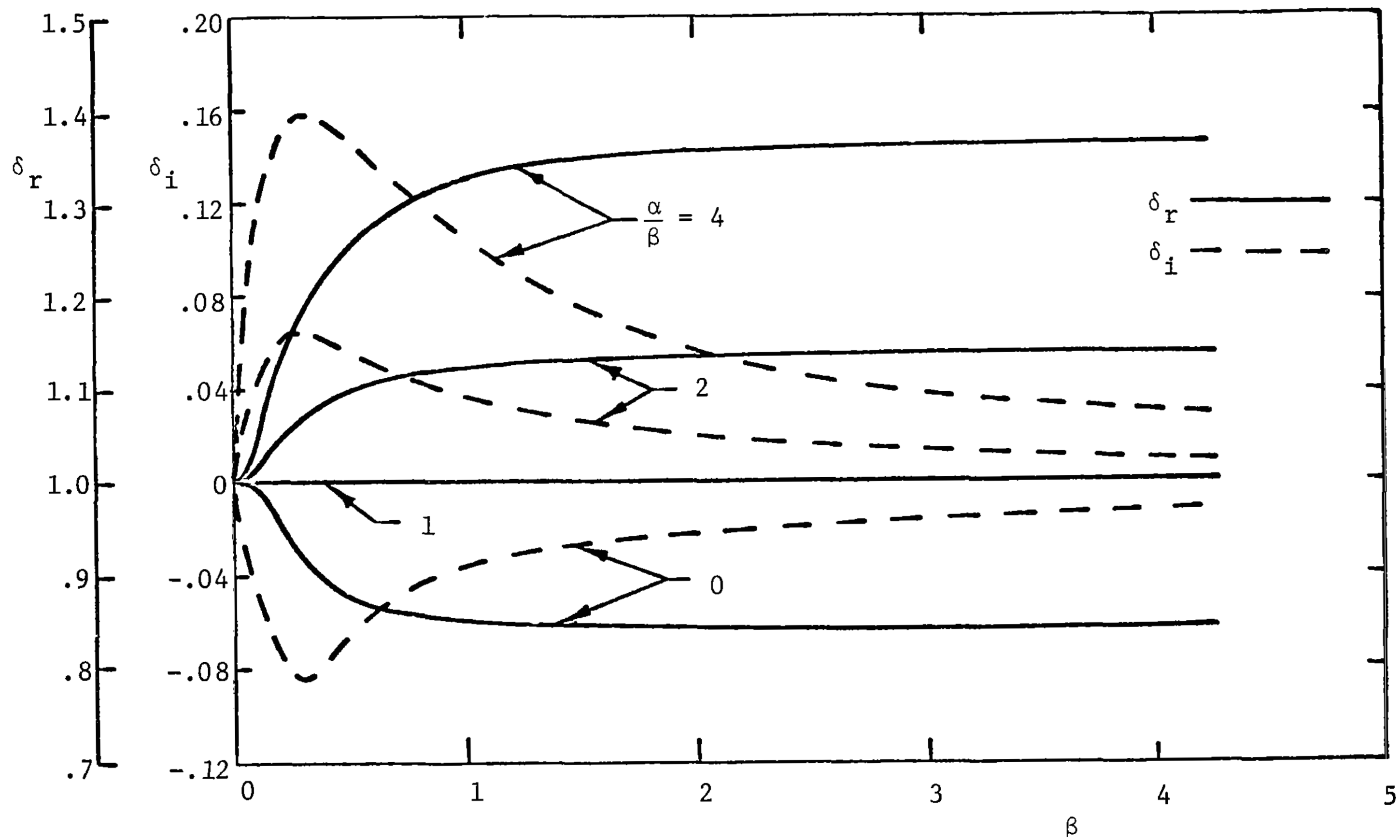

Figure 3. Complex eigenvalue for single 1-D radiating and heat generative gas with γ = 1.4 confined between rigid adiabatic walls.

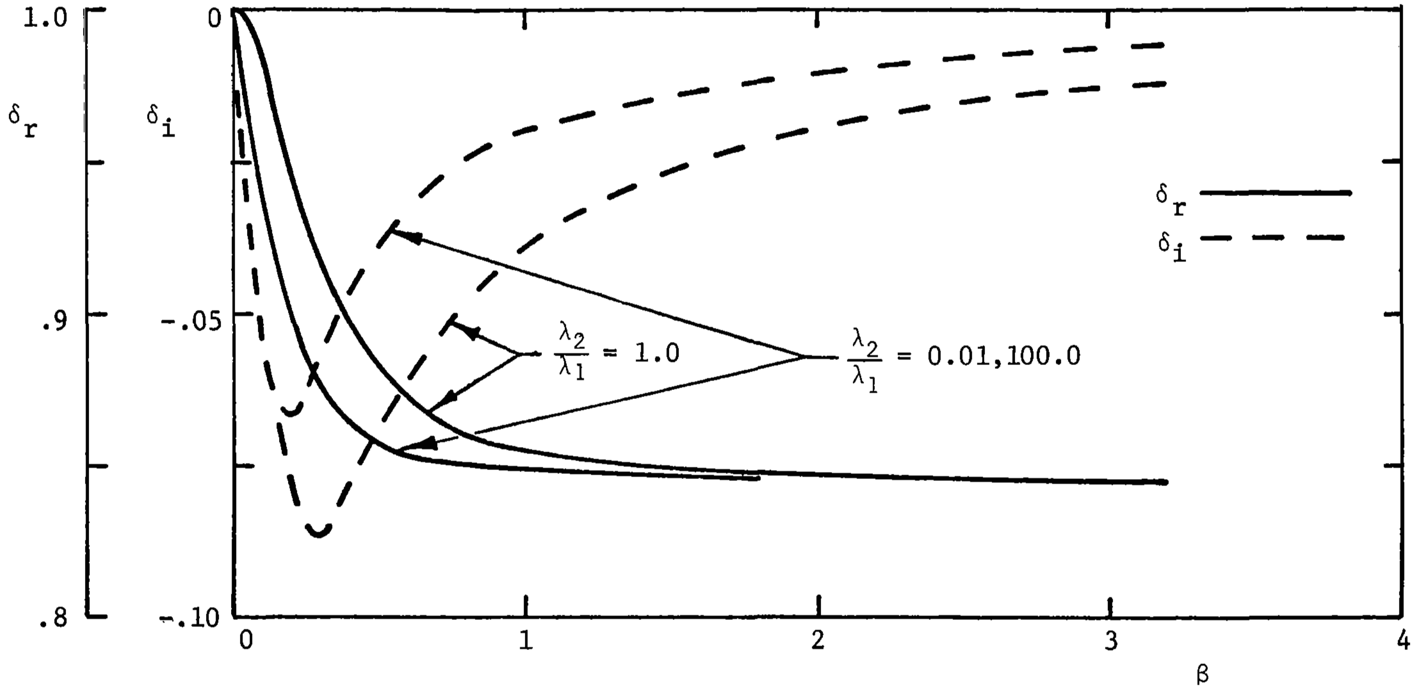

Figure 4. Complex eigenvalue for two 1-D adjacent radiating gases with $\gamma = 1.4$ confined between rigid adiabatic walls.

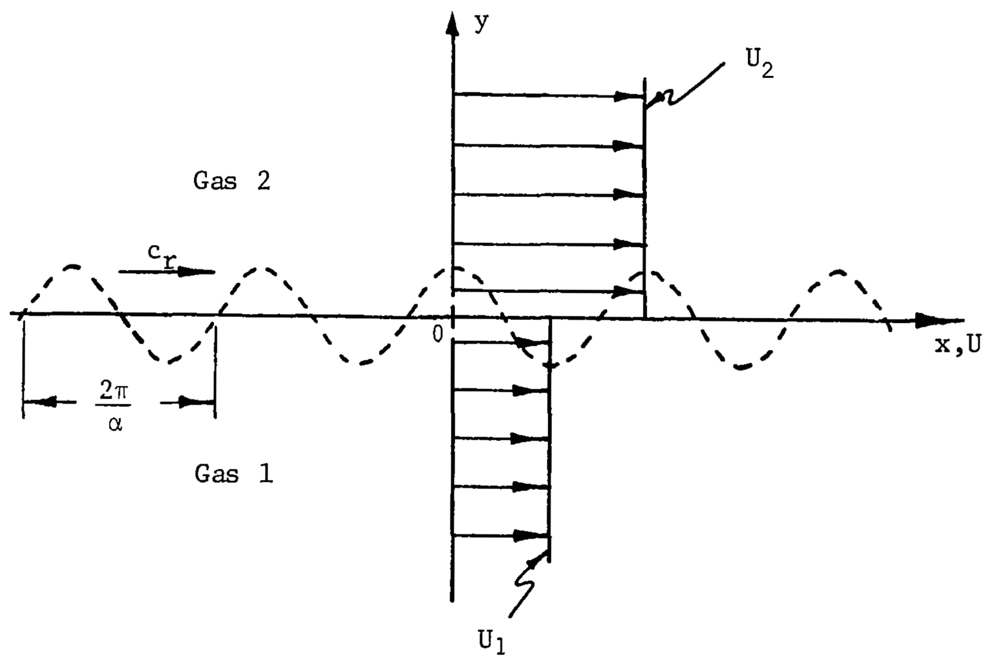

Figure 5. Schematic of disturbed 2-D vortex sheet separating semi-infinite gases.

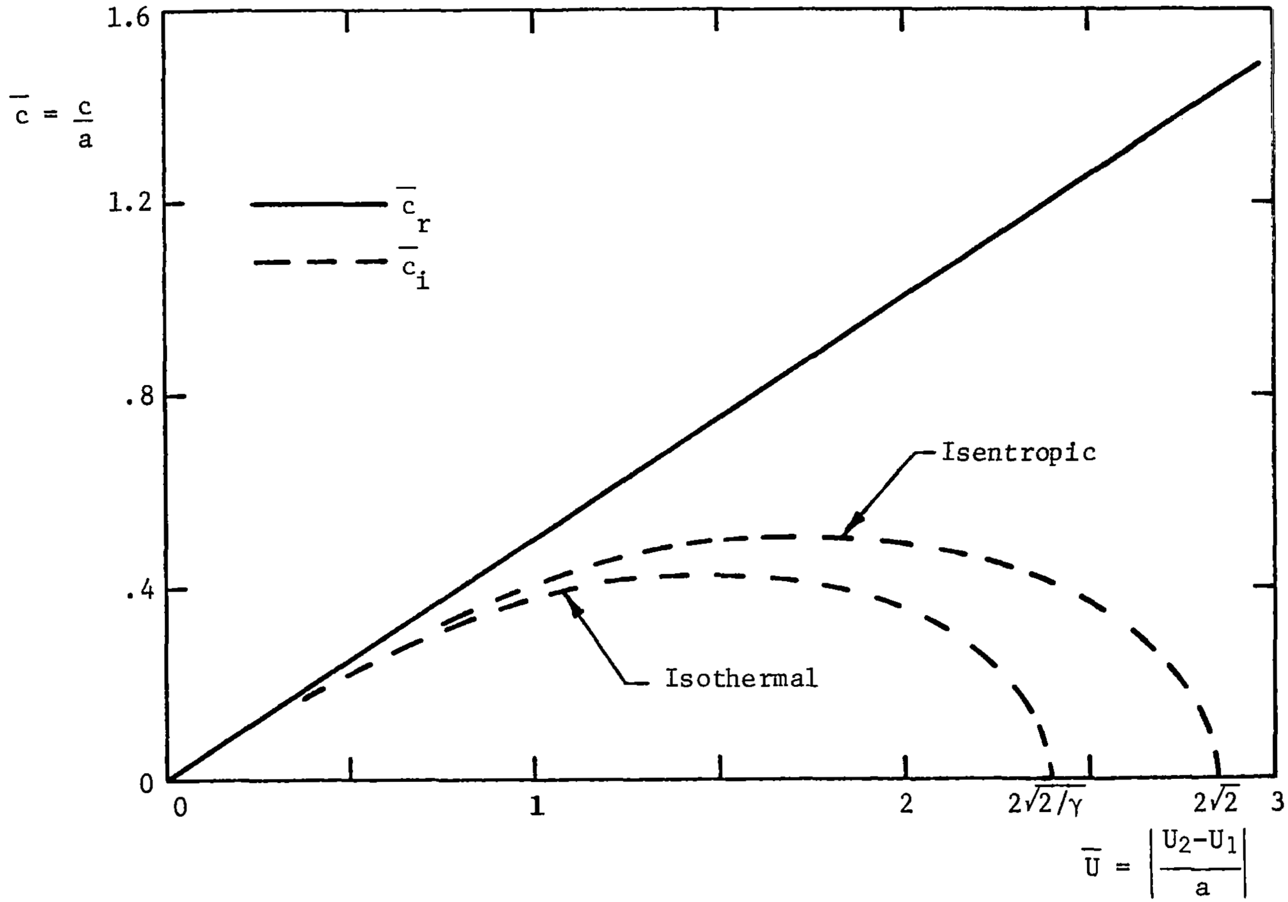

Figure 6. Complex eigenvalue for disturbed 2-D vortex sheet separating semi-infinite isentropic or isothermal gases with $\gamma_1 = \gamma_2 = 1.4$ and $a_1 = a_2$.

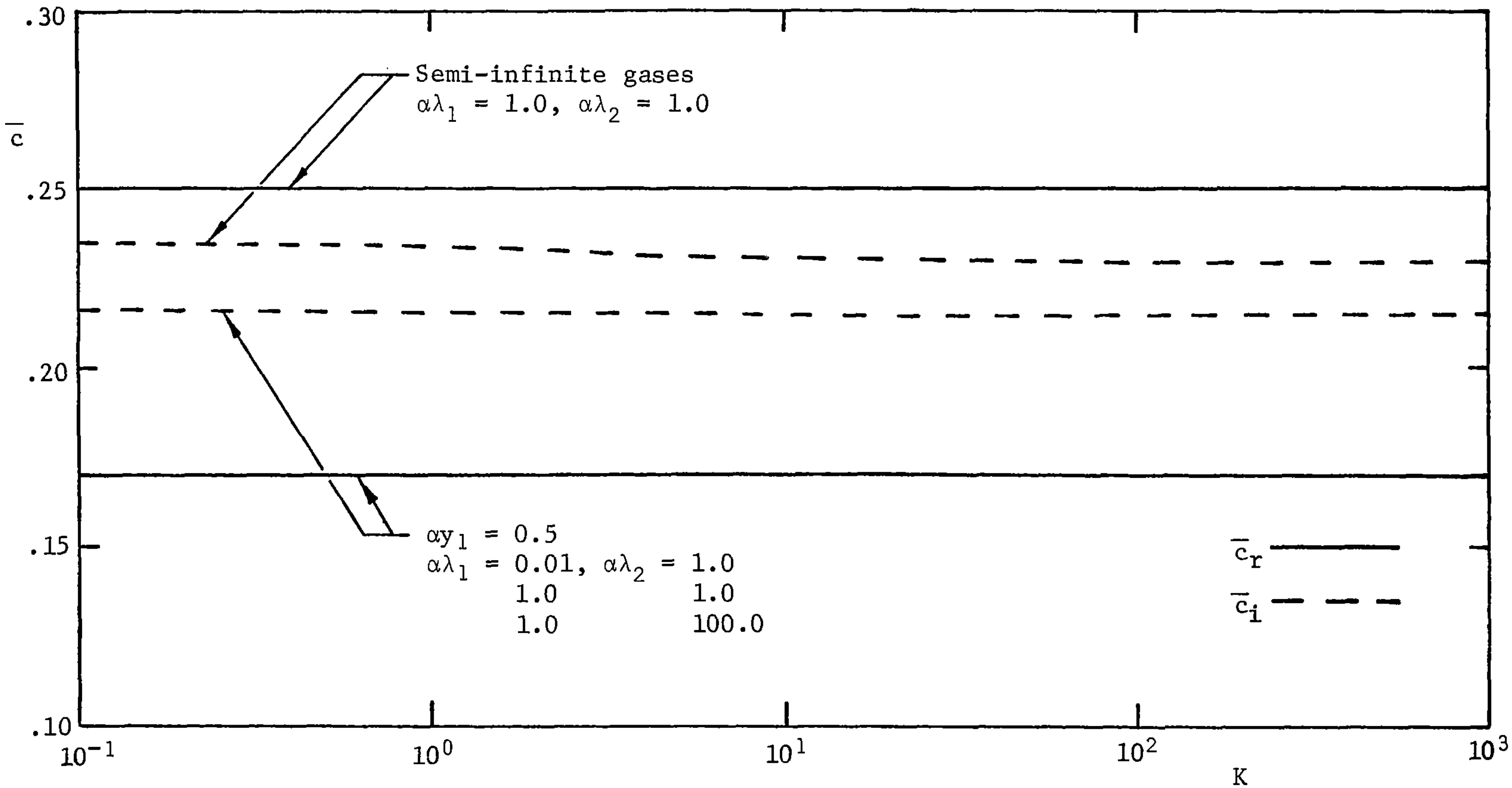

Figure 7. Complex eigenvalue for disturbed 2-D vortex sheet between semi-infinite radiating gases and near symmetry plane for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$ and $\overline{U} = 0.5$.

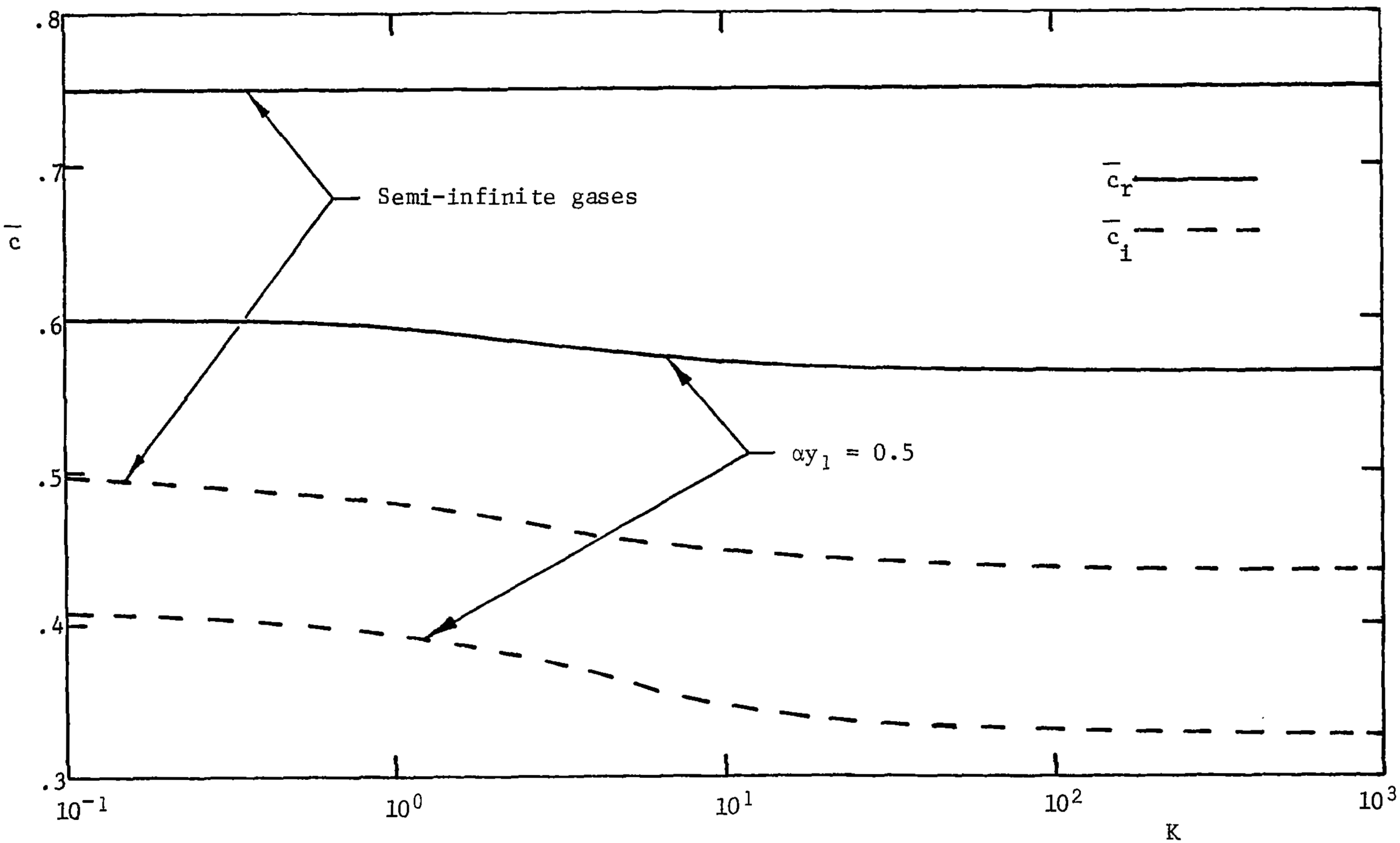

Figure 8. Complex eigenvalue for disturbed 2-D vortex sheet between semi-infinite radiating gases and near symmetry plane for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha\lambda_1 = \alpha\lambda_2 = 1.0$ and $\bar{U} = 1.5$.

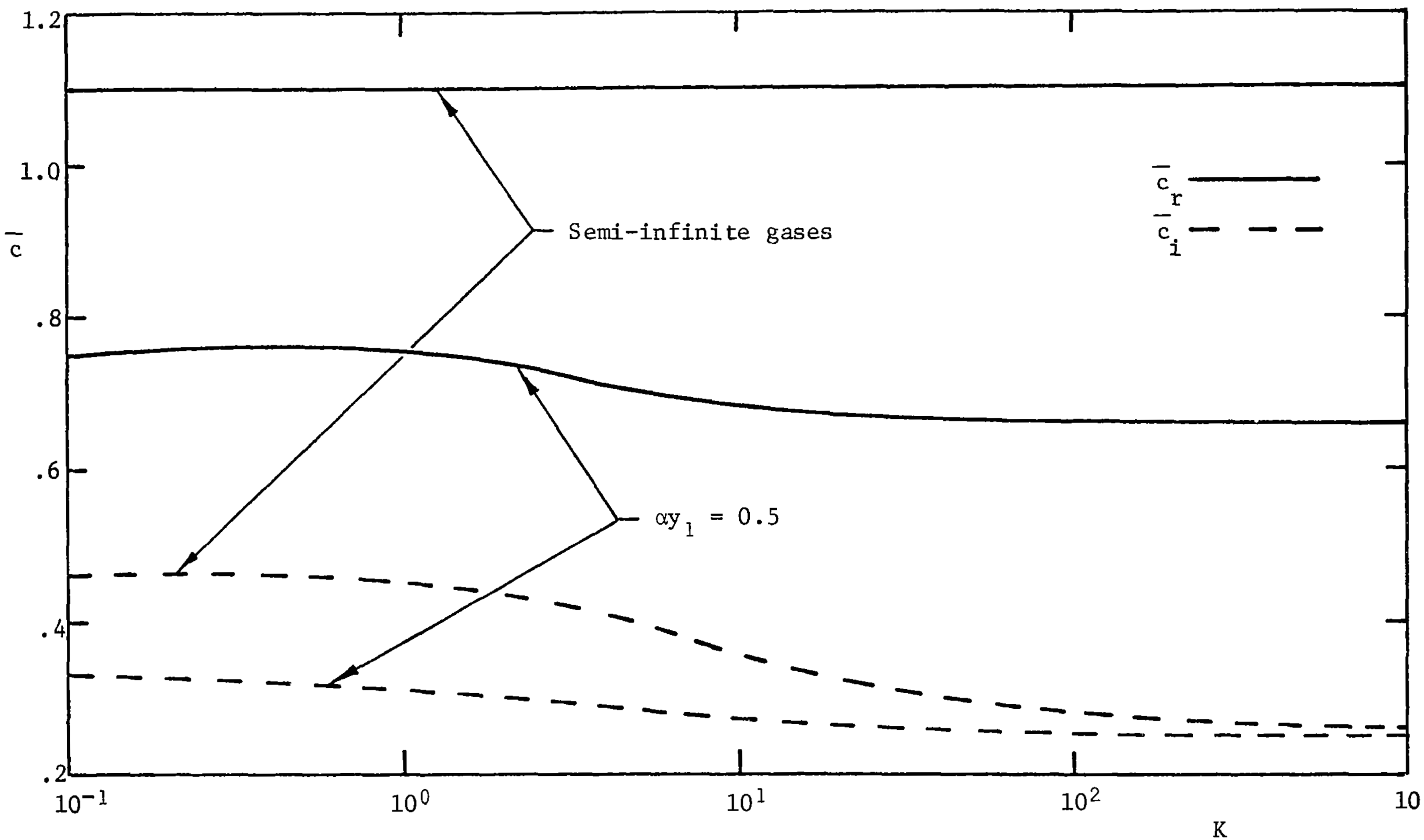

Figure 9. Complex eigenvalue for disturbed 2-D vortex sheet between semi-infinite radiating gases and near symmetry plane for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha\lambda_1 = \alpha\lambda_2 = 1.0$ and $\overline{U} = 2.2$.

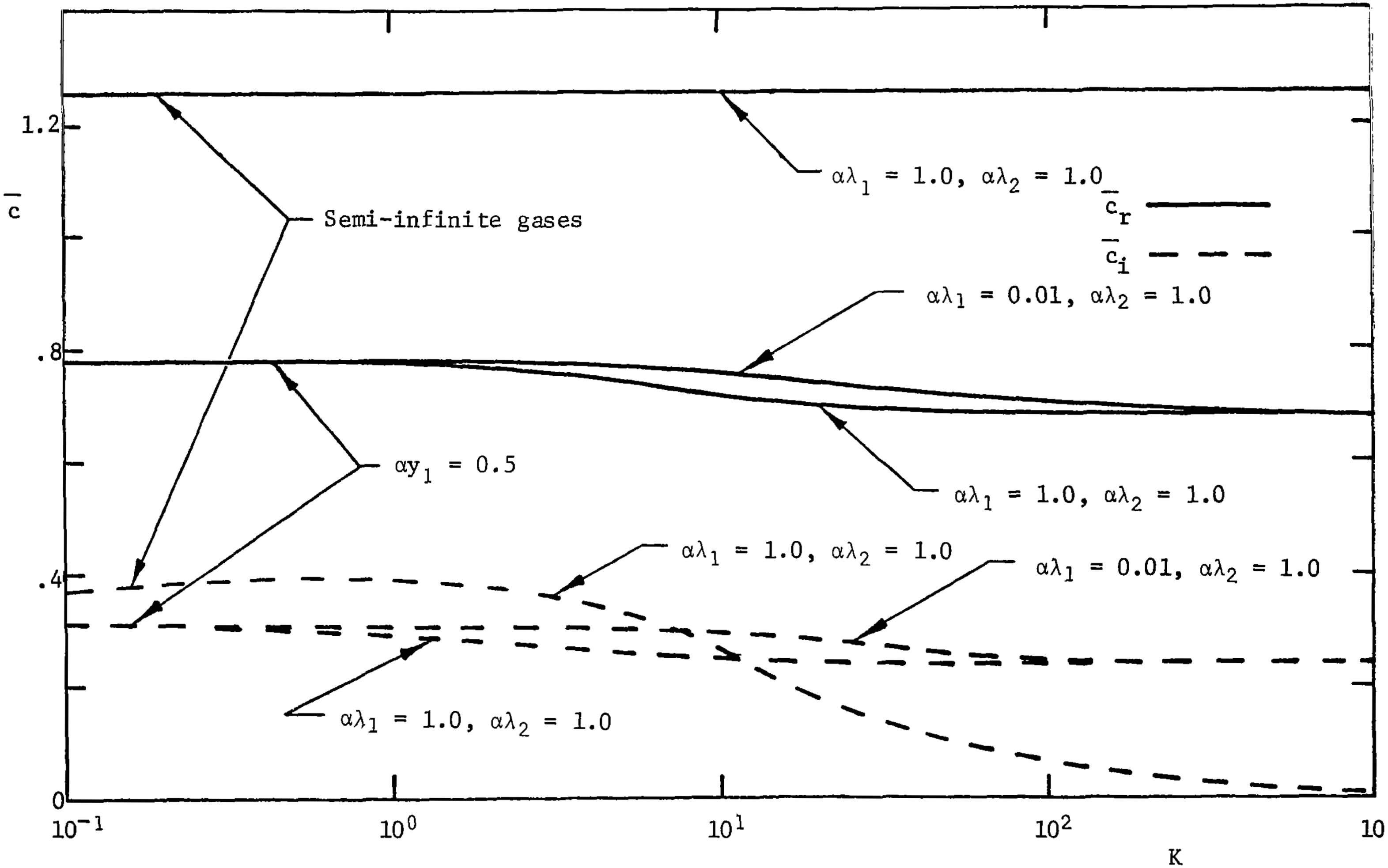

Figure 10. Complex eigenvalue for disturbed 2-D vortex sheet between semi-infinite radiating gases and near symmetry plane for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$ and $\overline{U} = 2.5$.

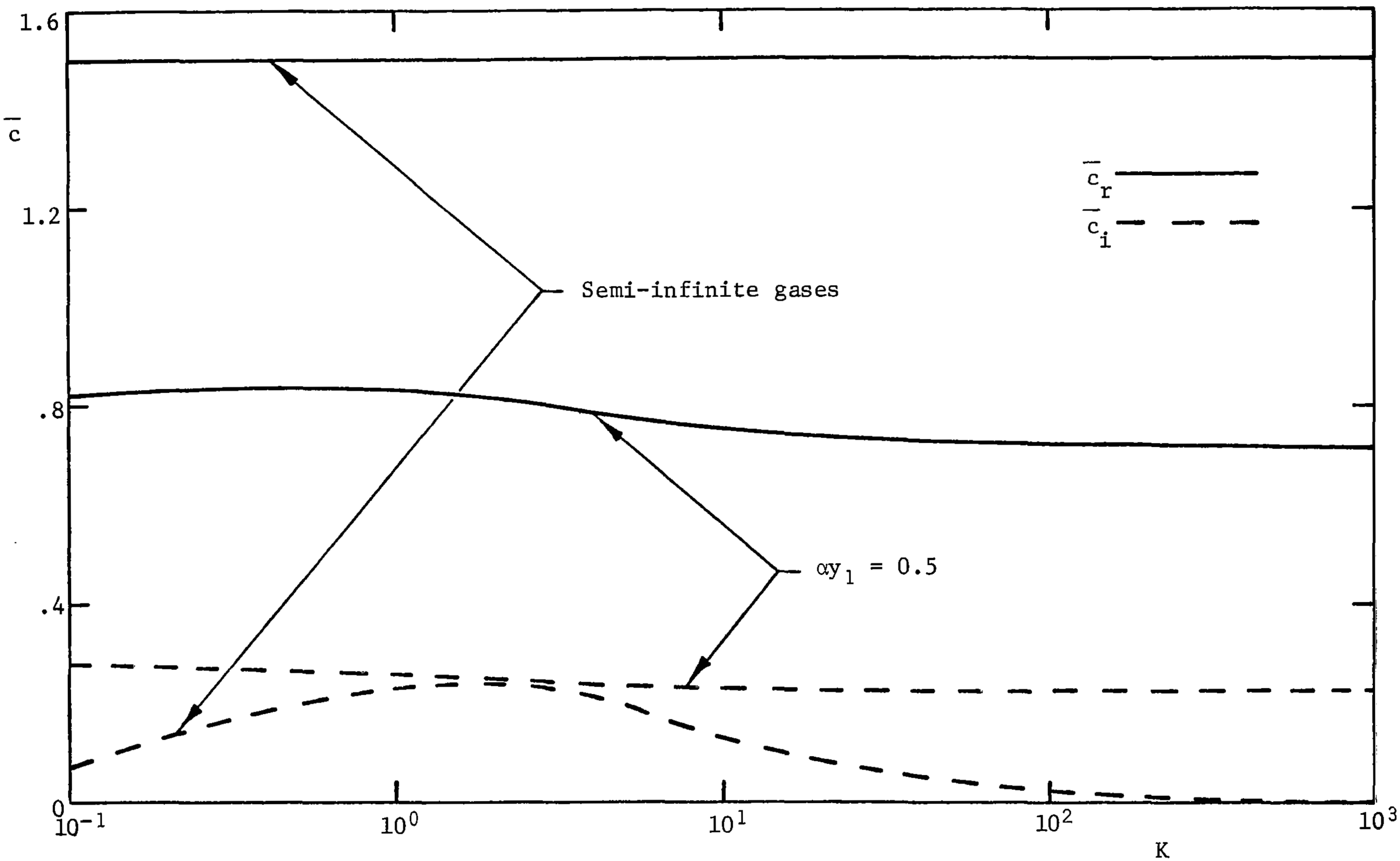

Figure 11. Complex eigenvalue for disturbed 2-D vortex sheet between semi-infinite radiating gases and near symmetry plane for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha\lambda_1 = \alpha\lambda_2 = 1.0$ and $\overline{U} = 3.0$.

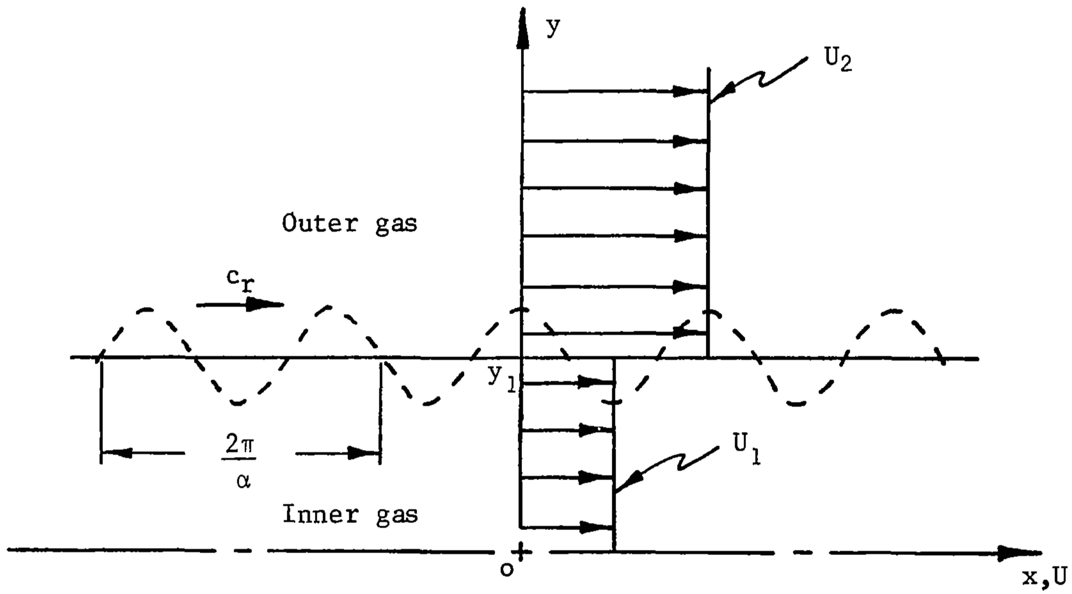

Figure 12. Schematic of disturbed 2-D vortex sheet near symmetry plane.

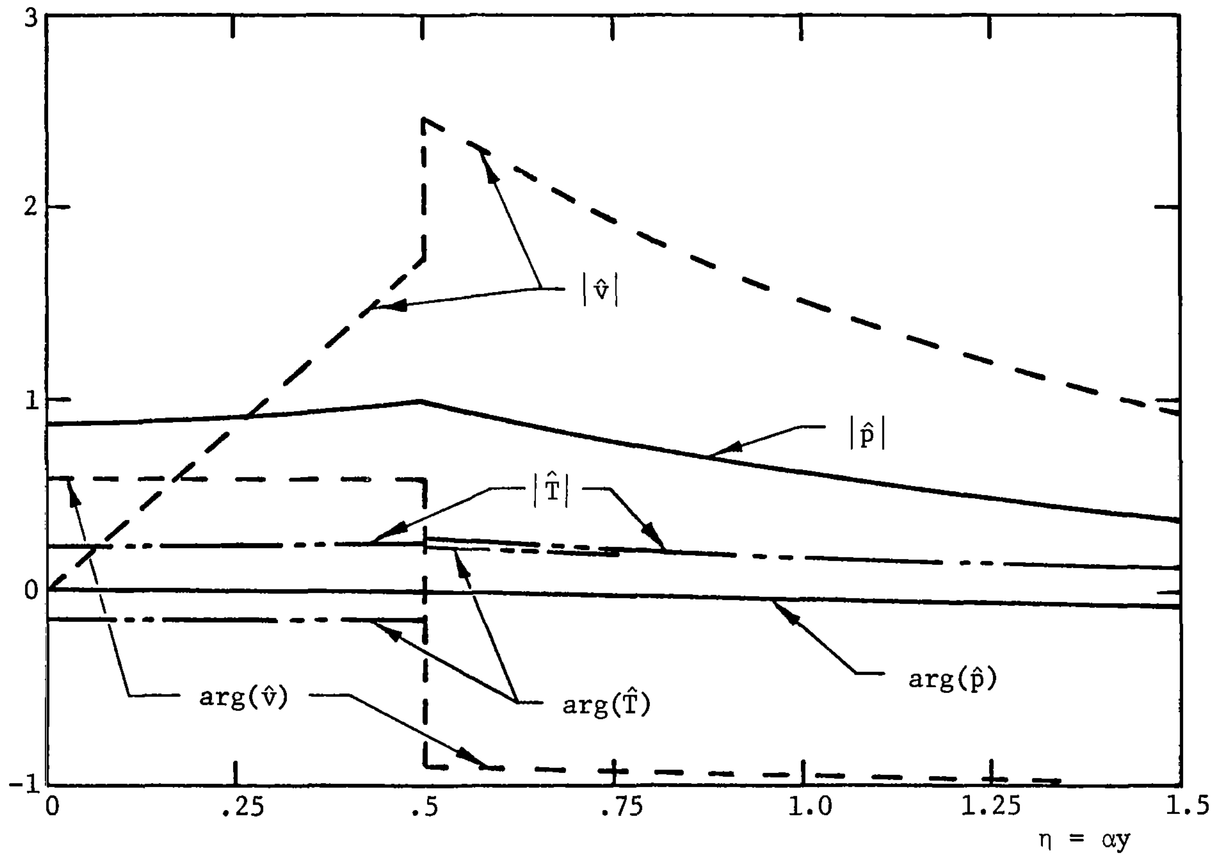

Figure 13. Disturbance profiles for radiating gases separated by disturbed 2-D vortex sheet near symmetry plane for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha y_1 = 0.5$, $\alpha\lambda_1 = \alpha\lambda_2 = 1.0$, $\overline{U} = 0.5$ and $K = 0.5$.

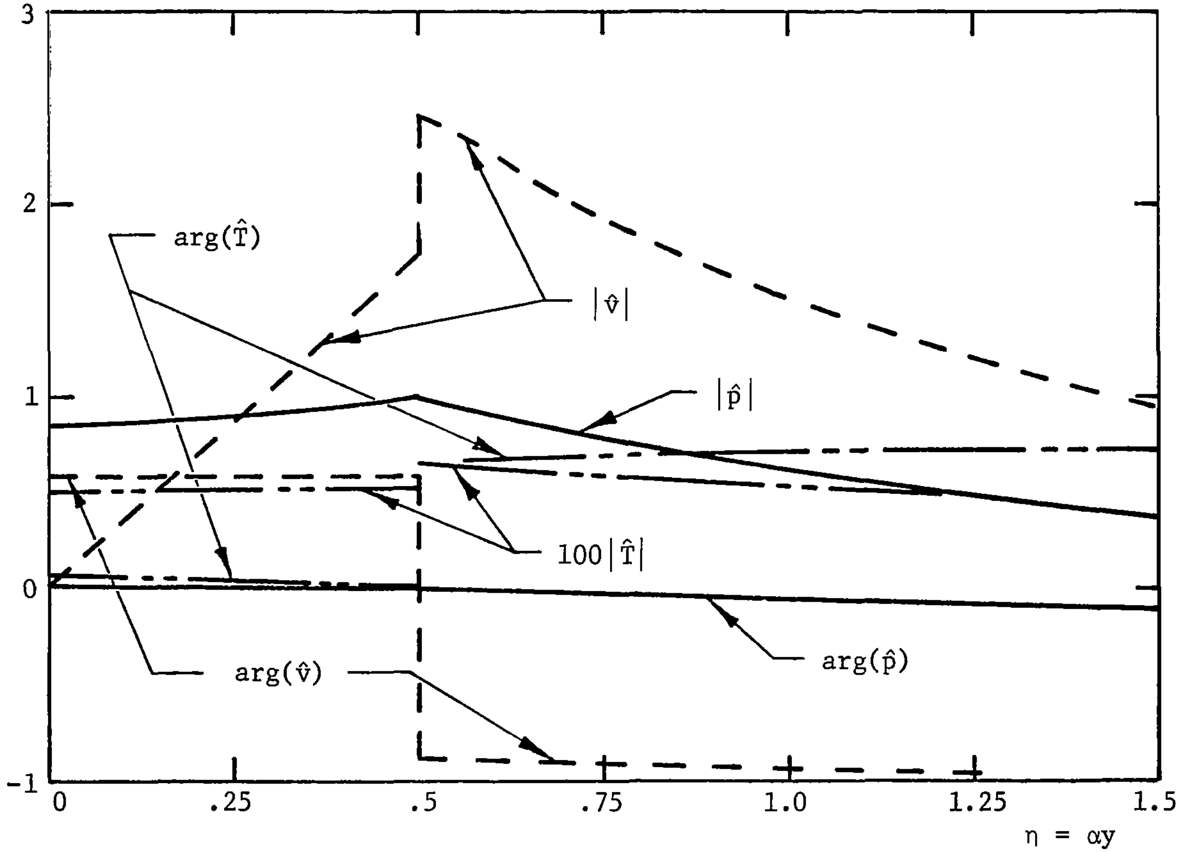

Figure 14. Disturbance profiles for radiating gases separated by disturbed 2-D vortex sheet near symmetry plane for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha y_1 = 0.5$, $\alpha\lambda_1 = \alpha\lambda_2 = 1.0$, $\overline{U} = 0.5$ and $K = 50.0$.

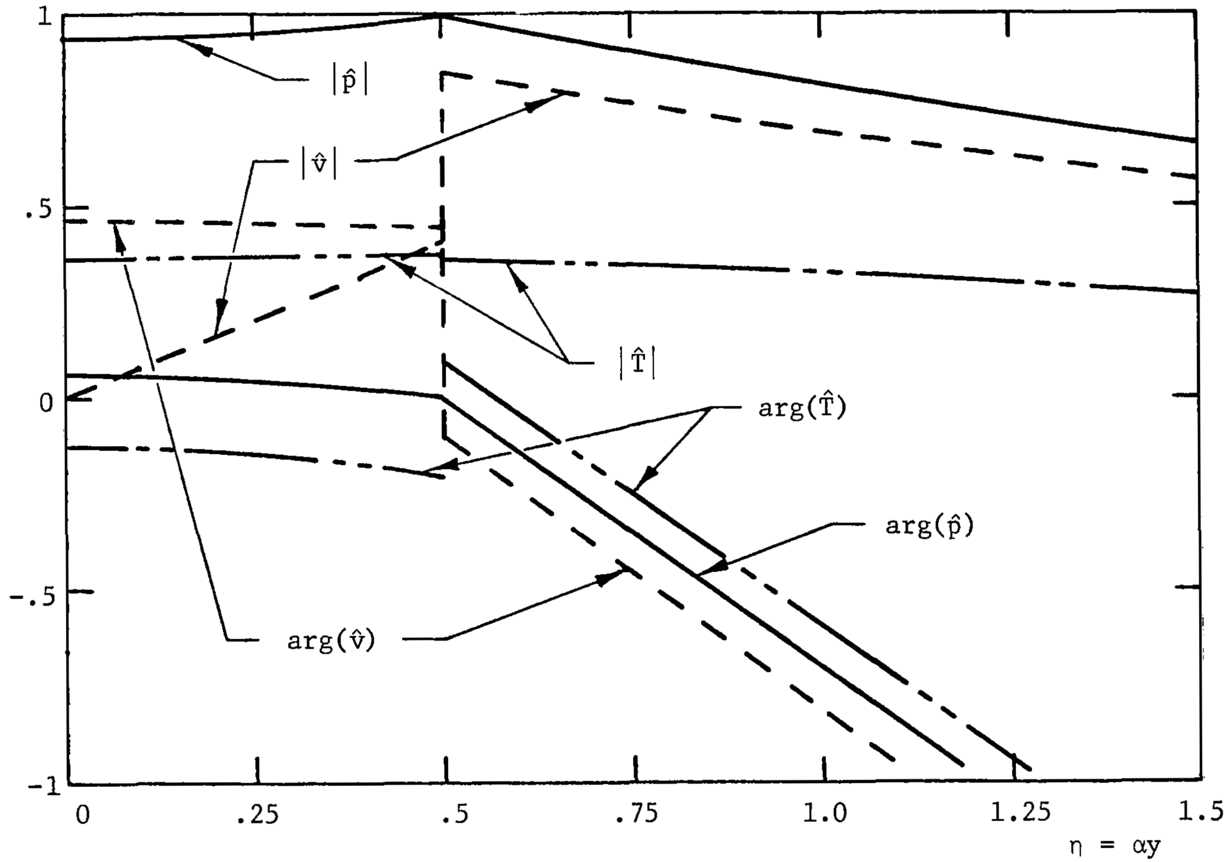

Figure 15. Disturbance profiles for radiating gases separated by disturbed 2-D vortex sheet near symmetry plane for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha y_1 = 0.5$, $\alpha\lambda_1 = \alpha\lambda_2 = 1.0$, $\overline{U} = 2.5$ and $K = 0.5$.

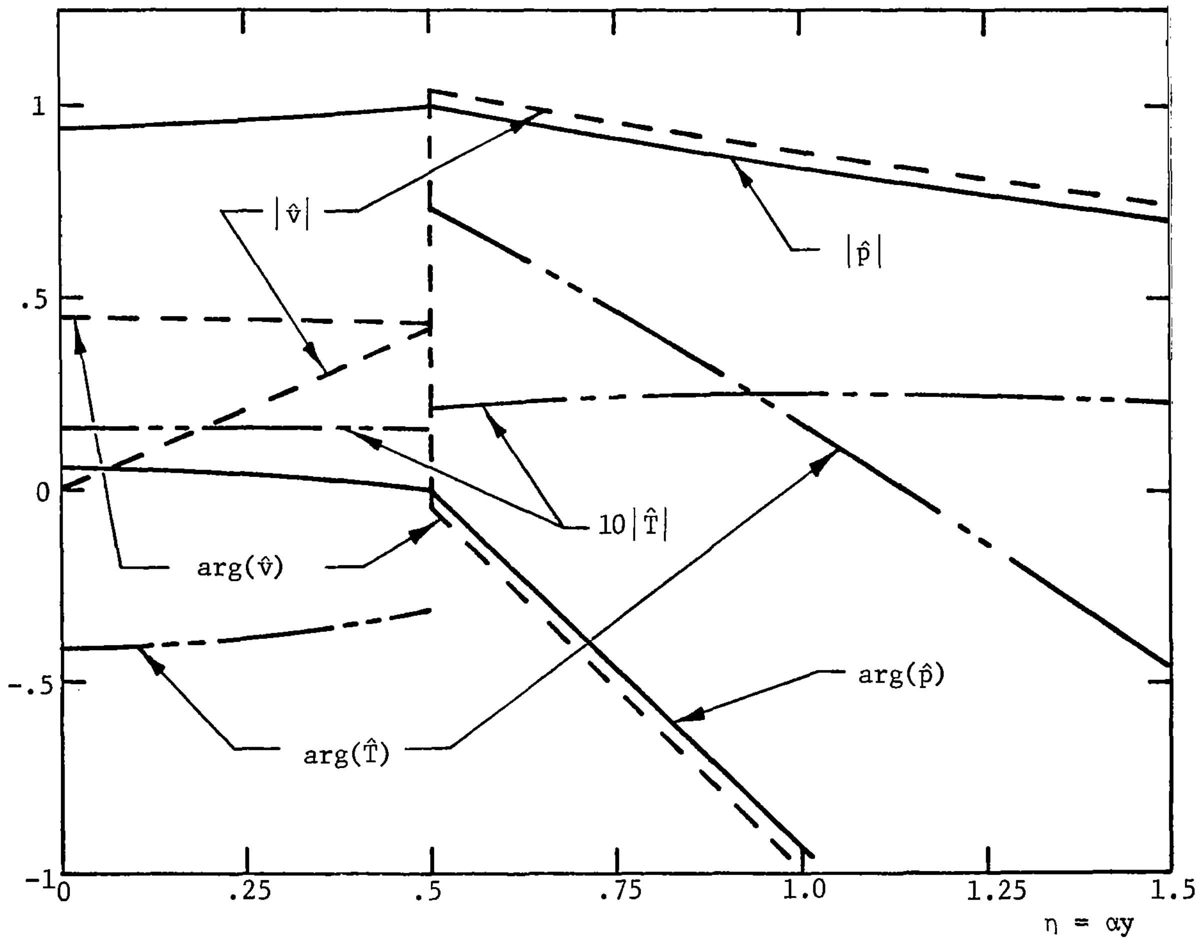

Figure 16. Disturbance profiles for radiating gases separated by disturbed 2-D vortex sheet near symmetry plane for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha y_1 = 0.5$, $\alpha\lambda_1 = \alpha\lambda_2 = 1.0$, $\overline{U} = 2.5$ and $K = 50.0$.

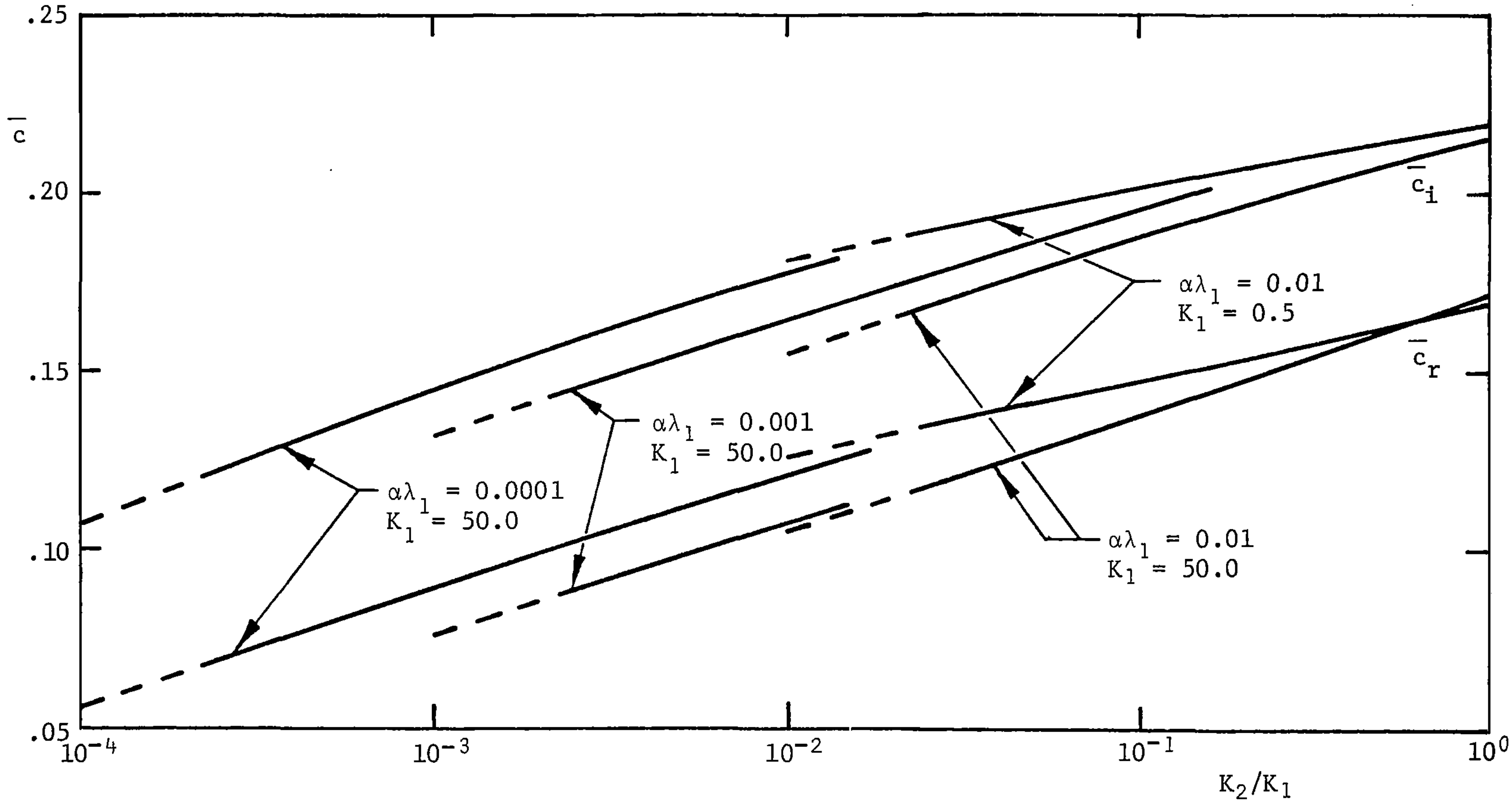

Figure 17. Complex eigenvalue for disturbed 2-D vortex sheet near symmetry plane in radiating gases with base flow heat transfer for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha y_1 = 0.5$, $\alpha\lambda_2 = 1.0$ and $\overline{U} = 0.5$.

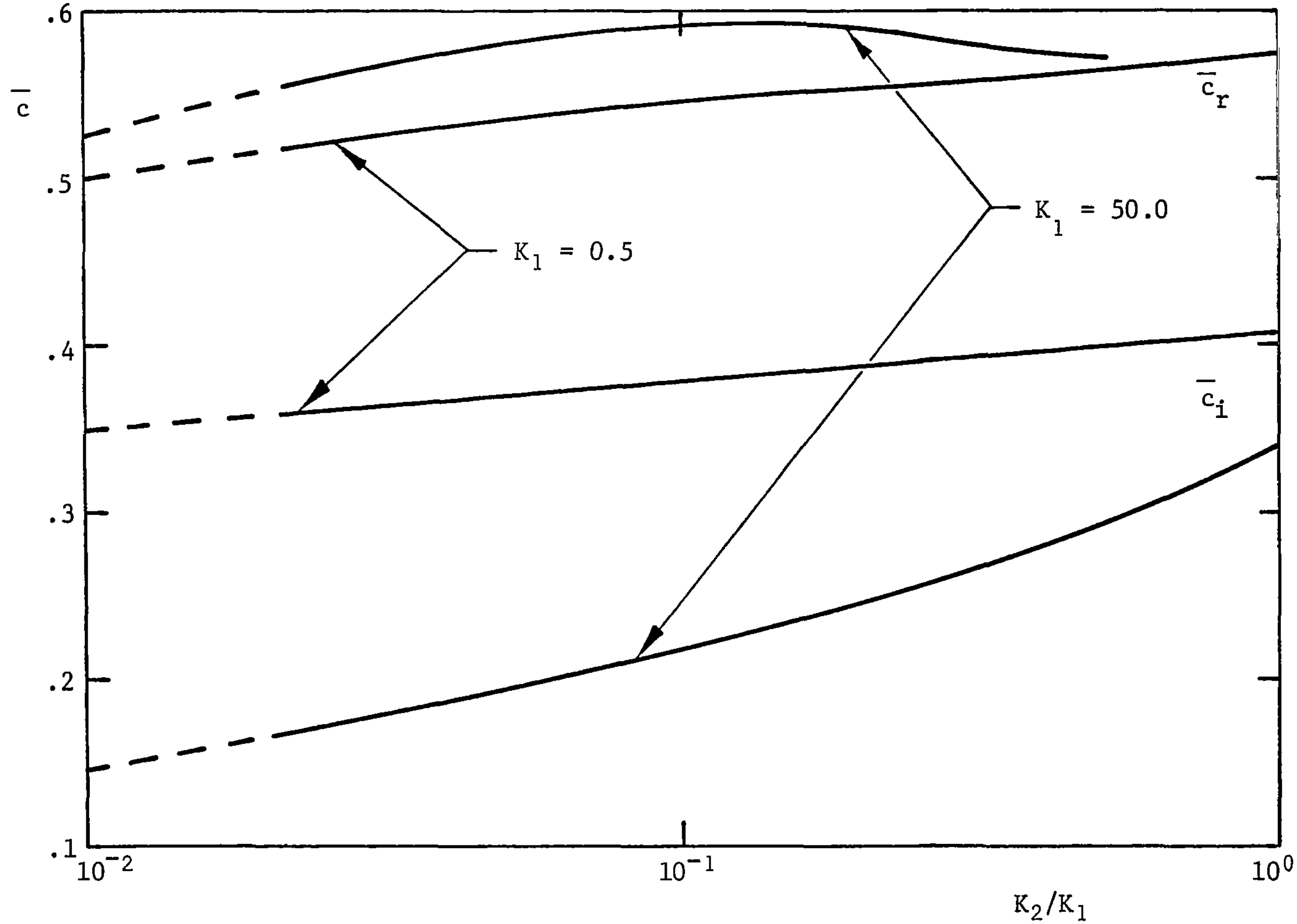

Figure 18. Complex eigenvalue for disturbed 2-D vortex sheet near symmetry plane in radiating gases with base flow heat transfer for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha y_1 = 0.5$, $\alpha\lambda_1 = 0.01$, $\alpha\lambda_2 = 1.0$ and $\overline{U} = 1.5$.

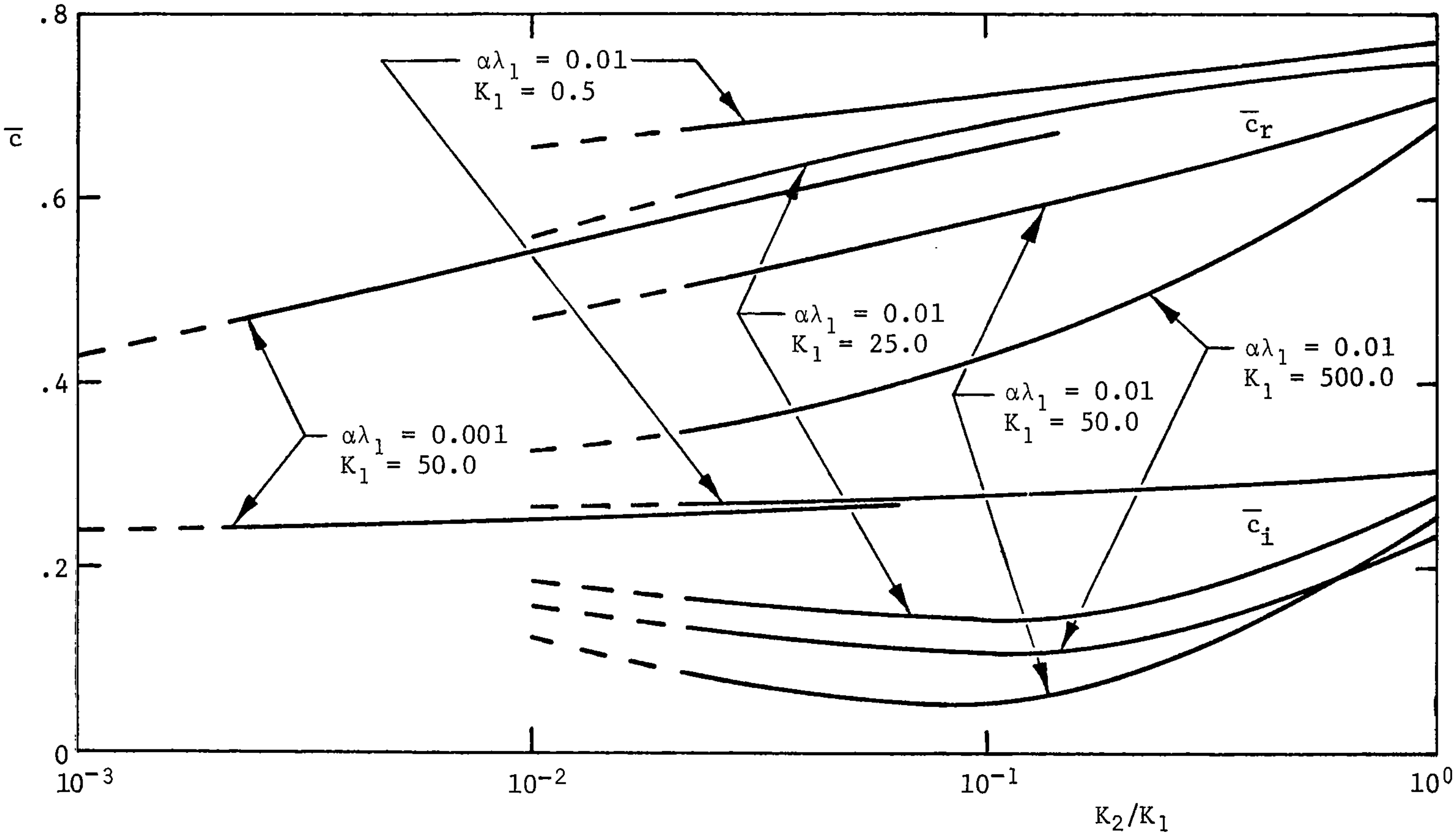

Figure 19. Complex eigenvalue for disturbed 2-D vortex sheet near symmetry plane in radiating gases with base flow heat transfer for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha y_1 = 0.5$, $\alpha\lambda_2 = 1.0$ and $\overline{U} = 2.5$.

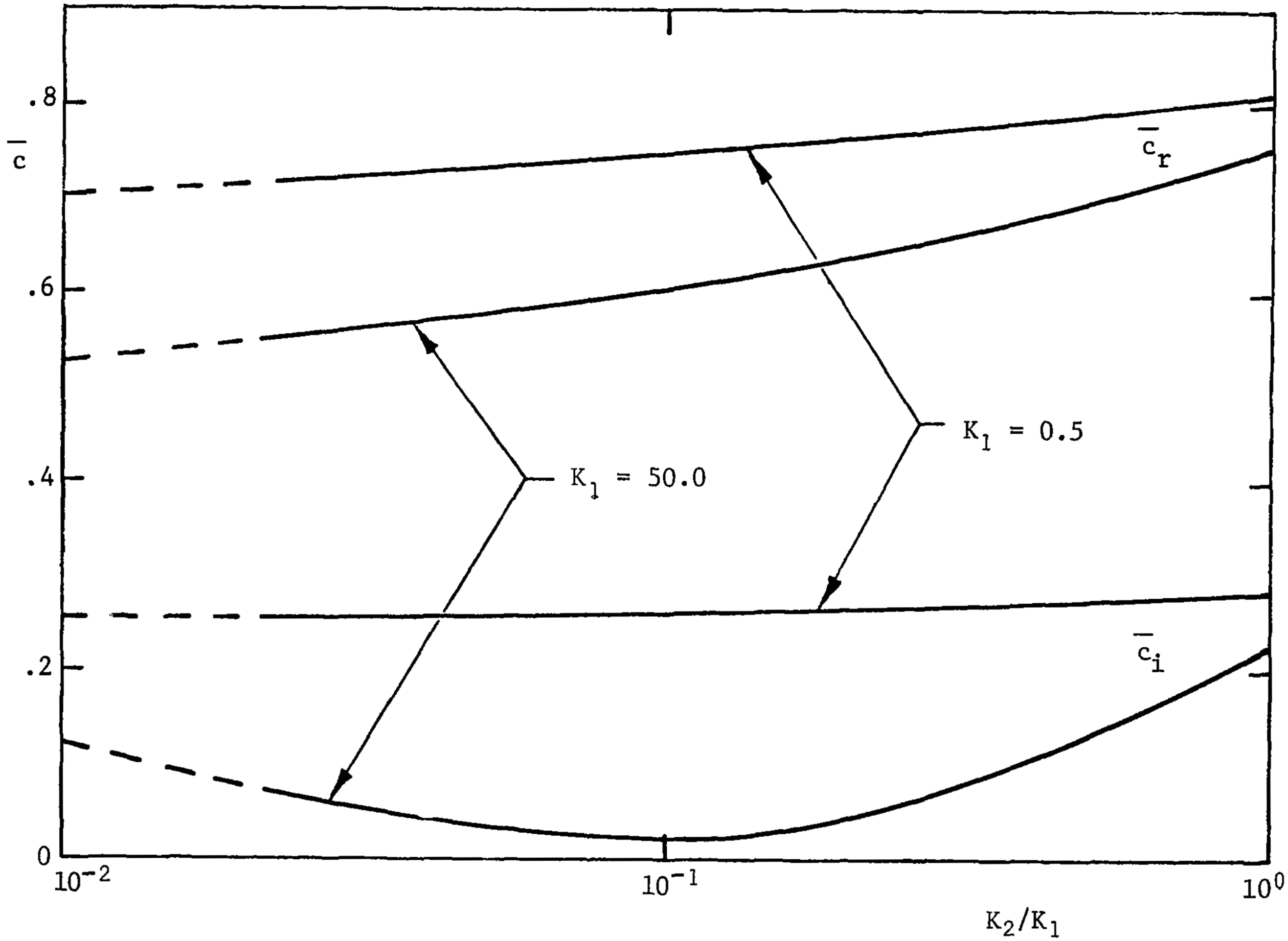

Figure 20. Complex eigenvalue for disturbed 2-D vortex sheet near symmetry plane in radiating gases with base flow heat transfer for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha y_1 = 0.5$, $\alpha\lambda_1 = 0.01$, $\alpha\lambda_2 = 1.0$ and $\overline{U} = 3.0$.

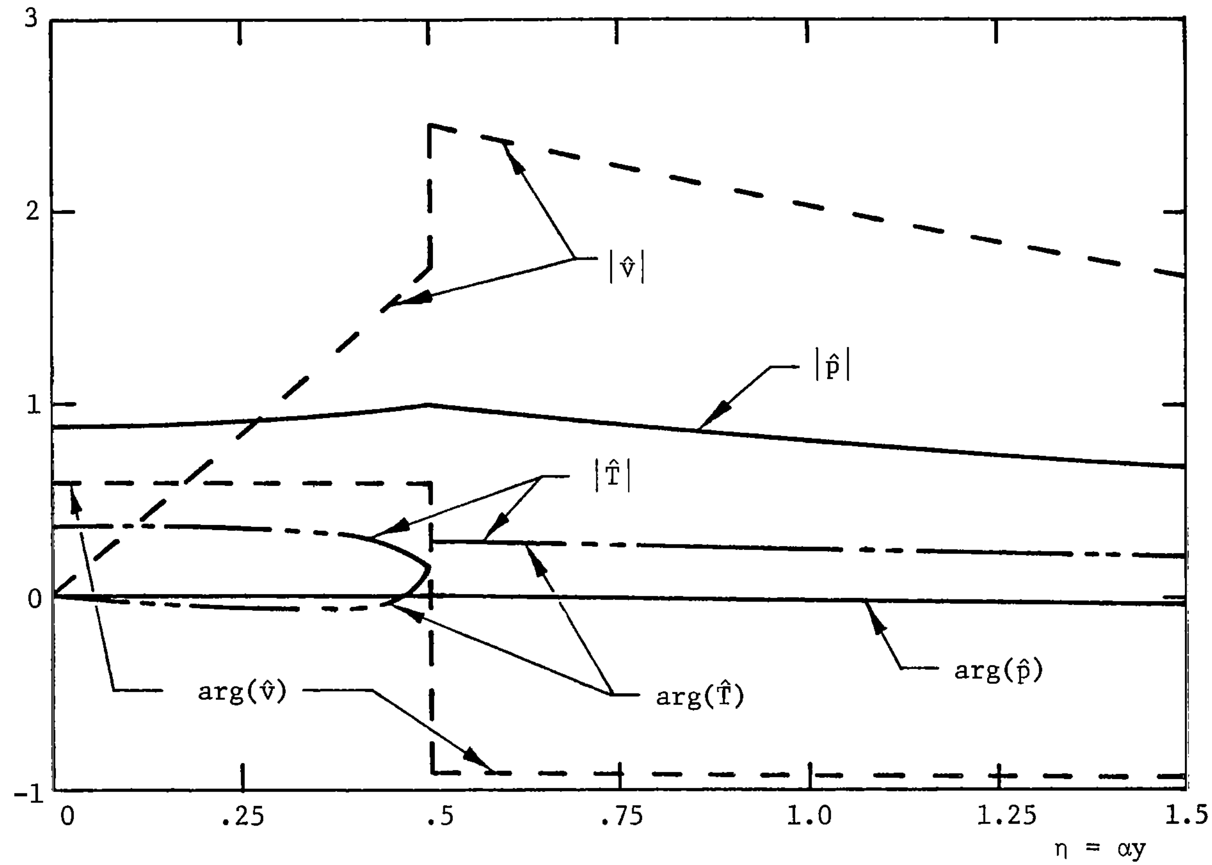

Figure 21. Disturbance profiles for radiating gases separated by disturbed 2-D vortex sheet near symmetry plane with base flow heat transfer for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha y_1 = 0.5$, $\alpha\lambda_1 = 0.01$, $\alpha\lambda_2 = 1.0$, $\overline{U} = 0.5$, $K_1 = 0.5$ and $K_2/K_1 = 1.0$.

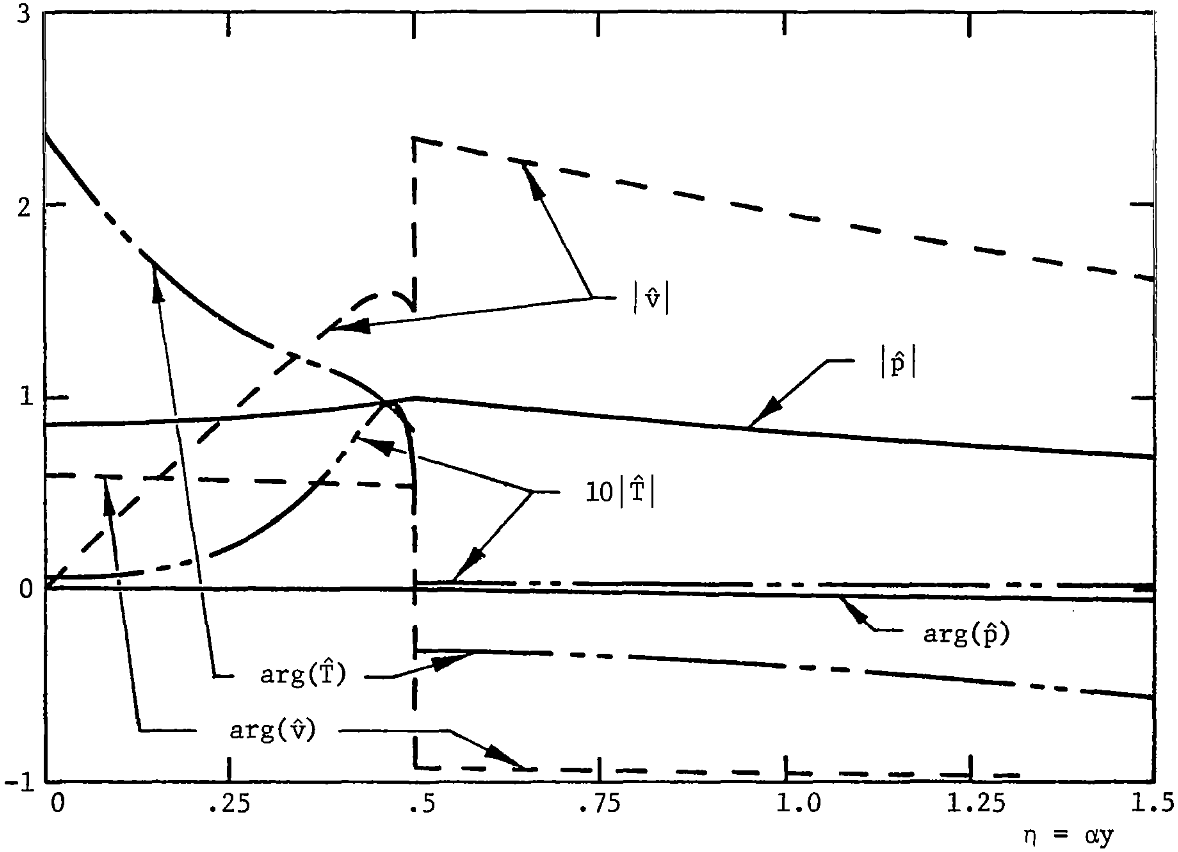

Figure 22. Disturbance profiles for radiating gases separated by disturbed 2-D vortex sheet near symmetry plane with base flow heat transfer for $\gamma_1 = \gamma_2 = 1.4$, $a_1 = a_2$, $\alpha y_1 = 0.5$, $\alpha\lambda_1 = 0.01$, $\alpha\lambda_2 = 1.0$, $\overline{U} = 0.5$, $K_1 = 0.5$ and $K_2/K_1 = 0.1$.

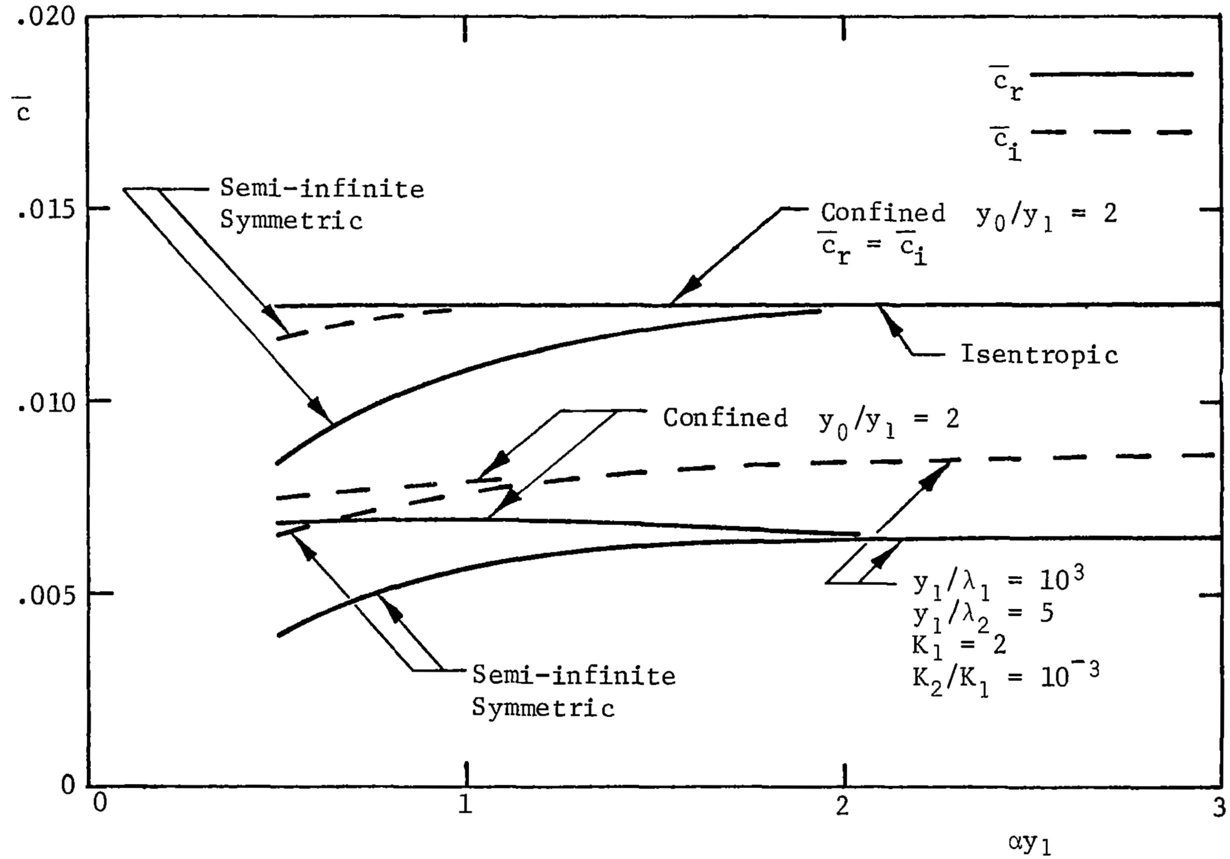

Figure 23. Comparison of complex eigenvalues for disturbed 2-D vortex sheet between confined isentropic gases and gases with radiation parameters approximating gaseous nuclear rocket application.

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