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THE PARAMETER ADAPTIVE CONTROL OF
UNKNOWN PLANTS

by

James W. Gilbert

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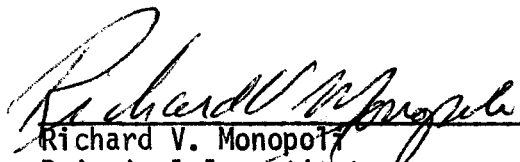
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ACCEPTANCE

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Richard V. Monopoli
Principal Investigator
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University of Massachusetts

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TABLE OF CONTENTS

	<u>Page</u>
ACKNOWLEDGEMENT	iii
LIST OF FIGURES	vf
LIST OF SYMBOLS	viii

Chapter

I	INTRODUCTION	1
	A. Background Information	1
	B. Summary of Previous Research on Gradient and Liapunov Methods	5
	C. Purpose of the Dissertation	7
	D. Layout of the Dissertation	8
II	IMPROVED CONVERGENCE IN THE PARAMETER ADAPTIVE CONTROL OF PLANTS WITH UNKNOWN CONSTANT PARAMETERS	10
	A. The Modified Liapunov Design	10
	B. Comparison of Convergence Properties	16
	C. Simulation Results	21
	Example 1	21
	Example 2	25
III	EXTERNAL PARAMETER ADJUSTMENT SCHEMES FOR PLANTS WITH UNKNOWN CONSTANT PARAMETERS	33
	A. Motivation	33
	B. Design Technique for Plants with Feedback and Prefilter Adjustable Gains	35
	1. Development	35
	2. Second Order Example	41

<u>Chapter</u>	<u>Page</u>
IV	PLANTS CONTAINING UNKNOWN TIME-VARYING PARAMETERS . . . 45
A.	Development for Plants without Input Differential Operators 45
B.	Plants with Differential Operators Acting on the Input 50
V	APPLICATION OF THE MODEL REFERENCE PARAMETER ADAPTIVE TECHNIQUE TO AN OPTICAL TRACKING MOUNT 66
A.	The Existing Servo Control System and Motivation for an Adaptive System 66
B.	Developing the Model Reference Adaptive Control Technique for the Tracking Mount 72
C.	Driving the Mount at Fixed Rates Utilizing the Adaptive Scheme 76
D.	Adaptive Compensation of Static Friction 81
VI	SUMMARY AND SUGGESTIONS 86
A.	Summary 86
B.	Suggestions for Future Research 87
APPENDIX A	 89
APPENDIX B	 91
BIBLIOGRAPHY	 93

LIST OF FIGURES

<u>Figure</u>		<u>Page</u>
1.1	Summary of Adaptive Control Techniques	2
1.2	A Parameter Adaptive Control System	4
1.3	An Adaptive Insensitive Control System	4
2.1	Adaptive Structure for Multi-Input Multi-Output Systems	11
2.2	Root Locus with Respect to K_1 for Linearized Error System	19
2.3	Root Locus with Respect to K_1 for Linearized Error System (Modified Rule)	19
2.4	Second Order Model Reference System	22
2.5	Time Response of $e(t)$ for Second Order System	22
2.6	Time Response of $\dot{e}(t)$ for Second Order System	24
2.7	Time Response of $K_c(t)$ for Second Order System . . .	24
2.8	First Order System with Adaptive Time Constant . . .	26
2.9	Error Response of First Order System	27
2.10	Time Response of $a_c(t)$ for First Order System	28
2.11	Figure 4 of Reference 14	30
2.12	Implementation of (2-31)	31
3.1	Design of Second Order System Using Feedback and Feedforward Adjustable Gains	42
3.2	Time Response of $e(t)$ for Second Order System Using Feedback and Feedforward Adjustable Gains	43
4.1	Plant and Model Responses for Second Order System with Time-Varying Parameters	64
4.2	Time Response of $e(t)$ for Second Order System with Time-Varying Parameters	65

<u>Figure</u>		<u>Page</u>
5.1	Block Diagram of Present Control System (One Axis) . .	68
5.2	D/A, Compensation, and Tachometer Outputs in Manual Velocity Mode	69
5.3	Analog Tach Loop	71
5.4	X-Axis Tachometer Output without Adaptive Scheme ($r = 13 \text{ mv}$)	73
5.5	Analog Tach Loop Including Bearing Friction and Control Input	75
5.6	Bearing Friction Characteristic	77
5.7	The Parameter Adaptive System for the Mount	78
5.8	Tachometer Output and Associated Control Input Using Adaptive Scheme	80
5.9	Tachometer Output and its Integral Utilizing Existing Scheme and Adaptive Scheme	80
5.10	Adaptive Compensation for Static Friction	83
5.11	$-k_3$, $\dot{x}_p$, and $\dot{e}$ Utilizing Adaptive Mechanism with Sinusoidal Input	84

LIST OF SYMBOLS

The list below contains symbols in general use throughout the dissertation. Additional specialized symbols of limited usage, such as those appearing in examples, are defined as they are encountered.

A_d	matrix of model dynamics
$A_p(t)$	matrix of plant dynamics
a_{ij}^d	<u>ijth</u> element of A_d
a_{ij}^p	<u>ijth</u> element of $A_p(t)$
a_{ip}	unknown constant plant parameters
B_d	model input matrix
$B_p(t)$	plant input matrix
b_{ij}^d	<u>ijth</u> element of B_d
b_{ij}^p	<u>ijth</u> element of $B_p(t)$
b_{ip}	unknown constant plant parameters
b_{ip}^*	unknown constant plant parameters
c_i	coefficients of $P(s)$
c_{ij}^a	unknown constant part of a_{ij}^p
c_{ij}^b	unknown constant part of b_{ij}^p
d_{ij}^a	unknown constant in a_{ij}^p whose sign is known
d_{ij}^b	unknown constant in b_{ij}^p whose sign is known
$\underline{d}$	vector of unknown constants
d_i	<u>ith</u> unknown constant
$d_i(t)$	<u>ith</u> unknown variation
D_i	bound on $ d_i(t) $

$\underline{e}$	error state vector
e_i	<u>i</u> th element of $\underline{e}$
$\underline{\Delta e}$	linearized error vector
Δe_i	<u>i</u> th element of $\underline{\Delta e}$
F_d	matrix of model dynamics
F_p	matrix of plant dynamics
f_{ij}^p	<u>ij</u> th element of F_p
$\underline{g}_d$	model input vector
$\underline{g}_p$	plant reference input vector
$\underline{g}_p^*$	plant control input vector
g_{pi}	<u>i</u> th element of $\underline{g}_p$
g_{pi}^*	<u>i</u> th element of $\underline{g}_p^*$
$\underline{h}_d$	model output vector
$\underline{h}_p$	plant output vector
h_{pi}	<u>i</u> th element of $\underline{h}_p$
I	$m \times m$ identity matrix
K_1	linear combination of the α_{ij}
K_2	linear combination of the β_{ij}
$\underline{k}$	vector of adjustable parameters
k_i	<u>i</u> th adjustable parameter
k_r	adjustable gain
k_{ij}^a	adjustable parameter in the <u>ij</u> th element of $A_p(t)$
k_{ij}^b	adjustable parameter in the <u>ij</u> th element of $B_p(t)$
ℓ	number of model zeros
m	number of plant zeros

$m_d(s)$	model numerator dynamics
$m_p(s)$	plant numerator dynamics
n	number of plant poles
$n_d(s)$	characteristic polynomial of the model
$n_p(s)$	characteristic polynomial of the plant
P	positive definite symmetric matrix
$P(s)$	polynomial with left half plane roots
p_{ij}	<u>ijth</u> element of P
Q	positive definite symmetric matrix
q	constant vector
q_{ij}	<u>ijth</u> element of Q
R	defined by K_1/K_2
$r(t)$	reference input
s	Laplace variable
t_0	time instant the adaptive mechanism is initiated
$u(t)$	control input
u_0	constant
V	Liapunov function
$\underline{x}_d$	model state vector
$\underline{x}_{do}$	constant vector
$\underline{x}_p$	plant state vector
x_{id}	<u>ith</u> element of $\underline{x}_d$
x_{ip}	<u>ith</u> element of $\underline{x}_p$
$\underline{z}_d$	model state vector
$\underline{z}_p$	plant state vector

z_{ip}	i th element of $\underline{z}_p$
α_{ij}	diagonal elements of Λ_α
β_{ij}	diagonal elements of Λ_β
β_i	indexed set of arbitrary positive constants
β_r	arbitrary positive constant
γ_{ij}	diagonal elements of Λ_γ
δ_{ij}	diagonal elements of Λ_δ
ε	bound on $\ \underline{e}\ $
λ_q	smallest eigenvalue of Q
$\Lambda_\alpha, \Lambda_\beta, \Lambda_\gamma, \Lambda_\delta, \Lambda_1, \Lambda_2 \}$	positive definite diagonal matrices
$\ \underline{e}\ $	defined by $(\underline{e}^T \underline{e})^{1/2}$

C H A P T E R I

INTRODUCTION

A. Background Information

The imposition of stringent performance requirements on systems which undergo wide parameter variations frequently necessitates the use of an adaptive control system. For example, adaptive techniques have been applied to devices such as autopilots [1] - [2] where satisfactory response characteristics are required in the presence of unpredictable changes in flight conditions along the aircraft's trajectory, causing parameter variations in the airframe's equations of motion. More recently, an extensive investigation into the possible application of adaptive techniques to the control of tactical missiles [3] was performed. Air targets require the missile autopilot to achieve rapid response to guidance commands in order to overcome target maneuvers. Other possible applications include the control of optical tracking mounts. The results of preliminary testing on one such mount are given in Chapter V. In this case, small tracking errors are required in the presence of changing mount inertia and bearing friction.

A brief summary of the various paths which research in adaptive control systems has taken is shown in Figure 1.1. The majority of the design techniques which have been developed can be classified as one of two types: parameter adaptive or adaptive insensitive. A parameter adaptive control system is an adaptive system in which the adaptive mechanism is fixed to a set of adjustable gains, which are either

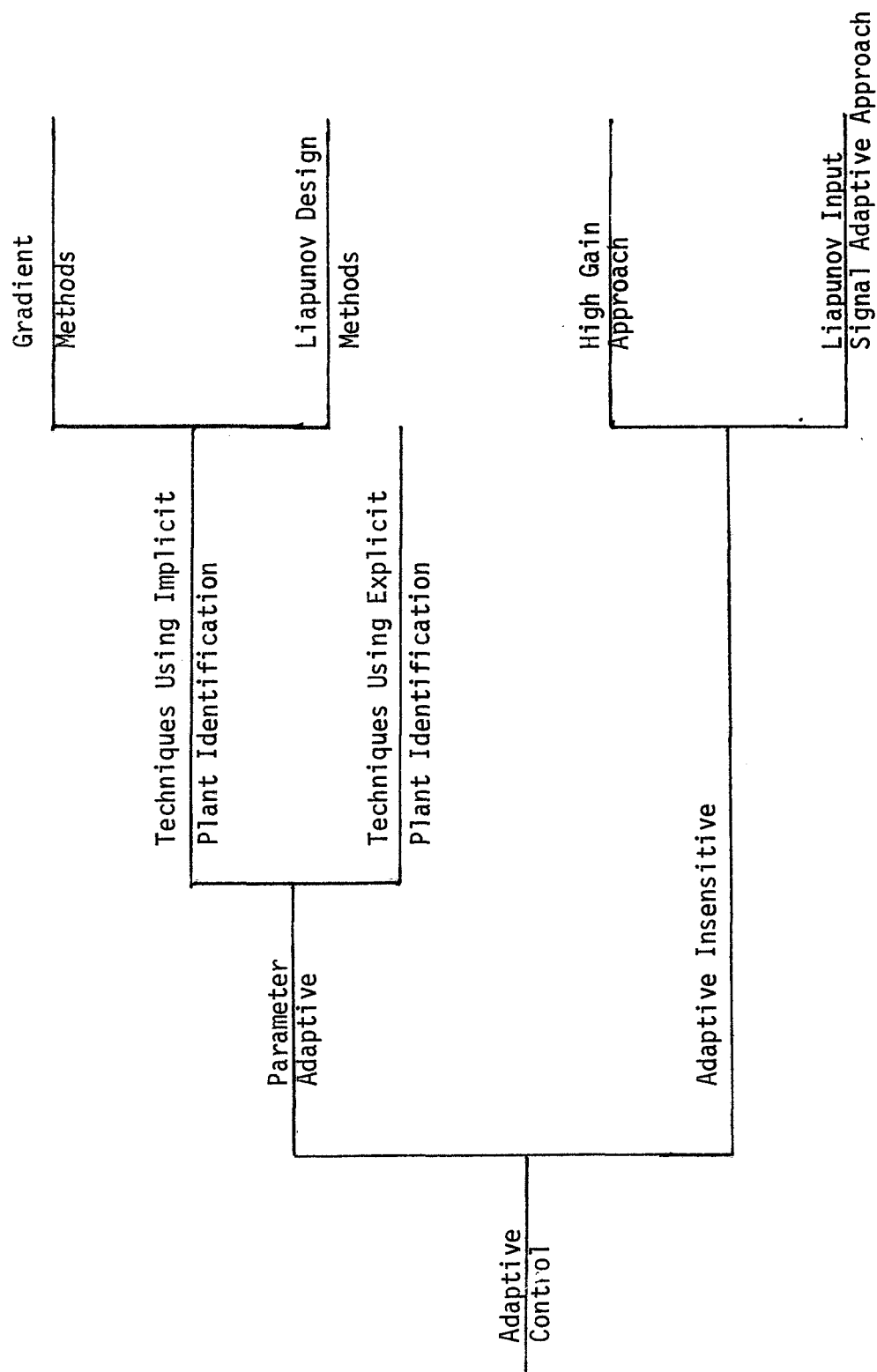


Figure 1.1. Summary of Adaptive Control Techniques

contained in the open loop plant dynamics or introduced externally. These gains are then adjusted according to a specified adaptation algorithm. An example of this kind of adaptive system is shown in Figure 1.2. In the figure, the plant outputs or state variables are elements of the vector $\underline{x}_p(t)$ while the vector $\underline{x}_d(t)$, the state vector of a model reference system, represents the desired plant trajectories. A measure of performance is given by the error vector $\underline{e}$ which is defined as $\underline{e} = \underline{x}_d - \underline{x}_p$. It is evident that plant compensation is to be accomplished via a vector, $\underline{k}(t)$, of adjustable parameters which are defined as feedback gains on the plant states. The parameter adjusting mechanism is to provide continuous adjustment of $\underline{k}(t)$ in a manner which guarantees that $\lim_{t \rightarrow \infty} \underline{e}(t) = \underline{0}$. On the other hand, an adaptive insensitive control system, as shown in Figure 1.3, incorporates a fixed controller structure to minimize the plant's sensitivity to unknown parameter variations. The simplest technique which fits the configuration of Figure 1.3 is the classical high gain feedback approach [4], [5] in which the controller is just a large negative gain. Liapunov theory has also been utilized [6] to design the controller in order to achieve a stable error system. In general, the techniques utilizing insensitive controllers obtain a fast convergence rate without a great deal of controller complexity. However, they usually result in larger control input signal magnitudes and are more likely to excite high order plant structural modes than the parameter adaptive scheme in Figure 1.2.

All of the parameter adjusting designs which have been developed employ either explicit or implicit plant identification in the process

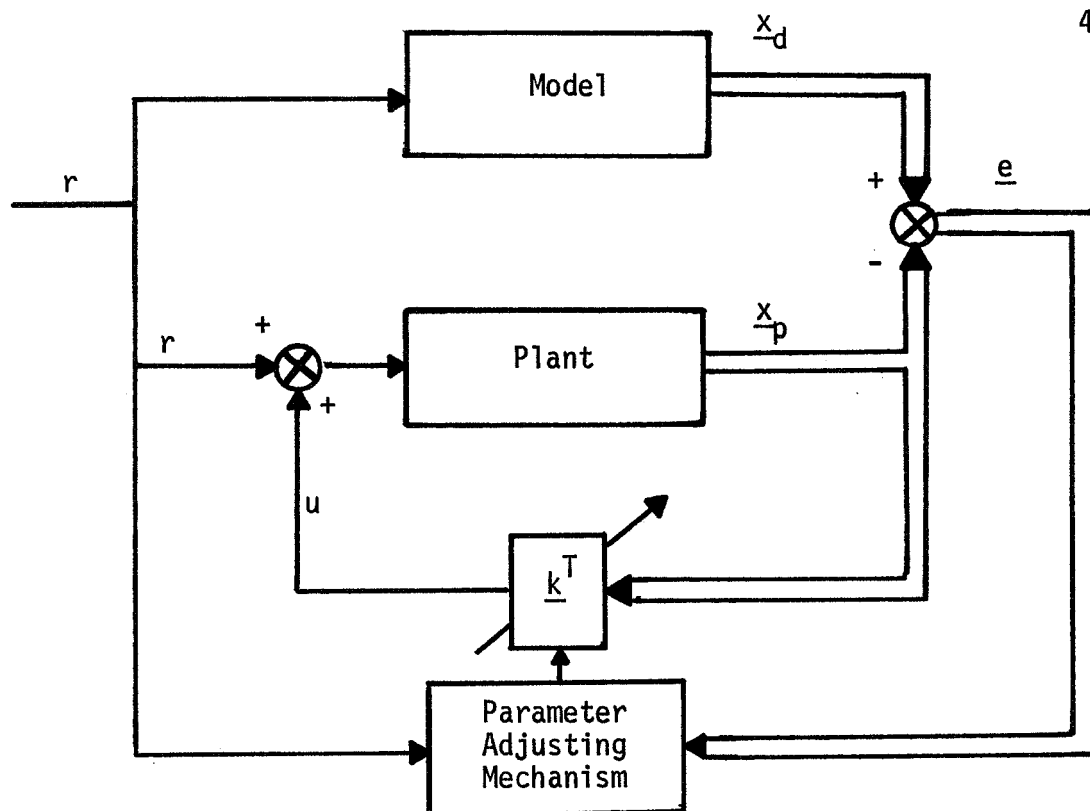


Figure 1.2. A Parameter Adaptive Control System

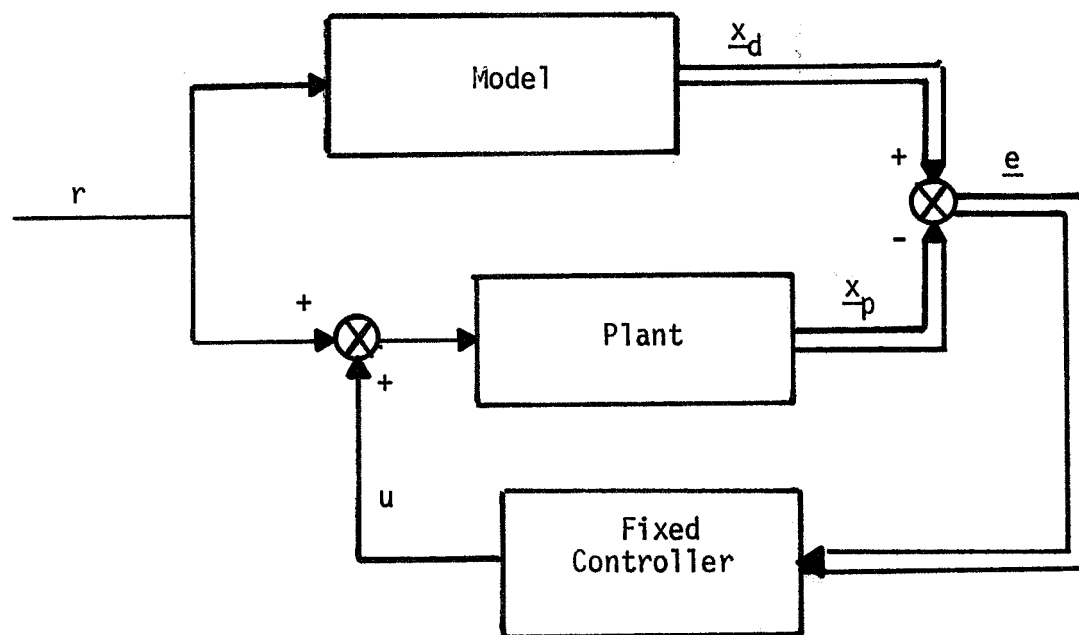


Figure 1.3. An Adaptive Insensitive Control System

of control. An adaptive scheme with explicit plant identification attempts to directly estimate the unknown plant parameters and then utilizes this information to adjust the adaptive gains. Techniques based on this approach are found in [7] and [8]. The implicit plant identification approach maintains continuous parameter adjustment based on minimizing the error between the plant outputs and desired responses. This method, while more complex than the adaptive insensitive schemes, does not require the degree of controller complexity necessitated by techniques which perform explicit plant identification.

B. Summary of Previous Research on Gradient and Liapunov Methods

The first parameter adaptive design utilizing implicit plant identification were gradient methods used to minimize a suitable performance index. The most popular of these was the "M.I.T." rule developed by Whitaker, Kezar, et al. [9]. It was shown in [10], however, that this approach led to analysis difficulties as well as possible instability for simple systems with step inputs. Subsequent techniques based on gradient methods [11] - [12] suffered from these problems as well. As an alternative method, it was suggested in [10] that adaptive loop synthesis be based on Liapunov's direct method. In [13] the adaptive laws were chosen in order to make the time derivative of a Liapunov function negative definite. Hence, global asymptotic stability was obtained as a byproduct of the design. Following its introduction in [10], [13], the technique was improved and extended by other

researchers, [14] - [16]. For the case of plants with an unknown constant gain, a more heavily damped error response was obtained in [14] with the addition of a subsidiary feedback loop. In [15], the technique was extended to plants with unknown time-varying gains and in [15] and [16] the number of derivative signals required for controller implementation was reduced in the case of plants with an unknown constant gain. In [17] the design technique was extended to a broader class of linear, time-invariant plants which contained as many as $n^2 + nr$ adjustable parameters (n th order plant with r inputs). However, there were several disadvantages with the technique in [17]. First, simulations show that large adaptive loop gains are necessary to achieve a satisfactory plant response. These large gains may be shown to lead to high frequency oscillations in the transient response of the plant if the error system is perturbed from its equilibrium point. Second, the method fails if plant parameters cannot be adjusted directly; i.e., feedback gains only can be adjusted. This is unfortunate since this is the case which occurs most frequently in practice. The problem which arises is that the adaptive laws for the feedback gains are generally nonunique. Third, the adaptive laws as presented in [17] require the generation of $n - 1$ output derivative signals (or n plant states). Hence, controller implementation necessitates building $(n - 1)$ th order physically unrealizable derivative networks. Finally, all of the work reported on the Liapunov technique in [10], [13] - [17], with the exception of [15], has been concerned with plants containing unknown parameters which are constant. In [15], a technique was developed for plants with unknown

time-varying gains only, and required a bound on the unknown parameter as well as its time derivative.

C. Purpose of the Dissertation

The adaptive technique discussed in this dissertation is of the parameter adaptive variety utilizing implicit plant identification. Due to the difficulties in analysis and possible instability associated with most gradient methods, Liapunov's direct method has been selected as the basis for synthesizing the adaptive laws. It is the purpose of this dissertation to eliminate several disadvantages which are associated with the Liapunov approach. Specifically, the goals are the following:

1. Develop adaptive laws which yield a rapidly convergent error response and eliminate the high frequency oscillations which are present in the transient response of both the plant outputs and the adjustable parameters.
2. Alleviate the problem of conflicting adaptive laws which results in the practical situation where the adjustable parameters are incorporated into the plant dynamics as feed-forward and feedback adjustable gains.
3. Develop a parameter adaptive configuration and associated adaptive laws which require a minimum number of physically unrealizable derivative networks for their implementation.
4. Extend the Liapunov technique to plants whose unknown parameters are time-varying.

Furthermore, all of the results above are to be obtained while maintaining the global stability of the resulting error system.

D. Layout of the Dissertation

In Chapter II, a "modified adaptive rule" is derived which results from the choice of a new Liapunov function. A local stability analysis of the modified system is performed which shows that a more rapidly convergent error response is attainable. It is also shown that the high frequency oscillations in the plant outputs and adjustable parameters are not present when using the modified adaptive laws. In addition, simulation results are presented which show the flexibility afforded by the modified design. For instance, by varying the fixed adaptive loop gains, a tradeoff between small error overshoot and minimum settling time is effected. Furthermore, system performance comparable to the previous design may be maintained with much smaller gains in the adaptive loop. Furthermore, the modified adaptive rule requires only one more summing amplifier per adjustable parameter as additional implementation.

Section A of Chapter III is devoted to the problem of conflicting adaptive laws which results when the Liapunov design technique is applied to single-input single-output plants whose adjustable parameters are introduced into the plant dynamics externally. The adaptive technique in Section III-B eliminates this problem, while requiring only $n - m - 1$ derivative networks on the plant output in contrast to the n states (or $n - 1$ derivative networks) required in [17] (n th order plant with m

zeros). Accordingly, if $m = n - 1$, no derivative networks are required even though the plant transfer function is not positive real (P.R.) as required in [10]. Furthermore, the result in [10] applied to plants whose transfer function was known to within a gain factor as did the derivative reduction technique in [16].

Chapter IV deals with the extension of the Liapunov parameter adaptive approach to plants with unknown time-varying parameters. Adaptive laws are derived which guarantee the global stability (in the sense of Lagrange) of the resulting system error. It is shown that a suitable norm of the error vector $\underline{e}$ may be driven into an arbitrarily small region of state space about the origin, the error bound being determined by certain fixed adaptive loop gains. It is required that bounds on the unknown parameters be known and if the plant contains differential operators acting on the input, bounds on certain of the time derivatives of the unknown parameters are required.

In Chapter V, the results of the adaptive control of an optical tracking mount are presented. It is shown that the adaptive compensation of changing mount inertia and bearing friction results in increased tracking accuracy.

C H A P T E R I I

IMPROVED CONVERGENCE IN THE PARAMETER ADAPTIVE CONTROL OF PLANTS WITH UNKNOWN CONSTANT PARAMETERS

A. The Modified Liapunov Design

In this chapter, an adaptive rule [18] is derived which is based on a certain Liapunov function, thereby assuring the global asymptotic stability of the resulting system equations. Since the adaptive laws developed here include the integral terms incorporated in the laws given in [17] plus the time derivative of these terms, they are termed the "modified adaptive laws." Local linearization and root locus techniques are utilized to illustrate the more rapid convergence and improved transient response of the modified design.

Consider the multi-input, multi-output plant described by the state equations (see Figure 2.1),

$$\dot{\underline{x}}_p = A_p(t)\underline{x}_p + B_p(t)\underline{u} \quad (2-1)$$

where the state vector $\underline{x}_p^T = (x_{1p}, x_{2p}, \dots, x_{np})$, the input vector $\underline{u}^T = (u_1, u_2, \dots, u_r)$ and $A_p(t)$ and $B_p(t)$ are, respectively, $n \times n$ and $n \times r$ matrices containing adjustable parameters as well as unknown parameters. For example, each of the elements a_{ij}^p and b_{ij}^p of A_p and B_p , respectively, are assumed to be of the form

$$\begin{aligned} a_{ij}^p &= c_{ij}^a + k_{ij}^a d_{ij}^a, & i = 1, n; j = 1, n \\ b_{ij}^p &= c_{ij}^b + k_{ij}^b d_{ij}^b, & i = 1, n; j = 1, r \end{aligned} \quad (2-2)$$

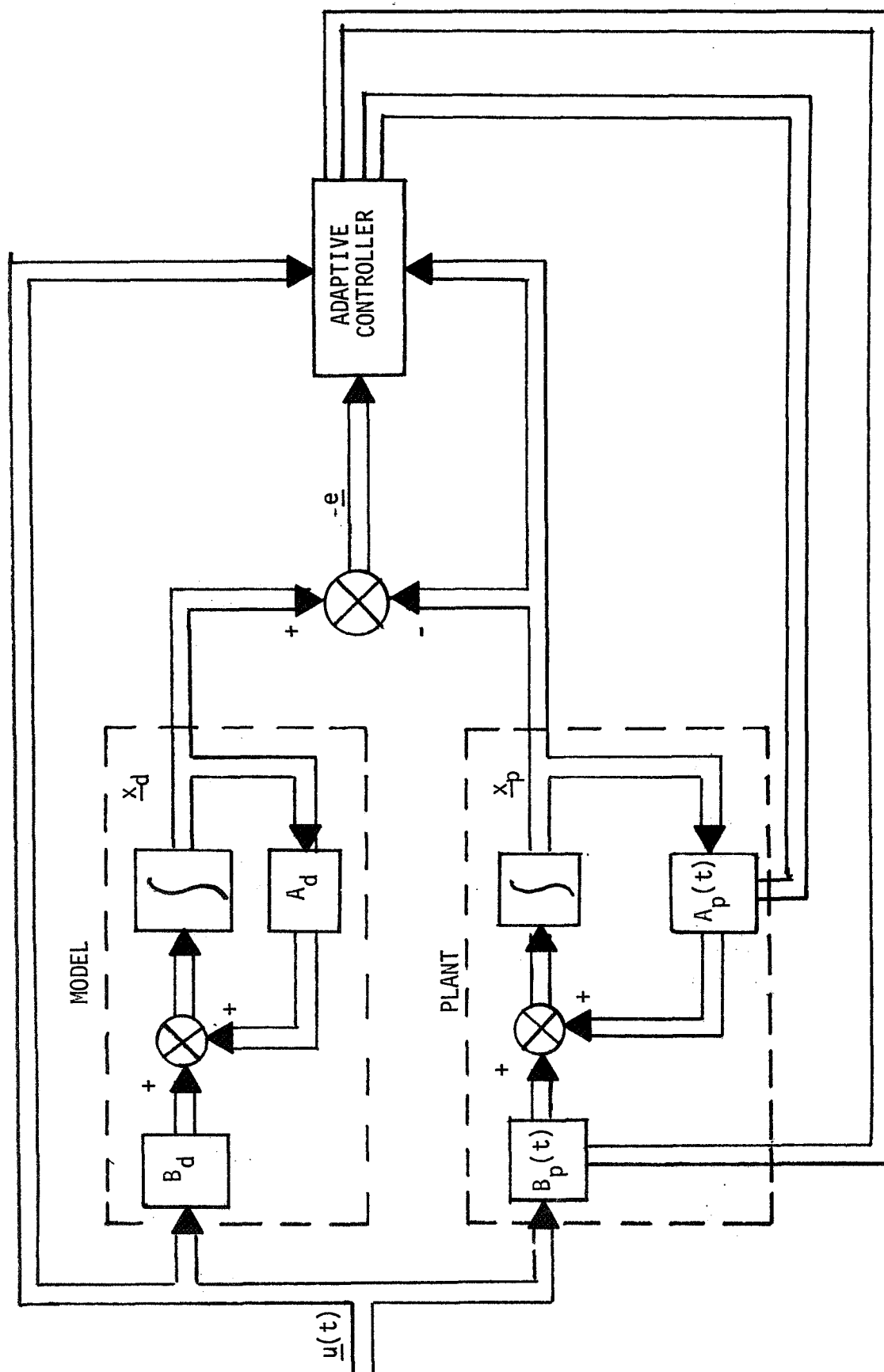


Figure 2.1. Adaptive Structure for Multi-Input Multi-Output Systems

where $c_{ij}^a, c_{ij}^b, d_{ij}^a, d_{ij}^b$ are constant unknown parameters with only the signs of d_{ij}^a and d_{ij}^b being known and each k_{ij}^a and k_{ij}^b is a unique parameter to be adjusted. Furthermore, it is assumed that the desired plant characteristics are represented in a model reference system described by

$$\dot{\underline{x}}_d = A_d \underline{x}_d + B_d \underline{u} \quad (2-3)$$

where $\underline{x}_d^T = [x_{1d}, x_{2d}, \dots, x_{nd}]$, A_d and B_d are known constant matrices of dimensions $n \times n$ and $n \times r$, respectively. If the error is defined as

$$\underline{e} = \underline{x}_d - \underline{x}_p \quad (2-4)$$

where $\underline{e} = [e_1, e_2, \dots, e_n]$ then (2-1) may be subtracted from (2-3) to yield the error equation

$$\dot{\underline{e}} = A_d \underline{e} + A \underline{x}_p + B \underline{u} \quad (2-5)$$

where $A = A_d - A_p$ and $B = B_d - B_p$.

The design procedure leading to the modified adaptive law begins by choosing the following potential Liapunov function:

$$\begin{aligned} V = & \underline{e}^T P \underline{e} + (\underline{a}^T + \underline{z}_x^T \Lambda_\beta) \Lambda_\alpha^{-1} (\underline{a} + \Lambda_\beta \underline{z}_x) \\ & + (\underline{b}^T + \underline{z}_u^T \Lambda_\delta) \Lambda_\gamma^{-1} (\underline{b} + \Lambda_\delta \underline{z}_u) \end{aligned} \quad (2-6)$$

In (2-6), the elements of the symmetric matrix P are denoted by p_{ij} and the column vectors $\underline{a}$, $\underline{b}$, $\underline{z}_x$, and $\underline{z}_u$ are of dimensionality n^2 , nr , n^2 , and nr , respectively, partitioned according to

$$\underline{a}^T = (\underline{a}_1^T, \underline{a}_2^T, \dots, \underline{a}_n^T)$$

$$\underline{b}^T = (\underline{b}_1^T, \underline{b}_2^T, \dots, \underline{b}_n^T)$$

$$\underline{z}_x^T = (\underline{z}_{x1}^T, \underline{z}_{x2}^T, \dots, \underline{z}_{xn}^T)$$

$$\underline{z}_u^T = (\underline{z}_{u1}^T, \underline{z}_{u2}^T, \dots, \underline{z}_{un}^T)$$

where $\underline{a}_i$ and $\underline{b}_i$ are the i th rows of the matrices A and B , respectively; i.e., the n elements of $\underline{a}_i$ are a_{ij} , $j = 1, n$ while the r elements of $\underline{b}_i$ are b_{ij} , $j = 1, r$. The n elements of the vectors $\underline{z}_{xi}$ ($n \times 1$) and $\underline{z}_{ui}$ ($r \times 1$) are given in (2-8).

$$\underline{z}_{xi} = \sum_{k=1}^n e_k p_{ki} x_{jp}, \quad j = 1, n$$

$$\underline{z}_{ui} = \sum_{k=1}^n e_k p_{ki} u_j, \quad j = 1, n$$

The matrices Λ_α^{-1} , ($n^2 \times n^2$), Λ_β , ($n^2 \times n^2$), Λ_γ^{-1} , ($nr \times nr$) and Λ_δ , ($nr \times nr$) are positive definite and diagonal. The n^2 nonzero elements of Λ_α , in the order of their position along the diagonal, are given by $1/\alpha_{1j}$, $j = 1, n$; $1/\alpha_{2j}$, $j = 1, n$; ... ; $1/\alpha_{nj}$, $j = 1, n$, while the diagonal elements of Λ_β , Λ_γ , and Λ_δ are, respectively, β_{ij} , $i, j = 1, n$; γ_{ij} , $i, j = 1, n$; and δ_{ij} , $j = 1, n$ which are ordered in the same manner as the elements of Λ_α . The second and third terms on the right-hand side of (2-6) differ from the corresponding terms in the Liapunov function used in [17] which was

$$V = \underline{e}^T P \underline{e} + \underline{a}^T \Lambda_\alpha^{-1} \underline{a} + \underline{b}^T \Lambda_\gamma^{-1} \underline{b} \quad (2-7)$$

The additional terms in (2-6) lead to direct feedback of the error

states to each adjustable parameter, resulting in a faster convergence rate and an improved transient response. The time derivative of (2-6) is

$$\begin{aligned} \dot{\underline{V}} = & \underline{e}^T (\underline{A}_d^T \underline{P} + \underline{P} \underline{A}_d) \underline{e} + 2 \left\{ \underline{e}^T \underline{P} \underline{A} \underline{x}_p + \underline{e}^T \underline{B} \underline{u} + \underline{a}^T \underline{\Lambda}_\alpha^{-1} \dot{\underline{a}} + \underline{a}^T \underline{\Lambda}_\alpha^{-1} \underline{\Lambda}_\beta \dot{\underline{z}}_x \right. \\ & + \dot{\underline{a}}^T \underline{\Lambda}_\alpha^{-1} \underline{\Lambda}_\beta \underline{z}_x + \underline{z}_x^T \underline{\Lambda}_\alpha^{-1} \underline{\Lambda}_\beta \dot{\underline{z}}_x + \underline{b}^T \underline{\Lambda}_\gamma^{-1} \dot{\underline{b}} \\ & \left. + \underline{b}^T \underline{\Lambda}_\gamma^{-1} \underline{\Lambda}_\delta \dot{\underline{z}}_u + \dot{\underline{b}}^T \underline{\Lambda}_\gamma^{-1} \underline{\Lambda}_\delta \underline{z}_u + \underline{z}_u^T \underline{\Lambda}_\gamma^{-1} \underline{\Lambda}_\delta \dot{\underline{z}}_u \right\} \end{aligned} \quad (2-8)$$

where the second and third terms in (2-8) may be written as

$$\begin{aligned} \underline{e}^T \underline{P} \underline{A} \underline{x}_p &= \underline{a}^T \underline{z}_x \\ \underline{e}^T \underline{P} \underline{B} \underline{u} &= \underline{b}^T \underline{z}_u \end{aligned} \quad (2-9)$$

The vectors $\dot{\underline{a}}$ and $\dot{\underline{b}}$ are now selected in order to make $\dot{\underline{V}}$ negative definite. They are chosen as

$$\dot{\underline{a}}^T = -\underline{\Lambda}_\alpha \underline{z}_x - \underline{\Lambda}_\beta \dot{\underline{z}}_x \quad (2-10)$$

$$\dot{\underline{b}}^T = -\underline{\Lambda}_\gamma \underline{z}_u - \underline{\Lambda}_\delta \dot{\underline{z}}_u \quad (2-11)$$

Since from (2-2) the elements $\dot{a}_{ij}$, $\dot{b}_{ij}$ of the vectors $\dot{\underline{a}}$ and $\dot{\underline{b}}$, respectively, are related to the adjustable parameters $\dot{k}_{ij}^a$ and $\dot{k}_{ij}^b$ according to $\dot{a}_{ij} = -\dot{a}_{ij}^p = -\dot{k}_{ij}^a d_{ij}^a$ and $\dot{b}_{ij} = -\dot{b}_{ij}^p = -\dot{k}_{ij}^b d_{ij}^b$, integrating both sides of (2-10) and (2-11) yields the adaptive laws

$$(\underline{k}^a)^T = \underline{\Lambda}_{d\alpha} \int_{t_0}^t \underline{z}_x^T(t) dt + \underline{\Lambda}_{d\beta} \underline{z}_x^T(t) + (\underline{k}^a(t_0))^T \quad (2-12)$$

$$(\underline{k}^b)^T = \Lambda_{d\gamma} \int_{t_0}^t \underline{z}_u^T(t) dt + \Lambda_{d\delta} \underline{z}_u^T(t) + (\underline{k}^b(t_0))^T \quad (2-13)$$

In (2-12) and (2-13), the vectors $\underline{k}^a$ and $\underline{k}^b$ are defined by

$$(\underline{k}^a)^T = ((\underline{k}_1^a)^T, (\underline{k}_2^a)^T, \dots, (\underline{k}_n^a)^T)$$

$$(\underline{k}^b)^T = ((\underline{k}_1^b)^T, (\underline{k}_2^b)^T, \dots, (\underline{k}_n^b)^T)$$

where the elements of $\underline{k}_i^a$ are given by k_{ij}^a , $j = 1, n$ and the elements of $\underline{k}_i^b$ are denoted by k_{ij}^b , $j = 1, n$. The elements of the positive definite diagonal matrices $\Lambda_{d\alpha}$, $\Lambda_{d\beta}$, $\Lambda_{d\gamma}$, and $\Lambda_{d\delta}$ are given by α_{ij}/d_{ij}^a , β_{ij}/d_{ij}^a , γ_{ij}/d_{ij}^b , and δ_{ij}/d_{ij}^b , respectively, positioned along the diagonal in the same manner as the elements $1/\alpha_{ij}$ are ordered along the diagonal of the matrix Λ_α . These elements are arbitrary positive or negative constants depending on the signs of d_{ij}^a and d_{ij}^b , and t_0 is the time instant the adaptive loop is closed. Substituting (2-10) and (2-11) into (2-8), $\dot{V}$ becomes, in view of (2-9),

$$\dot{V} = \underline{e}^T (A_d^T P + P A_d) \underline{e} - 2 \underline{z}_x^T \Lambda_\beta \underline{z}_x - 2 \underline{z}_u^T \Lambda_\delta \underline{z}_u \quad (2-14)$$

Now if A_d is a stable matrix, the positive definite symmetric matrix P may be determined as the unique solution [19] of

$$A_d^T P + P A_d = -Q \quad (2-15)$$

where Q is an arbitrary positive definite symmetric matrix. Since the last two summation terms in (2-14) are less than or equal to 0, $\dot{V}$, with this choice of Q , is a negative definite form over the space of $\underline{e}$ and $\underline{V}$

is a Liapunov function for the error equation, (2-5). Thus, asymptotic stability, in the sense that $\underline{e} \rightarrow \underline{0}$ as $t \rightarrow \infty$, is assured. The a_{ij} and b_{ij} may or may not converge to zero since $\dot{V}$ is only a negative semi-definite function of the variables e_k , a_{ij} , and b_{ij} . The plant output however, will always track the model output. Equations (2-12) and (2-13) are the modified adaptive laws for each adjustable parameter in the A_p and B_p matrices. Implementation of these laws requires only one more summer for each gain than implementation of the adaptive laws in [17] which can be obtained from (2-12) and (2-13) by setting $\beta_{ij} = \delta_{ij} = 0$, for all i, j .

B. Comparison of Convergence Properties

The convergence properties of this design and previously reported Liapunov designs will be compared by performing a local stability analysis when the adjustable parameters are implemented first according to the development in [17] and then according to the modified adaptive law. A comparison of the root loci of the characteristic equations of the perturbed linearized system equations for the two cases will show that the modified law speeds convergence. In order to clearly illustrate the details of this comparison, the local stability analysis is first performed on a second order system. Consider the plant with scalar input described by

$$\begin{bmatrix} \dot{x}_{1p} \\ \dot{x}_{2p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c_{21} + k_{21}^a d_{21}^a & c_{22}^a + k_{22}^a d_{22}^a \end{bmatrix} \begin{bmatrix} x_{1p} \\ x_{2p} \end{bmatrix} + \begin{bmatrix} 1 \\ c_2^b + k_2^b d_2^b \end{bmatrix} u \quad (2-16)$$

with x_{1p} being the plant output. The model is taken as

$$\begin{bmatrix} \dot{x}_{1d} \\ \dot{x}_{2d} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_{21}^d & a_{22}^d \end{bmatrix} \begin{bmatrix} x_{1d} \\ x_{2d} \end{bmatrix} + \begin{bmatrix} 1 \\ b_d \end{bmatrix} u \quad (2-17)$$

where a_{21}^d and a_{22}^d are negative constants. Hence, the error equation is

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_{21}^d & a_{22}^d \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{1p} \\ x_{2p} \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} u \quad (2-18)$$

where $a_{21} = a_{21}^d - c_{21}^a - k_{21}^a d_{21}^a$, $a_{22} = a_{22}^d - c_{22}^a - k_{22}^a d_{22}^a$, and $b_2 = b_{2d} - c_2^b - k_2^b d_2^b$.

The adaptive law in [17] requires that the time derivatives of the adaptive parameters be chosen as

$$\dot{k}_{2i}^a = -\alpha_{2i} \sum_{k=1}^2 e_k p_{k2} x_{ip}, \quad i = 1, 2; \quad \dot{k}_2^b = -\gamma_2 \sum_{k=1}^2 e_k p_{k2} u \quad (2-19)$$

where α_{21} , α_{22} , and γ_2 are arbitrary positive constants.

Equations (2-18) and (2-19) constitute a set of system equations.

An equilibrium point for this set of equations may be defined by taking

$$k_{2i}^a = k_{2i}^{a0}, \quad i = 1, 2; \quad k_2^b = k_2^{b0}, \quad \underline{e} = \dot{\underline{e}} = 0, \quad \underline{x}_p = \underline{x}_d^0 \quad \text{and} \quad u = u_0 \quad (2-20)$$

where k_{2i}^{a0} , $i = 1, 2$; k_2^{b0} ; and $u_0 = \lim_{t \rightarrow \infty} u(t)$ are constants and $\underline{x}_d^0$ is a constant vector. Note that it is not necessary to assume a_{21} , a_{22} and b_2 are all equal to zero for $k_{2i}^a = k_{2i}^{a0}$ and $k_2^b = k_2^{b0}$. The linearized

set of system equations given in (2-21) is derived assuming small perturbations from the equilibrium point. Thus,

$$\begin{aligned}
 \dot{\Delta e}_1 &= \Delta e_2 \\
 \dot{\Delta e}_2 &= a_{21}^d \Delta e_1 + a_{22}^d \Delta e_2 - \sum_{i=1}^2 x_{id}^0 d_{2i} \Delta k_{2i}^a - u_0 d_2^b \Delta k_2^b \\
 \Delta \dot{k}_{2i}^a &= (\alpha_{2i}/d_{2i}^a) \sum_{k=1}^2 \Delta e_k p_{k2} x_{di}, \quad i = 1, 2 \\
 \Delta \dot{k}_2^b &= (\gamma_2/d_2^b) \sum_{k=1}^2 \Delta e_k p_{k2} u_0
 \end{aligned} \tag{2-21}$$

where the definitions for the perturbed variables are given by

$$\begin{aligned}
 k_{2i}^a &= k_{2i}^{a0} + \Delta k_{2i}, \quad i = 1, 2; \quad k_2^b = k_2^{b0} + \Delta k_2^b, \\
 \underline{e} &= \underline{\Delta e}, \text{ and } \underline{x}_p = \underline{x}_d^0 + \underline{\Delta x}_p = \underline{x}_d^0 - \underline{\Delta e}
 \end{aligned} \tag{2-22}$$

where $\underline{\Delta e}^T = [\Delta e_1, \Delta e_2]$. Equation (2-21) is Laplace transformed and the characteristic equation in a form for root locus analysis for this system is found to be

$$1 + \frac{K_1(p_{12} + p_{22}s)}{s(s^2 - a_{22}^d s - a_{21}^d)} = 0 \tag{2-23}$$

where $K_1 = \sum_{i=1}^2 (x_{id}^0)^2 \alpha_{2i} + u_0^2 \gamma_2$ is a linear combination of the arbitrary positive gains α_{21} , α_{22} , γ_2 in (2-19). It is seen that the gains α_{21} , α_{22} , and α_2 do not act independently of one another in their effect on the linearized system dynamics. A typical root locus for (2-23) as a function of K_1 is shown in Figure 2.2. If the gains α_{21} , α_{22} and γ_2 are made large it is evident that the linearized system response becomes

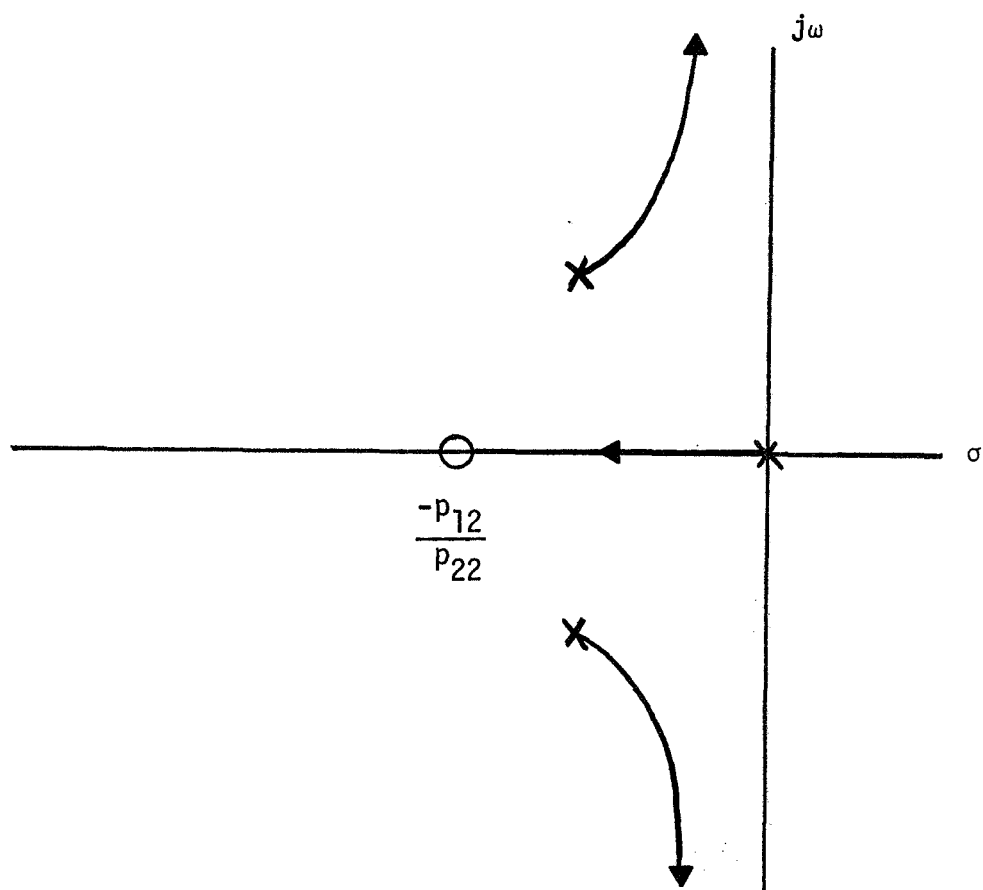


Figure 2.2. Root Locus with Respect to K_1 for Linearized Error System

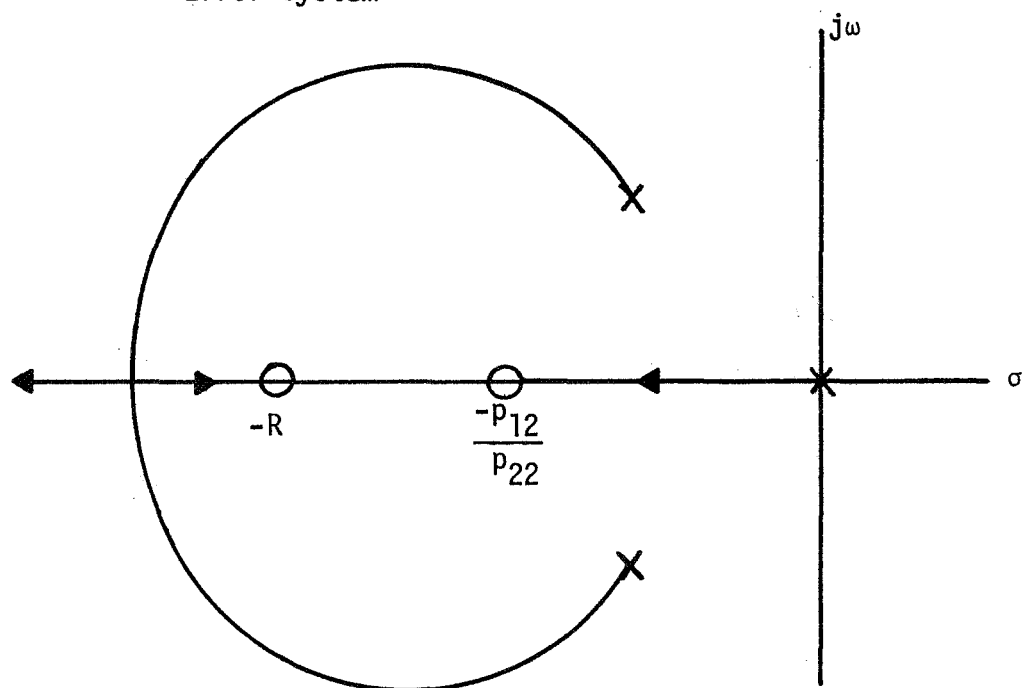


Figure 2.3. Root Locus with Respect to K_1 for Linearized Error System (Modified Rule)

highly oscillatory. This is verified in simulation results presented in Section B.

For comparison, the development (2-16) to (2-23) is repeated with the adaptive parameters defined by (2-10) and (2-11) instead of (2-19). Thus

$$\begin{aligned}\dot{k}_{2i}^a &= -\alpha_{2i} \sum_{k=1}^2 e_k p_{k2} x_{ip} - \beta_{2i} \frac{d}{dt} \left(\sum_{k=1}^2 e_k p_{k2} x_{ip} \right), \quad i = 1, 2 \\ \dot{k}_2^b &= -\gamma_2 \sum_{k=1}^2 e_k p_{k2} u - \delta_2 \frac{d}{dt} \left(\sum_{k=1}^2 e_k p_{k2} u \right)\end{aligned}\tag{2-24}$$

The characteristic equation for this case is

$$1 + \frac{K_1(p_{12} + p_{22}s)(1 + K_2s/K_1)}{s(s^2 - a_{22}^d s - a_{21}^d)} = 0\tag{2-25}$$

where $K_2 = \sum_{i=1}^2 (x_{id}^0)^2 \beta_{2i} + u_0^2 \delta_2$. The root locus for (2-25) is plotted as a function of K_1 in Figure 2.3 where $R (> 0)$ is defined by $K_1 = RK_2$. The effect of the additional terms in (2-24) has been to place a zero at $-R$, which eliminates the oscillatory behavior of the previous design. In addition, it is evident that one pole of the linearized system response may be made as heavily damped as necessary by increasing R and K_1 .

The root locus behavior in Figures 2.2 and 2.3 for large K_1 is typical for higher order systems of the form (2-2) and (2-3) as well. The characteristic equation for an nth order system analogous to (2-23) is given by

$$1 + \frac{K_1(s^n + d_1 s^{n-1} + \dots + d_n)}{s^2 |sI - A_d|} = 0\tag{2-26}$$

where K_1 is a linear combination of the α_{ij} and the d_i , $i = 1, n$ are functions of the a_{ij}^d , α_{ij} , β_{ij} , and q_{ij} and the roots of $s^n + d_1 s^{n-1} + \dots + d_n = 0$ are in the left half s -plane. From (2-26) it is seen that there are two more poles than zeros whose loci must approach $\pm 90^\circ$ asymptotes in the complex plane for large K_1 . On the other hand, using the modified adaptive rule, the characteristic equation becomes

$$1 + \frac{K_1(s^{n+1} + g_1 s^n + \dots + g_{n+1})}{s^2 |sI - A_d|} = 0 \quad (2-27)$$

where the g_i , $i = 1, n + 1$ are functions of the α_{ij} , β_{ij} , a_{ij}^d and q_{ij} . Again, the modified rule forces the locus back to the real axis by introducing an additional left half plane zero.

C. Simulation Results

Example 1. Consider the second-order system shown in Figure 2.4 where $u(t)$ is a scalar input, $K_V (> 0)$ is a constant unknown plant gain, K_C is the adjustable plant gain, and K (constant) is the known desired gain. The system can be represented in the form of (2-5) by the equations

$$\dot{\underline{e}} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \underline{e} + \begin{bmatrix} 0 \\ K - K_V K_C \end{bmatrix} u(t) \quad (2-28)$$

where $e_1(t) = e(t)$ and $e_2(t) = \dot{e}(t)$. In (2-15), Q , which may be any positive definite symmetric matrix, is chosen as

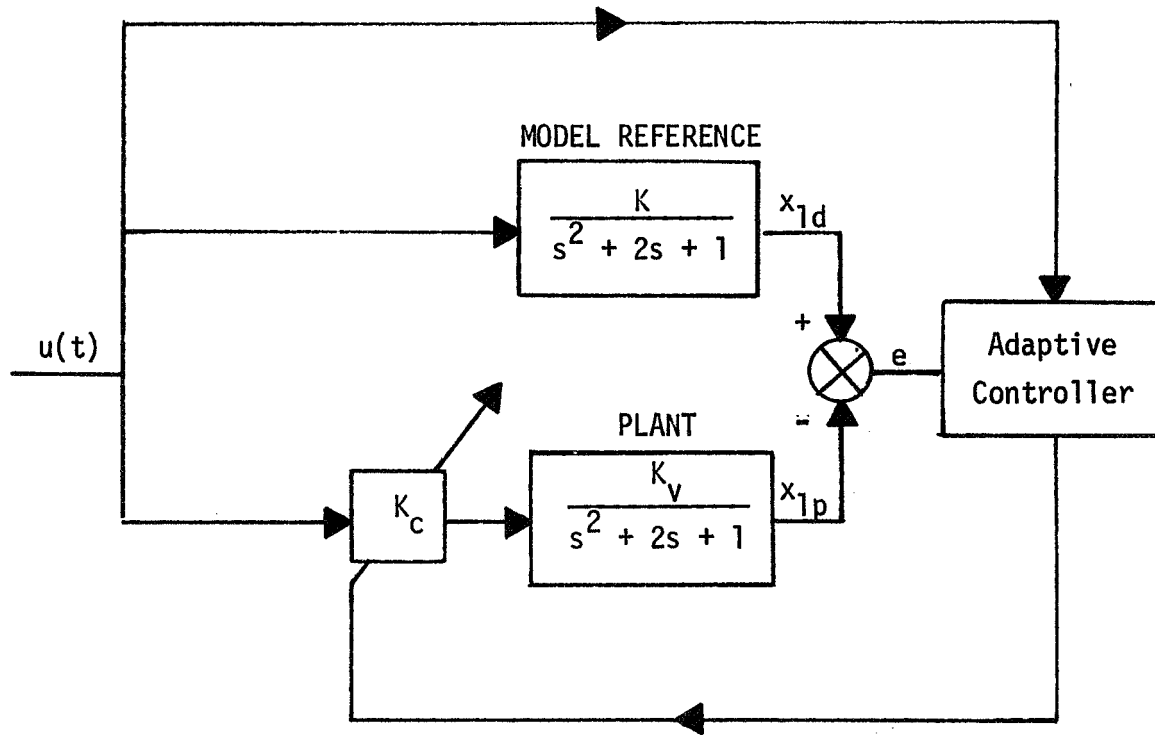


Figure 2.4. Second Order Model Reference System

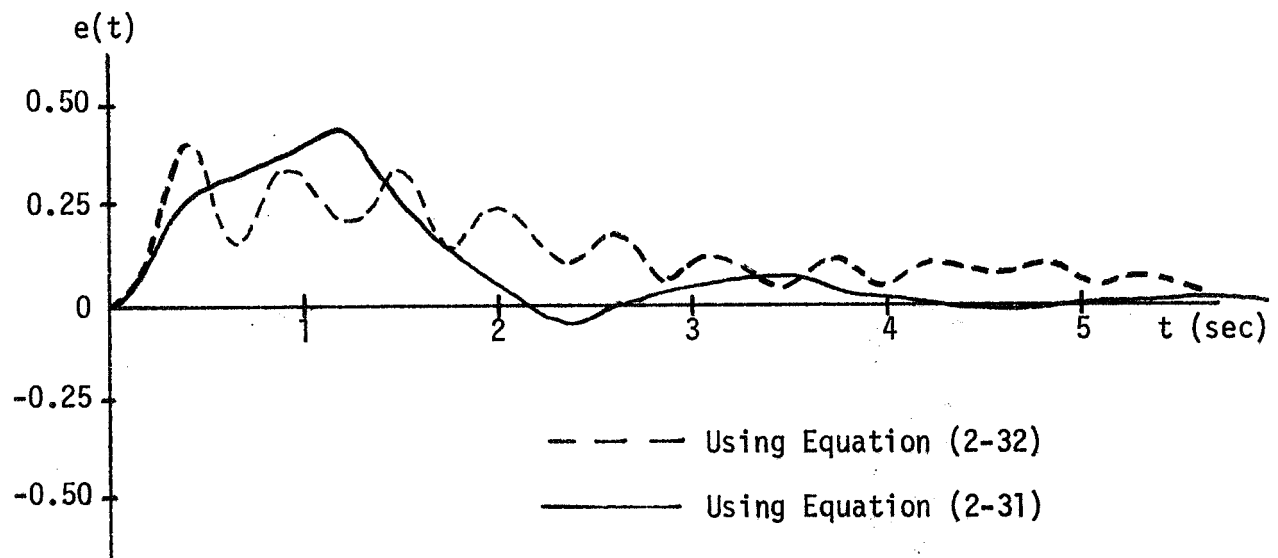


Figure 2.5. Time Response of $e(t)$ for Second Order System

$$Q = \begin{bmatrix} 7/4 & 1 \\ 1 & 3/4 \end{bmatrix} \quad (2-29)$$

in order to yield a P matrix which is

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad (2-30)$$

The application of (2-13) results in the following implementation of K_c .

$$K_c = \alpha \int_{t_0}^t (eu + \dot{e}u) dt + \beta(eu + \dot{e}u) + K_c(t_0) \quad (2-31)$$

where α (> 0) and β (> 0) are arbitrary constants. The application of Equation 9 in [17], using (2-30), yields

$$K_c = \gamma \left[\int_{t_0}^t (eu + \dot{e}u) dt \right] + K_c(t_0) \quad (2-32)$$

where γ is an arbitrary positive constant.

The dashed line in Figure 2.5 is the error response, $e(t)$, when the adaptive controller is implemented according to (2-32) with $K = 25$, $K_v = 1$, $u(t) = 1.5 \sin 2.8t$, $t_0 = 0$, $K_c(0) = 5$, $e(0) = \dot{e}(0) = 0$, and $\gamma = 93$. It was found that $\int_0^\infty |e(t)| dt = 0.63$ and any decrease in γ resulted in a larger integral absolute error. The solid line indicates $e(t)$ when K_c is implemented according to (2-31). It was found that the same integral absolute error could be obtained if $\alpha = \beta = 16$, about six times smaller than the gain requirement when using (2-32). Figure 2.6 shows the response of $\dot{e}(t)$ with the two different implementations of K_c . The rapid convergence of $\dot{e}(t)$ when using (2-31) is clearly evident. The

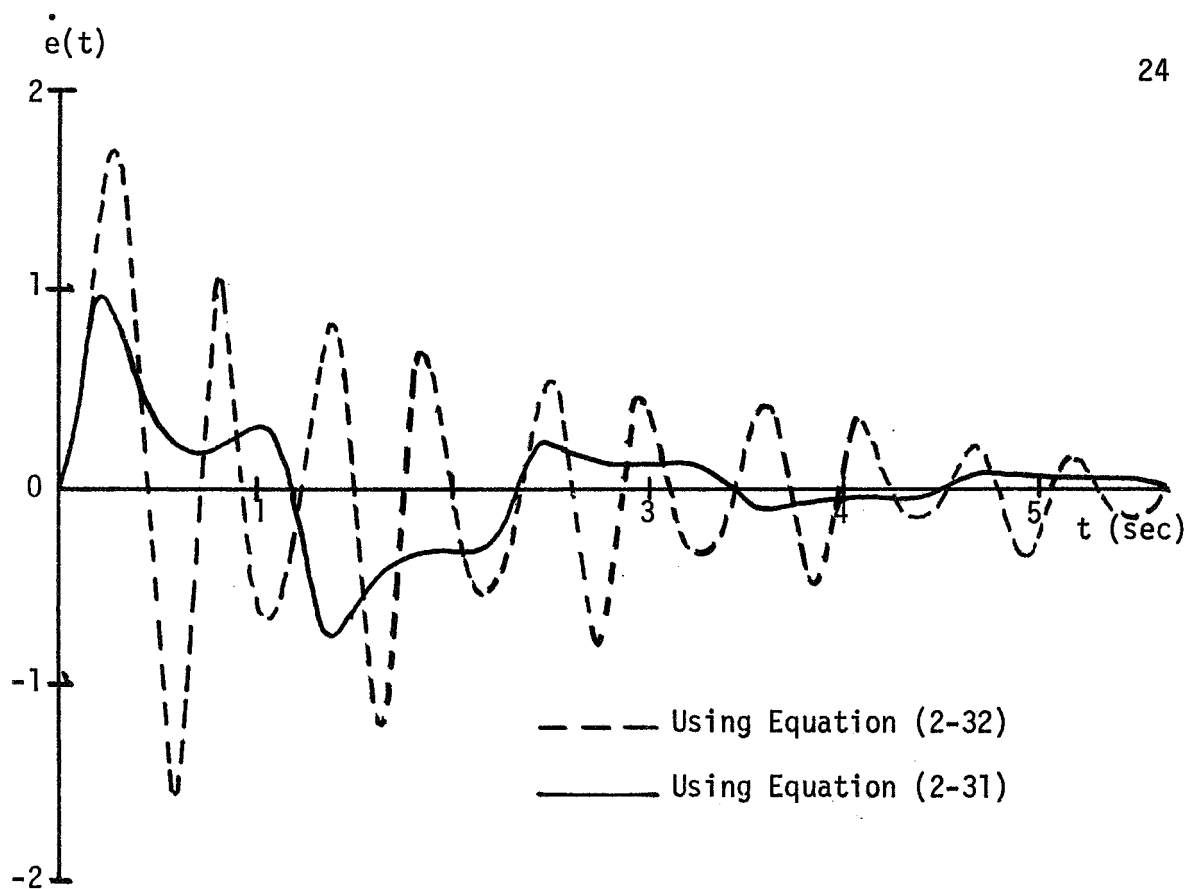


Figure 2.6. Time Response of $\dot{e}(t)$ for Second Order System

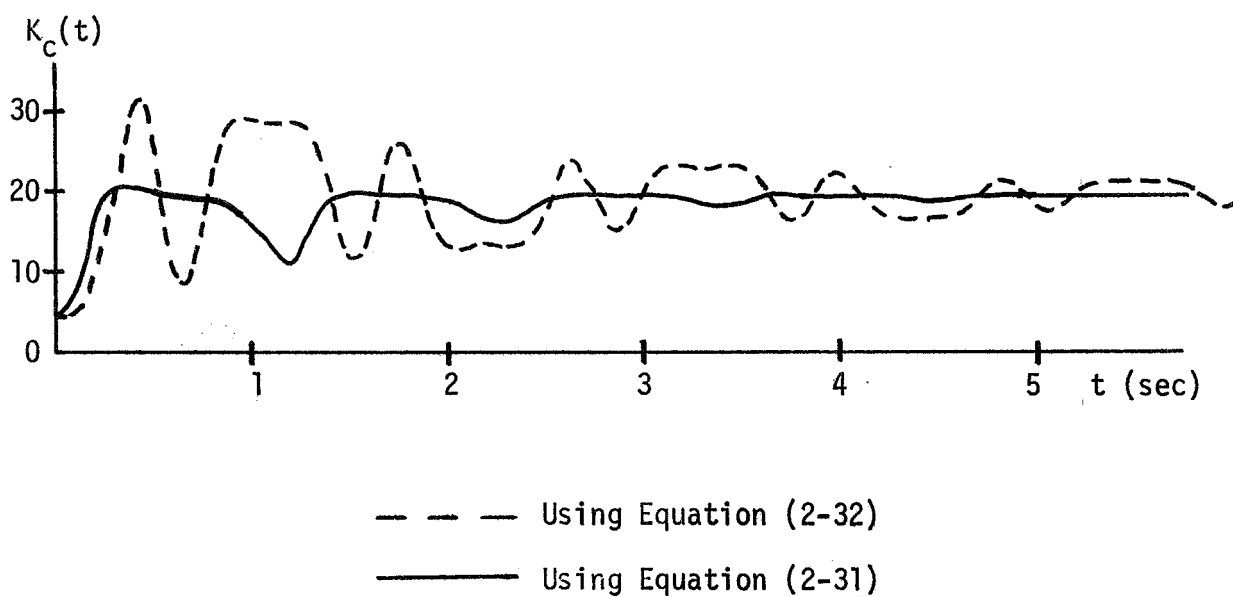


Figure 2.7. Time Response of $K_c(t)$ for Second Order System

time response of the adjustable parameters of (2-31) and (2-32) are compared directly in Figure 2.7, illustrating the smooth adaption afforded by (2-31).

Example 2. An illustration of the flexibility of the modified design is provided in this example. Consider the first order adaptive system in Figure 2.8 where $u(t)$ is a step input with magnitude 5 and $a_v (> 0)$ is an unknown constant. The adaptive controller must compensate for the difference in the location of the model and plant poles. The system can be represented in the form of (2-5) by the equation

$$\dot{e} = -4e + (a_c + a_v - 4)x_p \quad (2-33)$$

The matrices P and Q in (2-15) are scalars in this case and the choice of $Q = 8$ yields $P = 1$. The application of (2-12) yields the following adaptive law for $a_c(t)$

$$a_c(t) = \alpha \int_{t_0}^t e x_p dt + \beta e x_p + a_c(t_0) \quad (2-34)$$

where $\alpha (> 0)$ and $\beta (\geq 0)$ are arbitrary constants. The developments in [10] and [17] yield an adaptive law containing only the integral term in (2-34); hence, (2-34) represents the previous design if $\beta \equiv 0$.

Three simulations were performed on the system in Figure 2.8 with $a_v = 1$, $t_0 = 0$, $e(0) = 0$, $a_c(0) = 0$, and $\alpha = 100$. Figure 2.9 illustrates the error response for the system when the adaptive controller is implemented according to (2-34). The three curves correspond to three different values of β ; i.e., $\beta = 0$, which corresponds to the design in [10] and [17], and $\beta = 10$ and 90 , which correspond to the modified

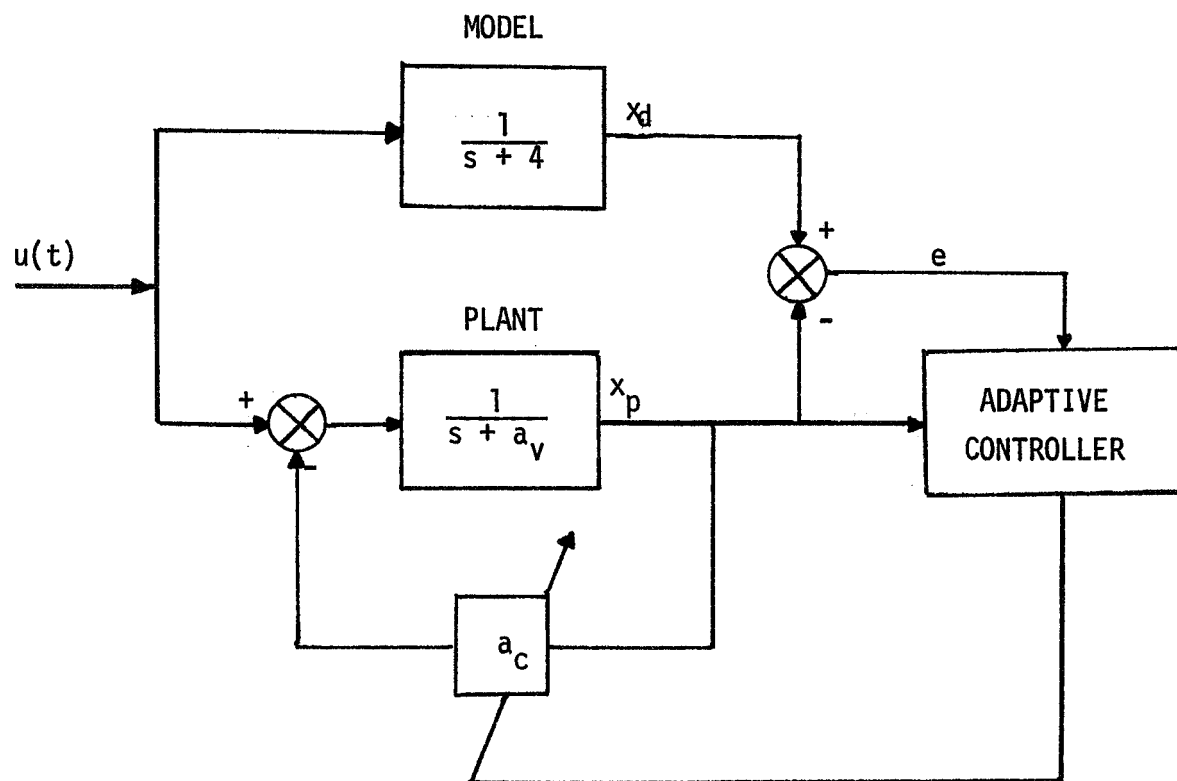


Figure 2.8. First Order System with Adaptive Time Constant.

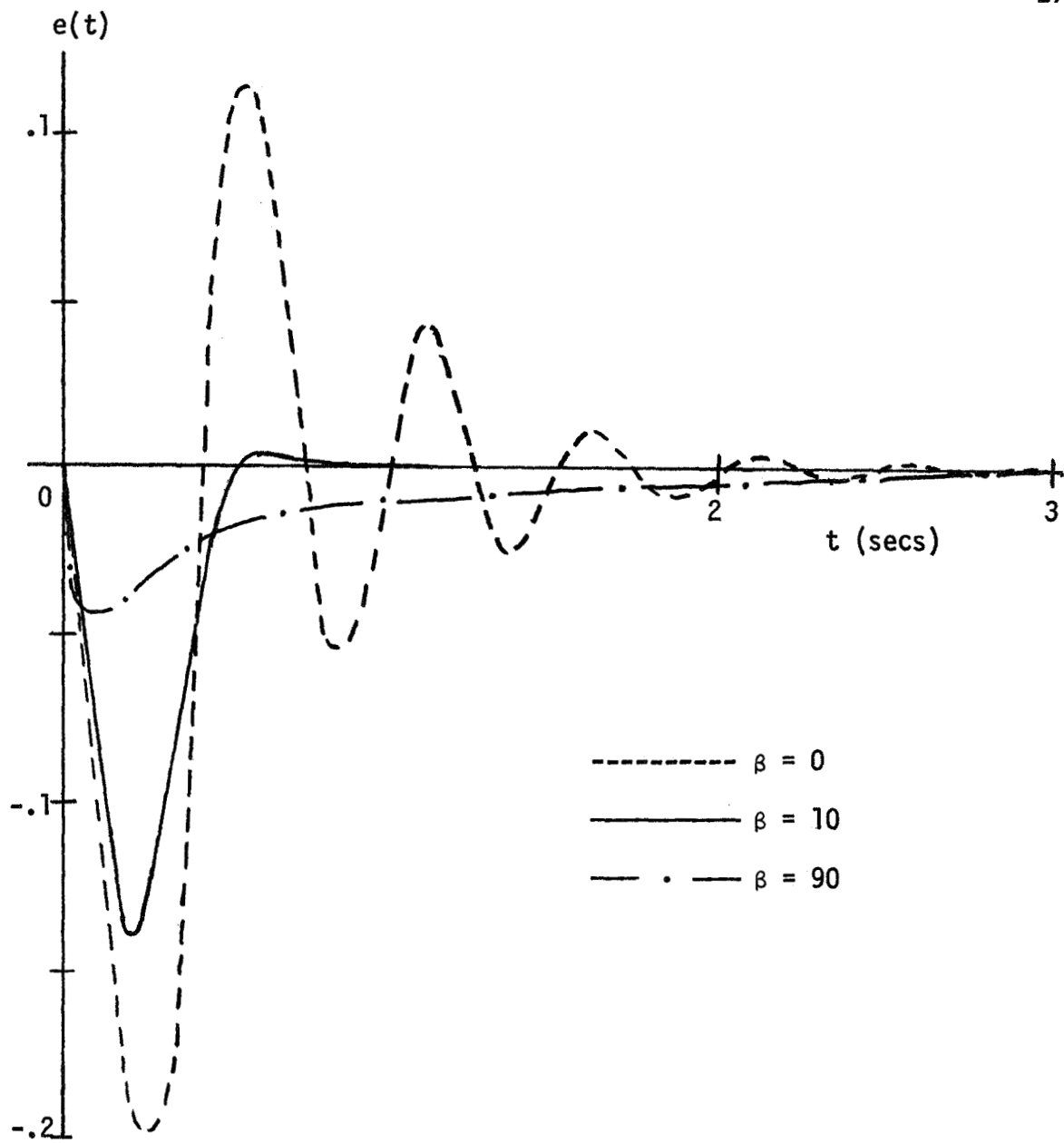


Figure 2.9. Error Response of First Order System.

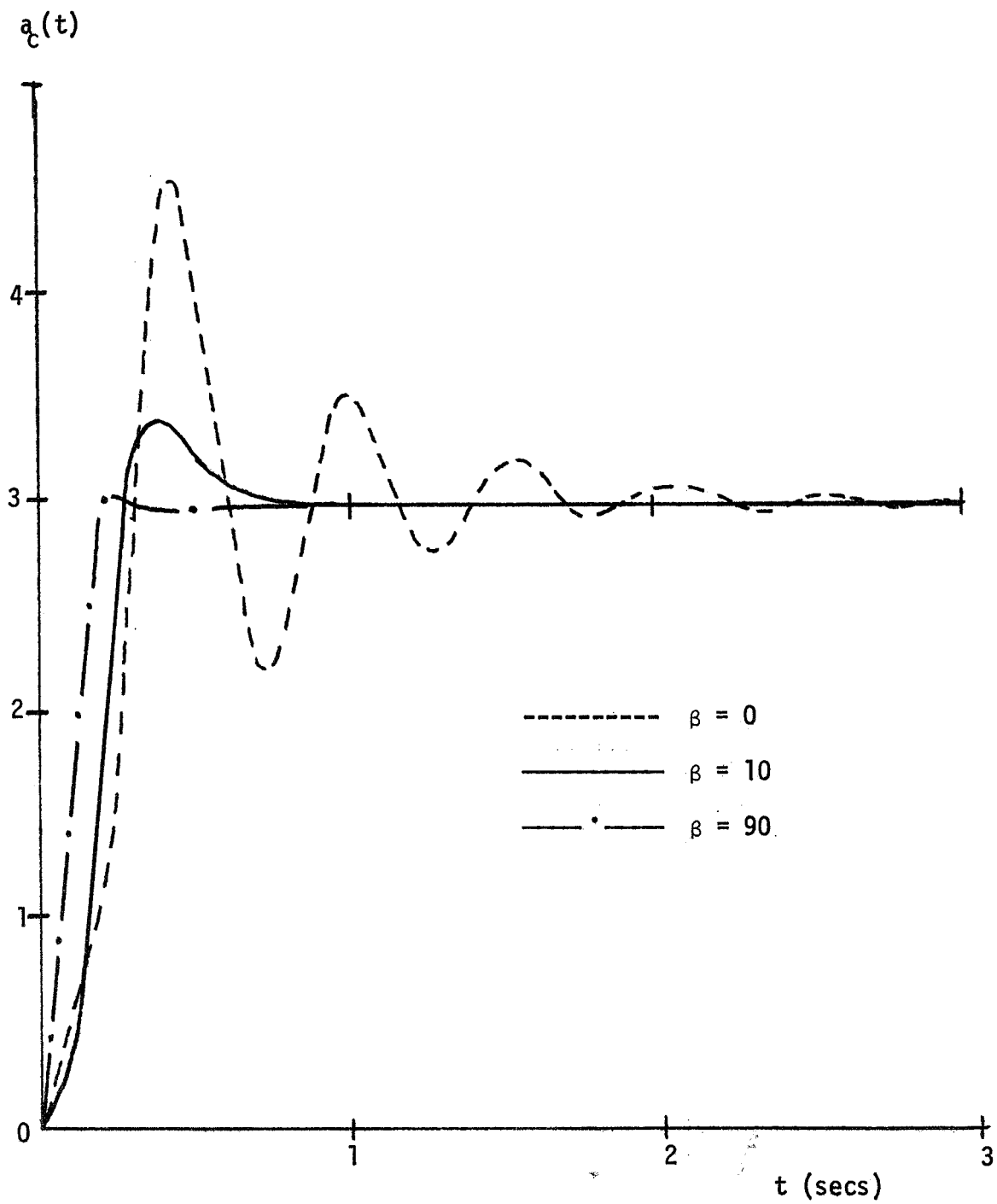


Figure 2.10. Time Response of $a_c(t)$ for First Order System.

design. It is seen that the modified design with $\beta = 90$ yields a peak value of $|e(t)|$ equal to 0.043, which is five times smaller than the peak value of the previous design (0.22). In addition, the modified design with $\beta = 10$ requires only 0.8 sec for $|e(t)|$ to reach and remain below 0.02% of the input, whereas the previous design requires almost four times as long (2.9 secs). Values of β between 10 and 90 in the modified design result in a tradeoff between settling time and peak absolute value of $e(t)$. Figure 2.9 illustrates the time response of $a_c(t)$ for the three simulations. The rapid adaption provided by the modified design for both values of β is evident. As in the previous example, the modified design does not contain the undesirable oscillations encountered in the designs in [2] and [9].

D. Remark

The design technique presented in [14] is actually a special case of the modified design presented here. (See [21].) For example, the configuration which results when the adaptive technique in [14] is applied to the system of Example 1 in Section C is shown in Figure 2.11 where γ and G are arbitrary positive constants. On the other hand, the implementation of the adaptive controller in Figure 2.4 according to the modified rule given by (2-31) is shown in Figure 2.12. It is easily seen that the configurations in Figures 2.11 and 2.12 are mathematically identical if α and β are related to γ and G by $\alpha = G$ and $\beta = \gamma G$. The use of the subsidiary input technique in [14] is, however, restricted to plants with unknown gains only and requires exact cancellation of the

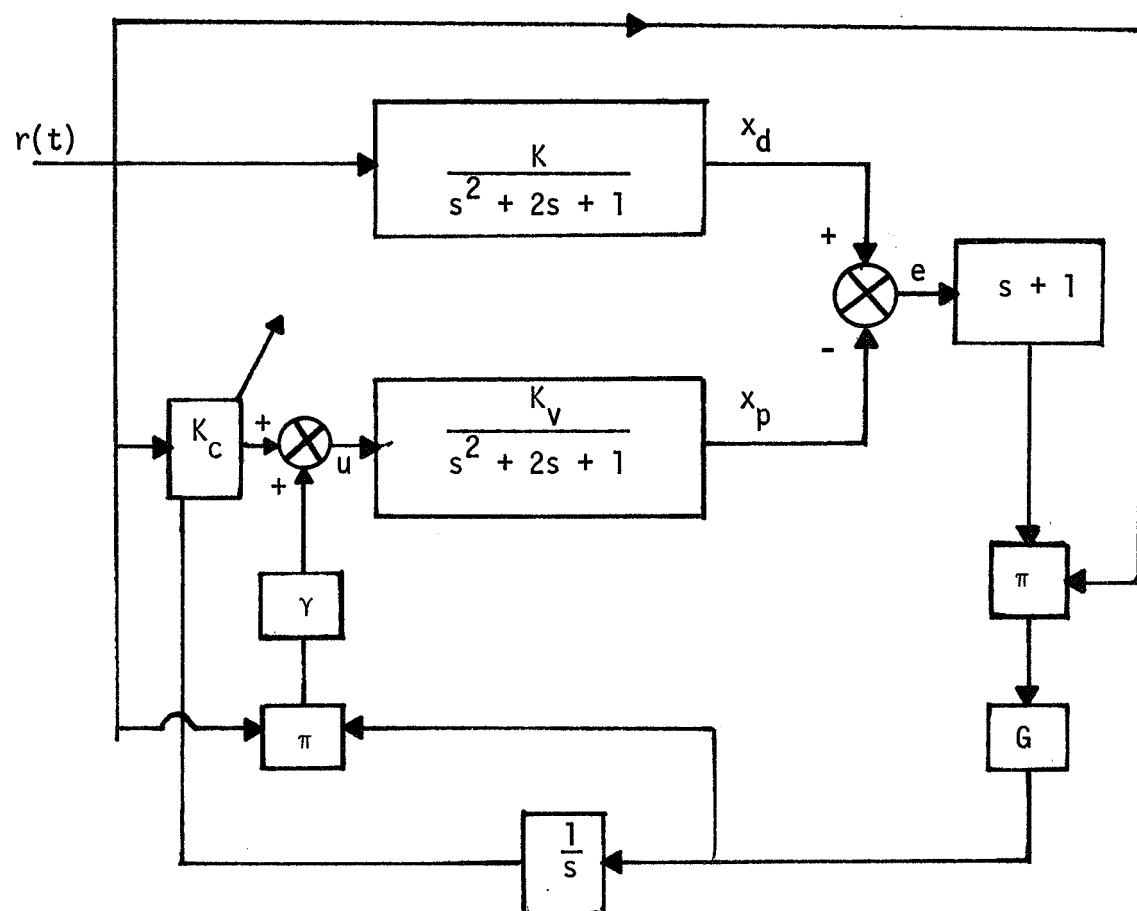


Figure 2.11. Figure 4 of Reference 14

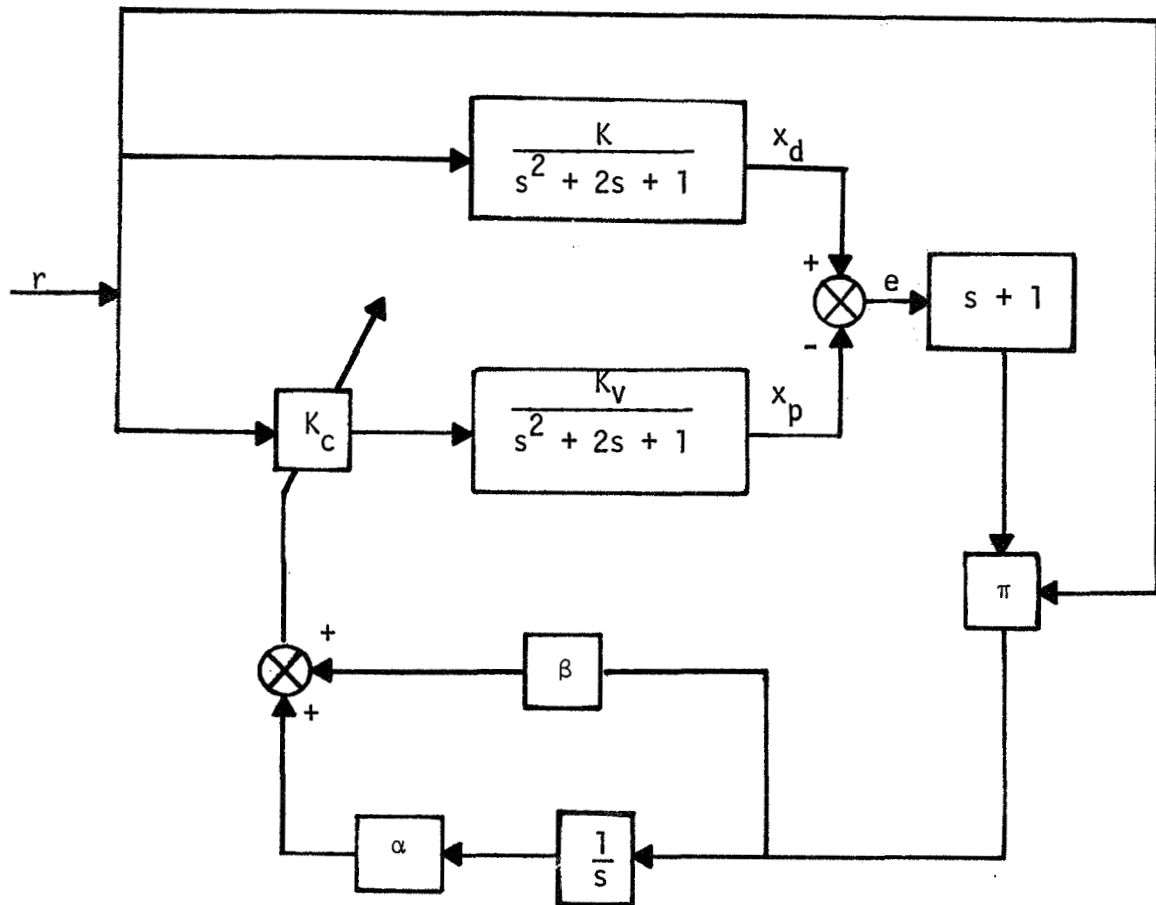


Figure 2.12. Implementation of (2-31)

plant zeros with a fixed prefilter prior to injection of the subsidiary input.

CHAPTER III

EXTERNAL PARAMETER ADJUSTMENT SCHEMES FOR PLANTS WITH UNKNOWN CONSTANT PARAMETERS

A. Motivation

In Chapter II, it was assumed that the adjustable parameters could be introduced into the plant dynamics in the manner described by (2-1) and (2-2). Oftentimes in practice this must be done by utilizing feedback and feedforward gains on the plant states and inputs, respectively. It was shown in [17], however, that the Liapunov approach suggested in [10] and [13] results in nonunique adaptive laws for single-input/single-output plants. To see this, consider the plant described by the equations:

$$\dot{\underline{z}}_p = F_p \underline{z}_p + \underline{g}_p r + \underline{g}_p^* u \quad (3-1)$$

$$x_p = \underline{h}_p^T \underline{z}_p \quad (3-2)$$

where $\underline{z}_p$ is an n -dimensional state vector defined by $\underline{z}_p^T = (z_{p1}, z_{p2}, \dots, z_{pn})$, $x_p(t)$ is the plant output, $r(t)$ is the command input, and $u(t)$ is the control input to be defined. Each element (f_{ij}^p) of the $n \times n$ matrix F_p and each element g_{pi} , g_{pi}^* , h_{pi} , respectively, of the $n \times 1$ dimensional vectors $\underline{g}_p$, $\underline{g}_p^*$ (not necessarily equal to $\underline{g}_p$) and $\underline{h}_p$ is an unknown constant parameter. The equations of the model are given by

$$\dot{\underline{z}}_d = F_d \underline{z}_d + \underline{g}_d r \quad (3-3)$$

$$x_d = \underline{h}_d^T \underline{z}_d \quad (3-4)$$

Plant compensation is provided by implementing the scalar control law in (3-5); i.e.,

$$u = \underline{k}^T \underline{x}_p + k_r r \quad (3-5)$$

where $\underline{k}^T = (k_1, k_2, \dots, k_n)$ and k_r are adjustable parameters. Substituting (3-5) into (3-1), the closed loop form of the plant becomes

$$\dot{\underline{z}}_p = [F_p - \underline{g}_p^* \underline{k}^T] \underline{z}_p + [\underline{g}_p - \underline{g}_p^* k_r] r \quad (3-6)$$

Equation (3-6) is now in the form of (2-1) with $A_p(t) = F_p - \underline{g}_p^* \underline{k}^T(t)$ and $B_p(t) = \underline{g}_p - \underline{g}_p^* k_r(t)$, where $B_p(t)$ is now a column vector. Each element, then, of $A_p(t)$ and $B_p(t)$ is defined in the form of (2-2); i.e.,

$$\begin{aligned} a_{ij}^p &= f_{ij}^p - g_{pi}^* k_j \\ b_i^p &= g_{pi} - g_{pi}^* k_r \end{aligned} \quad (3-7)$$

If (3-6) is subtracted from (3-3), an error equation in the form of (2-5) results with $\underline{e} = \underline{z}_d - \underline{z}_p$, $A = F_d - F_p + \underline{g}_p^* \underline{k}^T$ and $B = \underline{g}_d - \underline{g}_p + \underline{g}_p^* k_r$. It was assumed in [17] that $\underline{h}_d = \underline{h}_p$ and hence, if $\underline{e} = \underline{0}$, then, the desired result $e = x_d - x_p = \underline{h}_d^T \underline{e} = 0$ is achieved. For this system, it is seen that

$$\begin{aligned} \dot{a}_{ij}^p &= -\dot{a}_{ij}^p = g_{pi}^* \dot{k}_j, & i = 1, n \\ \dot{b}_i^p &= -\dot{b}_i^p = g_{pi}^* \dot{k}_r, & i = 1, n \end{aligned} \quad (3-8)$$

which, upon the application of (2-10) and (2-11) generally results in n conflicting definitions for each of the $\dot{k}_j$, $j = 1, n$ and $\dot{k}_p$. Thus, for systems in which one has access only to a set of feedback and feed-forward adjustable gains, the adaptive laws given by (2-10) and (2-13) are not directly applicable.

Several other authors [9], [11], [12], [23] have considered the parameter adaptive control of plants described by (3-1) and (3-2). The basis for synthesizing the adaptive loops in these techniques has been the minimization of some performance index via a gradient adjustment law. All of these design techniques, however, suffer from the disadvantage that the global asymptotic stability of the error e cannot be guaranteed. For example, it has been shown in [10] that the technique in [9] is unstable even for first order plants with step inputs. Furthermore, the approaches in [11] and [12] require extensive stability analyses and/or simulations in order to determine if the error will converge to zero. Finally, the method presented in [23] guarantees only local stability for stationary inputs.

B. Design Technique for Plants with Feedback and Prefilter Adjustable Gains

1. Development. The design technique presented in this section [22] alleviates the problem of conflicting adaptive laws for the class of plants described by (3-1) and (3-2). Unique adaptive laws are derived which result in a globally asymptotically stable error system for arbitrary inputs. In addition, it is not required that the elements

of the vector $\underline{h}_p$ in (3-2) be known or that the states of the system be available for measurement. The technique, however, does utilize the plant output and its first $n - m - 1$ derivatives (n th order plant with m zeros).

The class of plants to which the technique applies can be described by (3-1) and (3-2) which may be put into the form

$$x_p^{(n)} + \sum_{i=1}^n a_{ip} x_p^{(n-i)} = \sum_{i=0}^{\ell} b_{ip} r^{(m-i)} + \sum_{i=0}^m b_{ip}^* u^{(m-i)} \quad (3-9)$$

where the a_{ip} , $i = 1, n$; b_{ip} , $i = 1, \ell$; and b_{ip}^* , $i = 1, m$ are known constants with $b_{op}^* > 0$ and $\ell, m \leq n - 1$. The plant output is $x_p(t)$ and $x_p^{(i)}(t)$ denotes the i th derivative of $x_p(t)$ with $x_p^{(0)}(t) = x_p(t)$. The reference input is $r(t)$ and $u(t)$ is the control input through which the feedback and prefilter adjustable gains are to be introduced. The constants a_{ip} , $i = 1, n$; b_{ip} , $i = 1, \ell$; and b_{ip}^* , $i = 1, m$ in (3-9) can be related to the vectors $\underline{g}_p$, $\underline{g}_p^*$, $\underline{h}_p$ and the matrix F_p in (3-1) and (3-2) as follows. If $m_p(s) = \sum_{i=0}^n a_{ip} s^{(n-i)}$, $n_p(s) = \sum_{i=0}^{\ell} b_{ip} s^{(\ell-i)}$, and $n_p^*(s) = \sum_{i=0}^m b_{ip}^* s^{(m-i)}$, then

$$n_p(s)/m_p(s) = \underline{h}_p^T (sI - F_p)^{-1} \underline{g}_p \quad (3-10)$$

and

$$n_p^*(s)/m_p(s) = \underline{h}_p^T (sI - F_p)^{-1} \underline{g}_p^* \quad (3-11)$$

The desired plant characteristics are given by a stable model reference system described by

$$x_d^{(n)} + \sum_{i=1}^n a_{id} x_d^{(n-i)} = \sum_{i=0}^{\ell} b_{id} r^{(m-i)} \quad (3-12)$$

where the roots of $s^n + \sum_{i=1}^n a_{id}s^{(n-i)} = 0$ lie in the open left half s-plane. The a_{id} , $i = 1, n$ and b_{id} , $i = 1, \ell$ in (3-12) may be related to F_d , g_d , and h_d in (3-3) and (3-4) in the same manner as described by (3-10) and (3-11); i.e., if $m_d(s) = \sum_{i=1}^n a_{id}s^{(n-i)}$ and $n_d(s) = \sum_{i=1}^{\ell} b_{id}s^{(n-i)}$; then,

$$n_d(s)/m_d(s) = h_d^T(sI - F_d)^{-1}g_d \quad (3-13)$$

A differential equation for the error, $e = x_d - x_p$, is obtained by subtracting (3-9) from (3-12) which yields

$$e^{(n)} + \sum_{i=1}^n a_{id}e^{(n-i)} = w(t) \quad (3-14)$$

where

$$\begin{aligned} w(t) = & \sum_{i=1}^n (a_{ip} - a_{id})x_p^{(n-i)} + \sum_{i=0}^{\ell} (b_{id} - b_{ip})r^{(m-i)} \\ & - \sum_{i=0}^m b_{ip}^*u^{(m-i)} \end{aligned} \quad (3-15)$$

A reduction of the order of Equation (3-14) may be accomplished as follows. Take the Laplace transform of both sides of (3-15) and then divide both sides by the polynomial $P(s) = s^m + \sum_{i=1}^m c_i s^{m-i}$ where $c_i > 0$, $i = 1, m$, and the roots of $P(s) = 0$ are in the open left half s-plane. Since the coefficients a_{id} , $i = 1, n$, are real, the poles of the model transfer function are either real or they occur in complex conjugate pairs. Hence, the roots of $P(s) = 0$ may always be chosen equal to m of the model poles, i.e., the roots of $|sI - F_d| = 0$, if m is even. If m is odd and the model possesses no real poles, the root

of $P(s) = 0$ which does not cancel a root of $|sI - F_d| = 0$ is chosen in order to make A_d in (3-16) a stable matrix. For example, if $n = 2$, $m = 1$, and the two model poles are complex, $P(s)$ is given by $P(s) = s + c$ where c is chosen such that $0 < c < a_{1d}$. After the Laplace inverse is taken, the error equation may be written in phase variable form as

$$\dot{\underline{e}} = A_d \underline{e} + \underline{b}(\underline{d}^T \underline{f}_d - b_{op}^* u) \quad (3-16)$$

where $\underline{e}^T = (e, \dot{e}, \dots, e^{(n-m-1)})$, $\underline{b}^T = (0, 0, \dots, 0, 1)$, and A_d is a stable matrix (by the choice of $P(s)$ in standard form representation as defined in [24]. If $y_p(t) = \mathcal{L}^{-1}\{X_p(s)/P(s)\}$, $u_p(t) = \mathcal{L}^{-1}\{U(s)/P(s)\}$, $r_p(t) = \mathcal{L}^{-1}\{R(s)/P(s)\}$, and $p(t) = \mathcal{L}^{-1}\{1/P(s)\}$ where $X_p(s)$, $U(s)$, and $R(s)$ are the laplace transforms of x_p , u , and r , respectively; then the vector $\underline{f}_d$ is defined by

$$\underline{f}_d^T = (\underline{f}_1^T, \underline{f}_2^T, \dots, \underline{f}_5^T, e_s, r) \quad (3-17)$$

with

$$\begin{aligned} \underline{f}_1^T &= (x_p, \dot{x}_p, \dots, x_p^{(n-m-1)}) \\ \underline{f}_2^T &= (y_p, \dot{y}_p, \dots, y_p^{(m-1)}) \\ \underline{f}_3^T &= (u_p, \dot{u}_p, \dots, u_p^{(m-1)}) \\ \underline{f}_4^T &= (r_p, \dot{r}_p, \dots, r_p^{(m-1)}) \\ \underline{f}_5^T &= (p(t), \dot{p}(t), \dots, p^{(m-1)}(t)) \end{aligned} \quad (3-18)$$

and $e_s(t) = \mathcal{L}^{-1}\{E(s)/(s + s_1^0)\}$ where s_1^0 is the root of $P(s) = 0$ which does not cancel a model pole. If m is even or if the model transfer

function has a real pole, then $e_s(t)$ does not appear on the right-hand side of (3-16). The vector $\underline{d}$ is an $n + 3m + 2$ dimensional vector whose elements are comprised of plant and model parameters and initial conditions on x_p , x_d , and their $n - 1$ derivatives. For example, in the case where $n = 2$ and $m = 1$, the elements of $\underline{d}$ are

$$\begin{aligned} d_1 &= a_{1p} - a_{1d}, \quad d_2 = a_{2p} - a_{2d} - c_1(a_{1p} - a_{1d}), \\ d_3 &= b_{0p}c_1 - b_{1p}, \quad d_4 = b_{1d} - b_{1p} - c(b_{0d} - b_{0p}), \\ d_5 &= e(0) + (a_{1d} - c_1)e(0) - (a_{1p} - a_{1d})x_p(0) \\ &\quad - (b_{0d} - b_{0p})r(0) + b_{0p}u(0), \\ d_6 &= a_{1d}c_1 - a_{2d} - c_1^2, \quad d_7 = b_{0d} - b_{0p} \end{aligned} \quad (3-19)$$

The control input, u , in (3-16) is now defined as

$$u = \underline{k}^T \underline{f}_d \quad (3-20)$$

where $\underline{k}$ is a vector of adjustable parameters defined by $\underline{k}^T = (k_1, k_2, \dots, k_{n+3m+2})$ which is introduced to compensate for plant and model parameter differences represented by the vector $\underline{d}$. Substituting (3-20) into (3-16) yields

$$\dot{\underline{e}} = A_d \underline{e} + \underline{b}(\underline{d}^T - b_{0p} \underline{k}^T) \underline{f}_d \quad (3-21)$$

The system error equation, (3-21), is now in suitable form so that the development (2-6) to (2-15) may be applied to obtain the adaptive laws for the adjustable parameters

$$\underline{k} = \Lambda_1 \int_0^t \underline{f}_d \underline{e}^T \underline{q} dt + \Lambda_2 \underline{f}_d \underline{e}^T \underline{q} \quad (3-22)$$

In Equation (3-22), Λ_1 and Λ_2 are arbitrary positive definite diagonal matrices and $\underline{q} = P\underline{b}$ where P is the unique solution of $PA_d + A_d^T P = -Q$. The Liapunov theory guarantees that $\lim_{t \rightarrow \infty} \underline{e}(t) \rightarrow \underline{0}$ if the adaptive control system is implemented according to (3-21) and (3-22). However, in order to maintain a bounded control input, $u(t)$, it is necessary in practice to restrict the zeros of the plant to the open left half s -plane. In order to show this, the model equation (3-12) is alternatively described by

$$X_d(s) = (m_d(s)/n_d(s))R(s) + I_d(s)/n_d(s) \quad (3-23)$$

where $m_d(s)$ and $n_d(s)$ are defined above (3-13) and

$$I_d(s) = \sum_{j=0}^{n-1} s^{n-1-j} \sum_{i=0}^j a_{id} x_d^{(j-i)}(0). \text{ The plant equation (3-9) is}$$

similarly described by

$$X_p(s) = \frac{m_p(s)}{n_p(s)} R(s) + \frac{m_p^*(s)}{n_p(s)} U(s) + \frac{I_p(s)}{n_p(s)} \quad (3-24)$$

$$\text{where } m_p(s) = \sum_{i=0}^m b_{ip} s^{(m-i)}, \quad m_p^*(s) = \sum_{i=0}^m b_{ip}^* s^{(m-i)},$$

$$n_p(s) = s^n + \sum_{i=1}^n a_{ip} s^{(n-i)}, \text{ and } I_p(s) = \sum_{j=0}^{n-1} s^{n-1-j} \sum_{i=0}^j a_{ip} x_p^{(j-i)}(0).$$

The steady state transfer function, $U(s)/R(s)$, can be derived from the steady state condition $e(t) = x_d(t) - x_p(t) = 0$ which implies that $X_d(s) = X_p(s)$. For arbitrary inputs $r(t)$, this requires, from (3-23) and (3-24) that

$$U(s)/R(s) = \frac{m_d(s)n_p(s) - m_p(s)n_d(s) + n_p(s)I_d(s) - n_d(s)I_p(s)}{m_p^*(s)n_d(s)} \quad (3-25)$$

Hence, in order that $u(t)$ remain bounded, we require that the polynomials $m_p^*(s)$ and $n_d(s)$ be nonminimum phase. In other words, the model must be stable and the zeros of the plant transfer function $X_p(s)/U(s)$ must lie in the open left half plane.

2. Second-order example. The implementation of (3-20) for a second-order system (i.e., $n = 2$, $m = 1$) is shown in Figure 3.1 where, for convenience, b_{0p}^* and b_{1p}^* have been chosen equal to b_{0p} and b_{1p} , respectively. It is evident that the adaptive control system is building, on-line, an appropriate cascade compensator, feedback compensator, and prefilter to give the desired plant behavior. In addition, the adaptive scheme utilizes no derivative signals nor is it required that the plant transfer function be positive real (P.R.) as in [10]. Furthermore, the result in [10] applied to plants containing an unknown gain only, whereas the results here obtain for plants with $n + m + 1$ unknown parameters.

A simulation was performed on the system in Figure 3.1 using particular parameter values in order to verify the preceding theoretical results. Consider a plant in the form of (3-9) described by

$$\ddot{x}_p + \dot{x}_p + x_p = \dot{r} + 2r + \dot{u} + 2u \quad (3-26)$$

where $x_p(0) = \dot{x}_p(0) = 0$ and $r(t) = 1 + \sin t$, $t > 0$, $r(0) = 0$. The model reference, representing the desired plant characteristics, is

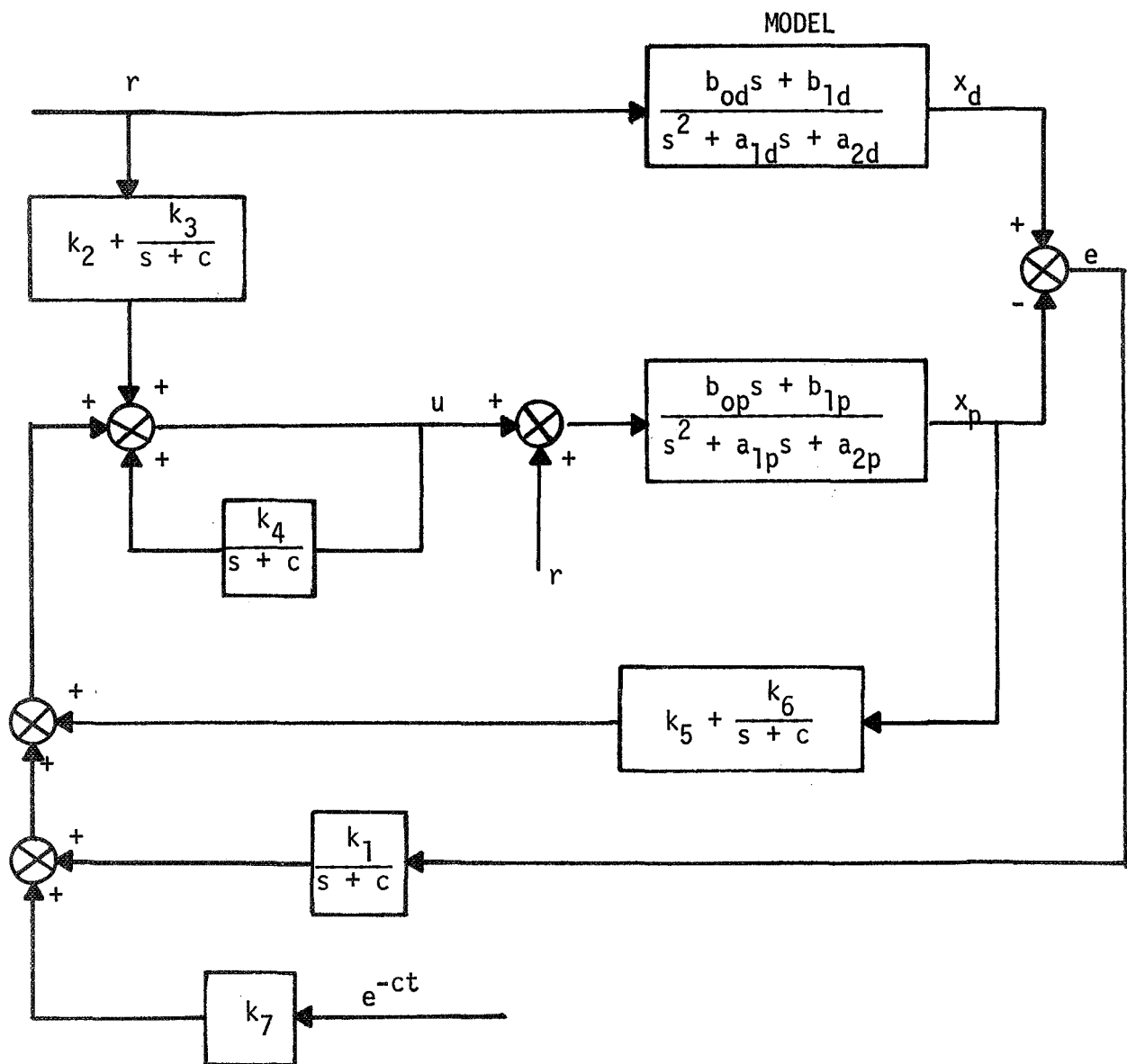


Figure 3.1. Design of Second Order System Using Feedback and Feedforward Adjustable Gains

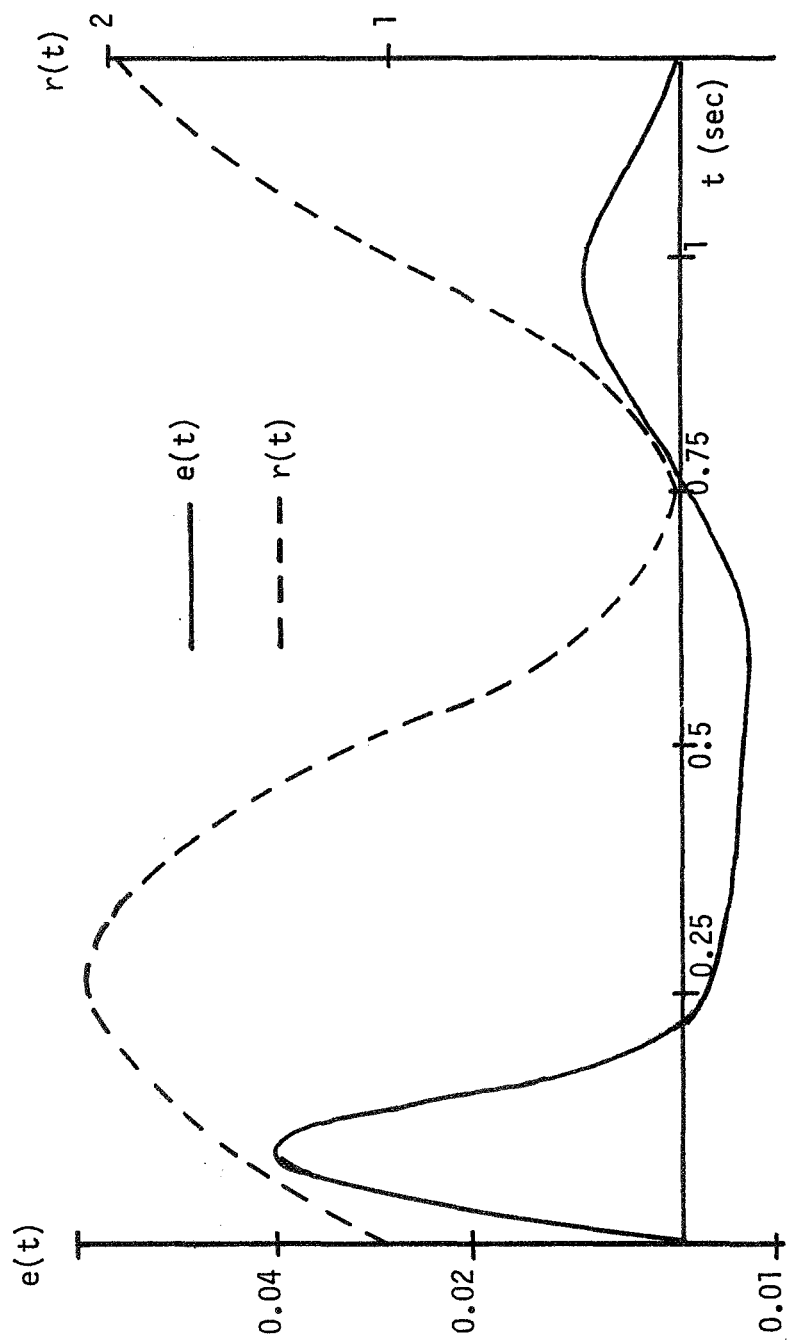


Figure 3.2. Time Response of $e(t)$ for Second Order System Using Feedback and Feedforward Adjustable Gains

defined in the form of (3-12) by

$$\ddot{x}_d + 3\dot{x}_d + 2x_d = 2\dot{r} + 6r \quad (3-27)$$

where $x_d(0) = \dot{x}_d(0) = 0$. Applying the development (3-14) to (3-22) with $P(s) = s + 1$ and $\gamma_i/b_{0p} = 100$, $\delta_i/b_{0p} = 10$ for $i = 1 - 4, 7$ yields the following adaptive laws for the adjustable parameters, k_i

$$k_i = 100 \int_0^t e f_{di} dt + 10 e f_{di}, \quad i = 1 - 4, 7 \quad (3-28)$$

where $k_i(0) = 0$, $i = 1 - 4, 7$. In this example, it is not necessary to introduce the adjustable parameters k_5 and k_6 since $d_5 = 0$ and $d_6 = 0$, respectively. It is not necessary to introduce k_5 since $e(0) = \dot{e}(0) = x_p(0) = r(0) = u(0) = 0$ and k_6 is not necessary since the root of $P(s) = 0$; i.e., -1 is one of the poles of the model. The error response, $e(t)$, and the input, $r(t)$, are plotted in Figure 3.2 for $0 \leq t \leq 1.2$ sec illustrating the rapid convergence provided by the adaptive scheme. After 1.2 sec it was found that the absolute value of the error remained below 0.003.

C H A P T E R I V

PLANTS CONTAINING UNKNOWN TIME-VARYING PARAMETERS

A. Development for Plants without Input Differential Operators

The research presented in Chapters II and III, as well as the work reported in [11] - [18], [20] - [22], dealt only with the control of plants whose unknown parameters were constant. In many applications, such as the control of tactical missiles, this could be a severe limitation [3]. The design technique to be developed here, which was summarized in [25], extends the Liapunov parameter adaptive approach to plants whose unknown parameters are time-varying.

The technique is perhaps best illustrated by its application to plants of the form given in (4-1). The case when the plant contains differential operators acting on the input is considered in Section B. Consider the single-input/single-output plant described by the equation

$$x_p^{(n)} + \sum_{i=1}^n a_{ip}(t) x_p^{(n-i)} = b_p(t)r + b_p^*(t)u \quad (4-1)$$

where instead of constant parameters as in (3-9), the $a_{ip}(t)$, $i = 1, n$, $b_p^*(t)$ (> 0), and $b_p(t)$ (not necessarily equal to $b_p^*(t)$) are unknown time-varying parameters bounded by $A_{\ell i} \leq a_{ip}(t) \leq A_{hi}$, $B_{\ell}^* \leq b_p^*(t) \leq B_h^*$, and $B_{\ell} \leq b_p(t) \leq B_h$, respectively. It is assumed as in (3-9) that the desired plant behavior is represented by a stable linear, time-invariant model described by

$$x_d^{(n)} + \sum_{i=1}^n a_{id} x_d^{(n-i)} = b_d r \quad (4-2)$$

where the a_{id} , $i = 1, n$ and b_d are known constants. A particular form for the differential equation of the error, $e = x_d - x_p$, is obtained by subtracting (4-1) from (4-2) and dividing by $b_p^*(t)$ (> 0), which yields

$$\begin{aligned} & \left(\frac{1}{b_p^*(t)} \right) e^{(n)} + \frac{1}{b_p^*(t)} \sum_{i=1}^n a_{id} e^{(n-i)} \\ &= \sum_{i=1}^n \left(\frac{a_{ip}(t) - a_{id}}{b_p^*(t)} \right) x_p^{(n-i)} + \left(\frac{b_d - b_p(t)}{b_p^*(t)} \right) r - u \end{aligned} \quad (4-3)$$

The control input, $u(t)$, is to be defined in a manner which insures that $\|e\| \leq \epsilon$ where $\|\cdot\|$ is a suitable norm and ϵ is a positive constant which can be made as small as necessary by adjusting certain fixed design parameters. This kind of stability is generally termed "strong practical stability." The design procedure is as follows. Rewrite the parameter differences on the right-hand side of (4-3) as the sums of constants plus time-varying unknowns.

$$\begin{aligned} \frac{a_{ip}(t) - a_{id}}{b_p^*(t)} &= f_{i0} + f_i(t), \quad i = 1, n \\ \frac{b_d - b_p(t)}{b_p^*(t)} &= g_0 + g(t) \end{aligned} \quad (4-4)$$

where the f_{i0} , $i = 1, n$ and g_0 are unknown constants, and the $f_i(t)$, $i = 1, n$, and $g(t)$ are unknown time-varying deviations about the f_{i0} , $i = 1, n$ and g_0 , respectively, with $|f_i(t)| \leq F_i$ and $|g(t)| \leq G$ for all t . The F_i , $i = 1, n$ and G may be determined from the upper and lower bounds on the $a_{ip}(t)$, $i = 1, n$, $b_p(t)$, and $b_p^*(t)$. The control input, $u(t)$, is now defined as

$$u = \sum_{i=1}^n k_i x_p^{(n-i)} + k_r r \quad (4-5)$$

where each of the adjustable parameters k_i , $i = 1, n$ and k_r may be written as the sum of two terms; i.e., $k_i = k_i^I + k_i^P$, $i = 1, n$ and $k_r = k_r^I + k_r^P$. The superscript I denotes the fact that k_i^I , $i = 1, n$ and k_r^I are to be integral functions of the error states while the superscript P in k_i^P , $i = 1, n$ and k_r^P is indicative of a proportional relationship to the error states. When Equations (4-4) and (4-5) are substituted into (4-3), and the error system is rewritten in phase variable canonic form, the result is Equation (4-6).

$$\left(\frac{1}{b_p} \right) \dot{\underline{e}} = \left(\frac{1}{b_p} \right) A_d \underline{e} + \underline{b} w(t) \quad (4-6)$$

where

$$\begin{aligned} w(t) = & \sum_{i=1}^n (f_{i0} - k_i^I) x_p^{(n-i)} + (g_0 - k_r^I) r \\ & + \sum_{i=1}^n (f_i(t) - k_i^P) x_p^{(n-i)} + \sum_{i=1}^n (g(t) - k_r^P) r \end{aligned} \quad (4-7)$$

and the pair $(A_d, \underline{b})$ are in phase variable canonic form defined by

$$A_d = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \\ -a_{n,d} & -a_{n-1,d} & \dots & -a_{1,d} \end{bmatrix} \underline{b}^T = (0, 0, \dots, 1) \quad (4-8)$$

The structure of the error equation (4-6) is such that adaptive laws for the k_i , $i = 1, n$ and k_r may be defined so that the k_i^I , $i = 1, n$

and k_r^I will compensate for the constant part of the unknown quantities while the k_i^P , $i = 1, n$ and k_r^P are to compensate for the rapid variations of these quantities. To this end, a potential Liapunov function is chosen, which is a positive definite function of $\underline{e}$ and includes $n + 1$ semidefinite forms in certain parameter differences. It is given by

$$V = \frac{1}{b_p} \underline{e}^T P \underline{e} + \sum_{i=1}^n \lambda_i (f_{i0} - k_i^I)^2 + \lambda_r (g_0 - k_r^I)^2 \quad (4-9)$$

where $\underline{e}^T = (e, \dot{e}, \dots, e^{(n-1)})$, P is a positive definite symmetric matrix to be determined and λ_i , $i = 1, n$ and λ_r are arbitrary positive constants.

The adjustable parameters are now defined by

$$\begin{aligned} k_i^I &= \alpha_i \int_0^t \underline{e}^T P \underline{b} x_p^{(n-i)} dt + k_i^I(0), \quad i = 1, n \\ k_r^I &= \alpha_r \int_0^t \underline{e}^T P \underline{b} r dt + k_r^I(0) \\ k_i^P &= M_i \text{ sat}[(\beta_i/M_i) \underline{e}^T P \underline{b} x_p^{(n-i)}], \quad i = 1, n \\ k_r^P &= N \text{ sat}[(\beta_r/N) \underline{e}^T P \underline{b} r] \end{aligned} \quad (4-10)$$

where the $\alpha_i = 1/\lambda_i$, $\beta_r = 1/\lambda_r$, β_i and β_r are arbitrary positive constants and $M_i \geq F_i$, $N \geq G$. The sat function in (4-10) is defined by $\text{sat}(z) = 1$ if $|z| > 1$ and $\text{sat}(z) = z$ if $|z| \leq 1$. The time derivative of (4-9) with the definitions in (4-10) becomes

$$\dot{V} = -(b_p^*/b_p^{*2}) \underline{e}^T P \underline{e} - (1/b_p^*) \underline{e}^T Q \underline{e} + y(t) \quad (4-11)$$

where $PA_d + A_d^T P = -Q$, Q positive definite, and $y(t)$ is given by

$$y(t) = 2z \sum_{i=1}^n x_p^{(n-i)} \left\{ f_i(t) - M_i \operatorname{sat} \left[\frac{\beta_i z x_p^{(n-i)}}{M_i} \right] \right\} \\ + 2zr \left\{ g(t) - N \operatorname{sat} \left[\frac{\beta_r z r}{N} \right] \right\} \quad (4-12)$$

with $z = \underline{e}^T P \underline{b}$. There are two cases for which $\dot{V}(t)$ in (4-11) must be examined: (1) $\dot{b}_p^* \geq 0$, and (2) $\dot{b}_p^* < 0$. In the first case, $\dot{V} < 0$ if $\underline{e}^T Q \underline{e} > b_p^* y(t)$. In the second case, it can be shown, by integrating both sides of (4-11), that

$$V(t) \leq \frac{V(0)b_p^*(0)}{B_\ell^*} \exp \left[- \int_0^t \left([(1/b_p^*) \underline{e}^T Q \underline{e} - y(t)]/V \right) dt \right] \quad (4-13)$$

Hence, V is bounded by a decreasing exponential if $\underline{e}^T Q \underline{e} > b_p^* y(t)$. In order to obtain a bound then on $\underline{e}^T Q \underline{e}$, an upper bound on $b_p^*(t)y(t)$ must be obtained. Consider the second term in (4-12). When $|\beta_r z r/N| > 1$, $\operatorname{sat}[\beta_r z r/N] = 1$ and since $|g(t)| \leq G < N$, it follows that

$$J(t) = 2zr(g(t) - N \operatorname{sat}[\beta_r z r/N]) < 0 \quad (4-14)$$

On the other hand, when $|\beta_r z r/N| < 1$, $\operatorname{sat}[\beta_r z r/N] = \beta_r z r/N$, and

$$J(t) = 2zr(g(t) - \beta_r z r) \quad (4-15)$$

In this case, setting $\partial J / \partial z r = 0$ yields the fact that $J(t)$ attains its maximum value when $z r = g(t)/2\beta_r$. This implies that for all t ,

$$J(t) \leq g^2(t)/2\beta_r \leq G^2/2\beta_r \quad (4-16)$$

The other n terms in (4-12) may be bounded in a similar manner and since

$b_p^*(t) \leq B_h^*$; then,

$$b_{op}^*(t)y(t) \leq B_h^*/2 \left(\sum_{i=1}^n D_i^2/\beta_i + G^2/\beta_r \right) \quad (4-17)$$

Hence, $V(t)$ will decrease until

$$\underline{e}^T Q \underline{e} \leq \frac{B_h^*}{2} \left(\sum_{i=1}^n \frac{D_i^2}{\beta_i} + \frac{G^2}{\beta_r} \right) \quad (4-18)$$

Therefore, (4-9) is a Liapunov function for (4-6). Since

$\underline{e}^T Q \underline{e} \geq \lambda_q \underline{e}^T \underline{e}$, where λ_q is the smallest eigenvalue of Q and an appropriate norm on the vector $\underline{e}$ is $\|\underline{e}\| = (\underline{e}^T \underline{e})^{1/2}$, it is seen from (4-18) that

$\|\underline{e}\| \leq \epsilon$ where $\epsilon = \left[(B_h^*/2\lambda_q) \left(\sum_{i=1}^n (D_i^2/\beta_i) + (G^2/\beta_r) \right) \right]^{1/2}$ can be made as

small as necessary by increasing the arbitrary positive gains β_i ,

$i = 1, n$ and β_r .

B. Plants with Differential Operators Acting on the Input

In this section, the design approach is extended to plants with mth order input differential operators. It is shown that the adaptive laws may be defined on a reduced state space error vector at the expense of requiring that certain derivatives of the unknown parameters be bounded. Consider the plant

$$\begin{aligned} x_p^{(n)} + \sum_{i=1}^n a_{ip}(t)x_p^{(n-i)} &= \sum_{i=0}^{\ell} b_{ip}(t)r^{(m-i)} \\ &+ \sum_{i=0}^m b_{ip}^*(t)u^{(m-i)} \end{aligned} \quad (4-19)$$

where the definitions of the $a_{ip}(t)$, x_p , r , and u are the same as in (4-1) and the $b_{ip}(t)$ and $b_{ip}^*(t)$ are time-varying unknown parameters bounded by $B_{li} \leq b_{ip}(t) \leq B_{hi}$, and $B_{li}^* \leq b_{ip}^*(t) \leq B_{hi}^*$ with $B_{l0}^* > 0$. The stable linear time-invariant model for (4-19) is given by

$$x_d^{(n)} + \sum_{i=1}^n a_{id} x_d^{(n-i)} = \sum_{i=0}^{\ell} b_{id} r^{(\ell-i)} \quad (4-20)$$

where the a_{id} , $i = 1, n$ and b_{id} , $i = 0, m$ are known constants. If the error $e = x_d - x_p$ is defined as before, and (4-19) is subtracted from (4-20), the error equation in (4-21) is obtained.

$$\begin{aligned} e^{(n)} + \sum_{i=1}^n a_{id} e^{(n-i)} &= \sum_{i=0}^{\ell} (b_{id} - b_{ip}) r^{(\ell-i)} \\ &+ \sum_{i=1}^n (a_{ip} - a_{id}) x_p^{(n-i)} \\ &- \sum_{i=0}^m b_{ip}^* u^{(m-i)} \end{aligned} \quad (4-21)$$

From here on, this approach deviates from the development (4-4) to (4-18) since this would lead to adaptive laws which require m th order derivatives of the reference input, $r(t)$. Instead, a reduction of the order of (4-21) is performed to eliminate its dependence on the derivatives of $r(t)$ and the $(n - m + 1)$ th to n th order derivatives of $e(t)$. In order to accomplish this reduction of order, the procedure is, as in Chapter III, to take the Laplace transform of (4-21) and divide both sides of the equation by the polynomial $P(s) = s^m + \sum_{i=1}^m c_i s^{m-i}$ where $c_i > 0$, $i = 1, n$, and the roots of $P(s) = 0$ lie on the negative real axis of the s -plane. After the Laplace inverse is taken, it is shown

in Appendix A that Equation (4-21) may be written as

$$\dot{\underline{e}} = A_d \underline{e} + \underline{b}(v_1 + v_2) \quad (4-22)$$

where $v_1(t)$ is defined by

$$\begin{aligned} v_1^{(m)} + \sum_{i=1}^m c_i v_1^{(m-i)} &= \sum_{i=0}^{\ell} (b_{id} - b_{ip}) r^{(\ell-i)} \\ &\quad + \sum_{i=1}^{m+1} (a_{ip} - a_{id}) x_p^{(n-i)} \\ &= \sum_{i=0}^m b_{ip}^* u^{(m-i)} \end{aligned} \quad (4-23)$$

and $v_2(t)$ is defined by its Laplace transform as

$$\begin{aligned} v_2(s) &= p^{-1}(s) \left[\sum_{i=0}^{m-1} \gamma_i s^i + \mathcal{L} \left(\sum_{i=m+2}^n (a_{ip} - a_{id}) x_p^{(n-i)} \right) \right] \\ &\quad - H(s)E(s) \end{aligned} \quad (4-24)$$

In (4-22), the pair $(A_d, \underline{b})$ is in phase variable canonic form and A_d is stable and in (4-24), $H(s)$ is a known transfer function which can be generated with $(n - m - 1)$ th order derivative networks (see Appendix A). The γ_i are functions of the initial conditions on the error and its derivatives and $\underline{b}$ and $\underline{e}$ are defined as in (4-6). In order to eliminate the dependence of $v_1(t)$ on the m th order derivatives of r and u and the $(n - m + 1)$ th to n th order derivatives of x_p , a particularly useful state-variable formulation in [26] is utilized. In [26] it is shown that (4-23) may be represented by the state equations

$$\begin{aligned}\dot{\underline{w}} &= A_c \underline{w} + \underline{z}_w \\ v_1(t) &= \underline{d}^T \underline{w}(t) + \phi_w\end{aligned}\quad (4-25)$$

where $\underline{z}_w(t) = \underline{g}_x x_p^{(n-m-1)} + \underline{g}_r r + \underline{g}_u u$ and $\phi_w(t) = (a_{1p} - a_{1d})x_p^{(n-m-1)} + (b_{od} - b_{op})r - b_{op}^* u$. The matrix A_c is the (stable) companion matrix of $P(s)$ in standard form representation which is defined, along with the vector $\underline{d}$ in (4-26).

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -c_m & -c_{m-1} & \dots & & -c_1 \end{bmatrix}, \quad \underline{d} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (4-26)$$

The quantities $\underline{g}_x$, $\underline{g}_r$, and $\underline{g}_u$ are m -dimensional column vectors whose elements are generated according to (4-27). These elements are

$$\begin{aligned}g_{ri} &= b_{id} - b_{ip}(t) - S_i(g_{rj}), & i &= 1, m \\ g_{ui} &= b_{ip}(t) - S_i(g_{uj}), & i &= 1, m \\ g_{xi} &= a_{i+1,p}(t) - a_{i+1,d} - S_i(g_{xj}), & i &= 1, m\end{aligned}\quad (4-27)$$

where

$$S_i(g_j) = \sum_{j=0}^{i-1} \sum_{k=0}^{i-j} c_{i-j-k} \left(\begin{matrix} n+k+i \\ n-1 \end{matrix} \right) \frac{d^k g_j}{dt^k}$$

with $g_{x0} = a_{1p} - a_{1d}$, $g_{r0} = b_{od} - b_{ip}$, $g_{u0} = -b_{op}^*$. It is assumed that bounds on the g_{ri} , g_{ui} , and g_{xi} exist and they are given by

$|g_{ri}| \leq G_{ri}$, $|g_{ui}| \leq G_{ui}$, and $|g_{xi}| \leq G_{xi}$. It follows from (4-25) that

$\underline{d}^T \underline{w}(t)$ may be rewritten as

$$\underline{d}^T \underline{w}(t) = \mathcal{L}^{-1}\{\underline{q}^T(s) \underline{z}_w(s)\} + \mathcal{L}^{-1}\{\underline{q}^T(s) \underline{w}(0)\} \quad (4-28)$$

where $\underline{q}^T(s) = \underline{d}^T(sI - A_c)^{-1}$. The elements w_i , $i = 1, m$ of the vector $\underline{w}(0)$ are defined from (4-22) and are given by

$$w_i(0) = -\phi^{(i)}(0) - \sum_{j=1}^{m-1} z_{wj}^{(m-i)}(0), \quad i = 1, m \quad (4-29)$$

Utilizing (4-25) to (4-28), the function $(v_1 + v_2)/b_{op}^*$ can be written as

$$\frac{v_1 + v_2}{b_{op}^*} = \left(\frac{a_{1p} - a_{1d}}{b_{op}^*} \right) x_p^{(n-m-1)} + \left(\frac{b_{od} - b_{op}}{b_{op}^*} \right) r - u - y \quad (4-30)$$

where

$$\begin{aligned} y(t) = & \frac{1}{b_{op}^*} \mathcal{L}^{-1}[\underline{q}^T(s) \mathcal{L}\{g_x x_p^{(n-m-1)} + g_r r - g_u u\}] \\ & + \frac{1}{b_{op}^*} \mathcal{L}^{-1}[H(s)E(s)] \\ & + \frac{1}{b_{op}^*} \mathcal{L}^{-1}\left\{p^{-1}(s) \mathcal{L}\left\{\sum_{i=m+2}^n (a_{ip} - a_{id}) x_p^{(n-i)}\right\}\right\} \\ & + I_c(t) \end{aligned} \quad (4-31)$$

and $I_c(t)$ is given by

$$I_c(t) = \frac{1}{b_{op}^*} \mathcal{L}^{-1}\left\{p^{-1}(s) \sum_{i=0}^{m-1} \gamma_i s^i + \underline{q}^T(s) \underline{w}(0)\right\} \quad (4-32)$$

The parameter differences in (4-30) can be decomposed in a manner similar to (4-4). For instance, the difference appearing in the first term on the right-hand side of (4-30) is written

$$\left(\frac{a_{1p} - a_{1d}}{b_{op}^*} \right) = a_1^c + a_1^v(t) \quad (4-33)$$

and similarly decomposing the unknown parameters in the first term of $y(t)$ yields

$$\frac{1}{b_{op}^*} = b_2^c + b_2^v(t) \quad (4-34)$$

$$g_{x1} = g_2^c + g_2^v(t)$$

where a_1^c , b_2^c , and g_2^c are unknown constants and $a_1^v(t)$, $b_2^v(t)$, and $g_2^v(t)$, respectively, are variations about these constants. Hence, it is seen that

$$\left(\frac{a_{1p} - a_{1d}}{b_{op}^*} \right) x_p^{(n-m-1)} = d_1^0 x_p^{(n-m-1)} + d_1(t) \quad (4-35)$$

where $d_1^0 = a_1^c$ and $d_1(t) = a_1^v(t) x_p^{(n-m-1)}$, while the first term on the right-hand side of (4-31) is written as

$$\begin{aligned} & \frac{1}{b_{op}^*} \mathcal{L}^{-1} \{ q_1(s) \mathcal{L}(g_{x1} x_p^{(n-m-1)}) \} \\ &= d_2^0 \mathcal{L}^{-1} \{ q_1(s) \mathcal{L}(x_p^{(n-m-1)}) \} + d_2(t) \end{aligned} \quad (4-36)$$

where d_2^0 is a constant given by $d_2^0 = b_2^c g_2^c$ and

$$\begin{aligned} d_2(t) &= b_2^c \mathcal{L}^{-1} \{ q_1(s) \mathcal{L}[g_2^v(t) x_p^{(n-m-1)}] \} \\ &+ (1/b_{op}^*) \mathcal{L}^{-1} \{ q_1(s) \mathcal{L}[g_2^v(t) x_p^{(n-m-1)}] \} + g_2^c b_2^v(t) \mathcal{L}^{-1} \{ q_1(s) \mathcal{L}[x_p^{(n-m-1)}] \}. \end{aligned}$$

The signal $I_c(t)$ is handled in the following manner. Since

$$q^T(s) = ADJ(sI - A_c)/P_s, \quad I_c(t) \text{ can be written as}$$

$$I_c(t) = \frac{1}{b_{op}^*} \mathcal{L}^{-1}\{P^{-1}(s)Q(s)\} \quad (4-37)$$

where $Q(s) = \sum_{i=0}^{m-1} \gamma_i s^i + ADJ(sI - A_c)w(0)$. Expanding $Q(s)/P(s)$ into partial fractions yields

$$Q(s)/P(s) = \sum_{i=1}^m A_i/(s + s_i^0) \quad (4-38)$$

if the s_i^0 , $i = 1, m$ are distinct. In (4-38), the A_i , $i = 1, m$ are unknown constants and $-s_i^0$, $i = 1, m$ are the roots of $P(s) = 0$ with the s_i^0 being real and positive due to the choice of $P(s)$. If the s_i^0 are not distinct, the right-hand side of (4-38) contains terms of the form $B_i/(s + s_i^0)^\ell$, $\ell = 2, 3, \dots$, where the B_i are unknown constants.

Utilizing (4-38), $I_c(t)$ becomes

$$I_c(t) = \frac{1}{b_{op}^*} \sum_{i=1}^m A_i e^{-s_i^0 t}; \quad t \geq 0 \quad (4-39)$$

and in lieu of the first equation in (4-34), Equation (4-39) can be written as

$$I_c(t) = \sum_{i=1}^m d_{j+i}^0 e^{-s_i^0 t} + \sum_{i=1}^m d_{j+i}(t); \quad t \geq 0 \quad (4-40)$$

where $j = n + 2m + 2$, $d_{j+i}^0 = b_2^c A_i$, and $d_{j+i}(t) = b_2^v(t) A_i$.

Since all of the terms in (4-30) and (4-31) can be written in a manner similar to (4-35), (4-36), or (4-40), the function $(v_1 + v_2)/b_{op}^*$ in (4-30) and (4-31) can be alternatively described by

$$\frac{v_1 + v_2}{b_{op}^*} = \sum_{i=1}^{n+3m+2} d_i^0 h_i + \sum_{i=1}^{n+3m+2} d_i(t) - u \quad (4-41)$$

where the h_i , $i = 1, n + 3m + 2$ are a sequence of filtered signals which are given by $h_1 = x_p^{(n-m-1)}$; $h_i = \mathcal{L}^{-1}\{q^T(s)\mathcal{L}(x_p^{(n-m-1)})\}$, $i = 2, m + 1$; $h_i = \mathcal{L}^{-1}\{p^{-1}(s)\mathcal{L}(x_p^{(n-i)})\}$, $i = m + 2, n$; $h_{n+1} = r$; $h_i = \mathcal{L}^{-1}\{q^T(s)R(s)\}$, $i = n + 2, n + m + 1$; $h_i = \mathcal{L}^{-1}\{q^T(s)U(s)\}$, $i = n + m + 2, n + 2m + 1$; $h_{n+2m+2} = \mathcal{L}^{-1}\{H(s)E(s)\}$; $h_i = e^{-s_0^i t}$, $i = n + 2m + 3, n + 3m + 2$.

The control input, $u(t)$, in (4-41) is defined by

$$u = \sum_{i=1}^{n+3m+2} k_i^I h_i + \sum_{i=1}^{n+3m+2} k_i^P h_i^! \quad (4-42)$$

where the $h_i^!$, $i = 1, n + 3m + 2$ are obtained from the h_i , $i = 1, n + 3m + 2$, respectively, by placing absolute value signs on the signal associated with that term. For example, $h_1^! = |x_p^{(n-m-1)}|$; $h_2^! = \mathcal{L}^{-1}\{q_1(s)\mathcal{L}(|x_p^{(n-m-1)}|)\}$, $\dots$, $h_{n+2m+2}^! = |\mathcal{L}^{-1}\{H(s)E(s)\}|$, $h_i^! = h_i$, $i = n + 2m + 3, n + 3m + 2$. Substituting (4-41) and (4-42) into (4-22) and dividing both sides of (4-22) by b_{op}^* yields an equation similar to (4-6); i.e.,

$$\left(\frac{1}{b_{op}^*}\right) \dot{\underline{e}} + \left(\frac{1}{b_{op}^*}\right) A_d \underline{e} = \underline{b}w(t) \quad (4-43)$$

where

$$w(t) = \sum_{i=1}^{n+3m+2} (d_i^0 - k_i^I) h_i + \sum_{i=1}^{n+3m+2} (d_i(t) - k_i^P h_i^!) \quad (4-44)$$

The development (4-7) through (4-18) can now be applied to (4-43), yielding the following adaptive laws:

$$\begin{aligned} k_i^I(t) &= \alpha_i \int_0^t \underline{e}^T P b h_i dt + k_i^I(0), \quad i = 1, n + 3m + 2 \\ k_i^P(t) &= M_i \text{ sat}[(\beta_i/M_i) \underline{e}^T P b h_i^!], \quad i = 1, n + 3m + 2 \end{aligned} \quad (4-45)$$

where the α_i , β_i , $k_i^I(0)$, and the matrix P are defined as in (4-9) and (4-10), $\underline{b}$ is defined in (4-8), and the $M_i \geq D_i$, the D_i , $i = 1, n + 2m + 3$ being defined below. In this case, the error is bounded by

$$\|e\| \leq \epsilon \quad (4-46)$$

where $\epsilon = \left(\frac{B_{h0}^*}{2\lambda q} \sum_{i=1}^{n+3m+2} D_i^2 / \beta_i \right)^{1/2}$. The D_i , $i = 1, n + 3m + 2$ are upper bounds obtained from the $d_i(t)$, $i = 1, n + 3m + 2$. For example, D_1 and D_2 are defined by (4-47) and (4-48) as follows:

$$|d_1(t)| \leq D_1 |x_p^{(n-m-1)}| = D_1 h_1^1(t) \quad (4-47)$$

and

$$|d_2(t)| \leq D_2 \int_0^t q_1(t - \tau) |x_p^{(n-m-1)}(\tau)| d\tau = D_2 h_2^1(t) \quad (4-48)$$

where $D_1 = \max_t a_1^v(t) \leq (a_{1d} - A_{\ell 1})/B_{\ell 0}^*$, and $D_2 = [\max_t g_{x1}(t)]/[\min_t b_{op}^*] = G_{x1}/B_{\ell 0}^*$. Equation (4-48) follows from the fact that $q_1(t) = \mathcal{L}^{-1}(q_1(s)) \geq 0$ for all time. This is shown in Appendix B. It is also shown in Appendix B that $q_i(t) = \mathcal{L}^{-1}(q_i(s))$, $i = 2, m$ and $p(t) = \mathcal{L}^{-1}(1/P(s))$ are greater than or equal to zero for all t . Therefore, following Equations (4-47) and (4-48), the $d_i(t)$, $i = 1, n + 3m + 2$ are bounded by

$$|d_i(t)| \leq D_i h_i^1(t) \quad (4-49)$$

Hence, the adaptive control of plants in the form of (4-19) may be accomplished utilizing $n - m - 1$ output derivatives. Bounds are required on some of the derivatives of the unknown parameters.

Similar to the case of plants with unknown constant parameters which was discussed in Chapter III, it is necessary to examine the boundedness of the control input u . From Equation (4-19), it is seen that

$$\sum_{i=0}^m b_{ip}^* u^{(m-i)} = x_p^{(n)} + \sum_{i=1}^n a_{ip} x_p^{(n-i)} - \sum_{i=0}^{\ell} b_{ip} r^{(m-i)} \quad (4-50)$$

At the completion of adaptation, $\underline{e}$ is bounded by (4-33) and

$$\begin{aligned} \sum_{i=0}^m b_{ip}^* u^{(m-i)} &= x_d^{(n)} + \sum_{i=1}^n a_{ip} x_d^{(n-i)} - \sum_{i=0}^{\ell} b_{ip} r^{(m-i)} \\ &+ \varepsilon(t) \end{aligned} \quad (4-51)$$

where $\varepsilon(t)$, $|\varepsilon(t)| < E$, arises from the difference between the model and plant outputs. Since r is bounded and the reference model is stable, to determine if $u(t)$ is bounded, it is necessary to ascertain the bounded input-bounded output stability (b.i.b.o.) of the equation

$$\sum_{i=0}^m b_{ip}^* u^{(m-i)} = \mu(t) \quad (4-52)$$

where $\mu(t) = x_d^{(n)} + \sum_{i=1}^n a_{ip} x_d^{(n-i)} - \sum_{i=0}^{\ell} b_{ip} r^{(m-i)} + \varepsilon(t)$. It is shown in [27] that (4-52) is b.i.b.o. stable if the solution of the homogeneous equation

$$\sum_{i=0}^m b_{ip}^* \phi^{(m-i)} = 0 \quad (4-53)$$

goes to zero at an exponential rate; i.e., $|\phi(t)| \leq M e^{-\lambda t}$ for some positive constants M and λ . For example, if $m = 1$, the solution of (4-53) becomes

$$\phi(t) = e^{-\int_0^t [b_{lp}^*(t)/b_{op}^*(t)]dt} \quad (4-54)$$

Since $b_{op}^*(t) > 0$ and $B_{\ell 1}^* < b_{lp}^*(t) < B_{h1}^*$, $\phi(t)$ approaches zero at an exponential rate if $b_{lp}^*(t) > 0$ except over a finite number of finite time intervals during which b_{lp}^* may be negative or zero. For arbitrary m , sufficient conditions for the exponential stability of (4-53) are given in [27]. First, however, it is seen that Equation (4-53) may be rewritten as m first order state equations

$$\dot{\underline{\phi}} = B(t)\underline{\phi} \quad (4-55)$$

where

$$B(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \\ \frac{-b_{mp}^*}{b_{op}^*} & \frac{-b_{m-1,p}^*}{b_{op}^*} & \dots & \dots & \frac{-b_{lp}^*}{b_{op}^*} \end{bmatrix} \quad (4-56)$$

and $\underline{\phi}^T = (\phi, \dot{\phi}, \dots, \phi^{(m-1)})$. Now, following Brockett, restrict the eigenvalues of $B(t)$ to the half-plane $\text{Re}(s) < 0$ for all t and define the Liapunov function

$$V = \underline{y}^T P(t) \underline{y} \quad (4-57)$$

where $P(t)$ ($m \times m$) is a positive definite symmetric matrix. The time derivative of (4-57) is given by

$$\dot{V} = -\underline{y}^T \underline{y} + \underline{y}^T \dot{P} \underline{y} \quad (4-58)$$

where $P(t)$ has been chosen as the unique solution of

$$PB + B^T P = -I \quad (4-59)$$

where I ($m \times m$) is an identity matrix. Now $\dot{P}(t)$ is a function of the b_{ip}^*/b_{op}^* , $i = 1, m$ and their time derivatives and the maximum eigenvalue, λ_M , of $\dot{P}(t)$ can be made less than one by placing suitable bounds on the time derivatives of the quantities, (b_{ip}^*/b_{op}^*) , $i = 1, m$. Hence, dividing both sides of (4-57) by $\underline{y}^T \underline{y}$ yields

$$\dot{V}/\underline{y}^T \underline{y} = -1 + (\underline{y}^T \dot{P} \underline{y}/\underline{y}^T \underline{y}) \quad (4-60)$$

and $\underline{y}^T \dot{P} \underline{y}/\underline{y}^T \underline{y} \leq \lambda_M$, where λ_M is the maximum eigenvalue of $\dot{P}(t)$. If $\lambda < 1$ as discussed above, then $\dot{V} < 0$ unless $\underline{y} = \underline{0}$. Hence (4-57) is a Liapunov function for (4-55) and (4-55) is exponentially stable. In general, then, it is necessary to restrict the eigenvalues of $B(t)$; i.e., the roots of $\sum_{i=0}^m b_{ip}^*(t)s^{(m-i)} = 0$ to the open left half s -plane for every t as in the case of constant parameters discussed in Chapter III. In addition, however, it is necessary to place bounds on the time derivatives of the ratios (b_{ip}^*/b_{op}^*) , $i = 1, m$.

C. Simulation Results

In this section, the adaptive control of the second order plant in the form of (4-19); i.e., $n = 2$, $m = 1$, is considered. In this case, the polynomial $P(s)$ is given by $P(s) = s + c$, $0 < c < a_{1d}$, and the error equation in the form of (4-22) becomes

$$\dot{e}/b_{op}^* + (c - a_{1d})(e/b_{op}^*) = \frac{v_1 + v_2}{b_{op}^*} \quad (4-61)$$

where $(v_1 + v_2)/b_{op}^*$ is defined by (4-30) as

$$\frac{v_1 + v_2}{b_{op}^*} = \left(\frac{a_{1p} - a_{1d}}{b_{op}^*} \right) x_p + \left(\frac{b_{od} - b_{op}}{b_{op}^*} \right) r - y - u \quad (4-62)$$

In (4-62), $y(t)$ is defined in (4-31) by

$$y(t) = \frac{1}{b_{op}^*} \mathcal{L}^{-1} \left\{ \frac{\mathcal{L}[g_x x_p + g_r r - g_u u]}{s + c} \right\} - \frac{1}{b_{op}^*} (a_{1d}c - a_{2d} - c^2)e^f \quad (4-63)$$

where $a_{1d}c - a_{2d} - c^2 = 0$ if $-c$ is fixed at one of the model pole locations. In (4-63), g_x , g_r , and g_u are scalars with the following definitions:

$$\begin{aligned} g_x(t) &= a_{2p} - a_{2d} + c(a_{1d} - a_{1p}) - \dot{a}_{1p} \\ g_r(t) &= b_{1d} - b_{1p} + c(b_{op} - b_{od}) + \dot{b}_{op} \\ g_u(t) &= b_{1p} - cb_{op} - \dot{b}_{op} \end{aligned} \quad (4-64)$$

The signal $e^f(t)$ in (4-63) is a filtered version of $e(t)$ given by

$$\dot{e}^f + ce^f = e \quad (4-65)$$

with $e^f(0) = 0$, and $\gamma(t)$ is defined by

$$\begin{aligned} \gamma(t) &= \frac{1}{b_{op}^*} [\dot{e}(0) + (a_{1d} - c)e(0) - (a_{1p}(0) - a_{1d})x_p(0) \\ &\quad - (b_{od} - b_{op}(0))r(0) + b_{op}^*(0)u(0)] \end{aligned} \quad (4-66)$$

The signals, $h_i(t)$, $i = 1, 7$ and the $h_i^f(t)$, $i = 1, 7$ in the control law of (4-42) become $h_1 = x_p$, $h_2 = x_p^f$, $h_3 = r$, $h_4 = r^f$, $h_5 = u^f$, $h_6 = e^f$, $h_7 = e^{-ct}U(t)$ and $h_1^f = |x_p|$, $h_2^f = |x_p|^f$, $h_3^f = |r|$, $h_4^f = |r|^f$, $h_5^f = |u|^f$, $h_6^f = |e|^f$, $h_7^f = h_7 = e^{-ct}U(t)$, where, for instance, $\dot{r}^f + cr^f = r$.

A simulation for this system using particular parameter values was performed. The plant parameter values selected were

$a_{1p}(t) = 2 + \sin(2t) + e^{-t}$, $a_{2p}(t) = 1.5 - 2 \cos(5t)$, $b_{op}(t) = b_{op}^*(t) = 2 + \sin(4t) + te^{-5t}$ and $b_{op}^*(t) = 1 + 0.5 \cos(\cos t)$ while the model parameters chosen were $a_{1d} = 2$, $a_{2d} = 1$, $b_{od} = 1.5$, and $b_{1d} = 1$.

Initial values for the plant and model signals were given by $x_p(0) = \dot{x}_p(0) = 0$, $x_d = \dot{x}_d(0) = 0$, $r(0) = 1$, $u(0) = 0$ and the reference input was $r(t) = U(t)$, a unit step function. The parameter c , in (4-65) was chosen equal to 1 which yields $a_{1d}c - a_{2d} - c^2 = 0$, eliminating the necessity for introducing the adjustable parameters k_6^I and k_6^P . Using these parameter values, the M_i , $i = 1, 7$ in the adaptive laws in (4-45) were computed as $M_1 = 1$, $M_2 = 6$, $M_3 = 5$, $M_4 = 6$, $M_5 = 5$, $M_6 = 0$, $M_7 = 5$. The arbitrary positive gains α_i and β_i in (4-45) were chosen to be $\alpha_i = \beta_i = 10$, $i = 1-5, 7$.

Figure 4-1 shows the response of the uncontrolled plant, the model reference system, and the controlled plant for 10 seconds. Figure 4-2 illustrates the error response, $e = x_d - x_p$ for the same time interval. It is seen that after the initial transient, the adaptive system results in an error which is bounded by $|e(t)| \leq 0.01$ for $0 \leq t \leq 10$ secs. After 10 secs, the absolute value of the error remained below 0.01 as well.

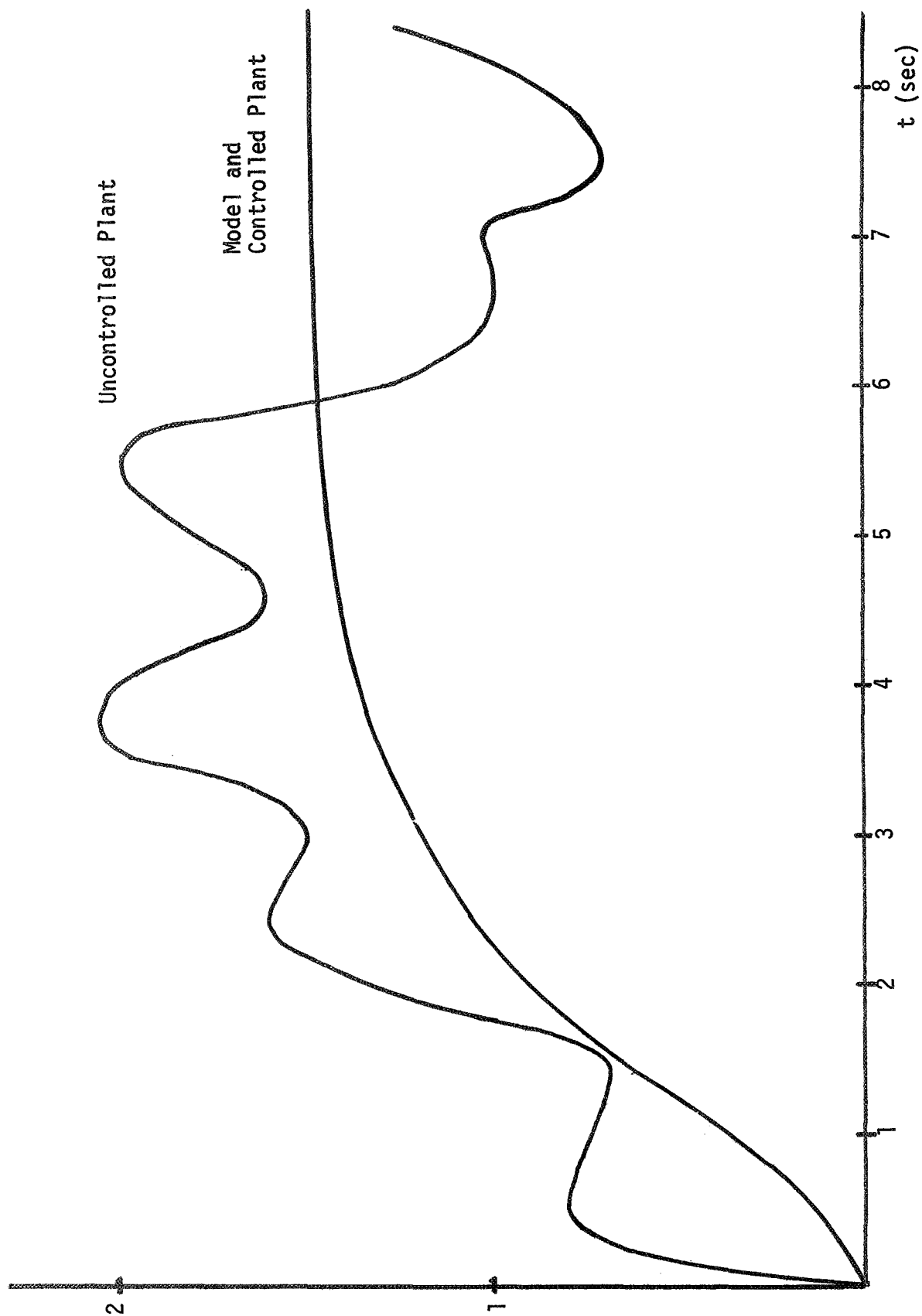


Figure 4.1. Plant and Model Responses for Second Order System with Time-Varying Parameters

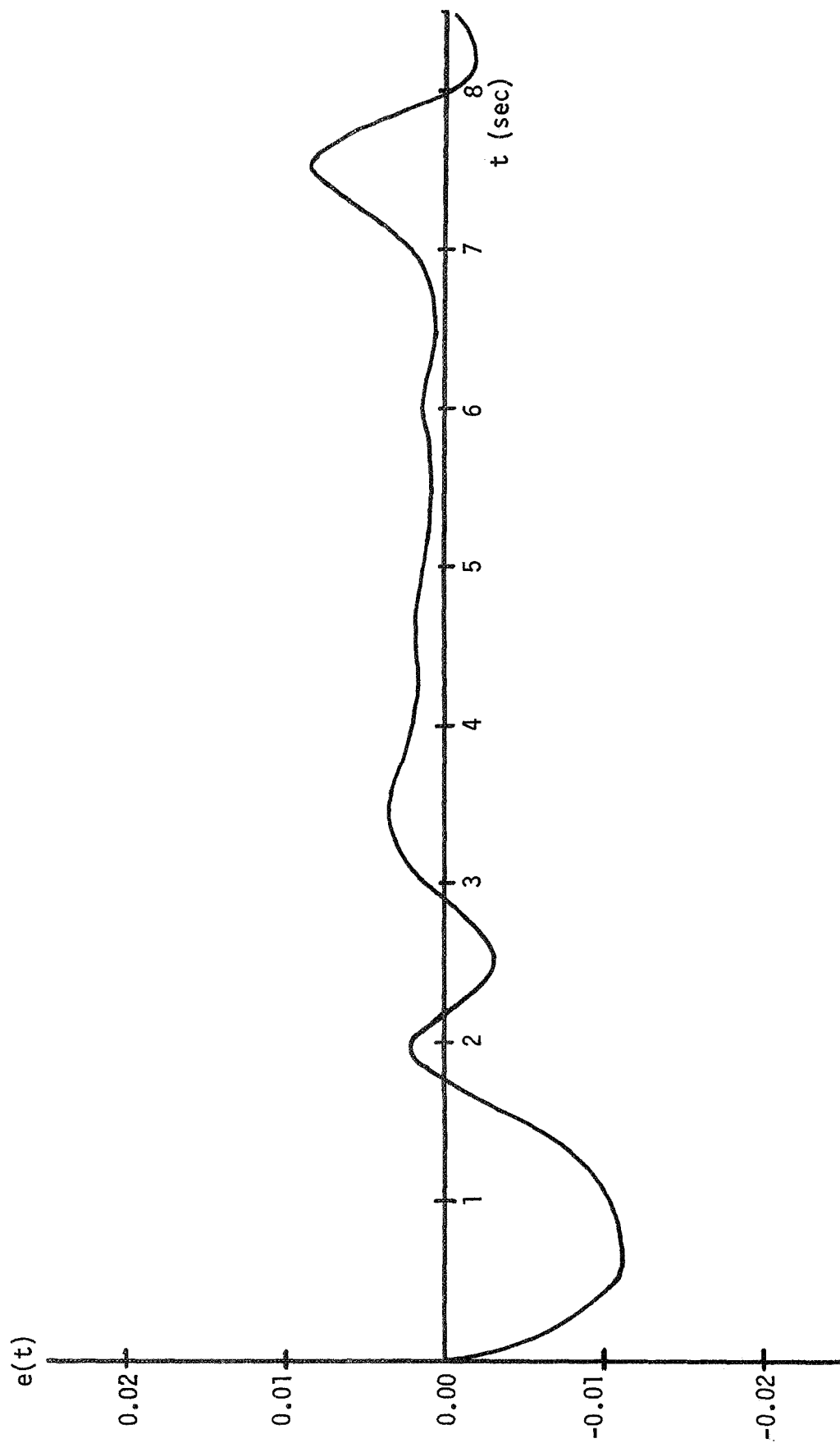


Figure 4.2. Time Response of $e(t)$ for Second Order System with Time-Varying Parameters

C H A P T E R V

APPLICATION OF THE MODEL REFERENCE PARAMETER ADAPTIVE TECHNIQUE TO AN OPTICAL TRACKING MOUNT

A. The Existing Servo Control System and Motivation for an Adaptive System

The National Aeronautics and Space Administration has recently developed a telescope at the Optical Research Facility of Goddard Space Flight Center, Greenbelt, Maryland, for research and development work in laser space communications and the tracking of deep space probes. The telescope consists of a tube, gimballed about two axes (X and Y), which is approximately 14 ft long and 3 ft in diameter. The 3,500 lb weight of the telescope must be pointed to an accuracy of 1 or 2 arc-sec (less than 0.001°) due to the narrow laser beamwidths which are required. In addition, the system must drive smoothly at rates ranging from 0.001 deg/sec to 5 deg/sec in order to track stars and low altitude satellites.

The control system configuration for one axis of the mount, as it presently exists, is depicted in Figure 5.1. The design philosophy is discussed thoroughly in [28] and will be treated only in brief here. In the figure, the input signal, $r[(n + 1)T]$ is a predictive position command consisting of an angle in digital form that indicates, for example, a satellite or star's position at time $t = (n + 1)T$, where n is the number of the current sample interval, and T is the sampling interval. In the course of operation, two system states, $x_p(t)$ and $\dot{x}_p(t)$, are monitored which represent the mount's continuous position and velocity,

respectively, in one axis. As seen in Figure 5.1, the control system for each axis of the mount incorporates three feedback loops. The inner one is an analog rate loop closed by a D.C. tachometer. This loop acts to minimize the effects of friction and to reduce the effective mechanical time constant. The second loop is an incremental position loop which drives the mount 0.001° for each pulse of the command train $\epsilon'(nT)$. The outer loop is a digital position loop which is differenced with the reference input thus generating a discrete position error, $\epsilon(nT) = r[(n+1)T] - x_p(nT)$. A rate signal, generated on the basis of this difference, drives the rate loop at the precise rate required to reduce the error to zero in the sampling interval. Finally the digital counter and analog compensation network are designed to provide zero steady state position, velocity, and acceleration errors, substantial gain and phase margins, and appropriate forward loop bandwidth.

Due to the variety of uses for which it was assembled, the mount is operational in several different modes. The first mode analyzed was the manual velocity mode, in which the mount is required to drive smoothly at low rates in order that slowly moving stars and satellites may be acquired within the field of view (about 20 arc-sec) of the automatic tracking system. In this mode, the position loop is open and a fixed command rate signal is applied directly to the incremental rate loop. Figure 5.2 depicts the outputs of the D/A converter, compensation network, and tachometer, respectively, while the mount was driving in the manual velocity mode. Although the rate selected was 0.005 deg/sec, the tachometer output plot shows that the mount rate was actually

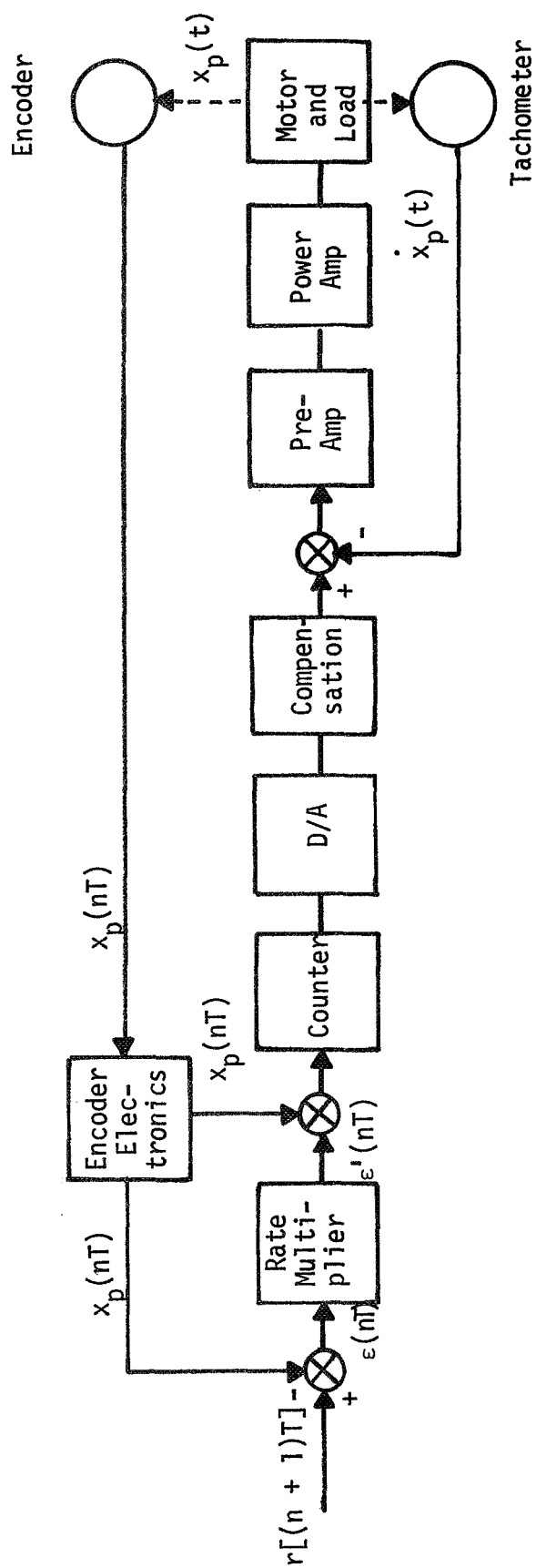


Figure 5.1. Block Diagram of Present Control System (One Axis)

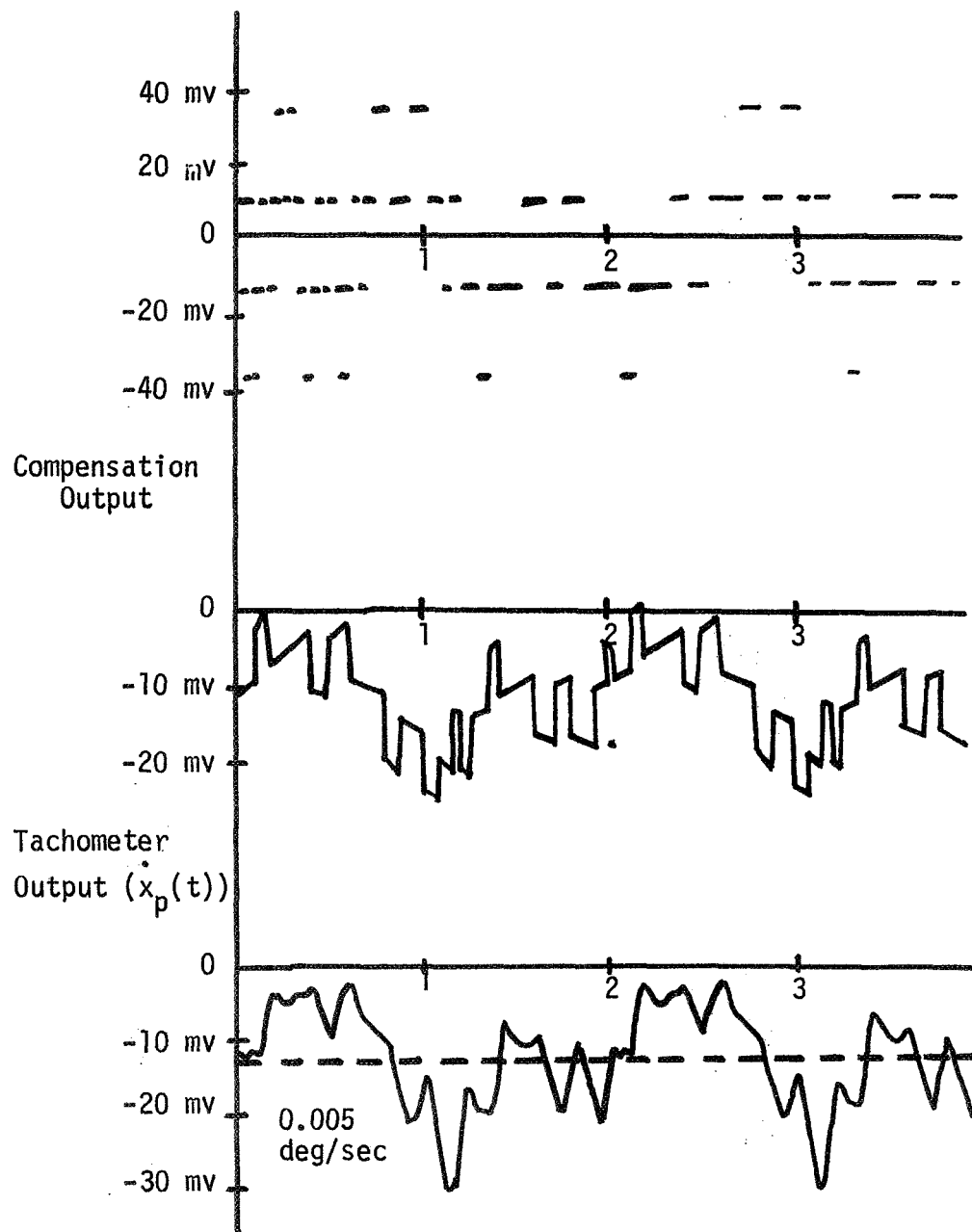


Figure 5.2. D/A, Compensation, and Tachometer Outputs in Manual Velocity Mode

fluctuating about the nominal rate from 0.001 deg/sec to 0.012 deg/sec. This fluctuation is apparently due to the discontinuous nature of the D/A output, which is smoothed over somewhat by the compensation network, but reaches the analog tach loop in the form shown in the second plot in Figure 5.2.

In order to drive the mount smoothly at given input rates, the following scheme was developed. The output of the compensation network in Figure 5.1 was disconnected and a constant reference signal was applied to the analog tach loop. In Figure 5.3, the resulting configuration is shown, along with the transfer functions of the different elements in the loop. The constants K_p and $-a_p$ are the gain and pole, respectively, of the transfer function of the combined D.C. motor and load, $r(t)$ is the reference input, and $\dot{x}_p(t)$ is the tachometer output. The transfer function of the motor and load neglects variations due to changing mount inertia and friction, but these effects will be discussed shortly. The closed loop transfer function is given by

$$\frac{\mathcal{L}[\dot{x}_p]}{R(s)} = \frac{1400 K_p}{s + a_p + 1400 K_p} \quad (5-1)$$

where $R(s) = \mathcal{L}[r(t)]$. Hence, if $r(t)$ is a step function with amplitude A and $\dot{x}_p(0) = 0$, then from (5-1),

$$\dot{x}_p(t) = \left(\frac{1400 K_p A}{a_p + 1400 K_p} \right) (1 - e^{-(a_p + 1400 K_p)t}) \quad (5-2)$$

Since the order of magnitude of K_p is 10^{-2} , the transient decays rapidly and $\dot{x}_p(t) \cong (1400 K_p A)/(a_p + 1400 K_p)$. Figure 5.4 illustrates the

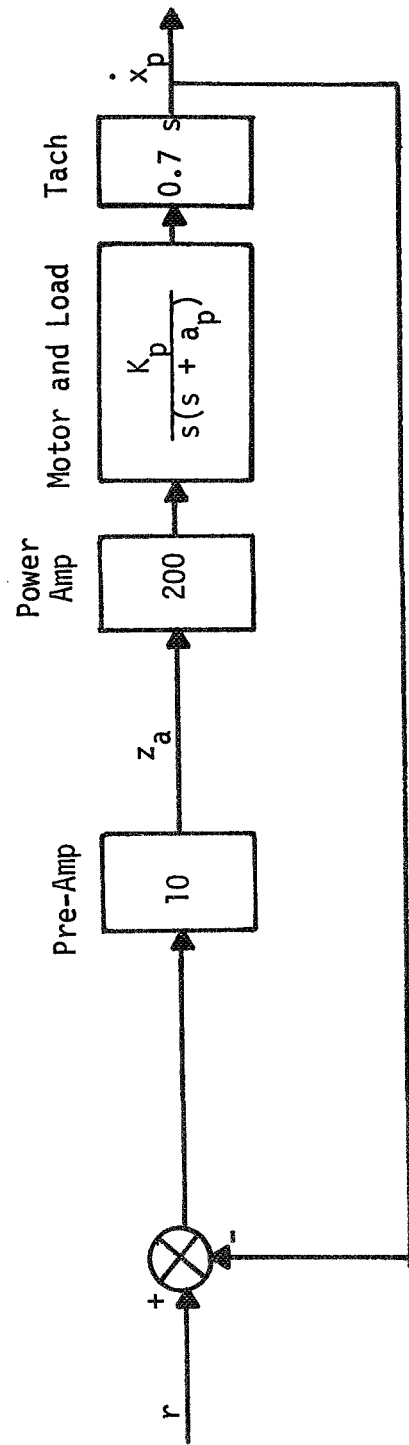


Figure 5.3. Analog Tach Loop

tachometer output when driving the X-axis with $r(t) = 13$ mv. This input voltage was selected so that the X-axis of the mount would move at 0.005 deg/sec when the Y-axis was fixed at 0 degrees and the X-axis was initially positioned at 0 degrees driving East to West. The coordinates $X = 0^\circ$, $Y = 0^\circ$ correspond to the mount's zenith position. The X-axis was then initially positioned at -80° , -40° , $+40^\circ$, $+80^\circ$, respectively, and the output rate was recorded at each point for 30 secs. The Y-axis was then fixed at $+80^\circ$ and the various X-axis rates were again recorded. The last two plots in Figure 5.4 show the output rates when the same procedure was followed with the mount driving in the opposite direction (West to East). It is apparent by comparing Figures 5.2 and 5.4 that far smoother tach output rates are obtained when applying a fixed voltage to the analog tach loop as in Figure 5.3 rather than to the incremental rate loop as shown in Figure 5.1. However, in Figure 5.4, it is also observed that, depending upon the positioning of the X and Y axes and the direction of motion of the X-axis, an input of 13 mv will drive the mount at various rates ranging from 0 deg/sec to 0.0094 deg/sec. It is apparent, then, that changing X-axis inertia, mount imbalance, and static friction cannot be neglected if the mount is to move smoothly at the desired rate over the entire field of view. It is here that the adaptive technique described in Chapters II and III can be of use.

B. Developing the Model Reference Adaptive Control Technique for the Tracking Mount

The adaptive technique presented here is an application of results

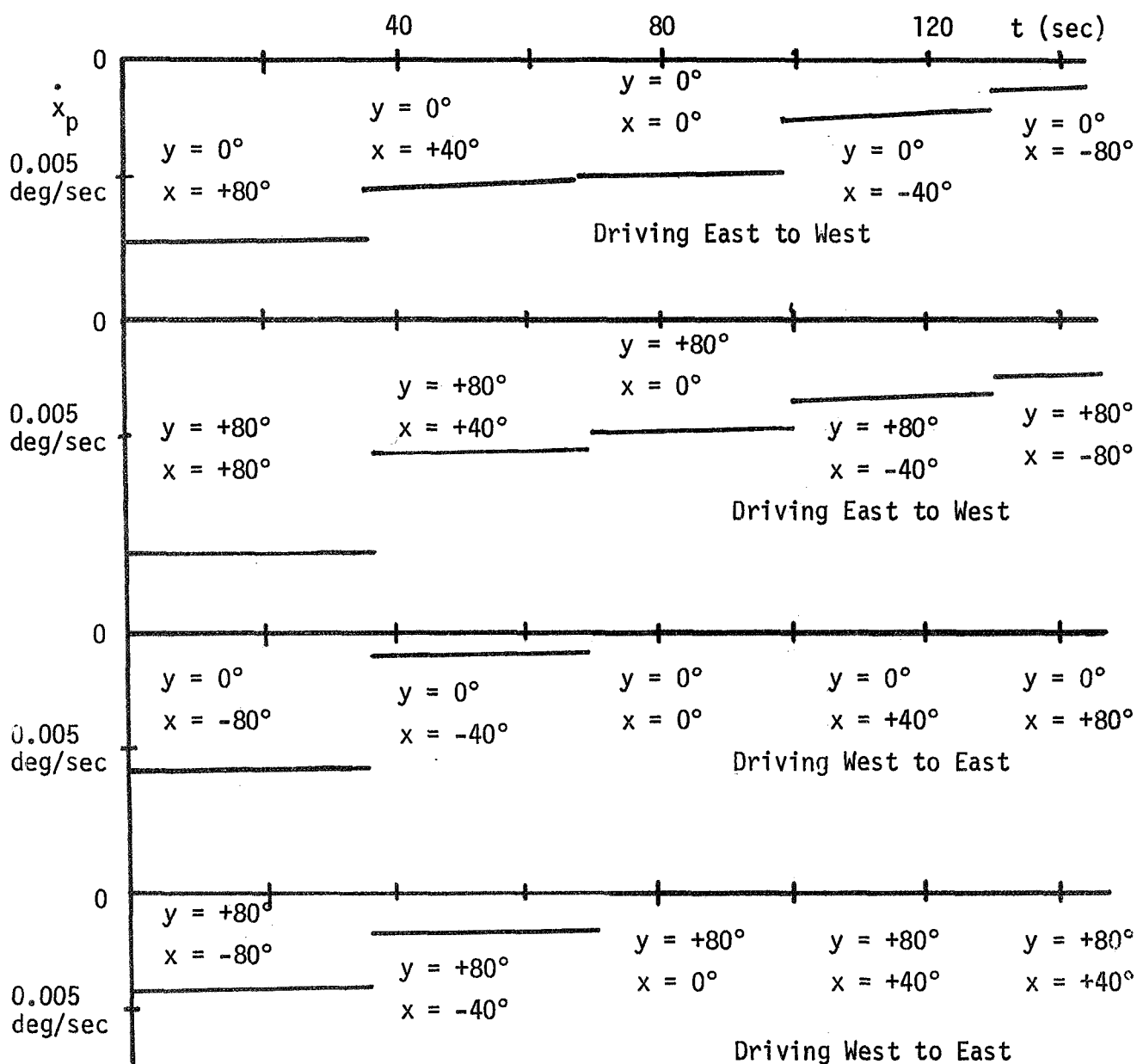


Figure 5.4. X-Axis Tachometer Output without Adaptive scheme ($r = 13$ mv).

presented in Chapters II and III, and includes a new adaptive term which compensates for the nonlinear bearing friction characteristic. Consider now a diagram of the analog tach loop in Figure 5.5. The differential equation relating $\dot{x}_p(t)$ to z_a which includes an approximate model (neglecting hysteresis) of static and coulomb friction as reported in [29], is given by

$$\ddot{x}_p + a_p \dot{x}_p = 140 K_p z_a + f(\dot{x}_p) + 140 K_p u \quad (5-3)$$

where $u(t)$ is the adaptive input to be defined and $-f(\dot{x}_p)$ is described in Figure 5.6. It is assumed that the plant parameters, K_p and a_p , as well as the coulomb friction level f_c , are slowly time-varying parameters, which are essentially constant over the adaptation period of the adaptive mechanism. This assumption will be justified shortly. An ideal model of the motor, load, and tachometer, representing nominal plant characteristics, is given by

$$\ddot{x}_d + 12\dot{x}_d = 0.25z_a \quad (5-4)$$

where $\dot{x}_d$ is the model or ideal velocity. Defining $\dot{e} = \dot{x}_d - \dot{x}_p$ and subtracting (5-3) from (5-4) yields the error equation

$$\ddot{e} + 12\dot{e} = (a_p - 12)\dot{x}_p + (140 K_p - 0.25)z_a - f(\dot{x}_p) - 140 K_p u \quad (5-5)$$

In order to provide adaptive parameters to compensate for the unknown quantities appearing on the right-hand side of Equation (5-5) the control input $u(t)$ is defined as

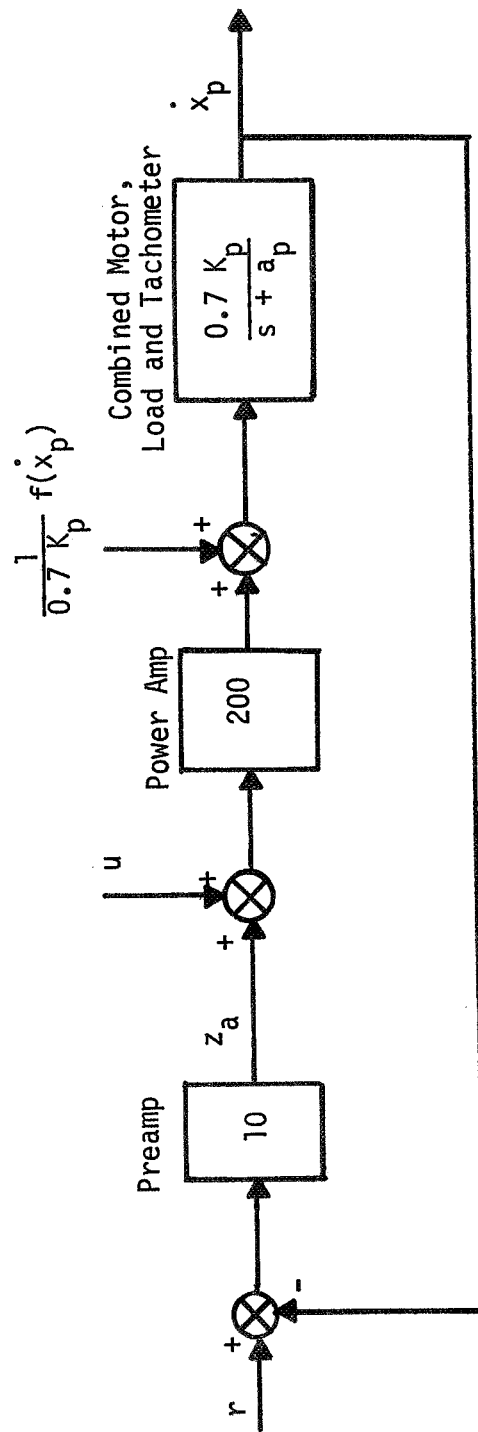


Figure 5.5. Analog Tach Loop Including Bearing Friction and Control Input

$$u = k_1 \dot{x}_p + k_2 z_a + k_3 \operatorname{sgn}(\dot{x}_p) \quad (5-6)$$

where $k_1(t)$, $k_2(t)$, and $k_3(t)$ are the adjustable parameters and $\operatorname{sgn} \dot{x}_p = +1$ for $\dot{x}_p > 0$, $\operatorname{sgn} \dot{x}_p = -1$ for $\dot{x}_p < 0$. The third term in (5-6) is new here and its purpose is to compensate for coulomb friction in the mount bearings. Substituting (5-6) into (5-5), and rewriting (5-5) yields an error equation in the form of (2-5), given by

$$\ddot{e} + 12\dot{e} = \sum_{i=1}^3 y_i g_i \quad (5-7)$$

where $y_1 = a_p - 12 - 140 K_p k_1$, $y_2 = 140 K_p - 0.25 - 140 K_p k_2$, $y_3 = f_c - 140 K_p k_3$, $g_1 = \dot{x}_p$, $g_2 = z_a$, and $g_3 = \operatorname{sgn} \dot{x}_p$. In Equation (5-7) it has been assumed, from Figure 5.6 that $f(\dot{x}_p) = -\operatorname{sgn} \dot{x}_p$. The error equation, (5-7), is now in an appropriate form so that the development (2-6) to (2-15) may be applied, which yields the adaptive laws

$$k_i(t) = \alpha_i \int_0^t \dot{e} g_i dt + \beta_i \dot{e} g_i, \quad i = 1, 2, 3 \quad (5-8)$$

where the α_i and β_i , $i = 1, 3$, are arbitrary positive gains. Hence, we are guaranteed that $\lim_{t \rightarrow \infty} \dot{e}(t) = 0$ and that $\dot{x}_p(t)$ will track the output, $\dot{x}_d(t)$, of a linear, time-invariant model.

C. Driving the Mount at Fixed Rates Utilizing the Adaptive Scheme

The implementation of the adaptive design is shown in Figure 5.7 with the k_i , $i = 1, 3$ defined in (5-8). In the figure, the power amplifier, motor, load, and tachometer have been combined as the "plant" and

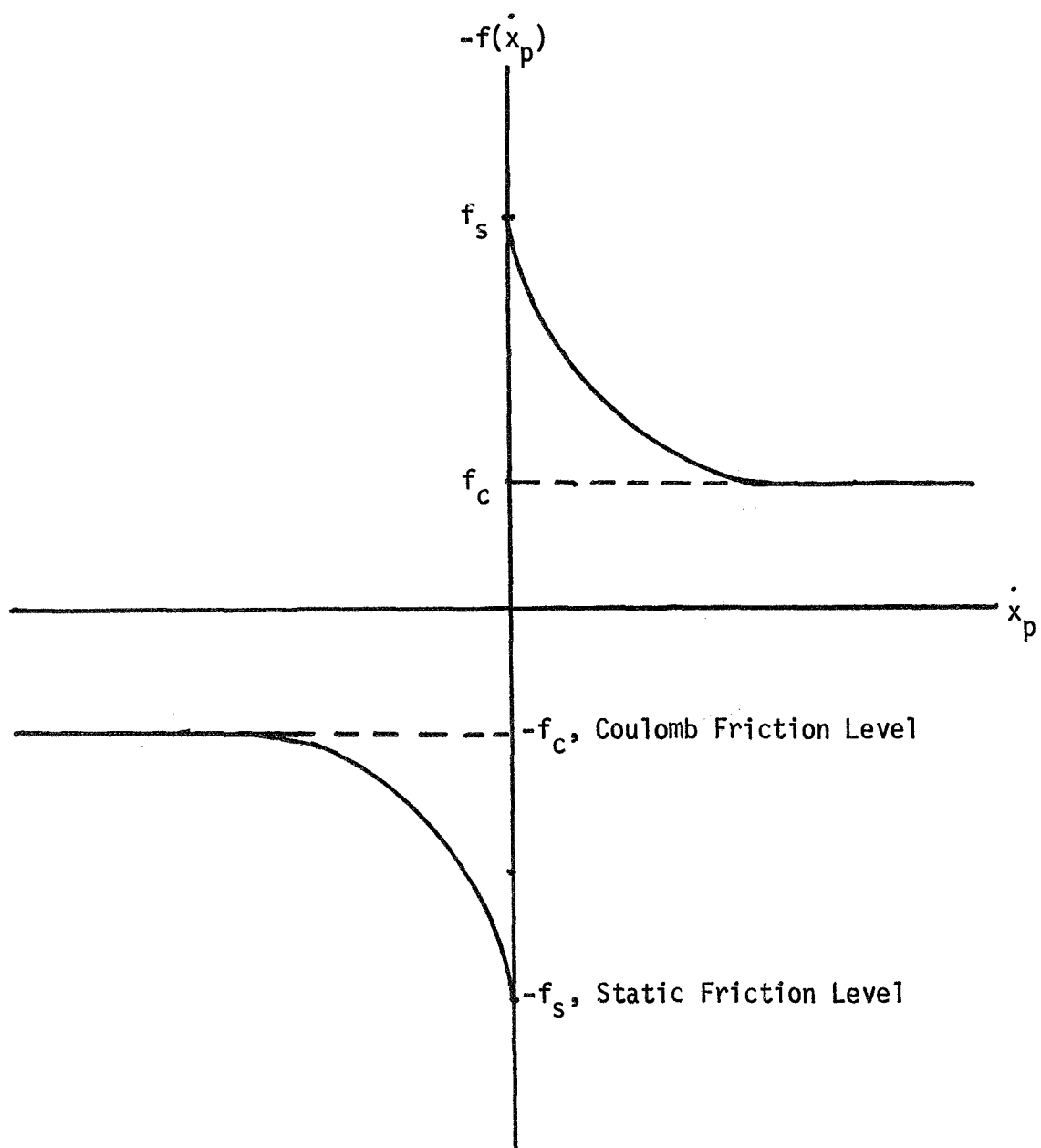


Figure 5.6. Bearing Friction Characteristic

the model system is represented by the transfer function for Equation (5-4). Figure 5.8 illustrates the results of using this design with values for the α_i and β_i in (5-8) of $\alpha_1 = \alpha_2 = 100$, $\beta_1 = \beta_2 = \alpha_3 = 10$, and $\beta_3 = 1$. The reference input ($r = 13$ mv) for Figure 5.8 was the same as the input for Figure 5.4. However, it is evident that the mount is driving at 0.005 deg/sec regardless of the initial positioning of the X and Y axes. In Figure 10, the mount was moving West to East with the Y-axis fixed at 0° . This corresponds to the third plot in Figure 5.4 which was obtained without utilizing the adaptive system. It is seen that when Y is fixed at 0° and X is initially positioned at -80° , the mount drives at nearly 0.006 deg/sec without the adaptive scheme. Hence, the control input (Figure 5.8) is negative, resulting in an opposing torque to slow the X-axis rate. However, for the next four runs (-40° , 0° , $+40^\circ$, $+80^\circ$), the mount will drive too slowly or not at all without the adaptive technique. In Figure 5.8 it is seen that the control input gets increasingly more positive as the X-axis is moved further East, supplying the additional control force necessary to drive the mount at the desired rate. A table illustrating the steady-state values of k_1 , k_2 , and k_3 for the 35-sec runs in Figure 5.8 is given below. The table shows the different configurations which the system in Figure 5.7 assumes as the X and Y coordinates of the mount vary.

Figure 5.9 is a concise illustration of what is to be gained by employing the system in Figure 5.7. In the figure, the tachometer output and its time integral, which is proportional to the mount's

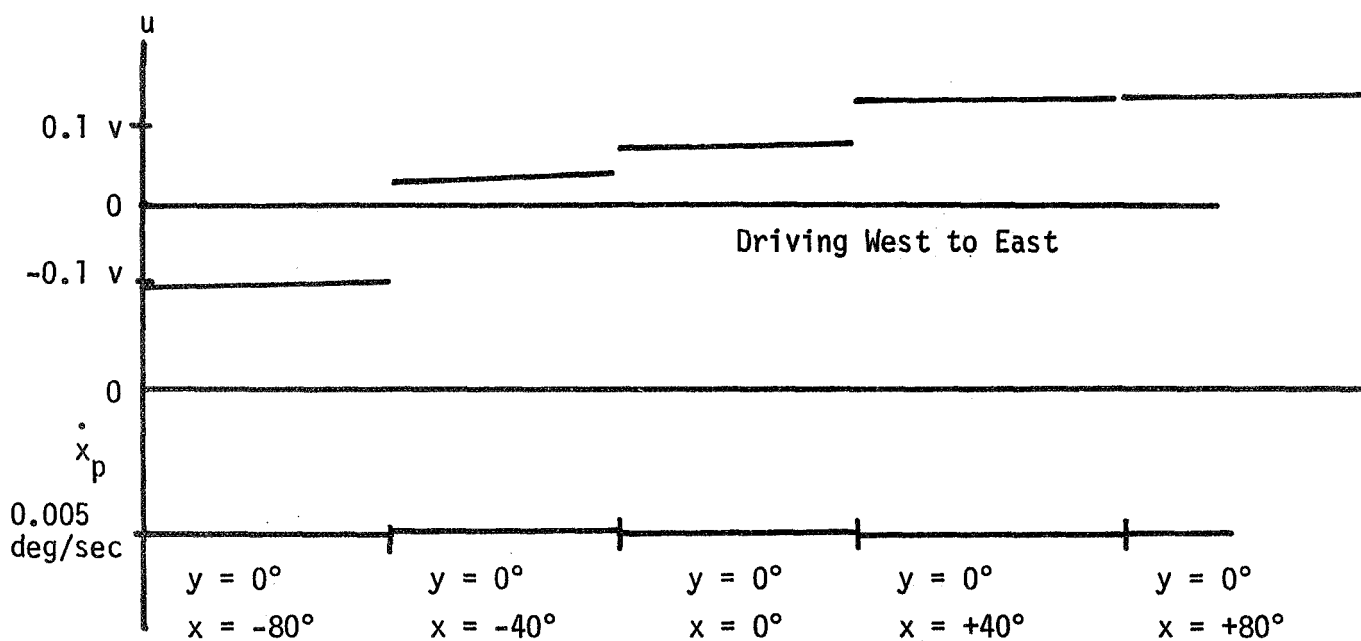


Figure 5.8. Tachometer Output and Associated Control Input Using Adaptive Scheme

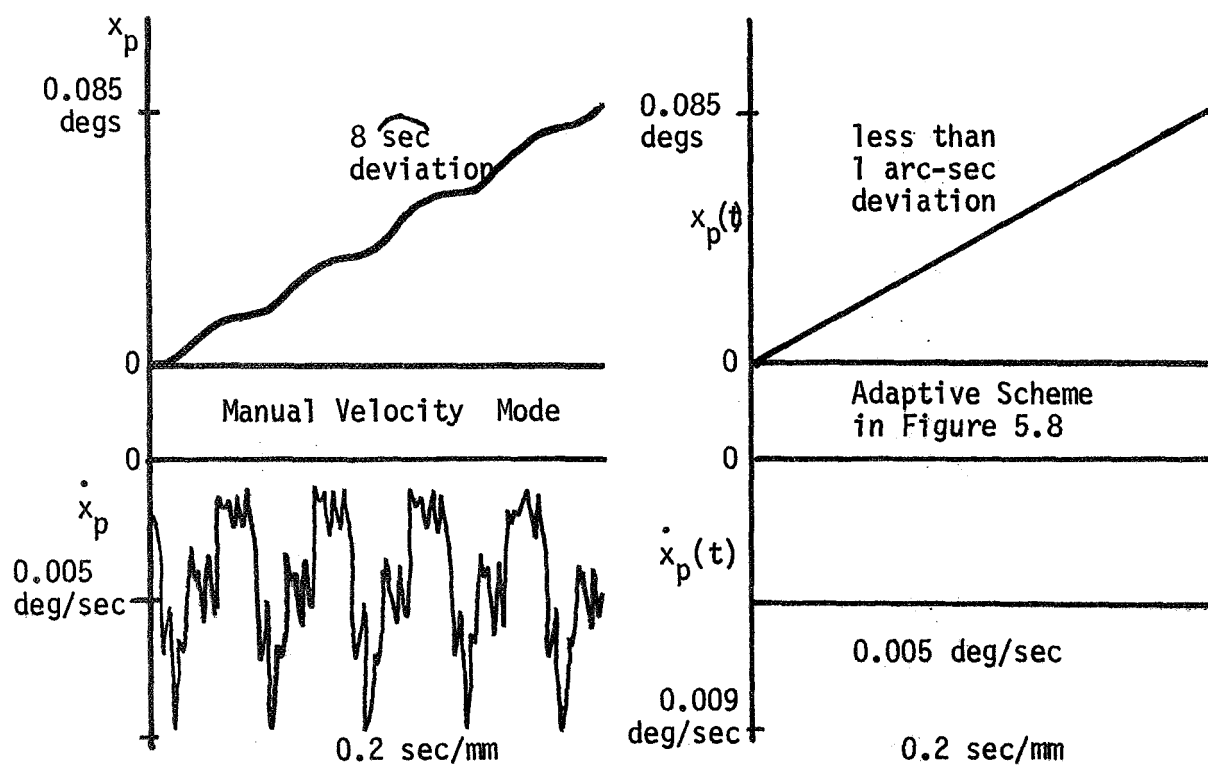


Figure 5.9. Tachometer Output and its Integral Utilizing Existing Scheme and Adaptive Scheme

Table 5-1: Steady-State Adaptive Gains Driving West to East (Y-Axis at 0°)

X-Axis Initially Located at	k_1	k_2	k_3
-80° (West)	0.65	-0.13	-0.33
-40°	-0.32	+0.02	+0.88
0°	-0.39	+0.31	+1.31
+40°	-0.65	+0.85	+0.88
+80° (East)	-0.89	+1.33	+1.54

X-axis position, are compared during a typical run when incorporating both the existing manual velocity mode depicted in Figure 5.1 and the adaptive technique in Figure 5.7. For both runs, the Y-axis was fixed at 0° and the X-axis was driving East to West beginning from 10° East. In the manual velocity mode, the tach output deviated more than ± 0.004 deg/sec or $\pm 80\%$ of its correct value, while the mount position deviated ± 8 arc-sec from its nominal position. On the other hand, utilizing the adaptive scheme with an analog tach loop input, the tach output deviated only ± 0.0002 deg/sec or $\pm 4\%$ of its correct value, while the mount was tracking to within an arc-second of its nominal position.

D. Adaptive Compensation of Static Friction

In order to illustrate the effectiveness of the adaptive scheme to compensate for static friction in the bearings, a tach loop input of $r = 0.013 \sin 0.01 t$ (volts) was chosen. This input approximates the

the velocity characteristic of the circular-orbit 100-mile high orbit. The graphs in Figure 5.10 were taken on separate runs with the Y-axis fixed at 80° North and the X-axis oscillating about 40° West. The input for all three runs is shown in the first graph. The second graph illustrates the tach output without utilizing the adaptive technique. The effect of the increasing retarding torque caused by increasing friction when the mount velocity passes through zero is clearly evident. The third plot in Figure 5.10 shows the mount velocity when the adaptive mechanism of Figure 5.7 is employed. It is seen that the resulting mount velocity is nearly free of static friction effects as well as being completely symmetrical about zero velocity. The last plot in Figure 5.10 shows the mount velocity when only two adjustable parameters in Figure 5.7 are incorporated, i.e., k_1 and k_2 . Although this represents an improvement over the second graph, the mount does not drive as smoothly as when the adjustable gain, k_3 , is utilized. The compensation afforded by utilizing the additional adjustable parameter is clearly illustrated in Figure 5.11. The first plot is the negative of the time variation of $k_3(t)$. It is evident that $k_3(t)$ is following the changing friction level as shown in Figure 5.6. Since $k_3(t)$ is multiplied by $\text{sgn}(\dot{x}_p)$ prior to injection into the power amplifier, as shown in Figure 5.7, the adaptive mechanism automatically provides a control signal of the proper sign, depending upon the sign of $\dot{x}_p(t)$. The third plot in Figure 5.11 is the time response of $\dot{e}(t)$, the error between the model reference and the "plant." The peaks in this curve occur around the instants of time when $\dot{x}_p(t)$ passes through zero and their absolute

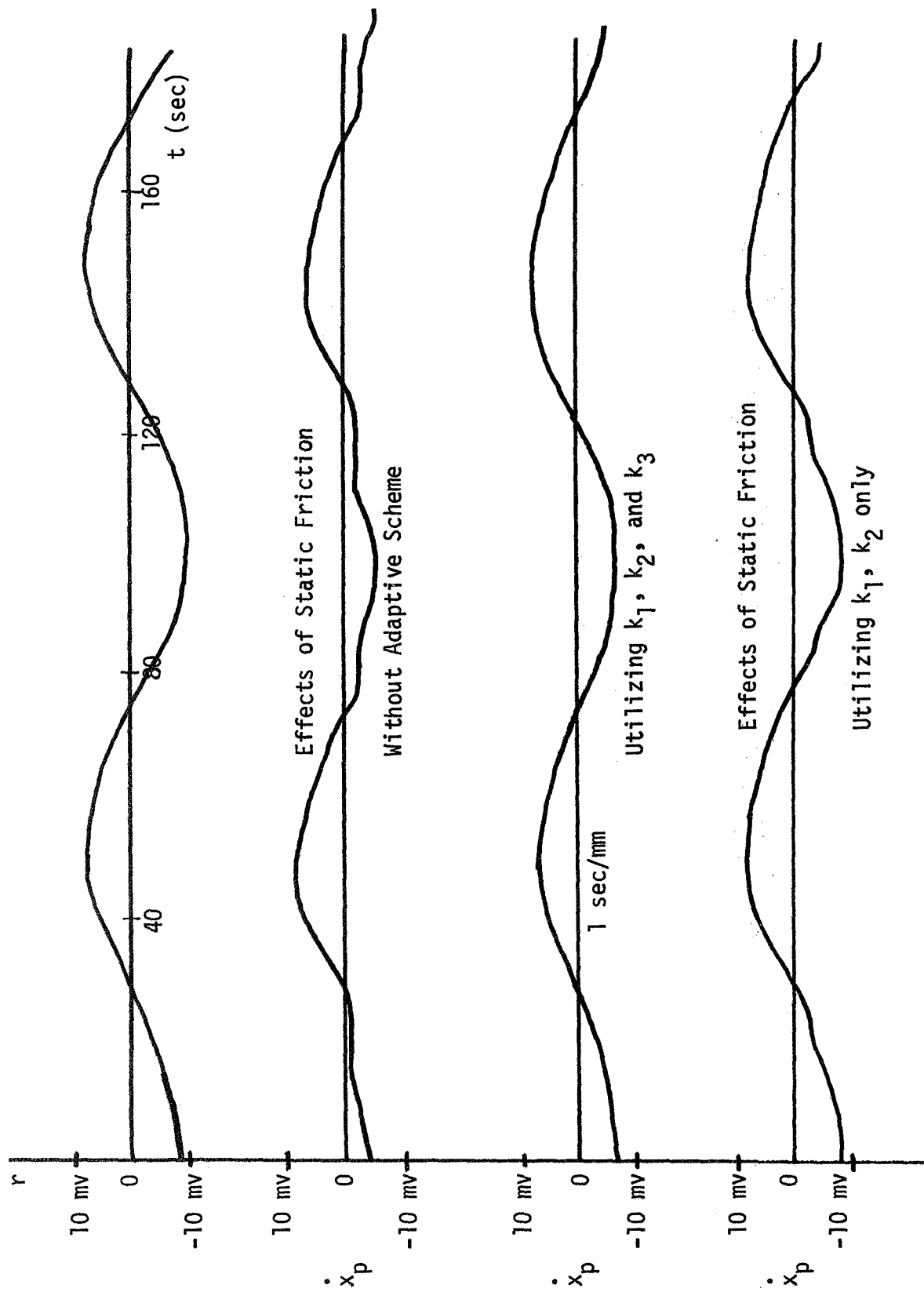


Figure 5.10. Adaptive Compensation for Static Friction

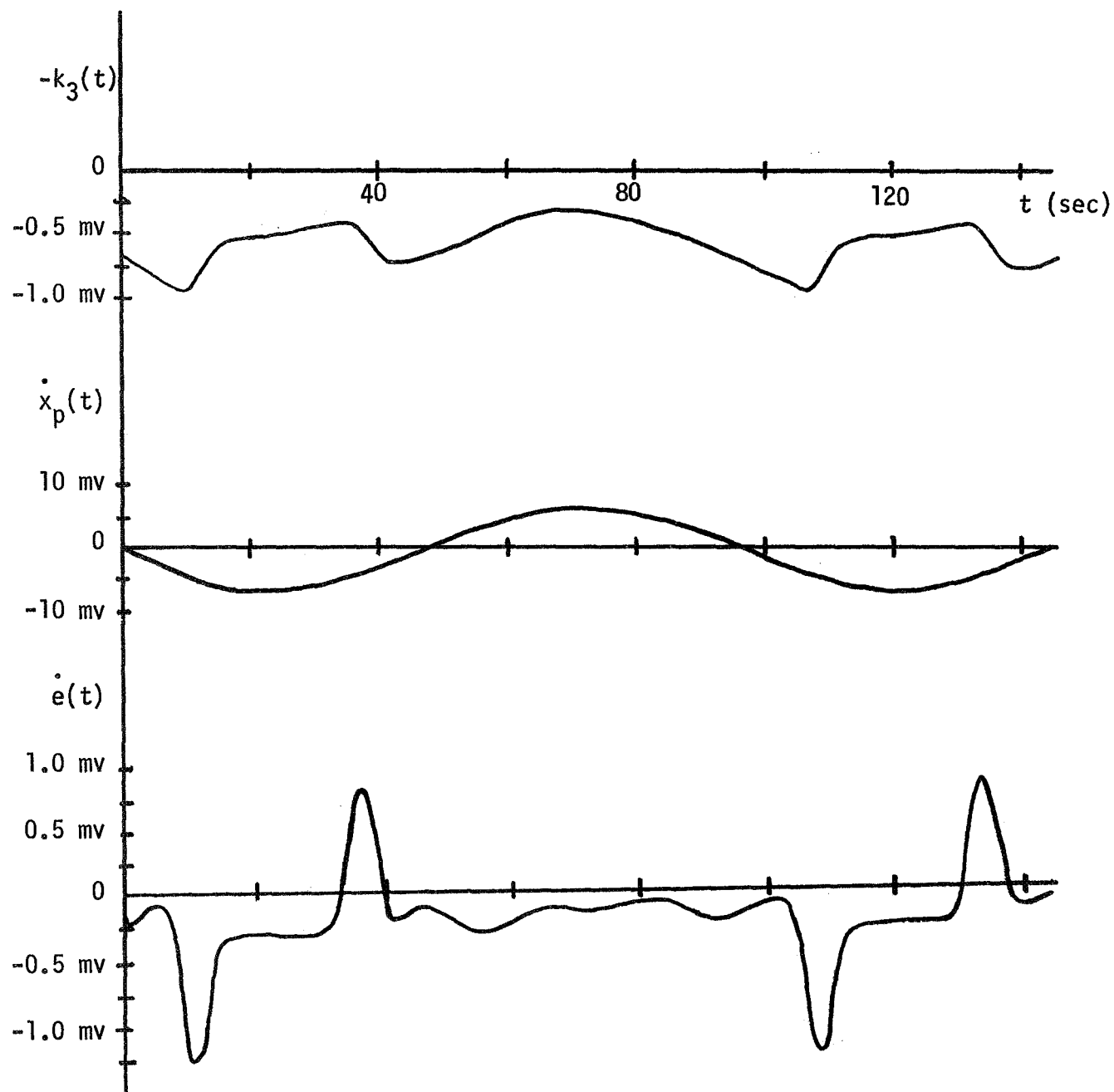


Figure 5.11. $-k_3$, $\dot{x}_p$, and $\dot{e}$ Utilizing Adaptive Mechanism with Sinusoidal Input

value is less than 1.3 mv.

CHAPTER VI

SUMMARY AND SUGGESTIONS

A. Summary

A modified Liapunov design technique for model referenced parameter adaptive control systems was presented. A modified adaptive rule for plants whose open loop transfer function contained adjustable parameters was developed from a new quadratic form Liapunov function. The modified design yielded a more rapidly convergent error system and an improvement in the transient response of the adjustable parameters. Simulations illustrated that, by varying certain fixed gains in the system, a trade-off between small error overshoot and rapid settling time could be effected. In addition, comparable system performance with previous designs could be maintained with smaller gain requirements. As additional implementation, the modified design required only one more summing amplifier for each adjustable parameter.

The case when the adaptive parameters must be introduced as feedback and feedforward adjustable gains was also treated. A technique was developed, utilizing the modified adaptive rule, which alleviated the problem of nonunique adaptive laws reported in the literature [17]. In addition to the adjustable gains, physically realizable, linear, time-invariant filter networks were incorporated in the design if the open loop plant transfer function contained zeros. Unique adaptive laws were derived which required the generation of only $n - m - 1$ derivative networks (n th order plant with m zeros) in contrast to the n states or

$n - 1$ derivative networks required in previous designs. Furthermore, in the case where $n = m - 1$, the plant transfer function was not required to be positive real as in [10]. In order to maintain a bounded plant input, it was necessary to restrict the plant zeros to the open left half plane.

The parameter adaptive approach was then extended to plants whose unknown parameters were time-varying. Adaptive laws were derived which guaranteed the Lagrange or practical stability of the resulting error system. Upper and lower bounds on the unknown parameters were required and if the open loop plant contained differential operators acting on the input, bounds on certain of the time derivatives of these unknown parameters were also required.

Finally, the results of the parameter adaptive control of an optical tracking mount were presented. Included in this application was the adaptive compensation of the nonlinear stiction-friction characteristic of a D.C. motor. It was shown that the mount's position error could be reduced to less than 1 arc-sec with the adaptive scheme whereas the existing servo control system yielded an error of 8 arc-secs.

B. Suggestions for Future Research

Future research in parameter adaptive control systems should include the control of plants whose open loop transfer function is nonminimum phase. It would seem that in order to accomplish this while maintaining a bounded plant input, one will have to be content with a

scheme which provides adaptive compensation for the plant poles only.

Subsequent endeavors should also be concerned with the effects of measurement noise on the adaptive schemes discussed here. Some results have already been reported in this area [30] for plants whose unknown parameters are constant.

Furthermore, it seems clear that discrete adaptive laws analogous to the continuous laws exist for the classes of plants treated in the dissertation. One problem which will have to be resolved in this regard is whether or not the multiplications, additions, and numerical integrations required for each iteration of the discrete adaptive laws can be accomplished in "real time." For example, these calculations will have to be performed in time $t' \leq T$ where T is the sampling interval.

Additional areas of research would also include extending the parameter adaptive approach to systems with more general nonlinearities than the one treated in Chapter V, as well as developing a technique which, when implemented, requires no derivative networks.

APPENDIX A

If the Laplace transform of both sides of (4-21) is taken, it is seen that

$$E(s) \left(s^n + \sum_{i=1}^n a_{id} s^{n-i} \right) = V_e(s) \quad (A-1)$$

where $V_e(s) = \mathcal{L} \left\{ \sum_{i=0}^{\ell} (b_{id} - b_{ip}) r^{(\ell-i)} + \sum_{i=1}^n (a_{ip} - a_{id}) x_p^{(n-i)} - \sum_{i=0}^m b_{ip}^* u^{(m-i)} \right\} + \sum_{j=0}^{n-1} s^{m-1-j} \sum_{i=0}^j a_{id} e^{(j-i)}(0)$. Dividing both sides of (A-1) by $P(s) = s^m + \sum_{i=1}^m c_i s^i$ where the roots of $P(s) = 0$ are in the left half s-plane yields

$$[P_a(s)/P(s)]E(s) = V_e(s) \quad (A-2)$$

where $P_a(s) = s^n + \sum_{i=1}^n a_{id} s^{n-i}$. The ratio of polynomials, $P_a(s)/P(s)$ can be written as follows:

$$P_a(s)/P(s) = s^{k_r} + \sum_{i=1}^{k_r} p_i s^{k_r-i} + (G(s)/P(s)) \quad (A-3)$$

where $k_r = n - m$, and p_i , $i = 1, k_r$ are given by

$$p_{i+1} = a_{i+1,d} - c_{i+1} - \sum_{j=1}^i c_j p_{i-j}$$

and the polynomial $G(s)$ in the remainder term $G(s)/P(s)$ is of order $m - 1$. Equation (A-3) can be rewritten as follows:

$$P_a(s)/P(s) = s^{k_r} + \sum_{i=1}^{k_r} a_{id}^r s^{k_r-i} + H(s) \quad (A-4)$$

where $H(s) = \sum_{i=1}^{k_r} (p_i - a_{id}^r) s^{k_r-i} + \frac{G(s)}{P(s)}$ and the roots of

$s^{k_r} + \sum_{i=1}^{k_r} a_{id}^r s^{k_r-i} = 0$ are in the left half s -plane. Substituting

(A-4) into (A-2) and taking the Laplace inverse of both sides of (A-2) yields

$$e^{(k_r)} + \sum_{i=1}^{k_r} a_{id}^r e^{(k_r-i)} = v_e = v_1 + v_2 \quad (A-5)$$

where $v_e(t) = \mathcal{L}^{-1}[V_e(s)]$ has been decomposed into the signals $v_1(t)$ and $v_2(t)$, defined in (4-23) and (4-24), respectively. Equation (A-5) may be represented by the state equations:

$$\dot{\underline{e}} = A_d \underline{e} + \underline{b}(v_1 + v_2) \quad (A-6)$$

where A_d and $\underline{b}$ are given by

$$A_d = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \\ -a_{k_r d}^r & -a_{k-1, d}^r & \dots & & -a_{1 d}^r \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad (A-7)$$

and the vector $\underline{e}$ is defined by $\underline{e}^T = (e, \dot{e}, \dots, e^{(n-m-1)})$.

APPENDIX B

The vector $\underline{q}(s)$ may be written as

$$\underline{q}^T(s) = \underline{d}^T(\text{ADJ}(sI - A_c))/P(s) \quad (\text{B-1})$$

since $(sI - A_c)^{-1} = \text{ADJ}(sI - A_c)/|sI - A_c|$ and $|sI - A_c| = P(s)$

$= s^m + \sum_{i=1}^m c_i s^{m-i}$. The m elements d_{ai} of the row vector

$\underline{d}_a^T(s) = \underline{d}^T \text{ADJ}(sI - A_c)$ are given by

$$d_{ai} = s^{m-i} + \sum_{j=1}^{m-i} c_j s^{m-i-j}, \quad i = 1, m \quad (\text{B-2})$$

The polynomial $P(s)$ may be written as a product of factors; i.e.,

$$P(s) = \prod_{i=1}^m (s + s_i^0) \quad (\text{B-3})$$

where the s_i^0 , $i = 1, m$ are real and positive. Each element in (B-2) can be rewritten in terms of the s_i^0 , $i = 1, m$ in the following manner.

$$\begin{aligned} d_{ai} = & s^{m-i} + s^{m-i-1} \sum_{i_1=1}^{m-i} s_{i_1}^0 + s^{m-i-2} \sum_{\substack{i_1, i_2=1 \\ i_1 \neq i_2}}^{m-i} s_{i_1}^0 s_{i_2}^0 \\ & + \dots + \prod_{j=1}^m s_j^0 \end{aligned} \quad (\text{B-4})$$

Dividing (B-4) by (B-3), it is seen that the elements $q_i(s)$ of $\underline{q}(s)$ are given by

$$q_m(s) = 1/P(s)$$

$$q_{m-1}(s) = 1/\prod_{i=2}^m (s + s_i^0) + \sum_{i=1}^m s_i^0/P(s)$$

$$\begin{aligned}
q_{m-2}(s) &= 1 / \prod_{i=3}^m (s + s_i^0) + \sum_{i=3}^m s_i^0 / \prod_{i=2}^m (s + s_i^0) \\
&\quad + \sum_{\substack{i,j=2 \\ i \neq j}}^m s_i^0 s_j^0 / p(s) \\
&\quad \vdots \\
q_1(s) &= \frac{1}{s + s_m^0} + \frac{s_m^0}{(s + s_m^0)(s + s_{m-1}^0)} + \dots + \frac{s_m^0 s_{m-1}^0 \dots s_2^0}{p(s)}
\end{aligned} \tag{B-5}$$

Since the s_i^0 , $i = 1, m$ are real and positive, it is evident from (B-5) that each element $q_i(t) = \mathcal{L}^{-1}[q_i(s)]$ is comprised of a sum of terms of the form $k_g g(t)$ where $g(t) = e^{-s_1^0 t} \otimes e^{-s_j^0 t} \otimes \dots$ for some $i, j, \dots$, and $k_g > 0$. Since $g(t)$ is a convolution of time functions which are greater than or equal to zero, we have that $g(t) \geq 0$ as well. Hence, $q_i(t) \geq 0$ for every i .

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