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THE TRANSFORMATION OF A SERIES EXPANSION IN SOLID SPHERICAL HARMONICS UNDER TRANSLATION AND ROTATION OF COORDINATES

by

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William Nelson Lee

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ABSTRACT

A formula for transformation of solid spherical harmonics under coordinate changes makes it possible to convert potential functions, expressed in a series of these harmonics, to representations valid in different coordinate systems

Such a formula has several geophysical applications when the potential of a celestial body has been determined relative to one particular set of coordinates A coordinate change may improve the convergence of the series in a given region, or such a change may be convenient for other reasons

There are also applications to micro-gravitational interactions among non-spherical, but rotationally symmetric bodies, allowing the analytic calculation of quantities which depend on potential functions, such as gravitational forces and torques

In this work the necessary transformation formulas are developed and applied to the task of analytically calculating the force between two homogeneous hemispheres Two special cases are worked out in detail as examples where the coordinate changes involve both translation and rotation

Thesis Supervisor. Stephen J Madden, Jr
Title Lecturer

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CHAPTER 1

INTRODUCTION

1 1 Motivation

The motivation for this research was the need for a satisfactory method of calculating forces and torques caused by gravitational attraction among bodies on the laboratory scale

Historically, the calculation of gravitational forces has been limited to relatively large forces - those involving celestial bodies, where all, or all but one of the masses, can effectively be treated as a point mass Under these conditions, the calculations reduce to a simple gradient operation on the scalar potential.

Laboratory experiments which measure the magnitude of gravitational forces and torques using test masses, such as the Cavendish experiment, have in the past involved at least one spherical test body of uniform density The problem again reduces to one of determining the potential and its gradient

Currently, however, attempts are being made to greatly improve the accuracy of the Cavendish measurement in order to make a relativistic check of the dependence of the gravitational constant on the potential The use of non-spherical test masses appears to be one way in which the sensitivity can be improved (Lee, 1970) Calculating the forces and torques in this case involves a triple integral, or the case of homogenous mass, a double integral over the volume or surface, respectively, of the passive mass,

$$\underline{F} = \rho_d \int_{\text{volume}} \underline{\nabla}\, V \, d\tau$$

$$= \rho_d \int_{\text{surface}} V \, \underline{n} \, dS, \qquad (1\text{-}1)$$

where ρ_d is the mass density, $\underline{n}$ is the outward normal to the surface, and V is the potential due to the gravitating mass(es) In general this calculation is formidable - difficult to carry out analytically, and computer approximation schemes do not efficiently yield the accuracy required for a relativistic measurement.

1 2 Objectives

The purpose of this investigation is to develop an analytical method for calculating the gravitational interactions among rotationally symmetric bodies of uniform density, permitting the evaluation of the forces to the desired accuracy Other quantities which depend on the potential, such as torques and partial derivatives thereof, can be treated in the same manner

1 3 Method

To facilitate the evaluation of the surface integral in equation (1-1), the coordinate system is oriented to take full advantage of the symmetry of the passive body The coordinates and coordinate system originally selected are best suited for expression of the potential function of the gravitating mass(es). Once the potential has been determined, the reference frame will, in general, have to be changed before permitting the integrations. Although the scalar potential remains unchanged under such a transformation, individual terms in a series expansion of the potential will change with the coordinate transformation

In this thesis a potential expansion in solid spherical harmonics is transformed under translation and rotation of coordinates, permitting the relatively easy evaluation of the surface integrals The transformation will involve one translation and one rotation, unless the translation is limited to the direction of the axis of symmetry of the gravitating mass, in which case two rotations are required in addition to the special translation.

The transformation of the expansion requires the transformation of the coefficients in the series, which in turn necessitates the transformation of the solid spherical harmonics under the coordinate change

1 4 Other Applications

The transformation of a series expansion in solid spherical harmonics appears to have several applications other than the one described in 1.1 The potentials of the earth and moon, for example, which are originally determined in terms of a particular coordinate system, can then be written in any other coordinate system desired This may facilitate its evaluation in some regions, either because the convergence is not satisfactory, or to capitalize on a particular geometrical configuration

CHAPTER 2

TRANSFORMATION OF SOLID SPHERICAL HARMONICS DUE TO TRANSLATION OF AXES

2 1 Coordinate Systems

A pure translation of the origin of the x, y, z coordinate system (Figure 2-1) to a new system (ξ,η,ζ) is represented by a translation vector $\underline{\rho}$ A field point P is located by the position vector $\underline{r}$ in the old system and a vector $\underline{R}$ in the new system

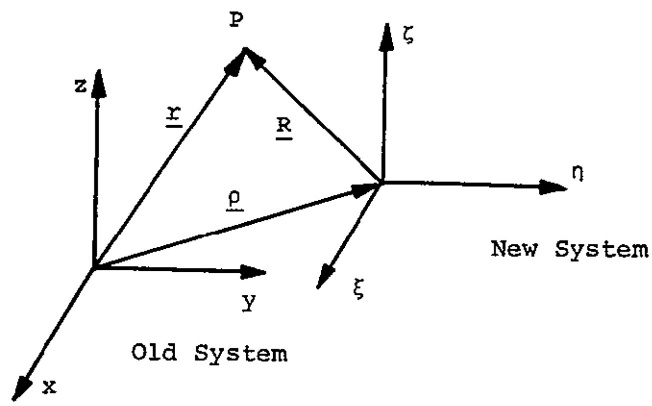


Figure 2-1 Relationship Between Old and New Coordinate Systems

In spherical coordinates, the angles θ, λ and α are the co-latitudes, ϕ, γ and β the longitudes, and r, ρ and R the magnitudes of the three vectors $\underline{r}$, $\underline{\rho}$ and $\underline{R}$, respectively The coordinates of $\underline{r}$ are indicated in Figure 2-2 as an example

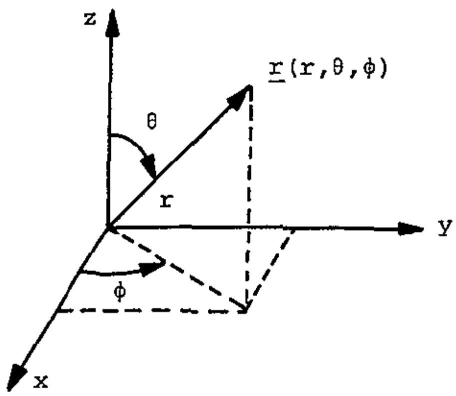


Figure 2-2 Definition of Spherical Coordinates

Thus,

$$\underline{r} = \underline{R} + \underline{\rho} \quad ,$$

where

$$\underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \sin\theta \cos\phi \\ r \sin\theta \sin\phi \\ r \cos\theta \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} R \sin\alpha \cos\beta \\ R \sin\alpha \sin\beta \\ R \cos\alpha \end{bmatrix}$$

$$\underline{\rho} = \begin{bmatrix} \rho \sin\lambda \cos\gamma \\ \rho \sin\lambda \sin\gamma \\ \rho \cos\lambda \end{bmatrix} \quad , \qquad (2\text{-}1)$$

and

$$0 \leq r,\ R,\ \rho$$

$$0 \leq \theta,\ \alpha,\ \lambda \leq \pi$$

$$0 \leq \phi,\ \beta,\ \gamma \leq 2\pi$$

2.2 Solid Spherical Harmonics

A solid spherical harmonic is defined as any homogeneous function $f(x,y,z)$ of degree n which also satisfies Laplace's equation,

$$\nabla^2 f = 0 \tag{2-2}$$

If harmonics of the form

$$r^p P_n^m(\cos\theta)e^{im\phi} \qquad n=0,1,2, \qquad m=-n,-n+1, \quad ,n-1,n$$

are considered, then p must assume one of two values,

$$p = n,-n-1,$$

since $P_n^m(\cos\theta)e^{im\phi}$ satisfies equation (2-2) (MacRobert, 1947, p 150)

The P_n^m's are known as associated Legendre polynomials, defined by

$$P_n^m(\nu) = (-1)^m(1-\nu^2)^{m/2}\frac{d^m}{d\nu^m}P_n(\nu), \tag{2-3}$$

where the P_n's are the familiar Legendre polynomials They can be determined from Rodrigues' formula

$$P_n(\nu) = \frac{1}{2^n n!} \frac{d^n}{d\nu^n} (\nu^2-1)^n.$$

Several important relationships among the P_n^m's are noted here for later use The first relates a polynomial of negative order to the corresponding polynomial of positive order (Jackson, 1966),

$$P_n^{-m}(\nu) = \frac{(-1)^m (n-m)!}{(n+m)!} P_n^m(\nu) \qquad m \geq 0 \tag{2-4}$$

Note that in the special case m = 0,

$$P_n^0(\nu) = P_n(\nu)$$

Other useful expressions are (Abramowitz, 1965)

$$P_n^m(\nu) = (-1)^{n+m} P_n^m(-\nu) \tag{2-5}$$

$$P_m^m(\nu) = \frac{(-1)^m (2m)!}{m! 2^m} (1-\nu^2)^{m/2}$$

$$P_{m+1}^m(\nu) = (2m+1)\nu P_m^m(\nu) , \tag{2-6}$$

and the recursive relations

$$(n-m)P_n^m(\nu) - (2n-1)\nu P_{n-1}^m(\nu) + (n+m-1)P_{n-2}^m(\nu) = 0 \tag{2-7}$$

$$(n-m)(n+m+1)P_n^m(\nu) + 2(m+1)\nu(1-\nu^2)^{-1/2}P_n^{m+1}(\nu) + P_n^{m+2}(\nu) = 0 \tag{2-8}$$

Several $P_n(\nu)$'s and $P_n^m(\nu)$'s are evaluated for later reference,

$$P_0(\nu) = 1 \qquad P_1^1(\nu) = -(1-\nu^2)^{1/2}$$

$$P_1(\nu) = \nu \qquad P_2^1(\nu) = -3(1-\nu^2)^{1/2}\nu$$

$$P_2(\nu) = \frac{1}{2}(3\nu^2-1) \qquad P_2^2(\nu) = 3(1-\nu^2)$$

$$P_3(\nu) = \frac{1}{2}(5\nu^3-3\nu) \qquad P_3^1(\nu) = -\frac{3}{2}(1-\nu^2)^{1/2}(5\nu^2-1)$$

$$P_4(\nu) = \frac{1}{8}(35\nu^4-30\nu^2+3) \qquad P_3^2(\nu) = 15(1-\nu^2)\nu$$

$$P_5(\nu) = \frac{1}{8}(63\nu^5-70\nu^3+15\nu) \qquad P_3^3(\nu) = -15(1-\nu^2)^{3/2}$$

A well known and useful generating function for harmonics involving the P_n's (Erdélyi, Vol 1, 1953, p 154) is

$$\frac{1}{(1+h^2-2\nu h)^{1/2}} = \begin{cases} \sum_{n=0}^{\infty} h^n P_n(\nu) & |h| < \text{smaller of } |\nu \pm (\nu^2-1)^{1/2}| \\ \sum_{n=0}^{\infty} h^{-n-1} P_n(\nu) & |h| > \text{larger of } |\nu \pm (\nu^2-1)^{1/2}| , \end{cases} \tag{2-9}$$

and finally, the addition theorem of spherical harmonics, involving the angle ψ between two vectors $\underline{r}$ and $\underline{r}'$ (Jackson, 1966, p.67), is

$$P_n(\cos\psi) = P_n(\cos\theta)P_n(\cos\theta')$$

$$+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi'), \tag{2-10}$$

where

$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

2.3 Transformation of Harmonics Involving Powers of r^n

In order to transform the coefficients in solid spherical harmonic expansions with powers of r^n and r^{-n-1}, the spherical harmonics themselves must be transformed in the opposite sense for each case. That is, the old harmonics must be found in terms of the new ones.

If the method developed by Aardoom (Aardoom, 1969) is adopted, use is made of the finite binomial expansion

$$(\underline{\bar{u}}^T\underline{r})^n = (\underline{\bar{u}}^T\underline{R}+\underline{\bar{u}}^T\underline{\rho})^n = \sum_{k=0}^{n}\binom{n}{k}(\underline{\bar{u}}^T\underline{R})^{n-k}(\underline{\bar{u}}^T\underline{\rho})^k, \tag{2-11}$$

where

$$\binom{n}{k} \equiv \frac{n!}{(n-k)!k!},$$

and $(\underline{\bar{u}}^T \underline{r})$ is the Hermitian scalar product of a special dummy complex vector $\underline{u}$, and $\underline{r}$. The bar over $\underline{u}$ indicates its complex conjugate and the superscript T stands for its transpose

The vector $\underline{u}$ is defined in terms of a free parameter t,

$$\underline{u} \equiv \begin{bmatrix} 1-t^2 \\ i(1+t^2) \\ -2t \end{bmatrix} \qquad t \neq 0 \ ,$$

and the Hermitian scalar products can be identified with the generating function in equation (A-11) of Appendix A Thus,

$$(\underline{\bar{u}}^T \underline{r})^n = t^n \sum_{m=-n}^{n} {}^{(1)}H_{n,m}(r, \cos\theta, \phi)\, t^m \ , \qquad (2\text{-}12)$$

where

$${}^{(1)}H_{n,m}(r, \cos\theta, \phi) \equiv \frac{(-1)^m (-2)^n n!}{(n+m)!} r^n P_n^m(\cos\theta) e^{im\phi}. \qquad (2\text{-}13)$$

Using equation (2-12) in three places in (2-11), and dividing by t^n,

$$\sum_{m=-n}^{n} {}^{(1)}H_{n,m}(\underline{r})\, t^m = \sum_{k=0}^{n} \binom{n}{k} \sum_{h=-n+k}^{n-k} {}^{(1)}H_{n-k,h}(\underline{R})\, t^h \sum_{j=-k}^{k} {}^{(1)}H_{k,j}(\underline{\rho})\, t^j \qquad (2\text{-}14)$$

With a redefinition of indices, (2-14) can be written as

$$\sum_{m=-n}^{n} {}^{(1)}H_{n,m}(\underline{r})\, t^m = \sum_{p=-n}^{n} t^p \sum_{k=0}^{n} \binom{n}{k} \sum_{\ell=L\left\{\begin{smallmatrix}-k \\ k+p-n\end{smallmatrix}\right.}^{S\left\{\begin{smallmatrix}k \\ -k+n+p\end{smallmatrix}\right.} {}^{(1)}H_{k,\ell}(\underline{R})\; {}^{(1)}H_{n-k,p-\ell}(\underline{\rho}), \qquad (2\text{-}15)$$

where

$$S\begin{cases} a \\ b \end{cases} \quad \text{and} \quad L\begin{cases} a \\ b \end{cases}$$

are taken as the smaller and the larger, respectively, of the two quantities (a,b) beside the bracket.

Since t is a free parameter, the coefficients of t^m in equation (2-15) can be equated In spherical coordinates, making use of equation (2-13), the result is

$$r^n P_n^m(\cos\theta)e^{im\phi} = \sum_{k=0}^{n} \sum_{\ell=L\begin{cases}-k \\ k+m-n\end{cases}}^{S\begin{cases}k \\ -k+n+m\end{cases}} {}^{(1)}K_{n,m}^{k,\ell}(\rho,\cos\lambda,\gamma)R^k P_k^\ell(\cos\alpha)e^{i\ell\beta} \quad , \tag{2-16}$$

where

$$^{(1)}K_{n,m}^{k,\ell}(\rho,\cos\lambda,\gamma) \equiv \binom{n+m}{k+\ell}\rho^{n-k}P_{n-k}^{m-\ell}(\cos\lambda)e^{i(m-\ell)\gamma} \tag{2-17}$$

Note that $K_{n,m}^{k,\ell}$ is a function only of the transformation variables ρ, λ and γ

2 4 Transformation of Harmonics Involving Powers of r^{-n-1}

The transformation of the harmonics involving inverse powers of r is treated in a manner analogous to that used in section (2 3), beginning this time with an infinite binomial expansion,

$$(\underline{\bar{u}}^T\underline{r})^{-n-1} = (\underline{\bar{u}}^T\underline{R}+\underline{\bar{u}}^T\underline{\rho})^{-n-1}$$

$$= \begin{cases} \sum_{k=0}^{\infty}(-1)^k\binom{n+k}{n}(\underline{\bar{u}}^T\underline{R})^{-n-k-1}(\underline{\bar{u}}^T\underline{\rho})^k & |\underline{\bar{u}}^T\underline{\rho}| < |\underline{\bar{u}}^T\underline{R}| \\ \\ \sum_{k=0}^{\infty}(-1)^k\binom{n+k}{n}(\underline{\bar{u}}^T\underline{\rho})^{-n-k-1}(\underline{\bar{u}}^T\underline{R})^k & |\underline{\bar{u}}^T\underline{\rho}| > |\underline{\bar{u}}^T\underline{R}| \end{cases} \tag{2-18}$$

Thus, the series takes one of two different forms, whichever will converge. The convergence criteria themselves are difficult to evaluate due to the involvement of the dummy vector $\underline{u}$ In the special case of a translation along the z axis ($\lambda=0$), it can be shown that the criteria reduce to the question of $\frac{\rho}{R}$ being less than or greater than one. This has not been directly shown to be true in general, except that an independent derivation by Madden (Madden, 1971) of the transformation of the harmonics yields the same results as will be obtained here, with the criteria

$$\rho \lessgtr R$$

holding in general Hence, these convergence criteria are adopted

The left-hand side of (2-18) can be identified with (A-13), so that

$$(\underline{\bar{u}}^T \underline{r})^{-n-1} = t^{-n-1} \sum_{m=-n}^{n} {}^{(2)}H_{n,m}(r,\cos\,\theta,\phi)t^m \quad , \tag{2-19}$$

where

$${}^{(2)}H_{n,m}(r,\cos\,\theta,\phi) \equiv \frac{(n-m)!}{(-2)^{n+1}n!} r^{-n-1}P_n^m(\cos\,\theta)e^{im\phi} \tag{2-20}$$

Making use of (2-12) and (2-19) and multiplying by t^{n+1}, equation (2-18) can be expressed as

$$\sum_{m=-n}^{n} {}^{(2)}H_{n,m}(\underline{r})t^m = \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{n} \sum_{h=-n-k}^{n+k} {}^{(2)}H_{n+k,h}\begin{bmatrix}\underline{R}\\ \underline{\rho}\end{bmatrix} t^h \sum_{j=-k}^{k} {}^{(1)}H_{k,j}\begin{bmatrix}\underline{\rho}\\ \underline{R}\end{bmatrix} t^j \quad , \tag{2-21}$$

where

$$\begin{bmatrix} a \\ b \end{bmatrix} \equiv \begin{cases} a & \rho < R \\ b & \rho > R \end{cases}$$

Changing indices and equating coefficients of t^m on each side of (2-21) gives

$$^{(2)}H_{n,m}(\underline{r}) = \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{n} \sum_{\ell=m-k}^{m+k} {}^{(2)}H_{n+k,\ell}\begin{bmatrix} \underline{R} \\ \underline{\rho} \end{bmatrix} {}^{(1)}H_{k,m-\ell}\begin{bmatrix} \underline{\rho} \\ \underline{R} \end{bmatrix}$$

Introducing spherical coordinates and expressions of the form of (2-13) and (2-20) yields

$$r^{-n-1}P_n^m(\cos\theta)e^{im\phi} = \sum_{k=0}^{\infty} \sum_{\ell=m-k}^{m+k} {}^{(2)}K_{n,m}^{k,\ell}(\rho,\cos\lambda,\gamma)$$

$$\times \begin{bmatrix} R^{-n-k-1} \\ R^k \end{bmatrix} P_{\begin{bmatrix} k+n \\ k \end{bmatrix}}^{\begin{bmatrix} \ell \\ m-\ell \end{bmatrix}}(\cos\alpha)e^{i\begin{bmatrix} \ell \\ m-\ell \end{bmatrix}\beta} , \tag{2-22}$$

where

$$^{(2)}K_{n,m}^{k,\ell} \equiv (-1)^{k+m+\ell}\binom{k+n-\ell}{n-m}\begin{bmatrix} \rho^k \\ \rho^{-n-k-1} \end{bmatrix} P_{\begin{bmatrix} k \\ k+n \end{bmatrix}}^{\begin{bmatrix} m-\ell \\ \ell \end{bmatrix}}(\cos\lambda)e^{i\begin{bmatrix} m-\ell \\ \ell \end{bmatrix}\gamma}, \tag{2-23}$$

and

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{cases} a & \rho < R \\ b & \rho > R \end{cases}$$

In the special case when the translation is in the positive z direction (λ=0), equations (2-16) and (2-22) reduce to :

$$r^n P_n^m(\cos\theta)e^{im\phi} = \sum_{k=0}^{n} \binom{n+m}{k+m} \rho^{n-k} R^k P_k^m(\cos\alpha)e^{im\beta}, \tag{2-24}$$

and

$$r^{-n-1} P_n^m(\cos\theta)e^{im\phi} = \begin{cases} \displaystyle\sum_{k=0}^{\infty} (-1)^k \binom{k+n-m}{k} \rho^k R^{-n-k-1} P_{n+k}^m(\cos\alpha)e^{im\beta} & \rho < R \quad (2\text{-}25) \\[2ex] \displaystyle\sum_{k=0}^{\infty} (-1)^{k+m} \binom{k+n}{k+m} \rho^{-n-k-1} R^k P_k^m(\cos\alpha)e^{im\beta} & \rho > R \quad (2\text{-}26) \end{cases}$$

CHAPTER 3

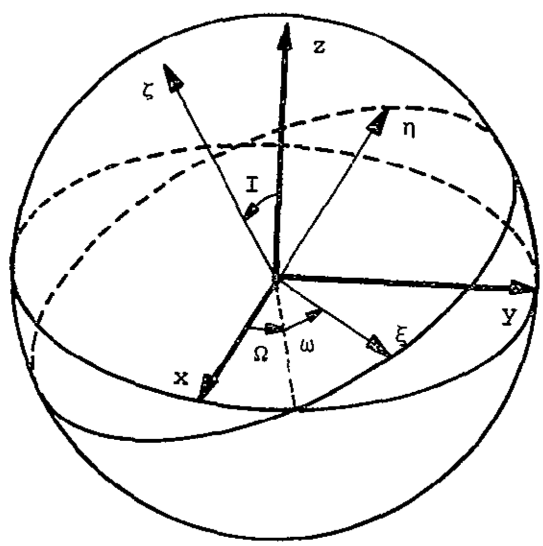
TRANSFORMATION OF SPHERICAL HARMONICS DUE TO ROTATION OF THE COORDINATE AXES

3.1 The Rotation Matrix

To effect a transformation of coefficients due to a rotation of coordinate axes (Chapter 4), the new harmonics are first determined in terms of the old ones. A particular rotation of axes defines a transformation matrix, M, such that

$$\underline{R}(R,\alpha,\beta) = M\ \underline{r}(r,\theta,\phi), \tag{3-1}$$

where M is formed by successive rotations through three Euler angles: 1) Ω about the z axis, 2) I about the newer x axis, and 3) ω about the new z axis, all in a right-handed sense (Figure 3-1).



$0 \leq \Omega \leq \pi$

$0 \leq I \leq \pi/2$

$0 \leq \omega \leq 2\pi$

Figure 3-1. Euler Angles Associated with a Rotation of Coordinate Axes

Adopting the method of Courant and Hilbert (Courant and Hilbert, Vol 1, pp 535-545), a skew symmetric matrix, A, is constructed in terms of three parameters, q_1, q_2 and q_3,

$$A = \begin{bmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{bmatrix} . \tag{3-2}$$

Three additional parameters, q_4, w and v, are introduced by

$$w = (q_1^2 + q_2^2 + q_3^2)^{1/2} \tag{3-3}$$

$$v = (q_1^2 + q_2^2 + q_3^2 + q_4^2)^{1/2} , \tag{3-4}$$

where q_1, q_2, q_3 and q_4 are all real. The immediate objective is to show that the orthogonal matrix M can be expressed as

$$M = (q_4 I_M + A)(q_4 I_M - A)^{-1} , \tag{3-5}$$

the exponent of minus one indicating the inverse of the matrix, and I_M representing the identity matrix

Note that $(q_4 I_M + A)$ and $(q_4 I_M - A)$ commute Hence, $(q_4 I_M + A)$ and $(q_4 I_M - A)^{-1}$ commute, since, in general, $D^{-1}C = CD^{-1}$, if $CD = DC$
It follows that

$$\begin{aligned} M^T &= (q_4 I_M - A^T)^{-1}(q_4 I_M + A^T) \\ &= (q_4 I_M - A)(q_4 I_M + A)^{-1} \\ &= M^{-1}, \end{aligned}$$

if A is skew symmetric (which has been assumed), *and if* the determinant of $(q_4 I_M - A)$ is different from zero, i e , $|q_4 I_M - A| \neq 0$

From (3-2), the characteristic equation of A can be determined,

$$\lambda^3 + w^2 \lambda = 0$$

Because a matrix satisfies its own characteristic equation,

$$A^3 + w^2 A = 0,$$

or

$$A = -\frac{A^3}{w^2}, \qquad w \neq 0 \tag{3-6}$$

From (3-5), again,

$$q_4 I_M + A = M(q_4 I_M - A), \tag{3-7}$$

or, using (3-6) and (3-4), the left-hand side of (3-7) is

$$q_4 I_M + A = q_4 I_M - \frac{A^3}{w^2}$$

$$= \frac{1}{v^2} (v^2 I_M + 2q_4 A + 2A^2)(q_4 I_M - A), \qquad v \neq 0$$

Comparing the above with (3-7), it is found that M can always be defined by

$$M = \frac{1}{v^2} (v^2 I_M + 2q_4 A + 2A^2), \qquad v > 0, \tag{3-8}$$

although it has yet to be shown that M is necessarily orthogonal when $|q_4 I_M - A| = 0$.

From the characteristic equation the eigenvalues of A are found to be 0, iw, and -iw For the case $|q_4 I_M - A| = 0$, q_4 must be an eigenvalue of A, but, since q_4 is real, it must be identically zero In this case, (3-8) becomes

$$M = I_M + \frac{2A^2}{w^2}, \qquad w > 0,$$

and, using (3-6), M can be solved for,

$$M = \begin{cases} I_M & A=0 \\ -I_M & A^2=-w^2 I_M \end{cases} \tag{3-9}$$

In either case, $M^T = M^{-1}$, and M is orthogonal This means that while (3-5) holds only for $|q_4 I_m - A| \neq 0$, equation (3-8) is valid for all $v > 0$

At this point it is necessary to examine the matrix M in more detail Since M is a rotation matrix, one of its eigenvalues (λ_1, for example) must be equal to unity The corresponding eigenvector coincides with the axis of rotation, since it is not altered in the rotation

It is also known (Hildebrand, 1965, p 51) that

$$\lambda_1\lambda_2\lambda_3 = |M|$$

$$= 1, \tag{3-10}$$

and that

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace } (M). \tag{3-11}$$

A second matrix, M', can be formed to perform the same rotation (Goldstein, 1950, p 123),

$$M' = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where ψ is defined as the angle of rotation Since only a similarity transformation is involved (Hildebrand, 1965, p 54),

$$\text{trace}(M) = \text{trace}(M')$$

$$= 1 + 2\cos\psi \tag{3-12}$$

From (3-9), (3-10), (3-11) and (3-12),

$$2\cos\psi = \lambda_2 + \frac{1}{\lambda_2}$$

Solving for λ_2 and λ_3, and using the results of (3-9),

$$\lambda_1 = 1, \qquad \lambda_2 = e^{i\psi}, \qquad \lambda_3 = e^{-i\psi}, \tag{3-13}$$

and, interestingly, it can be shown that the first eigenvector of M is simply

$$\underline{e}_1 = \frac{1}{w}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} .$$

From (3-8),

$$M = \frac{1}{v^2}\begin{bmatrix} q_4^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_1q_2-q_3q_4) & q_4^2+q_2^2-q_1^2-q_3^2 & 2(q_2q_3+q_1q_4) \\ 2(q_1q_3+q_2q_4) & 2(q_2q_3-q_1q_4) & q_4^2+q_3^2-q_1^2-q_2^2 \end{bmatrix} ,$$

(3-14)

so that

$$\mathrm{trace}(M) = \frac{4q_4^2 - v^2}{v^2}$$

Therefore, using equation (3-12),

$$\cos \psi = \frac{2q_4^2 - 1}{v^2}$$

$$\cos \frac{\psi}{2} = \pm \frac{q_4}{v}$$

$$\sin \frac{\psi}{2} = \pm(1 - q_4^2/v^2)^{1/2}, \qquad (3\text{-}15)$$

where the signs in the last two expressions depend on the quadrant, which is defined by the first equation.

The matrix M can also be expressed in terms of the Euler angles,

$$M = \begin{bmatrix} \cos\omega\cos\Omega & \cos\omega\sin\Omega & \sin\omega\sin I \\ -\cos I\sin\Omega\sin\omega & +\cos I\cos\Omega\sin\omega & \\ & & \\ -\sin\omega\cos\Omega & -\sin\omega\sin\Omega & \cos\omega\sin I \\ -\cos I\sin\Omega\cos\omega & +\cos I\cos\Omega\cos\omega & \\ & & \\ \sin I\sin\Omega & -\sin I\cos\Omega & \cos I \end{bmatrix}. \tag{3-16}$$

A comparison of (3-16) with (3-14) yields the values for the q's in terms of the Euler angles,

$$q_1 = v\sin(\tfrac{I}{2})\cos(\tfrac{\Omega-\omega}{2})$$

$$q_2 = v\sin(\tfrac{I}{2})\sin(\tfrac{\Omega-\omega}{2})$$

$$q_3 = v\cos(\tfrac{I}{2})\sin(\tfrac{\Omega+\omega}{2})$$

$$q_4 = v\cos(\tfrac{I}{2})\cos(\tfrac{\Omega+\omega}{2}) \tag{3-17}$$

Finally, it should be noted that the parameters ρ, σ and τ, used by Courant and Hilbert, and which will be used in section 3 3, are related to the Euler angles by

$$\sigma = \frac{\Omega-\omega}{2} - \frac{\pi}{2}$$

$$\rho = \frac{\Omega+\omega}{2}$$

$$\tau = \frac{I}{2} \quad , \tag{3-18}$$

or, alternatively,

$$\Omega = \sigma + \rho + \frac{\pi}{2}$$

$$\omega = \rho - \sigma - \frac{\pi}{2}$$

$$I = 2\tau \ .$$

3 2 The Transformation in Terms of a Set of Complex Polynomials, $S_{2n}^{m,k}(q_\nu)$

Two new complex variables, w_1 and w_2, are now introduced, and the coordinates x,y,z are mapped onto them by

$$w_1^2 = x + iy$$

$$w_2^2 = -x+iy$$

$$w_1 w_2 = z \quad , \tag{3-19}$$

where $x^2+y^2+z^2 = 0$ A transformation matrix, B, can, in turn, be constructed to linearly transform w_1 and w_2, just as M transforms x, y and z,

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = B \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{3-20}$$

The matrix B has two special properties (Goldstein, 1950, p 110),

$$B\,\bar{B}^T = I_M, \text{ and } |B| = +1,$$

such that if

$$B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix},$$

then

$$b_1b_4 - b_2b_3 = 1$$

$$b_1\bar{b}_1 + b_2\bar{b}_2 = 1$$

$$b_3\bar{b}_1 + b_4\bar{b}_2 = 0$$

From these, and using (3-1), (3-14), (3-19) and (3-20), it can be verified that

$$B = \frac{1}{v}\begin{bmatrix} q_4-iq_3 & -q_2+iq_1 \\ q_2+iq_1 & q_4+iq_3 \end{bmatrix} \qquad (3\text{-}21)$$

The new variables are also introduced into the left-hand side of equation (A-11) (through (3-19)) and expanded in a finite binomial series,

$$(-1)^{2n}(w_2+w_1t)^{2n} = \sum_{\ell=0}^{2n} \binom{2n}{\ell} w_2^{2n-\ell}(w_1t)^{\ell}$$

$$= t^n \sum_{\ell=0}^{2n} \binom{2n}{\ell} w_2^{2n-\ell}w_1^{\ell}t^{\ell-n}$$

Matching the coefficients of similar powers of t^n with those in (A-11) yields

$$^{(1)}H_{n,m}(\underline{r}) = \binom{2n}{n+m} w_1^{n+m} w_2^{n-m} \qquad (3\text{-}22)$$

Making use of (3-20) and (3-21), an expression analogous to (3-22) can be written as

$$\begin{aligned} ^{(1)}H_{n,m}(\underline{R}) &= \binom{2n}{n+m} w_1'^{\,n+m} w_2'^{\,n-m} \\ &= \binom{2n}{n+m} \left(\frac{w_2}{v}\right)^{2n} [-q_2+iq_1+(q_4-iq_3)s]^{n+m} \\ &\quad \times [q_4+iq_3+(q_2+iq_1)s]^{n-m} , \end{aligned} \qquad (3\text{-}23)$$

where $s \equiv w_1/w_2$

The next step is to introduce a new polynomial, $S_{2n}^{j,k}(q_\nu)$, and a dummy variable, t, through two expressions for a particular generating function, $G_{2n}(q_\nu,s,t)$,

$$\begin{aligned} G_{2n}(q_\nu,s,t) &= \sum_{j=0}^{2n} \sum_{k=0}^{2n} \binom{2n}{j} v^{2n} S_{2n}^{j-n,k-n}(q_\nu) t^j s^k \\ &= [iq_3(1-st)+iq_1(s+t)+q_2(s-t)+q_1(1+st)]^{2n}, \end{aligned} \qquad (3\text{-}24)$$

where, although it is not proved here (Courant and Hilbert, 1966, p 542),

$$\sum_{\nu=1}^{4} \frac{\partial^2}{\partial q_\nu^{\,2}} S_{2n}^{m,k}(q_\nu) = 0$$

The right-hand side of (3-24) can be expanded in a binomial series, and the coefficients of powers of t^j equated with those on the left-hand side,

$$\sum_{k=0}^{n} v^{2n} S_{2n}^{j-n,k-n}(q_\nu) s^k = [q_4+iq_3+s(q_2+iq_1)]^{2n-j}[-q_2+iq_1+s(q_4-iq_3)]^j \tag{3-25}$$

Since $s = w_1/w_2$, and letting $j = n+m$, the left-hand side of (3-25) is

$$\sum_{k=0}^{2n} v^{2n} S_{2n}^{m,k-n}(q_\nu) w_1^k w_2^{-k}$$

The right-hand side can be identified with part of (3-23), and, therefore, is equal to

$$^{(1)}H_{n,m}(\underline{R}) \frac{(w_2/v)^{-2n}}{\binom{2n}{n+m}}$$

Hence,

$$^{(1)}H_{n,m}(\underline{R}) = \binom{2n}{n+m} \sum_{k=0}^{2n} S_{2n}^{m,k-n}(q_\nu) w_1^k w_2^{2n-k} \tag{3-26}$$

From (3-22),

$$w_1^k w_2^{2n-k} = \frac{^{(1)}H_{n,k-n}(\underline{r})}{\binom{2n}{k}} ,$$

and inserting this into (3-26), followed by a change of summation index, gives

$$^{(1)}H_{n,m}(\underline{R}) = \sum_{k=-n}^{n} \frac{\binom{2n}{n+m}}{\binom{2n}{k+n}} S_{2n}^{m,k}(q_\nu) \; ^{(1)}H_{n,k}(\underline{r}) \tag{3-27}$$

The transformation formula for the spherical harmonics themselves can be easily obtained from (3-27), using (A-12) and the fact that R=r under pure rotation The transformation from the old harmonics to the new ones, in terms of the $S_{2n}^{m,k}$'s, is

$$P_n^m(\cos\alpha)e^{im\beta} = \sum_{k=-n}^{n} (-1)^{k-m} \frac{(n-k)!}{(n-m)!} S_{2n}^{m,k} P_n^k(\cos\theta)e^{ik\phi}$$

(3-28)

3.3 Evaluation of the Polynomial $S_{2n}^{m,k}$ as a Function of the Rotation Parameters

In section 3.2, the transformation of the spherical harmonics was formulated in terms of a set of polynomials, $S_{2n}^{m,k}(q_\nu)$'s, which have not, as yet, been evaluated. In this section, they will be determined as functions of the Euler angles associated with the rotation of coordinates.

Introduction of ρ, σ and τ, as defined by (3-18), allows the q's to be expressed as

$$q_1 = -v \sin\sigma \sin\tau$$

$$q_2 = v \cos\sigma \sin\tau$$

$$q_3 = v \sin\rho \cos\tau$$

$$q_4 = v \cos\rho \cos\tau$$

The generating function $G_{2n}(q_\nu;s,t)$ can then be written (from the last part of (3-24)) as

$$G_{2n}(\rho,\sigma,\tau,s,t) = v^{2n}[\cos\tau\, e^{i\rho}+s \sin\tau\, e^{-i\sigma} - t \sin\tau\, e^{i\sigma}+st \cos\tau\, e^{-i\rho}]^{2n}$$

This can be simplified by letting

$$s^* \equiv se^{-i(\rho+\sigma)} \ , \qquad t^* \equiv te^{-i(\rho-\sigma)} \ , \tag{3-29}$$

so that

$$G_{2n} = v^{2n} e^{i2\rho n} [\cos\tau \ (1+s^*t^*) + \sin\tau \ (s^*-t^*)]^{2n}$$

But, from the first part of (3-24), the same quantity can be expressed as

$$G_{2n} = \sum_{m=-n}^{n} \sum_{k=-n}^{n} \binom{2n}{n+m} v^{2n} S_{2n}^{m,k} t^{n+m} s^{k+n}$$

$$= \sum_{m=-n}^{n} \sum_{k=-n}^{n} \binom{2n}{n+m} v^{2n} e^{i(k+m)\rho} e^{i(k-m)\sigma} t^{*n+m} s^{*n+k} e^{2i\rho n} S_{2n}^{m,k}$$

Equating these two expressions yields

$$[(1+s^*t^*)\cos\tau + (s^*-t^*)\sin\tau]^{2n}$$

$$= \sum_{m=-n}^{n} \sum_{k=-n}^{n} \binom{2n}{m+n} S_{2n}^{m,k} e^{i(k+m)\rho} e^{i(k-m)\sigma} t^{*n+m} s^{*n+k} \ . \tag{3-30}$$

The left-hand side of (3-30) can be expanded in the familiar binomial series, and is

$$\sum_{p=-n}^{n} t^{*p+n} \binom{2n}{p+n} (s^* \cos\tau - \sin\tau)^{p+n} (\cos\tau + s^* \sin\tau)^{n-p}$$

The coefficients of powers of t^{*n+m} in the above series and in (3-30) are equated and then multiplied by $\cos^{n-m}\tau \ \sin^{m+n}\tau$, giving

$$\cos^{n-m}\tau \ \sin^{n+m}\tau \sum_{k=-n}^{n} s^{*n+k} S_{2n}^{m,k} e^{i(k+m)\rho} e^{i(k-m)\sigma}$$

$$= (\cos^2\tau + s^*\cos\tau\ \sin\tau)^{n-m}(s^*\cos\tau\ \sin\tau - 1 + \cos^2\tau)^{n+m} \tag{3-31}$$

To further simplify equation (3-31) and what follows, let

$$a \equiv \cos^2\tau$$

$$b \equiv \cos\tau\ \sin\tau$$

$$\chi \equiv s^* b\ ,$$

and apply Maclaurin's series in powers of χ to the right-hand side of (3-31),

$$\sum_{p=0}^{\infty} \left\{ \frac{\chi^p}{p!} \frac{d^p}{d\chi^p} [(a+\chi)^{n-m}(a+\chi-1)^{n+m}] \right\}_{\chi=0} \tag{3-32}$$

The evaluation of the quantity within the bracket is accomplished using a result of Hobson (Hobson, 1965, p 125),

$$\left[\frac{\partial^k}{\partial\chi^k} \phi(a+\chi) \right]_{\chi=0} = \frac{\partial^k}{\partial a^k}\ \phi(a)$$

In this case, let

$$\phi(a+\chi) \equiv (a+\chi)^{n-m}(a+\chi-1)^{n+m}\ ,$$

so that (3-32) becomes

$$\sum_{p=0}^{2n} \frac{s^{*p}}{p!} \cos^p\tau \sin^p\tau \frac{d^p}{d(\cos^2\tau)^p} [\cos^{2(n-m)}\tau (\cos^2\tau-1)^{n+m}] ,$$

since

$$\frac{d}{da} = \frac{\partial}{\partial a}$$

Substituting the above series into the right-hand side of equation (3-31) results in a power series in s* on both sides of the equation, so that the coefficients of s^{*n+k} can be equated With this, $S_{2n}^{m,k}$ can finally be solved for explicitly,

$$S_{2n}^{m,k}(\rho,\sigma,\tau) = e^{-i(k+m)\rho} e^{-i(k-m)\sigma} \cos^{k+m}\tau \sin^{k-m}\tau \frac{1}{(n+k)!} \frac{d^{n+k}}{d(\cos^2\tau)^{n+k}}$$

$$[\cos^{2(n-m)}\tau (\cos^2\tau-1)^{n+m}] \tag{3-33}$$

The derivatives in (3-33) can be related to the Jacobi polynomial J_n, which, in turn, can be expressed as a hypergeometric series, $_2F_1$ In general (Courant and Hilbert, Vol 1, pp 90,91),

$$J_n(p,q,x) \equiv \frac{x^{1-q}(1-x)^{q-p}}{q(q+1) \cdot (q+n-1)} \frac{d^n}{dx^n} [x^{q+n-1}(1-x)^{p+n-q}]$$

$$= {}_2F_1(-n,p+n,q,x) \qquad q > 0, \quad |x| < 1$$

Making the following identification of indices,

$$n+k \rightarrow n, \quad n+1-k \rightarrow p+n, \qquad 1-m-k \rightarrow q,$$

equation (3-33) becomes, for $m+k \leq 0$ and $\tau > 0$,

$$S_{2n}^{m,k} = (-1)^{n+m}\binom{n-m}{n+k} e^{-i(m+k)\rho} e^{-i(k-m)\sigma} \cos^{-m-k}\tau \ \sin^{m-k}\tau$$

$$\times \ {}_2F_1(-n-k, n+1-k, 1-m-k; \cos^2\tau), \tag{3-34}$$

where

$${}_2F_1(\alpha,\beta,\gamma,x) \equiv 1 + \frac{\alpha\ \beta}{1!\gamma}\, x + \frac{\alpha\ (\alpha+1)\ \beta\ (\beta+1)}{2!\gamma(\gamma+1)}\, x^2 + \cdots \quad . \tag{3-35}$$

It can be shown that

$$\bar{S}_{2n}^{m,k} = (-1)^{m+k} S_{2n}^{-m-k} \ , \tag{3-36}$$

so that for $m+k \geq 0$ and $\tau > 0$,

$$S_{2n}^{m,k} = (-1)^{n+k}\binom{n+m}{n-k} e^{-i(m+k)\rho} e^{-i(k-m)\sigma} \cos^{m+k}\tau \ \sin^{k-m}\tau$$

$$\times \ {}_2F_1(-n+k, n+1+k, 1+m+k, \cos^2\tau) \tag{3-37}$$

In the special case when $\tau = 0$, $S_{2n}^{m,k}$ can be evaluated directly from (3-31), all the polynomials are zero except for $k = m$, where

$$S_{2n}^{m,m}(\tau=0) = e^{-i2m\rho}$$

Using (3-18) again, the expressions for $S_{2n}^{m,k}$ are written in terms of the Euler angles Ω, I and ω,

$$
S_{2n}^{m,k} = \begin{cases}
(-1)^{n+m}\binom{n-m}{n+k} e^{-i(m-k)\pi/2} e^{-i(m\omega+k\Omega)} \cos^{-m-k}\frac{I}{2}\sin^{m-k}\frac{I}{2} \\
\times\ {}_2F_1(-n-k, n+1-k, 1-m-k, \cos^2\frac{I}{2}), \quad m+k \le 0,\ I \ne 0\ , & (3\text{-}38) \\[2ex]
(-1)^{n+k}\binom{n+m}{n-k} e^{-i(m-k)\pi/2} e^{-i(m\omega+k\Omega)} \cos^{m+k}\frac{I}{2}\sin^{k-m}\frac{I}{2} \\
\times\ {}_2F_1(-n+k, n+1+k, 1+m+k; \cos^2\frac{I}{2}), \quad m+k \ge 0,\ I \ne 0\ , & (3\text{-}39) \\[2ex]
e^{-im(\omega+\Omega)} \qquad\qquad k = m,\ I = 0, \\[2ex]
0 \qquad\qquad k \ne m,\ I = 0 & (3\text{-}40)
\end{cases}
$$

When k assumes the value of zero, $S_{2n}^{m,k}$ is independent of Ω (and vice-versa), and $S_{2n}^{m,k}$ reduces to a function involving a Legendre function From (3-39) for k=0 and $m \ge 0$,

$$
S_{2n}^{m,0} = (-1)^n i^{-m}\binom{n+m}{n} e^{-im\omega} \cot^m\frac{I}{2}\ {}_2F_1(-n, n+1, 1+m, \cos^2\frac{I}{2}) \tag{3-41}
$$

But in this case the hypergeometric function is of the form (Abramowitz, 1965, p 332)

$$
F(-n, n+1, 1-p, \frac{1-x}{2}) = \Gamma(1-p)\ \frac{(1-x)^{p/2}}{(1+x)^{p/2}}\ P_n^p(x)
$$

Letting x = -cos I and m = -p, this becomes

$$
F(-n, n+1, 1+m, \cos^2\frac{I}{2}) = m!\ \tan^m\frac{I}{2}\ P_n^{-m}(-\cos I)
$$

Then, using (2-4) and (2-5), this is

$$F(-n,n+1,1+m,\cos^2 \tfrac{I}{2}) = (-1)^n m! \tan^m \frac{I}{2} \frac{(n-m)!}{(n+m)!} P_n^m(\cos I) \tag{3-42}$$

Substituting (3-42) into (3-41) yields the special result for $k = 0$, which is valid for all m,

$$S_{2n}^{m,0} = i^{-m} \frac{(n-m)!}{n!} e^{-im\omega} P_n^m(\cos I) \tag{3-43}$$

Due to the lack of standardization of the notation in the literature, Table 3-1 has been included to assist anyone researching the subject

Table 3-1 Notation Used by Several Authors to Represent the Same Quantities

	Lee	Courant, Hilbert	Aardoom	Jeffreys	Goldstein
Subscripts					
order	n	n	n	ℓ	
degree	m	ℓ	s	m	
summation	k	r	k,ℓ	s	
Polynomials					
associated Legendre	P_n^m	$(-1)^{\ell} P_{n,\ell}$	P_n^s	$\frac{(-1)^{\ell}\ell'}{(\ell-m)'} P_{\ell}^m$	
transformation	$S_{2n}^{m,k}$	$S_{2n}^{\ell,r}$	$(\ 1)^{s+\ell}\bar{S}_{2n}^{\,n+s,n+\ell}$		
Euler Angles					
longitude of ascending node	Ω	$\sigma+\rho+\frac{\pi}{2}$	$\mu+\frac{\pi}{2}$	$\Omega, \eta+\frac{\pi}{2}$	ϕ
inclination	I	2τ	θ	I	θ
argument of perihelion	ω	$\rho-\sigma-\frac{\pi}{2}$	$\chi-\frac{\pi}{2}$	$\tilde{\omega}, \chi-\frac{\pi}{2}$	ψ
Old Coordinates					
colatitude	θ	ϕ	$\frac{\pi}{2}-\phi$	θ	
longitude	ϕ	λ	λ	λ	
radial distance	r		r	r	
New Coordinates					
colatitude	α	ϕ'	$\frac{\pi}{2}-\phi'$	θ'	
longitude	β	λ'	λ'	λ'	
radial distance	R		r'	r'	

CHAPTER 4

TRANSFORMATION OF A SERIES EXPANSION IN SOLID SPHERICAL HARMONICS

4.1 Explanation of the Transformation Formulas

In chapters two and three, transformation formulas were developed for transforming solid spherical harmonics under changes of coordinate axes Two classes of harmonics were treated because a general expansion of a harmonic function will involve either, or both, kinds A function, V, for example, in the old coordinate system, is of the form

$$V(r,\theta,\phi) = \frac{GM}{a} \sum_{s=0}^{\infty} \sum_{t=0}^{s} P_s^t(\cos\theta) \left\{ \left(\frac{a}{r}\right)^{s+1} [C_{st}\cos t\phi + S_{st}\sin t\phi] \right.$$

$$\left. + \left(\frac{r}{a}\right)^{s} [E_{st}\cos t\phi + F_{st}\sin t\phi] \right\}, \qquad (4\text{-}1)$$

where G, M and a are all constants, along with the coefficients C_{st}, S_{st}, E_{st} and F_{st} (see table 4-1) As implied by the notation, V can represent a potential function in free space (Kellogg, 1953, p 218) If V is a gravitational potential, G is the universal gravitational constant and M is the mass of the gravitating body The form of the expansion in (4-1) is probably the more familiar, but a slightly different, more symmetrical form will be used here,

$$V(r,\theta,\phi) = \frac{GM}{a} \sum_{s=0}^{\infty} \sum_{t=-s}^{s} P_s^t(\cos\theta) \left\{ \left(\frac{a}{r}\right)^{s+1} \mathrm{Re}[(A_{st}+iB_{st})e^{-it\phi}] \right.$$

$$\left. + \left(\frac{r}{a}\right)^{s} \mathrm{Re}[(T_{st}+iW_{st})e^{-it\phi}] \right\}, \qquad (4\text{-}2)$$

where Re stands for the real part of quantity enclosed by brackets

	Old Coefficients		New Coefficients	
Series	$(\frac{a}{r})^{s+1}$	$(\frac{r}{a})^{s}$	$(\frac{a}{R})^{n+1}$	$(\frac{R}{a})^{n}$
Asymmetric Form $0 \leq t \leq s$ $0 \leq m \leq n$	C_{st}, S_{st}	E_{st}, F_{st}	C'_{nm}, S'_{nm}	$E'_{nm}, F'_{nm}, \overset{*}{E}'_{nm}, \overset{*}{F}'_{nm}$
Symmetric Form $-s \leq t \leq s$ $-n \leq m \leq n$	A_{st}, B_{st}	T_{st}, W_{st}	J_{nm}, K_{nm}	$U_{nm}, V_{nm}, \overset{*}{U}_{nm}, \overset{*}{V}_{nm}$

Table 4-1 Notation Used for the Series Coefficients

Using (2-4), it can be seen from (4-2) that

$$A_{st}+iB_{st} = \frac{(-1)^{s}(s-t)!}{(s+t)!}(A_{s-t}-iB_{s-t})$$

Hence, the coefficients in the two representations are related simply by

$$\left.\begin{aligned} C_{st} &= 2A_{st} \\ S_{st} &= 2B_{st} \end{aligned}\right\} \quad t > 0$$

$$\left.\begin{aligned} C_{s0} &= A_{s0} \\ S_{s0} &= B_{s0} \end{aligned}\right\} \quad t = 0 \qquad (4\text{-}3)$$

Analogous relationships hold for the other four sets of coefficients in Table 4-1

After translation and/or rotation, the new series will be, in general,

$$V(R,\alpha,\phi) = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n^m(\cos\,\alpha) \left\{ \left(\frac{a}{R}\right)^{n+1} \mathrm{Re}[(J_{nm}+iK_{nm})e^{-im\beta}] \right.$$

$$\left. + \left(\frac{R}{a}\right)^{n} \mathrm{Re}\left\{[U_{nm}+\overset{*}{U}_{nm}+i(V_{nm}+\overset{*}{V}_{nm})]e^{-im\beta}\right\}\right\}. \qquad (4\text{-}4)$$

Once the coefficients are determined, they may be converted to the asymmetric series coefficients by (4-3), if so desired

	Asymmetric Form			Symmetric Form		
Old Coefficients	C_{st},S_{st}		E_{st},F_{st}	A_{st},B_{st}		T_{st},W_{st}
	$R>\rho$	$R<\rho$		$R>\rho$	$R<\rho$	
New Coefficients	C'_{nm},S'_{nm}	$\overset{*}{E}_{nm},\overset{*}{F}_{nm}$	E'_{nm},F'_{nm}	J_{nm},K_{nm}	$\overset{*}{U}_{nm},\overset{*}{V}_{nm}$	U_{nm},V_{nm}

Table 4-2 Conversion of Series Coefficients Under Translation

Nothing has been said, thus far, regarding the convergence of the various series expansions The original series, whether it is in powers of r^n or r^{-n-1}, or both, is assumed to converge For special cases, the constant "a" can be associated with a characteristic length of the gravitating body, and convergence of the series linked to the ratio of r/a

The primary concern here is the convergence of the resultant series In the case of a straight rotation of axes, as was seen in 3 2, there is no problem, as the A_{st},B_{st}'s transform into the J_{nm},K_{nm}'s and the T_{st},W_{st}'s into the U_{nm},V_{nm}'s With a translation, however, the T_{st},W_{st}'s transform into the U_{nm},V_{nm}'s, while the A_{st},B_{st}'s transform into $\overset{*}{U}_{nm}, \overset{*}{V}_{nm}$'s if $R<\rho$, or J_{nm}, K_{nm}'s if $R>\rho$ (See figure 4-1 and table 4-2)

In other words, the exact form of the series in the latter two cases depends on the ratio of the lengths of both the field and translation vectors The critical sphere separating the two regions of convergence is of radius ρ, centered on the new origin, and passes through the old origin

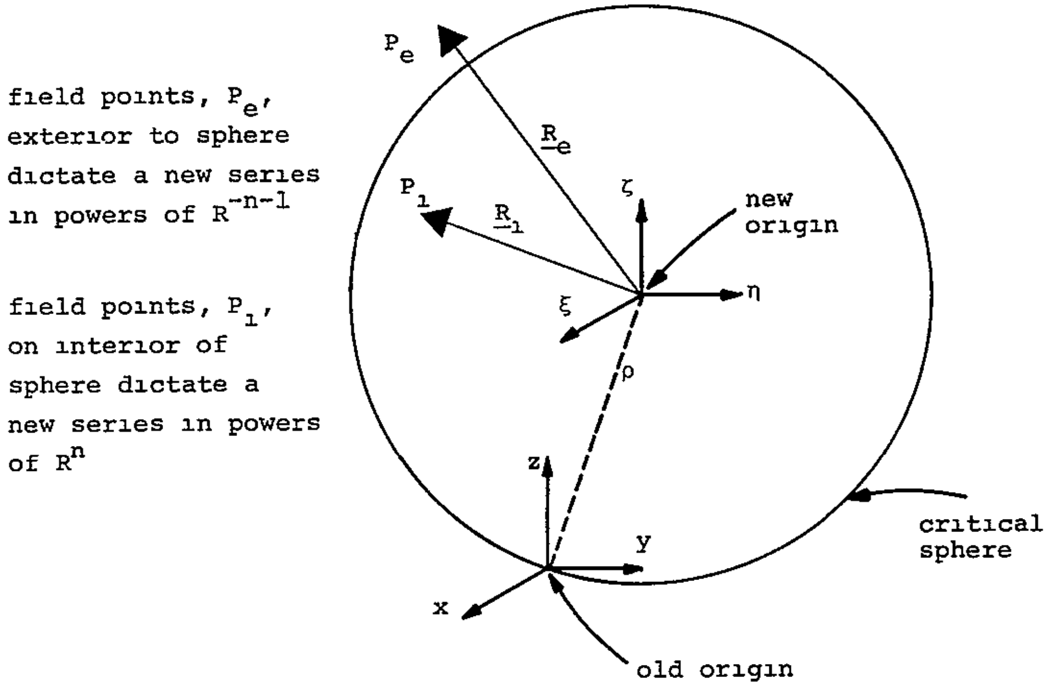


Figure 4-1 Critical Sphere for a Transformation Due to Translation of a Series Involving Powers of r^{-s-1}

The various cases are treated separately in the following four sections of this chapter

4 2 Translation Transformation of a Series Involving Powers of r^s, to One with Powers of R^n

In this section it is assumed that a function, $V(r,\theta,\phi)$, has been expressed in terms of convergent series of solid spherical harmonics of the following form

$$V = \frac{GM}{a} \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \left(\frac{r}{a}\right)^{s} P_s^t(\cos\theta)\,\mathrm{Re}[(T_{st}+iW_{st})e^{-it\phi}] \ , \qquad (4\text{-}5)$$

where the T_{st}'s and W_{st}'s are known With a pure translation of the origin the same function is expressed in a new series in the new coordinate system,

$$V(R,\alpha,\beta) = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\frac{R}{a}\right)^{n} P_n^m(\cos\alpha)\,\mathrm{Re}[(U_{nm}+iV_{nm})e^{-im\beta}] , \qquad (4\text{-}6)$$

where the U_{nm}'s and V_{nm}'s need to be evaluated

Equation (4-5) can be written as

$$V = \mathrm{Re}\left\{\frac{GM}{a} \sum_{s=0}^{\infty} \sum_{t=-s}^{s} P_s^t(\cos\theta)\left(\frac{r}{a}\right)^{s}(T_{st}+iW_{st})e^{-it\phi}\right\}$$

Using (2-16) and (2-17), which transform the harmonics to the new coordinate system, this becomes

$$V = \mathrm{Re}\left\{\frac{GM}{a} \sum_{s=0}^{\infty} \sum_{t=-s}^{s} (T_{st}+iW_{st})a^{-s} \sum_{k=0}^{\infty} \sum_{\ell=L\left\{\begin{smallmatrix}-k\\ k+t-s\end{smallmatrix}\right.}^{S\left\{\begin{smallmatrix}k\\ -k+s+t\end{smallmatrix}\right.} \right.$$

$$\left. \times \binom{s+t}{k+\ell}\rho^{s-k}e^{-i(t-\ell)\gamma}P_{s-k}^{t-\ell}(\cos\lambda)R^{k}e^{-i\ell\beta}P_k^{\ell}(\cos\alpha)\right\} \qquad (4\text{-}7)$$

Switching the inner two summations to the outside, and changing notation, (4-7) can be written as

$$V = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{s=n}^{\infty} \sum_{t=n+m-s}^{s-n+m} \binom{s+t}{n+m} a^{-n} \left(\frac{\rho}{a}\right)^{s-n} R^n$$

$$\times P_n^m(\cos\alpha) P_{s-n}^{t-m}(\cos\lambda) \operatorname{Re}[(T_{st} + iW_{st}) e^{-i(t-m)\gamma} e^{-im\beta}] \tag{4-8}$$

The real part of the bracketed portion of (4-8) is

$$\cos m\beta[T_{st}\cos(t-m)\gamma + W_{st}\sin(t-m)\gamma]$$

$$+\sin m\beta[-T_{st}\sin(t-m)\gamma + W_{st}\cos(t-m)\gamma] \tag{4-9}$$

Inspection of equation (4-6), the other expression for the potential, shows that the harmonics of degree n, order m, in the same variables, must match those in (4-8) The coefficients are then equated, where the real portion of the brackets in (4-6) is

$$U_{nm}\cos m\beta + V_{nm}\sin m\beta$$

Thus, (4-8) is

$$\begin{bmatrix} U_{nm} \\ V_{nm} \end{bmatrix} = \sum_{s=n}^{\infty} \sum_{t=n+m-s}^{s-n+m} \binom{s+t}{n+m} \left(\frac{\rho}{a}\right)^{s-n} P_{s-n}^{t-m}(\cos\lambda)$$

$$\times \begin{bmatrix} T_{st}\cos(t-m)\gamma + W_{st}\sin(t-m)\gamma \\ -T_{st}\sin(t-m)\gamma + W_{st}\cos(t-m)\gamma \end{bmatrix},$$

or using (2-4) and regrouping the quantities within the brackets,

$$U_{nm}+iV_{nm} = \sum_{s=n}^{\infty} \sum_{t=n+m-s}^{s-n+m} \frac{(-1)^{m-t}(s+t)!}{(m-n+s-t)!(n+m)!} \left(\frac{\rho}{a}\right)^{s-n}$$

$$\times P_{s-n}^{m-t}(\cos\lambda)e^{i(m-t)\gamma}(T_{st}+iW_{st}) \tag{4-10}$$

4 3 Translation Transformation of a Series Involving Powers of r^{-s-1} to One with Powers of R^n, $R<\rho$

In this case, the coefficients A_{st} and B_{st} are assumed to be known in the expansion

$$V(r,\theta,\phi) = \frac{GM}{a} \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \left(\frac{a}{r}\right)^{s+1} P_s^t(\cos\theta)\,\mathrm{Re}\left[(A_{st}+iB_{st})e^{-it\phi}\right] \tag{4-11}$$

The translation of the origin transforms the series into the form of (4-6), with the $\overset{*}{U}_{nm}$'s and $\overset{*}{V}_{nm}$'s unknown

The method of solving for the coefficients is analogous to that of section 4 2, using equations (2-22) and (2-23) to substitute for the "old" solid harmonics in (4-11) The result is

$$\overset{*}{U}_{nm}+i\overset{*}{V}_{nm} = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \frac{(-1)^{n+m+t}(n+s-m+t)!}{(s-t)!(n+m)!}$$

$$\times \left(\frac{a}{\rho}\right)^{s+n+1} P_{n+s}^{m-t}(\cos\lambda)e^{i(m-t)\gamma}(A_{st}+iB_{st}) \tag{4-12}$$

4 4 Translation Transformation of a Series Involving Powers of r^{-s-1} to One with Power of R^{-n-1}, $R>\rho$

Here the coefficients in the old series (4-11) are known, but the J_{nm}'s and K_{nm}'s are to be determined for the new series

$$V(R,\alpha,\beta) = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\frac{a}{R}\right)^{n+1} P_n^m(\cos\,\alpha)\,\mathrm{Re}[(J_{nm}+iK_{nm})e^{-im\beta}] \tag{4-13}$$

Again, the procedure is the same used in 4 2 and 4 3, equations (3-22) and (2-23) provide the transformation formulas for the solid harmonics The result is

$$J_{nm}+iK_{nm} = \sum_{s=0}^{n} \sum_{t=L\begin{cases}-s\\ s+m-n\end{cases}}^{S\begin{cases}s\\ n+m-s\end{cases}} \frac{(-1)^{n-s}(n-m)!}{(s-t)(n+m-s-t)!}$$

$$\times \left(\frac{\rho}{a}\right)^{n-s} P_{n-s}^{m-t}(\cos\,\lambda)e^{i(m-t)\gamma}(A_{st}+iB_{st}) \tag{4-14}$$

4 5 <u>Rotation</u>

A slightly different approach is taken in the case of a rotation, since the new harmonics have been determined in terms of the old ones (3-28), instead of vice-versa, as for the translation The two approaches are equally correct, the appropriate choice depends entirely on the direction in which the harmonics are transformed

Because the magnitude of $\underline{r}$ remains unchanged under rotation, there is only one form for the transformation, as opposed to the three for the translation Series involving A_{st}, B_{st}'s and T_{st}, W_{st}'s are, therefore, transformed by the same formula The first case is worked out here

The A_{st}'s and B_{st}'s are assumed to be known in the expansion

$$V(r,\theta,\phi) = \frac{GM}{a} \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \left(\frac{a}{r}\right)^{s+1} P_s^t(\cos\,\theta)\,\mathrm{Re}[(A_{st}+iB_{st})e^{-it\phi}] \tag{4-15}$$

But, as pointed out in 4 1, V can be considered as a potential function Hence, it can be represented by an integral over the body generating the potential,

$$V(r,\theta,\phi) = G\int_M \frac{dm'}{|\underline{r}-\underline{r}'|},$$

where the primes indicate variables of integration, dm' being an element of mass, in the case of a gravitational potential

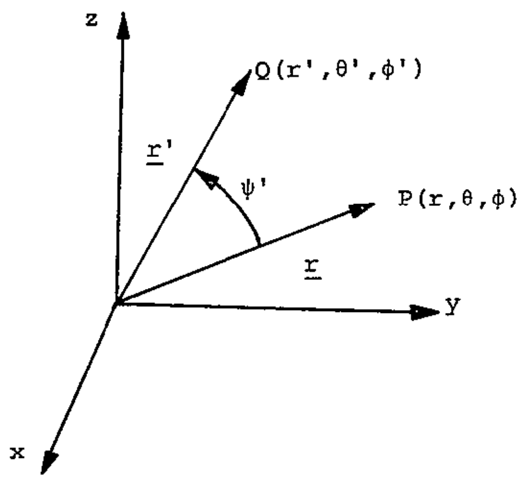


Figure 4-2 Definition of the Variables of Integration, r',θ',ϕ',ψ', for an Element of Mass at Q and a Field Point at P

Using the law of cosines in terms of the angle ψ' between the field vector and the integration vector,

$$|\underline{r}-\underline{r}'| = (r^2+r'^2-2rr'\cos\psi')^{1/2}$$

Then, applying the generating function for Legendre polynomials (2-9), followed by the addition theorem (2-10), and using (2-4), the potential is

$$V(r,\theta,\phi) = G\int_M dm' \sum_{n=0}^{\infty} r_n' \sum_{k=-n}^{n} \frac{(n-k)!}{(n+k)!} P_n^k(\cos\theta)$$

$$\times P_n^k(\cos\theta')\,\mathrm{Re}[e^{-ik(\phi-\phi')}], \tag{4-16}$$

where

$$r_n' \equiv \begin{cases} \dfrac{r'}{r^{n+1}} & \dfrac{r'}{r} < 1 \\[2ex] \dfrac{r^n}{r'^{\,n+1}} & \dfrac{r'}{r} > 1 \end{cases} \tag{4-17}$$

Equating the coefficients of each harmonic $P_s^t(\cos\theta)$ in (4-15) and (4-16) yields

$$\frac{a}{M}\int_M dm' r_s' P_s^t(\cos\theta') e^{it\phi'} = \frac{(s+t)!}{(s-t)!}\,(A_{st}+iB_{st}) \tag{4-18}$$

After the rotation, the "new" potential, $V(R,\alpha,\beta)$, is of the form of (4-13) But it, too, can be written in terms of a volume integral in exactly the same way as $V(r,\theta,\phi)$ Thus,

$$V(R,\alpha,\beta) = \int_M dm' \sum_{n=0}^{\infty} R_n' \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\alpha')$$

$$\times P_n^m(\cos\alpha)\,\mathrm{Re}[e^{-im(\beta-\beta')}],$$

or, equating the coefficients of the harmonics, $P_n^m(\cos\alpha)$,

$$J_{nm}+iK_{nm} = \frac{a}{M}\frac{(n-m)!}{(n+m)!}\int_M dm' R_n' P_n^m(\cos\alpha') e^{im\beta'}$$

Substituting for $P_n^m(\cos\alpha')e^{im\beta'}$ from (3-28), this becomes

$$J_{nm}+iK_{nm} = \frac{a}{M}\sum_{k=-n}^{n}(-1)^{k-m}\frac{(n-k)!}{(n+m)!}S_{2n}^{m,k}\int_M dm' R_n' \, P_n^k(\cos\theta')e^{ik\phi'} \tag{4-19}$$

But, since R=r and R'=r', the integral (4-18) can be substituted for the integral in (4-19), resulting in

$$J_{nm}+iK_{nm} = \sum_{k=-n}^{n}(-1)^{k-m}\frac{(n+k)!}{(n+m)!}S_{2n}^{m,k}(A_{nk}+iB_{nk}) \tag{4-20}$$

As pointed out earlier, the relationship between the U_{nm}, V_{nm}'s and the T_{st},W_{st},'s is the same,

$$U_{nm}+iV_{nm} = \sum_{k=-n}^{n}(-1)^{k-m}\frac{(n+k)!}{(n+m)!}S_{2n}^{m,k}(T_{nk}+iW_{nk}), \tag{4-21}$$

and note that only coefficients of the same degree (n) are used to determine the new ones

Finally, as a matter of completeness, it is possible to indirectly solve for the old spherical harmonics in terms of the new ones (under rotation) using (4-20) and (3-28) In chapter three the harmonics were transformed in the opposite order - the new ones in terms of the old

With the method developed in 4 2, it can be easily verified that

$$P_s^t(\cos\theta)e^{it\phi} = \sum_{m=-s}^{s}(-1)^{t-m}\frac{(s+t)!}{(s+m)!}\overline{S}_{2s}^{m,t}P_s^m(\cos\alpha)e^{im\beta} \tag{4-22}$$

Comparison of this with (3-28) reveals their similarity, except that in this case the complex conjugate of $S_{2s}^{m,t}$ is involved, and it is summed over the first superscript as opposed to the second in (3-28)

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CHAPTER 5

THE FORCE BETWEEN TWO HOMOGENEOUS HEMISPHERES

5 1 Potential Due to a Hemisphere

One of the primary objectives of this thesis is to demonstrate the applicability of the transformations developed in chapters 2-4 to problems involving volume integrals of bodies possessing rotational symmetry As an example, the force between two bodies A and B can be calculated from

$$\underline{F} = \rho_d \int_{\substack{\text{volume}\\ \text{of B}}} \underline{\nabla} V \, d\tau,$$

where ρ_d is the mass density of A, and V is the gravitational potential of A If B is homogeneous this can be expressed, using the gradient theorem, as a surface integral involving V,

$$\underline{F} = \rho_d \int_{\substack{\text{surface}\\ \text{of B}}} V \, \underline{n} \, dS, \tag{5-1}$$

where $\underline{n}$ is the outward normal to the surface

In the evaluation of the integral in (5-1) it is desirable to make the integrand as simple as possible This is accomplished by selecting the appropriate type of coordinates - in this case spherical - and orienting the coordinate system to take full advantage of the symmetry of body B But V also has to be determined in the coordinate system best suited for body A Hence, once V has been calculated, the coordinates are translated and rotated for the evaluation of the integral in (5-1) As shown in chapter 4, the potential V is expressed in the new system with a set of transformed coefficients.

This method is ideally suited for two homogeneous hemispheres, due to their simple shapes The potential of a hemisphere along its axis of rotation can be found from (see Figure 5-1),

$$V = \rho_d G \int_{\phi=0}^{2\pi} \int_{\theta=\pi/2}^{\pi} \int_{R=0}^{a} \frac{d\tau}{s'} \qquad (5\text{-}2)$$

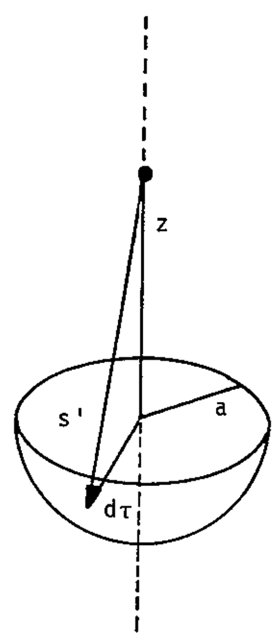


Figure 5-1 Calculation of the Potential Due to a Hemisphere at a Point on Its Polar Axis

Evaluation of (5-2) yields

$$V(z) = \frac{2}{3} \pi \rho_d G \frac{a^3}{z} \left\{ 1 + \frac{3}{2}\left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^3 - \left[1 + \left(\frac{z}{a}\right)^2\right]^{3/2} \right\} \qquad (5\text{-}3)$$

The off-axis potential is determined by expanding (5-3) in powers of $(z/a)^n$ or $(z/a)^{-n-1}$ and substituting $(r/a)^n P_n(\cos\theta)$ or $(r/a)^{-n-1} P_n(\cos\theta)$, respectively (MacMillan, 1958, pp 360-362) This gives

$$V(r,\cos\theta) = \frac{2}{3}\pi\rho_d G a^2 \left\{ \frac{3}{2} - \frac{3}{2}\frac{r}{a}P_1(\cos\theta) + \left(\frac{r}{a}\right)^2 P_2(\cos\theta) \right.$$

$$-\frac{3}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + 3\sum_{n=2}^{\infty} \frac{(-1)^n (2n-3)!}{2^{2n-1}(n+1)!(n-2)!}$$

$$\left. \times\left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta) \right\}, \qquad 0 \leq r \leq a, \qquad (5\text{-}4)$$

or

$$V(r,\cos\theta) = \frac{2}{3}\pi\rho_d G a^2 \left\{ \frac{a}{r} - \frac{3}{8}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) \right.$$

$$+ \frac{1}{16}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) - 3\sum_{n=2}^{\infty} \frac{(-1)^n (2n-1)!}{2^{2n+1}(n+2)!(n-1)!}$$

$$\left. \times\left(\frac{a}{r}\right)^{2n+2} P_{2n+1}(\cos\theta) \right\}, \qquad r \geq a \qquad (5\text{-}5)$$

5 2 Force Between Two Hemispheres

The force exerted by a hemisphere on a second identical one is calculated using (5-1), (5-4) and (5-5) The center of the face of the second is located by the vector $\underline{\rho}$ relative to the first (Figure 5-2(a))

The force vector $\underline{F}$ is determined relative to the new coordinate system ξ, η, ζ, or R, α, β In this system $\underline{n}\, dS$ is

$$\underline{n}\, dS = \begin{cases} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} R \sin\alpha \, dR \, d\beta & \text{flat surface } (\alpha = \pi/2) \\ \begin{bmatrix} \sin\alpha \cos\beta \\ \sin\alpha \sin\beta \\ \cos\alpha \end{bmatrix} R^2 \sin\alpha \, d\alpha \, d\beta & \text{spherical surface } (R=a), \end{cases}$$

so that (5-1) is comprised of two integrals

$$\underline{F} = \underline{F}_f + \underline{F}_s$$

$$= \rho_d \int_{\substack{\text{flat}\\ \text{surface}}} V(\alpha = \pi/2) \begin{bmatrix} 0 \\ 0 \\ -R \end{bmatrix} dR \, d\beta$$

$$+ \rho_d a^2 \int_{\substack{\text{spherical}\\ \text{surface}}} V(R=a) \begin{bmatrix} \sin^2\alpha \cos\beta \\ \sin^2\alpha \sin\beta \\ \sin\alpha \cos\alpha \end{bmatrix} d\alpha \, d\beta \qquad (5\text{-}6)$$

From (4-4) the potential of hemisphere A in the two cases, after coordinate transformations, is

$$V(\alpha = \pi/2) = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n^m(0) \left\{ \left(\frac{a}{r}\right)^{n+1} [J_{nm} \cos m\beta + K_{nm} \sin m\beta] \right.$$

$$\left. + \left(\frac{R}{a}\right)^n [U_{nm} + \overset{*}{U}_{nm}) \cos m\beta + (V_{nm} + \overset{*}{V}_{nm}) \sin m\beta] \right\} , \qquad (5\text{-}7)$$

and

$$V(R=a) = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n^m(\cos\ \alpha) [(J_{nm}+U_{nm}+\overset{*}{U}_{nm}) \cos\ m\beta$$

$$+ (K_{nm}+V_{nm}+\overset{*}{V}_{nm}) \sin\ m\beta] \qquad (5\text{-}8)$$

Although the extreme limits of the integration variables in (5-6) are specified,

$$0 \leq R \leq a$$

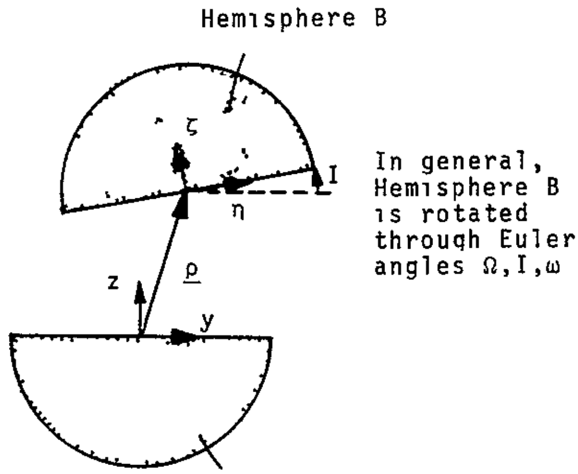
$$0 \leq \alpha \leq \pi/2$$

$$0 \leq \beta \leq 2\pi,$$

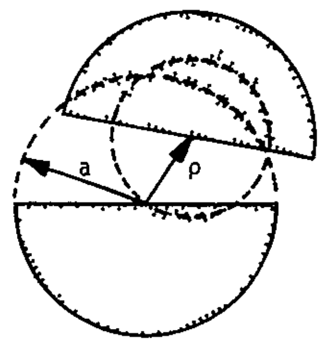
the particular limits corresponding to the region of convergence for determination of the coefficients in (5-7) and (5-8) is somewhat complicated, depending on the relative positions of the two bodies

In general there are five distinct cases for evaluating (5-6) The first is the case when the two hemispheres are face-to-face with no separation, i e , $\underline{\rho} = \underline{0}$ Since all surface points relative to the old origin, r_s, are equal to the radius, and since all R_s are greater than ρ, the integration of (5-6) involves only the coefficients of the U_{nm}, V_{nm}'s or the J_{nm}, K_{nm}'s, but not both

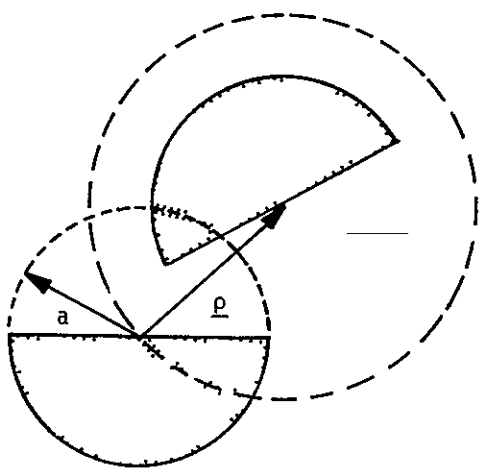
The second case is distinct because for every surface point, either $r_s < a$ or $R_s > \rho$ The integral, therefore, involves only U_{nm}, V_{nm}'s and J_{nm}, K_{nm}'s This includes all ρ up to ρ_1, where ρ_1 is the radius of the sphere centered at the new origin, passing through the old origin, and reaching out to the first intersection of the surface of hemisphere



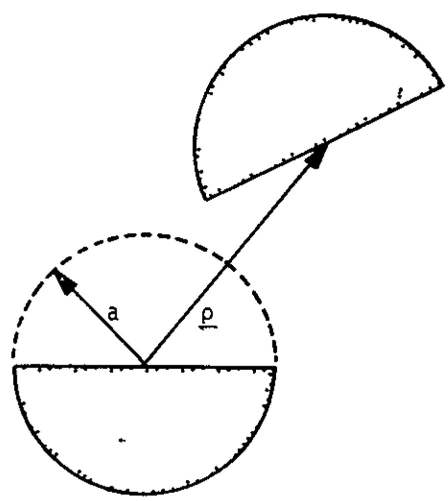
5-2(a) Coordinates



5-2(b) Case 2



5-2(c) Case 4



5-2(d) Case 5

Figure 5-2 Geometry for Two Hemispheres

Case	Criteria	Valid for These ρ	Either Flat or Spherical Surface Integrals, or Both, Involve These Coefficients
I	$r_s=a$ <u>and</u> $R_s \geq \rho$	$\rho=0$	U_{nm},V_{nm}'s or J_{nm},K_{nm}'s
II	$r_s<a$ <u>or</u> $R_s>\rho$	$0<\rho\leq\rho_1$	U_{nm},V_{nm}'s and J_{nm},K_{nm}'s
III	general case	$\rho_1<\rho\leq\rho_2$	U_{nm},V_{nm}'s and J_{nm},K_{nm}'s and $\overset{*}{U}_{nm},\overset{*}{V}_{nm}$'s
IV	$r_s<a$ <u>or</u> $R_s<\rho$	$\rho_2<\rho\leq\rho_3$	U_{nm},V_{nm}'s and $\overset{*}{U}_{nm},\overset{*}{V}_{nm}$'s
V	all $r_s>a$	$\rho_3<\rho$	$\overset{*}{U}_{nm},\overset{*}{V}_{nm}$'s

<u>Critical ρ's</u>

$a/2 \leq \rho_1 \leq a/\sqrt{2}$

$a \leq \rho_2 \leq 2a$

$a \leq \rho_3 \leq 2a$

Table 5-1 Summary of the Five Cases Involved in Evaluation of the Force Integrals

B with the sphere of radius a centered at the old origin (Figure 5-2(b)) Examination of the geometry shows that $0 < \rho \leq \rho_1$ and $a/2 \leq \rho_1 \leq a/\sqrt{2}$

The third case will be discussed after cases four and five The fourth case is special because it includes integrals of only the U_{nm}, V_{nm}'s and $\overset{*}{U}_{nm}, \overset{*}{V}_{nm}$'s From Figure 5-2(c) it can be seen that hemisphere B must lie within the sphere of radius a centered at the old origin, or within the sphere of radius ρ centered at the new origin In this case either $r_s < a$ or $R_s < \rho$. From Figure 5-4, $\rho_2 < \rho \leq \rho_3$, where

$$a \leq \rho_2 \leq 2a \quad \text{and} \quad a \leq \rho_3 \leq 2a$$

Due to the overlapping of the possible limits of integration, this case may not always be applicable

The fifth case applies when all $R_s \leq \rho$. The integral, therefore, involves only the coefficients $\overset{*}{U}_{nm}, \overset{*}{V}_{nm}$'s, and $\rho_3 < \rho$

Case three is the general case and includes everything not covered in cases 1,2,4 and 5 In order to evaluate integrals over both the flat and the spherical surfaces, all the coefficients are used

5.3 Evaluation of the Force when the Limits of Integration Have No Azimuthal Dependence

For a particular orientation of the hemispheres the force can be calculated after determining the appropriate limits of integration from the geometry In this section it is assumed that there is no dependence on β, the azimuthal angle This includes two cases which will be worked out in detail - separation of the hemispheres along the z axis, without

2

rotation, and a case involving rotation when $\underline{\rho} \geq 2a\hat{\underline{i}}_z$, where $\hat{\underline{i}}_z$ represents a unit vector in the positive z direction In this case, β can be integrated from 0 to 2π, and (5-6) becomes

$$\underline{F}_f = -\rho_d \int_R \int_{\beta=0}^{2\pi} \frac{GM}{a} R \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n^m(0) \left\{ \left(\frac{a}{r}\right)^{n+1} [J_{nm}\cos m\beta + K_{nm}\sin m\beta] \right.$$

$$\left. + \left(\frac{R}{a}\right)^n [(U_{nm}+\overset{*}{U}_{nm})\cos m\beta + (V_{nm}+\overset{*}{V}_{nm})\sin m\beta] \right\} \underline{i}_\zeta \, d\beta \, dR.$$

All of the terms involving mβ integrate to zero, except for m=0, leaving

$$\underline{F}_f = -2\pi\rho_d GM \,\hat{\underline{i}}_\zeta \sum_{n=0}^{\infty} P_n(0) \int_{R=R_{min}}^{R_{max}} [(\tfrac{a}{R})^n J_{nm} + (\tfrac{R}{a})^{n+1} (U_{nm}+\overset{*}{U}_{nm})] dR \qquad (5\text{-}9)$$

The polynomials $P_n(0)$ are obtained by Robin (1957, p 75),

$$P_{2n}(0) = \begin{cases} 1 & n=0 \\ \dfrac{(-1)^n (2n-1)!}{2^{2n-1} n! (n-1)!} & n=1,2,3 \quad , \end{cases} \qquad (5\text{-}10)$$

where the polynomials of odd degree are all zero Those of even degree can be evaluated from

$$P_{2n}(0) = -\frac{(2n-1)}{2n} P_{2n-2}(0), \qquad n \geq 2$$

With this, equation (5-9) becomes

$$\underset{\sim}{F}_f = -2\pi\rho_d GM\hat{\underset{\sim}{i}}_\zeta \left\{ RJ_{00} + \frac{R^2}{2a}(U'_{00}+\overset{*}{U}_{00}) \right.$$

$$\left. + \sum_{n=1}^{\infty} P_{2n}(0)\left[\frac{(\frac{a}{R})^{2n-1} J_{2n0}}{1-2n} + \frac{(\frac{R}{a})^{2(n+1)}}{2(n+1)}(U_{2n0}+\overset{*}{U}_{2n0})\right]\right\}_{R_{min}}^{R_{max}} . \qquad (5\text{-}11)$$

Over the spherical surface (5-6) is

$$\underset{\sim}{F}_s = a^2\rho_d \int_\alpha\int_\beta \frac{GM}{a}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} P_n^m(\cos\alpha)[J_{nm}+U_{nm}+\overset{*}{U}_{nm})\cos m\beta$$

$$+(K_{nm}+V_{nm}+\overset{*}{V}_{nm})\sin m\beta]\begin{bmatrix}\sin^2\alpha\ \cos\beta\\ \sin^2\alpha\ \sin\beta\\ \sin\alpha\ \cos\alpha\end{bmatrix} d\beta\ d\alpha$$

This time all of the terms with $|m| > 1$ drop out when β is integrated from 0 to 2π Consequently,

$$\underset{\sim}{F}_s = 2\pi\rho_d GMa \sum_{n=0}^{\infty}\int_\alpha d\alpha \begin{bmatrix} P_n^1(\cos\alpha)\sin^2\alpha(J_{n1}+U_{n1}+\overset{*}{U}_{n1})\\ P_n^1(\cos\alpha)\sin^2\alpha(K_{n1}+V_{n1}+\overset{*}{V}_{n1})\\ P_n(\cos\alpha)\sin\alpha\ \cos\alpha(J_{n0}+U_{n0}+\overset{*}{V}_{n0})\end{bmatrix} \qquad (5\text{-}12)$$

When the hemispheres are separated along the z axis, or for rotations when $\rho \geq 2a$, equation (5-12) is integrated from $0 \leq \alpha \leq \pi/2$ This involves two separate integrals of the form

$$\int_0^{\pi/2} P_n^1(\cos\alpha)\sin^2\alpha \, d\alpha \tag{5-13}$$

and

$$\int_0^{\pi/2} P_n(\cos\alpha)\sin\alpha\cos\alpha \, d\alpha \tag{5-14}$$

The integral in (5-14) is evaluated by Robin (1957, p 26) If C_n is defined as the integral (5-14), then

$$C_{2n} = \begin{cases} 1/2 & 2n = 0 \\ 1/3 & 2n = 1 \\ 1/8 & 2n = 2 \\ \dfrac{-(-1)^n(2n-3)!}{2^{2n-1}(n+1)!(n-2)!} & n = 2,3, \end{cases} \tag{5-15}$$

or recursively,

$$C_{2n} = \frac{3-2n}{2(n+1)} C_{2n-2}, \qquad n \geq 3$$

The integral in (5-13) can be expressed as

$$\int_0^1 (1-\nu^2)^{1/2} P_n^1(\nu)d\nu, \tag{5-16}$$

but from (2-3) it can be seen that

$$P_n^1(\nu) = -(1-\nu^2)^{1/2} \frac{d}{d\nu} P_n(\nu),$$

so that (5-16) is

$$-\int_0^1 \frac{d}{d\nu} P_n(\nu)d\nu + \int_0^1 \nu^2 \frac{d}{d\nu} P_n(\nu)d\nu$$

Evaluation of the integral above yields

$$\int_0^1 (1-\nu^2)^{1/2} P_{2n}(\nu)d\nu = P_{2n}(0) - 2C_{2n}$$

The force due to the integration over the spherical surface (5-12) can now be evaluated, and is equal to

$$\underline{F}_s = 2\pi\rho_d GMa \left\{ \begin{bmatrix} -\frac{2}{3}(J_{11}+U_{11}+\overset{*}{U}_{11}) \\ -\frac{2}{3}(K_{11}+V_{11}+\overset{*}{V}_{11}) \\ \frac{J_{00}}{2} + \frac{U_{00}}{2} + \frac{\overset{*}{U}_{00}}{2} + \frac{1}{3}(J_{10} + U_{10} + \overset{*}{U}_{10}) \end{bmatrix} \right.$$

$$\left. + \sum_{n=1}^{\infty} \begin{bmatrix} (P_{2n}(0)-2C_{2n})(J_{2n1}+U_{2n1}+\overset{*}{U}_{2n1}) \\ (P_{2n}(0)-2C_{2n})(K_{2n1}+V_{2n1}+\overset{*}{V}_{2n1}) \\ C_{2n}(J_{2n0}+U_{2n0}+\overset{*}{U}_{2n0}) \end{bmatrix} \right\} \qquad (5\text{-}17)$$

5 4 Calculation of the Force for Two Special Cases

The results of 5 3 are applied to two cases - when the hemispheres are separated along the polar axis, and when a rotation of hemisphere

B is performed in the Y-Z plane through an inclination I (Figure 5-2) For these positions case 4 (from Table 5-1) does not apply, and the first case is a derivative of the second, so that only one of cases two, three and five needs to be evaluated, depending on the value of ρ Because the rotation will be carried out for $\rho = 2a$, only case three is involved The equations developed in 5 3 are therefore applicable, as the integration over α and β are independent of the translation and rotation

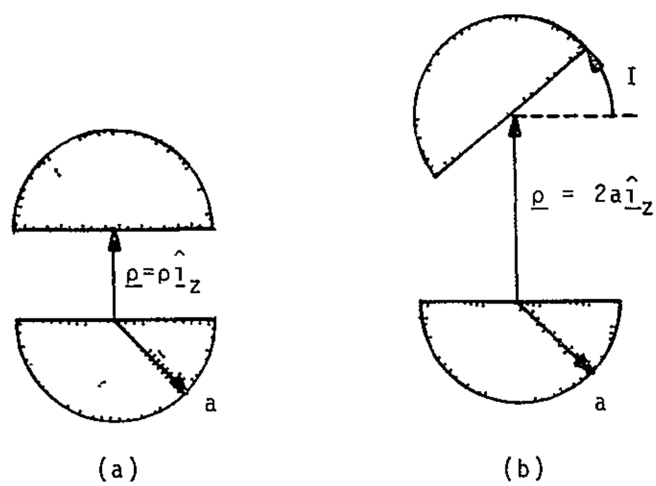


Figure 5-3 Positions of the Hemispheres for the Two Special Cases

Except in the case of the rotation, the force will have only a ζ component, hence the total force may be broken down as

$$\underline{F} = (F_{s_2} + F_{s_3} + F_{f_2} + F_{f_3} + F_{f_5})\hat{\underline{i}}_\zeta + \underline{F}_{s_5}$$

The geometry dictates that the critical values of ρ are $\rho_1 = a/\sqrt{2}$ and $\rho_2 = a$, and the limits of integration for the various cases are

$R = 0$, $(a^2-\rho^2)^{1/2}$ and a Thus, from equation (5-17) and Table (5-1),

$$F_{s_2} = \pi M\rho_d a[J_{00} + \frac{2}{3} J_{10} + 2 \sum_{n=1}^{\infty} C_{2n}J_{2n0}],$$

$$F_{s_3} = F_{s_2}$$

$$F_{s_5} = \pi GM\rho_d a \begin{bmatrix} -\frac{4}{3} \overset{*}{V}_{11} \\ -\frac{4}{3} \overset{*}{V}_{11} \end{bmatrix} \left. \right\} +2 \sum_{n=1}^{\infty} C_{2n}\overset{*}{V}_{2n0} \\ \overset{*}{U}_{00} + \frac{2}{3} \overset{*}{U}_{10} + 2 \sum_{n=1}^{\infty} C_{2n}\overset{*}{U}_{2n0}$$

Similarly, the forces evaluated over the flat surface are, from (5-11) and letting $R_1 = (a^2-\rho^2)^{1/2}$,

$$F_{f_2} = -\pi GM\rho_d a \Big\{ J_{00}(1- \frac{2R_1}{a})+(\frac{R_1}{a})^2 U_{00}- \frac{2}{3} J_{10}$$

$$-2 \sum_{n=1}^{\infty} J_{2n0}[C_{2n}+ \frac{P_{2n}(0)}{2n-1} (1-(\frac{a}{R_1})^{2n-1})]- \frac{P_{2n}(0)(\frac{R_1}{a})^{2n+2}}{2n+2} U_{2n0}\Big\},$$

$$F_{f_3} = -\pi GM\rho_d a \Big\{ -\frac{2}{3} J_{10}+ (1-2 \frac{\rho}{a})J_{00}+ (\frac{R_1}{a})^2 U_{00}$$

$$+ [(\frac{\rho}{a})^2 + (\frac{R_1}{a})^2]\overset{*}{U}_{00}+2 \sum_{n=1}^{\infty}$$

$$- C_{2n}J_{2n0}- \frac{P_{2n}(0)}{2n-1} J_{2n0}[1-(\frac{a}{\rho})^{2n-1}]+ \frac{P_{2n}(0)(\frac{R_1}{a})^{2n+2}}{2n+2} U_{2n0}$$

$$+ \frac{P_{2n}(0)}{2n+2} [(\frac{\rho}{a})^{2n+2} -(\frac{R_1}{a})^{2n+2}]\overset{*}{U}_{2n0} \Big\}$$

and

$$F_{f_5} = -2\pi GM\rho_d a \Big\{ -\frac{1}{3} \overset{*}{U}_{10}+ \sum_{n=1}^{\infty} \overset{*}{U}_{2n0}[\frac{P_{2n}(0)}{2n+2} -C_{2n}]\Big\}$$

The preceding equations were programmed for a digital computer and evaluated for various ratios of ρ/a The results are tabulated in Appendix B and agree very well with the results of an approximation scheme previously developed by the author (Lee, 1970) The first non-dimensionalization of the force is included because it follows directly from the equations The second is independent of the size of the masses V in this instance denotes the volume of one hemisphere

For the case of the rotation (ρ=2a), the components of force were evaluated and then transformed back to the x,y,z coordinate system by the matrix multiplication

$$\underline{F}(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{bmatrix} \underline{F}(\xi,\eta,\zeta)$$

The results are plotted in Figure 5-7 and tabulated in Appendix B Note that due to symmetry the x component is always zero and

$$F_y(-I) = -F_y(I), \qquad F_z(-I) = F_z(I), \text{ and}$$

$$F_y(\tfrac{\pi}{2} + I) = F_y(\tfrac{\pi}{2} - I)$$

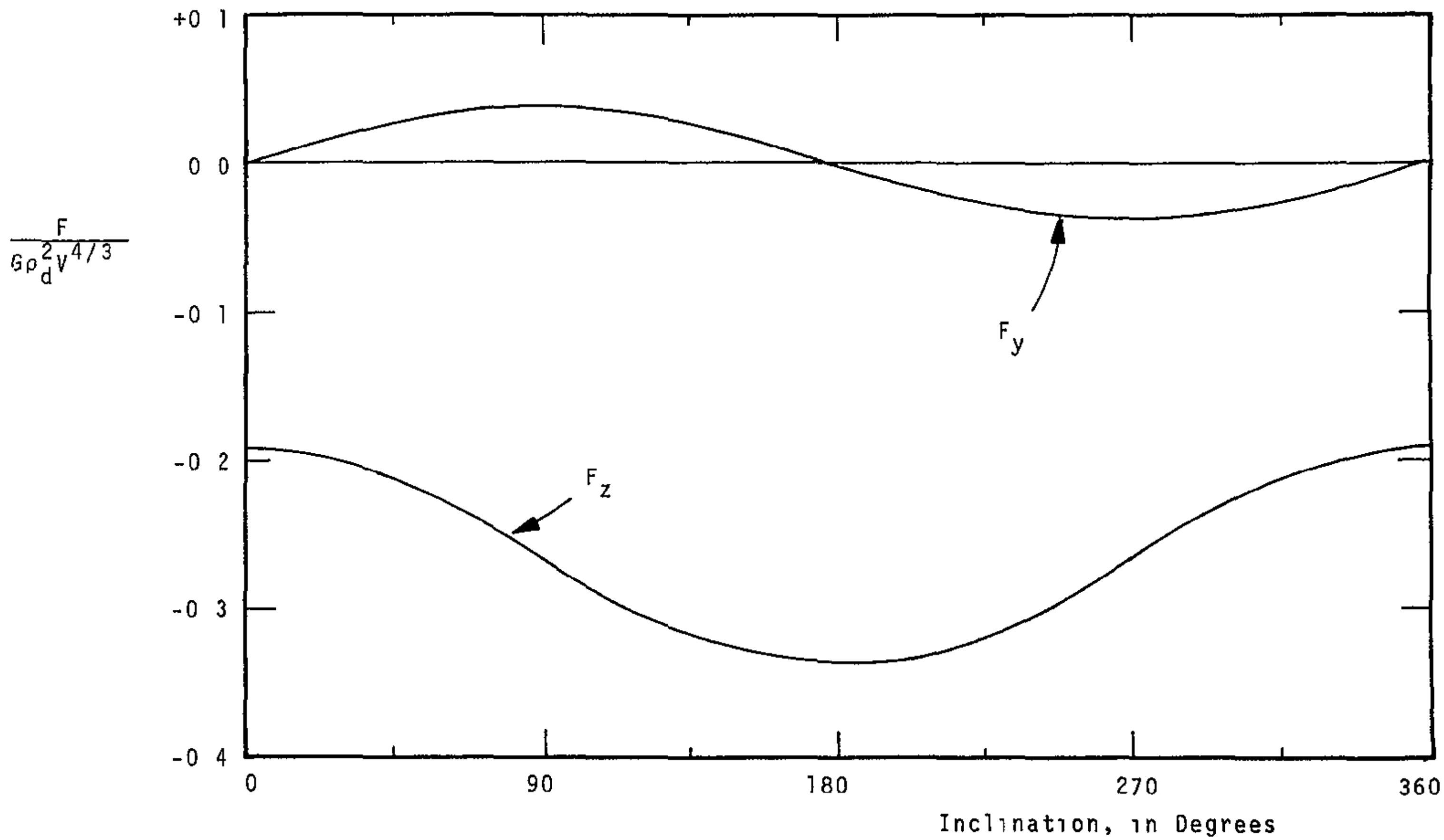


Figure 5-4 Normalized Components of Force as a Function of Angle of Rotation when $\underline{\rho} = 2a\hat{\underline{i}}_z$

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

As is demonstrated in chapter 5 with the example of two hemispheres, the force between two bodies can be efficiently and accurately calculated after performing the necessary integrations analytically Except for relatively simple cases, however, the integrations still involve a considerable amount of work The method of this thesis therefore appears to be best suited for a particular problem, where a high degree of accuracy is required, as opposed to applying it to situations with widely varying geometrical parameters

It should be noted that only potential functions expressed in spherical harmonics have been considered here, but analogous procedures may be applicable to functions expressed in spheroidal or ellipsoidal harmonics This is a question for further investigation

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APPENDIX A

GENERATION OF THE SOLID SPHERICAL HARMONICS ${}^{(1)}H_n^m(r,\cos\theta,\phi)$ AND ${}^{(2)}H_n^m(r,\cos\theta,\phi)$

Because all solid spherical harmonics satisfy Laplace's equation

$$\nabla^2\Psi = 0 \ , \tag{A-1}$$

it is useful to examine expressions which satisfy equation (A-1) in order to generate a particular set of solid spherical harmonics In this case, solutions of the form

$$\Psi_\ell = (ax+by+cz)^\ell \qquad \ell = 0,1,2,$$

are examined

Applying (A-1), it is apparent that Ψ_ℓ is a solution whenever the constants a,b,c satisfy

$$a^2+b^2+c^2 = 0 \tag{A-2}$$

By substitution into (A-2), it can be easily verified that the following choice of a,b,c satisfies (A-2)

$$a = 1-t^2$$

$$b = -i(1+t^2)$$

$$c = -2t \ ,$$

where t is a free parameter Hence,

$$\Psi_\ell = [(x-iy)-(x+iy)t^2-2zt]^\ell \tag{A-3}$$

Consistent with Figure (2-2), x,y,z are related to r,θ,ϕ by

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

With these substitutions, equation (A-2) becomes

$$\Psi_\ell = [r \sin \theta e^{-i\phi}-r \sin \theta e^{i\phi}t^2-2rt \cos \theta]^\ell \tag{A-4}$$

With the definition of three new variables s,ω,μ, (Courant, Hilbert, 1966, Appendix to Ch 7)

$$\left.\begin{aligned} s &= te^{i\phi} \\ s &= e^{i(\omega+\pi/2)} \\ \mu &\equiv \cos \theta \end{aligned}\right\} \qquad t = e^{i(\omega+\pi/2-\phi)}, \tag{A-5}$$

equation (A-4) is

$$\Psi_\ell = (\frac{rt}{s})^\ell [i(\mu^2-1)^{1/2}(1+e^{2i\omega})-2i\mu e^{i\omega}]^\ell ,$$

or

$$\Psi_\ell = (-2rt)^\ell f_\ell(\omega), \tag{A-6}$$

where

$$f_\ell(\omega) \equiv [\mu - \cos\omega(\mu^2-1)^{1/2}]^\ell. \tag{A-7}$$

Since $f_\ell(\omega)$ is an even function of ω, and for ℓ integer, (A-7) can be expanded in the following Fourier series

$$f_\ell(\omega) = \frac{a_0}{2} + \sum_{m=1}^{\ell} a_m \cos m\omega \ ,$$

where

$$a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} f_\ell(\omega)\, d\omega$$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f_\ell(\omega) \cos m\omega \, d\omega \qquad m = 1,2,3, \quad ,\ell$$

By Laplace's first and second integrals for $\mu \geq 0$, $n = 0,1,2,$ (MacRobert, 1947, pp. 125-129),

$$\int_0^{\pi} [\mu-(\mu^2-1)^{1/2}\cos\omega]^n \cos m\omega \, d\omega = \frac{i^{-m}\pi n!\, P_n^m(\mu)}{(n+m)!} \ , \tag{A-8}$$

and

$$\int_0^{\pi} [\mu-(\mu^2-1)^{1/2}\cos\omega]^{-n-1} \cos m\omega \, d\omega = \frac{i^{m}\pi (n-m)!\, P_n^m(\mu)}{n!} \tag{A-9}$$

These are the two integral equations which are necessary for the Fourier expansion of $f_\ell(\omega)$ when $\ell = n$ in the first case, and $\ell = -n-1$ in the second case

The result of applying equation (A-8) to (A-7) ($\ell = n$) is

$$^{(1)}f_n(\omega) = P_n(\mu) + 2\sum_{m=1}^{n} i^{-m} \frac{n!}{(n+m)!} P_n^m(\mu) \cos m\omega \tag{A-10}$$

The range of subscripts can be extended to include negative values ($-n \leq m \leq n$), noting that (Jackson, 1962, p 65)

$$P_n^{-m}(\mu) = \frac{(-1)^m (n-m)!}{(n+m)!} P_n^m(\mu),$$

and thus ${}^{(1)}f_n(\omega)$ is an even function with respect to m. Further, since

$$\sin m\omega = -\sin(-m\omega),$$

and

$$P_n^0(\mu) = P_n(\mu),$$

equation (A-6) can be written as

$${}^{(1)}\psi_n = (-2rt)^n \sum_{m=-n}^{n} \frac{i^{-m} n!}{(n+m)!} P_n^m(\cos\theta) e^{im\omega}$$

But

$$te^{i\phi} = e^{i\omega} e^{i\pi/2} ,$$

or

$$e^{im\omega} = t^m e^{im\phi} i^{-m}$$

So, finally,

$$[(x-iy) - (x+iy)t^2 - 2zt)^n$$

$$= t^n \sum_{m=-n}^{n} {}^{(1)}H_{n,m}(r,\cos\theta,\phi) t^m, \qquad \text{(A-11)}$$

where

$$^{(1)}H_{n,m}(r,\cos\theta,\phi) \equiv \frac{(-2)^n(-1)^m n!}{(n+m)!} r^n P_n^m(\cos\theta)e^{im\phi} \quad \text{(A-12)}$$

For the set of solid spherical harmonics involving powers of r^{-n-1},

$$\ell = -n-1,$$

and equation (A-9) yields the coefficients in the Fourier expansions, such that

$$[\mu-(\mu^2-1)^{1/2}\cos\omega]^{-n-1}$$

$$= P_n(\mu)+2\sum_{m=1}^{n}\frac{i^m(n-m)!}{n!}P_n^m(\mu)\cos m\omega$$

With treatment analogous to that for the first case, it is found that

$$[(x-iy)-(x+iy)t^2-2zt]^{-n-1}$$

$$= t^{-n-1}\sum_{m=-n}^{n}{}^{(2)}H_{n,m}(r,\cos\theta,\phi)t^m, \quad \text{(A-13)}$$

where

$$^{(2)}H_{n,m}(r,\cos\theta,\phi) \equiv \frac{(n-m)!}{(-2)^{n+1}n!} r^{-n-1}P_n^m(\cos\theta)e^{im\phi} \quad \text{(A-14)}$$

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APPENDIX B

NUMERICAL RESULTS OF THE CALCULATION OF THE FORCE BETWEEN TWO HEMISPHERES

B 1 Separation Along the Polar Axis

Normalized Separation	Normalized Force	
ρ/a	$F_z/GM\rho_d a$	$F_z/G\rho_d^2 V^{4/3}$
0 007937	-1 554	-1 214
0 01	-1 549	-1.211
0 01587	-1 537	-1 201
0 02381	-1 521	-1 189
0 05002	-1 468	-1 148
0 1000	-1 376	-1 075
0 2	-1 214	-0 9485
0 3996	-0 963	-0 753
0 5	-0 87	-0 68
1 998	-0 2473	-0 1933
10 0	-0 01799	-0 01406

B 2 Separation Along the Polar Axis and Rotation of One of the Hemispheres about the Center of its Base, in the Y-Z Plane

Angle of Rotation	Normalized Components of Force			
I (in deg)	$\frac{F_y}{GM\rho_d a}$	$\frac{F_z}{GM\rho_d a}$	$\frac{F_y}{G\rho_d^2 V^{4/3}}$	$\frac{F_z}{G\rho_d^2 V^{4/3}}$
0 0	0 0	-0 2470	0 0	-0 1930
22 5	0 01698	-0 2529	0.01327	-0 1977
45 0	0 03314	-0 2711	0 02590	-0 2119
67 5	0 04597	-0 3019	0.03593	-0 2359
90 0	0 05110	-0 3416	0 03994	-0.2670
112 5	0 04597	-0 3814	0 03593	-0.2981
135 0	0 03314	-0 4121	0 02590	-0 3221
157 5	0.01698	-0 4304	0 01327	-0 3364
180 0	0 0	-0 4363	0 0	-0 3410

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NEW TECHNOLOGY APPENDIX

This report is published in the belief that it constitutes an improvement in the state of the art In particular, pages 9 through 82 are referenced