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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

KALMAN FILTER INITIALIZATION USING PRIOR INFORMATION APPLIED TO APOLLO MIDCOURSE NAVIGATION

by
James Merrill Habbe
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James Merrill Habbe

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ABSTRACT

The application of Kalman filtering techniques to estimate the position and velocity of a vehicle in space flight requires specifying the initial state and the initial covariance matrix of the estimation errors. These initial conditions reflect prior knowledge of the state and often are not known very accurately. As a result, a transient period exists until the error covariance matrix becomes properly correlated and beneficial state updates achieved. A practical technique of forming a correlated matrix to initiate the filtering process is presented and applied to Apollo navigation during the trans-earth phase of the mission. Results from a digital computer simulation of the midcourse navigation problem utilizing the Monte Carlo approach show that the transient period associated with producing a good estimate is reduced when the correlated matrix, as opposed to a diagonal matrix, is used in initialization.

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NOTATION

Symbol	Definition
$\underline{r}(t)$	Position vector of spacecraft referenced to non-rotating system
x, y, z	Components of $\underline{r}$
$\underline{v}(t)$	Velocity vector of spacecraft referenced to non-rotating system
μ_P	Gravitation parameter of primary attracting body
μ_Q	Gravitation parameter of secondary attracting body
μ_S	Gravitation parameter of sun
$\underline{a}_{dQ}$	Disturbing acceleration due to secondary body
$\underline{a}_{dS}$	Disturbing acceleration due to sun
$\underline{r}_{osc}(t)$	Position vector associated with osculating orbit
$\underline{v}_{osc}(t)$	Velocity vector associated with osculating orbit
$\underline{r}_0$	$\underline{r}_{osc}(0)$
$\underline{v}_0$	$\underline{v}_{osc}(0)$
$\underline{\delta}(t)$	Position deviation of spacecraft from osculating orbit
$\underline{\nu}(t)$	Velocity deviation of spacecraft from osculating orbit
$\underline{r}_{PQ}$	Position of secondary body with respect to primary body
$\underline{r}_{PS}$	Position of sun with respect to primary body
$\underline{r}_{QC}$	Distance of spacecraft from secondary body
$\underline{r}_{SC}$	Distance of spacecraft from sun
$\underline{x}(t)$	Six dimensional state vector $\begin{bmatrix} \underline{r} \\ \underline{v} \end{bmatrix}$

NOTATION (cont.)

$\underline{\epsilon}(t)$	Three component error in position estimate
$\underline{\eta}(t)$	Three component error in velocity estimate
$\underline{e}(t)$	Six dimensional error vector $\begin{bmatrix} \underline{\epsilon}(t) \\ \underline{\eta}(t) \end{bmatrix}$
$E(t)$	Covariance matrix of estimation errors
Q	Measured angle in star-horizon measurement
ΔQ	Difference between the measured quantity and the best estimate of what that quantity should be
$\underline{\omega}_n$	Weighting vector for n^{th} measurement
$\hat{\underline{x}}_n$	Kalman filter estimate of $\underline{x}$ at time of n^{th} measurement based on previous $n-1$ measurements
$\hat{\underline{x}}'_n$	Kalman filter estimate of $\underline{x}$ at time of n^{th} measurement based on all n measurements
E_n	Covariance matrix of estimation errors at time of n^{th} measurement based on all n measurements
E'_n	Covariance matrix of estimation errors at time of n^{th} measurement based on previous $n-1$ measurements
$\underline{b}_n$	Six dimensional geometry vector associated with n^{th} measurement
α_n^2	Apriori value of variance of n^{th} measurement error
$\phi_{n,n-1}$	State transition matrix from time at $(n-1)$ st measurement to time of n^{th} measurement
$G(t)$	Gradient of the gravity vector with respect to the components of the estimated position vector of spacecraft

NOTATION (cont.)

W	Error transition matrix (square root of E)
$\underline{n}$	Six dimensional vector of random numbers
C	Correlation matrix of initial state errors
P	Matrix of orthonormal eigenvectors of C
λ_i	Eigenvalues of matrix C
δx_I	Initial deviation of state in inertial system
δx_{LV}	Initial deviation of state in local vertical coordinate system
$\underline{l}$	Unit vector in direction of star
$\underline{m}$	Unit vector in direction of horizon
$\underline{d}$	Position of horizon with respect to spacecraft
$\underline{p}$	Unit vector in plane of measurement and perpendicular to $\underline{d}$
δQ	Variation in Q resulting from a variation in components of state vector
a	Semi-major axis of spacecraft's orbit
$\underline{b}_E$	Geometry vector for energy pseudo-measurement
$\underline{b}_h$	Geometry vector for magnitude of angular momentum pseudo-measurement
$\overline{\alpha_E^2}$	Variance of error in energy pseudo-measurement
$\overline{\alpha_h^2}$	Variance of error in angular momentum pseudo-measurement
$\underline{h}$	Angular momentum of spacecraft's orbit

CHAPTER 1

INTRODUCTION

The application of Kalman filtering techniques to space problems requires that initial conditions on both the state and the error covariance matrix be specified. A practical technique of forming the initial covariance matrix for a space navigation problem is presented here using the Apollo navigation system as an example during the midcourse phase of the mission, which is the part from trans-earth insertion at the back side of the moon to re-entry into the earth's atmosphere. Since small insertion errors produce large errors in the trajectory at a later time, guidance is necessary to control the vehicle and insure the success of the mission. The problem then is to estimate the actual trajectory of the spacecraft and from this estimate determine the necessary velocity correction required to place the spacecraft on a proper trajectory for re-entry.

Kalman filtering techniques are applied whereby observations on the state are made, and processed with their associated noise statistics in a manner that produces an optimum estimate of the state. The state of the spacecraft, which is comprised of the three components of position and the three components of velocity, is maintained in the Apollo Guidance Computer (AGC). Since this estimate will be in error, it is necessary to also maintain the statistics associated with the errors as a measure of the quality of the estimate. Therefore the covariance matrix of estimation errors is also maintained in the AGC.

One of the fundamental problems associated with filtering theory is supplying the initial conditions to start the filtering process. Information describing the initial state is often poorly known and is usually supplied somewhat arbitrarily. Since the effect of starting the filter with incorrect initial conditions diminishes as measurements are processed, the actual values used are to a certain extent

not critical. However, the ability to produce a good estimate with a relatively small number of measurements requires better initial conditions.

The procedure currently used to start the filtering process in the midcourse phase is to assume that the initial errors in the state are uncorrelated and spherically distributed, which implies that the initial covariance matrix of estimation errors is a diagonal matrix. Since the true errors in the spacecraft's position and velocity are correlated, the initial covariance matrix does not accurately represent the true situation. As measurements are subsequently processed, the covariance matrix becomes properly correlated and tends to become more correct. The ability to produce beneficial improvements in the state estimate depends on the proper correlation between the state errors. Since this takes time, the ideal situation is to initialize with a covariance matrix that is already properly correlated. As a practical matter, the capability of accomplishing this seldom exists but can be achieved partially if more information concerning the initial state is known.

At the time of initialization in the midcourse phase, certain parameters pertaining to the spacecraft's orbit are possibly known quantities that are not reflected by a diagonal covariance matrix. Examples of such quantities are the total energy or the angular momentum of the orbit. These quantities, if known, provide information about the initial state and may be used to determine better initial values to start the filtering process. This study compares the results obtained using this additional information with the results using only the initial diagonal covariance matrix by making Monte Carlo runs on a digital computer simulation of the midcourse flight.

CHAPTER 2

LINEARIZED EQUATIONS OF MOTION

2.1 Coordinate Systems

In the Apollo program, various coordinate systems are used depending on the specific phase of the mission. This study deals with only the midcourse phase (transearth) and requires only one of the coordinate systems. The position and velocity of the spacecraft during the transearth phase is referenced to a non-rotating rectangular coordinate system that is either moon centered or earth centered. When the spacecraft is near enough to the moon that the moon can be considered the primary attracting body, with the earth being a disturbing body, the spacecraft is said to be within the lunar sphere of influence⁽¹⁾ and the coordinate system is moon centered. Conversely, when the spacecraft is outside the lunar sphere of influence the earth is considered to be the primary attracting body and the coordinate system is earth centered. The orientation of the coordinate system is defined by the line of intersection of the earth's mean equational plane with the plane of the earth's orbit (ecliptic). The x-axis is directed along this line with the positive sense toward the vernal equinox as defined at the beginning of the Besselian year which starts January 0.525, 1969. The z-axis is along the mean north pole and the y-axis completes a right-handed orthogonal system. The coordinate system is illustrated in Figure 2.1.

2.2 Osculating Orbit

The problem considered in this study deals with the navigation of the Apollo spacecraft during the midcourse phase of the mission. It is assumed that a desired or nominal trajectory for the transearth flight is known and that the vehicle has been given the necessary position and velocity to achieve this trajectory with the exception of

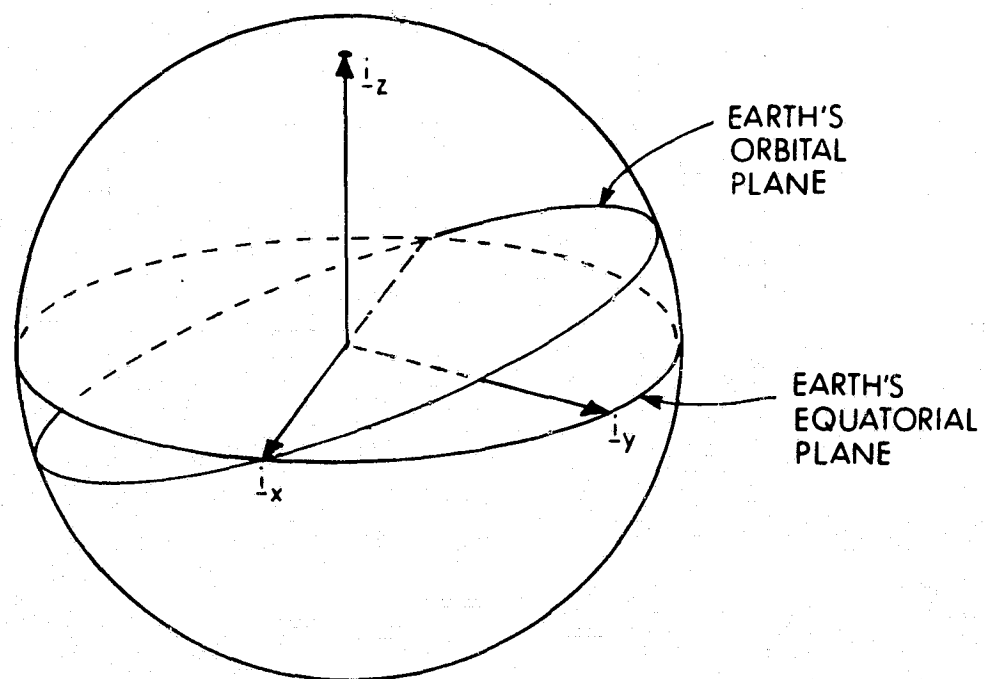


Figure 2.1 Schematic of the inertial non-rotating coordinate system

small initial position and velocity perturbations at the time of the trans-earth insertion from the back side of the moon. Furthermore, the only accelerations experienced by the spacecraft during the flight are assumed to be of a gravitational nature with the exception of brief accelerations associated with midcourse velocity corrections.

The position and velocity estimates of the spacecraft are maintained in an on-board computer by integrating the state equations with respect to time from one observation time to the next. The basic equation which must be integrated is

$$\frac{d^2 \underline{r}(t)}{dt^2} + \frac{\mu_P \underline{r}(t)}{r^3} = \underline{a}_d \quad (2.2.1)$$

where $\underline{r}(t)$ is the position vector of the spacecraft referenced to the nonrotating coordinate system, μ_P is the gravitational constant of the primary body, and $\underline{a}_d$ is a disturbing acceleration which causes the vehicle to deviate from a precise conical orbit with the focus at the primary body. The form of the disturbing acceleration depends on the position of the spacecraft relative to the lunar sphere of influence and is discussed in a later section. A procedure used to integrate the state, known as Encke's method,⁽¹⁾ takes advantage of the case where the disturbing acceleration is small compared to the central force field of the primary attracting body. At any particular time t_0 , the corresponding position and velocity vectors $\underline{r}_0$ and $\underline{v}_0$ completely specify a conic orbit that the spacecraft would move in if the disturbing acceleration were zero. The instantaneous conic orbit associated with the time t_0 is called the osculating orbit. The position and velocity vectors associated with the osculating orbit at any time t are obtained from solutions to the two-body differential equation.

$$\frac{d^2 \underline{r}_{osc}}{dt^2} + \frac{\mu_P \underline{r}_{osc}(t)}{r_{osc}^3} = 0 \quad (2.2.2)$$

using the initial conditions

$$\underline{r}_{osc}(t_0) = \underline{r}_0 \quad (2.2.3)$$

$$\underline{v}_{osc}(t_0) = \underline{v}_0 \quad (2.2.4)$$

Since the disturbing acceleration is not zero the actual path of the spacecraft deviates from the osculating orbit. Subtracting equation 2.2.2 from 2.2.1 and defining

$$\underline{\delta}(t) = \underline{r}(t) - \underline{r}_{osc}(t) \quad (2.2.5)$$

yields the differential equation of motion for this deviation:

$$\frac{d^2 \underline{\delta}}{dt^2} + \mu_P \left(\frac{\underline{r}}{r^3} - \frac{\underline{r}_{osc}}{r_{osc}^3} \right) = \underline{a}_d \quad (2.2.6)$$

Since the term in parenthesis causes numerical difficulties if $\underline{r}$ is approximately equal to $\underline{r}_{osc}$, a technique discussed in Reference 1 is used to avoid this difficulty. The result, $\underline{\delta}(t)$, is obtained from the differential equation

$$\frac{d^2 \underline{\delta}}{dt^2} - \frac{\mu_P}{r_{osc}^3} \underline{\delta} = \underline{a}_d - \frac{\mu_P}{r_{osc}^3} f(q) \underline{r} \quad (2.2.7)$$

subject to the initial conditions

$$\underline{\delta}(0) = 0 \quad (2.2.8)$$

$$\frac{d\delta}{dt}(0) = 0 \quad (2.2.9)$$

where

$$q = \frac{\delta \cdot (\delta - 2r)}{r^2} \quad (2.2.10)$$

and

$$f(q) = \frac{q(3 + 3q + q^2)}{1 + (1 + q)^{3/2}} \quad (2.2.11)$$

The right hand side of equation 2.2.6 is a function only of the position of the spacecraft. A numerical integration technique used that exploits this fact is known as Nystrom's Method⁽⁴⁾. It is of the same form as the well known Runge-Kutta Method but requires a lesser amount of calculations for a given desired accuracy. A third order algorithm giving fourth order accuracy is used to numerically integrate equation 2.2.6. This determines the amount that the true position and velocity vectors have deviated from those associated with the osculating orbit. The estimate of the state along the true trajectory is calculated from

$$\underline{r}(t) = \underline{\delta}(t) + \underline{r}_{osc}(t) \quad (2.2.12)$$

$$\underline{v}(t) = \underline{v}(t) + \underline{v}_{osc}(t) \quad (2.2.13)$$

where $\underline{v}(t)$ is defined to be $\frac{d\delta(t)}{dt}$.

The quantities $\underline{r}_{osc}(t)$ and $\underline{v}_{osc}(t)$ at any particular time can be obtained by solving Kepler's equation and need not be obtained by integrating equation 2.2.2.⁽¹⁾ Therefore accurate values for

these quantities may be obtained since the propagation of errors associated with numerical integration is avoided. The main advantage of Encke's Method is that errors in $\underline{\delta}(t)$ or $\underline{\nu}(t)$ won't produce appreciable errors in $\underline{r}(t)$ and $\underline{v}(t)$ until these errors become large enough to affect the least significant digits of $\underline{r}_{osc}(t)$ and $\underline{v}_{osc}(t)$. To maintain the efficiency of this method a new osculating orbit must be defined periodically from which to calculate the deviations. This procedure is known as rectification. When rectification occurs at time t , the new osculating orbit is defined by the current values of $\underline{r}(t)$ and $\underline{v}(t)$ and new values for $\underline{\delta}(t)$ and $\underline{\nu}(t)$ are computed from equation 2.2.7 with their initial conditions again set equal to zero.

2.3 Disturbing Accelerations

During midcourse flight the gravitational attractions of the sun and the secondary body, denoted by subscripts S and Q respectively, are taken into account. When the spacecraft is inside the lunar sphere of influence, the coordinate system is located at the center of the moon and the moon is considered the primary body with the earth being the secondary attracting body. Conversely when the spacecraft is outside the lunar sphere of influence, the origin of the coordinate system is at the center of the earth and the earth is the primary body with the moon being the secondary body.

The disturbing acceleration due to the secondary body and the sun are respectively⁽⁴⁾

$$\underline{a}_{dQ} = \frac{-\mu_Q}{r_{QC}^3} [f(q_Q) \underline{r}_{PQ} + \underline{r}] \quad (2.3.1)$$

$$\underline{a}_{dS} = \frac{-\mu_S}{r_{SC}^3} [f(q_S) \underline{r}_{PS} + \underline{r}] \quad (2.3.2)$$

where $\underline{r}_{PQ}$ and $\underline{r}_{PS}$ are the position vectors of the secondary body and the sun with respect to the primary body, r_{QC} and r_{SC} are the distances of the spacecraft from the secondary body and the sun and the arguments q_Q and q_S are computed from

$$q_Q = \frac{(\underline{r} - 2\underline{r}_{PQ}) \cdot \underline{r}}{r_{PQ}^2} \quad (2.3.3)$$

$$q_S = \frac{(\underline{r} - 2\underline{r}_{PS}) \cdot \underline{r}}{r_{PS}^2} \quad (2.3.4)$$

and the functions $f(q_Q)$ and $f(q_S)$ are calculated as in equation 2.2.9.

CHAPTER 3

LINEAR ESTIMATION THEORY APPLIED TO MIDCOURSE NAVIGATION

3.1 Theory of the Optimal Filter

Statistical filter theory is employed during the midcourse phase of the mission to estimate the position and velocity of the spacecraft. During prolonged periods of coasting flight filtering errors associated with the computational techniques used eventually reach a point where they rapidly degrade the estimate. Furthermore the mathematical model used to compute the force on the spacecraft is not exact. Therefore it is necessary to make observations or measurements of the state of the vehicle periodically to improve the estimate. The problem then becomes one of utilizing measurements that are corrupted with sensor errors and then processing them recursively in an on-board computer to give an "optimal" estimate of the state of the vehicle. The application of Kalman filtering techniques to the midcourse phase of the Apollo mission is given in detail in Reference 2. A summary is included here to aid in the understanding of the results of this study.

The state of the spacecraft is a six-dimensional vector consisting of three components of position and three components of velocity and is denoted by

$$\underline{x} = \begin{bmatrix} \underline{r} \\ \underline{v} \end{bmatrix} \quad (3.1.1)$$

The state is referenced to the non-rotating coordinate system that is either earth-centered or moon-centered as previously defined. Since the state vector as maintained by the on-board computer is only an estimate of the true values of position and velocity of the spacecraft

and will be in error, it is necessary from a filtering standpoint to also maintain the statistics associated with these errors. The linear estimator generates the covariance matrix of the estimation errors of the state as well as the state estimate itself. If the six-dimensional error vector is denoted by

$$\underline{e}(t) = \begin{bmatrix} \underline{\epsilon}(t) \\ \underline{\eta}(t) \end{bmatrix} \quad (3.1.2)$$

where $\underline{\epsilon}(t)$ is the three component error in the position estimate and $\underline{\eta}(t)$ is the three component error in the velocity estimate, then the covariance matrix of the estimation errors is the 6×6 matrix given by

$$E = \overline{\underline{e} \underline{e}^T} = \begin{bmatrix} \overline{\underline{\epsilon} \underline{\epsilon}^T} & \overline{\underline{\epsilon} \underline{\eta}^T} \\ \overline{\underline{\eta} \underline{\epsilon}^T} & \overline{\underline{\eta} \underline{\eta}^T} \end{bmatrix} \quad (3.1.3)$$

where the bar over a quantity implies an ensemble average. The trace of the upper left 3×3 matrix gives the mean-squared value of the estimated position error and the trace of the lower right 3×3 matrix gives the mean-squared value of the estimated velocity error. On the basis of information obtained from taking the measurement, the estimate of the state is updated to give a new estimate. The old estimate is first extrapolated to the measurement time which gives the best estimate prior to the incorporation of the new information. Using this value, an estimate of the quantity to be measured is computed. Given the value of the measurement itself, a new best estimate using the n th measurement is computed by the relationship

$$\hat{\underline{x}}_n = \hat{\underline{x}}'_n + \underline{\omega}_n \Delta Q_n \quad (3.1.4)$$

where ΔQ is the difference between the measured quantity and the computed estimate of what that quantity should be, $\underline{\omega}_n$ is a vector

that weights the new information, and the prime indicates quantities at the measurement time but prior to the incorporation of the measurement. Thus new best estimates are generated recursively as measurements are taken.

For the case where the measurements are scalar quantities and no process noise is included in the state equation, the weighting vector for the n^{th} measurement is given by Kalman filtering theory;

$$\underline{\omega}_n = \frac{\underline{E}'_n \underline{b}_n}{\underline{b}_n^T \underline{E}'_n \underline{b}_n + \alpha_n^2} \quad (3.1.5)$$

where $\underline{E}'_n$ is the covariance matrix based on $n-1$ measurements propagated to the time of the n^{th} measurement, α_n^2 is the apriori value of the variance of the measurement error, and $\underline{b}$ is a geometry vector associated with the measurement.

The covariance matrix is propagated to the measurement time by

$$\underline{E}'_n = \phi_{n,n-1} \underline{E}_{n-1} \phi_{n,n-1}^T \quad (3.1.6)$$

where $\phi_{n,n-1}$ is the state-transition matrix for the system between the times of the $n-1^{\text{st}}$ and the n^{th} measurements. The covariance matrix is updated by

$$\underline{E}_n = (\underline{I} - \underline{\omega}_n \underline{b}_n^T) \underline{E}'_n \quad (3.1.7)$$

The six-dimensional geometry vector associated with the measurement represents, to a first order approximation, the variation in the measured quantity Q which would result from variations in the components of the state vector. Thus

$$\underline{b} = \left(\frac{\partial Q}{\partial \underline{r}} \quad \frac{\partial Q}{\partial \underline{v}} \right)^T \quad (3.1.8)$$

Since measured quantities are scalars the denominator of equation 3.1.5 is also a scalar instead of a matrix as it would be if the measured quantities were vectors. Therefore the weighting vector is easily computed since the numerical difficulties of matrix inversion is avoided.

For a linear system or if the system equations are linearized about a nominal or reference path, the state transition matrix is a part of the problem statement and is assumed known. If the types of measurements that are to be made during the mission and their corresponding times are specified, the filter gains can be computed prior to the mission and stored for use whenever needed. Therefore real time implementation of the filter requires only the computation of

$$\hat{\underline{x}}_n = \phi_{n,n-1} \hat{\underline{x}}_{n-1} \quad (3.1.9)$$

and equation 3.1.4. However to allow for greater versatility in the types of measurements to be taken and at times that are not completely predetermined, these quantities are computed in the spacecraft computer as required.

The state transition matrix satisfies the matrix differential equation

$$\frac{d\phi}{dt} = \begin{bmatrix} 0 & I \\ G(t) & 0 \end{bmatrix} \phi \quad (3.1.10)$$

subject to the initial condition that $\phi_{n,n}$ equals the six dimensional identity matrix and where $G(t)$ is the gradient of the gravity vector with respect to the components of the estimated position vector of the spacecraft. Therefore the "reference" trajectory about

which the equations of motion are linearized is the trajectory obtained from the on-board estimate of the position and velocity vectors. Thus the elements of the gravity gradient matrix cannot be precomputed prior to launch but must be evaluated along the estimated trajectory as the mission progress.

3.2 Square Root Formulation of the Recursive Navigation Problem⁽⁴⁾

From theoretical considerations the covariance matrix of the estimation errors must remain positive definite. However as measurements are incorporated it decreases in a positive definite sense. Due to accumulated numerical inaccuracies for a large number of calculations it can fail to remain positive definite. To avoid this difficulty, the problem is reformulated using the $W(t)$ matrix called the error transition matrix, such that

$$E(t) = W(t) W(t)^T \quad (3.2.1)$$

and is, in a matrix sense, the square root of $E(t)$. If needed $E(t)$ may be calculated and is guaranteed to be at least positive semi-definite since it is the product of a matrix with its transpose. The error transition matrix offers other advantages in that certain computational requirements are significantly reduced. The extrapolation of the error transition matrix from one measurement time to the next is accomplished by numerically integrating the differential equation

$$\frac{dW(t)}{dt} = \begin{pmatrix} 0 & I \\ G(t) & 0 \end{pmatrix} W(t) \quad (3.2.2)$$

where $G(t)$ for the midcourse phase is given by

$$\begin{aligned}
G(t) = & \frac{\mu_P}{r^5(t)} \left[3 \underline{r}(t) \underline{r}(t)^T - r^2(t) I \right] \\
& + \frac{\mu_Q}{r_{QC}^5(t)} \left[3 \underline{r}_{QC}(t) \underline{r}_{QC}(t)^T - r_{QC}^2(t) I \right]
\end{aligned}
\tag{3.2.3}$$

The equations for incorporating a measurement are

$$\hat{\underline{x}}_n = \hat{\underline{x}}'_n + \underline{\omega}_n \Delta Q_n \tag{3.2.4}$$

and

$$\underline{W}_n = \underline{W}'_n - \frac{\underline{\omega}_n \underline{z}_n^T}{1 + \sqrt{\frac{\alpha_n^2}{\underline{z}_n^2 + \alpha_n^2}}} \tag{3.2.5}$$

where

$$\underline{\omega}_n^T = \frac{1}{\underline{z}_n^2 + \alpha_n^2} \underline{z}_n^T \underline{W}'_n{}^T \tag{3.2.6}$$

and

$$\underline{z}_n = \underline{W}'_n{}^T \underline{b}_n \tag{3.2.7}$$

A convenient method of computing a square root of a 6 dimensional matrix is given in Appendix A.

3.3 Initialization

In applying the linear filter to a specific system the dynamic model to be used must be specified, namely the state-transition matrix ϕ , the geometry vectors $\underline{b}$ for the measurements q , and possibly the system noise. Also the statistics of the measurement errors σ_k^2 , and the initial conditions of the state $\hat{\underline{x}}_0$ and E_0 must be specified. $\hat{\underline{x}}_0$ and E_0 reflect prior knowledge of the state and the statistics associated with the errors in the initial state and as such are usually not very well known if indeed they are known at all. As a result these quantities are generally specified somewhat arbitrarily. Since the effect of prior data is eventually diminished after the incorporation of a sufficient number of observations, the initial conditions specified are to a certain extent not critical. However a larger number of observations are required before the filter is capable of "learning" the true state with a sufficiently small uncertainty to make the state estimate useful. For many applications of interest it is necessary to produce a good estimate in a short period of time which implies that the filter must converge to the true state after the incorporation of only a small amount of data. That is, the covariance matrix must become properly correlated so that beneficial updates of the state are made with each observation.

One method would be to choose more effective measurements of the state such that more or better information is available to the filter. Assuming that this has already been accomplished and a measurement schedule has been specified, the achievement of a good estimate after only a few measurements requires more accurate initial conditions.

A similar problem of choosing initial values exists if it becomes necessary to reinitialize the covariance matrix at any time during the mission. Here, as in initialization, the true state is not known. The reinitialized covariance matrix does not accurately represent true conditions and a transient period will exist until the filter converges to the true state. 16

3.4 Filter Divergence and Methods of Preventing Divergence

The performance of a linear filter when applied to a system is degraded if an inexact filter model is used. In effect, the filter converges to the wrong state after the incorporation of data and the difference between the true state and the "optimal estimate" may grow without bound or if bounded may grow to the extent that the estimate is no longer useful. This is referred to as filter divergence. The problem of divergence is magnified if the system noise and the measurement errors are small for then the filter is capable of learning the wrong state very well. The covariance matrix of the estimation errors becomes very small, the filter gain is also small, and further measurements tend to be ignored.

Filter divergence can also occur if linear filtering theory is applied to a non-linear system. The state equations are typically linearized about a nominal or reference trajectory and the deviations from this path are assumed small. If in the course of the problem these deviations become large enough to violate the restrictions imposed by the linearization the filter again converges to the wrong state and divergence can occur.

In practice when filter divergence occurs, the measurement residuals (the difference between the actual measurements and the best estimate of what that measurement should be) become inconsistent with their expected statistics. Since the measurement residual is a quantity that is used in the filtering process and is therefore easily monitored, it can be used to detect filter divergence. Once it is known that the filter is diverging, some methods to prevent divergence with varying degrees of success are available. In Reference 3 the subject of compensating techniques is discussed in some detail and further references are given. A few of these methods will be discussed briefly to better understand the significance of the results of this study.

When filter divergence occurs, the difference between the true state and the estimated state grows to a value where the estimate is no longer useful. In other words the covariance matrix becomes

too optimistic in its estimate of the errors and subsequent observations are weighted less heavily. Therefore divergence can be prevented to a certain extent if the filter gains are kept at a sufficiently high value. One method that accomplishes this is to include a fictitious system noise. The addition of this noise prevents the filter from learning the true state very well. Hence the covariance matrix is less conservative and the filter gains are increased and current observations are processed with greater weighting. This has the effect of compensating for an inexact model of the system and the amount of noise required is usually determined by a trial and error procedure until a satisfactory filter performance is achieved. For the midcourse phase of the Apollo mission, the error transition matrix is extrapolated instead of the covariance matrix thus this method is not directly applicable.

Other methods used to overweight the more recent data is to fix the value of the covariance matrix to be the value obtained after a given amount of data had been incorporated. Or alternatively an estimate could be computed that is a linear combination of the estimate based on all previous information and the estimate based on the current observation alone.

A method of preventing divergence that is used in various phases of the Apollo mission is by reinitialization of the covariance matrix. Periodically the covariance matrix is reinitialized to a larger value, in a positive definite sense, which has the effect of increasing the weighting on the more recent data. The basic problem associated with this method is determining what the covariance matrix should be reinitialized to. Typically the covariance matrix used in reinitialization and also in initialization is diagonal and the variances used are determined from Monte Carlo simulations of the problem. A diagonal matrix implies that the state errors are uncorrelated. This in general is not the case, thus information must be fed into the filter by taking measurements of the state before the covariance matrix becomes properly correlated. Since this takes time to

develop a correlated matrix, a transient period exists during which the measurements are processed improperly and the updates may degrade the estimate. To obtain an accurate estimate of the state with little or no transient, better initial conditions must be supplied. In other words, better or more information must be contained in the covariance matrix prior to starting or restarting the filtering process.

CHAPTER 4

INITIALIZATION USING PRIOR INFORMATION

4.1 Incorporation of Prior Information Using Pseudo-Measurements

In a given problem, such as the midcourse phase of the Apollo mission considered in this study, certain quantities pertaining to the state at the time of initialization may be known. Examples are the energy of the spacecraft's orbit, the angular momentum associated with the orbit, or perhaps direct information concerning the position and velocity. This information when supplied to the initial covariance matrix should reduce the transient in acquiring a good state estimate. Furthermore should reinitialization be required at any point in the mission, one of the aforementioned quantities, or others, may be known quite accurately. A means of conveying this information to the covariance matrix is by the use of the information matrix and pseudo-measurements on the specific quantities.

Given a set of m measurements with associated measurement vectors, $\underline{b}$, and uncorrelated measurement errors α_m to be batch processed, the information matrix becomes

$$E^{-1} = (\underline{b}_1 \underline{b}_2 \dots \underline{b}_m) \begin{pmatrix} \frac{1}{\alpha_1^2} & 0 & 0 & \dots \\ 0 & \frac{1}{\alpha_2^2} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & \dots & \dots & \frac{1}{\alpha_m^2} \end{pmatrix} \begin{pmatrix} \underline{b}_1^T \\ \underline{b}_2^T \\ \vdots \\ \underline{b}_m^T \end{pmatrix} \quad (4.1)$$

After multiplication, equation 4.1 can be written

$$E^{-1} = \sum_{i=1}^m \frac{1}{\alpha_i^2} \underline{b}_i \underline{b}_i^T \quad (4.2)$$

Hence, the general recursion relation for updating the information matrix is

$$E^{(m)} - 1 = E^{(m-1)} - 1 + \frac{1}{\alpha_m^2} \underline{b}_m \underline{b}_m^T \quad (4.3)$$

where the superscript m in parenthesis denotes the information matrix after m measurements have been incorporated. Since the matrix $\underline{b}_m \underline{b}_m^T$ is nonnegative definite, the information matrix becomes more positive definite as data is incorporated. Correspondingly, the covariance matrix becomes less positive definite and the uncertainty decreases, thus uncertainty is inversely related to information. A simple calculation is given in Appendix B to show the reduction in the volume of the equiprobability ellipsoid as new information is incorporated.

The above concept can be extended to the problem where the measurements are taken at different times on a dynamic system and are processed as they are taken. Assuming no system noise, the state and the covariance matrix are extrapolated from time t_{k-1} to time t_k by the relations

$$\underline{x}'_k = \phi_{k,k-1} \underline{x}_{k-1} \quad (4.4)$$

$$E'_k = \phi_{k,k-1} E_{k-1} \phi_{k,k-1}^T \quad (4.5)$$

The prime denotes quantities at the measurement times indicated by the subscripts but prior to the incorporation of the measurement data. Since the state transition matrix is nonsingular

$$E_k'^{-1} = \phi_{k-1, k}^T E_{k-1}^{-1} \phi_{k-1, k} \quad (4.6)$$

Rewriting equation 4.6 in terms of the initial value of the information matrix, E_o^{-1} , gives

$$E_k'^{-1} = \phi_{o, k}^T E_o^{-1} \phi_{o, k} + \sum_{i=1}^{k-1} \phi_{i, k}^T \frac{b_i b_i^T}{\alpha_i^2} \phi_{i, k} \quad (4.7)$$

The information matrix at time t_k based on the previous $k-1$ measurements does not depend on the actual data taken (for a fixed measurement schedule) but only depends on the system model used and the initial information matrix. Therefore, if the information contained in E_o^{-1} is increased, a decrease in the corresponding uncertainty in the state estimate at time t_k is implied.

Equation 4.2 is a convenient way of computing an initial covariance matrix if various quantities pertaining to the initial state are independently known. At the time of initialization, the measurement vector for each known quantity may be computed and an information matrix formed. Inverting the information matrix gives an initial covariance matrix of estimation errors, based on this knowledge of the state, with which to start the filtering process. Since these quantities are not actually measured with on-board instruments, the term pseudo-measurements is used to distinguish them from the star-horizon measurements made using a space sextant. The matrix inversion requires the information matrix to be of rank six which can be achieved by rewriting equation 4.2 as

$$E_{o+}^{-1} = E_o^{-1} + \sum_{i=1}^n \frac{\underline{b}_i \underline{b}_i^T}{\alpha_i^2} \quad (4.8)$$

where E_o^{-1} is a matrix of full rank and $\underline{b}_i$ is the pseudo-measurement vector for the i^{th} quantity.

CHAPTER 5

COMPUTER SIMULATION OF THE MIDCOURSE NAVIGATION PROBLEM

5.1 Monte Carlo Technique

The navigation problem associated with midcourse phase of the spaceflight was simulated on a digital computer where a stochastic model of the dynamical system was used. The digital computer simulation generates the trajectory of the spacecraft from the time of trans-earth insertion (TEI) occurring near the back side of the moon to the time of reentry into the earth's atmosphere. The actual trajectory is computed along with the trajectory estimated by the Apollo Guidance Computer (AGC) using linear estimation theory which was presented in Chapter 3. The errors associated with the initial position and velocity at TEI are assumed to be random quantities, normally distributed with a specified standard deviation and a mean of zero. TEI occurs at approximately 149.3 hours (ground elapsed time) into the mission. The state is then extrapolated for one hour to 150.3 hours at which time the error transition matrix is initialized in preparation for the first batch of observations.

Since a statistical model is assumed for the various error quantities, a series of flights are made and the performance of the navigation system is inferred by statistically averaging the results of the simulations. Thus one Monte Carlo run consists (for this study) of 25 midcourse flights from the moon to the earth.

The system model used to calculate the estimated trajectory will in general differ from the actual nominal values used to calculate the true or actual trajectory of the spacecraft. Since both trajectories, the "true" and the estimated are calculated in the digital simulation, two separate files are stored from which the various parameters used in calculating the trajectories are taken. Some of the results of this study were obtained by setting the two files equal and zeroing many of the biases and rms errors associated with quantities such as the sextant error, moon and earth horizon errors etc. The various files used are given in Appendix D.

5.2 Generation of Initial Position and Velocity Dispersions

The initial dispersions in position and velocity for the Monte Carlo simulations were generated from information supplied by the NASA Manned Spacecraft Center, Houston, Texas, in the form of a correlation matrix of initial state errors. The deviations from a nominal or reference state were obtained by generating six random numbers

$$\underline{n} = \begin{bmatrix} n_1 \\ n_2 \\ \cdot \\ \cdot \\ \cdot \\ n_6 \end{bmatrix} \quad (5.2.1)$$

from a normal distribution with mean values of zero and standard deviations determined by the square root of the eigenvalues of the correlation matrix C. Let P be the matrix whose columns are the orthonormal eigenvectors of C and let the characteristic values of C be denoted by $\lambda_1, \lambda_2, \dots, \lambda_6$. Then the random

number n_i has a mean value of zero and a standard deviation of $\sqrt{\lambda_i}$ and the initial deviation of the state from a nominal state is given by

$$\underline{\delta x}_I = P \underline{n} \quad (5.2.2)$$

The correlation matrix C giving the statistics associated with the initial errors in the state at TEI is given in Table 5.1 where the units on position and velocity are feet and feet per second respectively. The matrix as given is not positive definite since one of the eigenvalues is negative. In generating the errors this eigenvalue was first set to zero since an imaginary standard deviation is nonphysical. Also this correlation matrix is given in a local vertical coordinate system defined at the time of TEI. Since the initial errors are referenced to the inertial coordinate system, C must also be transformed to the inertial system before computing P and the eigenvalues. The definition of the local vertical coordinate system and the matrix of transformation is given in Appendix C. Applying equation 5.2.2 then generates the initial deviation of the state at TEI resolved in the inertial coordinate system.

The errors in the estimate of the state maintained in the AGC at TEI were generated using the matrix of eigenvectors and eigenvalues. Therefore for any one flight the estimate of the state at TEI would not be the same as the true state but a statistical average over an ensemble of flights would be the same.

Table 5.1 Error Covariance Matrix at TEI

$$E_o = \begin{bmatrix} 1.6 \times 10^7 & -7.43 \times 10^7 & 30.5 & 6.47 \times 10^4 & -1.2 \times 10^4 & 1.18 \\ -7.43 \times 10^7 & 5.81 \times 10^8 & 190 & -5.03 \times 10^5 & 4.96 \times 10^5 & -7.76 \\ 30.5 & 190 & 2.17 \times 10^6 & -.213 & -.0552 & 9.61 \times 10^3 \\ 6.47 \times 10^4 & -5.03 \times 10^5 & -.213 & 436 & -43.1 & 6.31 \times 10^{-3} \\ -1.2 \times 10^4 & 4.96 \times 10^5 & -.0552 & -43.1 & 9.11 & -1.02 \times 10^{-3} \\ 1.18 & -7.76 & 9.61 \times 10^3 & 6.31 \times 10^{-3} & -1.02 \times 10^{-3} & 81 \end{bmatrix}$$

5.3 Measurement Schedule and Types of Measurements

An on-board estimate of the state of the spacecraft is computed from linear filtering theory and stored in the AGC. The errors in the estimate, if propagated over an extended period of time, degrade rapidly. Therefore to maintain a useful estimate it becomes necessary to make periodic measurements of the state and supply this information to the navigation filter. The sensor used to obtain the data is a space sextant which allows the astronaut to measure the apparent elevation of a star above the horizon of a near body or to measure the angle subtended by a star and a known landmark. By knowing the direction to the star and the position of the near body, the position of the spacecraft in the plane of the measurement and perpendicular to the line of sight from the spacecraft to the horizon (or to the landmark) can be estimated. ⁽²⁾ By rotating the measurement plane, that is by choosing stars in various directions, additional information can be obtained to estimate the vehicles position. For this study the data consisted entirely of star-moon horizon or star-earth horizon measurements although both near and far horizons were considered in the measurement schedule. The six-dimensional measurement vector associated with the angle measurements represents, to a first order approximation, the variation in the measured quantity which would result from variations in the components of the state vector. From Figure 5.1 the measured quantity Q satisfies the relation

$$\underline{m} \cdot \underline{l} = \cos Q \quad (5.3.1)$$

treating all changes as first order differentials and assuming that the unit vector to the star is constant.

$$\delta \underline{m} \cdot \underline{l} = - \sin Q \delta Q \quad (5.3.2)$$

But

$$\delta \underline{m} = \frac{1}{d} \delta \underline{d} - \frac{1}{d^2} \delta d \underline{d} \quad (5.3.3)$$

Noting that $\delta \underline{d} = -\delta \underline{r}$ and writing δd as

$$\delta d = \frac{\underline{d} \cdot \delta \underline{d}}{d} \quad (5.3.4)$$

equation 5.3.3 becomes

$$\delta \underline{m} = -\frac{\delta \underline{r}}{d} + \frac{\underline{m}}{d} (\underline{m} \cdot \delta \underline{r}) \quad (5.3.5)$$

Substituting equation 5.3.5 into equation 5.3.2 and solving for δQ gives

$$\delta Q = -\frac{1}{\sin Q} \left[\frac{\underline{m} (\underline{m} \cdot \delta \underline{r}) - \delta \underline{r}}{d} \right] \cdot \underline{l} \quad (5.3.6)$$

Rearranging terms and using equation 5.3.1 gives

$$\delta Q = \frac{1}{d} \left(\frac{\underline{l} - \cos Q \underline{m}}{\sin Q} \right) \cdot \delta \underline{r} \quad (5.3.7)$$

The vector in parenthesis is the unit vector $\underline{p}$ shown in the figure and is perpendicular to the line of sight from the spacecraft to the near body horizon and in the plane determined by the unit vectors $\underline{m}$ and $\underline{l}$. The measurement vector, independent of the components of velocity is

$$\underline{b} = \frac{1}{d} \begin{bmatrix} \underline{p} \\ 0 \end{bmatrix} \quad (5.3.8)$$

A perfect measurement would fix the position of the spacecraft in space along a direction given by the vector $\underline{p}$.

The measurement schedule used in this study corresponds to the schedule used in the Apollo 14 mission. A summary of the measurement schedule is given in Table 5.2. Periodically during the flight a batch of measurements are taken consisting of either nine or fifteen observations occurring at three minute intervals. The times given in Table 1 refer to the beginning of a batch and are referenced to the time of launch from Cape Kennedy. A typical trajectory is given in Figure 5.2 and the position of the spacecraft at the times of each batch are indicated.

During the midcourse phase of the Apollo mission the spacecraft is in free fall flight between the moon and the earth with the exception of brief accelerations associated with midcourse velocity corrections. As a result the trajectory deviates only slightly from a straight line between the earth and the moon. Since only moon and earth horizon measurements are taken the measurement vectors are for the most part perpendicular to this line. As various stars are used the direction of the measurement vector changes but is restricted to move only in a plane perpendicular to the spacecraft's trajectory. Most of the information contained in the measurements tends to reduce the uncertainty of the spacecraft's position and velocity in directions lying in this plane. In other words, the equiprobability ellipsoids which indicates the distribution of the position error vector, becomes elongated in the direction perpendicular to the measurement plane.

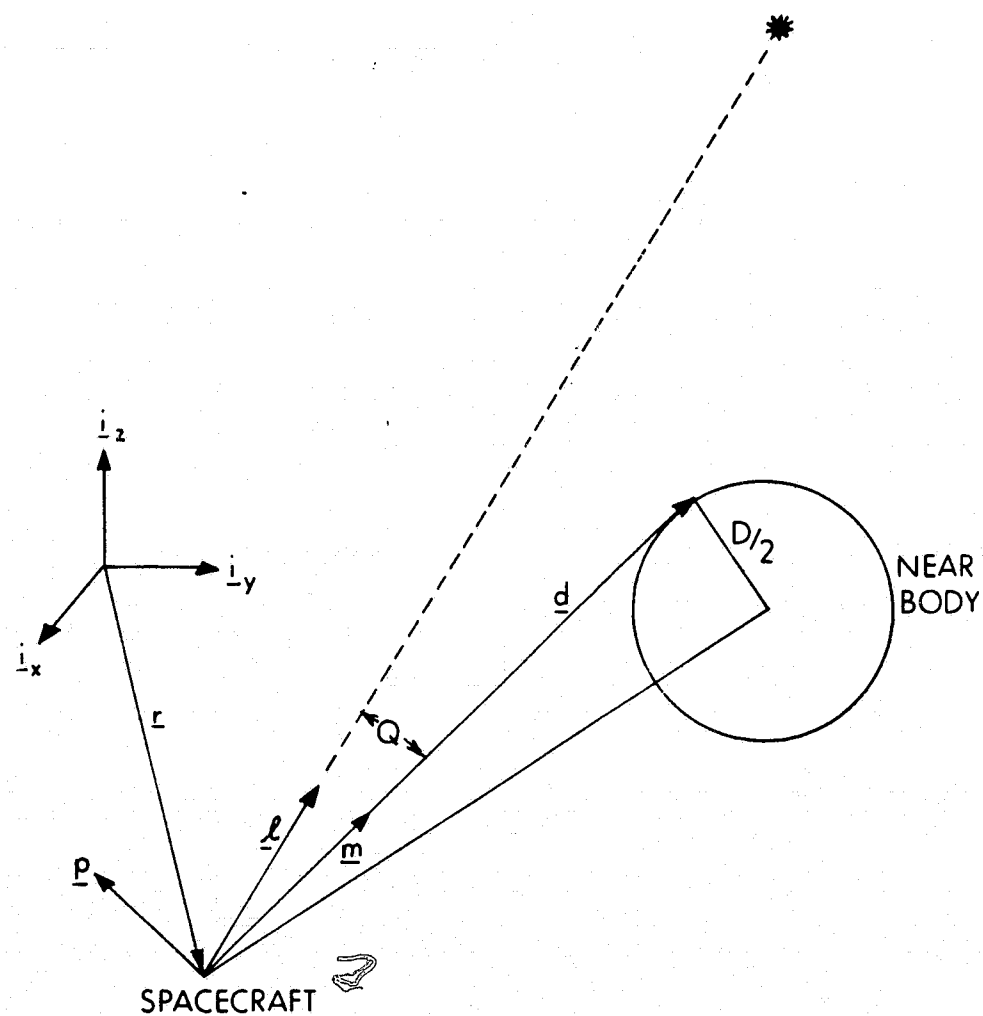


Figure 5.1 Geometry of a star-horizon measurement

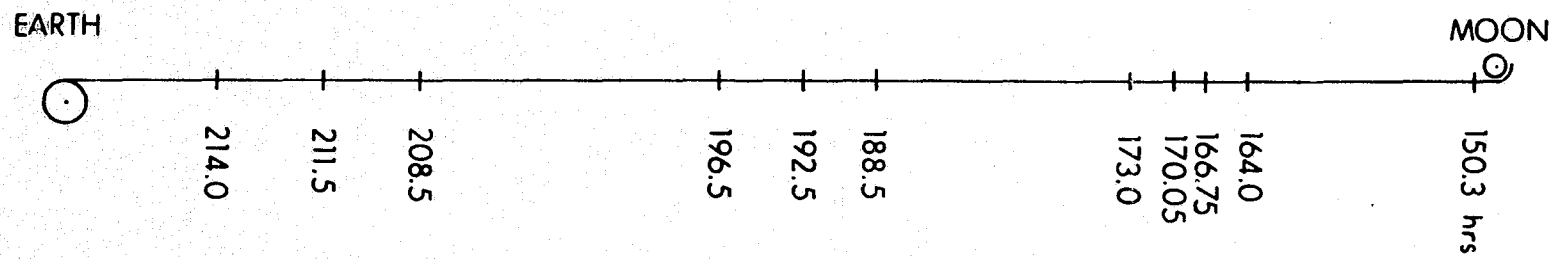


Figure 5.2 Sketch of a trajectory from the moon to the earth showing the approximate position and times of the measurement batches.

TIME (HOURS SINCE LAUNCH)	HORIZON	NUMBER OF MEASUREMENTS
149.28	Trans - Earth Insertion	
150.30	Moon	15
164.00	Earth	9
165.00	Midcourse Velocity Correction	
166.75	Moon	9
170.05	Earth	9
173.00	Earth	9
188.50	Earth	9
192.50	Earth	9
196.50	Earth	9
205.00	Midcourse Velocity Correction	
208.50	Earth	9
211.50	Moon	15
214.00	Moon	9
215.00	Midcourse Velocity Correction	

Table 5.2 Summary of the Measurement Schedule Used
in the Digital Simulation.

CHAPTER 6

RESULTS OF SIMULATION STUDY

The midcourse phase of the Apollo mission begins when the spacecraft is given the necessary velocity to place it on a return trajectory to intercept the earth. For the Apollo 14 mission this trans-earth insertion burn was scheduled to occur after a mission ground-elapsed time of approximately 149.3 hours. At 150.3 hours a batch of measurements, fifteen in all, were to be taken using various stars and the moon's horizon. Also at 150.3 hours and prior to the incorporation of any data the covariance matrix of the estimation errors was scheduled to be initialized. (The error transition matrix as defined in section 3.2 is actually initialized and extrapolated from one measurement time to the next however, for the purpose of presenting the results of this study, the actual covariance matrix will be referred to in hopes that less confusion will result). The initial covariance matrix at 150.3 hours used in the Apollo 14 mission was diagonal where the variance of each component of the position error was $(30,000 \text{ ft})^2$ and the variance of each component of the velocity error was $(30 \text{ ft/sec})^2$. This implies that the errors in the state have no preferred direction, or in other words, that the surfaces of equiprobability are spherical. This also implies that the velocity error in one direction is not correlated with the position error in any direction. In actuality this correlation does exist and is eventually developed after the processing of many observations. Furthermore since an angle measurement at time t_k gives information pertaining only to the position error of the spacecraft at t_k , no information concerning the error in the velocity of the spacecraft exists until the covariance matrix becomes reasonably correlated. The basic problem

associated with initializing the covariance matrix is to determine the correlation between the elements of the state vector errors. The more information that is known at the time of initialization the faster the estimate will converge to the true state. Therefore it is advantageous to incorporate a priori knowledge of the state before beginning the filtering process. Use of the information matrix and pseudo-measurements of the a priori known quantities as discussed in Chapter 4 is a way of incorporating this information and developing a correlated initial covariance matrix. This technique was applied in the simulation of the midcourse phase of the Apollo 14 mission. Certain parameters of the spacecraft's orbit at the time of initialization were assumed known and this information was incorporated into the initial diagonal covariance matrix. The results obtained were then compared to the results using the diagonal matrix only. Some of the Monte Carlo computer runs were made assuming that the system model agreed with the actual or simulated environment of the spacecraft and many of the biases and rms errors were set to zero so that these effects would not mask the effect of prior information. Since the model errors and biases do effect the state estimation, the results obtained using these files may be compared directly to the results using the diagonal covariance matrix on the same files. The various files for the system model and the actual environment used in the digital simulation are given in Appendix D.

One of the parameters that might be known at the time of initialization is the total energy of the spacecraft's orbit. If the energy (i. e. total energy per unit mass) were to be measured directly the geometry vector for this measurement would represent to first order the variation of the energy with respect to the elements of the state vector or

$$\underline{b}_E^T = \left(\frac{\partial Q}{\partial \underline{r}} \mid \frac{\partial Q}{\partial \underline{v}} \right) \quad (6.1)$$

where the hypothetically measured quantity $Q = -\frac{\mu_p}{2a}$ and a is the semimajor axis of the orbit. Since the energy in terms of the position and velocity is given by

$$Q = -\frac{\mu_p}{r} + \frac{v^2}{2} \quad (6.2)$$

the six dimensional geometry vector becomes

$$\underline{b}_E = \begin{bmatrix} \left(\frac{\mu_p}{r^2} \right) & \underline{i}_r \\ (v) & \underline{i}_v \end{bmatrix} \quad (6.3)$$

where $\underline{i}_r$ and $\underline{i}_v$ are unit vectors along the position and velocity vectors of the spacecraft respectively. The information can be incorporated into the initial covariance matrix using equation 4.8. The updated information matrix becomes

$$E_{o+}^{-1} = E_o^{-1} + \frac{\underline{b}_E \underline{b}_E^T}{\alpha_E^2} \quad (6.4)$$

where a plus sign as a subscript implies that additional information has been incorporated. After the matrix inversion, E_{o+} can be used to start the filtering process. In the Monte Carlo runs the true trajectory is calculated so that the true error in the state estimate can be computed and compared to the estimated error given by the trace of the positional part of the covariance matrix maintained in the AGC. These errors are plotted in Figures 6.5 - 6.8 for the case where the energy pseudo-measurement was

incorporated and a Monte Carlo run was made using the simplified data files of the system model and the actual environment. The error variance of the energy, $\overline{\alpha_E^2}$, was chosen arbitrarily to be $25 v^2$ where the nominal velocity in miles/hour at 150.3 hours was used for the calculation, that is the velocity of the spacecraft assuming no position or velocity errors at TEI. The error covariance matrix, E_0 , for all results given unless otherwise stated, will imply the diagonal matrix

$$E_0 = \begin{pmatrix} (5.68 \text{ miles})^2 & 0 & 0 & 0 & 0 \\ 0 & (5.68 \text{ miles})^2 & 0 & 0 & 0 \\ 0 & 0 & (5.68 \text{ miles})^2 & 0 & 0 \\ 0 & 0 & 0 & (20.45 \text{ miles/hr})^2 & 0 \\ 0 & 0 & 0 & 0 & (20.45 \text{ miles/hr})^2 \end{pmatrix} \quad (6.5)$$

where 5.68 miles corresponds to 30,000 feet and 20.45 miles per hour corresponds to 30 feet per second.

The comparable case using E_0 only is given in Figures 6.1 - 6.4. Only plots for the first and second batches of measurements will be given since little or no difference of significance was observed throughout the rest of the flight. That is, practically all the runs made converged to approximately the same values during the third batch of measurements. This conforms with theory since the effect of prior information diminishes if enough observations are processed. However, as can be seen from the plots, the estimate converges to the true errors more rapidly if E_{0+} is used than when E_0 is used. During the thirteen hour period between the first and second batch of measurements the errors grow considerably since no measurements are being taken

and the errors are just extrapolated in an open loop fashion. At the end of the second batch of measurements the first midcourse velocity correction on the return trip is scheduled. Therefore any significant reduction in the errors at this time would be useful because it would allow a more accurate velocity correction.

Another parameter of the orbit that was used in this study was the magnitude of the angular momentum. The angular momentum satisfies the vector relationship

$$\underline{h} = \underline{r} \times \underline{v} \quad (6.6)$$

If the components of $\underline{r}$, $\underline{v}$ and $\underline{h}$ are denoted by

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (6.7)$$

$$\underline{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \quad (6.8)$$

$$\underline{h} = \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} \quad (6.9)$$

the measurement vector becomes after performing the partial differentiation,

$$\underline{b}_h = \frac{1}{h} \begin{bmatrix} \underline{v} \times \underline{h} \\ \underline{h} \times \underline{r} \end{bmatrix} \quad (6.10)$$

An initial covariance matrix E_{o+} was computed using the value $625 \text{ miles}^4 / \text{hours}^2$ for α_h^2 . The results given in Figures 6.9 - 6.12 show that a considerable improvement is made at the end of the second batch of measurements even though little difference is made for the first few measurements of the first batch.

The best results overall were obtained when an initial covariance matrix E_{o+} was computed using information from pseudo measurements on both the energy and the magnitude of the angular momentum. The information matrix for this case is given by

$$E_{o+}^{-1} = E_o^{-1} + \frac{b_E b_E^T}{25v^2} + \frac{b_h b_h^T}{625} \quad (6.11)$$

Figures 6.13 - 6.16 show that a considerable reduction in the difference between the true errors and the estimated errors is made and furthermore the magnitude of the errors is reduced. Comparing these plots with the plots using E_o (Figures 6.1-6.4) alone show that the true errors in the position uncertainty has been reduced from 86 miles to 20 miles by the end of the second batch of measurements and the true errors in the velocity uncertainty has been reduced from 9 miles/hour to 2 miles/hour. Furthermore, the AGC's indicated velocity uncertainty is equal to the true velocity uncertainty which should reflect a more accurate midcourse velocity correction at 165 hours into the mission.

All of the results previously mentioned were obtained using simplified data files for the system model and the actual environment. Various errors and biases were then introduced in certain quantities such as the horizon of the moon and the earth,

the gravitational parameters of the moon and the earth, and other quantities as identified in Table D-2. Monte Carlo runs were then made to obtain a comparison between the state errors obtained using the diagonal initial covariance matrix E_0 with those obtained using the diagonal matrix plus pseudo-measurements on the energy and the magnitude of the angular momentum. The results using E_0 are given in Figures 6.17-6.20 and the results using the additional information are given in Figures 6.21-6.24. Here, as in the case with the simplified data files, the additional information is quite effective in producing a better estimate of the true errors. The true error in the estimate of the position at 164.4 hours has been reduced from 87 miles to 37 miles and the true error in the estimate of the velocity has been reduced from 9 miles/hour to 4 miles/hour.

The ability of the filter to converge rapidly to the true state depends to a large extent on the initial conditions supplied. If the initial conditions are not well known, a large amount of data must be processed before the covariance matrix becomes properly correlated such that beneficial updates can be made. The same condition prevails if the initial conditions reflect erroneous information concerning the initial state. Since the components of the state error vector are correlated, a diagonal initial covariance matrix reflects, to a certain extent, erroneous information as this implies no correlation. Adjoining additional information, if known, to the initial covariance matrix aids in developing a correlated matrix. An attempt was made to determine the relative effect of the information contained in the diagonal matrix with that contained in the pseudo-measurements on the known quantities. Figures 6.21-6.24 gave the results obtained when the initial covariance matrix was computed using equation 6.11 where E_0 is given in equation 6.5. The same case was run with the

diagonal elements increased to 10^4 miles² and 10^4 miles²/hours² corresponding to unrealistically large uncertainties. The results of these runs are given in Figures 6.25-6.28. Comparing these plots with those in Figures 6.21 - 6.24 shows that very little difference exists in the quality of the position estimate after seven or eight measurements. The velocity uncertainty at the same time is much larger although beneficial updates rapidly decrease the uncertainty. At the end of the second batch of measurements both the position and the velocity estimates are identical to those in Figures 6.27 and 6.28 even though the initial covariance matrix was much larger, in a positive definite sense. The rapid convergence of the filter can be attributed to the correlation introduced by adjoining the energy and angular momentum information to the initial covariance matrix.

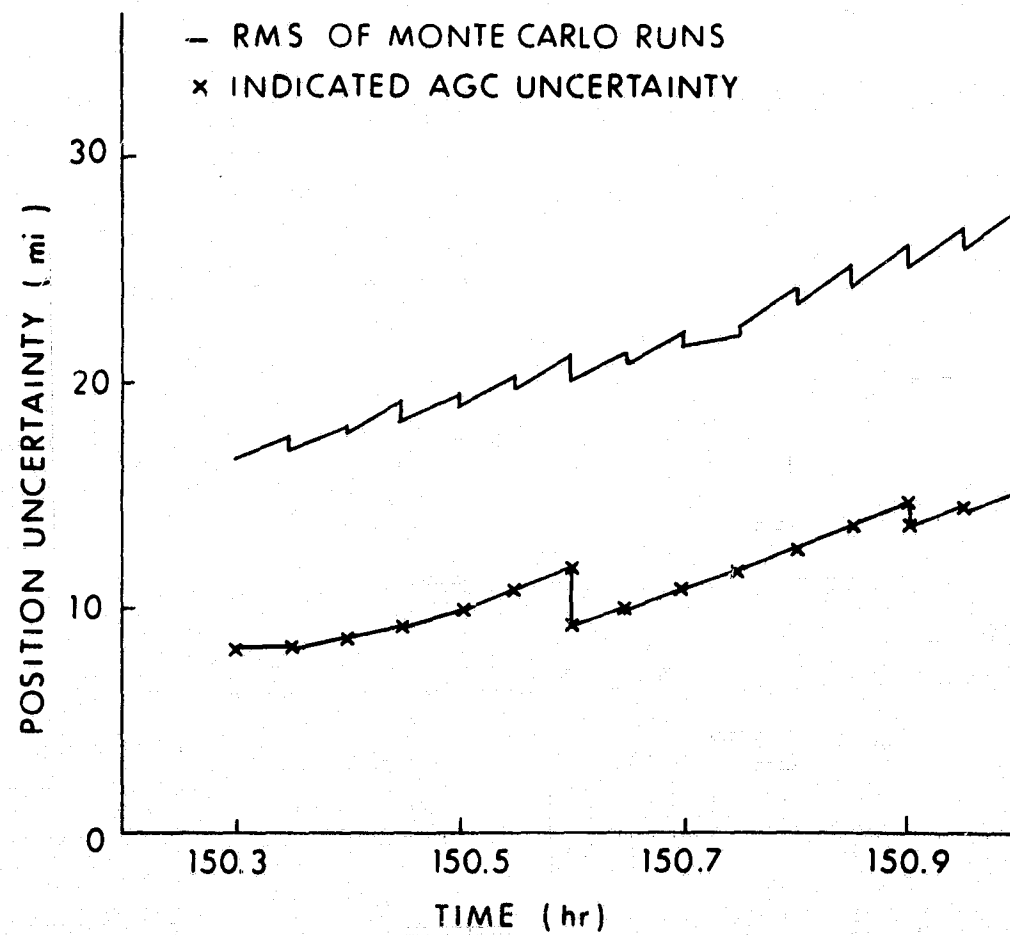


Figure 6.1 Position Uncertainty Using Diagonal Initial Covariance Matrix

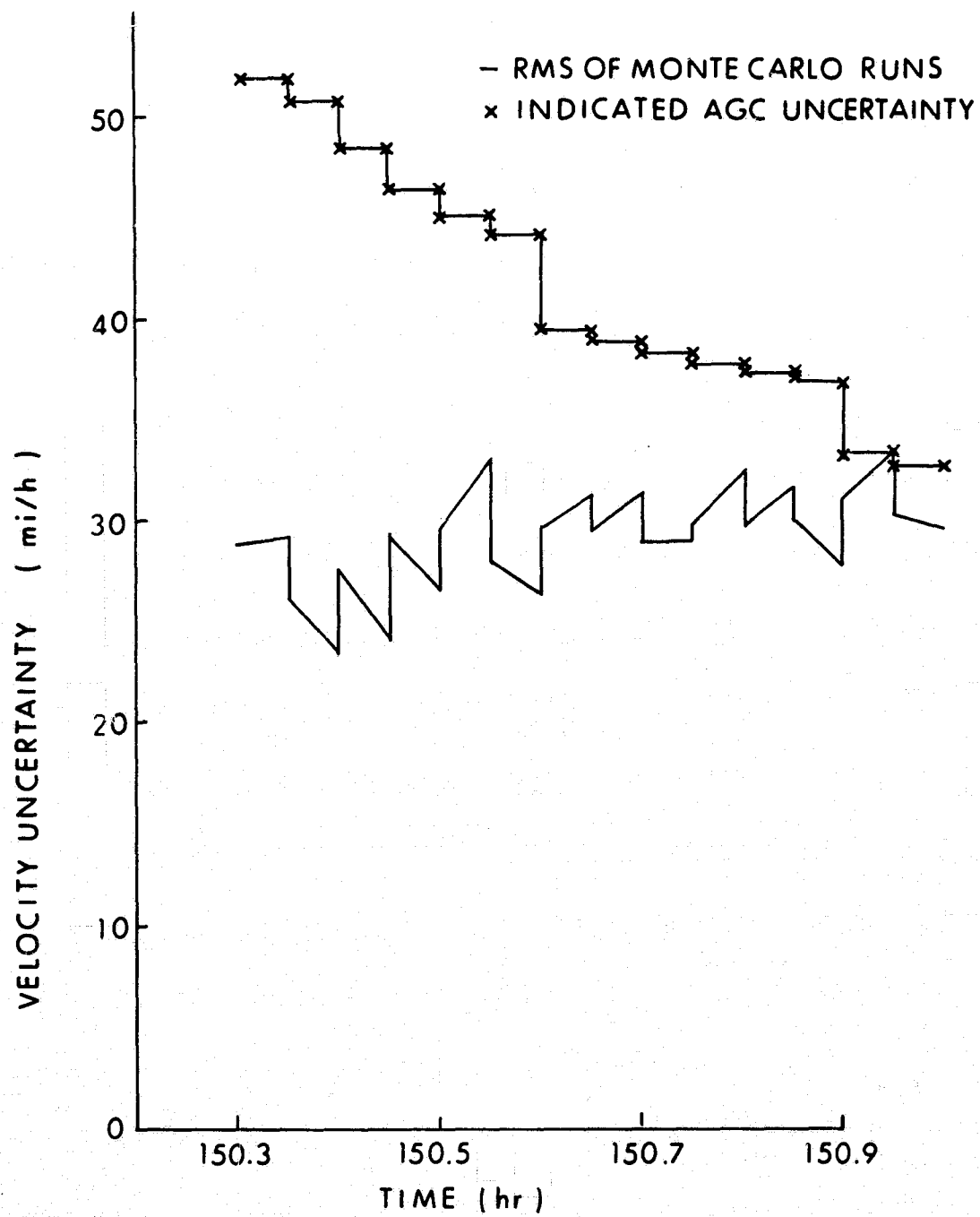


Figure 6.2 Velocity Uncertainty Using Diagonal Initial Covariance Matrix

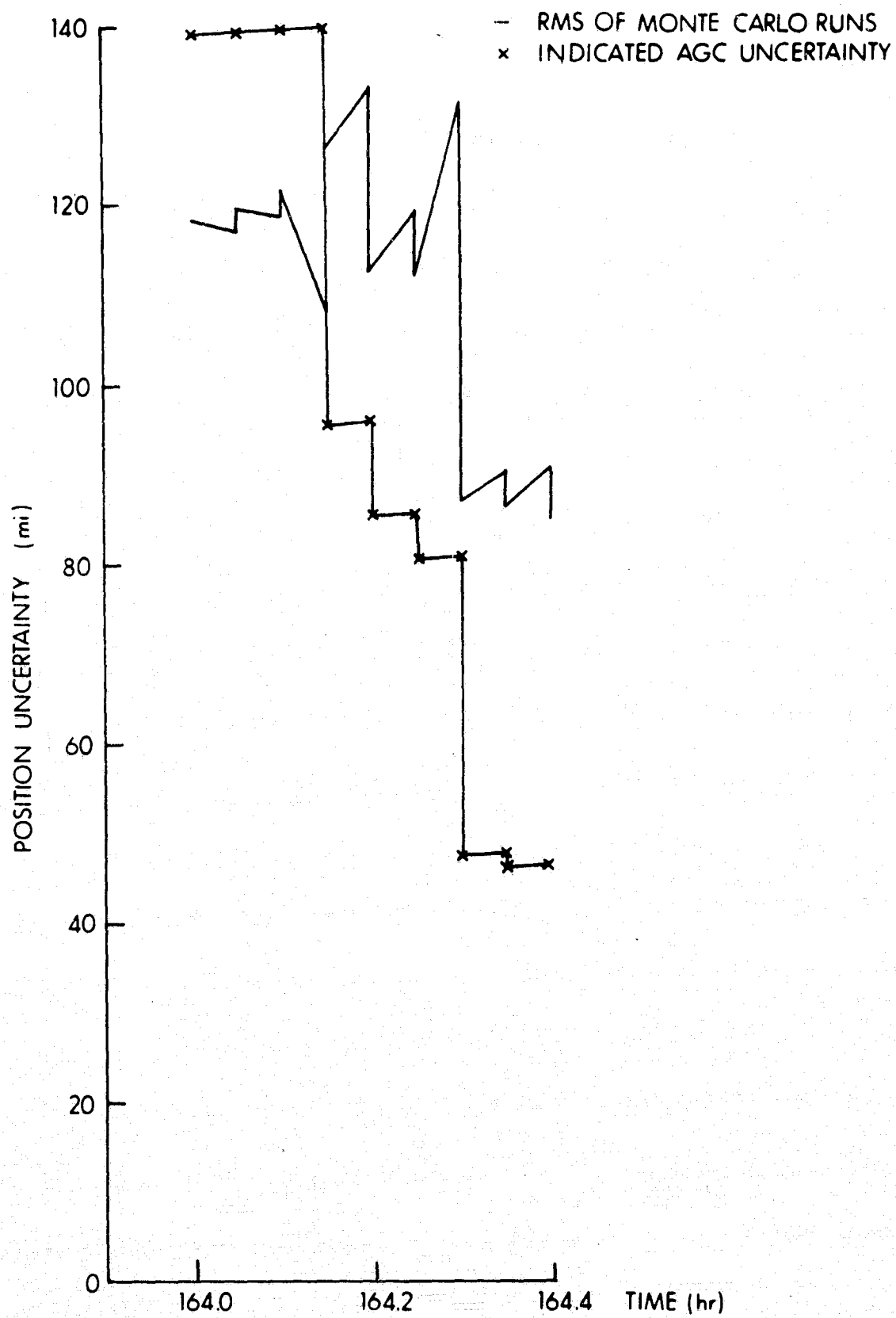


Figure 6.3 Position Uncertainty Using Diagonal Initial Covariance Matrix

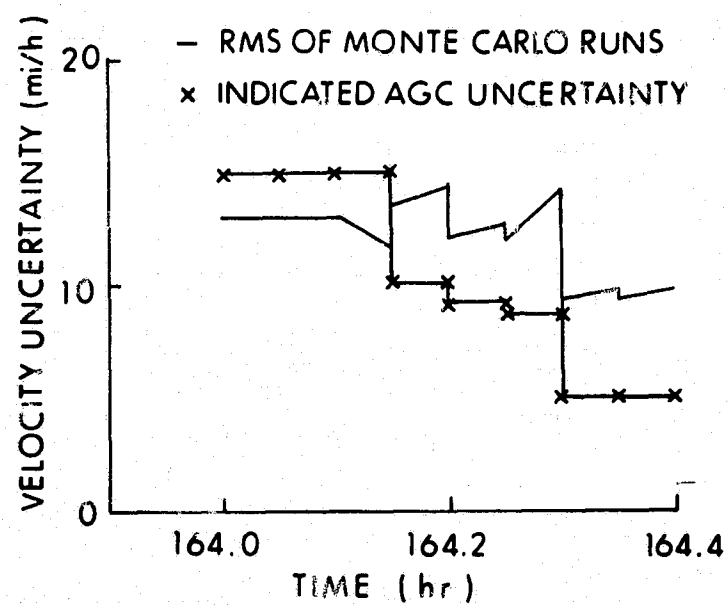


Figure 6.4 Velocity Uncertainty Using Diagonal Initial Covariance Matrix

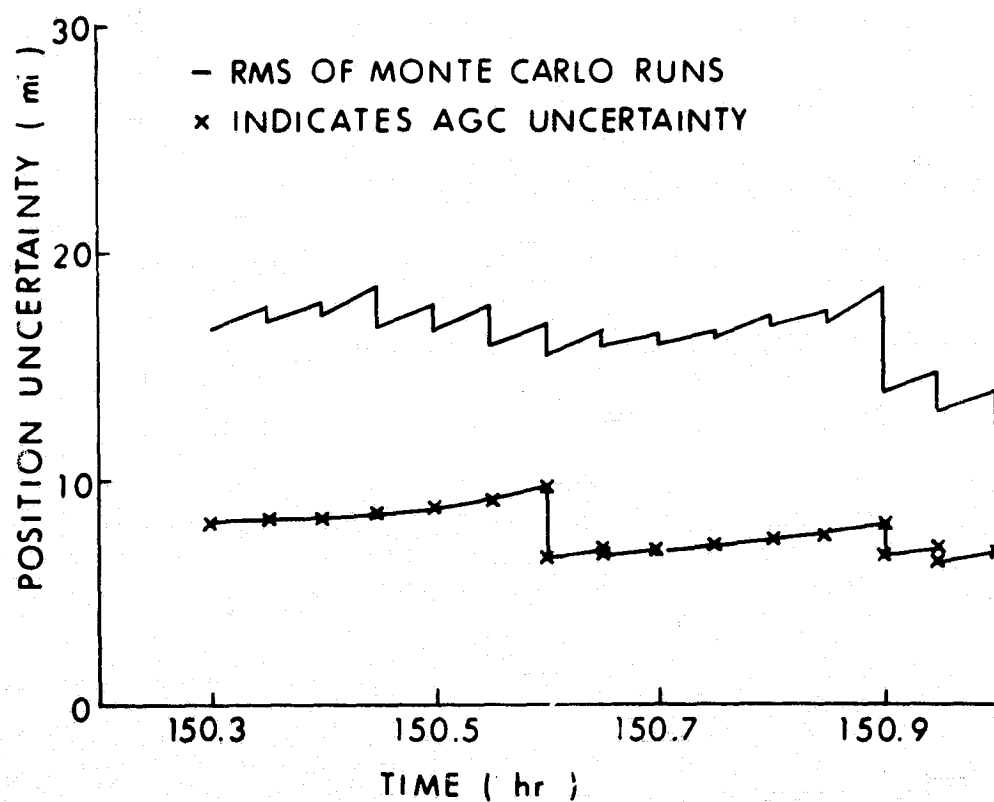


Figure 6.5 Position Uncertainty Using Diagonal Initial Covariance Matrix With Energy Adjoined

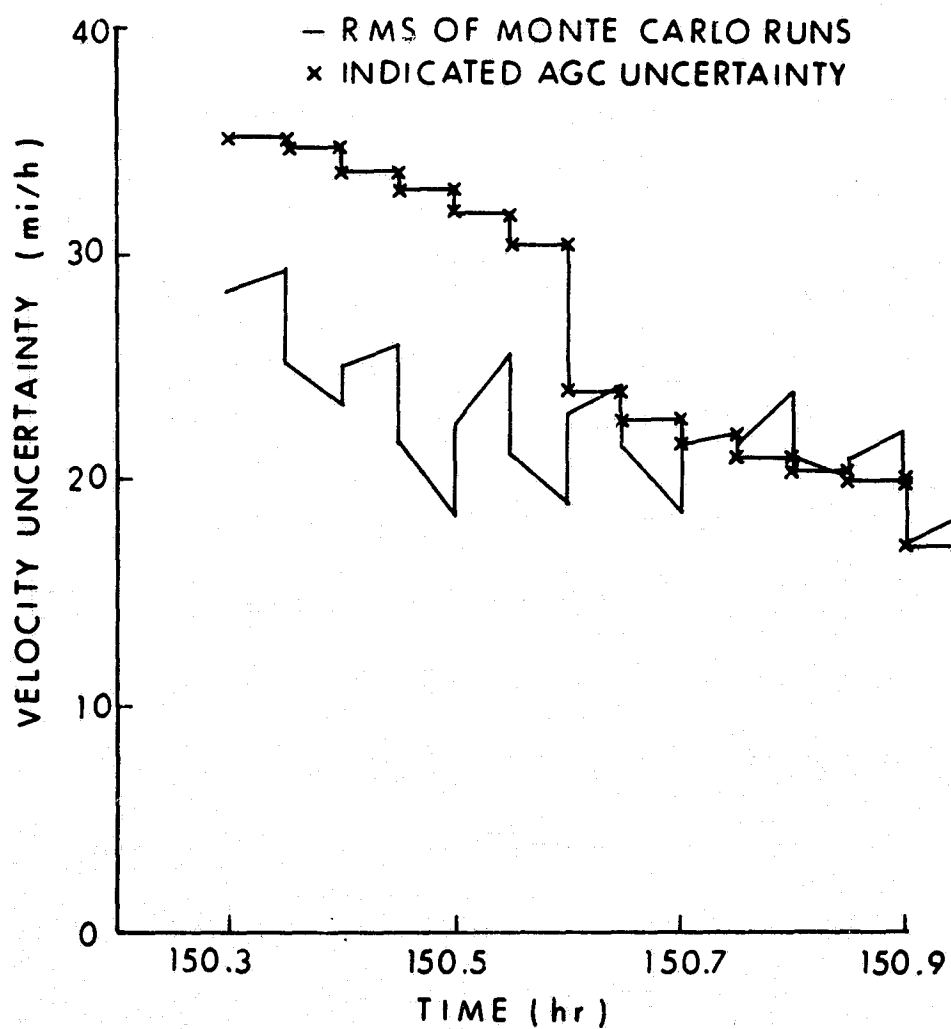


Figure 6.6 Velocity Uncertainty Using Diagonal Initial Covariance Matrix With Energy Adjoined

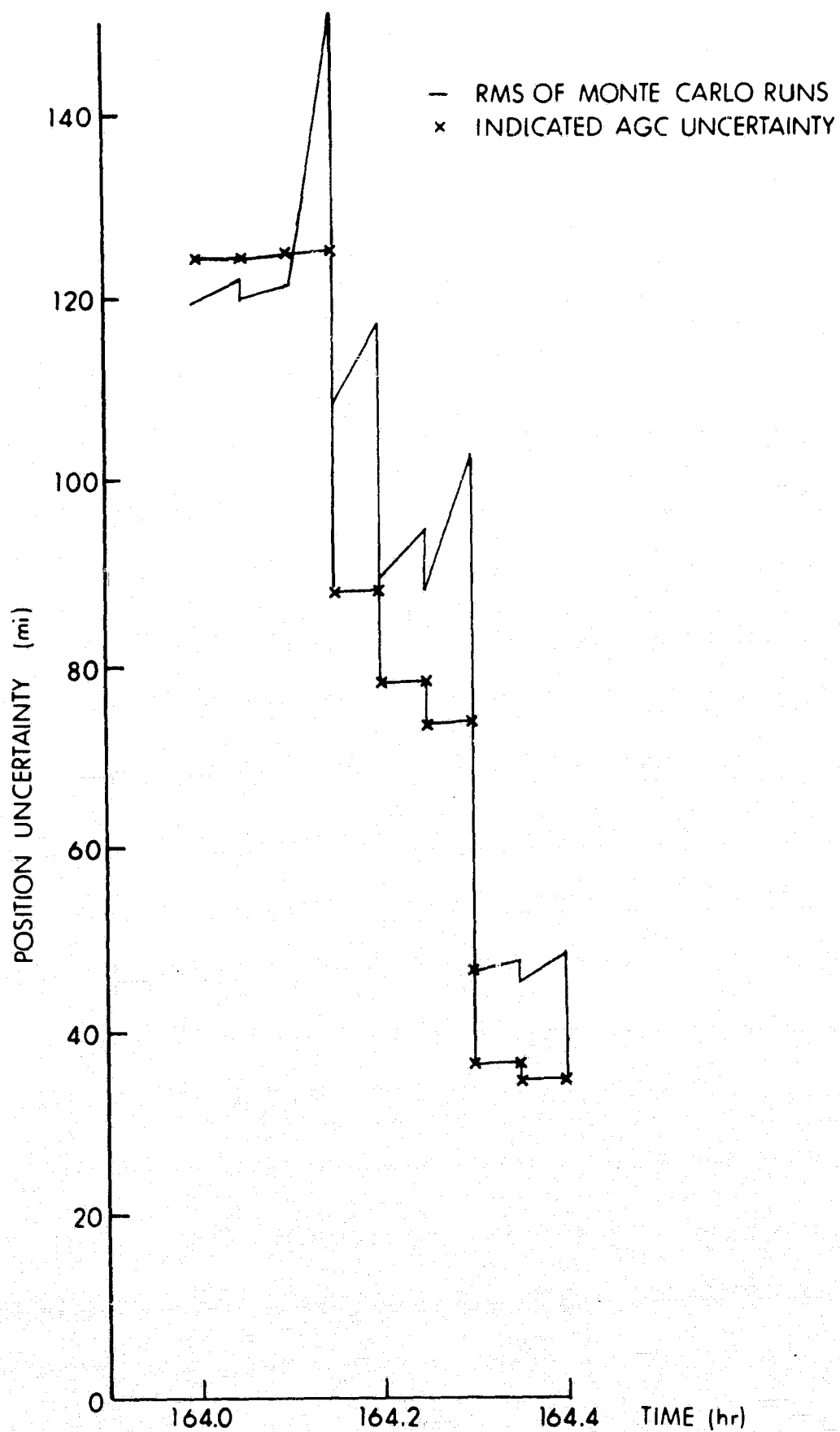


Figure 6.7 Position Uncertainty Using Diagonal Initial Covariance Matrix With Energy Adjoined

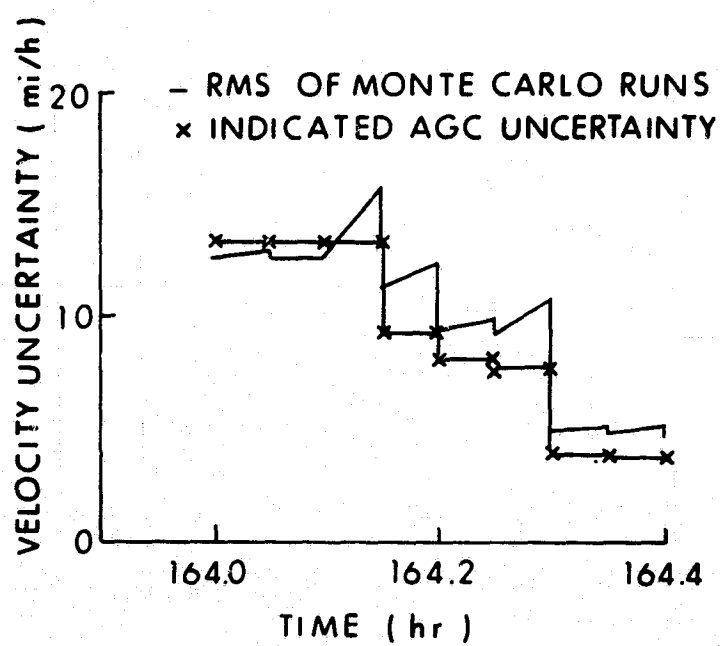


Figure 6.8 Velocity Uncertainty Using Diagonal Initial Covariance Matrix With Energy Adjoined

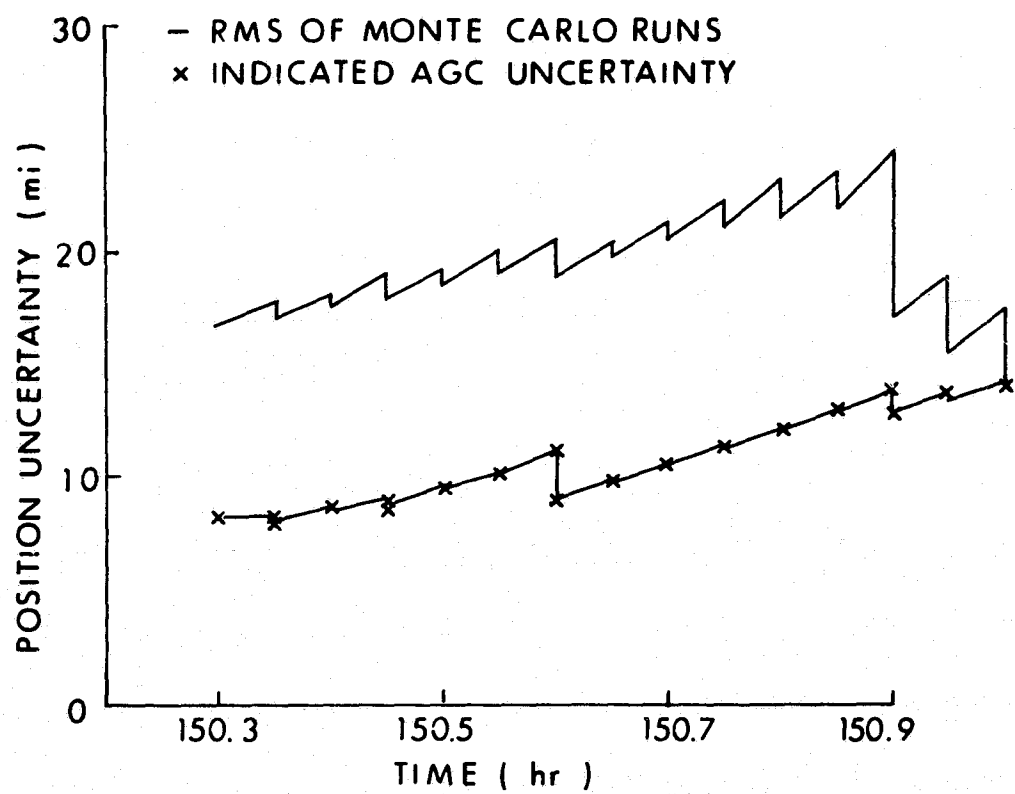


Figure 6.9 Position Uncertainty Using Diagonal Initial Covariance Matrix With Angular Momentum Adjoined

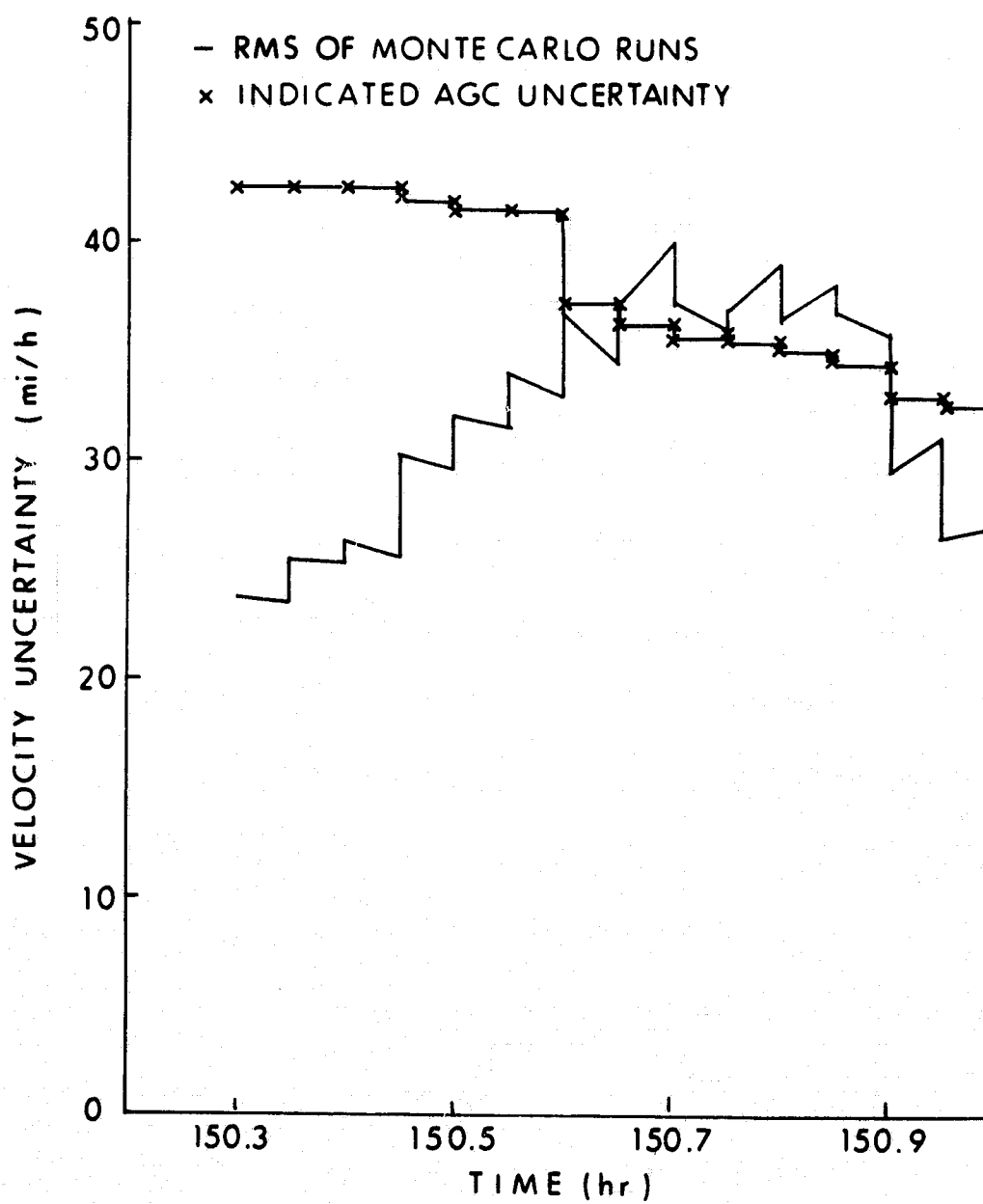


Figure 6.10 Velocity Uncertainty Using Diagonal Initial Covariance Matrix With Angular Momentum Adjoined

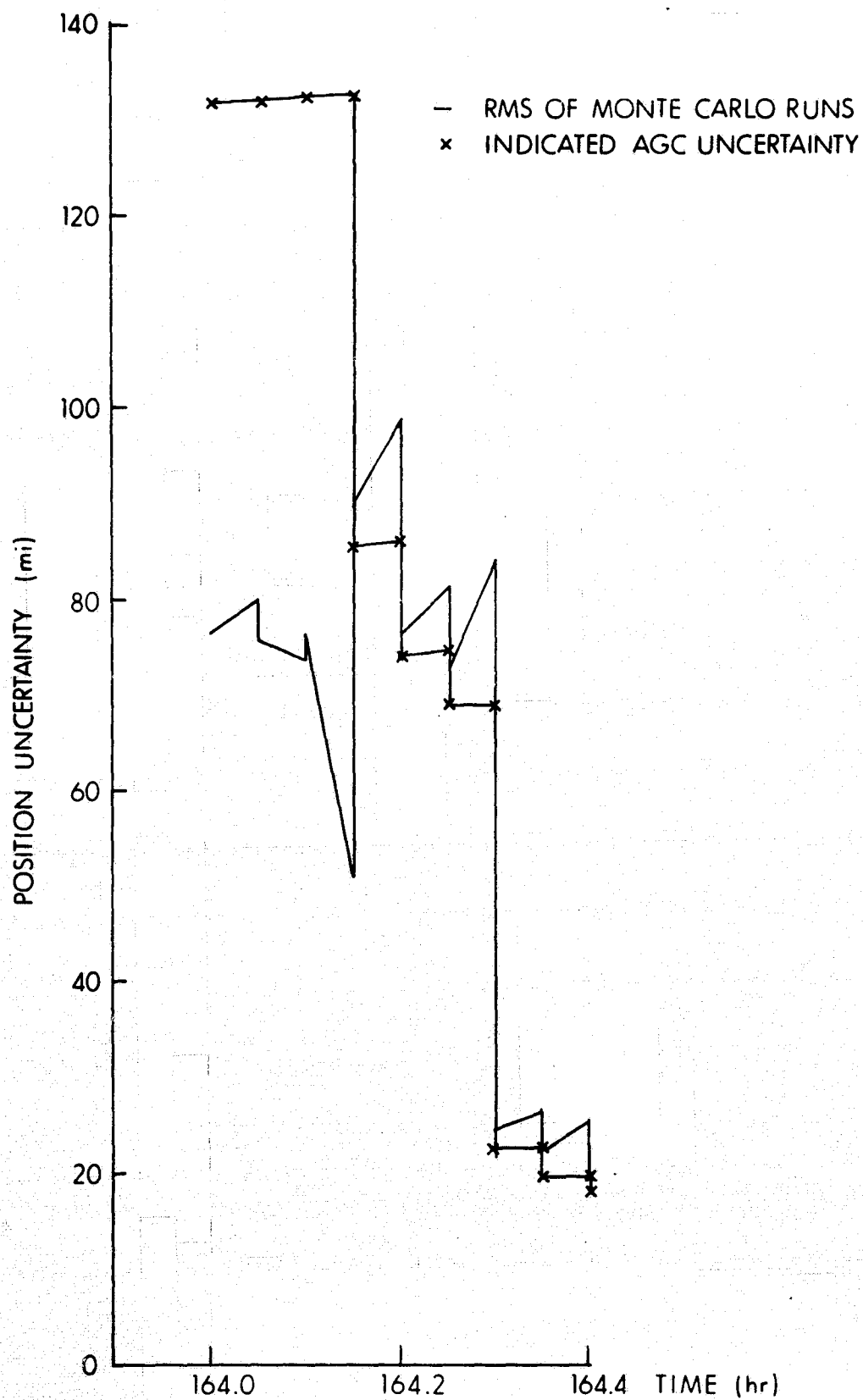


Figure 6.11 Position Uncertainty Using Diagonal Initial Covariance Matrix With Angular Momentum Adjoined

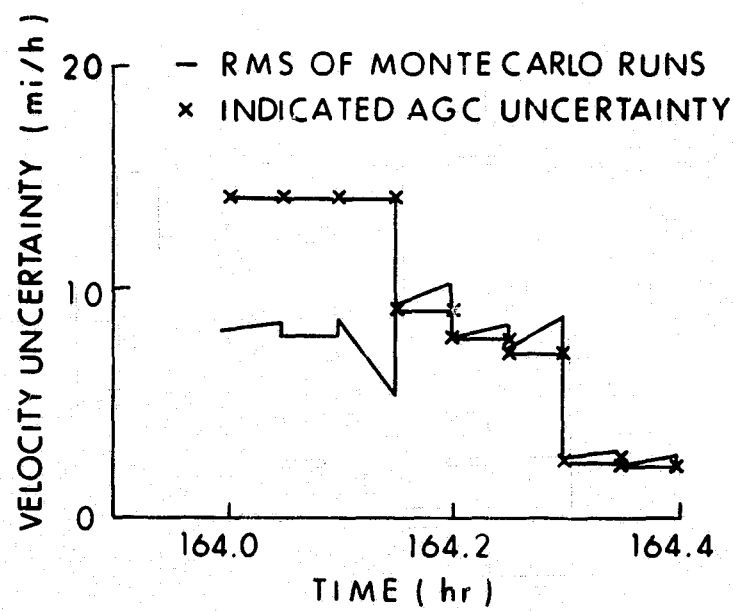


Figure 6.12 Velocity Uncertainty Using Diagonal Initial Covariance Matrix With Angular Momentum Adjoined

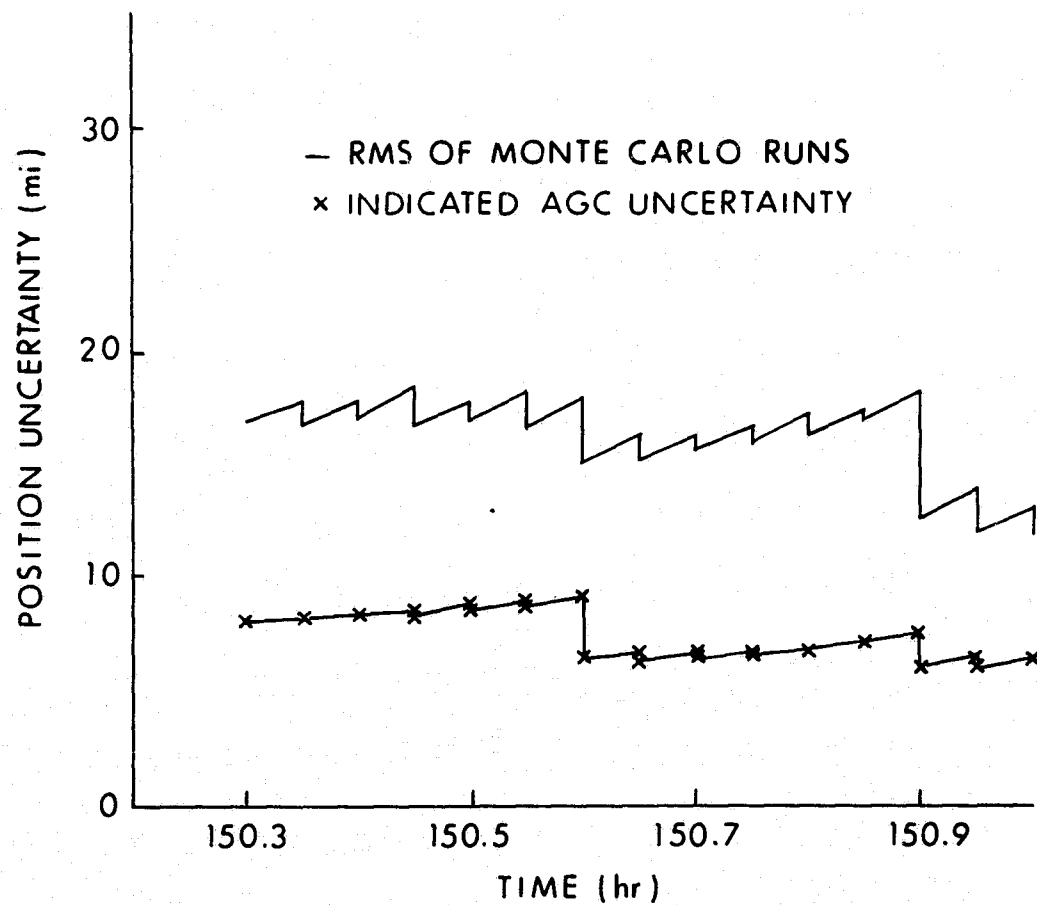


Figure 6.13 Position Uncertainty Using Diagonal Initial Covariance Matrix With Angular Momentum and Energy Adjoined

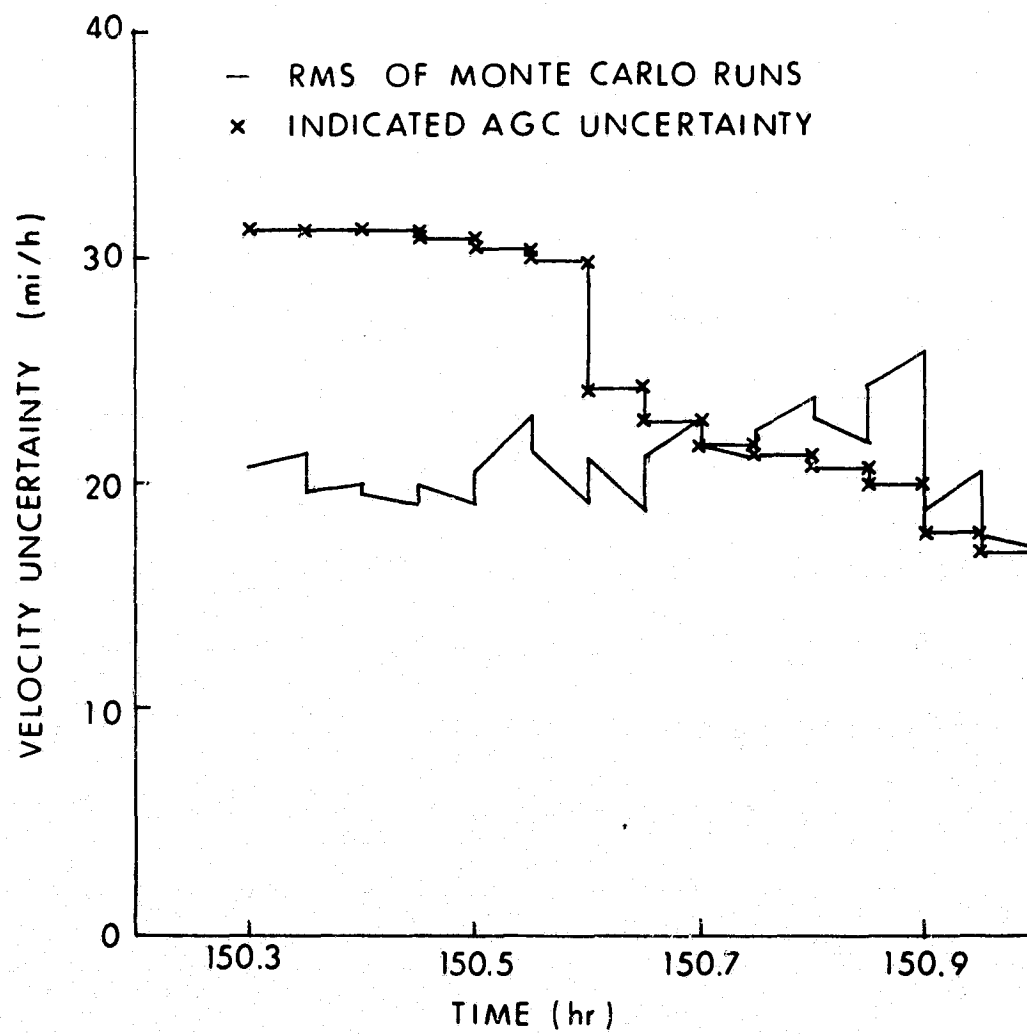


Figure 6.14 Velocity Uncertainty Using Diagonal Initial Covariance Matrix With Angular Momentum and Energy Adjoined

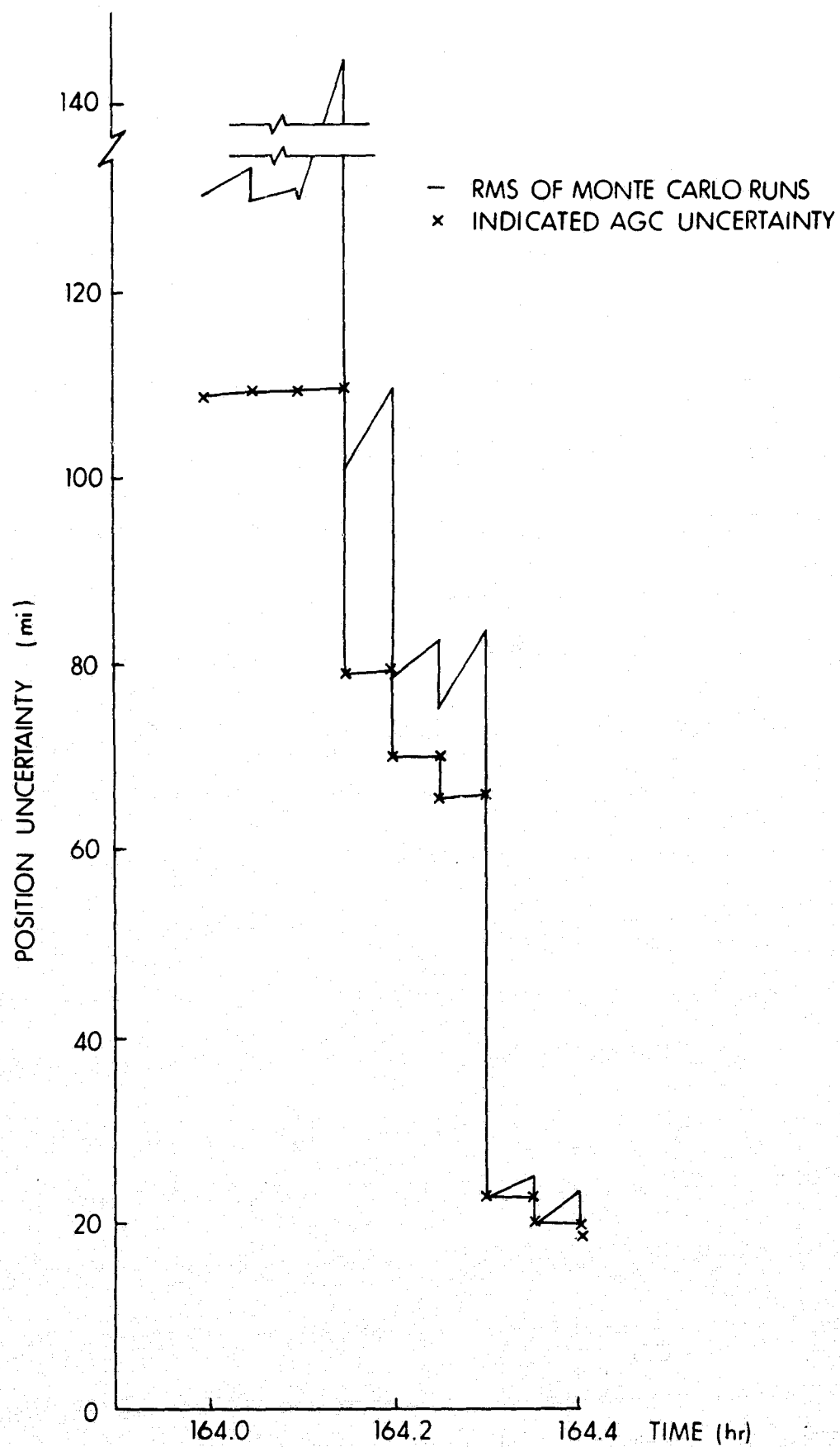


Figure 6.15 Position Uncertainty Using Diagonal Initial Covariance Matrix With Angular Momentum and Energy Adjoined

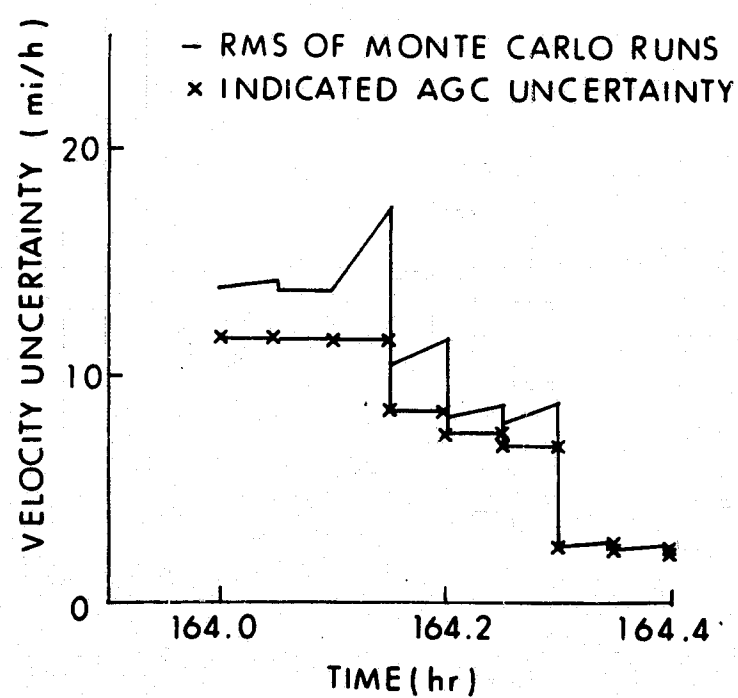


Figure 6.16 Velocity Uncertainty Using Diagonal Initial Covariance Matrix With Angular Momentum and Energy Adjoined

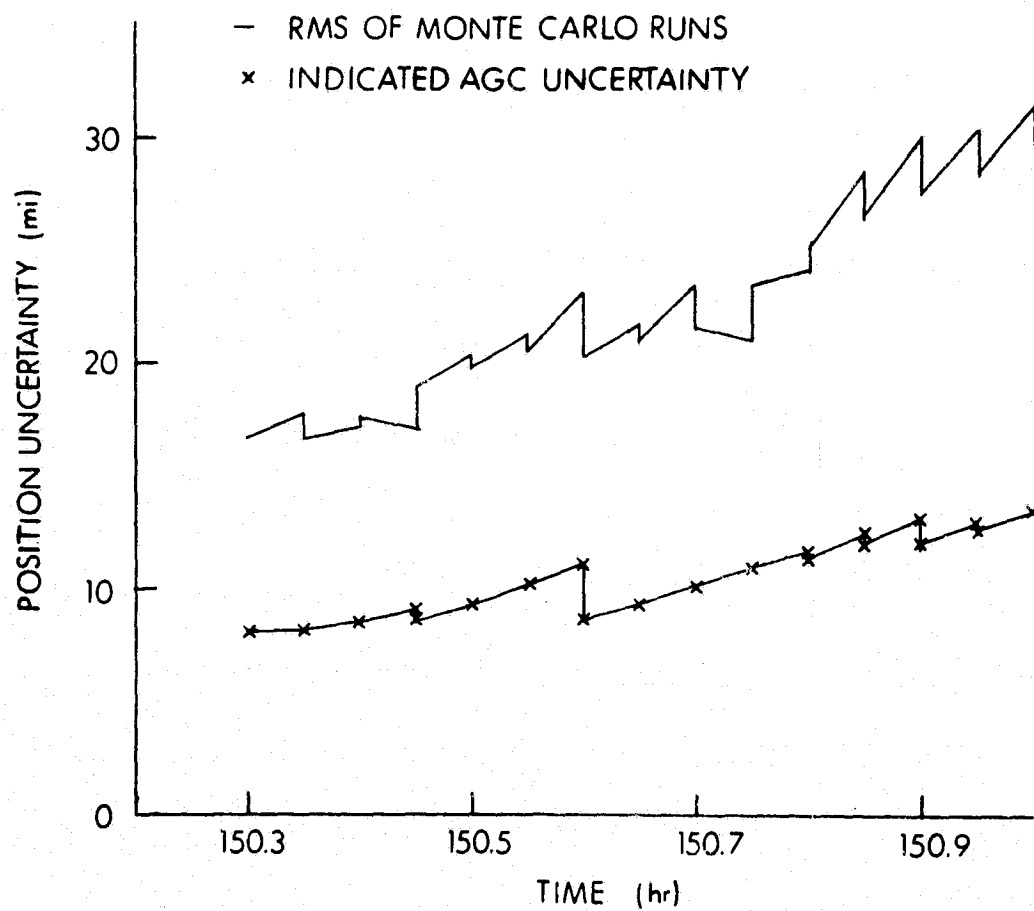


Figure 6.17 Position Uncertainty Using Diagonal Initial Covariance Matrix

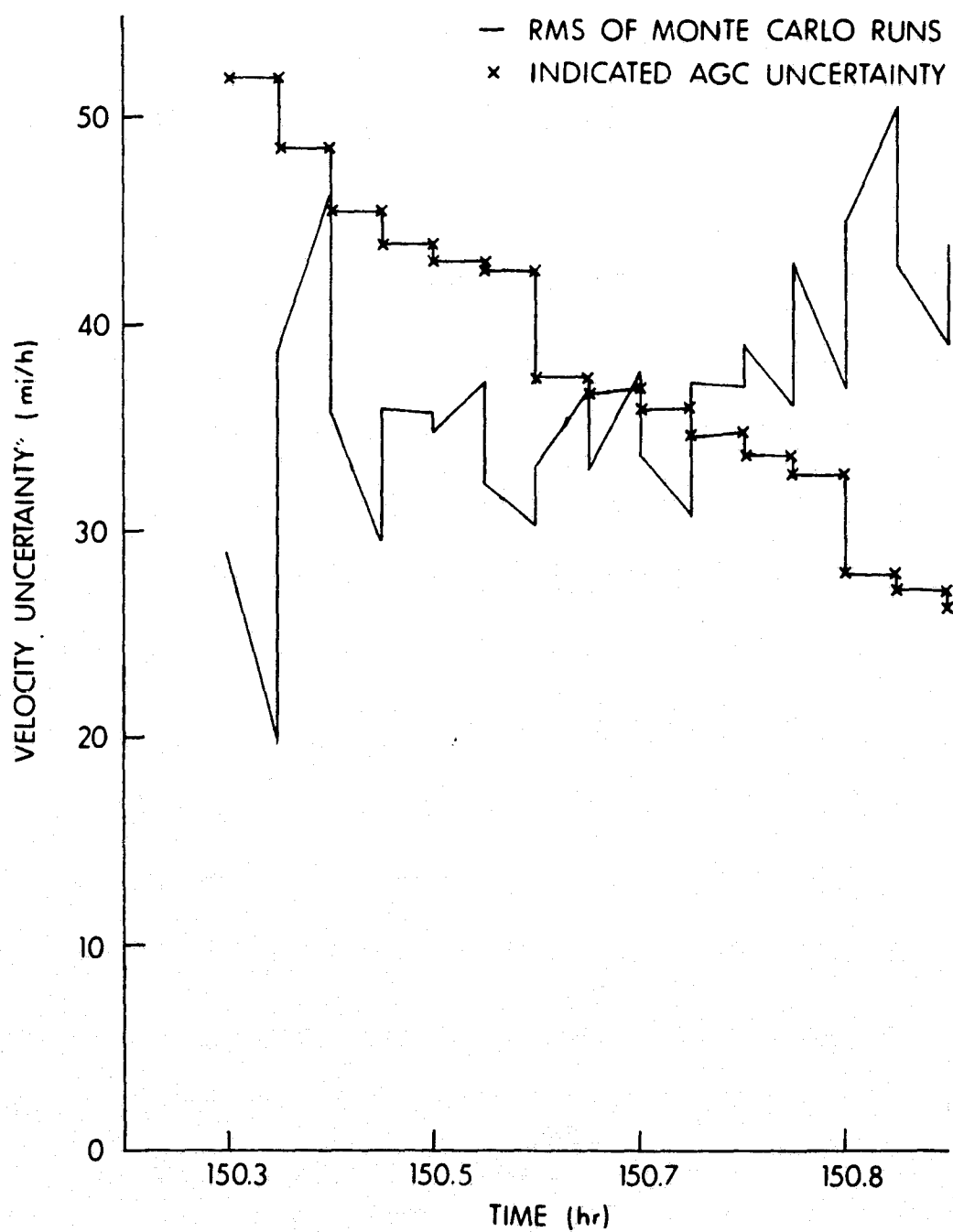


Figure 6.18 Velocity Uncertainty Using Diagonal Initial Covariance Matrix

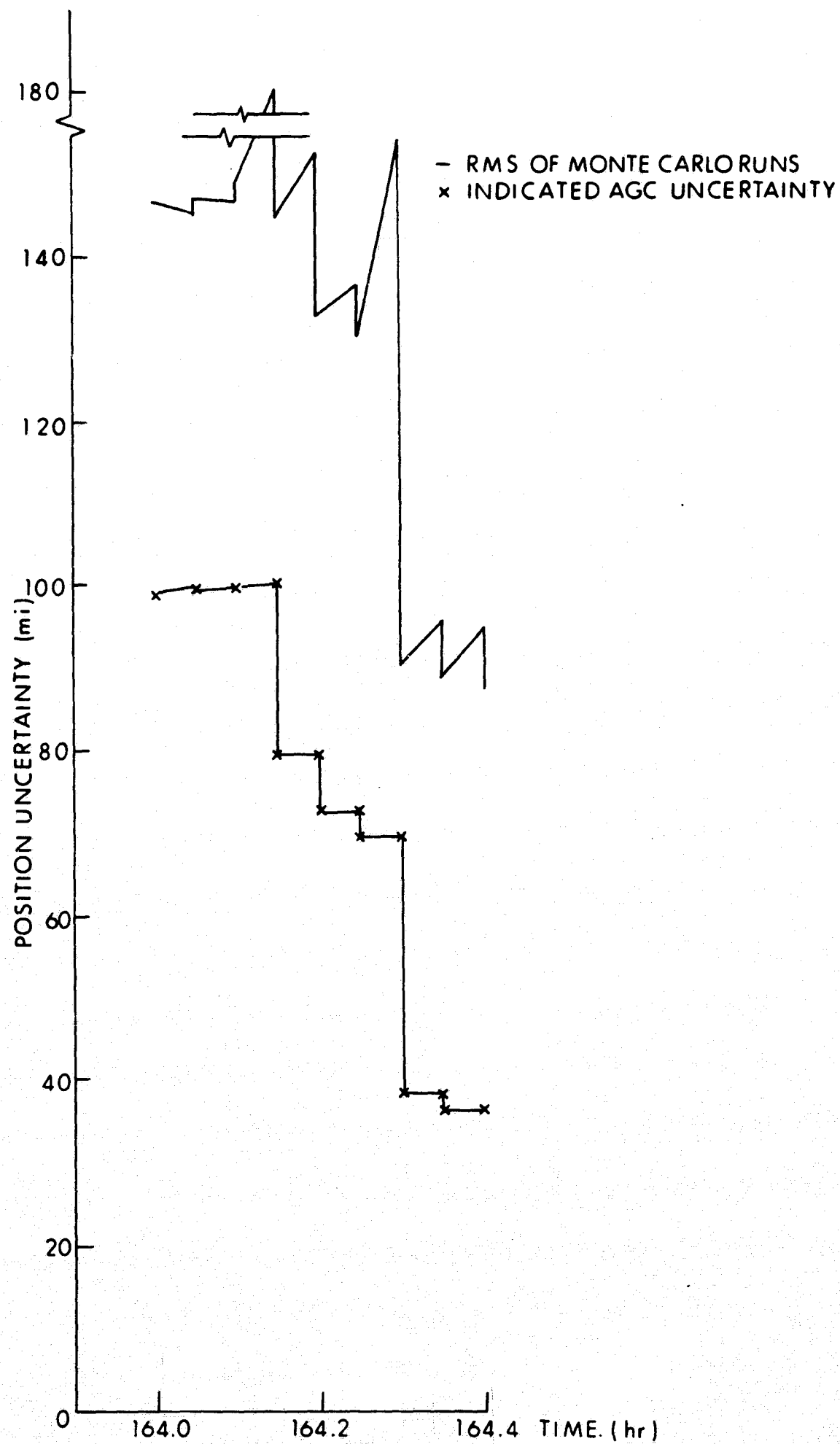


Figure 6.19 Position Uncertainty Using Diagonal Initial Covariance Matrix

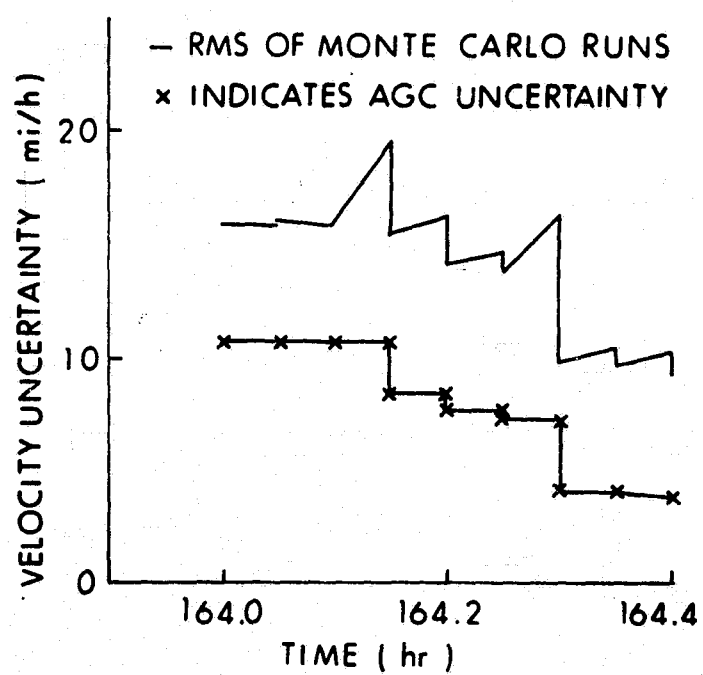


Figure 6.20 Velocity Uncertainty Using Diagonal Initial Covariance Matrix

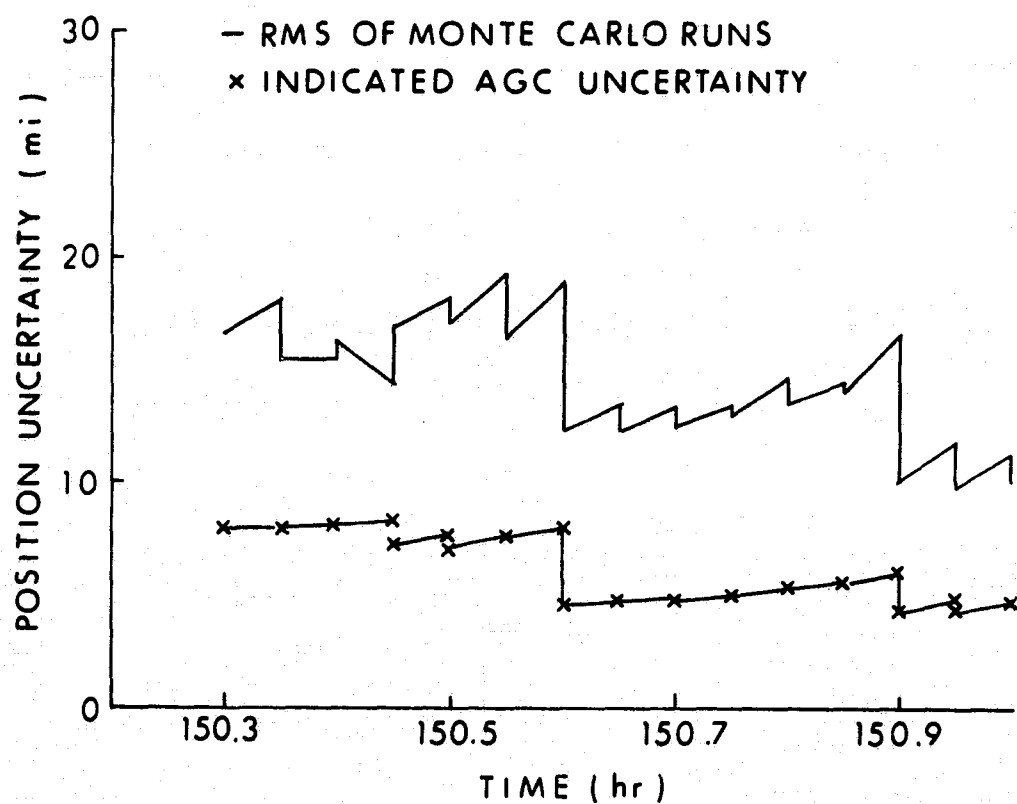


Figure 6.21 Position Uncertainty Using Diagonal Initial Covariance Matrix with Angular Momentum and Energy Adjoined

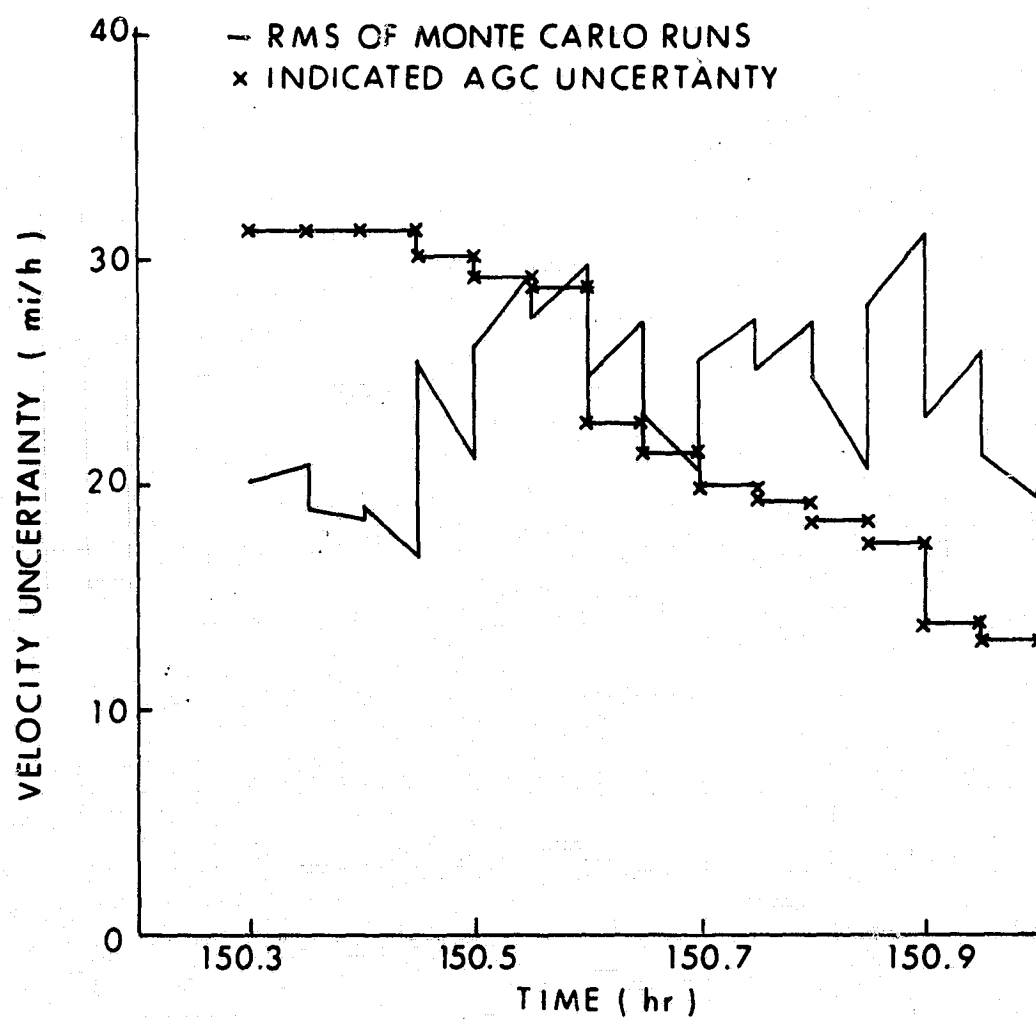


Figure 6.22 Velocity Uncertainty Using Diagonal Initial Covariance Matrix with Angular Momentum and Energy Adjoined

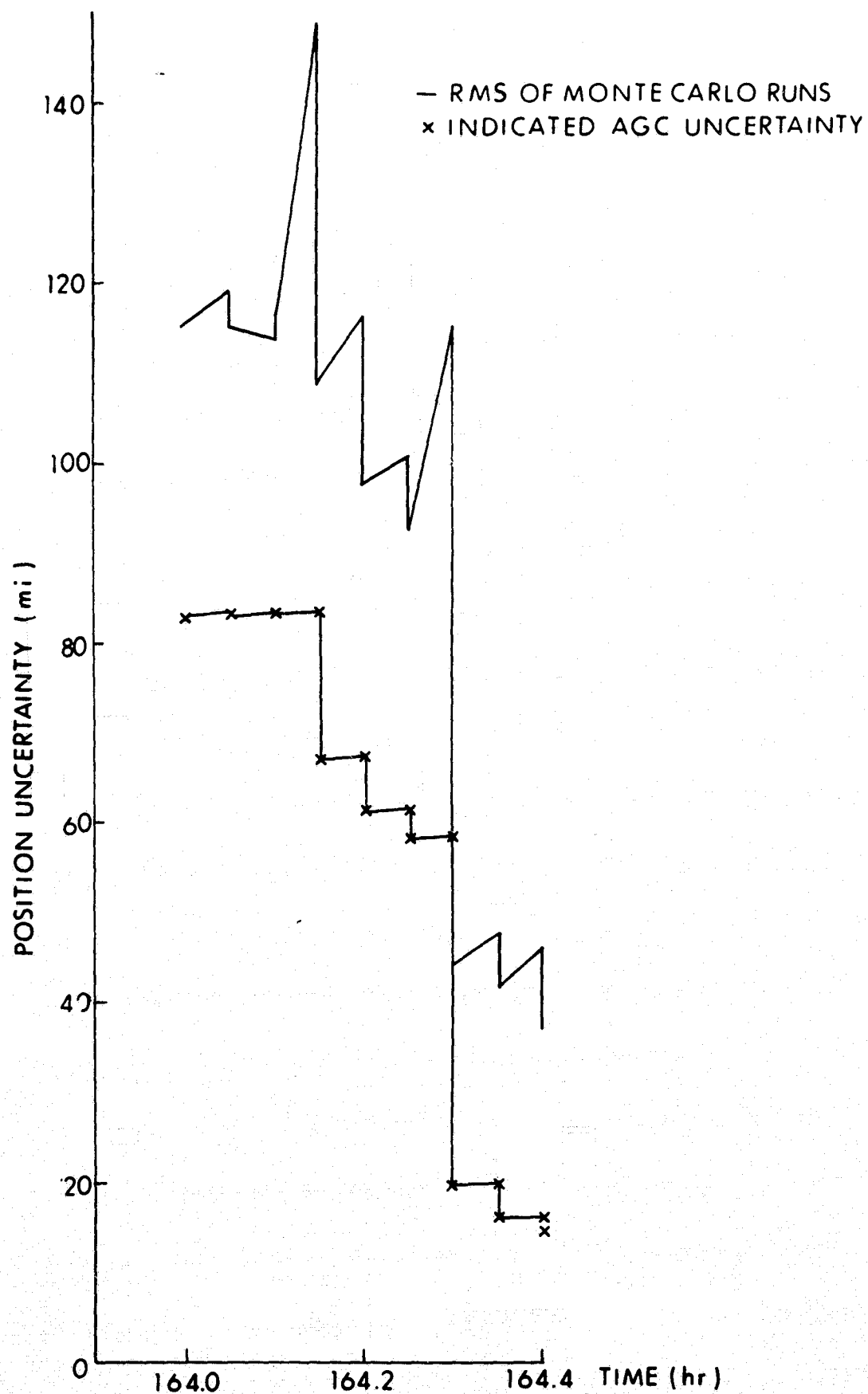


Figure 6.23 Position Uncertainty Using Diagonal Initial Covariance Matrix with Angular Momentum and Energy Adjoined

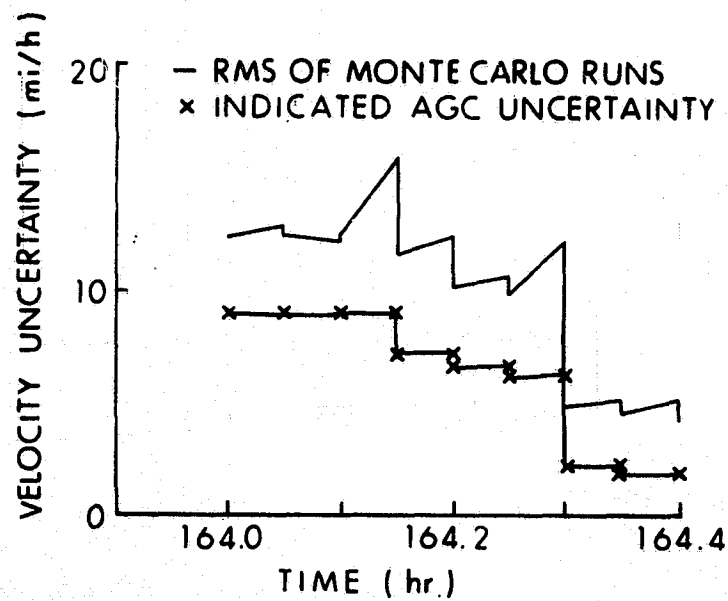


Figure 6.24 Velocity Uncertainty Using Diagonal Initial Covariance Matrix With Angular Momentum and Energy Adjoined

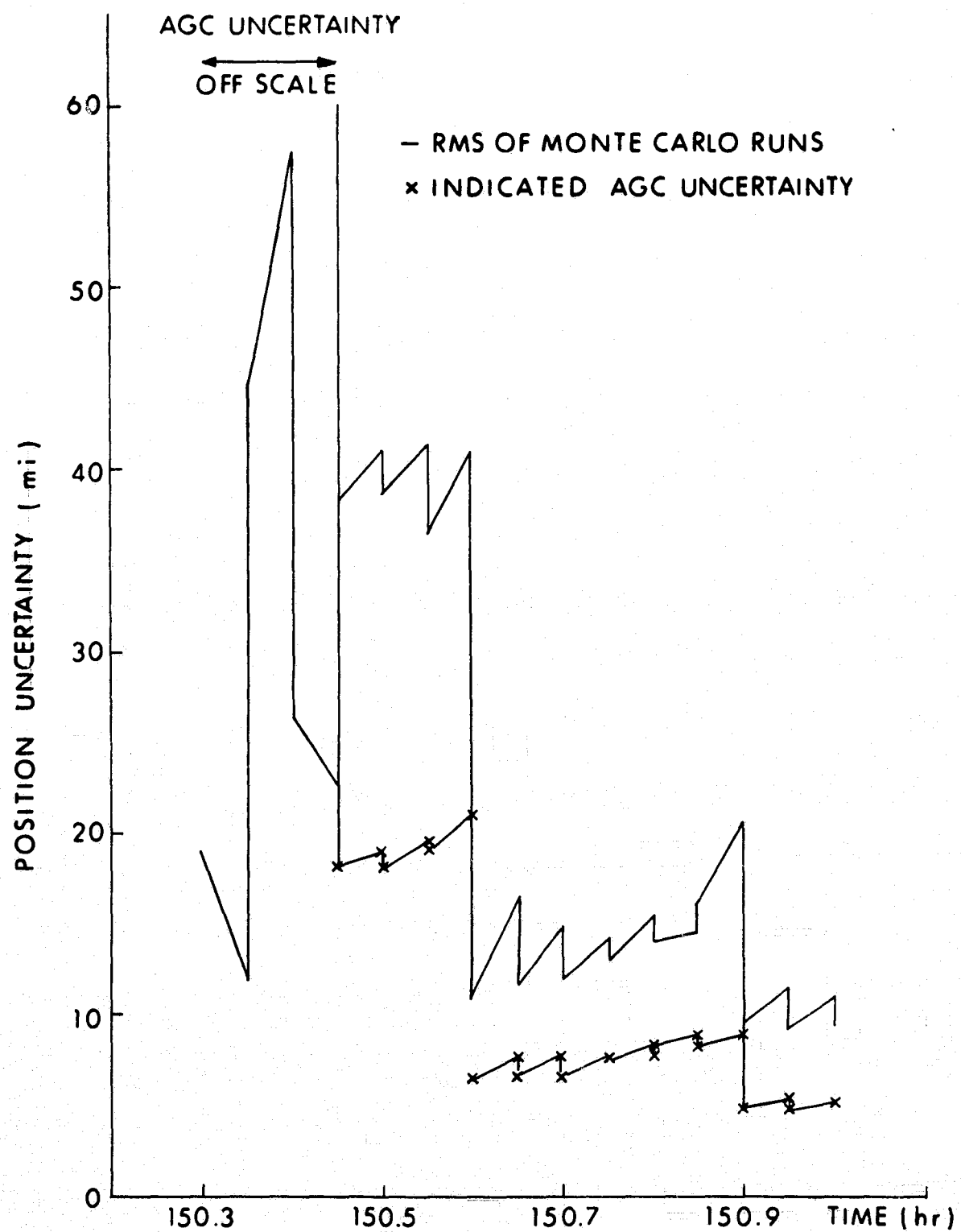


Figure 6.25 Position Uncertainty Using a Large Diagonal Initial Covariance Matrix with Angular Momentum and Energy Adjoined

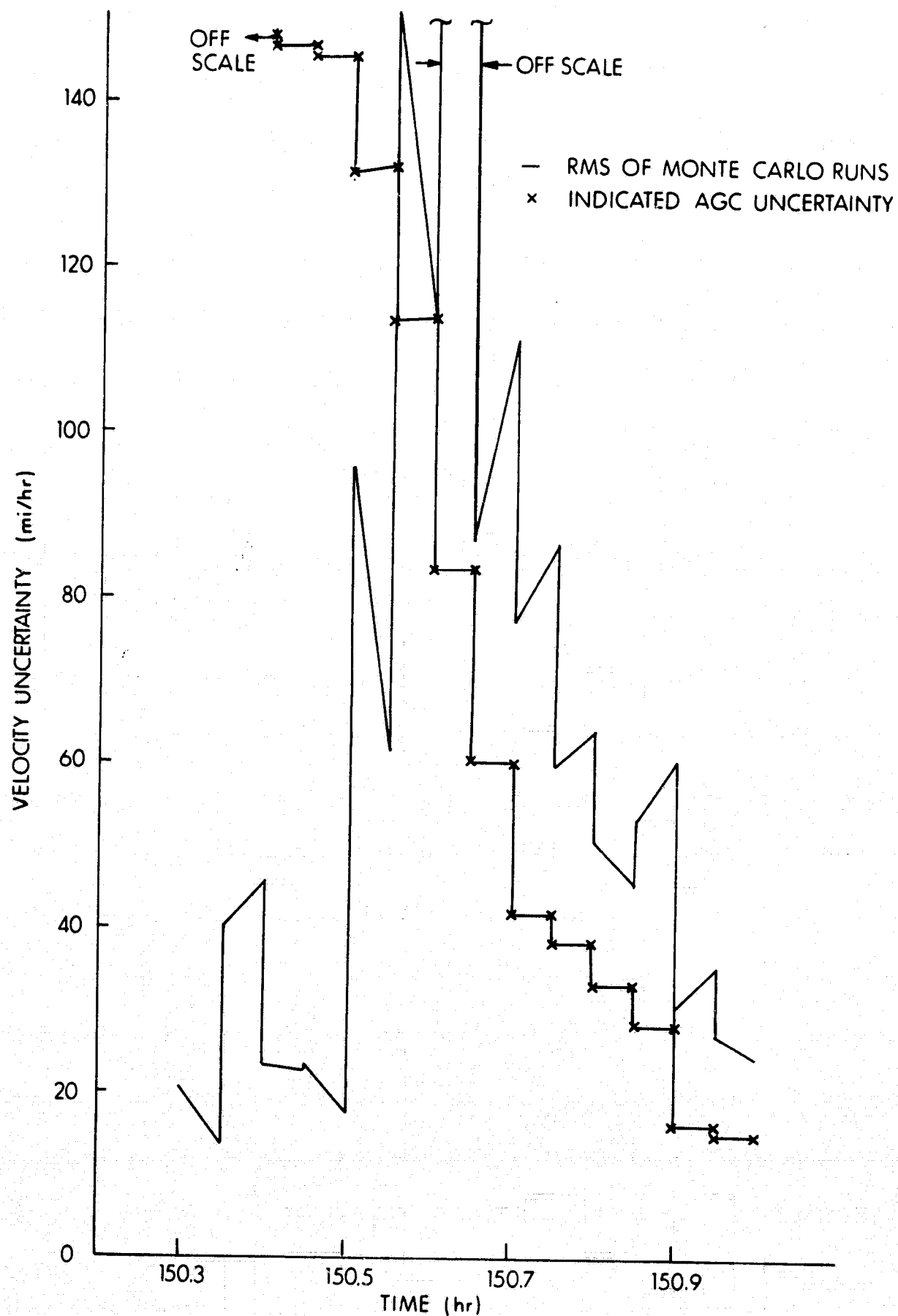


Figure 6.26 Velocity Uncertainty Using a Large Diagonal Initial Covariance Matrix with Angular Momentum and Energy Adjoined

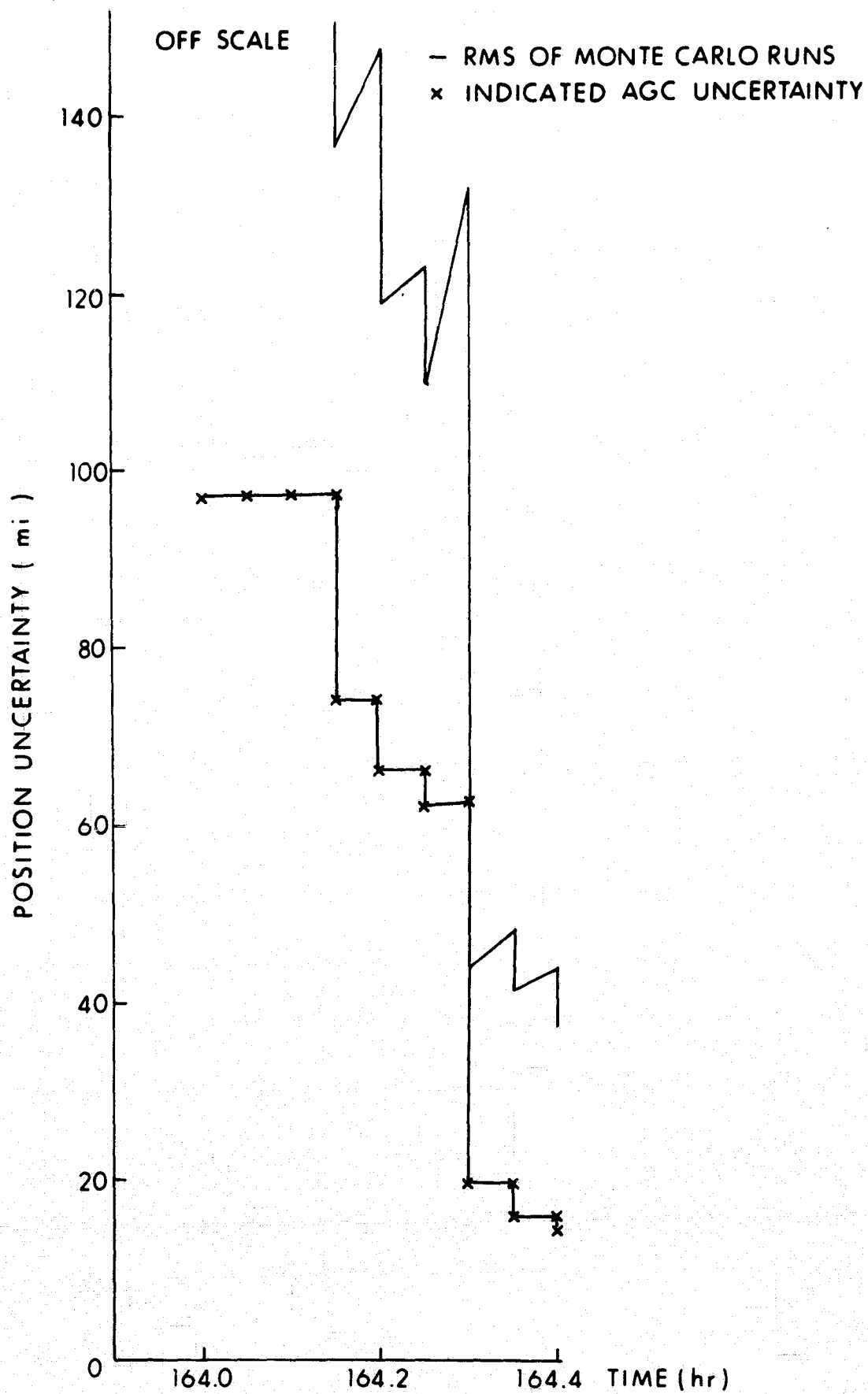


Figure 6.27 Position Uncertainty Using a Large Diagonal Initial Covariance Matrix with Angular Momentum and Energy Adjoined

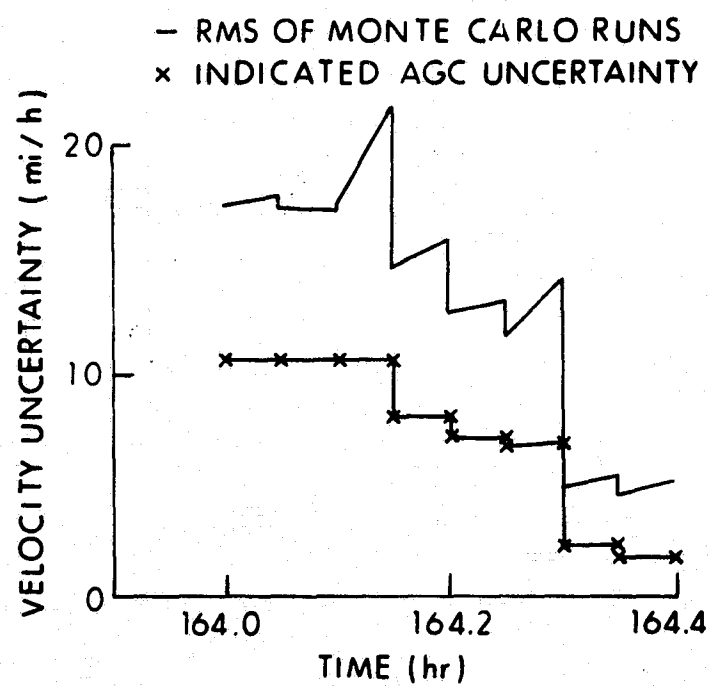


Figure 6.28 Velocity Uncertainty Using a Large Diagonal Initial Covariance Matrix with Angular Momentum and Energy Adjoined

CHAPTER 7

CONCLUSIONS

The results given in Chapter 6 show that the filtered estimate of the state converges to the true state significantly faster if the filtering process is begun with a properly correlated covariance matrix of the estimation errors than with a diagonal matrix. A method to generate a correlated matrix, if various quantities pertaining to the problem are known, is through the use of the information matrix and pseudo-measurements on these quantities. This method is applied to the midcourse phase of the Apollo 14 mission where the known quantities were assumed to be the energy and the magnitude of the angular momentum of the orbit at the time of filter initialization. The knowledge of either quantity when incorporated into the initial covariance matrix gave a substantially better estimate during the first and second batches of measurements after TEI than did the currently used diagonal matrix. The best results were obtained when pseudo-measurements on both quantities were used.

The degree of accuracy with which the quantities are known is reflected by the variance of the error in the pseudo-measurements. For this study, these variances were chosen somewhat arbitrarily and no attempt was made to determine the optimum values. Furthermore, only the above mentioned quantities were used. Others that might possibly be known are the eccentricity or the parameter of the orbit. Also, if available, range or range rate information from an external source could be utilized in a similar manner.

The additional information adjoined to the initial covariance matrix gave a substantial improvement in the estimate of the state during the first two batches of measurements. Since a midcourse velocity correction was scheduled to occur at the end of the second batch, the improved estimate would allow a more accurate correction to be performed. For the remainder of the flight little or no difference in the quality of the estimate was observed in comparison with the results obtained using a diagonal covariance matrix. The reason is that for tracking periods of long duration the initial covariance matrix is not critical since sufficient time is available to build up the proper correlation needed to make accurate updates. However, a common problem associated with long periods of navigation is that the filter gains become quite small and subsequent measurements tend to be ignored. If model errors exist, the filter becomes improperly correlated and incorrect updates are made causing filter divergence. One method of preventing this is to reinitialize the filtering process with a diagonal matrix that is larger in a positive definite sense than the old matrix. This increases the filter gains allowing current measurements to be processed with greater weighting and also eliminates the incorrect correlation resulting from errors in the system model. The problems associated with this is that the diagonal matrix, if large compared to the actual errors, causes a transient period to exist until the errors have been reduced to an acceptable level. Also, by reinitializing with a diagonal matrix, some useful information pertaining to the errors contained in the previous matrix is lost. From a filtering standpoint it is desirable to carry as much information as possible through the reinitialization phase so that the transient period is decreased. Even though the gains in the covariance matrix become wrong and

need to be reinitialized, certain parameters of the problem may be known quite accurately at the time of reinitialization from physical constraints on the system; insight into the problem, or actually calculated from the best estimate of the state. This information could be used, by the technique discussed, to develop a reasonably correlated matrix with which to restart the problem.

The problem area where this technique could be used with great advantage is for tracking periods of short duration where accurate estimates are required to perform a thrusting maneuver. For this problem the effect of starting with an incorrect covariance matrix is much more critical since sufficient time to build up proper correlation may not be available. Hence it is important to decrease the transient period associated with initializing with incorrect values. For unpowered flight, the quantities such as the energy and the angular momentum of the spacecraft's orbit are essentially constants of the motion, changing only by the presence of a disturbing acceleration. As functions of the position and velocity of the spacecraft, the geometry vector for pseudo-measurements on these quantities could be calculated from the current best estimate of the state vector at the time of reinitialization. Incorporating these pseudo-measurements into the reinitialized covariance matrix gives information pertaining to the correlation of the state vector errors that is not achieved using a diagonal matrix.

APPENDIX A

DETERMINING A SQUARE ROOT MATRIX

Numerical inaccuracies associated with a large number of calculations can cause the covariance matrix of estimation errors to become negative definite. To avoid this difficulty the navigation problem was reformulated using the error transition matrix which is, in a matrix sense, the square root of the covariance matrix. To start the filtering process the elements of the square root matrix, W , must first be determined. If the initial covariance matrix is diagonal, the calculation of W is a trivial matter. However, for a general covariance matrix this is not the case. The solution to the equation

$$E = W W^T \quad (A. 1)$$

is not unique and various methods to obtain these solutions are given in Reference 5. Since E is positive semi-definite, in theory, it can be expressed as

$$E = P D P^T \quad (A. 2)$$

where the columns of P are the orthonormal eigenvectors of E and D is a diagonal matrix whose diagonal elements are the eigenvalues of E . One form of W that satisfies equation A.1 is then

$$W = P D^{1/2} \quad (A. 3)$$

This method has the disadvantage that the diagonalization of E may be quite difficult for matrices of higher dimensions. One simple way of avoiding this difficulty that is easily extended to higher dimensions is to compute a triangular square root matrix. This method was used in this study to compute the initial error transition matrix from the various nondiagonal covariance matrices used in the initialization procedure. The algebraic equations from which to compute the elements of the 6×6 triangular square root matrix are derived below. Assuming W to be of the form

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & W_{16} \\ 0 & 0 & 0 & 0 & W_{25} & W_{26} \\ 0 & 0 & 0 & W_{34} & W_{35} & W_{36} \\ 0 & 0 & W_{43} & W_{44} & W_{45} & W_{46} \\ 0 & W_{52} & W_{53} & W_{54} & W_{55} & W_{56} \\ W_{61} & W_{62} & W_{63} & W_{64} & W_{65} & W_{66} \end{bmatrix} \quad (A. 4)$$

where the subscript implies the element in the i^{th} row and j^{th} column. After multiplication by W^T the element in the i^{th} row and the j^{th} column can be computed from the following relations.

$$\left. \begin{aligned} W_{16} &= \sqrt{E_{11}} \\ W_{i6} &= E_{1i} / W_{16} \quad i = 2, 3, \dots, 6 \end{aligned} \right\} \quad (A. 5)$$

$$\left. \begin{aligned} W_{25} &= \sqrt{E_{22} - W_{26}^2} \\ W_{i5} &= (E_{2i} - W_{26} W_{i6}) / W_{25} \quad i = 3, 4, \dots, 6 \end{aligned} \right\} \quad (A. 6)$$

$$W_{34} = \sqrt{E_{33} - W_{35}^2 - W_{36}^2} \quad (A. 7)$$

$$W_{i4} = (E_{3i} - W_{35} W_{i5} - W_{36} W_{i6}) / W_{34} \quad i = 4, 5, 6$$

$$W_{43} = \sqrt{E_{44} - W_{44}^2 - W_{45}^2 - W_{46}^2} \quad (A. 8)$$

$$W_{i3} = (E_{4i} - W_{44} W_{i4} - W_{45} W_{i5} - W_{46} W_{i6}) / W_{43} \quad i = 5, 6$$

$$W_{52} = \sqrt{E_{55} - W_{53}^2 - W_{54}^2 - W_{55}^2}$$

$$W_{62} = (E_{56} - W_{53} W_{63} - W_{54} W_{64} - W_{55} W_{65} - W_{56} W_{66}) / W_{52} \quad (A. 9)$$

$$W_{61} = \sqrt{E_{66} - W_{62}^2 - W_{63}^2 - W_{64}^2 - W_{65}^2 - W_{66}^2} \quad (A. 10)$$

The equations must be solved in the order given since the element W_{ij} depends on the element group $W_{i, j+1}$.

APPENDIX B

REDUCTION OF THE ERROR ELLIPSOID BY INCORPORATING A MEASUREMENT

Section 4-1 discussed an initialization technique to add a priori information into the initial covariance matrix. The effect of this is to reduce the volume in six dimensional state space which is a statistical model of the state error vector. Assuming the components of the error vector $\underline{e}$ to be a set of jointly normal random variables with mean zero, the frequency function becomes

$$p(\underline{e}) = \frac{1}{\sqrt{(2\pi)^6 / E}} e^{-\frac{1}{2} \underline{e}^T E^{-1} \underline{e}} \quad (\text{B. 1})$$

Letting $\underline{\epsilon}$ and $\underline{\delta}$ denote the position and velocity errors respectively, E is given by

$$\overline{\underline{e} \underline{e}^T} = \begin{pmatrix} \overline{\underline{\epsilon} \underline{\epsilon}^T} & \overline{\underline{\epsilon} \underline{\delta}^T} \\ \overline{\underline{\delta} \underline{\epsilon}^T} & \overline{\underline{\delta} \underline{\delta}^T} \end{pmatrix} \quad (\text{B. 2})$$

The surfaces of equiprobability are obtained by setting the exponent of equation B. 1 equal to a constant or

$$\underline{e}^T E^{-1} \underline{e} = k^2 \quad (\text{B. 3})$$

If the initial errors are assumed to be spherically distributed with variances σ_p^2 and σ_v^2 (subscripts p and v denote position and velocity), the initial information matrix becomes

$$E_o^{-1} = \begin{pmatrix} \frac{1}{\sigma_p^2} I & 0 \\ 0 & \frac{1}{\sigma_v^2} I \end{pmatrix} \quad (B. 4)$$

After substituting into equation B. 3 and multiplying out gives the scalar equation of an ellipsoid in six dimensions,

$$\begin{aligned} \left(\frac{\epsilon_1}{\sigma_p} \right)^2 + \left(\frac{\epsilon_2}{\sigma_p} \right)^2 + \left(\frac{\epsilon_3}{\sigma_p} \right)^2 + \left(\frac{\delta_1}{\sigma_v} \right)^2 + \left(\frac{\delta_2}{\sigma_v} \right)^2 \\ + \left(\frac{\delta_3}{\sigma_v} \right)^2 = k^2 \end{aligned} \quad (B. 5)$$

The lengths of the semi-axis are given by the standard deviations of the errors in that direction. After incorporating information from a measurement with a geometry vector $\underline{b}^T = (b_1 \ b_2 \ b_3 \ 000)$ and a measurement variance σ_m^2 , the updated information matrix becomes

$E^{-1} =$

$$\begin{pmatrix}
 \frac{1}{\sigma_p^2} + \frac{b_1^2}{\sigma_m^2} & \frac{b_1 b_2}{\sigma_m^2} & \frac{b_1 b_3}{\sigma_m^2} & 0 & 0 & 0 \\
 \frac{b_2 b_1}{\sigma_m^2} & \frac{1}{\sigma_p^2} + \frac{b_2^2}{\sigma_m^2} & \frac{b_2 b_3}{\sigma_m^2} & 0 & 0 & 0 \\
 \frac{b_3 b_1}{\sigma_m^2} & \frac{b_3 b_2}{\sigma_m^2} & \frac{1}{\sigma_p^2} + \frac{b_3^2}{\sigma_m^2} & 0 & 0 & 0 \\
 0 & 0 & 0 & \frac{1}{\sigma_v^2} & 0 & 0 \\
 0 & 0 & 0 & 0 & \frac{1}{\sigma_v^2} & 0 \\
 0 & 0 & 0 & 0 & 0 & \frac{1}{\sigma_v^2}
 \end{pmatrix}$$

(B. 6)

The surface of equiprobability after the measurement is given by the equation

$$\begin{aligned}
k^2 = & \frac{\epsilon_1^2}{\sigma_p^2} \left(1 + \frac{b_1^2 \sigma_p^2}{\sigma_m^2} \right) + \frac{\epsilon_2^2}{\sigma_p^2} \left(1 + \frac{b_2^2 \sigma_p^2}{\sigma_m^2} \right) \\
& + \frac{\epsilon_3^2}{\sigma_p^2} \left(1 + \frac{b_3^2 \sigma_p^2}{\sigma_m^2} \right) + \frac{2\epsilon_1\epsilon_2 b_1 b_2}{\sigma_m^2} + \frac{2\epsilon_1\epsilon_3 b_1 b_3}{\sigma_m^2} \\
& + \frac{2\epsilon_2\epsilon_3 b_2 b_3}{\sigma_m^2} + \left(\frac{\delta_1}{\sigma_v} \right)^2 + \left(\frac{\delta_2}{\sigma_v} \right)^2 + \left(\frac{\delta_3}{\sigma_v} \right)^2
\end{aligned}
\tag{B. 7}$$

This is seen to be the general equation of an ellipsoid where the position components are no longer along the principal axes. For the specific case where $b_2 = b_3 = 0$ which implies that the measurement pertains to the x component of the position only, equation B. 7 reduces to

$$k^2 = \frac{\epsilon_1^2}{\sigma_p^2 / \left(1 + \frac{b_1^2 \sigma_p^2}{\sigma_m^2} \right)} + \frac{\epsilon_2^2}{\sigma_p^2} + \frac{\epsilon_3^2}{\sigma_p^2} + \frac{\delta_1^2}{\sigma_v^2} + \frac{\delta_2^2}{\sigma_v^2} + \frac{\delta_3^2}{\sigma_v^2}
\tag{B. 8}$$

Comparing this with equation B. 5 shows that the semiaxes of the ellipsoid in the x direction has been reduced in length from

$$\sigma_p \text{ to } \sigma_p / \sqrt{1 + \frac{b_1^2 \sigma_p^2}{\sigma_m^2}} \text{ which reduces the volume of the}$$

error ellipsoid and consequently the uncertainty in the state. As the variance of the measurement error approaches zero the uncertainty in the x position also approaches zero.

APPENDIX C

LOCAL VERTICAL COORDINATE SYSTEM AND THE MATRIX OF TRANSFORMATION TO THE INERTIAL SYSTEM

The local vertical coordinate system referred to in Section 5.2 is defined using the inertial position and velocity vectors. If the x , y , and z axes are the inertial axes with corresponding unit vectors $\underline{i}_x$, $\underline{i}_y$, $\underline{i}_z$, and if the position and velocity vectors coordinatized in the inertial system are given by

$$\underline{r} = r_x \underline{i}_x + r_y \underline{i}_y + r_z \underline{i}_z \quad (C.1)$$

$$\underline{v} = v_x \underline{i}_x + v_y \underline{i}_y + v_z \underline{i}_z \quad (C.2)$$

A local vertical coordinate system with axes ξ , η , ζ is defined by

$$\underline{i}_\xi = \frac{\underline{r}}{r} = \underline{i}_r \quad (C.3)$$

$$\underline{i}_\zeta = \frac{\underline{r} \times \underline{v}}{r v} = \underline{i}_r \times \underline{i}_v \quad (C.4)$$

and

$$\underline{i}_\eta = \underline{i}_\zeta \times \underline{i}_\xi \quad (C.5)$$

In words, ξ is along the position vector $\underline{r}$, ζ is perpendicular to the plane containing $\underline{r}$ and $\underline{y}$, and η is defined to give a right-handed orthogonal system.

The matrix of transformation, A, that transforms the coordinates of a vector in the ξ, η, ζ system to the x, y, z system is given by the direction cosine matrix

$$A = \begin{bmatrix} \underline{i}_x \cdot \underline{i}_\xi & \underline{i}_x \cdot \underline{i}_\eta & \underline{i}_x \cdot \underline{i}_\zeta \\ \underline{i}_y \cdot \underline{i}_\xi & \underline{i}_y \cdot \underline{i}_\eta & \underline{i}_y \cdot \underline{i}_\zeta \\ \underline{i}_z \cdot \underline{i}_\xi & \underline{i}_z \cdot \underline{i}_\eta & \underline{i}_z \cdot \underline{i}_\zeta \end{bmatrix} \quad (C.6)$$

$$\text{Since } \underline{i}_\xi = \frac{r_x}{r} \underline{i}_x + \frac{r_y}{r} \underline{i}_y + \frac{r_z}{r} \underline{i}_z \quad (C.7)$$

$$\underline{i}_x \cdot \underline{i}_\xi = \frac{r_x}{r}$$

$$\underline{i}_y \cdot \underline{i}_\xi = \frac{r_y}{r} \quad (C.8)$$

$$\underline{i}_z \cdot \underline{i}_\xi = \frac{r_z}{r}$$

The first column of the matrix A is made up of the components of the unit vector $\underline{i}_\xi$. In a similar fashion the second and third columns can be shown to be $\underline{i}_\eta$ and $\underline{i}_\zeta$ respectively giving

$$A = (\underline{i}_\xi \ \underline{i}_\eta \ \underline{i}_\zeta) \quad (C.9)$$

as the matrix of transformation from the local vertical to the inertial system.

The correlation matrix of initial state errors supplied by NASA is given in the local vertical system defined at the time of trans-earth insertion using the unperturbed position and velocity vectors. Using the notation that the initial state deviation coordinatized in the local vertical is given by

$$\delta \underline{x}_{LV} = \begin{bmatrix} \delta \underline{r}_{LV} \\ \delta \underline{v}_{LV} \end{bmatrix} \quad (C.10)$$

the state deviation coordinatized in the inertial system is given by

$$\delta \underline{x}_I = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \delta \underline{x}_{LV} \quad (C.11)$$

where the zeros imply 3×3 zero matrices. By definition, the correlation matrix in the inertial system is

$$C_I = \overline{\delta \underline{x}_I \delta \underline{x}_I^T} \quad (C.12)$$

where the overbar indicates an ensemble average. Substituting equation C.11 into equation C.12 gives the transformation of the correlation matrix of initial errors from the local vertical system to the inertial system, or

$$C_I = T C_{LV} T^T \quad (C.13)$$

where T is defined to be the 6×6 matrix

$$T = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad (C.14)$$

APPENDIX D

DATA FILES OF THE SYSTEM AND SIMULATED ENVIRONMENT

The Monte Carlo technique was used in the digital simulation of the navigation problem whereby both the estimated trajectory and the "true" or actual trajectories were computed. In an actual mission the estimated trajectory will differ from the true trajectory due to limitations and inaccuracies in the mathematical model of the system. To simulate these effects, the parameter values used to propagate the state along the estimated trajectory differed from those to compute the simulated true trajectory. The values used are summarized in the following tables. Table D-1 gives the simplified data files corresponding to the results given in Figures 6.1 - 6.16. Table D - 2 corresponds to Figures 6.17-6.28.

Parameter	Value Used in the Filter Model	Value Used to Simulate Actual Environment
μ_E	$1.239364713 \times 10^{12} \text{ mi}^3/\text{hr}^2$	$1.239364713 \times 10^{12} \text{ mi}^3/\text{hr}^2$
μ_M	$1.524405706 \times 10^{12} \text{ mi}^3/\text{hr}^2$	$1.524405706 \times 10^{12} \text{ mi}^3/\text{hr}^2$
μ_S	$4.126480656 \times 10^{17} \text{ mi}^3/\text{hr}^2$	$4.126480656 \times 10^{17} \text{ mi}^3/\text{hr}^2$
RMS error in μ_E	0	0
RMS error in μ_M	0	0
Earth semi-major axis	3963.2086 mi	3963.2086 mi
Earth semi-minor axis	3949.9226 mi	3949.9226 mi
RMS error in earth axes	0	0
Moon radius	1081.5 mi	1081.5 mi
RMS error in moon radius	0	0
RMS earth horizon error	1.2 mi	1.2 mi
RMS moon horizon error	0.9 mi	0.9 mi
Sextant RMS error	$5.0 \times 10^{-5} \text{ rad}$	$5.0 \times 10^{-5} \text{ rad}$
Sextant bias	0	0

Table D-1: Simplified Data Files Corresponding to Results Given in Figures 6.1 - 6.16.

Parameter	Value Used in the Filter Model	Value Used to Simulate Actual Environment
μ_E	$1.239364713 \times 10^{12} \text{ mi}^3/\text{hr}^2$	$1.239364713 \times 10^{12} \text{ mi}^3 / \text{hr}^2$
μ_M	$1.524405706 \times 10^{12} \text{ mi}^3/\text{hr}^2$	$1.524405706 \times 10^{12} \text{ mi}^3 / \text{hr}^2$
μ_S	$4.126480656 \times 10^{17} \text{ mi}^3/\text{hr}^2$	$4.126480656 \times 10^{17} \text{ mi}^3 / \text{hr}^2$
RMS error in μ_E	0	$1.24 \times 10^7 \text{ mi}^3 / \text{hr}^2$
RMS error in μ_M	0	$1.52 \times 10^5 \text{ mi}^3 / \text{hr}^2$
Earth semi-major axis	3963.2086 mi	3963.2086 mi
Earth semi-minor axis	3949.9226 mi	3949.9226 mi
RMS error in earth axes	0.2 mi	0.2 mi
Moon radius	1081.5 mi	1081.5 mi
RMS error in moon radius	0.2 mi	0.5 mi
RMS earth horizon error	1.2 mi	6.2 mi
RMS moon horizon error	0.9 mi	1.9 mi
Sextant RMS error	$5.0 \times 10^{-5} \text{ rad}$	$7.4 \times 10^{-5} \text{ rad}$
Sextant bias	0	$4.16 \times 10^{-5} \text{ rad}$

Table D-2: Data Files Corresponding to Results Given in Figures 6.17 - 6.28.

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- 4.) "Guidance System Operations Plan to Manned CM Earth Orbital And Lunar Missions Using Program Colossus 2C (Comanche 67)" M. I. T. Instrumentation Laboratory, Cambridge, Massachusetts, July 1969.
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