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GREENBELT, MARYLAND

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ON THE TIDAL EFFECTS IN THE MOTION OF ARTIFICIAL SATELLITES

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ABSTRACT

The general perturbations in the elliptic and vectorial elements of a satellite as caused by the tidal deformations of the non-spherical Earth are developed into trigonometric series in the standard ecliptical arguments of Hill-Brown lunar theory and in the equatorial elements ω and Ω of the satellite.

The integration of the differential equations for variation of elements of the satellite in this theory is easy because all arguments are linear or nearly linear in time.

The trigonometrical expansion permits a judgment about the relative significance of the amplitudes and periods of different tidal "waves" over a long period of time.

Graphs are presented of the tidal perturbations in the elliptic elements of the BE-C satellite which illustrate long term periodic behavior. The tidal effects are clearly noticeable in the observations and their comparison with the theory permits improvement of the "global" Love numbers for the Earth.

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BASIC NOTATIONS

ℓ, ℓ', F, D and Γ - the arguments of the lunar theory,

G - the gravitational constant,

M - the mass of the Earth,

m' - the mass of the Moon,

m'' - the mass of the Sun,

$\bar{r}$ - the geocentric position vector of the satellite,

$r = |\bar{r}|$,

$\bar{r}^\circ$ - the unit vector in the direction of $\bar{r}$,

$\bar{r}'$ - the geocentric position vector of the Moon,

$r' = |\bar{r}'|$,

$\bar{r}'^\circ$ - the unit vector in the direction of $\bar{r}'$,

$\bar{r}''$ - the geocentric position vector of the Sun,

$r'' = |\bar{r}''|$,

$\bar{r}''^\circ$ - the unit vector in the direction of $\bar{r}''$,

a' - the mean distance of the Moon from the Earth. Its numerical value is defined in such a manner that the constant part in the expansion of the parallax a'/r' is equal to one.

a'' - the mean geocentric distance of the Sun,

S' - the angle between $\bar{r}$ and $\bar{r}'$,

S'' - the angle between $\bar{r}$ and $\bar{r}''$,

$\lambda, \mu, \nu, \lambda', \mu', \nu', \lambda'', \mu'' \nu''$ - the components of $\bar{r}^\circ$, $\bar{r}'^\circ$ and $\bar{r}''^\circ$ respectively in the equatorial system,

e - the eccentricity of the orbit of the satellite,

i - the orbital inclination of the satellite, In the beginning of the exposition these symbols designate the osculating elements. Toward the end they designate the mean elements.

Ω - the longitude of the ascending node of the satellite,

ω - the argument of perigee of the satellite,

π - the longitude of the perigee of the satellite, $\omega + \Omega$,

c = cos i/2,

s = sin i/2,

δM - the perturbations in the mean anomaly of the satellite caused by tides,

$\delta\Omega$ - the perturbations in Ω caused by tides,

$\delta\pi$ - the perturbations in π,

δi - the perturbations in δi caused by tides,

n - the mean motion of the satellite,

a - the semi-major axis of the satellite's orbit,

$\overline{R}$ - unit vector normal to the orbital plane of the satellite,

u_1, u_2, u_3 - components of $\overline{R}$ in the equatorial system,

$\delta u_1, \delta u_2, \delta u_3$ - perturbations in u_1, u_2, u_3 respectively caused by tides,

ϵ - the obliquity of the ecliptic,

ε - the eccentricity of the Earth's meridian,

$k_2, k_3, \cdots$ - Love numbers,

R - the equatorial radius of the Earth,

$\alpha = R/a$

$\alpha' = R/a'$

INTRODUCTION

The precision of observations of artificial satellites has now reached the stage where consideration must be given to the effects in their motion as caused by the tidal deformations of the Earth. The comparison of tidal theory with observations will permit us to determine the "global" Love numbers for the Earth. They appear as coefficients in the expansion of the tidal potential and describe the average elastic properties of the Earth as "seen" from outer space. In connection with this topic the works by (Kozai, 1965), (Fisher and Felsentreger, 1966) and (Newton, 1967) must be mentioned. The tidal effects in the motion of satellites are very small, a few seconds of arc, and their computation, as well as the determination of Love numbers, requires a great care in the expansion of the tidal disturbing potential and in the numerical computations.

We discard the usual trigonometric expansion of the tidal potential with the angular equatorial elements of the Moon as arguments. This type of expansion is very simple in form, but the analytical integration of the resulting differential equations is difficult because the mean equatorial node of the lunar orbit does not change linearly with time, but oscillates between two fixed limits.

We suggest instead the expansion of perturbations into trigonometric series with the <u>ecliptical</u> arguments ℓ, ℓ', F, D and Γ of the lunar theory and with the <u>equatorial</u> elements ω and Ω of the satellite.

The arguments ℓ, ℓ', F, D and Γ are very nearly linear with respect to time and, as a consequence, the integration becomes much simpler and will be valid for a longer interval of time.

We expand the tidal potential into a sum of products of two harmonic functions, the first depending on the position of the satellite and the second on the position of the Moon (Sun).

In order to facilitate the programming we represent the harmonic functions in Maxwellian form as polynomials in equatorial components of unit vectors.

We obtain the trigonometric expansions of the functions λ', μ', ν' and $(a'/r')^3$ which appear in the tidal potential by making use of the standard developments of the lunar parallax a'/r', latitude $\beta_{☾}$ and $\delta\lambda_{☾}$, related to the lunar longitude $\lambda_{☾}$ by

$$\lambda_{☾} = \ell' + D + \Gamma + \delta\lambda_{☾}$$

into Fourier series (Brown, 1919), (Eckert, 1966), (Eckert et al., 1969), (Siry, 1970), (Amer. Eph., 1972). The accuracy of these series presently is sufficient to treat the tidal effects even for a relatively long period of time.

The equatorial components of the lunar unit vector are

$$\lambda' = \cos\lambda_{☾} \cos\beta_{☾}$$

$$\mu' = \sin\lambda_{☾} \cos\beta_{☾} \cos\epsilon - \sin\beta_{☾} \sin\epsilon$$

$$\nu' = \sin\beta_{☾} \cos\epsilon + \sin\lambda_{☾} \cos\beta_{☾} \sin\epsilon$$

where ϵ is the obliquity of the ecliptic. Expressions for $\sin\lambda_{☾}$, $\cos\lambda_{☾}$, $\sin\beta_{☾}$ and $\cos\beta_{☾}$ are obtained from

$$\sin\delta\lambda_{☾} = \delta\lambda_{☾} - \frac{1}{6}(\delta\lambda_{☾})^3 + \cdots$$

$$\cos\delta\lambda_{☾} = 1 - \frac{1}{2}(\delta\lambda_{☾})^2 + \frac{1}{24}(\delta\lambda_{☾})^4 - \cdots$$

$$\cos \lambda_{☾} = \cos(\ell' + D + \Gamma) \cos \delta\lambda_{☾} - \sin(\ell' + D + \Gamma) \sin \delta\lambda_{☾}$$

$$\sin \lambda_{☾} = \sin(\ell' + D + \Gamma) \cos \delta\lambda_{☾} + \cos(\ell' + D + \Gamma) \sin \delta\lambda_{☾}$$

$$\sin \beta_{☾} = \beta_{☾} - \frac{1}{6}\beta_{☾}^3 + \cdots$$

$$\cos \beta_{☾} = 1 - \frac{1}{2}\beta_{☾}^2 + \cdots .$$

It is sufficient to retain in $\delta\lambda_{☾}$, $\beta_{☾}$ and a'/r' only the terms with amplitudes larger than 0.5×10^{-5} so that the resulting accuracy of the expansion of the harmonic functions depending on the Moon's position will be 1×10^{-4}. The accuracy of the expansion of the tidal potential is then 6×10^{-12} and the formal accuracy of perturbations is about 6×10^{-9} or 0".001.

The expansion of the tidal potential is obtained on the machine. PL/I was used by one of us (R. E.) to program the operations with Fourier series.

The Fourier series depending on the moon's position have purely numerical coefficients and the decision concerning the relative importance of terms is handled by programming logic. All terms with amplitudes too small to be observed are rejected automatically. The machine calculations, in addition to standard checks for numerical accuracy, are subject to other chocks on procedure and accuracy by the formal properties of the expressions.

Only the long period tidal effects can be easily observed and, consequently, we suppress all the terms with periods nearly equal to or less than the period of revolution of the satellite by averaging the force function over the satellite's orbit. The coefficients of the averaged expansion are simple rational functions in e, $\sqrt{1 - e^2}$, cos i/2, and sin i/2.

To analytically continue the tidal potential into extraterrestrial space we need a suitable approximation to the Earth's surface.

Rather then assumming sphericity, as is usually done, we select an ellipsoid of rotation as this approximation. It is of interest to note that oblateness will produce near resonance effects in the neighborhood of some critical angles, like 46°.4, 58°.1, 63°.4, 69°.0 and 73°.1.

In observations these resonances cannot be separated from the similar resonances caused by the direct lunar attraction. In addition, they are hardly noticeable when the eccentricity is small. It is also of some interest to note that the oblateness causes a term of the form

$$\frac{GM}{r} \varepsilon^2 T,$$

to appear in the extraterrestrial tidal potential, where ε designates the eccentricity of the meridian and T is a harmonic function of the Moon's position.

The term is very small, but it can gain importance as the requirements put on the accuracy of the determination of GM gradually increase.

The aggregate of terms associated with k_2 and not containing $\pi = \omega + \Omega$ in the arguments constitutes the largest and the most important portion of the exterior tidal potential. It defines the so-called "main problem" in the theory of tidal effects. Obviously the main problem in the present exposition is not associated with the spherical approximation to the Earth surface, as the oblateness enters into the coefficients of the expansion. The semi-major axis and the eccentricity remain invariant under the influence of the tidal forces defined by the main problem (Kozai, 1965).

The lag of tides in time can be determined from the shift in phase of the equatorial node Ω' of the lunar orbit (Kozai, 1965).

However, in the process of expansion into Fourier series this shift can be associated with the element Ω, because Ω' always appears in combination $\Omega - \Omega'$. A similar shift exists in the arguments ℓ, F and D.

THE EXTERIOR TIDAL POTENTIAL

The expansion of the lunar tidal force function on the surface of the Earth is:

$$V' = Gm' k_2 \frac{r^2}{r'^3} \left(\frac{3}{2}\cos^2 S' - \frac{1}{2}\right) + Gm' k_3 \frac{r^3}{r'^4} \left(\frac{5}{2}\cos^3 S' - \frac{3}{2}\cos S'\right) + \cdots \tag{1}$$

Similarly for the Sun we have:

$$V'' = Gm'' k_2 \frac{r^2}{r''^3} \left(\frac{3}{2}\cos^2 S'' - \frac{1}{2}\right) + Gm'' k_3 \frac{r^3}{r''^4} \left(\frac{5}{2}\cos^3 S'' - \frac{3}{2}\cos S''\right) + \cdots \tag{1'}$$

Making use of the expansion of the disturbing function for the direct perturbations in the motion of an artificial satellite into a series of products of spherical harmonics (Musen, Bailie and Upton, 1961) we can re-write (1) in the form:

$$V' = \frac{Gm'}{a'^3} k_2 \{a_{20} C'_{20} + (a_{21} C'_{21} + b_{21} S'_{21}) + (a_{22} C'_{22} + b_{22} S'_{22})\}$$

$$+ \frac{Gm'}{a'^4} k_3 \{(a_{31} C'_{31} + b_{31} S'_{31}) + b_{32} S'_{32} \qquad (2)$$

$$+ (a_{33} C'_{33} + b_{33} S'_{33}) + (a_{34} C'_{34} + b_{34} S'_{34})\} + \cdots\cdots,$$

where we let

$$a_{20} = +\frac{1}{4} r^2 (1 - 3\nu^2)$$

$$a_{21} = +\frac{3}{4} r^2 (\lambda^2 - \mu^2), \qquad b_{21} = \frac{3}{2} r^2 \lambda \mu,$$

$$a_{22} = +3 r^2 \mu \nu, \qquad b_{22} = +3 r^2 \lambda \nu,$$

$$a_{31} = +\frac{3}{8} r^3 \lambda (1 - 5\nu^2), \qquad b_{31} = +\frac{3}{8} r^3 \mu (1 - 5\nu^2),$$

$$b_{32} = +\frac{1}{4} r^3 \nu (3 - 5\nu^2),$$

$$a_{33} = +\frac{5}{8} r^3 \lambda (\lambda^2 - 3\mu^2), \qquad b_{33} = +\frac{5}{8} r^3 \mu (3\lambda^2 - \mu^2),$$

$$a_{34} = +\frac{15}{2} r^3 \lambda \mu \nu, \qquad b_{34} = +\frac{15}{4} r^3 \nu (\lambda^2 - \mu^2),$$

and

$$C'_{20} = \left(\frac{a'}{r'}\right)^3 (1 - 3\nu'^2)$$

$$C'_{21} = \left(\frac{a'}{r'}\right)^3 (\lambda'^2 - \mu'^2), \qquad S'_{21} = 2\left(\frac{a'}{r'}\right)^3 \lambda' \mu',$$

$$C'_{22} = \left(\frac{a'}{r'}\right)^3 \mu' \nu', \qquad S'_{22} = \left(\frac{a'}{r'}\right)^3 \lambda' \nu',$$

$$C'_{31} = \left(\frac{a'}{r'}\right)^4 \lambda' (1 - 5\nu'^2), \qquad S'_{31} = \left(\frac{a'}{r'}\right)^4 \mu' (1 - 5\nu'^2),$$

$$S'_{32} = \left(\frac{a'}{r'}\right)^4 \nu' (3 - 5\nu'^2),$$

$$C'_{33} = \left(\frac{a'}{r'}\right)^4 \lambda' (\lambda'^2 - 3\mu'^2), \qquad S'_{33} = \left(\frac{a'}{r'}\right)^4 \mu' (3\lambda'^2 - \mu'^2),$$

$$C'_{34} = 2\left(\frac{a'}{r'}\right)^4 \lambda' \mu' \nu', \qquad S'_{34} = \left(\frac{a'}{r'}\right)^4 \nu' (\lambda'^2 - \mu'^2)$$

The extensions A_{ij}, B_{ij} of the harmonic functions a_{ij}, b_{ij} into extraterrestrial space are respectively

$$A_{20} = +\frac{1}{6}\varepsilon^2 \frac{R^3}{r} + \left(1 - \frac{55}{42}\varepsilon^2\right) \frac{R^5}{a^3} A_{200} + \varepsilon^2 \frac{R^7}{a^5} A_{201},$$

$$A_{200} = +\frac{1}{4}\left(\frac{a}{r}\right)^3 (1 - 3\nu^2),$$

$$A_{201} = +\frac{3}{56}\left(\frac{a}{r}\right)^5 (3 - 30\nu^2 + 35\nu^4),$$

$$A_{21} = \left(1 - \frac{5}{14}\varepsilon^2\right) \frac{R^5}{a^3} A_{210} + \varepsilon^2 \frac{R^7}{a^5} A_{211}$$

$$A_{210} = +\frac{3}{4}\left(\frac{a}{r}\right)^3 (\lambda^2 - \mu^2)$$

$$A_{211} = + \frac{15}{56} \left(\frac{a}{r}\right)^5 (\lambda^2 - \mu^2)(1 - 7\nu^2),$$

$$B_{21} = \left(1 - \frac{5}{14}\varepsilon^2\right) \frac{R^5}{a^3} B_{210} + \varepsilon^2 \frac{R^7}{a^5} B_{211},$$

$$B_{210} = + \frac{3}{2} \left(\frac{a}{r}\right)^3 \lambda\mu,$$

$$B_{211} = + \frac{15}{28} \left(\frac{a}{r}\right)^5 \lambda\mu(1 - 7\nu^2),$$

$$A_{22} = + \left(1 - \frac{15}{14}\varepsilon^2\right) \frac{R^5}{a^3} A_{220} + \varepsilon^2 \frac{R^7}{a^5} A_{221}$$

$$A_{220} = + 3 \left(\frac{a}{r}\right)^3 \mu\nu$$

$$A_{221} = + \frac{45}{14} \left(\frac{a}{r}\right)^3 \mu\nu \left(1 - \frac{7}{3}\nu^2\right)$$

$$B_{22} = + \left(1 - \frac{15}{14}\varepsilon^2\right) \frac{R^5}{a^3} B_{220} + \varepsilon^2 \frac{R^7}{a^5} B_{221},$$

$$B_{220} = + 3 \left(\frac{a}{r}\right)^3 \lambda\nu,$$

$$B_{221} = + \frac{45}{14} \left(\frac{a}{r}\right)^5 \lambda\nu \left(1 - \frac{7}{3}\nu^2\right),$$

$$A_{31} = + \frac{R^7}{a^4} A_{310},$$

$$A_{310} = + \frac{3}{8} \left(\frac{a}{r}\right)^4 \lambda(1 - 5\nu^2),$$

$$B_{31} = +\frac{R^7}{a^4} B_{310},$$

$$B_{310} = +\frac{3}{8} \left(\frac{a}{r}\right)^4 \mu (1 - 5\nu^2),$$

$$A_{32} = 0,$$

$$B_{32} = +\frac{R^7}{a^4} B_{320},$$

$$B_{320} = +\frac{1}{4} \left(\frac{a}{r}\right)^4 \nu (3 - 5\nu^2),$$

$$A_{33} = +\frac{R^7}{a^4} A_{330},$$

$$A_{330} = +\frac{5}{8} \left(\frac{a}{r}\right)^4 \lambda (\lambda^2 - 3\mu^2),$$

$$B_{33} = +\frac{R^7}{a^4} B_{331},$$

$$B_{331} = +\frac{5}{8} \left(\frac{a}{r}\right)^4 \mu (3\lambda^2 - \mu^2),$$

$$A_{34} = +\frac{R^7}{a^4} A_{340},$$

$$A_{340} = +\frac{15}{2} \left(\frac{a}{r}\right)^4 \lambda\mu\nu,$$

$$B_{34} = +\frac{R^7}{a^4} B_{340},$$

$$B_{340} = +\frac{15}{4} \nu (\lambda^2 - \mu^2).$$

The correctness of these extensions can be checked directly.

Replacing a_{ij}, b_{ij} in (2) by the corresponding A_{ij}, B_{ij} and taking

$$n^2 a^3 = GM$$

into consideration we obtain the following decomposition of the tidal potential acting on the satellite:

$$V' = \frac{m'}{M} \varepsilon^2 \alpha'^3 k_2 W'_{00} \frac{GM}{r}$$

$$+ n^2 a^2 \frac{m'}{M} \alpha^2 \alpha'^3 k_2 W'_{20}$$

$$+ n^2 a^2 \frac{m'}{M} \alpha^4 \alpha'^3 \varepsilon^2 k_2 W'_{21}$$

$$+ n^2 a^2 \frac{m'}{M} \alpha^3 \alpha'^4 k_3 W'_{30},$$

where

$$W'_{00} = + \frac{1}{6} C'_{20}$$

$$W'_{20} = \left(1 - \frac{55}{42} \varepsilon^2\right) A_{200} C'_{20}$$

$$+ \left(1 - \frac{5}{14} \varepsilon^2\right) (A_{210} C'_{21} + B_{210} S'_{21})$$

$$+ \left(1 - \frac{15}{14} \varepsilon^2\right) (A_{220} C'_{22} + B_{220} S'_{22}),$$

$$W'_{21} = A_{201} C'_{20} + (A_{211} C'_{21} + B_{211} S'_{21})$$

$$+ (A_{221} C'_{22} + B_{221} S'_{22}),$$

$$W'_{30} = (A_{310} C'_{31} + B_{310} S'_{31}) + B_{320} S'_{32}$$

$$+ (A_{330} C'_{33} + B_{330} S'_{33}) + (A_{340} C'_{34} + B_{340} S'_{34}).$$

It is of interest to note the presence of the factor GM/r in the first term. The expansion of the factor associated with GM/r consists of a constant term and a set of long period terms depending upon the arguments ℓ, ℓ', F, D and Γ of the lunar theory. Consequently, the contribution of the oblateness to the tidal potential is a term which may influence the accurate determination of the basic quantity GM.

The main part of V associated with the "main problem" of the determination of tidal effects is, of course, W'_{20}. Because of the additional parallactic factor in W'_{30} we have used only the spherical approximation to the Earth's surface to obtain the extension of terms depending upon k_3 into extra-terrestrial space.

We have:

$$[W'_{20}] = \left[+ \frac{1}{4} \left(1 - \frac{55}{42} \varepsilon^2\right) (1 - 6 s^2 c^2) C'_{200} \right.$$

$$+ \frac{3}{2} \left(1 - \frac{5}{14} \varepsilon^2\right) s^2 c^2 C'_{210}$$

$$\left. + 3 \left(1 - \frac{15}{14} \varepsilon^2\right) sc (c^2 - s^2) C'_{220} \right] (1 - e^2)^{-3/2},$$

where

$$C'_{200} = C'_{20},$$

$$C'_{210} = C'_{21} \cos 2\Omega + S'_{21} \sin 2\Omega,$$

$$C'_{220} = C'_{22} \cos \Omega - S'_{22} \sin \Omega,$$

are the trigonometrical series in ℓ, ℓ', F, D, Γ and Ω,

$$s = \sin\frac{i}{2}, \quad c = \cos\frac{i}{2},$$

and the brackets designate the averaged values. Similarly,

$$[W'_{21}] = \left[+ \frac{3}{56}\left(1 + \frac{3}{2}e^2\right)(3 - 60 s^2 c^2 + 280 s^4 c^4) C'_{2010} \right.$$

$$+ \frac{9}{224} e^2 (60 s^2 c^2 - 280 s^4 c^4) C'_{2011}$$

$$+ \frac{45}{224} e^2 c^4 (1 - 14 s^2 + 28 s^4) C'_{2111}$$

$$+ \frac{45}{224} e^2 s^4 (15 - 42 s^2 + 28 s^4) C'_{2112}$$

$$+ \frac{15}{56}\left(1 + \frac{3}{2}e^2\right) s^2 c^2 (9 - 42 s^2 + 42 s^4) C'_{2113}$$

$$+ \frac{135}{56} e^2 s c^3 \left(1 - 7 s^2 + \frac{28}{3} s^4\right) C'_{2211}$$

$$+ \frac{45}{56} e^2 s^3 c (10 - 35 s^2 + 28 s^4) C'_{2212}$$

$$\left. + \frac{45}{14}\left(1 + \frac{3}{2}e^2\right) s c (1 - 9 s^2 + 21 s^4 - 14 s^6) C'_{2213} \right] (1 - e^2)^{-7/2}$$

where

$$C'_{2010} = + C'_{20},$$

$$C'_{2011} = + C'_{20} \cos (2\pi - 2\Omega),$$

$$C'_{2111} = + C'_{21} \cos 2\pi + S'_{21} \sin 2\pi,$$

$$C'_{2112} = + C'_{21} \cos (2\pi - 4\Omega) - S'_{21} \sin (2\pi - 4\Omega),$$

$$C'_{2113} = + C'_{21} \cos 2\Omega + S'_{21} \sin 2\Omega,$$

$$C'_{2211} = - C'_{22} \cos (2\pi - \Omega) + S'_{22} \sin (2\pi - \Omega),$$

$$C'_{2212} = + C'_{22} \cos (2\pi - 3\Omega) + S'_{22} \sin (2\pi - 3\Omega),$$

$$C'_{2213} = + C'_{22} \cos \Omega - S'_{22} \sin \Omega,$$

and

$$[W'_{30}] = \Big[+ \frac{3}{8} c^2 (1 - 10 s^2 + 15 s^4) C'_{3101}$$

$$- \frac{3}{8} s^2 (6 - 20 s^2 + 15 s^4) C'_{3102}$$

$$+ \frac{3}{2} s c (1 - 5 s^2 + 5 s^4) C'_{3201}$$

$$+ \frac{15}{8} s^2 c^4 C'_{3301}$$

$$+ \frac{15}{8} s^4 c^2 C'_{3302}$$

$$+ \frac{15}{4} s c^3 (1 - 3 s^2) C'_{3401}$$

$$+ \frac{15}{4} s^3 c (2 - 3 s^2) C'_{3402} \Big] e (1 - e^2)^{-5/2},$$

where

$$C'_{3101} = C'_{31} \cos \pi + S'_{31} \sin \pi,$$

$$C'_{3102} = C'_{31} \cos (\pi - 2\Omega) - S'_{31} \sin (\pi - 2\Omega),$$

$$C'_{3201} = S'_{32} \sin (\pi - \Omega),$$

$$C'_{3301} = C'_{33} \cos (\pi + 2\Omega) + S'_{33} \sin (\pi + 2\Omega),$$

$$C'_{3302} = C'_{33} \cos (\pi - 4\Omega) - S'_{33} \sin (\pi - 4\Omega),$$

$$C'_{3401} = C'_{34} \cos (\pi + \Omega) - S'_{34} \sin (\pi + \Omega),$$

$$C'_{3402} = C'_{34} \cos (\pi - 3\Omega) + S'_{34} \sin (\pi - 3\Omega).$$

$[W'_{21}]$ and $[W'_{30}]$ are trigonometric series in the arguments of the lunar theory and in π and Ω. The presence of π can be a source of resonances at the orbital inclinations previously mentioned.

These resonances are of the same type as in the direct lunar effects and both effects cannot be separated from each other by means of harmomic analysis.

We propose that in future planning the close neighborhood of the resonance zones be avoided.

THE MAIN PROBLEM

Making use of Lagrange equations:

$$\frac{d\,\delta i}{dt} = -\frac{1}{n a^2 \sqrt{1 - e^2} \sin i} \frac{\partial V}{\partial \Omega},$$

$$\frac{d\,\delta\Omega}{dt} = +\frac{1}{n a^2 \sqrt{1 - e^2} \sin i} \frac{\partial V}{\partial i},$$

$$\frac{d\,\delta\pi}{dt} = + \frac{1 - \cos i}{n a^2 \sqrt{1 - e^2} \sin i} \frac{\partial V}{\partial i} + \frac{\sqrt{1 - e^2}}{n a^2 e} \frac{\partial V}{\partial e} ,$$

$$\frac{d\,\delta M}{dt} = - \frac{1 - e^2}{n a^2 e} \frac{\partial V}{\partial e} - \frac{2}{n a^2} \left(a \frac{\partial V}{\partial a} \right)$$

and setting

$$V = n^2 a^2 \frac{m'}{M} \alpha^2 \alpha'^3 k_2 \, [W'_{20}]$$

$$= n^2 a^2 \frac{m'}{M} \alpha^2 \alpha'^3 k_2 \times$$

$$\left[+ \frac{1}{4} \left(1 - \frac{55}{42} \varepsilon^2\right) \left(1 - \frac{3}{2} \sin^2 i\right) C'_{200} \right. \tag{3}$$

$$+ \frac{3}{8} \left(1 - \frac{5}{14} \varepsilon^2\right) \sin^2 i \, C'_{210}$$

$$\left. + \frac{3}{2} \left(1 - \frac{15}{14} \varepsilon^2\right) \sin i \cos i \, C'_{220} \right],$$

we obtain

$$\frac{d\,\delta i}{dt} = - \frac{n}{(1 - e^2)^2} \cdot \frac{m'}{M} \alpha^2 \alpha'^3 k_2 \times$$

$$\left[+ \frac{3}{8} \left(1 - \frac{5}{14} \varepsilon^2\right) \sin i \cdot \frac{\partial C'_{210}}{\partial \Omega} \right. \tag{3^{I}}$$

$$\left. + \frac{3}{2} \left(1 - \frac{15}{14} \varepsilon^2\right) \cos i \cdot \frac{\partial C'_{220}}{\partial \Omega} \right],$$

$$\frac{d\,\delta\Omega}{dt} = + \frac{n}{(1-e^2)^2} \cdot \frac{m'}{M} \alpha^2 \alpha'^3 k_2 \times$$

$$\left[-\frac{3}{4} \left(1 - \frac{55}{42}\varepsilon^2\right) \cos i \cdot C'_{200} \right.$$

$$+ \frac{3}{4} \left(1 - \frac{5}{14}\varepsilon^2\right) \cos i \cdot C'_{210} \tag{3 II}$$

$$\left. + \frac{3}{2} \left(1 - \frac{15}{14}\varepsilon^2\right) \frac{\cos 2i}{\sin i} C'_{220} \right],$$

$$\frac{d\,\delta\pi}{dt} = + \frac{n}{(1-e^2)^2} \cdot \frac{m'}{M} \alpha^2 \alpha'^3 k_2 \times$$

$$\left\{ + \frac{3}{8} \left(1 - \frac{55}{42}\varepsilon^2\right) (-1 - 2\cos i + 5\cos^2 i)\, C'_{200} \right.$$

$$+ \frac{3}{8} \left(1 - \frac{5}{14}\varepsilon^2\right) (1 - \cos i)(3 + 5\cos i)\, C'_{210} \tag{3 III}$$

$$\left. + \frac{3}{2} \left(1 - \frac{15}{14}\varepsilon^2\right) (-1 + 3\cos i + 5\cos^2 i)\, \mathrm{tg}\,\frac{i}{2}\, C'_{220} \right\},$$

$$\frac{d\,\delta M}{dt} = + \frac{n}{(1-e^2)^{3/2}} \cdot \frac{m'}{M} \alpha^2 \alpha'^3 k_2 \times$$

$$\left\{ + \frac{3}{4} \left(1 - \frac{55}{42}\varepsilon^2\right)\left(1 - \frac{3}{2}\sin^2 i\right) C'_{200} \right.$$

$$+ \frac{9}{8} \left(1 - \frac{5}{14}\varepsilon^2\right) \sin^2 i\, C'_{210} \tag{3 IV}$$

$$\left. + \frac{9}{2} \left(1 - \frac{15}{14}\varepsilon^2\right) \sin i \cos i\, C'_{220} \right\}$$

It is of interest to know the variation of the vector

$$\bar{c} = \sqrt{1 - e^2}\,\bar{R},$$

and of the Laplace vector

$$\bar{e} = e\,\bar{P},$$

where $\bar{R}$ is the unit vector normal to the osculating orbital plane, $\bar{P}$ the unit vector along the line of apsides directed toward the perigee, and $\bar{Q} = \bar{R} \times \bar{P}$ completes the right-handed system.

The differential equations of Milankovic (1939) for the variation of vectorial elements in our case become:

$$\frac{d\,\delta\bar{c}}{dt} = \frac{1}{n\,a^2}\left(\bar{c} \times \frac{\partial V}{\partial \bar{c}} + \bar{e} \times \frac{\partial V}{\partial \bar{e}}\right),$$

$$\frac{d\,\delta\bar{e}}{dt} = \frac{1}{n\,a^2}\left(\bar{e} \times \frac{\partial V}{\partial \bar{c}} + \frac{1 - e^2}{e^2}\,\bar{c} \times \frac{\partial V}{\partial \bar{e}}\right).$$

Taking into account that in the main problem

$$\delta e = 0, \quad \frac{\partial V}{\partial \bar{e}} = 0$$

the last system becomes:

$$\frac{d\,\delta\bar{R}}{dt} = \frac{1}{n\,a^2\sqrt{1 - e^2}}\,\bar{c} \times \frac{\partial V}{\partial \bar{c}} \tag{4}$$

$$\frac{d\,\delta\bar{P}}{dt} = \frac{1}{n\,a^2}\,\bar{P} \times \frac{\partial V}{\partial \bar{c}}\,. \tag{5}$$

The mean values of the elements can be substituted into the right hand sides of these equations. Taking into account

$$u_1 = + \sin i \sin \Omega$$

$$u_2 = - \sin i \cos \Omega$$

$$u_3 = + \cos i$$

$$u_1^2 + u_2^2 + u_3^2 = 1$$

we obtain from (3):

$$V = \frac{n^2 a^2}{c^3} \cdot \frac{m'}{M} \alpha^2 \alpha'^3 k_2 \times$$

$$\left\{ + \frac{1}{8} \left(1 - \frac{55}{42} \varepsilon^2\right) (u_3^2 - u_1^2 - u_2^2) \right. \tag{6}$$

$$+ \frac{3}{8} \left(1 - \frac{5}{14} \varepsilon^2\right) \left[C'_{21} (u_2^2 - u_1^2) - 2 S'_{21} u_1 u_2\right]$$

$$\left. - \frac{3}{2} \left(1 - \frac{15}{14} \varepsilon^2\right) (C'_{22} u_2 + S'_{22} u_1) u_3 \right\}.$$

Making use of

$$\frac{\partial c}{\partial \bar{c}} = \bar{R}, \quad \frac{\partial \bar{R}}{\partial \bar{c}} = \frac{1}{c} (I - \bar{R}\bar{R})$$

where I is the idemfactor, we have

$$\frac{\partial V}{\partial \bar{c}} = \frac{1}{c} \frac{\partial V}{\partial \bar{R}} \cdot (I - \bar{R}\bar{R}) - \frac{3}{c} \bar{R} V$$

and, because

$$\frac{\partial V}{\partial \bar{R}} \cdot \bar{R} = +2V,$$

we have finally:

$$\frac{\partial V}{\partial \bar{c}} = \frac{1}{c}\left(\frac{\partial V}{\partial \bar{R}} - 5\bar{R}\,V\right).$$

Inserting this decomposition into (4) and (5), we obtain

$$\frac{d\,\delta\bar{R}}{dt} = \frac{1}{n\,a^2\sqrt{1-e^2}}\left(\bar{R}\times\frac{\partial}{\partial\bar{R}}\right)V, \tag{7}$$

$$\frac{d\,\delta\bar{P}}{dt} = \frac{1}{n\,a^2\sqrt{1-e^2}}\left[\left(\bar{P}\times\frac{\partial}{\partial\bar{R}}\right)V + 5\bar{Q}\,V\right]. \tag{8}$$

Substituting into (7)

$$\bar{R}\times\frac{\partial}{\partial\bar{R}} = \begin{pmatrix} u_2\ \ \partial/\partial u_3 - u_3\ \ \partial/\partial u_2 \\ u_3\ \ \partial/\partial u_1 - u_1\ \ \partial/\partial u_3 \\ u_1\ \ \partial/\partial u_2 - u_2\ \ \partial/\partial u_1 \end{pmatrix}$$

and taking (6) into account we deduce:

$$\frac{d\,\delta u_1}{dt} = \frac{n}{(1-e^2)^2}\cdot\frac{m'}{M}\,\alpha^2\,\alpha'^3\,k_2\,\times$$

$$\left\{+\frac{3}{4}\left(1-\frac{55}{42}\varepsilon^2\right)C'_{200}\,u_2\,u_3\right.$$

$$+\frac{3}{4}\left(1-\frac{5}{14}\varepsilon^2\right)(-C'_{21}\,u_2 + S'_{21}\,u_1)\,u_3$$

$$\left.+\frac{3}{2}\left(1-\frac{15}{14}\varepsilon^2\right)\left[+C'_{22}\,(u_3^2-u_2^2) - S'_{22}\,u_1\,u_2\right]\right\},$$

$$\frac{d\,\delta u_2}{dt} = \frac{n}{(1-e^2)^2} \cdot \frac{m'}{M} \alpha^2 \alpha'^3 k_2 \times$$

$$\left\{ -\frac{3}{4}\left(1 - \frac{55}{42}\varepsilon^2\right) u_1 u_3 \right.$$

$$-\frac{3}{4}\left(1 - \frac{5}{14}\varepsilon^2\right) (+C'_{21} u_1 + S'_{21} u_2) u_3$$

$$\left. +\frac{3}{2}\left(1 - \frac{15}{14}\varepsilon^2\right) [+C'_{22} u_1 u_2 + S'_{22} (u_1^2 - u_3^2)] \right\},$$

$$\frac{d\,\delta u_3}{dt} = \frac{n}{(1-e^2)^2} \cdot \frac{m'}{M} \alpha^2 \alpha'^3 k_2 \times$$

$$\left\{ +\frac{3}{4}\left(1 - \frac{5}{14}\varepsilon^2\right) [2C'_{21} u_1 u_2 + S'_{21} (u_2^2 - u_1^2)] \right.$$

$$\left. -\frac{3}{2}\left(1 - \frac{15}{14}\varepsilon^2\right) (C'_{22} u_1 - S'_{22} u_2) u_3 \right\},$$

or, in a form more convenient for integration,

$$\frac{d\,\delta u_1}{dt} = +\frac{n}{(1-e^2)^2} \cdot \frac{m'}{M} \alpha^2 \alpha'^3 k_2 \times$$

$$\left\{ -\frac{3}{4}\left(1 - \frac{55}{42}\varepsilon^2\right) C'_{200} \cos\Omega \sin i \cos i \right.$$

$$+\frac{3}{4}\left(1 - \frac{5}{14}\varepsilon^2\right) \sin i \cos i \,(-C'_{21} \cos\Omega + S'_{21} \sin\Omega)$$

$$+\frac{3}{2}\left(1 - \frac{15}{14}\varepsilon^2\right) \left[+\left(1 - \frac{3}{2}\sin^2 i\right) C'_{22} \right.$$

$$\left.\left. +\frac{1}{2}\sin^2 i \,(-C'_{22} \cos 2\Omega + S'_{22} \sin 2\Omega) \right] \right\},$$

$$\frac{d\,\delta u_2}{dt} = + \frac{n}{(1-e^2)^2} \cdot \frac{m'}{M}\, \alpha^2\, \alpha'^3\, k_2 \times$$

$$\left\{ -\frac{3}{4}\left(1 - \frac{55}{42}\varepsilon^2\right) \sin i \cos i\; C'_{200} \sin \Omega \right.$$

$$+\frac{3}{4}\left(1 - \frac{5}{14}\varepsilon^2\right) \sin i \cos i\; (-C'_{21} \sin \Omega + S'_{21} \cos \Omega)$$

$$-\frac{3}{2}\left(1 - \frac{15}{14}\varepsilon^2\right) \left[+\frac{1}{2} \sin^2 i\; (+C'_{22} \sin 2\Omega + S'_{22} \cos 2\Omega) \right.$$

$$\left.\left. + S'_{22}\left(1 - \frac{3}{2}\sin^2 i\right)\right]\right\},$$

$$\frac{d\,\delta u_3}{dt} = + \frac{n}{(1-e^2)^2} \cdot \frac{m'}{M}\, \alpha^2\, \alpha'^3\, k_2 \times$$

$$\left\{ +\frac{3}{4}\left(1 - \frac{5}{14}\varepsilon^2\right) \sin^2 i\; (-C'_{21} \sin 2\Omega + S'_{21} \cos 2\Omega) \right.$$

$$\left. -\frac{3}{2}\left(1 - \frac{15}{14}\varepsilon^2\right) \sin i \cos i\; (+C'_{22} \sin \Omega + S'_{22} \cos \Omega) \right\}.$$

Tables I–IV display the terms of the Fourier expansions of the tidal potential, eq. (3), whose coefficients exceed 10^{-4}. The effect of the time dependence of the obliquity, which must be taken into account for accurate long term prediction, manifests itself in the sixth decimal digit of the expansion coefficients.

To obtain the total effect of the earth tides on an artificial satellite the differential equations of the Main Problem (3^{I}) - (3^{IV}) must be supplemented by the indirect effect of the tides in the oblateness perturbations (Kozai, 1965)

TABLE I. Development of C'_{200}

Coef. of Cosine × 10^4	ℓ	ℓ'	F	D	Γ	Ω
7570	0	0	0	0	0	0
205	0	0	0	2	0	0
-7	1	0	-2	0	0	0
15	0	1	-1	3	1	0
-983	0	1	-1	1	1	0
-9	1	1	1	-1	1	0
-17	1	-1	1	-3	-1	0
17	1	0	2	0	0	0
35	1	-1	-1	-3	-1	0
25	2	1	1	1	1	0
72	0	2	0	4	2	0
10	0	2	-2	2	2	0
85	1	-2	0	-4	-2	0
-66	1	-2	0	-2	-2	0
59	2	2	0	2	2	0
-1	2	-2	0	-4	-2	0
-7	2	2	0	0	2	0
-1	0	2	0	3	2	0
3	0	0	0	4	0	0
1	1	0	-2	-2	0	0
-3	1	-1	-1	1	-1	0
-80	1	-1	1	-1	-1	0
1	0	2	0	6	2	0
2	2	2	0	4	2	0
1	2	-2	0	-6	-2	0
13	0	1	0	-2	0	0
2	1	3	0	0	2	0
-2	1	3	0	2	2	0
1	0	1	0	1	0	0
-2	1	-1	0	-2	0	0
2	1	-1	0	2	0	0
1239	1	0	0	0	0	0
6	1	0	0	-4	0	0
32	1	0	0	2	0	0
1	2	0	2	0	0	0
5	3	2	0	2	2	0
-2	1	-1	1	1	-1	0
10	1	1	0	-2	0	0
1	2	-1	0	0	0	0
11	2	0	0	-2	0	0
4	2	0	0	2	0	0
8	3	0	0	0	0	0

Coef. of Cosine × 10^4	ℓ	ℓ'	F	D	Γ	Ω
-6	0	0	2	-2	0	0
91	0	0	2	0	0	0
-17	0	1	1	-1	1	0
972	0	1	1	1	1	0
-27	1	-1	-1	-1	-1	0
3	1	1	-1	3	1	0
-3	2	1	1	-1	1	0
30	0	1	1	3	1	0
186	1	1	1	1	1	0
2346	0	2	0	2	2	0
2	0	2	0	0	2	0
-17	1	2	0	0	2	0
449	1	2	0	2	2	0
17	1	2	0	4	2	0
-8	0	3	0	2	2	0
7	0	1	0	2	2	0
-7	0	0	0	1	0	0
-1	0	2	0	1	2	0
1	0	0	2	2	0	0
-13	0	1	-1	-1	1	0
-80	1	1	-1	1	1	0
5	1	1	1	3	1	0
2	1	-2	0	-6	-2	0
2	0	3	0	0	2	0
2	0	1	0	4	2	0
2	1	1	0	2	2	0
1	1	-1	0	-4	-2	0
-2	0	1	0	0	0	0
-2	0	1	0	2	0	0
8	1	-1	0	0	0	0
-15	1	1	-1	-1	1	0
-7	2	1	-1	1	1	0
238	1	0	0	-2	0	0
-2	2	-1	-1	-1	-1	0
2	3	1	1	1	1	0
-1	1	0	0	1	0	0
-7	2	-1	1	-1	-1	0
-6	1	1	0	0	0	0
4	2	0	0	-4	0	0
101	2	0	0	0	0	0
-1	2	1	0	0	0	0

TABLE II. Development of C'_{210}

Coef. of Cosine x 10^4	Multiples of ℓ	ℓ'	F	D	Γ	Ω
17	(0	2	0	2	2	2)
-9	(0	2	0	3	2	-2)
7	(0	2	0	0	2	-2)
-4	(0	2	2	0	2	-2)
19	(0	1	0	4	2	-2)
28	(0	1	0	2	2	-2)
-3	(1	-3	0	-4	-2	2)
5	(1	1	0	4	2	-2)
-7	(1	-2	-2	-2	-2	2)
-67	(1	2	0	0	2	-2)
3	(1	2	0	2	2	2)
-257	(1	-2	0	-2	-2	2)
67	(1	2	0	4	2	-2)
15	(1	-2	0	-4	-2	2)
4	(2	1	0	2	2	-2)
-27	(2	2	0	0	2	-2)
230	(2	2	0	2	2	-2)
-3	(2	3	0	2	2	-2)
26	(3	2	0	2	2	-2)
3	(1	-2	2	-2	-2	2)
11	(0	0	0	2	0	2)
5	(0	0	2	0	0	2)
12	(1	0	0	-2	0	2)
64	(1	0	0	0	0	2)
2	(1	0	0	2	0	2)
5	(2	0	0	0	0	2)
9	(0	1	1	-1	1	-2)
-339	(0	1	1	1	1	-2)
3	(1	1	1	-1	1	-2)
-65	(1	1	1	1	1	-2)
-15	(0	1	-1	1	1	2)
-1	(1	1	-1	1	1	2)
6	(1	-1	1	-3	-1	2)
-1	(1	-1	1	-1	-1	-2)
-12	(1	-1	-1	-3	-1	2)
-9	(2	1	1	1	1	-2)
5	(1	1	-1	-1	1	-2)
9081	(0	2	0	2	2	-2)
278	(0	2	0	4	2	-2)
7	(0	2	0	6	2	-2)
37	(0	2	-2	2	2	-2)
-31	(0	3	0	2	2	-2)
-4	(0	3	0	4	2	-2)
16	(1	1	0	2	2	-2)
4	(1	2	-2	2	2	-2)
13	(1	-2	0	-6	-2	2)
330	(1	-2	0	-4	-2	2)
1739	(1	2	0	2	2	-2)
-2	(1	2	0	3	2	-2)
-3	(1	3	0	0	2	-2)
-15	(1	3	0	2	2	-2)
8	(2	-2	0	-6	-2	2)
-5	(2	-2	0	-4	-2	2)
11	(2	2	0	4	2	-2)
-6	(3	2	0	0	2	-2)
3	(4	2	0	2	2	-2)
786	(0	0	0	0	0	2)
11	(0	0	0	2	0	-2)
5	(0	0	2	0	0	-2)
12	(1	0	0	-2	0	-2)
64	(1	0	0	0	0	-2)
2	(1	0	0	2	0	-2)
5	(2	0	0	0	0	-2)
15	(0	1	1	1	1	2)
9	(1	-1	-1	-1	-1	2)
3	(1	1	1	1	1	2)
-5	(0	1	-1	3	1	-2)
342	(0	1	-1	1	1	-2)
28	(1	1	-1	1	1	-2)
28	(1	-1	1	-1	-1	2)
-10	(0	1	1	3	1	-2)
-3	(1	1	1	3	1	-2)
5	(0	1	-1	-1	1	-2)
2	(2	-1	1	-1	-1	2)

TABLE III. Development of C'_{220}

Coef. of Cosine $\times 10^4$	Multiples of l	l'	F	D	Γ	Ω
-1	(0	1	1	-1	1	1)
52	(0	1	1	1	1	1)
10	(1	-1	-1	-1	-1	1)
3	(1	1	1	-1	1	-1)
-68	(1	1	1	1	1	-1)
-52	(0	1	-1	1	1	1)
-4	(1	1	-1	1	1	1)
6	(1	-1	1	-3	-1	1)
-4	(1	-1	1	-1	-1	-1)
-11	(0	1	1	3	1	-1)
2	(1	-1	-1	-3	-1	-1)
1	(2	1	1	1	1	1)
5	(0	1	-1	-1	1	-1)
2	(2	-1	1	-1	-1	1)
-1884	(0	2	0	2	2	-1)
-58	(0	2	0	4	2	-1)
-4	(0	1	0	4	2	-1)
-6	(0	1	0	2	2	-1)
-3	(1	-2	0	-6	-2	1)
-69	(1	-2	0	-4	-2	1)
16	(1	2	0	2	2	1)
53	(1	-2	0	-2	-2	1)
-14	(1	2	0	4	2	-1)
3	(1	3	0	2	2	-1)
2	(2	2	0	2	2	1)
-2	(2	2	0	4	2	-1)
1811	(0	0	0	0	0	1)
25	(0	0	0	2	0	-1)
11	(0	0	2	0	0	-1)
2	(0	1	0	-2	0	-1)
28	(1	0	0	-2	0	-1)
148	(1	0	0	0	0	-1)
4	(1	0	0	2	0	-1)
1	(1	1	0	-2	0	-1)
1	(2	0	0	-2	0	-1)
12	(2	0	0	0	0	-1)
2	(1	0	2	0	0	-1)
1	(2	1	-1	1	1	-1)
1	(0	2	0	3	2	-1)

Coef. of Cosine $\times 10^4$	Multiples of l	l'	F	D	Γ	Ω
10	(0	1	1	-1	1	-1)
-355	(0	1	1	1	1	-1)
-1	(1	-1	-1	-1	-1	-1)
10	(1	1	1	1	1	1)
-5	(0	1	-1	3	1	-1)
359	(0	1	-1	1	1	-1)
30	(1	1	-1	1	1	-1)
29	(1	-1	1	-1	-1	1)
2	(0	1	1	3	1	1)
-13	(1	-1	-1	-3	-1	1)
-3	(1	1	1	3	1	-1)
-9	(2	1	1	1	1	-1)
6	(1	1	-1	-1	1	-1)
81	(0	2	0	2	2	1)
2	(0	2	0	4	2	1)
-8	(0	2	-2	2	2	-1)
6	(0	3	0	2	2	-1)
-3	(1	1	0	2	2	-1)
14	(1	2	0	0	2	-1)
3	(1	-2	0	-4	-2	-1)
-361	(1	2	0	2	2	-1)
-2	(1	-2	0	-2	-2	-1)
-3	(1	-1	0	-4	-2	1)
6	(2	2	0	0	2	-1)
-48	(2	2	0	2	2	-1)
-5	(3	2	0	2	2	-1)
25	(0	0	0	2	0	1)
11	(0	0	2	0	0	1)
2	(0	1	0	-2	0	1)
28	(1	0	0	-2	0	1)
148	(1	0	0	0	0	1)
4	(1	0	0	2	0	1)
1	(1	1	0	-2	0	1)
1	(2	0	0	-2	0	1)
12	(2	0	0	0	0	1)
2	(1	0	2	0	0	1)
-1	(2	1	-1	1	1	1)
-1	(0	2	0	3	2	1)

TABLE IV.

Development of C''_{200}

Coef. of Cosine $\times 10^4$	l	l'	F	D	Γ	Ω
7629	0	0	0	0	0	0
2372	0	2	0	0	2	0
383	0	1	0	0	0	0
10	0	2	0	0	0	0
-20	0	1	0	0	2	0
139	0	3	0	0	2	0
3	0	4	0	0	2	0

Development of C''_{210}

Coef. of Cosine $\times 10^4$	l	l'	F	D	Γ	Ω
-77	0	1	0	0	2	-2
20	0	1	0	0	0	2
17	0	2	0	0	2	2
538	0	3	0	0	2	-2
792	0	0	0	0	0	2
20	0	1	0	0	0	-2
9185	0	2	0	0	2	-2
22	0	4	0	0	2	-2

Development of C''_{220}

Coef. of Cosine $\times 10^4$	l	l'	F	D	Γ	Ω
16	0	1	0	0	2	-1
46	0	1	0	0	0	1
1	0	2	0	0	0	1
82	0	2	0	0	2	1
5	0	3	0	0	2	1
-5	0	4	0	0	2	-1
1826	0	0	0	0	0	1
46	0	1	0	0	0	-1
1	0	2	0	0	0	-1
-1906	0	2	0	0	2	-1
-112	0	3	0	0	2	-1

$$\frac{d\,\delta\Omega}{dt} = -\dot{\Omega}\tan i\,(\delta i),$$

$$\frac{d\,\delta\pi}{dt} = (5\sin i - \tan i)\,\dot{\Omega}(\delta i),$$

$$\frac{d\,\delta M}{dt} = 3\dot{\Omega}\sin i(\delta i),$$

where δi is the tidal perturbation given by equation (3^I) and (Brouwer, 1959)

$$\dot{\Omega} = n\left\{-\frac{3}{2}\frac{J_2 R^2 \cos i}{a^2(1-e^2)^2}\cdot\right.$$

$$+\frac{3 J_2^2 R^4 \cos i}{32 a^4 (1-e^2)^4}\left[4 - 9e^2 + 12\sqrt{1-e^2} - (40 - 5e^2 + 36\sqrt{1-e^2})\cos^2 i\right]$$

$$\left. - \frac{15 J_4 R^4 (2+3e^2)\cos i}{32 a^4 (1-e^2)^4}(3 - 7\cos^2 i)\right\}.$$

We have integrated the resulting set of differential equations with mean elements obtained for the BE-C satellite at the epoch, 1970 June 19.22547, and predictions for the tidal perturbations, neglecting the secular terms, are presented in Figures I-IV. While an accurate determination of tidal parameters will require a longer span of data, comparison with available inclination residuals over a three month interval show k_2 to be in the neighborhood of .31.

CONCLUSION

The theory and the Fourier expansion of the tidal effects in the motion of an artificial satellite presented in this article permit a deeper insight into the nature of the tidal perturbations and allow a judgment about the relative significance of the amplitudes and periods of the terms in the series.

The tidal effects are clearly noticeable in the observations and their comparison with the theory will permit improvement of the "global" Love numbers for the Earth.

In our work we found inspiration in works by Kozai (1965) and Newton (1968) and also in work by D. Smith and R. Kolenkiewicz*, but we changed the method of expansion by resorting to the standard form of the lunar theory. In the present expansion all arguments are nearly linear in time. Thus the integration is easy and prediction for a longer interval of time is possible.

We made use of the existing standard lunar theory because it is, presently, sufficient for our purposes. Any change in the coefficients, however, can be easily incorporated.

The future addition to theory as the accumulation of observations and experience proceed can go along several lines: more Love numbers can be included into the theory and the dependence of Love numbers on the longitude and latitude (Kaula, 1969) can be incorporated.

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*private communication.

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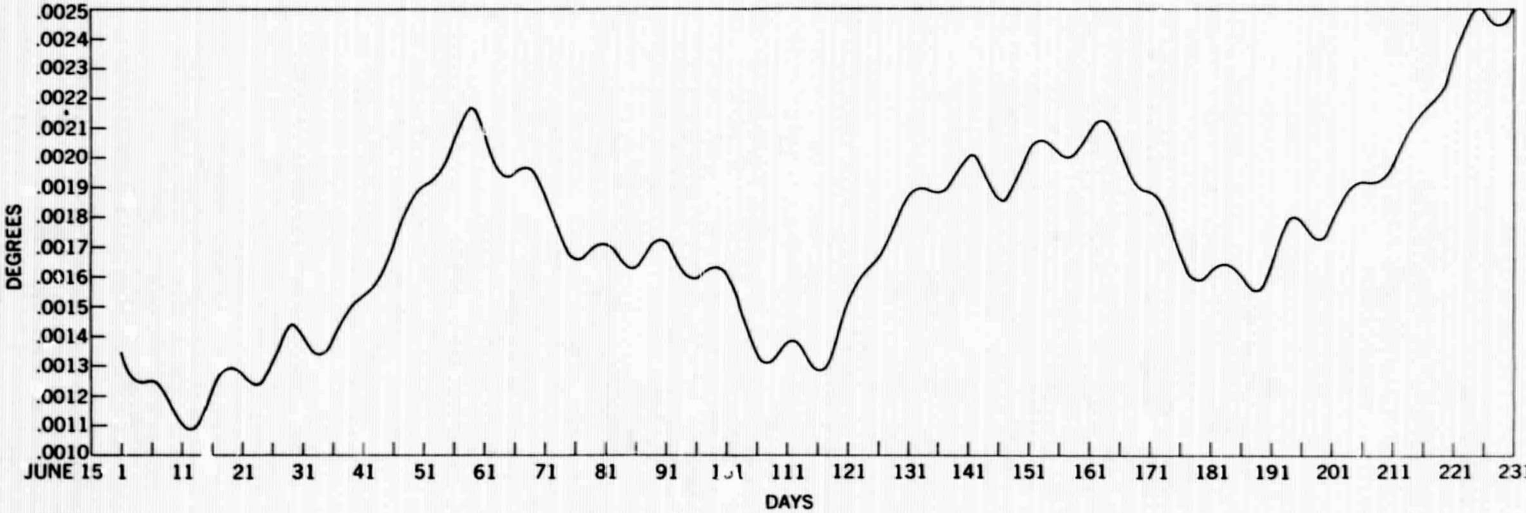

Figure 1. $\delta\Omega$ for K_2 =.3 at the epoch June 19.22547, 1970 for the BE-C satellite.

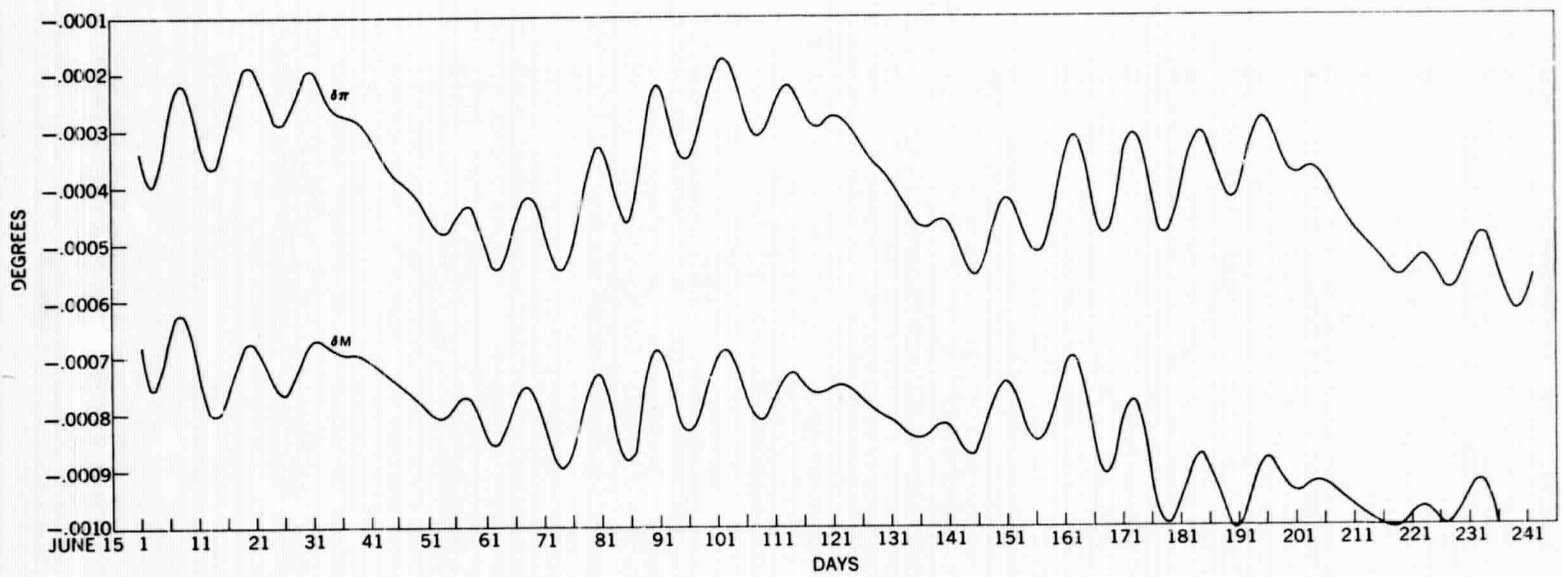

Figure 2. $\delta\pi$ and δM for $K_2 = .3$ at the epoch June 19.22547, 1970 for the BE-C satellite.

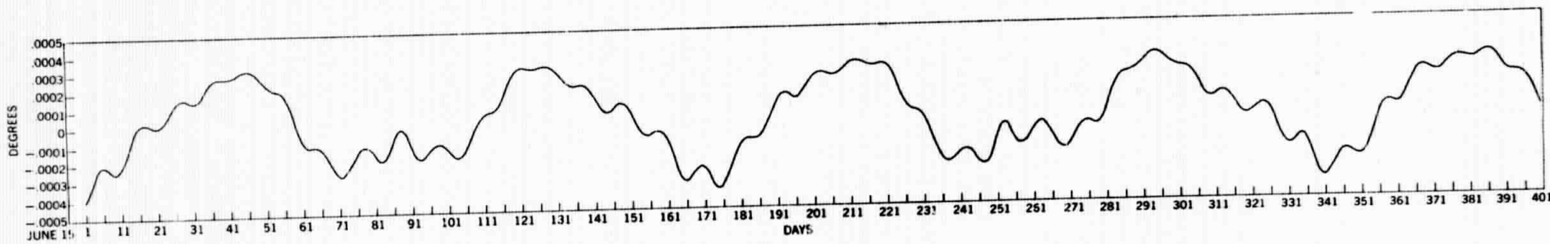

Figure 3. δi for K_2 =.3 at the epoch June 19.22547, 1970 for the BE-C satellite.

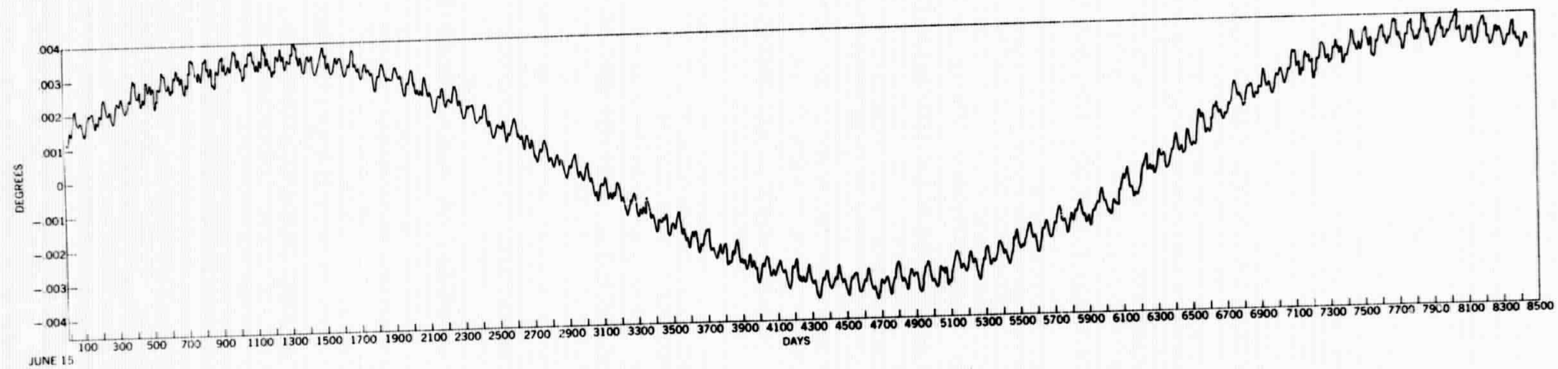

Figure 4. $\delta\Omega$ for K_2 =.3 at the epoch June 19.22547, 1970 for the BE-C satellite.